Section 2

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Introduction

In this section, we will discuss:

- Reasoning by Representation
- Derivation path of distribution representations
- Poisson Processes

Reasoning by Representation

Some of the important terms and definitons of Chapter 3 is as follow:

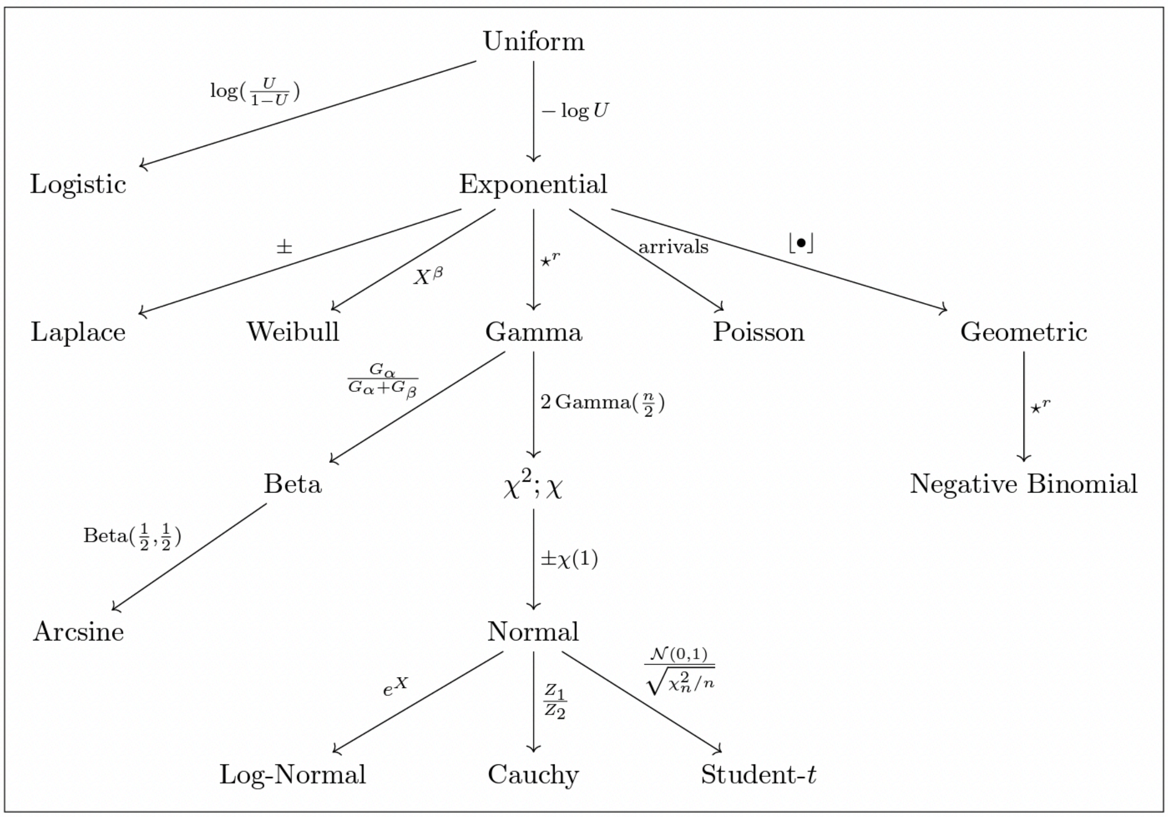
1. **Bernoulli**: A random variable Y has the *Bernoulli distribution* with parameter p, denoted $Y \sim Bern(p)$, if $P(Y=1)=p$ and $P(Y=0)=1-p$
2. **Binomial**: sum of n iid Bernoulli random variables
3. **Uniform**: $U=\sum_{i=1}^{\infty} \frac{B_i}{2^i} \sim$ Unif where B_i iid $Bern(0.5)$.
4. **PIT sampling**: Let F be *any* CDF with quantile function F^{-1}, let $U \sim Unif$ then $F^{-1}(U) \sim F$.
5. **PIT pivoting**: Let F be a continuous CDF, and $Y \sim F$, then $F(Y) \sim Unif$.
6. **Exponential**: Let $U \sim$ Unif. Then $X=-\log U \sim$ Expo.
7. **Gamma**: For a positive integer n, $G_n = X_1 + X_2 + \cdots + X_n \sim Gamma(n)$, where $X_i \overset{i.i.d}{\sim}$ Expo.
8. **Laplace** : $L \sim$ Laplace if $L \sim SX$ where S is random sign and $X \sim Expo$.
9. **Weibull**: $W = X^{\beta} \sim$ Wei(β) where $X \sim Expo$ and $\beta > 0$.
10. **Beta**: $B=\frac{G_a}{G_a+G_b} \sim Beta(a,b)$ where $ G_a Gamma(a)$, $G_b \sim Gamma(b)$ and $G_a \perp\!\!\!\perp G_b$; note that $U^{1/\alpha} \sim Beta(\alpha, 1)$.
11. **Beta-Gamma**: Let $G_a \sim Gamma(a) \perp\!\!\!\perp G_b \sim Gamma(b)$, $B=\frac{G_a}{G_a+G_b}$ and $T = G_a + G_b \sim Gamma(a+b)$. Then $T \perp\!\!\!\perp B$.
12. **Chi-squared**: $G \sim \chi_n^2$ if $G \sim 2Gamma\left(\frac{n}{2}\right)$.

13. **Normal**: $Z = SX \sim \mathcal{N}(0,1)$ where S is random sign and $X \sim \chi_1$.
14. **Box-Muller**: $U_1, U_2 \overset{i.i.d}{\sim} \text{Unif.}$ Then $\sqrt{-2\ln U_2}\cos(2\pi U_1), \sqrt{-2\ln U_2}\sin(2\pi U_1) \overset{i.i.d}{\sim} \mathcal{N}(0,1)$
15. **Student t-distribution**: $T = \frac{Z}{\sqrt{V_n/n}} \sim t_n$ where $Z \sim N(0,1) \perp\!\!\!\perp V_n \sim \chi^2_n$.
16. **Cauchy**: $C = \frac{Z_1}{Z_2} \sim \text{Cauchy} \sim t_1$ where $Z_1, Z_2 \sim_{i.i.d} \mathcal{N}(0,1)$.

For ease of referencing, Eric Zhang have summarized how to move from one distribution to next in a figure, in which we will discuss in the next section.

Derivation path of distribution representations:

Below is a pitcure showing derivation path of distribution representations, kindly created by Eric Zhang:



Pencil 3.3.2

Show that if $U \sim Unif$, then $1 - U \sim Unif$. Also, show that $2U - 1 \sim SU$ for S a random sign independent of U (so $2U - 1$ is symmetric about 0, while U is symmetric about 1/2; see Section 3.10 for more information.)

Solution

If $U \sim Unif$, $1 - U = 1 + (0 - 1)U$ is Uniform between 1 and 0 so $1 - U \sim Unif$ as well.
$2U - 1 = -1 + (1 - (-1))U$ is Uniform between -1 and 1, and SU is also Uniform between -1 and 1 by definition, so $2U - 1 \sim SU$

Pencil 3.4.12

Let Y_1 and Y_2 be r.v.s (possibly deifnied on different Ω's) with CDFs F_1 and F_2 respectively. A commonly used partial order on distributions (and thus on r.v.s), *stochastic domination* is defined by the relation $Y_1 \preceq Y_2$ iff $F_1(y) \geq F_2(y)$ for all $y \in \mathbb{R}$.

Find an example of Y_1 and Y_2 on the same space with $Y_1 \preceq Y_2$ but $P(Y_1 > Y_2) \geq 0.95$

Solution

As in the previous clock example, we can let Y_1 and Y_2 be discrete Uniform on a clock with $\lfloor 1/0.95 \rfloor$ numbers, so that $Y_1 \sim Y_2$ when $Y_1 \preceq Y_2$. Then, $P(Y_1 > Y_2) \geq P(Y_1 = Y_2 + 1) \geq 0.95$

Pencil 3.5.10

Let $W_1 \sim Wei(\beta_1), W_2 \sim Wei(\beta_2)$. Show that $(W_1 | W_1 \geq 1) \preceq (W_2 | W_2 \geq 1)$ iff $\beta_1 \leq \beta_2$.

Solution

$(W_1 | W_1 \geq 1) \preceq (W_2 | W_2 \geq 1)$ implies that $F_1(w) \geq F_2(w)$, which by definition implies that $1 - exp(-w^{1/\beta_1}) \geq 1 - exp(-w^{1/\beta_2})$. This further implies that $exp(-w^{1/\beta_2}) \geq exp(-w^{1/\beta_1})$, which implies that $-w^{1/\beta_2} \geq -w^{1/\beta_1}$, whcih is similar to $w^{1/\beta_1} \geq w^{1/\beta_2}$, which further implies than when $w \geq 1$, $\frac{1}{\beta_1} \geq \frac{1}{\beta_2}$, thus we show that $\beta_1 \leq \beta_2$.