EXAM 1

HONOR CODE PLEDGE:
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not all white noise is normally dist.

1 A) A WHITE NOISE SERIES IS SIMPLY

$$x_t = w_t$$

WHERE $w_t \overset{iid}{\sim} N(\mu, \sigma_w^2)$. IT IS JUST A SERIES OF RANDOM OBSERVATIONS FROM A NORMAL DISTRIBUTION. IT IS STATIONARY, WITH CONSTANT μ AND

$$\gamma_w(h) = \begin{cases} \sigma_w^2 & h=0 \\ 0 & h \neq 0 \end{cases}.$$

c) THEORETICALLY, WE KNOW w_t (MEAN 0, sd 5) IS STATIONARY SINCE

$$\mu_{w_t} = E(w_t) = 0$$

AND

$$\gamma_w(h) = cov[w_t, w_t]$$

$$= \begin{cases} 5 & h=0 \\ 0 & h \neq 0 \end{cases}$$

SINCE μ_{w_t} IS CONSTANT AND γ_w IS A FUNCTION ONLY OF LAG h, w_t IS STATIONARY.

GRAPHICALLY, WE CAN OBSERVE THAT w_t IS STATIONARY SINCE IT APPEARS THAT THE SERIES HAS MEAN 0. FURTHERMORE, LOOKING AT THE PLOT FOR THE SAMPLE ACF SHOWS US THAT $\hat{\rho}(h)$ FOLLOWS $\rho(h)$ (ALTHOUGH $\hat{\rho}(h)$ DEVIATES FROM 0 FOR $h \neq 0$, IT IS STATISTICALLY INSIGNIFICANT).

② b) MOVING AVERAGES ARE USEFUL IN EVALUATING TRENDS OVERTIME, ESPECIALLY FOR DAILY OR MONTHLY DATA.

c) $E(v_t) = E\left(\frac{1}{3}(w_{t-1} + w_t + w_{t+1})\right)$

$= \frac{1}{3}E(w_{t-1}) + \frac{1}{3}E(w_t) + \frac{1}{3}E(w_{t+1})$

$= 0 + 0 + 0$

$= 0$

FIND AUTOCOVARIANCE:

$\underline{h=0}$: $\gamma_v(t,t) = cov\left[\frac{1}{3}(w_{t-1}+w_t+w_{t+1}), \frac{1}{3}(w_{t-1}+w_t+w_{t+1})\right]$

$= \frac{1}{9}cov(w_{t-1}, w_{t-1}) + \frac{1}{9}cov(w_t, w_t) + \frac{1}{9}cov(w_{t+1}, w_{t+1})$

$= \frac{1}{9}\sigma_w^2 + \frac{1}{9}\sigma_w^2 + \frac{1}{9}\sigma_w^2$

$= \frac{1}{3}\sigma_w^2$

WLOG, $\underline{h=\pm 1}$: $\gamma_v(t,t+1) = cov\left[\frac{1}{3}(w_{t-1}+w_t+w_{t+1}), \frac{1}{3}(w_t + w_{t+1} + w_{t+2})\right]$

$= \frac{1}{9}cov(w_t, w_t) + \frac{1}{9}cov(w_{t+1}, w_{t+1})$

$= \frac{2}{9}\sigma_w^2$

WLOG, $\underline{h=\pm 2}$: $\gamma_v(t,t+2) = cov\left[\frac{1}{3}(w_{t-1}+w_t+w_{t+1}), \frac{1}{3}(w_{t+1}, w_{t+2}, w_{t+3})\right]$

$= \frac{1}{9}cov(w_{t+1}, w_{t+1})$

$= \frac{1}{9}\sigma_w^2$

CLEARLY, FOR $|h|>2$, $\gamma_v(h) = 0$. THUS, SINCE μ_{v_t} IS CONSTANT AND γ_v IS A FUNCTION ONLY OF LAG h, v_t IS STATIONARY.

② c) *CONTINUED*

$$\rho_v(h) = \frac{\gamma_v(h)}{\gamma_v(0)}$$

so, $\rho_v(0) = 1$.

$$\rho_v(\pm 1) = \frac{\gamma_v(1)}{\gamma_v(0)}$$

$$= \frac{2/9\,\sigma_w^2}{1/3\,\sigma_w^2} = \frac{2}{9}\sigma_w^2 \cdot 3\sigma_w^2$$

$$= \frac{6}{9}\sigma_w^2 = \frac{2}{3}\sigma_w^2$$

$$\rho_v(\pm 2) = \frac{\gamma_v(2)}{\gamma_v(0)}$$

$$= \frac{1/9\,\sigma_w^2}{1/3\,\sigma_w^2}$$

$$= \frac{1}{3}\sigma_w^2$$

$$\rho_v(h) = 0 \quad \forall |h| > 2$$

IN SUMMARY,

$$\boxed{\rho_v(h) = \begin{cases} 1 & h = 0 \\ 2/3\,\sigma_w^2 & h = \pm 1 \\ 1/3\,\sigma_w^2 & h = \pm 2 \\ 0 & |h| > 2 \end{cases}}$$

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③ $x_t = \delta + x_{t-1} + w_t$

A) PROCEED BY INDUCTION.

BASIS ($t=1$): $x_1 = \delta + x_0 + w_1$
$= \delta + 0 + w_1$

$= t\delta + \sum_{k=1}^{1} w_k$

STATEMENT HOLDS FOR $t=1$.

INDUCTION: SUPPOSE FOR SOME $t \geq 1$ THAT

$$x_t = t\delta + \sum_{k=1}^{t} w_t$$

THEN,

$$x_{t+1} = \delta + \left(t\delta + \sum_{k=1}^{t} w_t\right) + w_{t+1}$$
$$= (t+1)\delta + \sum_{k=1}^{t+1} w_t$$

HENCE, THE STATEMENT HOLDS FOR $t+1$, GIVEN $t \geq 1$.

BY INDUCTION, WE HAVE PROVEN THAT

$$x_t = t\delta + \sum_{k=1}^{t} w_k. \blacksquare$$

B) $\mu_{xt} = E(x_t) = E\left[\delta t + \sum_{k=1}^{t} w_k\right]$
$= E(\delta t) + \sum_{k=1}^{t} E(w_t)$
$= \delta t + 0$
$= \delta t$

$\gamma_x(s,t) = E\left[(x_s - \mu_{xs})(x_t - \mu_{xt})\right]$
$= E\left[\left(\delta s + \sum_{k=1}^{s} w_s\right)\left(\delta t + \sum_{k=1}^{t} w_k\right)\right]$
$= E\left[st\delta + \delta s \sum_{k=1}^{t} w_k + \delta t \sum_{k=1}^{s} w_k + \sum_{k=1}^{s} w_k \sum_{k=1}^{t} w_k\right]$
$= E(st\delta) + \delta s E\left(\sum_{k=1}^{t} w_k\right) + \delta t E\left(\sum_{k=1}^{s} w_k\right) + E\left(\sum_{k=1}^{s} w_k \sum_{k=1}^{t} w_k\right)$
$= st\delta + 0 + 0 + \text{MIN}\{s,t\}\sigma_w^2$
$= st\delta + \text{MIN}\{s,t\}\sigma_w^2$

how ??

③ D) ALTHOUGH $\mu_{x_t} = 0$ FOR $\sigma = 0$, THE AUTOCOVARIANCE FOR ANY σ CANNOT BE EXPRESSED AS A FUNCTION OF LAG. HENCE, x_t IS NOT STATIONARY.

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④ CROSS-COVARIANCE:

$\underline{h=0}$: $\gamma_{xy}(0) = \text{cov}[x_{t+0}, y_t] = \text{cov}[w_t + w_{t-1}, w_t - w_{t-1}]$

$= \text{cov}(w_t, w_t) + -\text{cov}(w_{t-1}, w_{t-1})$

$= \sigma_w^2 - \sigma_w^2$

$= 0$

$\underline{h=1}$: $\gamma_{xy}(1) = \text{cov}[x_{t+1}, y_t] = \text{cov}[w_{t+1} - w_t, w_t - w_{t-1}]$

$= -\text{cov}(w_t)$

$= -\sigma_w^2$

$\underline{h=-1}$: $\gamma_{xy}(-1) = \text{cov}(x_{t-1}, y_t) = \text{cov}[w_{t-1} - w_{t-2}, w_t - w_{t-1}]$

$= -\text{cov}(w_{t-1}, w_{t-1})$

$= -\sigma_w^2$

NOTE THAT $\gamma_{xy}(h) = 0$ FOR ALL $h \neq 0, -1, 1$.

THUS, THE CCF IS:

$$\rho_{xy}(h) = \frac{\gamma_{xy}(h)}{\sqrt{\gamma_x(0)\gamma_y(0)}}$$

SO,

$$\rho_{xy}(0) = \frac{0}{\sqrt{\sigma_w^4}} = 0$$

-1

$$\rho_{xy}(1) = \frac{-\sigma_w^2}{\sigma_w^2} = -1$$

check

$$\rho_{xy}(-1) = \frac{-\sigma_w^2}{\sigma_w^2} = -1$$

IN SUMMARY,

$$\rho_{xy}(h) = \begin{cases} -1 & h = \pm 1 \\ 0 & h \neq \pm 1 \end{cases}$$

SINCE BOTH x_t AND y_t ARE CLEARLY STATIONARY AND THE CCF IS A FUNCTION ONLY OF LAG, x_t AND y_t ARE JOINTLY STATIONARY.

(5) 1. THE PERIODICITY OF THE SOI ACF FUNCTION INDICATES THAT THE ACF IS A FUNCTION OF ONLY LAG. ASSUMING THE MEAN VALUE FUNCTION IS CONSTANT, SOI SEEMS TO BE STATIONARY

2. RECRUITMENT'S ACF APPEARS TO ONLY DEPEND ON LAG, AND IT APPEARS TO BE ROUGHLY 0 FOR $h \geq 2$. ASSUMING THE MEAN VALUE FUNCTION IS CONSTANT, REC APPEARS TO BE STATIONARY AS WELL.

3. WITH THIS CCF, WE WANT TO BE ABLE TO SEE IF THE TWO SERIES ARE JOINTLY STATIONARY. THE CCF IS NOT SYMMETRIC, WHICH IS EXPECTED, AS CCF'S ARE NOT GENERALLY SYMMETRIC (UNLIKE ACF'S). FURTHERMORE, WE OBSERVE PERIODICITY IN THE CCF, INDICATING IT'S A FUNCTION ONLY OF LAG. THIS, IN ADDITION TO SOI AND REC APPEARING TO BE STATIONARY, DEMONSTRATES THAT THEY'RE LIKELY JOINTLY STATIONARY

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(6) B) $var(x_t) = \frac{\sigma_w^2}{1-\phi^2}$

$$\gamma_x(0) = cov[\phi x_{t-1} + w_t, \phi x_{t-1} + w_t]$$
$$= \phi^2 var[x_{t-1}] + cov(w_t)$$
$$= \phi^2 var[x_{t-1}] + \sigma_w^2$$
$$= \frac{\phi^2 \sigma_w^2}{1-\phi^2} + \frac{\sigma^2(1-\phi^2)}{1-\phi^2}$$
$$= \frac{1}{1-\phi^2}$$

3) $\rho(h) = \dfrac{\gamma_x(h)}{\gamma_x(0)}$

$=$

Thus, $\rho_x(h) = \phi^h$.