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Digital Signal Processing
Lecture 4

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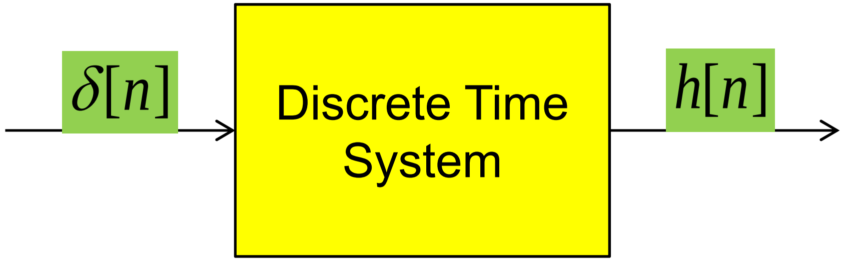
Linear Time-Invariant (LTI) Systems

Impulse Response



$$y[n] = \sum_{k=0}^{\infty} x[n-k]$$

$$h[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

$$h[n] = u[n]$$

$$y[n] = \sum_{l=-\infty}^{n} x[l]$$

$$h[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$h[n] = u[n]$$

$$y[n] = a_1 x[n] + a_2 x[n-1] + a_3 x[n-2]$$

$$h[n] = a_1 \delta[n] + a_2 \delta[n-1] + a_3 \delta[n-2]$$

Linear Time-Invariant (LTI) Systems

$$\delta[n] \rightarrow h[n]$$

$$\delta[n-k] \rightarrow h[n-k]$$

$$\underbrace{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]}_{x[n]} \rightarrow \underbrace{\sum_{k=-\infty}^{\infty} x[k]h[n-k]}_{y[n]}$$

Convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

LTI Systems: Convolution

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y[n] = x[n] * h[n]$$

h[n]: impulse response of LTI system

$$y[n-n_0] = \sum_{k=-\infty}^{\infty} x[k]h[n-n_0-k]$$

$$= x[n] * h[n-n_0]$$

Example

$$x[n] = (0.2)^n u[n]$$

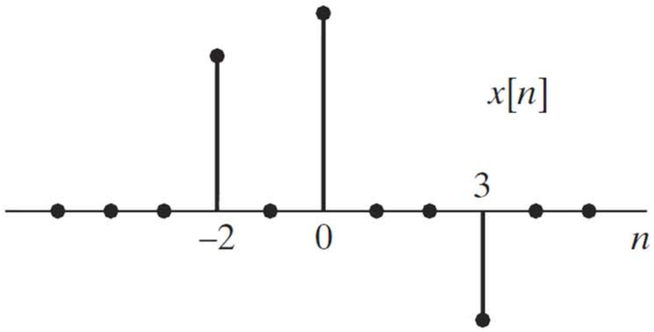
$$h[n] = (0.6)^n u[n]$$

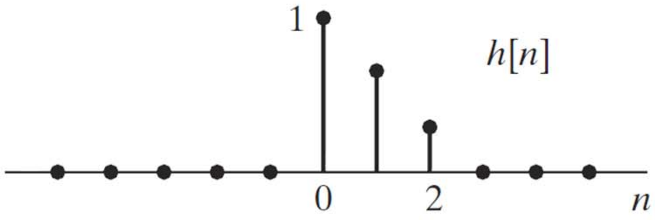
$$y[n] = x[n] * h[n]?$$

$$y[n] = 2.5(0.6^{n+1} - 0.2^{n+1})u[n]$$

Convolution Sum-Example

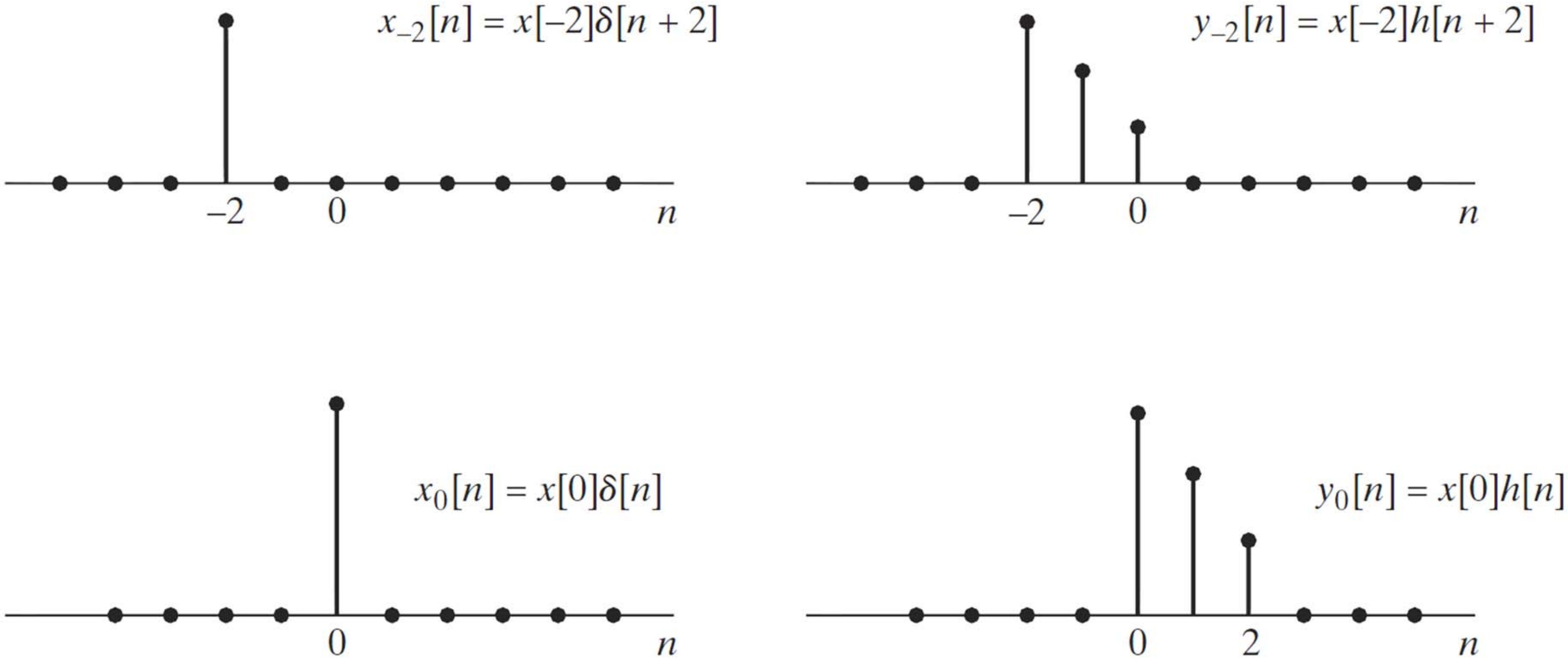
The output of an LTI system can be obtained as the superposition of responses to individual samples of the input. This approach is shown to estimate y[n] in the case of x[n] and h[n] given in the following:



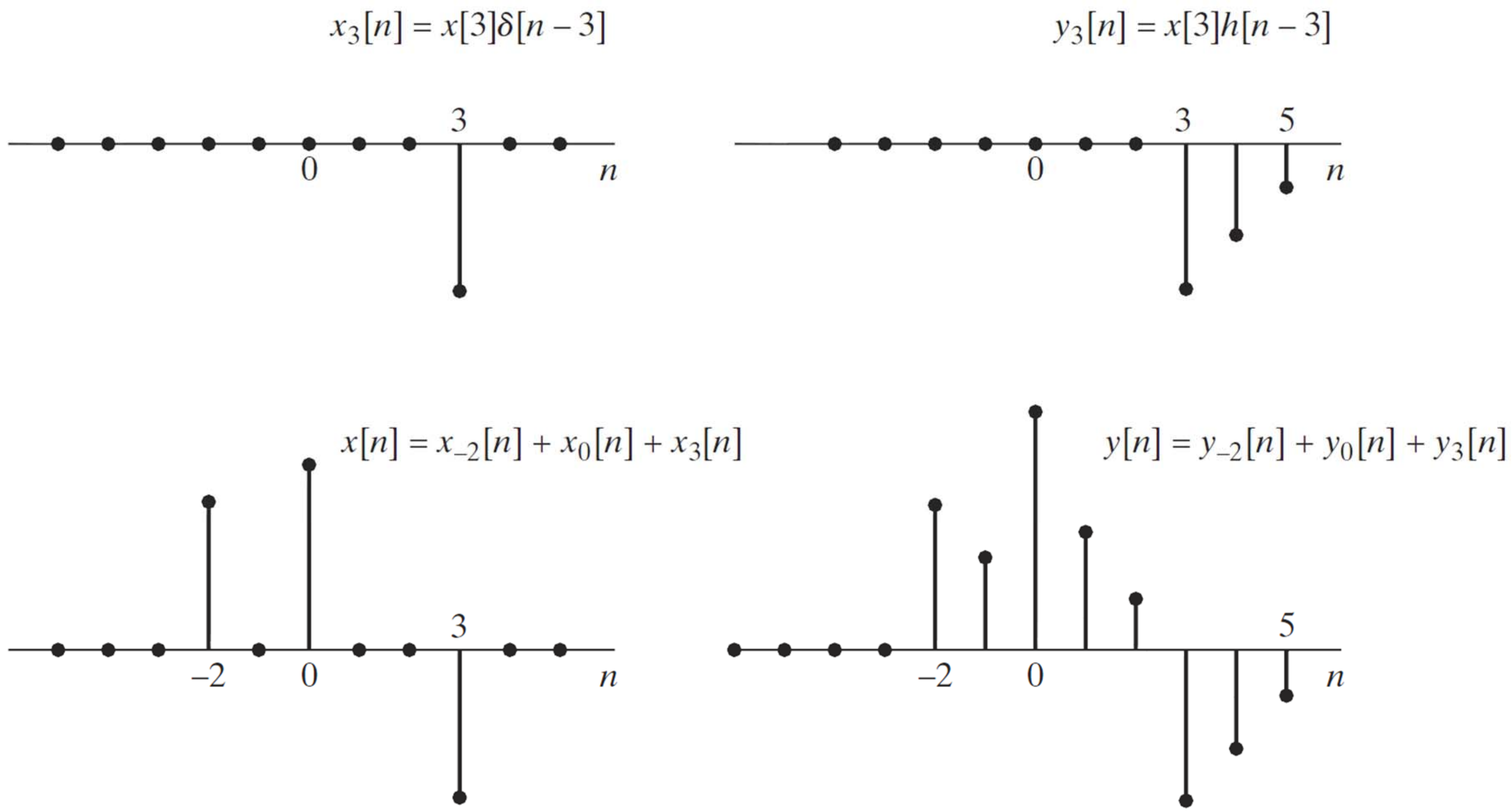


y[n]?

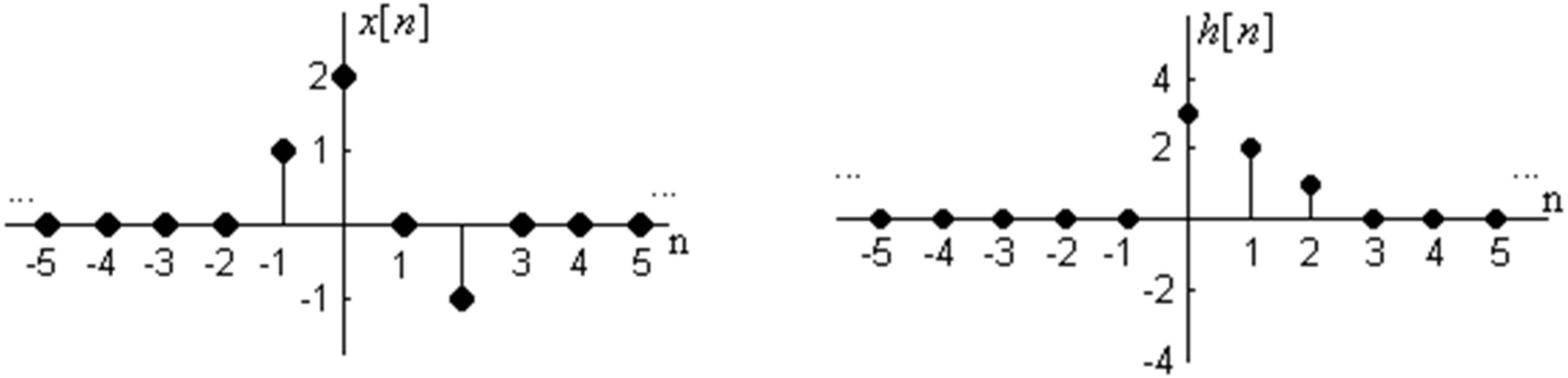
Convolution Sum-Example-cont



Convolution Sum-Example-cont

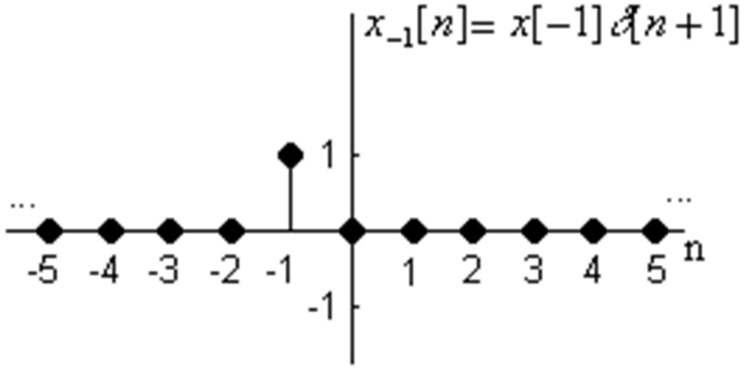


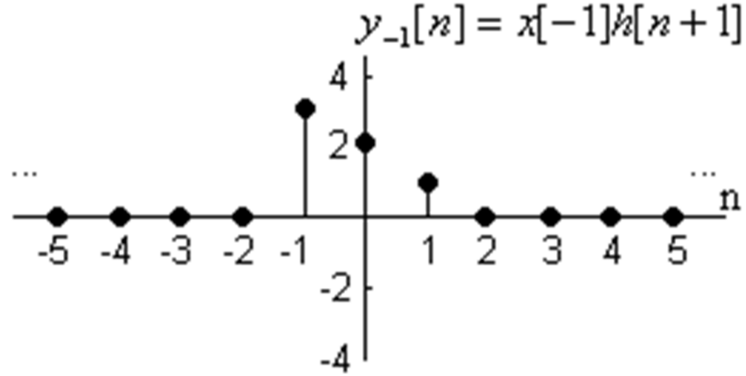
Example

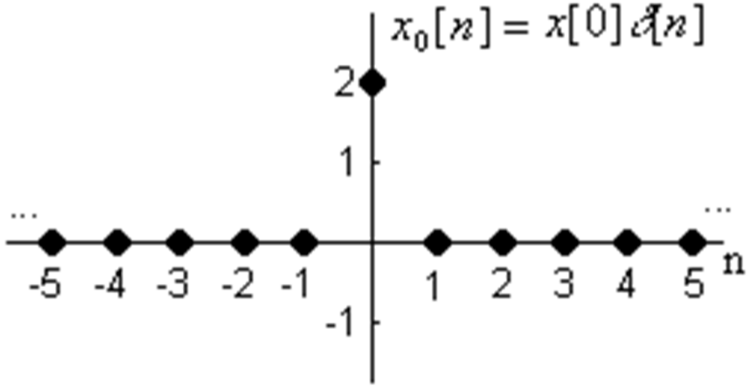


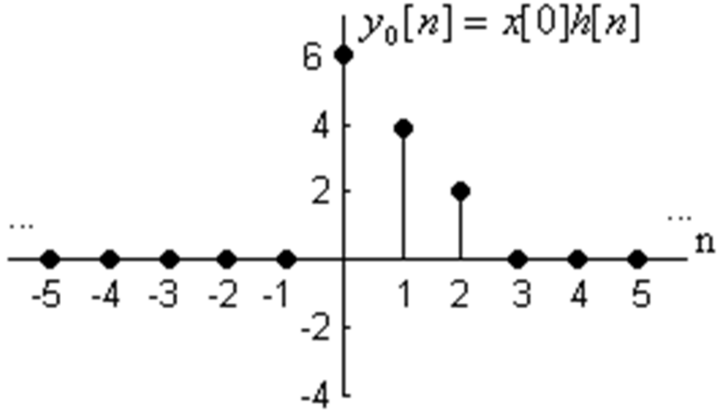
y[n]?

Example-cont

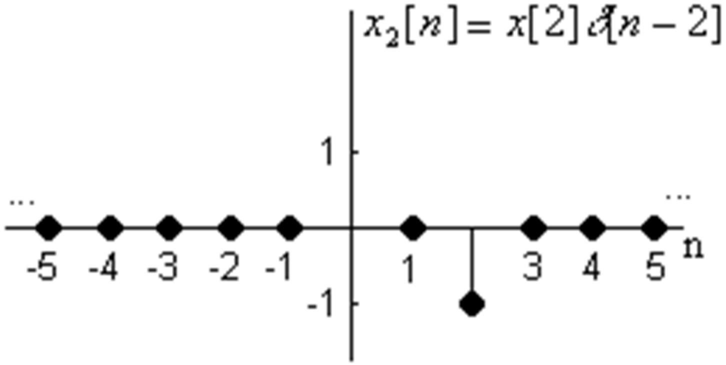


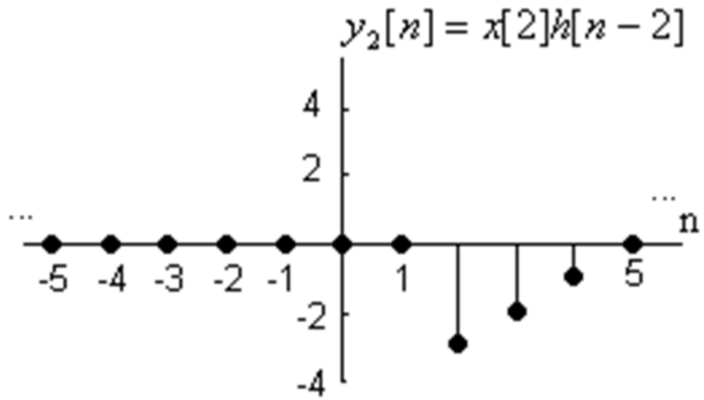


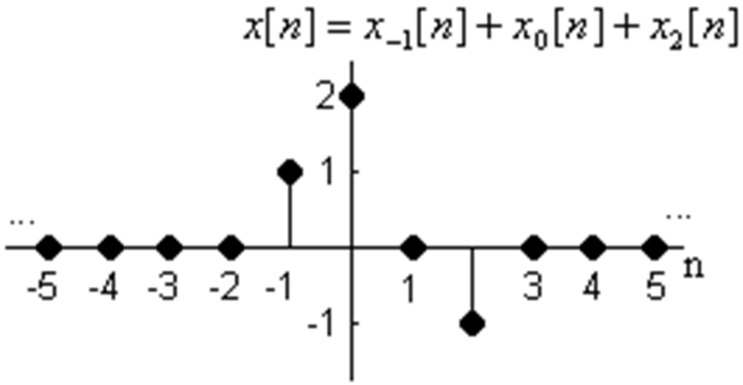


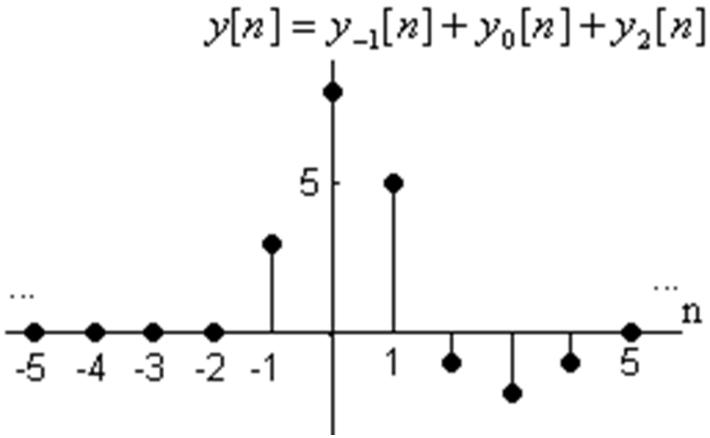


Example-cont









LTI Systems-Convolution

- ✓ The output sequence y[n] can be also obtained by multiplying the input sequence (expressed as a function of k) by the sequence whose values are $h[n-k]$, $-\infty < k < \infty$ for any fixed value of n, and then summing all the values of the products $x[k]h[n-k]$, with k a counting index in the summation process.

- ✓ Therefore, the operation of convolving two sequences involves doing the computation specified by

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

for each value of n, thus generating the complete output sequence y[n], $-\infty < n < \infty$.

LTI Systems-Convolution

- ✓ The process of computing the convolution between x[n] and h[n] involves the following steps:

  1. Express x[n] and h[n] as a function of k and obtain x[k] and h[k]

  2. Obtain h[-k] from h[k].

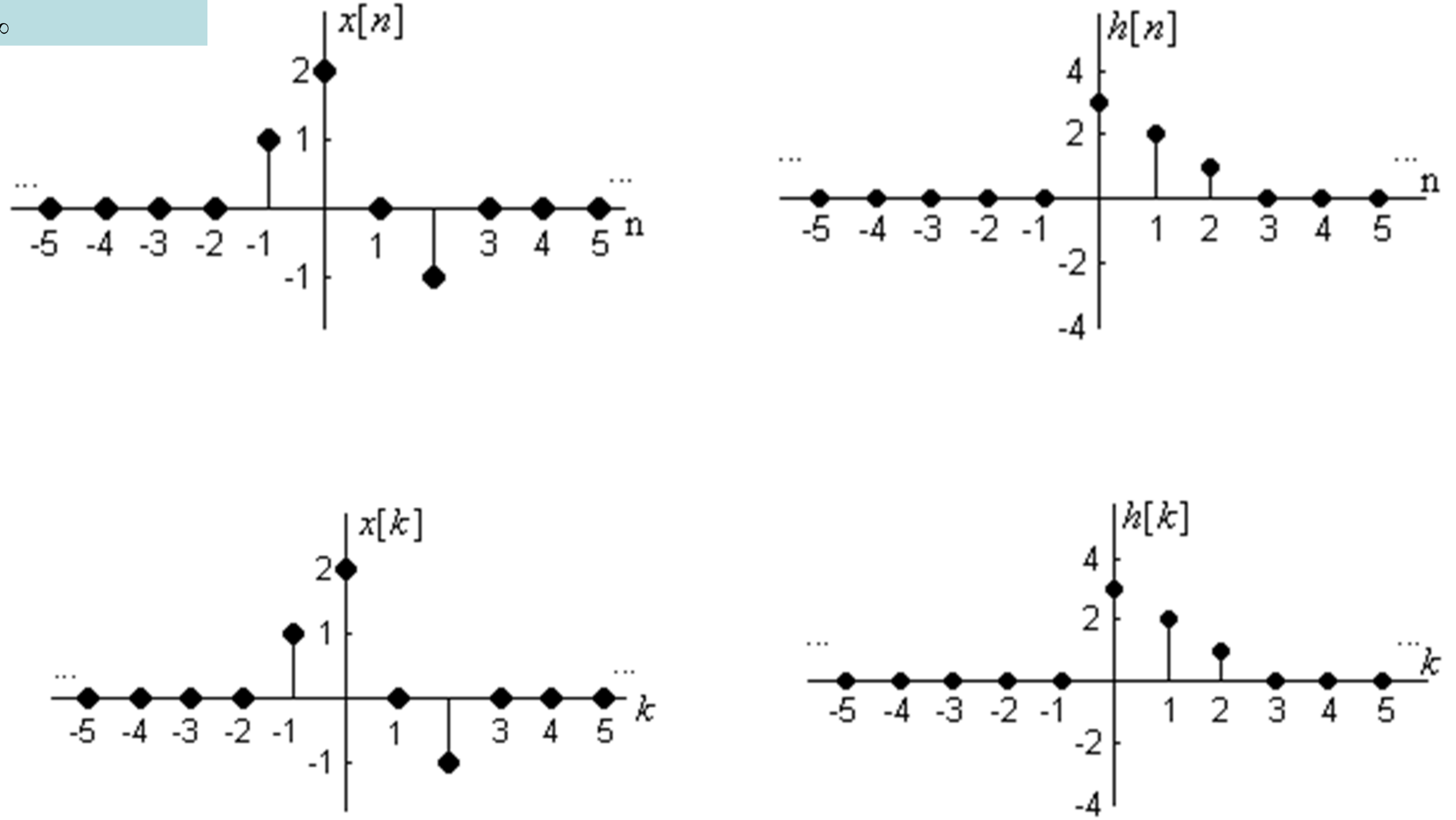
  3. Shift h[-k] by n_0 to the right if n is positive to obtain h[n_0 – k] (shift h[- k] by n_0 to the left if n is negative).

  4. Multiply x [k] by h[n_0 -k] to obtain the product sequence.

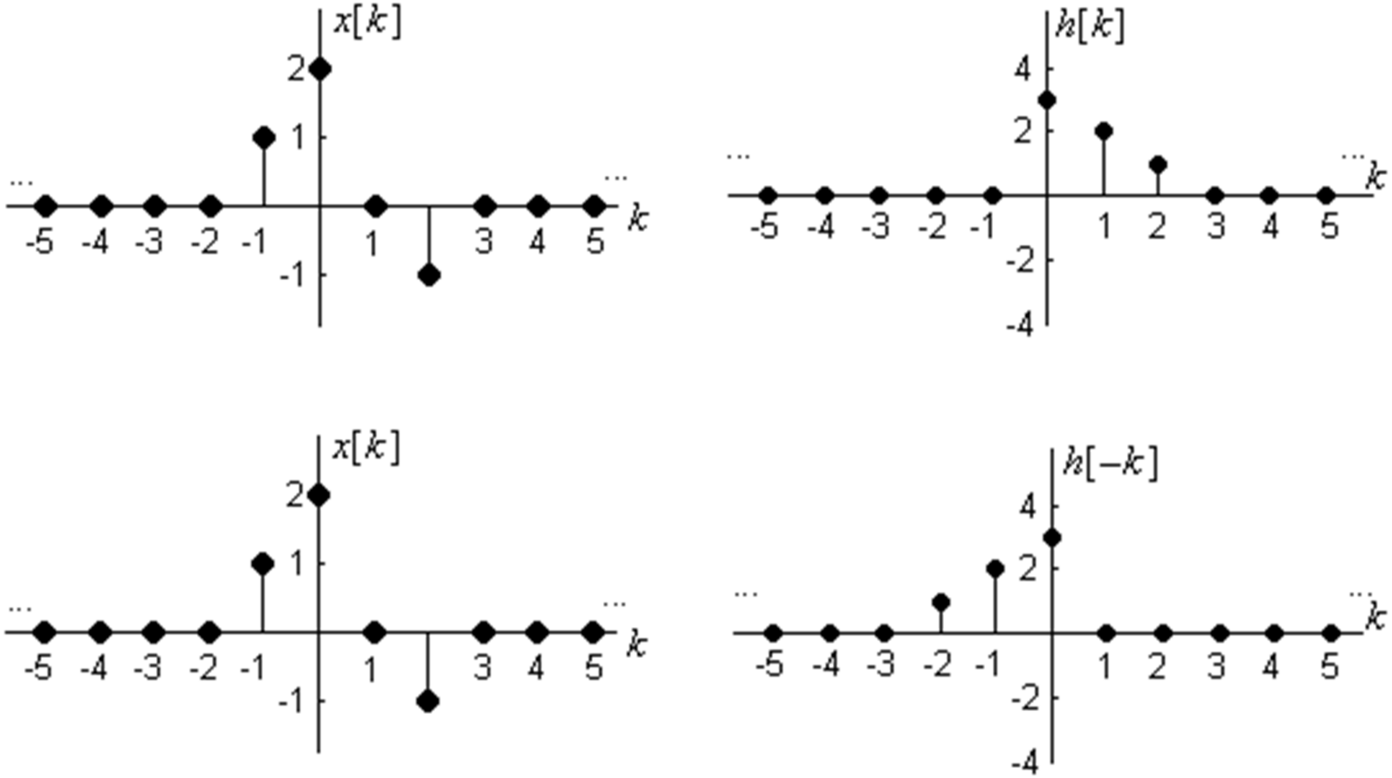
  5. Sum all the values of the product sequence to obtain the value of the output at time n= n_0.

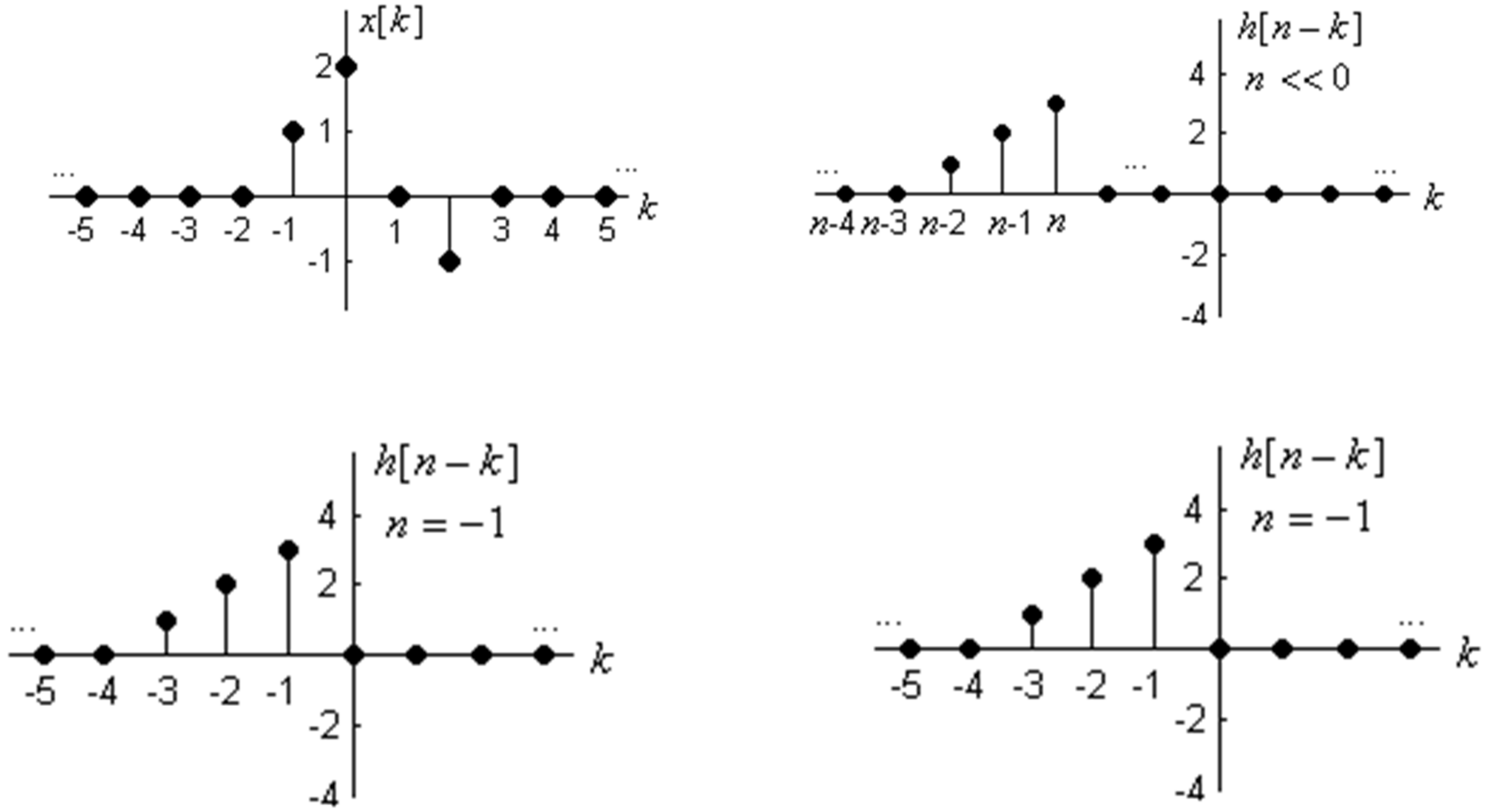
- ✓ The steps from 3 to 5 should be repeated for each values of n_0 (i.e., for any fixed value of n).

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



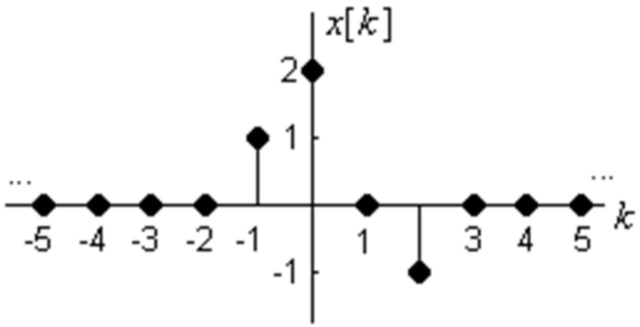
LTI Systems-Convolution

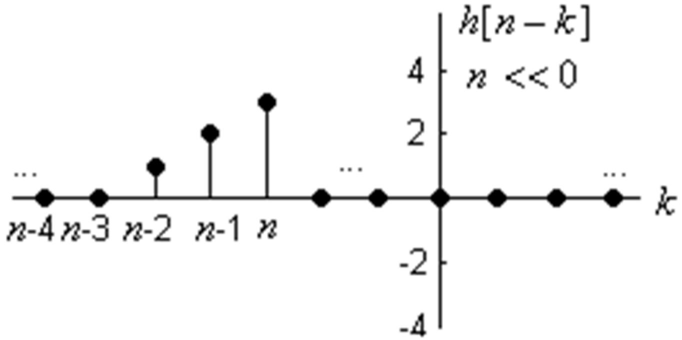


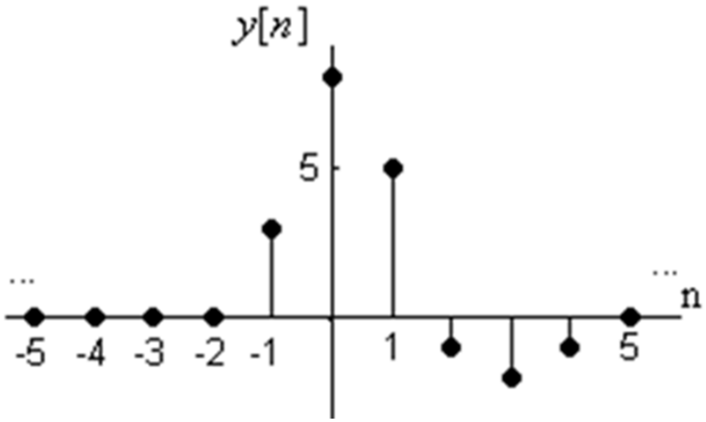


For each n value, estimate h[n-k], and then multiply with x[k] to estimate y[n].

LTI Systems-Convolution

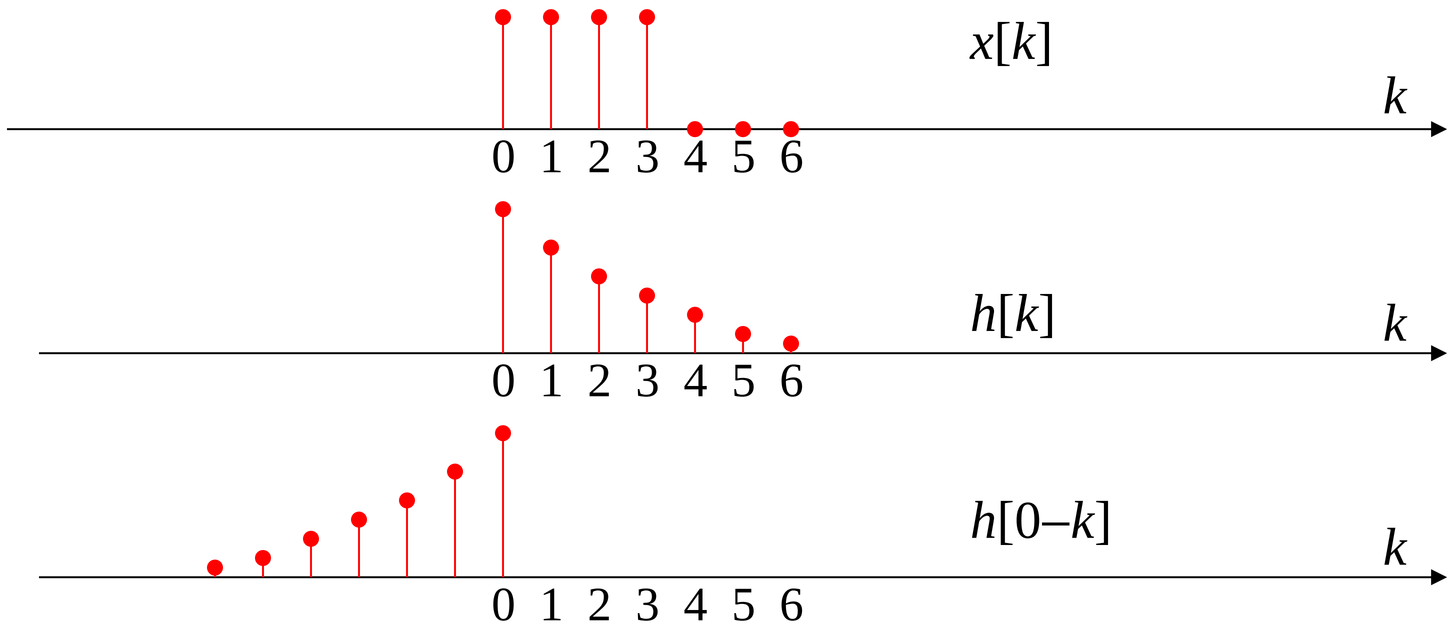




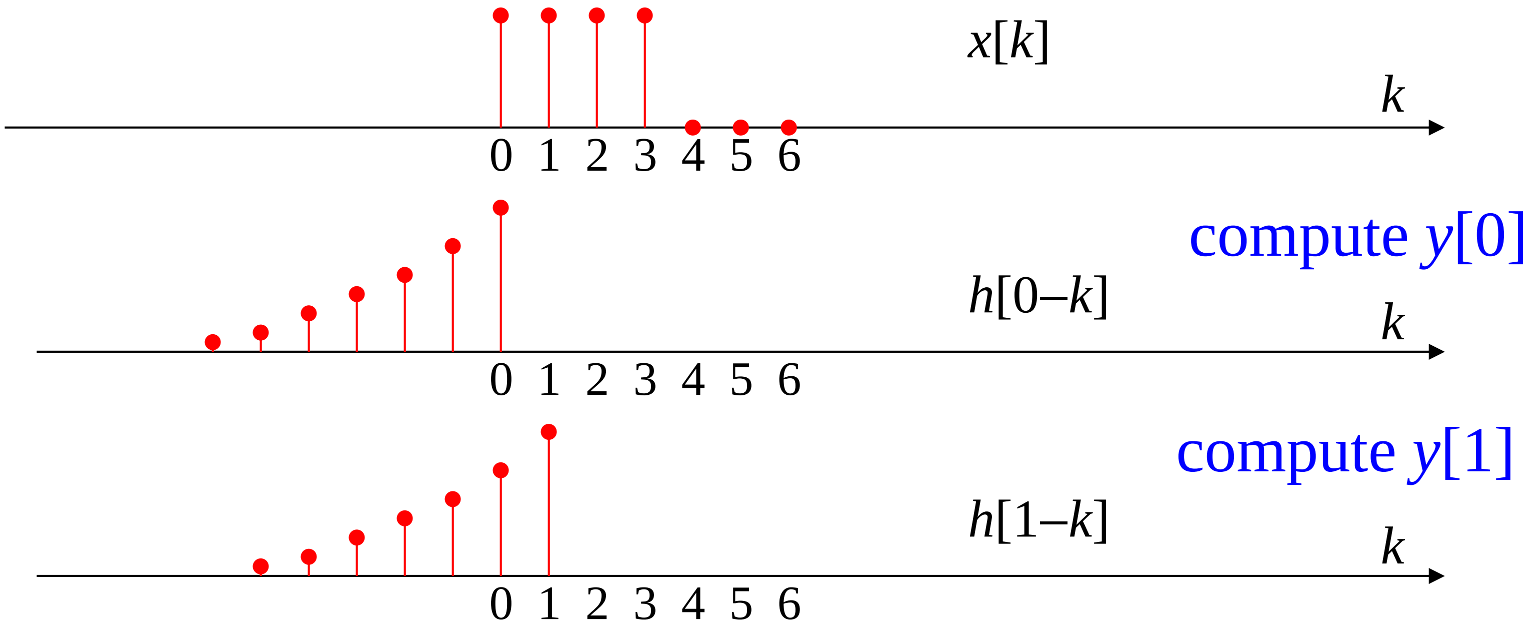


LTI Systems-Convolution

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



LTI Systems-Convolution

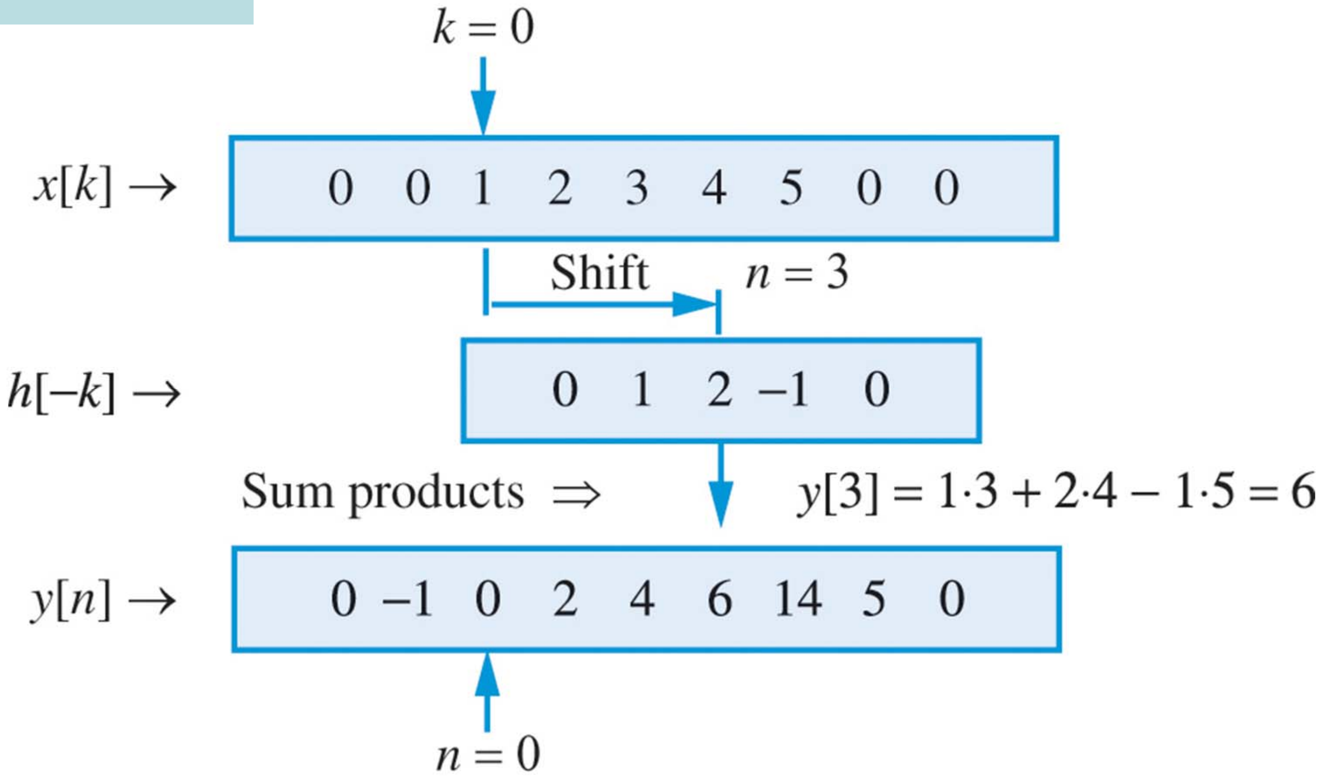


How to compute $y[n]$?

For each n value, estimate h[n-k], and then multiply with x[k] to estimate y[n].

$$x[n] = \left\{ \underset{\uparrow}{1} \;\; 2 \;\; 3 \;\; 4 \;\; 5 \right\}$$

$$h[n] = \left\{ -1 \;\; \underset{\uparrow}{2} \;\; 1 \right\} \qquad y[n] = ?$$

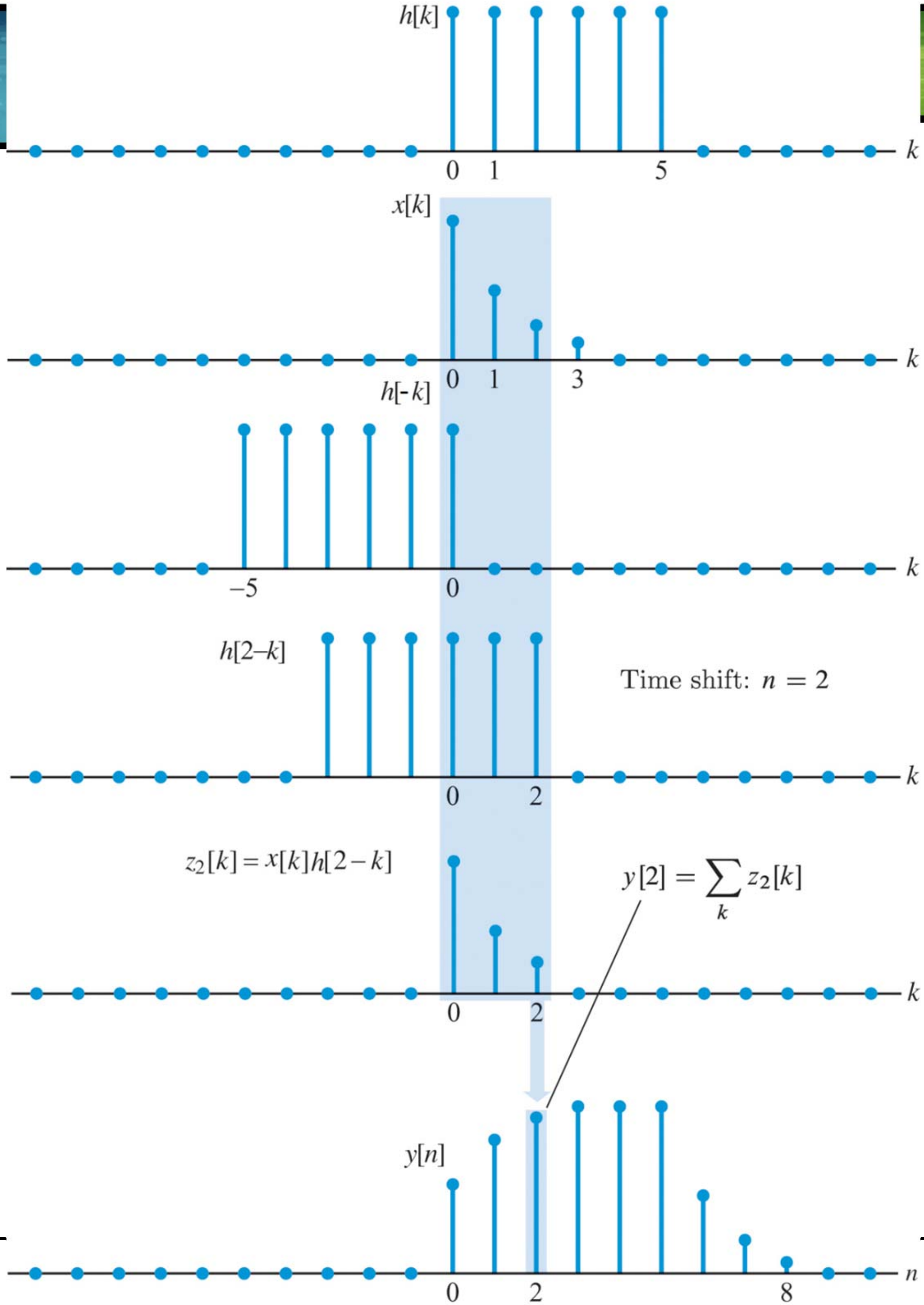


Example

$$h[n]=\left\{\underset{\uparrow}{1}\ 1\ 1\ 1\ 1\ 1\right\}$$

$$x[n]=\left\{\underset{\uparrow}{1}\ 0.5\ 0.25\ 0.125\right\}$$

$y[n]$?



Convolution Features

commutative: $x_1[n]*x_2[n]=x_2[n]*x_1[n]$

associative: $x_1[n]*x_2[n]*x_3[n]=x_1[n]*\left(x_2[n]*x_3[n]\right)$

distributive: $x_1[n]*\left(x_2[n]+x_3[n]\right)=x_1[n]*x_2[n]+x_1[n]*x_3[n]$

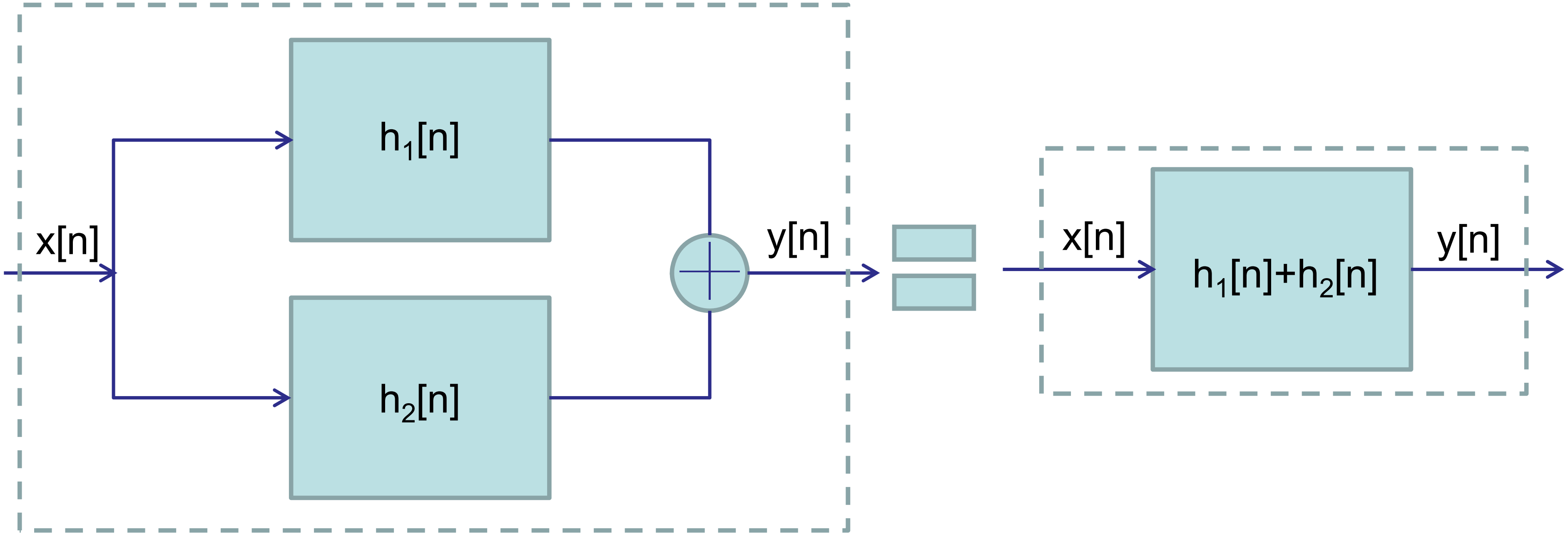
$$x_1[n]*x_2[n]=c_2[n]\Rightarrow$$
$$x_1[n-m]*x_2[n-k]=c_2[n-m-k]$$

if $x_1[n]$ has m samples and $x_2[n]$ k samples,

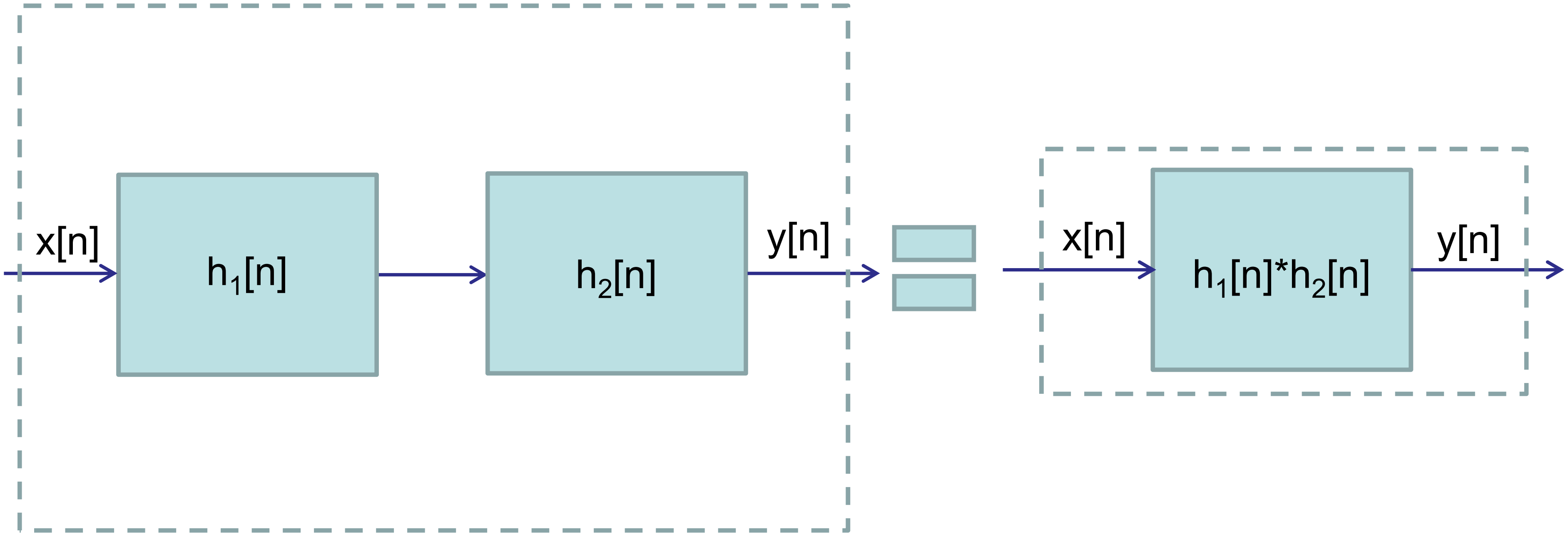
$c[n]=x_1[n]*x_2[n]$ will have $m+k$ -1 samples.

$$x_1[n]*\delta[n]=x_1[n]$$

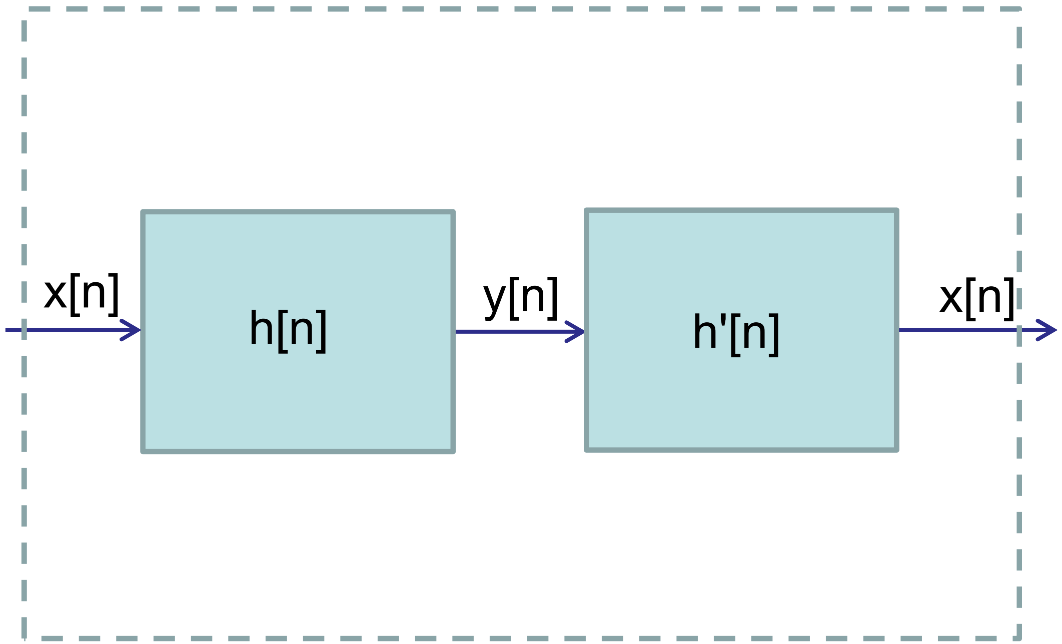
LTI Systems-Parallel Connection of Systems



LTI Systems-Cascade Connection of Systems



LTI Systems-Inverse Systems



$$h[n] * h'[n] = \delta[n]$$

Example-Inverse Systems

$$y[n] = \sum_{k=0}^{\infty} x[n-k]$$

$$h[n] = \sum_{k=0}^{\infty} \delta[n-k] = u[n]$$

$$h'[n] = ?$$

$$h'[n] * \sum_{k=0}^{\infty} \delta[n-k] = \delta[n]$$

$$\sum_{k=0}^{\infty} h'[n-k] = \delta[n]$$

$$h'[n] + h'[n-1] + h'[n-2] + \ldots = \delta[n]$$

$$h'[0] = 1 \quad h'[1] + h'[0] = 0 \quad h'[1] = -1$$

$$h'[2] = 0, h'[n] = 0, n > 1$$

$$h'[n] = \delta[n] - \delta[n-1]$$

Stability of Systems

All LTI systems are BIBO stable if and only if h[n] is absolutely summable:

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

This is due to the fact that:

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k] x[n-k] \right| \leq B_y < \infty,$$

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k] x[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |h[k]| |x[n-k]|$$

if $x[n]$ is bounded ($|x[n]| \leq B_x$), so that

$$|y[n]| \leq B_x \sum_{k=-\infty}^{\infty} |h[k]|, \quad \sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Stability of Systems-Example

$$h[n] = \alpha^n u[n] \text{ is stable?}$$

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=0}^{\infty} |a|^k$$

$$\text{if } |\alpha| < 1 \text{ the system is stable}$$

$$\text{if } |\alpha| \geq 1 \text{ the system is NOT stable}$$

Stability of Systems-Example

$$h[n] = \sum_{k=-\infty}^{n} \delta[k]?$$

$$= \begin{cases} 1 & n \geq 0 \\ 0 & n<0 \end{cases}$$

$$= u[n]$$

The system is NOT stable

Stability of Systems-Example

$$h[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} \delta[n-k]?$$

$$= \begin{cases} \dfrac{1}{M_1 + M_2 + 1} & M_1 \leq n \leq M_2 \\ 0 & \text{otherwise} \end{cases}$$

The system is stable

Causality of Systems

A LTI system is causal if an only if

$$h[n] = 0 \text{ for } n < 0$$

Causality of Systems

$$y[n] = x[n - n_0]$$

Is $h[n]$ causal?

$$h[n] = \delta[n - n_0]$$

Since h[n] includes only one sample at $n=n_0$, h[n] is causal.
It is stable because it is absolutely summable.

$$y[n] = x[n+1] - x[n]$$

Is $h[n]$ causal?

$$h[n] = \delta[n+1] - \delta[n]$$

Since h[n] includes sample at $n=-1$, h[n] is not causal,
however it is stable because it is absolutely summable.

Causality of Systems

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} x[n-k]$$

Is $h[n]$ causal?

$$h[n] = \frac{1}{N}\sum_{k=0}^{N-1} \delta[n-k] = \begin{cases} \dfrac{1}{N} & 0 \le n \le N-1 \\ 0 & n < 0, n > N-1 \end{cases}$$

Since h[n] does not include samples when at *n<0* , h[n] is causal.
It is stable because it is absolutely summable.

Causality of Systems

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Is $h[n]$ causal?

$$h[n] = \sum_{k=-\infty}^{n} \delta[k] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$h[n] = u[n]$$

Since h[n] does not include samples when at *n<0* , h[n] is causal.
It is NOT stable because it is NOT absolutely summable.

FIR and IIR Systems

- ✓ Finite Impulse Response (FIR) Systems:
  - Systems with only a finite length of nonzero values in h[n] are called FIR systems.
  - Examples: Ideal delay, moving average filter…
  - FIR systems are always STABLE
- ✓ Infinite Impulse Response (IIR) Systems:
  - Systems with infinite length of nonzero values in h[n] are called IIR systems.
  - Examples:Accumulator, filters …
  - IR systems can be STABLE or UNSTABLE

FIR and IIR Systems

FIR Systems:

$$h[n] = \delta[n - n_0]$$

$$h[n] = \delta[n+1] - \delta[n]$$

$$h[n] = \frac{1}{N}\sum_{k=0}^{N-1}\delta[n-k]$$

IIR Systems:

$$h[n] = u[n]$$

$$h[n] = a^n u[n]$$