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Digital Signal Processing
Lecture 3

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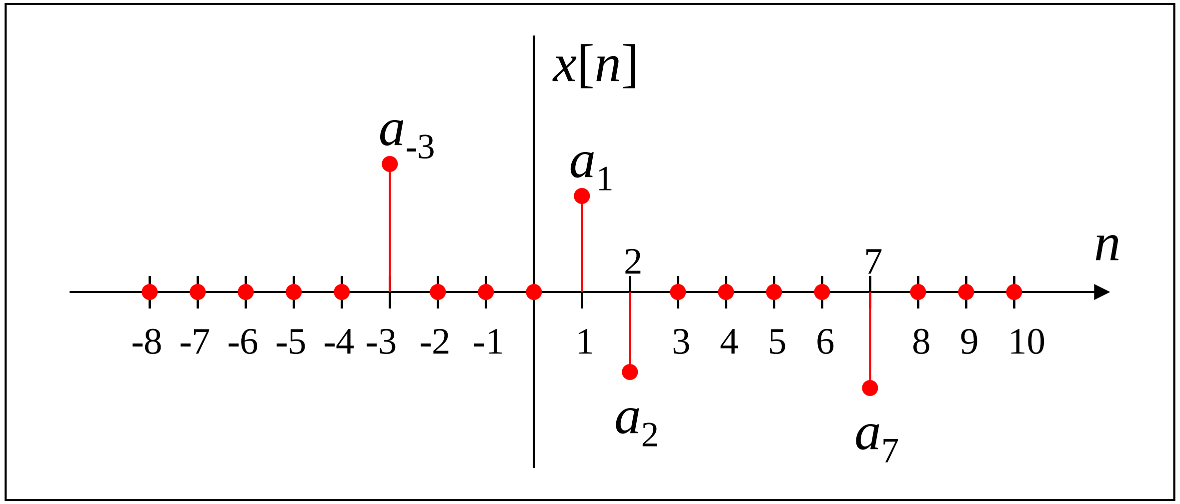
Brief summary

Sequence representation using delay unit:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$



$$x[n] = a_{-3}\delta[n+3] + a_1\delta[n-1] + a_2\delta[n-3] + a_7\delta[n-7]$$

Brief summary

1-If a discrete time signal is written as a sequence of numbers inside braces, the location of the sample value associated with time index n=0 is indicated by an arrow under it.

$$\{x[n]\} = \{\ldots, 0.35, 1, \underset{\uparrow}{1.5}, -0.6, -2, \ldots\}$$

2- $\boxed{x[n] = u[n] + u[-n] \Rightarrow x[n] = 1 + \delta[n] \quad \forall n}$

3- $\boxed{x[n] = u[n] + u[-n-1] = 1 \quad \forall n}$

Brief summary

Discrete-time sinusoids whose freq. are separated by an integer multiple of 2π are identical. Proof:

$$\cos[(\omega + 2\pi r)n] = \cos(\omega n + 2\pi rn)$$

$$\cos(a \pm b) = \cos(a).\cos(b) \mp \sin(a)\sin(b)$$

$$\text{let } \alpha = \omega n \ \& \ b = 2\pi rn$$

$$= \cos(\omega n)\cos(2\pi rn) - \sin(\omega n)\sin(2\pi rn)$$

$$\cos(2\pi rn) = 1 \forall r \quad \sin(2\pi rn) = 0 \forall r$$

$$\cos((\omega + 2\pi r)n) = \cos(\omega n)$$

Digital Signals

- ✓ **Bounded signal:** $|x[n]| \le B_x < \infty \quad \forall n$
- ✓ **Absolutely summable sequence:** $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$
- ✓ **Square summable sequence** (Finite Energy sequence): $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$
- ✓ **Energy of the sequence:** $E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$
- ✓ **Power:** $P_x = \lim_{K \to \infty} \frac{1}{2K+1} \sum_{n=-K}^{K} |x[n]|^2$

Examples

Find the energy of the sequence given below:

$$x[n] = \begin{cases} 2^{-n}, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

According to the definition, the energy is given by:

$$E = \sum_{n=0}^{\infty} 2^{-2n} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n$$

Geometric series expansion is required for the solution:

$$\sum_{k=M}^{N} r^k = \frac{r^{N+1} - r^M}{r-1} \quad , \ r \neq 1 \qquad \Rightarrow \qquad E = \frac{1}{1 - 1/4} = \frac{4}{3}$$

Examples

- In discrete-time signals and systems we will always encounter the summation of exponential samples, that is
$$\sum_{n=N_1}^{n_2} r^n, \quad N_1 \le N_2$$
which is equivalent to the integral of exponential signal in continuous time.
- **Basic result**:
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad \text{provided } |r| < 1$$
- **Additional Results**:
$$\sum_{n=N_1}^{\infty} r^n = \frac{r^{N_1}}{1-r}, \quad \text{provided } |r| < 1$$
$$\sum_{n=0}^{N-1} r^n = \frac{1-r^N}{1-r}, \quad \text{no condition on } r$$
$$\sum_{n=N_1}^{N_2-1} r^n = \frac{r^{N_1}-r^{N_2}}{1-r}, \quad \text{no condition on } r$$

Examples

Estimate the energy and power of x[n] given below:

$$x[n] = u[n]$$

$$E = \sum_{n=-\infty}^{\infty} \left| x[n] \right|^2 = \sum_{n=0}^{\infty} 1^2 = \infty$$

$$\begin{aligned}
P &= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left| x[n] \right|^2 \\
&= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} 1^2 \\
&= \lim_{N \to \infty} \frac{N+1}{2N+1} \\
&= \lim_{N \to \infty} \frac{1+1/N}{2+1/N} \\
&= \frac{1}{2}
\end{aligned}$$

apply L'Hopital's rule to solve

Examples

Is $x[n] = \alpha^n u[n]$ an energy or power signal?

$a)\ |\alpha| < 1$

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=0}^{\infty} |a^n|^2 = \sum_{n=0}^{\infty} a^{2n}$$

Geometric series expansion is required for the solution:

$$\sum_{k=M}^{N} r^k = \frac{r^{N+1} - r^M}{r-1} \quad , \ r \neq 1$$

$$E_x = \lim_{N\to\infty} \sum_{n=0}^{N} \left(a^2\right)^n = \lim_{N\to\infty} \frac{a^{2(N+1)} - a^0}{a^2 - 1}$$

$$\lim_{N\to\infty} a^{2(N+1)} = 0 \ , \ |a| < 1$$

$$E_x = \frac{-1}{a^2 - 1} = \frac{1}{1 - a^2}$$

Examples

Is $x[n]=\alpha^n u[n]$ an energy or power signal?

$b)\ |\alpha|=1$

$$E_x=\sum_{n=0}^{\infty}\left|x[n]\right|^2=\sum_{n=0}\left|\alpha^n\right|^2=\sum_{n=0}^{\infty}\alpha^{2n}$$

$$E_x=\sum_{n=0}^{\infty}\alpha^{2n}=\sum_{n=0}^{\infty}1, \text{thus not an energy signal.}$$

$$\mathrm{P}=\lim_{\mathrm{N}\to\infty}\frac{1}{2N+1}\sum_{n=0}^{N}1=\lim_{\mathrm{N}\to\infty}\frac{N+1}{2N+1}$$

apply L'Hospital's rule to solve

$$P=\frac{1}{2}$$

Digital Signals

- ✓ A discrete time signal maybe a finite length or an infinite length sequence.

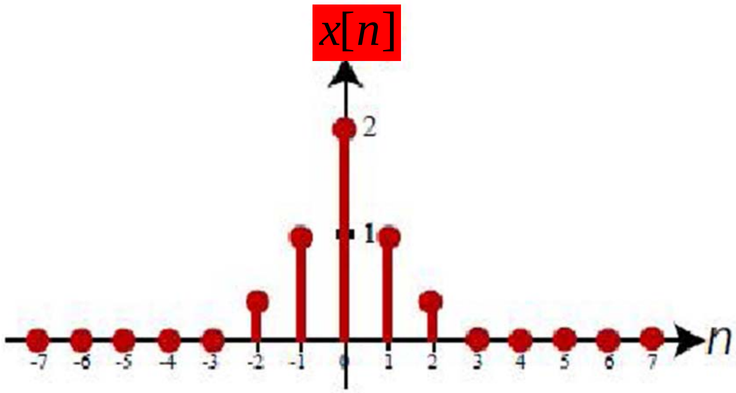
- ✓ An infinite-length sequence can be:
  - Right Sided Signal: the sequence has zero-valued samples for $n < N_1$ where N_1 is a finite integer that can be positive or negative.
  - Left Sided Signal: the sequence has zero-valued samples for $n > N_2$ where N_2 is a finite integer that can be positive or negative.
  - Two sided signal that is the combination of left and right sided.

Digital Signals

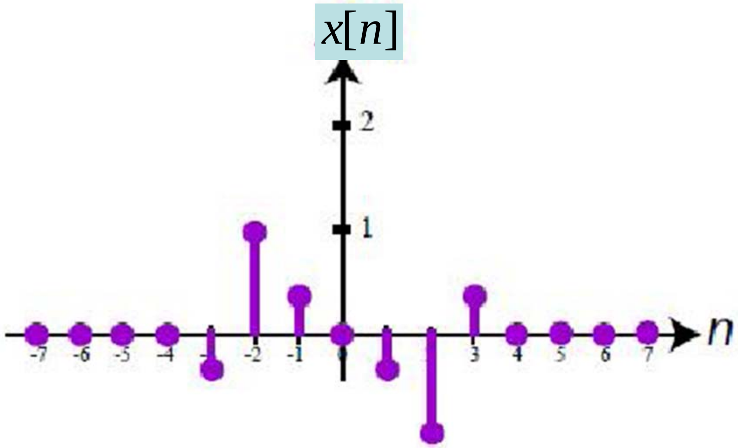
✓ Sequences can be even or odd or general:

✓ If $x[n] = x[-n]$, it is an even signal.

✓ If $x[n] = -x[-n]$, it is an odd signal.



even signal



odd signal

Digital Signals

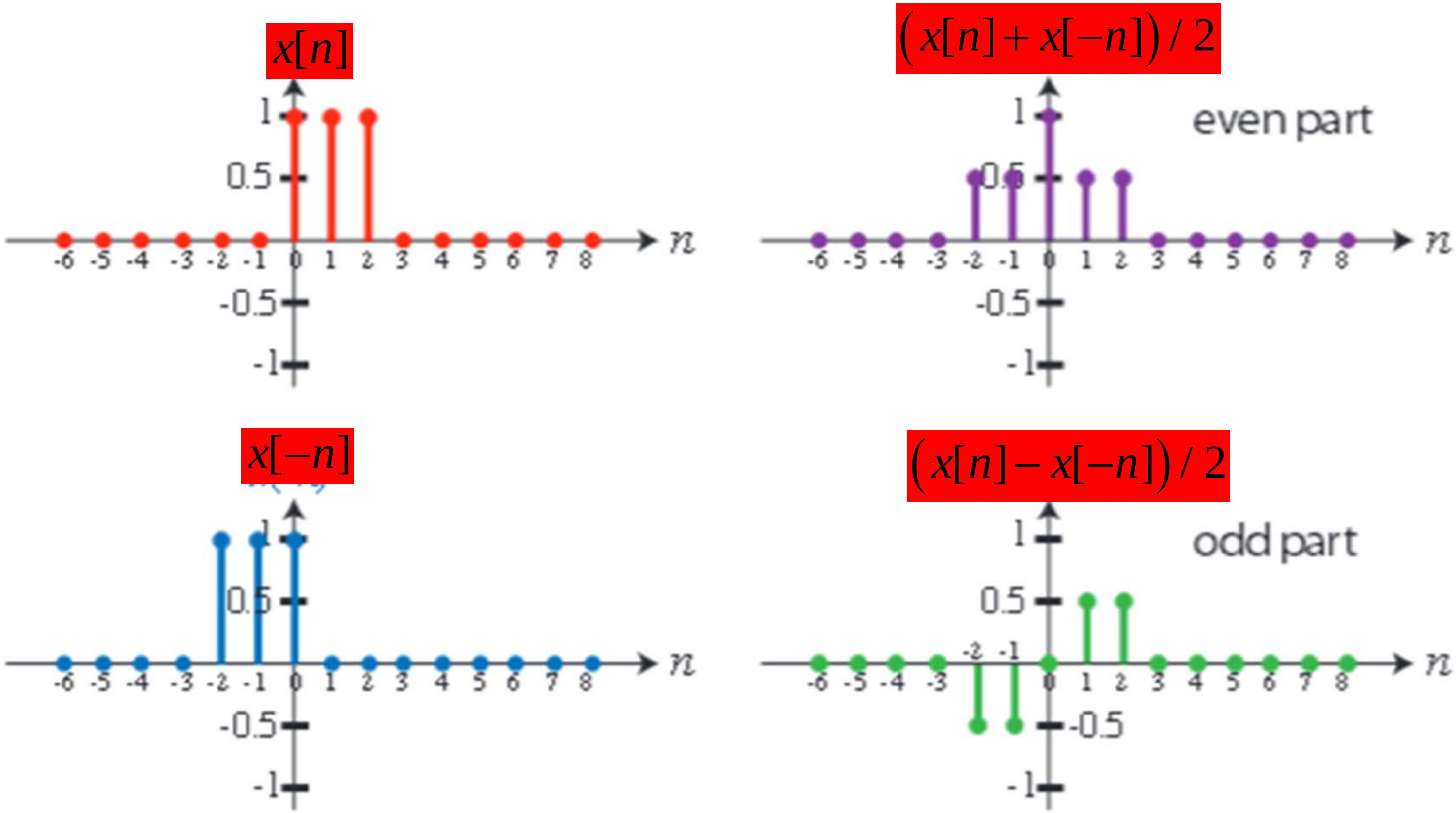
- ✓ Any sequence can be expressed as a sum of its even part and its odd part:

$$x[n] = x_{even}[n] + x_{odd}[n]$$

where

$$x_{even}[n] = \frac{1}{2}\left(x[n] + x[-n]\right) \qquad x_{odd}[n] = \frac{1}{2}\left(x[n] - x[-n]\right)$$

Digital Signals



Example

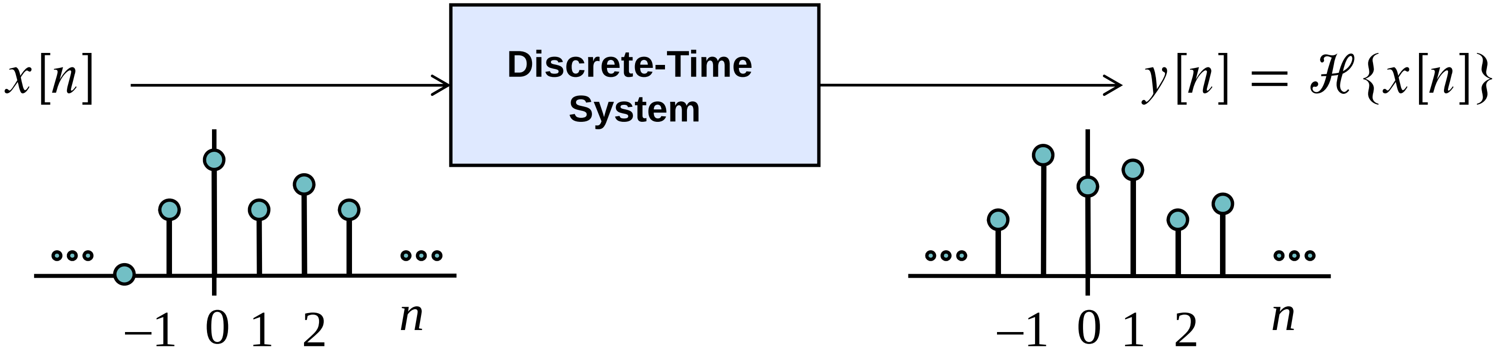
Find the even and the odd components of the discrete-time signal

$$x[n] = \begin{cases} 4 - n & 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

$$x_e[n] = 0.5\big(x[n] + x[-n]\big) \Rightarrow x_e[n] = \begin{cases} 2 + 0.5n & -4 \le n \le -1 \\ 4 & n = 0 \\ 2 - 0.5n & 1 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

$$x_o[n] = 0.5\big(x[n] - x[-n]\big) \Rightarrow x_o[n] = \begin{cases} -2 - 0.5n & -4 \le n \le -1 \\ 0 & n = 0 \\ 2 - 0.5n & 1 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Digital Systems



- A discrete-time system is a computational process or algorithm that transforms a sequence $x[n]$, called the input signal, into another sequence $y[n]$, called the output signal
- Implementation
  - Equation or algorithm on paper
  - Software on a general purpose computer
  - Software on a special purpose Digital Signal Processor
  - Dedicated hardware

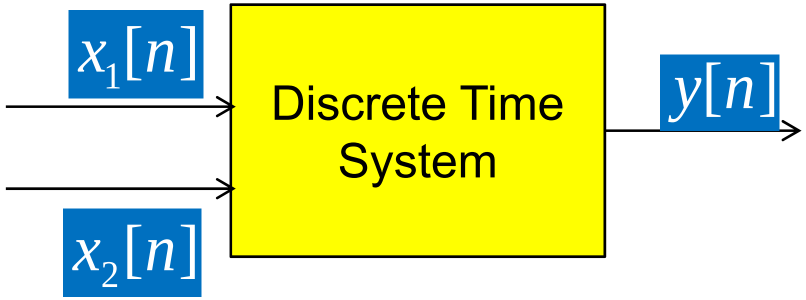
Digital Systems

Examples

1- $y[n] = x[n - n_0], -\infty < n < \infty$

→ Sample shifting

✓ 2- Multiplexer:



$$y[n] = \begin{cases} x_1[n/2] & , \ n \text{ even} \\ x_2[(n-1)/2], & n \text{ odd} \end{cases}$$

3- $$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} x[n-k]$$

↓

N point moving average system

Examples

4- Accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$= x[n] + y[n-1]$$

or

$$y[n] = \sum_{k=-\infty}^{-1} x[k] + \sum_{k=0}^{n} x[k]$$

initial condition

$$= y[-1] + \sum_{k=0}^{n} x[k]$$

Memoryless Systems

✓ If the output of the system depends only on the current values of the input, the system is known as memoryless.

$$y[n] = x[n] + x^2[n]$$ → Memoryless

$$y[n] = x[n] + x[n+5]$$ → Not Memoryless

$$y[n] = x[n] + 5u[n-1]$$ → Memoryless

$$y[n] = a^{n-1}x[n]$$ → Memoryless

Causal Systems

- ✓ A system is causal if the output of the system depends on only the current and past values of the input signal. In other words, the system does not depend on the future values of the input.

$$y[n] = x[n] + x^2[n] \longrightarrow \text{Causal}$$

$$y[n] = x[n] + x[n+5] \longrightarrow \text{Not-causal}$$

$$y[n] = x[n] + 5u[n-1] \longrightarrow \text{Causal}$$

$$y[n] = a^{n-1}x[n] \longrightarrow \text{Causal}$$

Stable Systems and Passivity of system

- ✓ A system is stable if every bounded input sequence produces a bounded output sequence.

$$|x[n]| \le B_x < \infty$$

$$|y[n]| \le B_y < \infty$$

- ✓ A system is passive if the energy of the output can not exceed the energy of the input:

$$\sum_{n=-\infty}^{\infty} |y[n]|^2 \le \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Stable Systems

$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

$$|x[n]| \le B_x < \infty$$

$$|y[n]| \le \frac{1}{M}\sum_{k=0}^{M-1} B_x$$

$$|y[n]| \le B_y$$

The system is BIBO stable.

Stable Systems

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = \sum_{k=-\infty}^{n} u[k] = \begin{cases} 0 & , n < 0 \\ n+1 & , n \geq 0 \end{cases}$$

Output has no finite upper bound. Therefore, the system gives unbounded output for bounded signal

Linear Systems

✓ A system is linear if the system obey additivity and homogeneity (scaling):

- Additivity:

$$x_1[n] \rightarrow y_1[n] \text{ and } x_2[n] \rightarrow y_2[n]$$
$$x_1[n] + x_2[n] \rightarrow y_1[n] + y_2[n]$$

- Scaling:

$$x_1[n] \rightarrow y_1[n]$$
$$ax_1[n] \rightarrow ay_1[n]$$

Superposition=additivity+scaling

$$x_1[n] \rightarrow y_1[n] \text{ and } x_2[n] \rightarrow y_2[n]$$
$$ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n]$$

Linear Systems

$$y[n] = x[n] + x[n+5]$$

$$y_1[n] = x_1[n] + x_1[n+5]$$
$$y_2[n] = x_2[n] + x_2[n+5]$$
$$y_3[n] = \alpha y_1[n] + b y_2[n]$$

$$x_3[n] = \alpha x_1[n] + b x_2[n]$$
$$y_3[n] = x_3[n] + x_3[n+5]$$
$$y_3[n] = \left(\alpha x_1[n] + b x_2[n]\right) + \left(\alpha x_1[n+5] + b x_2[n+5]\right)$$

$$=$$

LINEAR

Linear Systems

$$y[n] = x[n] + 5u[n-1]$$

$$y_1[n] = x_1[n] + 5u[n-1]$$
$$y_2[n] = x_2[n] + 5u[n-1]$$
$$y_3[n] = \alpha y_1[n] + b y_2[n]$$

$$x_3[n] = \alpha x_1[n] + b x_2[n]$$
$$y_3[n] = x_3[n] + 5u[n-1]$$
$$y_3[n] = \alpha x_1[n] + b x_2[n] + 5u[n-1]$$

$$\neq$$

Non-LINEAR

Linear Systems

$$y[n] = \sum_{l=-\infty}^{n} x[l]$$

LINEAR

$$y[n] = y[-1] + \sum_{l=0}^{n} x[l]$$

Non-LINEAR

For linearity, zero input should result in zero output!

Linear Systems

✓ Examples:

$$y[n] = x[n] + x^2[n] \longrightarrow \text{Non Linear}$$

$$y[n] = x[n - n_0] \longrightarrow \text{Linear}$$

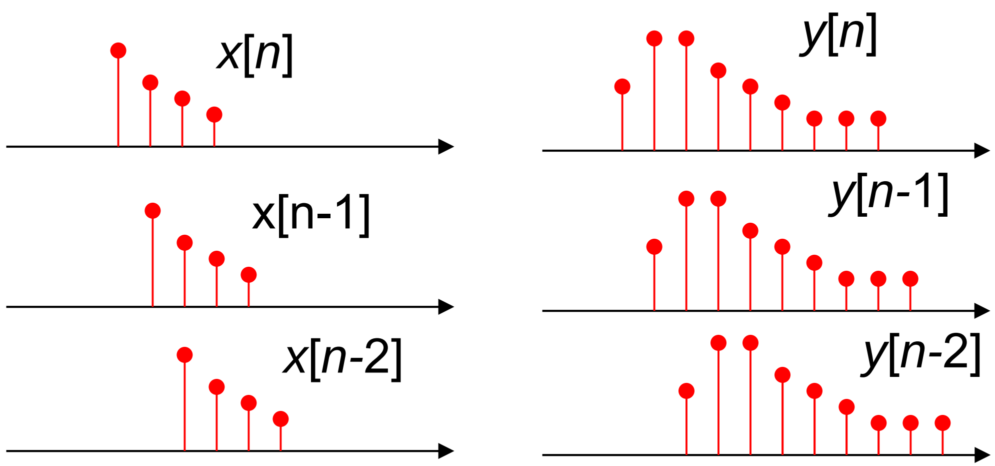
$$y[n] = x[n] + 1 \longrightarrow \text{Non Linear}$$

$$y[n] = a^{n-1}x[n] \longrightarrow \text{Linear}$$

Time Invariant Systems

✓ A system is time invariant if a shift in the input signal produce the same shift in the output signal. In other words, a system is called time-invariant if its input-output characteristics do not change with time.

$$x[n] \rightarrow y[n]$$
$$x[n-n_0] \rightarrow y[n-n_0]$$



✓ An up sampler system is time invariant?

Time Invariant Systems

$$y[n] = x[n] + x[n+5]$$

$$y_1[n] = x_1[n] + x_1[n+5]$$
$$y_1[n-n_0] = x_1[n-n_0] + x_1[n-n_0+5]$$

$$x_2[n] = x_1[n-n_0]$$
$$y_2[n] = x_2[n] + x_2[n+5]$$
$$y_2[n] = x_1[n-n_0] + x_1[n-n_0+5]$$

$$=$$

Time Invariant

Time Invariant Systems

$$y[n] = x[n] + 5u[n-1]$$

$$y_1[n] = x_1[n] + 5u[n-1]$$
$$y_1[n-n_0] = x_1[n-n_0] + 5u[n-n_0-1]$$

$$x_2[n] = x_1[n-n_0]$$
$$y_2[n] = x_2[n] + 5u[n-1]$$
$$y_2[n] = x_1[n-n_0] + 5u[n-1]$$

$$\neq$$

Not Time Invariant

Time Invariant Systems

$$y[n] = x[n] + x^2[n]$$ → Time invariant

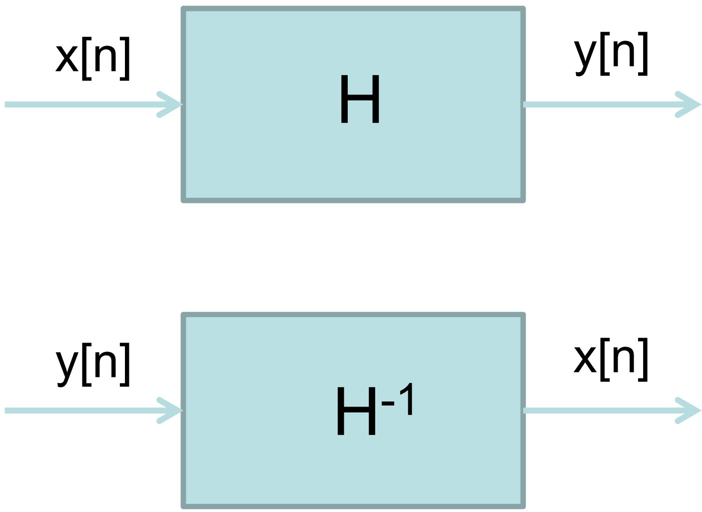
$$y[n] = x[n - n_0]$$ → Time invariant

$$y[n] = x[2n]$$ → Time variant

$$y[n] = a^{n-1}x[n]$$ → Time variant

Invertible Systems

A system is invertible if the input sequence is reconstituted using a system that takes y[n] the as input.



Example

$$y[n] = x[-n]$$

linear

non-causal

time varying

bounded

passive