Elliptic Integral

$$E(K,\phi) = \int_0^{\phi} \sqrt{1 - K^3 \sin^2 \phi}\; d\phi$$ → Elliptic integral of II kind

→ $K^2 < 1$

$$F(K,\phi) = \int_0^{1} \frac{1}{\sqrt{1-K^2\sin^2\phi}}\; d\phi$$ → Elliptic integral of I st kind

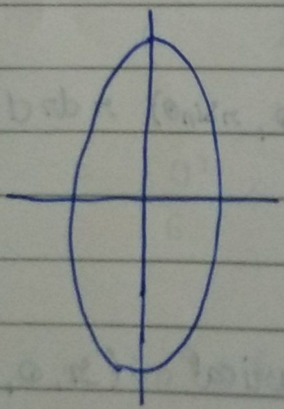
→ K is modulus

ϕ is amplitude

Q. Find the arc length of the ellipse :

$$x = a\cos\phi \quad , \quad y = b\sin\phi \qquad a < b$$

$$a^2 = b^2(1-e^2) \qquad e^2 < 1$$



$$AP = \int_0^{\phi} \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dx}\right)^2}\; d\phi$$

$$= \int_0^{\phi} \sqrt{(a\sin\phi)^2 + (b\cos\phi)^2}\; d\phi$$

$$= \int_0^{\phi} \sqrt{b^2(1-e^2)\sin^2\phi + b^2(1-\sin^2\phi)}\; d\phi$$

$$= b\int_0^{\phi} \sqrt{1 - e^2\sin^2\phi}\; d\phi$$

$$= bE(e, \phi)$$

Perimeter of Ellipse

$$4b \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2\phi}\, d\phi$$

↳ COMPLETE ELLIPTIC INTEGRAL

$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2\phi}}\, d\phi$$ [complete elliptic integral of I^{st} kind]

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\phi}$$ [complete elliptic integral of II^{nd} kind]

$$= b \int_0^{\phi} \sqrt{1 - e^2 \sin^2\phi}\, d\phi$$

$$= bE(e, \phi)$$

Elliptic Integral

$\sin\phi = x$ in eqn ① and ②

$$\phi = \sin^{-1} x$$

$$d\phi = \frac{1}{\sqrt{1-x^2}}\, dx$$

$$F(k, \sin^{-1} x) = \int_0^{\sin^{-1} x} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \qquad k^2 < 1$$

$$E(k, \sin^{-1} x) = \int_0^{\sin^{-1} x} \sqrt{\frac{1-kx^2}{1-x^2}}\, dx \qquad k^2 < 1$$

Jacobian

- ELLIPTIC INTEGRAL

Ex. 1 : $\int_0^{\pi/2} \sqrt{\cos x}\, dx$

Let $\cos x = \cos^2 \phi$

$$x = \cos^{-1} \cos^2 \phi$$

$$dx = \pm \frac{2\cos\phi \sin\phi}{\sqrt{1-\cos^4\phi}}\, d\phi$$

$$= \frac{2\cos\phi \sin\phi}{\sqrt{1-\cos^2\phi}\sqrt{1+\cos^2\phi}}\, d\phi$$

$$= \int_0^{\pi/2} \frac{(1+\cos^2\phi)^{-1}}{\sqrt{1+\cos^2\phi}}\, d\phi$$

JACOBIAN

$$= 2\left[\int_0^{\pi/2} \frac{1+\cos^2\phi}{\sqrt{1+\cos^2\phi}}\,d\phi - \int_0^{\pi/2} \frac{1}{\sqrt{1+\cos^2\phi}}\,d\phi\right]$$

$$= 2\int_0^{\pi/2} \sqrt{1+\cos^2\phi}\,d\phi \quad - \quad 2\int_0^{\pi/2} \frac{1}{\sqrt{1+\cos^2\phi}}\,d\phi$$

$$= 2\sqrt{2}\int_0^{\frac{\pi}{2}} \sqrt{1-\frac{1}{2}\cos^2\phi}\,d\phi - \frac{2}{\sqrt{2}}\int_0^{\pi/2} \frac{1}{\sqrt{1-\frac{1}{2}\sin^2\phi}}\,d\phi$$

ANS.

ELLIPTIC $= 2\sqrt{2}\,E\left(\frac{1}{\sqrt{2}}\right) - \sqrt{2}\,K\left(\frac{1}{\sqrt{2}}\right)$

Ex. 2

$$\int_0^{\pi/2} \frac{dx}{\sqrt{2-\cos x}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{2-\left(2\cos^2\frac{x}{2}-1\right)}}$$

$$= \int_0^{\pi/2} \frac{dx}{\sqrt{3-2\cos^2\frac{x}{2}}}$$

$$\frac{x}{2} = \frac{\pi}{2} - \phi$$

$x = 0 \qquad \phi = \frac{\pi}{2}$

$x = \pi/2 \qquad \phi = \pi/4$

$$= -2\int_{\pi/2}^{\pi/4} \frac{d\phi}{\sqrt{1-\frac{2}{3}\sin^2\phi}} = \frac{2}{\sqrt{3}}\int_{\pi/4}^{\pi/2} \frac{d\phi}{\sqrt{1-\frac{2}{3}\sin^2\phi}}$$

$$= \frac{2}{\sqrt{3}}\left[\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-\frac{2}{3}\sin^2\phi}} - \int_0^{\pi/4} \frac{d\phi}{\sqrt{1-\frac{2}{3}\sin^2\phi}}\right]$$

$$= \boxed{\frac{2}{\sqrt{3}} K\left(\sqrt{\frac{2}{3}}\right) - \frac{2}{\sqrt{3}} F\left(\sqrt{\frac{2}{3}}, \frac{\pi}{4}\right)}$$

Ex.

$$\int_0^{\pi/2} \frac{dx}{\sqrt{1+3\sin^2 x}}$$

Let $x = \pi/2 - \phi$

$$\Rightarrow -\int_{\pi/2}^{0} \frac{d\phi}{\sqrt{1+3\underbrace{\cos^2\phi}_{(1-\sin^2\phi)}}}$$

$$= \frac{1}{2} K\left(\frac{\sqrt{3}}{2}\right)$$

Ex.

$$\int_0^{\pi/6} \frac{dx}{\sqrt{\sin x}}$$

$$\sin x = \cos^2\phi$$

$$x = \sin^{-1} \cos^2\phi$$

$$dx = -\frac{2\cos\phi}{\sqrt{1-\cos^4\phi}}$$

$$\sqrt{1-\cos^2\phi}\,\sqrt{1+\cos^2\phi}$$

$$= -\int_{\pi/2}^{\pi/4} \frac{2\cos\phi}{\sqrt{1+\cos^2\phi}\ \cos\phi}\, d\phi$$

$$1-\sin^2\phi$$

Ex. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^4}}$

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2}\,\sqrt{1+x^2}}$$

Put $x = \cos\phi$

Ex

Prove $\displaystyle\int_0^{\pi/2} \frac{dx}{\sqrt{2ax-x^2}\,\sqrt{a^2-x^2}} = \frac{2}{3a} K\left(\frac{1}{3}\right)$

Put $x = \frac{a}{2}(1-\sin\theta)$

SIMPLEX ALGORITHM

Ex 1. Max $z = 29x_1 + 45x_2$

Set, $x_1 + 4x_2 \le 30$

$x_1 + x_2 \le 15$,

$x_1, x_2 \ge 0$

SIMPLEX ALGORITHM

[i] Convert the LPP to a system of linear equation (using slack ~~a~~ variables). Slack variables are used to convert ~~the~~ inequality ~~in~~ into equality

$x_1 + 4x_2 + x_3 = 30$

$x_1 + x_2 + 0x_3 + x_4 = 15$

$x_i \ge 0$

[ii] Set up the initial table (augmented matrix)

B.V.	x_1	x_2	x_3	x_4	b
x_3	1	4	1	0	30
x_4	1	1	0	1	15
z_j	$-c_1$	$-c_2$	$-c_3$	$-c_4$	0
	-29	-45	0	0	0

[iii] Select a pivot element. Look at the last row, from this row chose the -ve number with largest magnitude (excluding bi column). Its column is the pivot column

~~[illegible]~~

4] If all the numbers in this row are zero or positive then optimal condition is satisfied.

5] Select the pivot element in the pivot column. In the 'a' part of this step for each +ve entry a_{ij} in the pivot column compute the Test ratio b_i/a_{ij}

b part of this step, out of these ratios, chose the smallest one corresponding a_{ij} is the pivot element.

5] Use the pivot element to clear the pivot column in normal manner. This gives the next table or the augmented matrix.

6] Repeat steps 2-5 until there are no negative number on the last row

	x_1	x_2	x_3	x_4	b	b_i/a_{ij}
x_3	$1/4$	1	$1/4$	0	7.5	$7.5/(3/4) = 30$
x_4	$3/4$	0	$-3/4$	1	7.5	$7.5/3/4 = 10$
	$-51/4$	0	$45/4$	0	$675/2$	

$R_2\ (3/4)$ (I)

$R_1 - \frac{1}{4} R_2$ (II)

$R_3 + \frac{91}{4} R_2$ (III)

x_2

x_1

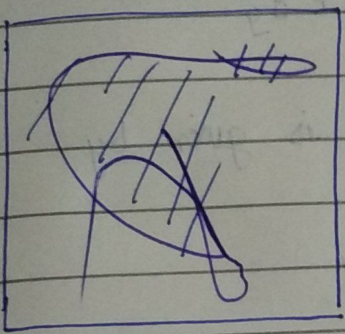
x_4

Ex 2 Max $z = 3x_1 + 5x_2 + 4x_3$

st $2x_1 + 3x_2 \leq 8$

$3x_1 + 2x_2 + 4 \leq 15$

$2x_2 + 5x_3 \leq 10$



$x_1, x_2, x_3 \geq 0$

Physical Application of multiple integrals

[~~Mass, Volume~~, Centre of gravity and measurement of inertia

calculation of mass, for a plane lamina, let $\rho = f(x,y)$ be surface density per unit area at the point $P(x,y)$ then the element mass at any point $P(x,y)$ in $\rho\, \delta x\, \delta y$

The total mass M of the lamina is given by

$$M = \iint_A \rho\, dx\, dy = \iint f(x,y)\, dx\, dy$$

where the double integral is being taken

[ii] ~~For~~ In polar cordinate $\rho = f(r, \theta)$. Given at any point $P(r,\theta)$, then the total mass is given by

$$M = \iint_A f(r,\theta)\, r\, dr\, d\theta$$

[iii] For a solid volume V, f the density at the point $P(x,y,z)$ be $\rho = f(x,y,z)$. When the total mass M is given by

$$M = \iiint_V \rho\, dx\, dy\, dz = \iiint_V \partial(x,y,z)\, dx\, dy\, dz$$

where the integral is ~~been~~ being taken over the volume V

PROBLEMS:

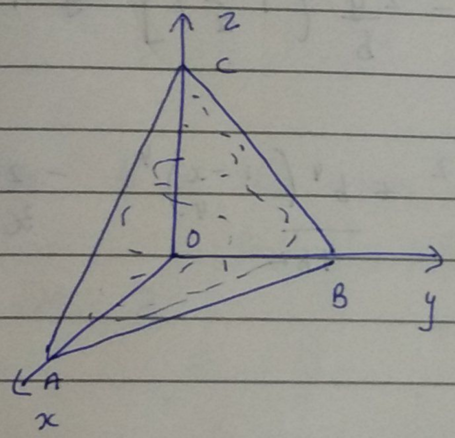
Q1 Find the mass of tetrahedron bounded by the co-ordinate planes and the planes $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

the variable density $\rho = xyz$

Soln The elementary mass at any point $P(x,y,z)$ is $\boxed{M\,xyz\,\delta x\,\delta y\,\delta z}$

The ~~total~~ mass $M = \iiint_V M\,xyz\,dx\,dy\,dz$

where V is the volume of the tetrahedron OABC



The tetrahedron's plate is bounded by the planes

$x=0,\ y=0,\ z=0$ — (1)

Solving and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Solving (ii) for z, we get

$$z = c\left(1-\frac{x}{a}-\frac{y}{b}\right)$$

∴ The required total mass is

$$M = \int_{x=0}^{a}\int_{y=0}^{b(1-\frac{x}{a})}\int_{z=0}^{z=c(1-\frac{x}{a}-\frac{y}{b})} M\,xyz\,dz\,dy\,dx$$

$$= \int_{x=0}^{a} \int_{0}^{y=b(1-\frac{x}{a})} xy \left[\frac{z^2}{2}\right]_{z=0}^{z=c(1-\frac{x}{a}-\frac{y}{b})} dy\,dx$$

$$= \frac{c^2}{2} \int_{x=0}^{a} \int_{y=0}^{y=b(1-\frac{x}{a})} dy \left(\left(1-\frac{x}{a}\right) - \frac{y}{b}\right)^2 dy\,dx$$

$$= M\frac{c^2}{2} \int_{x=0}^{a} x\left[\left(1-\frac{x}{a}\right)^2 y + \frac{y^2}{b^2} - 2\frac{y^2}{b}\left(1-\frac{x}{a}\right)\right] dy\,d$$

$$= \frac{Mc^2}{2} \int_{x=0}^{a} x\left[\left(1-\frac{x}{a}\right)^2 \frac{y}{2}\left(1-\frac{x}{a}\right)^2 + \frac{b^4\left(1-\frac{x}{a}\right)^4}{4b^2} - \frac{2}{36} x b^3\left(1-\frac{x}{a}\right)^3 \left(1-\frac{x}{a}\right)\right] dx$$

ANS. $\frac{Ma^2b^2c^2}{72c}$

Q2. Find the mass of a plate in the shape at the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$, the density being given $\rho = Kxy$

HINT: The total mass is the mass plate in the first quadrant

$M = 4 \iint_A \rho\, dx\,dy$ where A is the mass in the first quadrant

$$M = 4Y \int_{y=0}^{b} \int_{x=0}^{a\left[1-\left(\frac{y}{b}\right)^{2/3}\right]^{3/2}} K xy\,dx\,dy = 4k \int_{y=0}^{b} y\left[\frac{x^2}{2}\right]_{x=0}^{x=a\left[1-\left(\frac{y}{b}\right)^{2/3}\right]^{3/2}} dx$$

$$= 2K \int_0^b a^2\left(1-\left(\frac{y}{b}\right)^{2/3}\right]^3 y\,dy$$

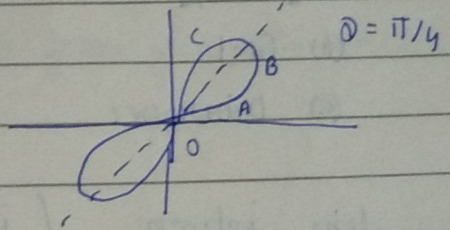
Put $y = b\sin^3\theta$

Q3. Find the mass of a plate in form of one loop of the loop of the leminicate of Bernaulli $r^2 = a^2 \sin 2\theta$ where $\rho = kr^2$

HINT : The required mass is given by

$$M = 2 * \text{Mass of area } OAB$$

$$= 2 * \iint_R \rho\, r\, dr\, d\theta$$



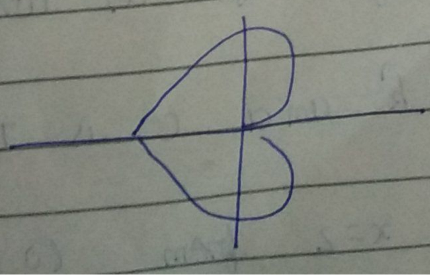
where R is the area of the region OAB

$$M = 2 \int_0^{\pi/4} \int_0^{r = a\sqrt{\sin 2\theta}} k\, r^2\, r^2\, dr\, d\theta$$

ANS. $\dfrac{k\pi a^4}{16}$

Q4. Find the mass of a plate in the form of a loop of the leminocate of Bernauli of $r^2 = a^2 \cos 2\theta$ if the density of any point variable as the square at the distance of the point from the plate

Q5. A wire is in the form of a cardiod $r = a(1+\cos\theta)$ is the density at any point of the wire is porportional to its distance from the pole then find the mass of the wire



VECTOR CALCULUS

- Vector Algebra
- Vector Calculus
  - Scalar point fn
  - Vector point fn

A : (a) Gradient
(b) Curl
(c) Divergence

B : line integral $\int_C \vec{P}\, d\vec{r}$

where C is the closed curve along which the line integration being taken

$$\int_C \vec{P} \times d\vec{r} \;/\; \oint_C \vec{r}\, d\vec{r}$$

NOTE :

Work done by a force $\vec{F}$ during a small displacement δx of the particle on C in the line integral $\int_C \vec{P}\, d\vec{r}$

PROBLEMS

① Evaluate $\int_C \vec{F}\, d\vec{r}$ where $\vec{F} = xy\, i + (x^2+y^2)\, j$, curve C is the arc of $y = x^2 - 4$ from $(2, 10)$ to $(4, 10)$

② Evaluate the line integral $\int \vec{F} \cdot d\vec{r}$ where $\vec{F} = x^2 y\, \vec{i} + (x - z)\, \hat{j} + xyz\, \hat{k}$ and C is the arc of the curve

$y = x^2$, $x = 2$ from $(0, 0, 2)$ to $(1, 1, 2)$

③ Evaluate $\int_c \vec{F} \cdot d\vec{r}$ where $\vec{F} = (x^2 + y^2)\,\hat{i} - 2xy\,\hat{j}$ and curve C is the rectangle in the xy plane bounded $\int_c \vec{P} \cdot d\vec{x}$ where $\vec{P} = xy\,\hat{i} + (x^2 + y^2)\,\hat{j}$ and C is the x-axis from $x = 2$ to $x = y$ and the line $x = 4$ from $y = 0$ to $y = 12$

Soln of problem ①

Given: $\vec{P} = xy\,\hat{i} + (x^2 + y^2)\,\hat{j}$

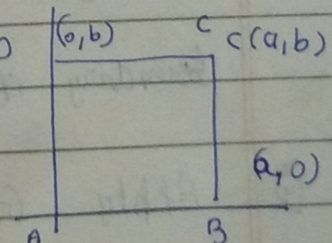
$$\int_c \vec{F} \cdot d\vec{x} = \int_c xy\,dx + \int (x^2 + y^2)\,dy$$

$$= \int_{x=2}^{4} x(x^2 - y)\,dx + \int_{y=0}^{12} (y + 4 + y^2)\,dy$$

Soln of Ex 3.

$$\int \vec{F} \cdot d\vec{r} = \int_c (x^2 + y^2)\,dr - 2xy\,dy$$

(i) along the line AB



$x = a$ and $y = 0$

$$\int_0^A (x^2 + 0)\,dx \quad - \underbrace{2x + 0 + dy}_{\text{zero}}$$

$$= \left[\frac{x^3}{3}\right]_{x=0}^{a} = \frac{a^3}{3}$$

Green's Theorem in Plane :

If C be a closed curve bounding the region R in xy plane and if, M and N are continuous function of x and y, and and $\frac{\partial x}{\partial y}$ and $\frac{\partial x}{\partial y}$

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial y} \right) dx\, dy$$

Problems ①

① Verify Green's theorem for $\int (x - 2y)\, dx + x\, dy$ where C is the circle $x^2 + y^2 = 1$.

② Verify Green's theorem for $\int_C (x-y)\, dx + 2xy\, dy$ where C is in the boundry at the region ending

$x^2 = 4y$ and $y^2 = 4x$

③ Verify Green's theorem in the xy plane for

$\iint_C (xy^2 - 2)\, dx + (x^2 y + 3)\, dy$ around the boundary of the region enclosed by $y^2 = 8x$ and $x = 2$

④ Apply Green's theorem to evalute $\oint [y - \sin x]\, dx + \cos x\, dy]$ where C is the plane triangle enclosed by the lines

HINT to Q1.

$$l.h.a = \int_C M dx + N dy \quad \text{where } M = x - 2y, \; y = x, \; \frac{\partial x}{\partial x} = 1,$$

$$\frac{\partial x}{\partial y} = -2$$

$$r.h.a \quad 4 \int_{x=0}^{1} \int_{y=0}^{y=\sqrt{1-x^2}} [1 - (-2)] \, dy \, dx$$

$$= 4 \int_{x=0}^{x=1} \int_{y=0}^{\sqrt{1-x^2}} 2 \, dy \, dx = 4 \times 3 \int_{x=0}^{1} \Big[y \Big]_{y=0}^{y=\sqrt{1-x^2}} dx$$

$$= 12 \int_{x=0}^{x=1} \sqrt{1-x^2} \, dx$$

<u>l.h.a</u> :→ Along the curve $x = \cos\theta$, $y = \sin\theta$, $0 \le \theta \le 2\pi$

$$l.h.a = \int_{\theta=0}^{2\pi} (\cos\theta - 2\sin\theta)(-\sin\theta) \, d\theta + \cos\theta(\cos\theta \, d\theta)$$

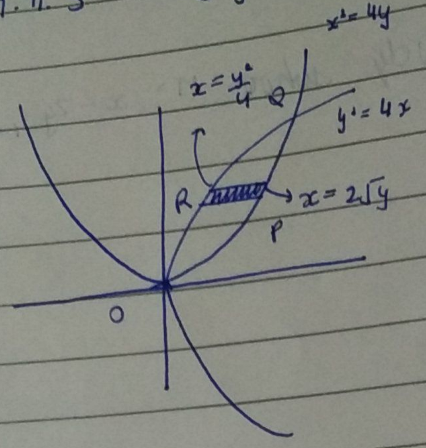
Soln of problem 2 :

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \, dy$$

Here $M = x - y$ $\qquad N = 3xy$

$$\frac{\partial N}{\partial x} = 3y \qquad \frac{\partial M}{\partial y} = -1$$

r.h.s $= \iint (3y+1)\, dx\, dy$



$$= \int_{y=0}^{4} \int_{x=\frac{y^2}{4}}^{x=2\sqrt{y}} (3y+1)\, dx\, dy$$

$$= \int_{y=0}^{y=4} (3y+1)\, [x]_{\frac{y^2}{4}}^{x=2\sqrt{y}}\, dy$$

$$= \int_{y=0}^{4} (3y+1)\left(2\sqrt{y} - \frac{y^2}{4}\right) dy$$

$$= \frac{512}{15}$$

l.h.s $= \int_C M\, dx + N\, dy$

$$= \int (x-y)\, dx + 3xy\, dy$$

$$= \int_{OPQ} (\quad) + \int (\quad)$$

$x^2 = 4y \Rightarrow y = \frac{x^2}{4}$

$x^2 = 4y \Rightarrow y = \frac{x^2}{4}$

$\frac{dy}{dx} = \frac{2x}{4} = \frac{x}{2}$

$$\int_{OPQ} \left(x - \frac{x^2}{4}\right) dx + \int_{OPQ} 2\sqrt{y} \cdot y \, dy$$

OR $$\int_{x=0}^{4} \left(x - \frac{x^2}{4}\right) dx + 3x \times \frac{x^2}{4} \times \frac{x}{2} dx$$

Along the curve $y^2 = 4x$

$$2y \frac{dy}{dx} = 4$$

OR

$$2y\,dy = 4dx$$

$$y\,dy = 2dx$$

$$\int (x - 2\sqrt{x}) dx + 3x + 2\sqrt{x} + \frac{2dx}{2\sqrt{x}}$$

$$= \int_0^4 \left[(x - 2\sqrt{x}) + 6x\right] dx$$

STOKE'S THEOREM :

Let 's' be a given surface bounded by a simple closed curve C and $F(x, y, z) = F_x(x, y, z)\hat{i} + F_y(x, y, z)\hat{j} + F_z(x, y, z)\hat{k}$ to be a vector function having a continuous first order partial derivatives in a domain containing S Then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n}\, ds$$

where $\vec{n}$ is the unit outward normal vector to S and C is taken in anticlockwise direction

Surface Integral : $\iint_S \vec{F}\, d\vec{s}$: any integral which is to be evaluated over a surface S (say) is called the surface integral

NOTE 1 : Surface integral of $\vec{F}$ over S, written as

$$\iint_S \vec{F}\, d\vec{s} \quad \text{OR} \quad \iint \vec{F}\, \vec{n}\, d\bar{s} \rightarrow \text{given the area of surface}$$

let $\vec{F} = F_i\hat{i} + F_y\hat{j} + F_z\hat{k}$ and $\vec{b} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$

where α, β, γ are the angles made ~~in~~ by the unit vector $\vec{b}$ along x, y, and z axes respectively

Here $\cos\alpha = \hat{x} \cdot \hat{i}$, $\cos\beta = \hat{x}\ \vec{y}$, $\cos\gamma = \hat{x}\ \vec{k}$

*my*COMPANION

$$\iint_S \vec{F}\cdot\hat{x}\,k \; ds = \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, ds$$

$$= \iint_S F_1 (\cos\alpha)\, ds + \iint_S F_2 (\cos\beta) ds + \iint_S F_3 (\cos\gamma) ds$$

where $(\cos\alpha)\, ds = dy\,dz$ OR $ds = dy\,dz / (\hat{x}\cdot\hat{i})$ ①

$(\cos\beta)\, ds = dz\,dx$ OR $ds = dz\,dx / (\hat{n}\cdot\hat{y})$ ②

and $(\cos\gamma)\, ds = dx\,dy$ OR $ds = dx\,dy\,(\hat{x}\cdot\hat{k})$ ③

NOTE : 2 Let R_1, R_2 and R_3 be the projection of S in the xy, yz and zx planes respectively

Then using ①, ② and ③ above, we can write the surface integral as follows.

$$\iint_S \vec{F}\cdot\hat{x}\, ds = \iint_R \vec{F}\cdot\hat{x} \frac{dx\,dy}{\hat{x}\cdot\hat{k}}$$

$$= \iint_{R_2} \vec{F}\cdot\hat{x} \frac{dy\,dz}{\hat{x}\cdot\hat{i}}$$

$$= \iint_{R_3} \vec{F}\cdot\hat{x} \frac{dz\,dx}{\hat{x}\cdot\hat{j}}$$

NOTE 3 : Other forms of surface integral are :

$$\iint_S \vec{F}\, ds, \quad \iint_S \vec{F}\times\hat{x}\, ds \quad \text{and} \quad \iint_S f\,\hat{n}\, ds$$

where $\vec{F}$ is a vector field and f is a scalar field and $\hat{n}$ is the unit normal vector to the surface S

PROBLEMS ① Evaluate the surface integral $\iint_S \vec{F} \cdot \hat{n} \, ds$ where $\vec{F} = z^2 \hat{i} + xy \hat{j} + y^2 \hat{k}$ and S is the surface of the cylinder $x^2 + y^2 = 25$, $0 \le z \le 3$ included in the first ~~octant~~ octant

ANS . 170

② Evaluate $\iint_S \vec{F} \cdot \hat{n} \, ds$ where $\vec{F} = x\hat{i} + y\hat{j} - 2z\hat{k}$ and in the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy plane

ANS :

③ Evaluate $\iint_S \vec{F} \, ds$ where $\vec{F} = xy\hat{i} - x^2\hat{j} + (x+z)\hat{k}$ and in the region of the plane $2x + 2y + z = 6$, bounded in the first octant.

ANS : $\frac{27}{4}$

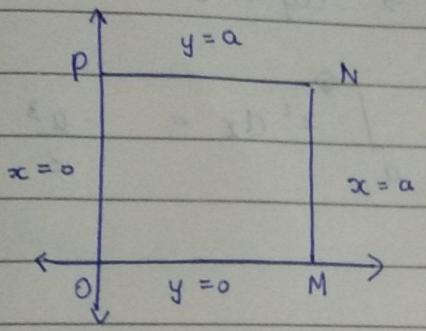
④ Evaluate $\iint_S \vec{F} \cdot \hat{n} \, ds$ where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2x^2\hat{k}$ and S is the outer surface at the cylinder $x^2 + y^2 = 9$, $0 \le z \le 4$

ANS. 144π

⑤ Verify stoke's theorem for upper half of the surface of the sphere of unit radius $x^2 + y^2 + z^2 = 1$

Q6. Verify stokes theorem for the function $\vec{F} = x^2\hat{i} + xy\hat{j}$ integrated round the square whose sides are $x=0, y=0, x=a$ and $y=a$ in the plane $z=0$

A6.



$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot\vec{n}\, ds$$

$$\int_C (x^2\hat{i} + xy\hat{j})\cdot d\vec{r} = \iint_S \text{curl } (x^2\hat{i} + xy\hat{j})$$

LHS :

Line integral along OM :

$$\int_C \vec{F}\cdot d\vec{r} = \int (x^2\hat{i} + xy\hat{j})(\hat{i}\, dx + \hat{j}\, dy)$$

$$= \int_0^a x^2\, dx + xy\, dy$$

$y = 0$, $dy = 0$ x varies from 0 to a

$$= \int_0^a x^2\, dx = \left[\frac{x^3}{3}\right]_0^a$$

(ii) line integral along MN Here $x = a \therefore dx = 0$ and y varies from 0 to a

$$\int_{MN} x^2 dx + xy\, dy = \int_0^a a^2 y\, dy = \frac{1}{2} a^3$$

(iii) Along NP, $y = a \Rightarrow dy = 0$

$$\int_{NP} x^2 dx + xy\, dy = \int_a^0 x^2 dx = -\frac{a^3}{3}$$

(iv) Along line PO, $x = 0 \Rightarrow dx = 0$

$$\int_a^0 x^2 dx + xy\, dy = 0$$

Adding (i), (ii), (iii), (iv)

$$\int \vec{F} \cdot d\vec{r} = \frac{1}{2} a^3$$

RHS

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$$= y\hat{k}$$

For open surface, we choose square OMNP so that $\hat{n} = \hat{k}$

curl $\vec{F} \cdot \vec{n} = y \hat{k} \cdot \hat{k} = y$

$$\iint_S \text{curl } \vec{F} \cdot \hat{n}\, ds = \int_{y=0}^{a} \int_{x=0}^{a} y\, dx\, dy = \frac{a^2}{2}$$

Sol^n of problem ①

Let $f(x, y, z) = x^2 + y^2 - 25$

then $\nabla f = 2x\,\hat{i} + 2y\,\hat{j}$ and $\left\{ \vec{n} = \frac{\nabla f}{|\nabla f|} \right\}$

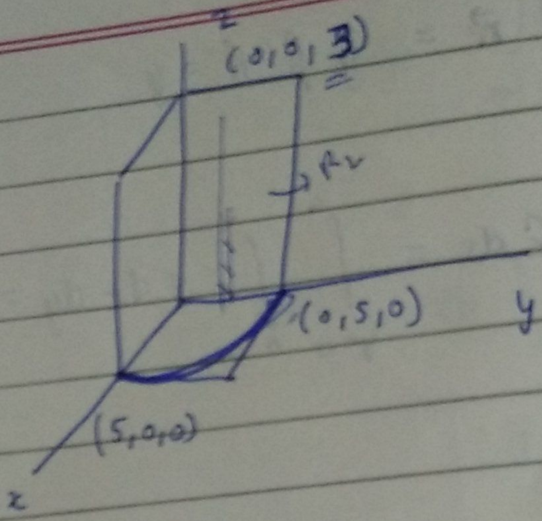
$$= \frac{2(x\,\hat{i} + y\,\hat{j})}{\sqrt{4(x^2+y^2)}} = \frac{x\,\hat{i} + y\,\hat{j}}{\sqrt{x^2+y^2}}$$

$$\hat{n} = \frac{1}{\sqrt{25}}(x\,\hat{i} + y\,\hat{j}) = \frac{x\,\hat{i} + y\,\hat{j}}{5}$$

$$\vec{F} \cdot \vec{n} = (z^2\,\hat{i} + xy\,\hat{j} - y^2\,\hat{k}) \cdot \frac{1}{5}(x\,\hat{i} + y\,\hat{j})$$

$$= \frac{1}{5}(xz^2 + xy^2)$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{R_2} \vec{F} \cdot \hat{n} \frac{dy\,dz}{\vec{n} \cdot \hat{i}}$$



By ~~projecting~~ S ~~on~~ ~~xy~~ yz plane, we have

$$\iint_R \frac{1}{5}(xz^2 + xy^2)\frac{dy\,dz}{x/5} \qquad \vec{n}\cdot\vec{i} = \frac{x}{5}$$

$$= \int_{z=0}^{3}\int_{y=0}^{5}(y^2+z^2)\,dy\,dz$$ As R_2 is a rectangle