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Digital Signal Processing
Lecture 6

Quote of the Day

The profound study of nature is the most fertile source of mathematical discoveries.

Joseph Fourier

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Discrete Time Fourier Transform

- ✓ The Discrete time Fourier transform (DTFT) of a discrete time sequence x[n] is a representation of the sequence in terms of complex exponential sequence.

- ✓ The Fourier transform representation of a sequence, if it exists, is unique and the original sequence can be computed from its transform representation by an inverse transform operation.

Discrete Time Fourier Transform

Discrete Time Fourier Transform:

$$X\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$X\left(e^{j\omega}\right)=X_{re}\left(e^{j\omega}\right)+jX_{im}\left(e^{j\omega}\right)$$
$$=\left|X\left(e^{j\omega}\right)\right|e^{j\angle X\left(e^{j\omega}\right)}$$

It is also known as Fourier spectrum/ Freq .Spectrum/Spectrum

Inverse transform:

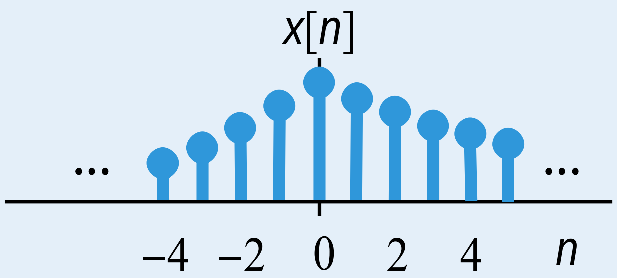
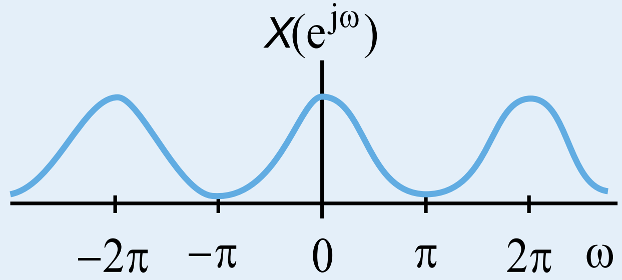
$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right)e^{j\omega n}d\omega$$

Periodicity:

$$X\left(e^{j(\omega+2\pi r)}\right)=\sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2\pi r)n}=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}e^{-j2\pi rn}=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}=X\left(e^{j\omega}\right)$$

Frequency Domain Representation

Discrete-time signals

Time-domain	Frequency-domain
	

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \xrightarrow{\text{DTFT}}$$

$$\xleftarrow{\text{IDTFT}} x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Discrete and aperiodic	Continous and periodic

Example:

$$x[n] = \delta[n]$$

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n}$$

$$= \delta[0] e^{-j\omega 0}$$

$$= 1$$

Example:

$$x[n] = 3\delta[n+2] + \delta[n] + 2\delta[n-1] - \delta[n-3]$$

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} \left(3\delta[n+2] + \delta[n] + 2\delta[n-1] - \delta[n-3]\right) e^{-j\omega n}$$

$$= 3e^{-j\omega(-2)} + e^{-j\omega 0} + 2e^{-j\omega 1} - e^{-j\omega 3}$$

$$= 3e^{j2\omega} + 1 + 2e^{-j\omega} - e^{-j3\omega}$$

Frequency Domain Representation

Frequency Response:

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

$$H(e^{j\omega}) = H_{re}(e^{j\omega}) + jH_{im}(e^{j\omega})$$

$$H(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{j\angle H(e^{j\omega})}$$

Phase function

Magnitude function

Example:

✓ What is the frequency response of a system whose input output relation is: $y[n] = x[n - n_0]$

$$h[n] = \delta[n - n_0]$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \delta[k - n_0] e^{-j\omega k} = e^{-j\omega n_0}, \left|H(e^{j\omega})\right| = 1, \ \angle H(e^{j\omega}) = -\omega n$$

Example:

$$h[n] = \begin{cases} 1 \quad , & 0 \le n \le N-1 \\ 0 \quad , & \text{otherwise} \end{cases}$$

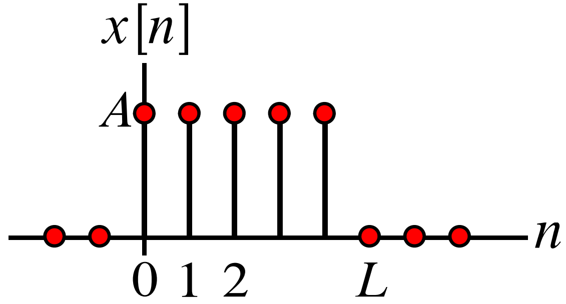
$$\Rightarrow$$

$$H(e^{j\omega}) = ?$$

$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} = \sum_{n=0}^{N-1} e^{-j\omega n} = \sum_{n=0}^{N-1} \left(e^{-j\omega}\right)^n$$

$$H\left(e^{j\omega}\right) = \sum_{n=0}^{N-1} \left(e^{-j\omega}\right)^n = \frac{e^{-j\omega N} - e^{-j\omega 0}}{e^{-j\omega} - 1} = \frac{e^{-j\omega N} - 1}{e^{-j\omega} - 1} = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

DTFT of Rectangular Pulse Sequence



$$x[n] = \begin{cases} A, & 0 \le n \le L-1 \\ 0, & \text{elsewhere} \end{cases}$$

Since $\sum_{n=-\infty}^{\infty} |x[n]| = L|A| < \infty$, its DTFT exists with $E_x = L|A|^2$

Then $X(e^{j\omega}) = \sum_{n=0}^{\infty} A e^{-j\omega n} = A \dfrac{1 - e^{-j\omega L}}{1 - e^{-j\omega}} = A e^{-j(\omega/2)(L-1)} \dfrac{\sin(\omega L/2)}{\sin(\omega/2)}$

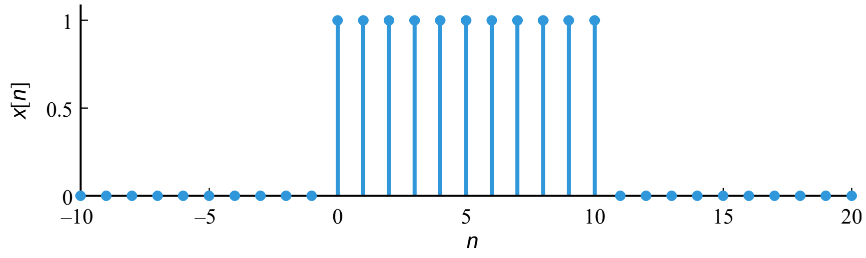
Hence

$$|X(e^{j\omega})| = \begin{cases} |A| L, & \omega = 0 \\ |A| \left| \dfrac{\sin(\omega L/2)}{\sin(\omega/2)} \right|, & \omega \ne 0 \end{cases}$$

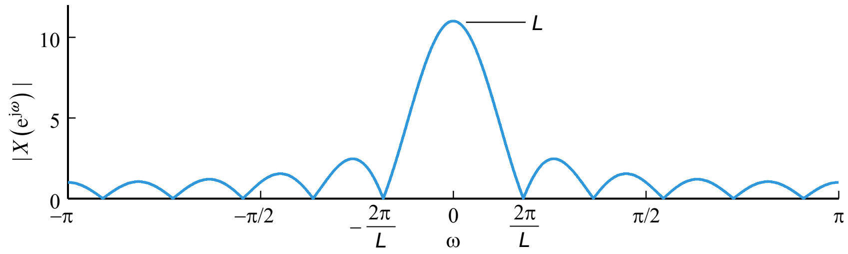
$$\angle\{X(e^{j\omega})\} = \angle\{A\} - \frac{\omega}{2}(L-1) + \angle\left\{ \frac{\sin(\omega L/2)}{\sin(\omega/2)} \right\}$$

Rectangular Pulse Sequence: DTFT Plots

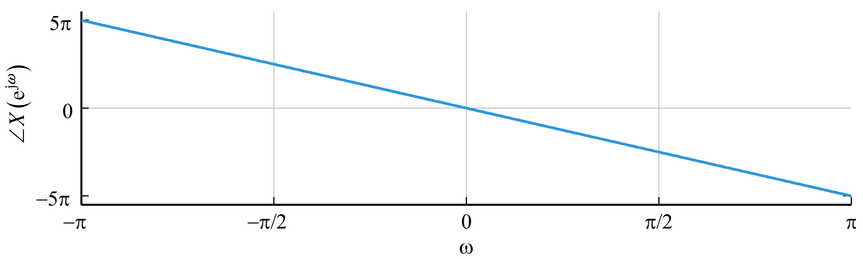
If L=11 and A=1:



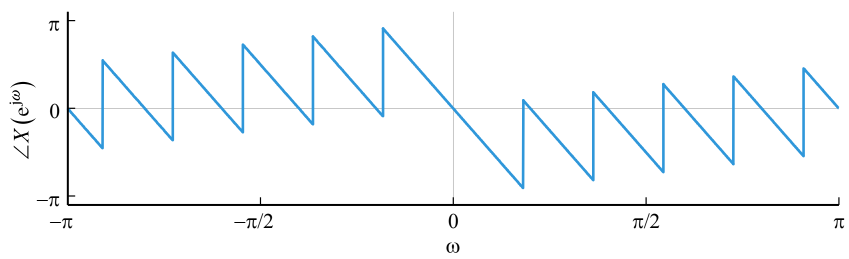
(a)



(b)



(c)



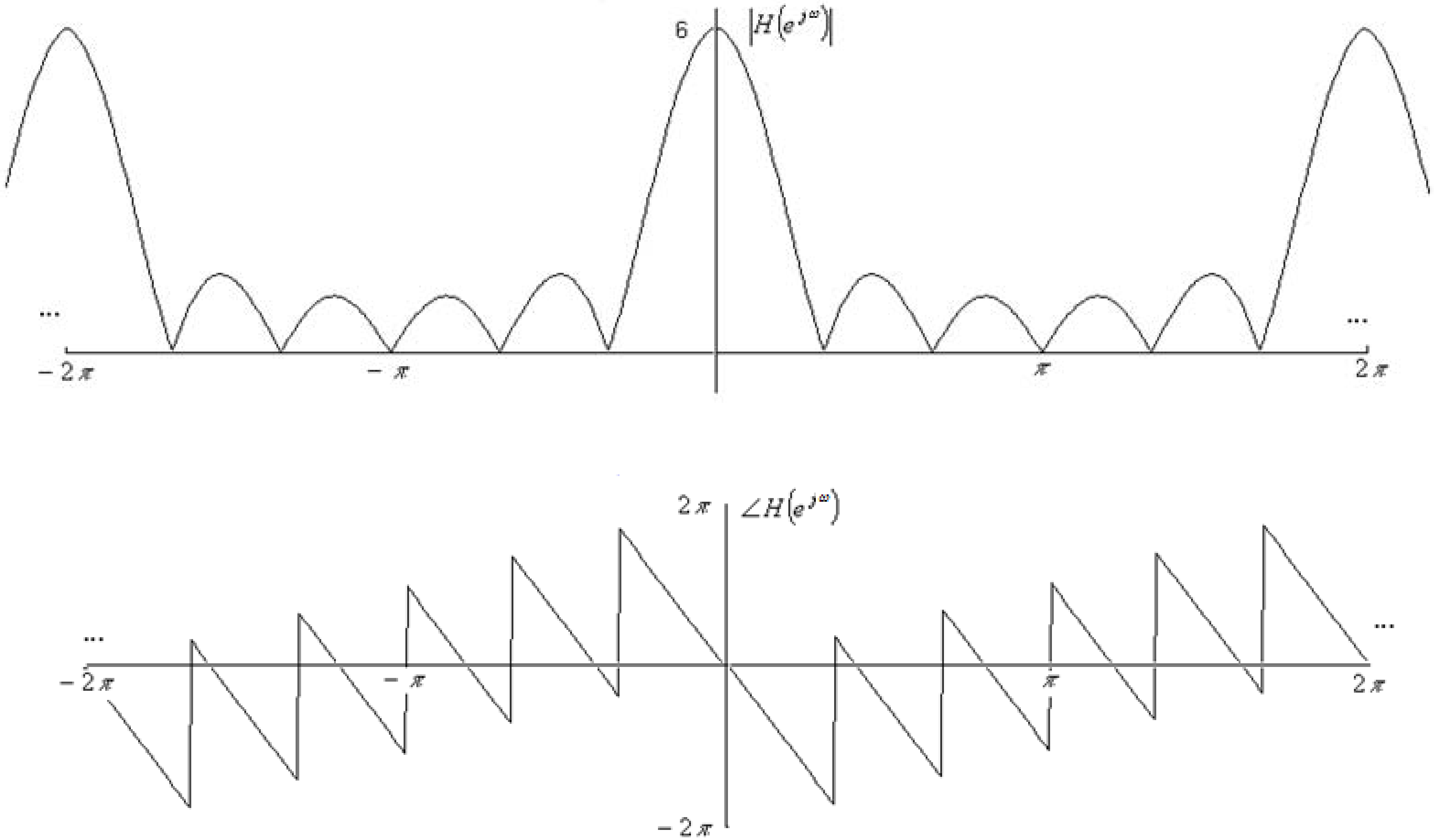
(d)

The phase obtained by using directly the equation

The phase obtained by using Matlab 'angle' function

Example-Cont:

for N=6



Example:

$x[n] = a^n u[n]$ $\quad$ $X(e^{j\omega}) = ?$

$$X\left(e^{j\omega}\right) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N} \left(ae^{-j\omega}\right)^n = \lim_{N\to\infty} \frac{\left(ae^{-j\omega}\right)^{N+1} - 1}{ae^{-j\omega} - 1}$$

$$\left|ae^{-j\omega}\right| < 1 \;\rightarrow\; \left|e^{-j\omega}\right| = 1 \;\Rightarrow\; X\left(e^{j\omega}\right) = \frac{1}{1 - ae^{-j\omega}}, |a| < 1$$

Example-Cont:

- DTFT of the real sequence $x[n]=a^n u[n]$

$$X\left(e^{j\omega}\right)=\frac{1}{1-ae^{-j\omega}} \qquad \text{if } |a|<1$$

- Some properties are

$$X\left(e^{j\omega}\right) = \frac{1}{1-ae^{-j\omega}}$$

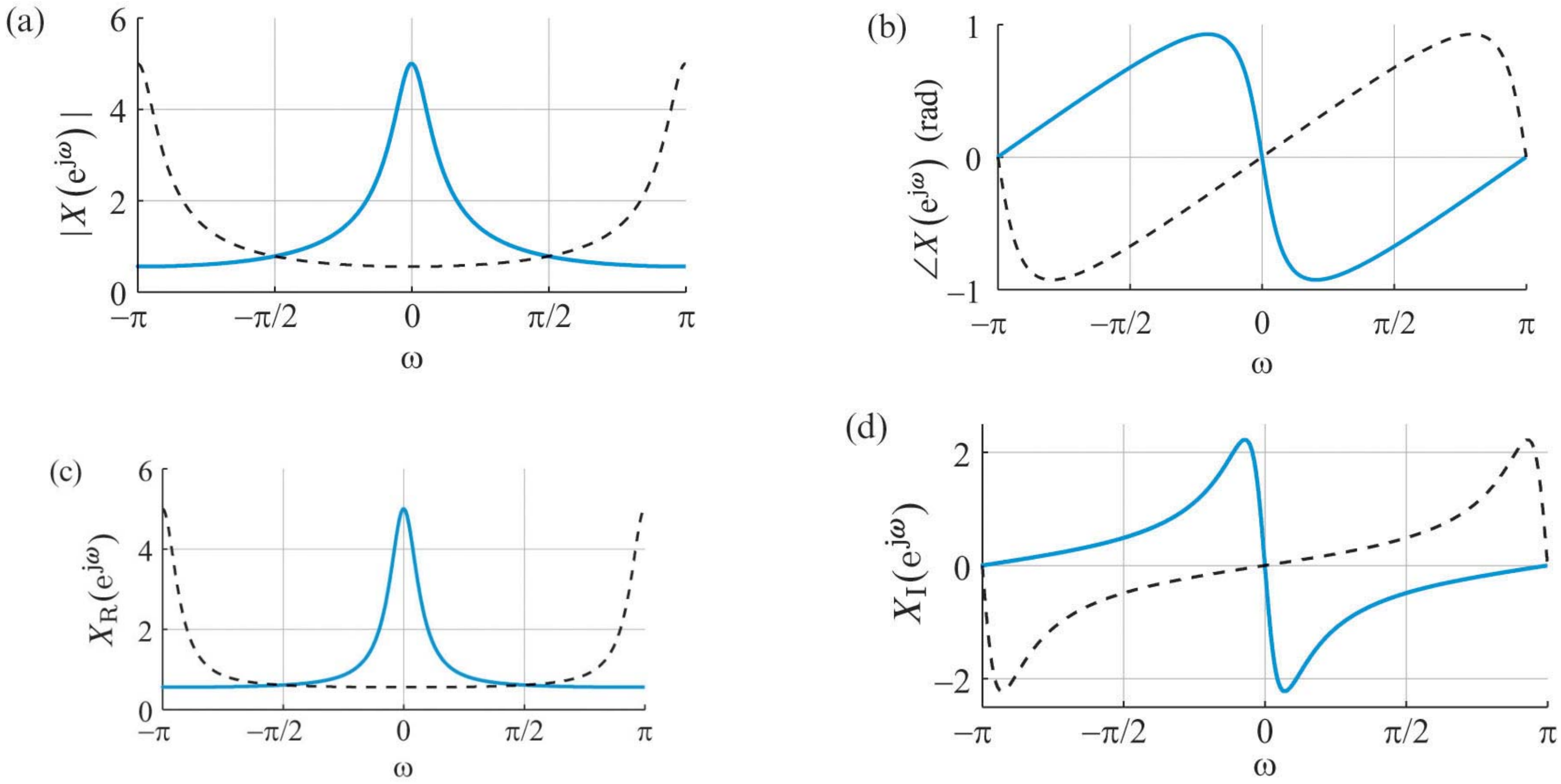
$$X_R\left(e^{j\omega}\right) = \frac{1-a\cos\omega}{1+a^2-2a\cos\omega}$$

$$X_I\left(e^{j\omega}\right) = \frac{-a\sin\omega}{1+a^2-2a\cos\omega}$$

$$\left|X\left(e^{j\omega}\right)\right| = \frac{1}{\sqrt{1+a^2-2a\cos\omega}}$$

$$\angle X\left(e^{j\omega}\right) = \tan^{-1}\left(\frac{-a\sin\omega}{1-a\cos\omega}\right)$$

Example-Cont:



- ✓ The solid line correspond to a=0.8
- ✓ The dashes line correspond to a=-0.8

Convergence Condition of DTFT

Since $X(e^{j\omega})$ is a infinite sum, it may not converge. We need the convergence result, that is

$$\left|X(e^{j\omega})\right| < \infty$$

Then,

$$\left|X(e^{j\omega})\right| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right| \le \sum_{n=-\infty}^{\infty} \left|x[n]\right|\left|e^{-j\omega n}\right| \le \sum_{n=-\infty}^{\infty} \left|x[n]\right| < \infty$$

Thus absolute summability is sufficient condition for the existence of $X(e^{j\omega})$.

In other words if x[n] is absolutely summable it's FT exists.

Uniform Convergence Condition

The Fourier transform $X(e^{j\omega})$ of x[n] is said to exist if it converges in some sense. Let

$$X_M(e^{j\omega}) = \sum_{n=-M}^{M} x[n] e^{-j\omega n}$$

denote the partial sum of the weighted complex exponentials. Then the uniform convergence :

$$\lim_{M \to \infty} \left| X_M(e^{j\omega}) - X(e^{j\omega}) \right| \to 0$$

Some sequences are not absolutely summable, thus they do not achieve uniform convergence. For example, infinite length sequences may or may not converge uniformly.

Mean Square Convergence Condition

- Absolutely summable sequences always have finite energy. However, finite energy sequences are not necessary absolutely summable.

- Some sequences are square summable that achieve mean-square convergence.

- Convergence in terms of means square error (MSE):

$$\lim_{M\to\infty} \underbrace{\int \left| X(e^{j\omega}) - X_M(e^{j\omega}) \right|^2 d\omega}_{\text{energy of the error}} = 0$$

- The total energy of the error must approach zero, not an error itself.

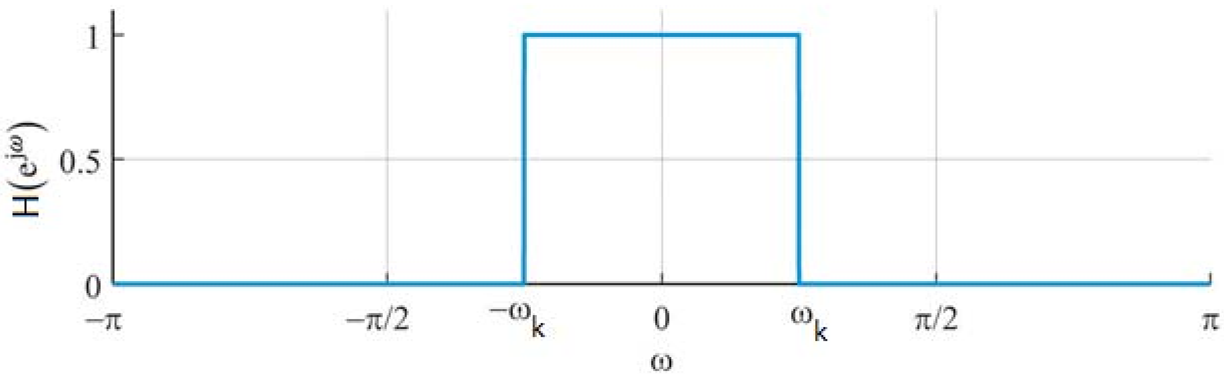
- In this case, we say that FT of x[n] exists.

Example:

Ideal Low Pass Filter:

$$H\left(e^{j\omega}\right)=\begin{cases}1 \quad , & |\omega|<\omega_k \\ 0 \quad , & \omega_k<|\omega|<\pi\end{cases}$$

$$h[n]=?$$



Example-Cont:

$$h[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}H\left(e^{j\omega}\right)e^{j\omega n}d\omega=\frac{1}{2\pi}\int_{-\omega_k}^{\omega_k}e^{j\omega n}d\omega=\frac{1}{2\pi}\left(\frac{e^{j\omega n}}{jn}\right)\Bigg|_{-\omega_k}^{\omega_k}$$

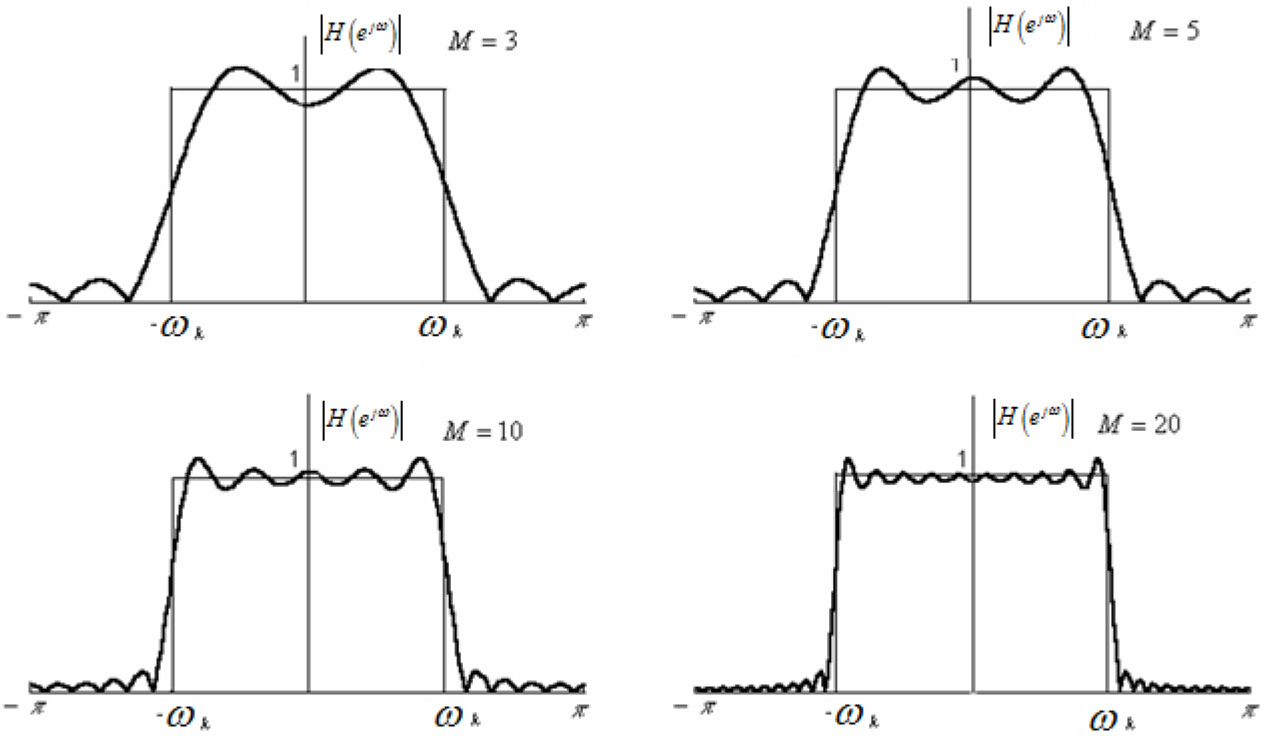
$$h[n]=\frac{1}{2\pi}\left(\frac{e^{j\omega n}}{jn}\right)\Bigg|_{-\omega_k}^{\omega_k}=\frac{1}{2\pi}\left(\frac{e^{j\omega_k n}-e^{-j\omega_k n}}{jn}\right)$$

$$h[n]=\frac{1}{2\pi}\left(\frac{e^{j\omega_k n}-e^{-j\omega_k n}}{jn}\right)=\frac{1}{2\pi}\frac{2j\sin\omega_k n}{jn}=\frac{\sin\omega_k n}{\pi\ n}\quad,-\infty<n<\infty$$

$$H\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}\frac{\sin\omega_k n}{\pi\ n}e^{-j\omega n}$$

Example-Cont:

$$H_M\left(e^{j\omega}\right) = \sum_{n=-M}^{M} \frac{\sin \omega_k n}{\pi\ n} e^{-j\omega n}$$



Convergence Conditions

- Consider the sequence $x[n]$ with the ideal low-pass DTFT spectrum

$$x[n] = ? \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

  - The inverse DTFT is

$$x[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega = \frac{1}{j2\pi n}\, e^{j\omega n}\Big|_{-\omega_c}^{\omega_c} = \frac{\sin \omega_c n}{\pi n}, \quad n \neq 0$$

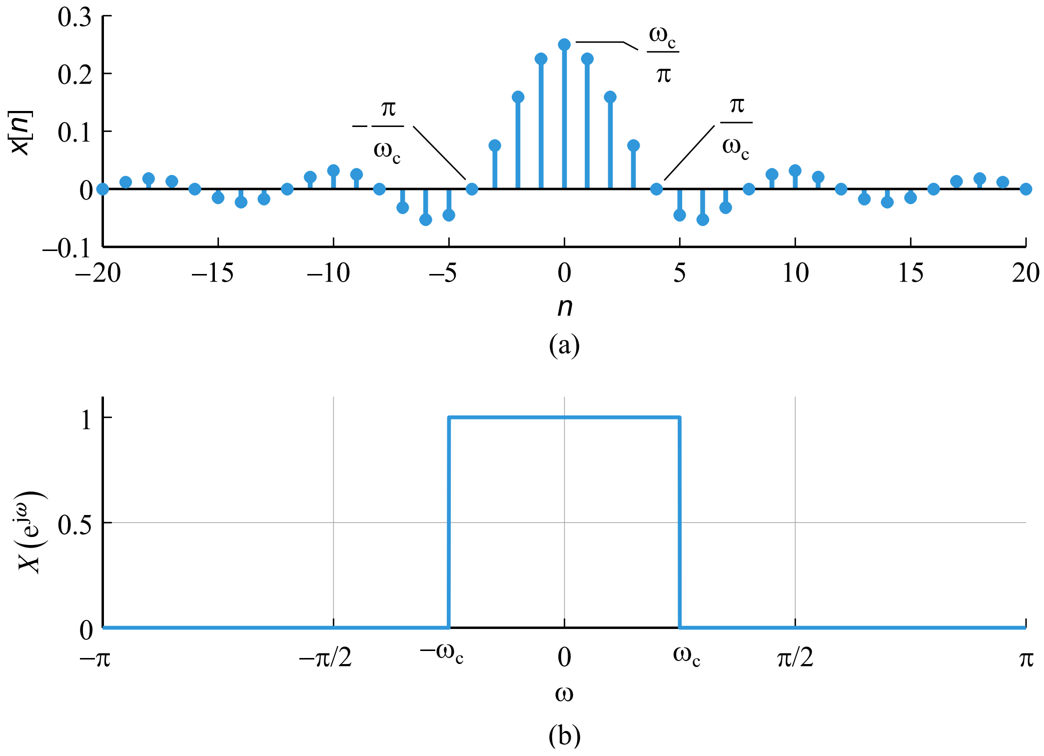
$$x[0] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi}$$

  - Theoretical challenges (Gibbs Phenomenon)

$$\sum_{n=-\infty}^{\infty}\left|\frac{\sin \omega_c n}{\pi n}\right| = \infty \text{ and } \sum_{n=-\infty}^{\infty}\left|\frac{\sin \omega_c n}{\pi n}\right|^2 < \infty \Rightarrow \text{Converges in MSE}$$

DTFT of Ideal Low-Pass Sequence

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\mathrm{e}^{\mathrm{j}\omega})\,\mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\omega_\mathrm{c}}^{\omega_\mathrm{c}} (1)\,\mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\omega \\
&= \begin{cases} \dfrac{\omega_c}{\pi}, & n = 0 \\[2ex] \dfrac{\sin(\omega_\mathrm{c} n)}{\pi n}, & n \neq 0 \end{cases} \\
&= \frac{\omega_\mathrm{c}}{\pi}\,\frac{\sin(\omega_\mathrm{c} n)}{\omega_c n} \\
&= 2 f_\mathrm{c} \operatorname{sinc}(2 f_\mathrm{c} n); \\
&\qquad 0 < \omega_\mathrm{c} = 2\pi f_\mathrm{c} \leq \pi
\end{aligned}
$$



Example-Cont :

- ✓ It can be seen from the figures that there are ripples, independently from the number of terms in the sum.

- ✓ The number of ripples increase when M increases, with the height of largest ripple remaning the same for all values of M.

- ✓ If M goes to infinity, convergence will be achived in sense of mean square.

- ✓ The oscillatory behaviour in the plots is commonly known as Gibbs phenomenon.

Convergence Condition of DTFT

- The Fourier transform can be also defined for a certain class of sequences that are neither absolutely summable nor square summable. Example of such sequences:

  - Unit step sequence
  - Sinusoidal sequence
  - Complex exponentiel sequence

- For this type of sequences a Fourier transform representation is possible by using Dirac delta functions.

- A dirac delta function, also called an ideal impulse function, $\delta(\omega)$ is a function of ω with infinite height, zero width and unit area.

$$\int_{-\infty}^{\infty} \delta(\omega) d\omega = 1, \qquad \delta(\omega) = 0, \omega \neq 0$$

Sequence	DTFT
$\delta[n-n_o]$	$e^{-j\omega n_o}$
1	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
$a^n u[n] \quad \lvert a\rvert<1$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_o n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_o+2\pi k)$
$\cos(\omega_o n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_o+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_o+2\pi k)]$

Discrete Time Fourier Transform-Symmetry Prop

Let's consider a complex sequence x[n]:

$$x[n] = x_{even}[n] + x_{odd}[n]$$

Complex conjugate

$$x_{even}[n] = \frac{1}{2}\left(x[n] + x^{*}[-n]\right) \qquad x_{odd}[n] = \frac{1}{2}\left(x[n] - x^{*}[-n]\right)$$

$$x_{even}[n] = x^{*}_{even}[-n] \qquad x_{odd}[n] = -x^{*}_{odd}[-n]$$

In the case of a complex sequence,

- even sequence is also called as a conjugate symmetric sequence;
- odd sequence is also called as a conjugate antisymmetric sequence.

Discrete Time Fourier Transform-Symmetry Prop

Similarly, symmetry properties can be extended to FT of a signal:

$$X\left(e^{j\omega}\right) = X_{odd}\left(e^{j\omega}\right) + X_{even}\left(e^{j\omega}\right)$$

$$X_{even}\left(e^{j\omega}\right) = \frac{1}{2}\left[X\left(e^{j\omega}\right) + X^{*}\left(e^{-j\omega}\right)\right]$$

$$X_{odd}\left(e^{j\Omega}\right) = \frac{1}{2}\left[X\left(e^{j\omega}\right) - X^{*}\left(e^{-j\omega}\right)\right]$$

$$X_{even}\left(e^{j\omega}\right) = X_{even}^{*}\left(e^{-j\omega}\right)$$

$$X_{odd}\left(e^{j\omega}\right) = -X_{odd}^{*}\left(e^{-j\omega}\right)$$

Discrete Time Fourier Transform-Symmetry Prop

Symmetry relation of the discreate-time Fourier transform for a complex sequence

Sequence $x[n]$	Discrete-Time Fourier Transform $X(e^{j\Omega})$
$x^*[n]$	$X^*(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$Re\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part)
$jIm\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part)
$x_e[n]$	$X_R(e^{j\omega}) = Re\{X(e^{j\omega})\}$
$x_o[n]$	$jX_I(e^{j\omega}) = jIm\{X(e^{j\omega})\}$

Discrete Time Fourier Transform-Symmetry Prop

For the real signals:

$$x[n] = x^*[n] \xrightarrow{DTFT} X\left(e^{j\omega}\right) = X^*\left(e^{-j\omega}\right)$$

$$X^*\left(e^{-j\omega}\right) = \left(\sum_{n=-\infty}^{\infty} x[n]e^{j\omega n}\right)^* = \sum_{n=-\infty}^{\infty} \left(x[n]\right)^* \left(e^{j\omega n}\right)^*$$

we assumed that x[n] is a real sequence, so

$$X^*\left(e^{-j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X\left(e^{j\omega}\right)$$

Discrete Time Fourier Transform-Symmetry Prop

Symmetry relation of the discreate-time Fourier transform for a real sequence

Sequence $x[n]$	Discrete-Time Fourier Transform $X(e^{j\omega})$
Any real $x[n]$	$X(e^{j\omega})=X^{*}(e^{-j\omega})$ (conjugate symmetric)
Any real $x[n]$	$X_R(e^{j\omega})=X_R(e^{-j\omega})$ (real part is even)
Any real $x[n]$	$X_I(e^{j\omega})=-X_I(e^{-j\omega})$ (imaginary part is odd)
Any real $x[n]$	$\lvert X(e^{j\omega})\rvert=\lvert X(e^{-j\omega})\rvert$ (magnitude is even)
Any real $x[n]$	$\angle X(e^{j\omega})=-\angle X(e^{-j\omega})$ (phase is odd)
$x_e[n]$	$X_R(e^{j\omega})$
$x_o[n]$	$jX_I(e^{j\omega})$

Example

- DTFT of the real sequence $x[n]=a^n u[n]$

$$X\left(e^{j\omega}\right)=\frac{1}{1-ae^{-j\omega}} \qquad \text{if } |a|<1$$

- Some properties are

$$X\left(e^{j\omega}\right) = \frac{1}{1-ae^{-j\omega}} = X^*\left(e^{-j\omega}\right)$$

$$X_R\left(e^{j\omega}\right) = \frac{1-a\cos\omega}{1+a^2-2a\cos\omega} = X_R\left(e^{-j\omega}\right)$$

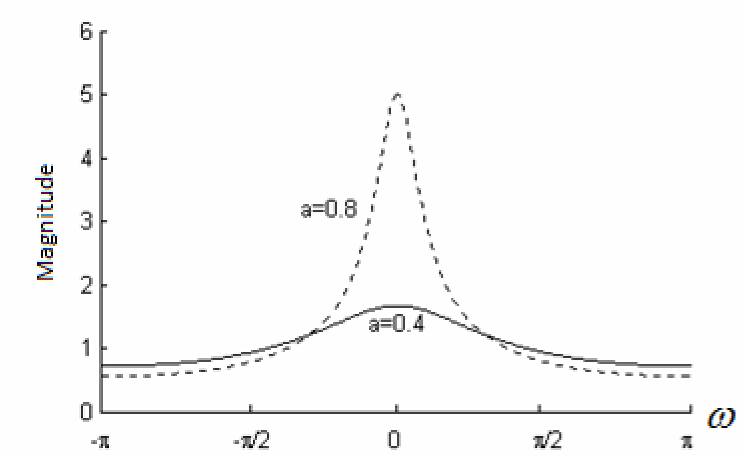
$$X_I\left(e^{j\omega}\right) = \frac{-a\sin\omega}{1+a^2-2a\cos\omega} = -X_I\left(e^{-j\omega}\right)$$

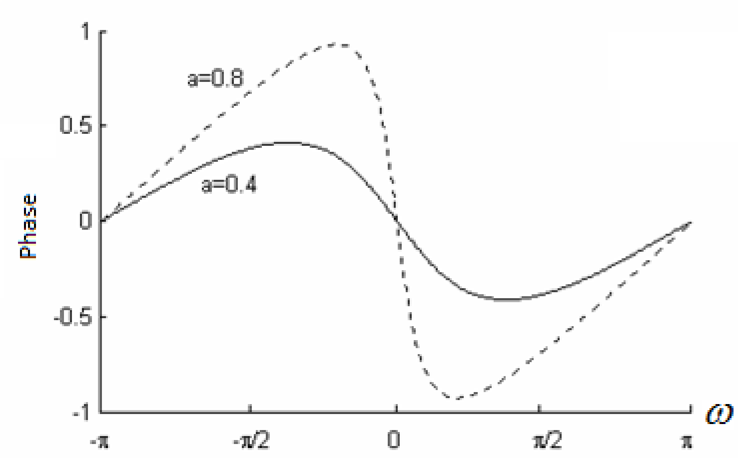
$$\left|X\left(e^{j\omega}\right)\right| = \frac{1}{\sqrt{1+a^2-2a\cos\omega}} = \left|X\left(e^{-j\omega}\right)\right|$$

$$\angle X\left(e^{j\omega}\right) = \tan^{-1}\left(\frac{-a\sin\omega}{1-a\cos\omega}\right) = -\angle X\left(e^{-j\omega}\right)$$

Example-Cont

$x[n] = a^n u[n]$ $\Rightarrow$ $X\left(e^{j\omega}\right) = \dfrac{1}{1 - ae^{-j\omega}}, |a| < 1$





Symmetry Properties of DTFT

Sequence $x[n]$	Transform $X(e^{j\omega})$
	Complex signals
$x^*[n]$	$X^*(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$x_R[n]$	$X_e(e^{j\omega}) \triangleq \frac{1}{2}\left[X(e^{j\omega}) + X^*(e^{-j\omega})\right]$
$jx_I[n]$	$X_o(e^{j\omega}) \triangleq \frac{1}{2}\left[X(e^{j\omega}) - X^*(e^{-j\omega})\right]$
$x_e[n] \triangleq \frac{1}{2}(x[n] + x^*[-n])$	$X_R(e^{j\omega})$
$x_o[n] \triangleq \frac{1}{2}(x[n] - x^*[-n])$	$jX_I(e^{j\omega})$
	Real signals
Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $X_R(e^{j\omega}) = X_R(e^{-j\omega})$ $X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ $\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$
$x_e[n] = \frac{1}{2}(x[n] + x[-n])$ Even part of $x[n]$	$X_R(e^{j\omega})$ real part of $X(e^{j\omega})$ (even)
$x_o[n] = \frac{1}{2}(x[n] - x[-n])$ Odd part of $x[n]$	$jX_I(e^{j\omega})$ imaginary part of $X(e^{j\omega})$ (odd)

DTFT Teorems: Linearity

$$x_1[n] \overset{DTFT}{\Leftrightarrow} X_1\left(e^{j\omega}\right) \qquad x_2[n] \overset{DTFT}{\Leftrightarrow} X_2\left(e^{j\omega}\right)$$

$$\Downarrow$$

$$ax_1[n] + bx_2[n] \overset{DTFT}{\Leftrightarrow} aX_1\left(e^{j\omega}\right) + bX_2\left(e^{j\omega}\right)$$

Example:

$$x[n] = \left(0.2^n + 0.4^n\right)u[n] \rightarrow X(e^{j\omega}) = ?$$

DTFT Teorems -Time shifting

$$x[n] \overset{DTFT}{\Leftrightarrow} X\left(e^{j\omega}\right)$$

$$\Downarrow$$

$$x[n-n_0] \overset{DTFT}{\Leftrightarrow} e^{-j\omega n_0} X\left(e^{j\omega}\right)$$

Example:

$$x[n] = \left(0.2^n\right) u[n-2] \quad \rightarrow \quad X(e^{j\omega}) = ?$$

DTFT Teorems-Frequency Shifting

$$x[n] \overset{DTFT}{\Leftrightarrow} X\left(e^{j\omega}\right)$$

$$e^{j\omega_0 n} x[n] \overset{DTFT}{\Leftrightarrow} X\left(e^{j(\omega-\omega_0)}\right)$$

Example:

$$x[n] = \cos\left(\frac{\pi n}{10}\right) 0.2^n u[n] \rightarrow X(e^{j\omega}) = ?$$

DTFT Teorems-Time Reversal

$$x[n] \overset{DTFT}{\Leftrightarrow} X\left(e^{j\omega}\right)$$

$$x[-n] \overset{DTFT}{\Leftrightarrow} X\left(e^{-j\omega}\right)$$

Example:

$$x[n] = 2^n u[-n] \rightarrow X\left(e^{j\omega}\right) = ?$$

DTFT Teorems -Differentiation in Frequency

$$x[n] \overset{DTFT}{\Leftrightarrow} X\left(e^{j\omega}\right)$$

$$nx[n] \overset{DTFT}{\Leftrightarrow} j\frac{dX\left(e^{j\omega}\right)}{d\omega}$$

Example:

$$x[n] = n0.2^n u[n] \rightarrow X\left(e^{j\omega}\right) = ?$$

DTFT Teorems-Convolution

$$x[n] \overset{DTFT}{\Leftrightarrow} X\left(e^{j\omega}\right) \qquad h[n] \overset{DTFT}{\Leftrightarrow} H\left(e^{j\omega}\right)$$

$$y[n] = x[n] * h[n] \overset{DTFT}{\Leftrightarrow} Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) H\left(e^{j\omega}\right)$$

Example:

$$x[n] = 0.2^n u[n]$$
$$h[n] = \delta(n-2) + \delta(n-1)$$

$\rightarrow$ $Y(e^{j\omega}) = ?$

Convolution Theorem

Theorem: $y[n] = x[n] * h[n] \xleftrightarrow{\text{DTFT}} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$

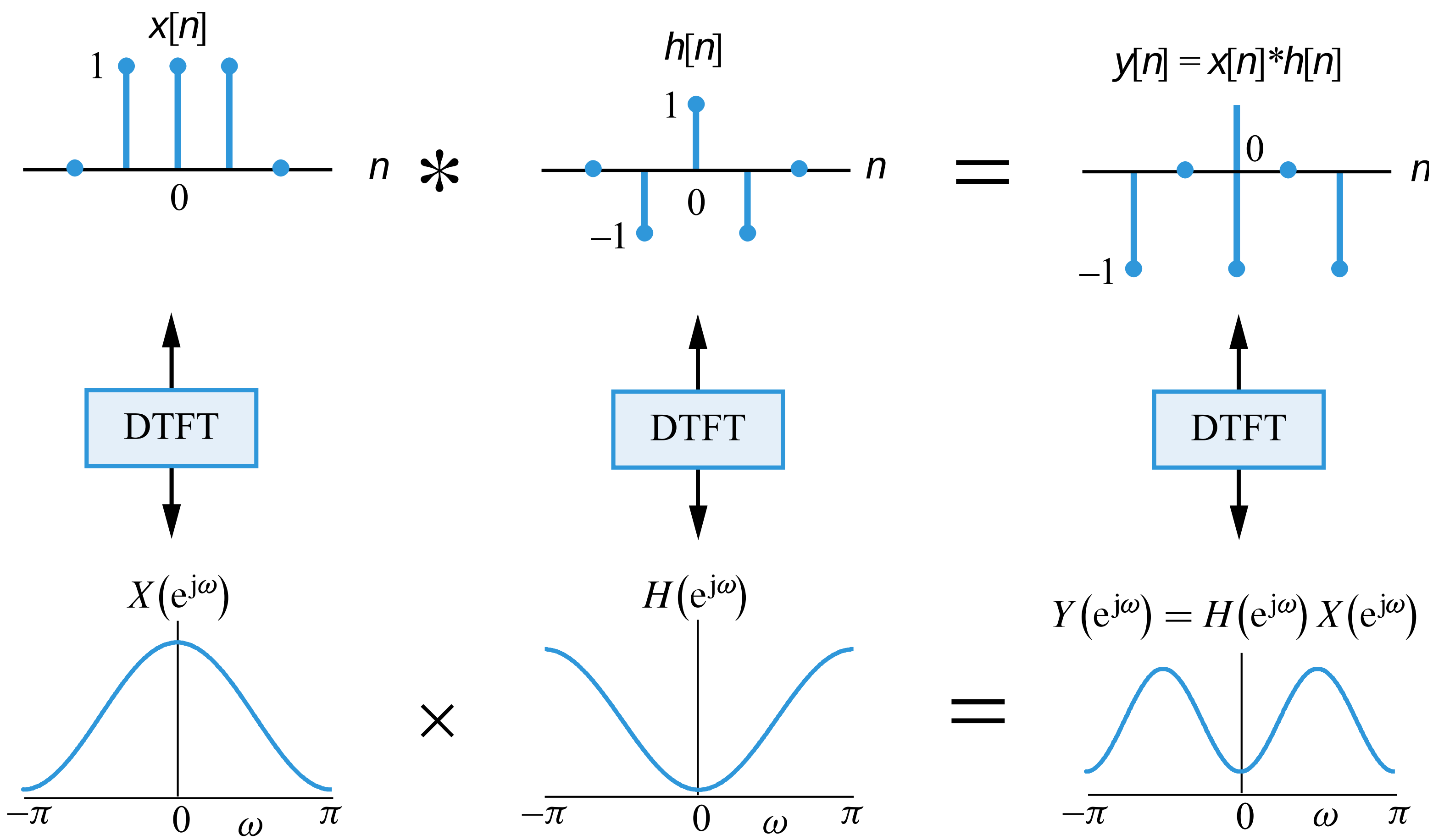
Example: Let $x[n] = \{1, \underset{\uparrow}{1}, 1\}$ and $h[n] = \{-1, \underset{\uparrow}{1}, -1\}$. Then

$$
\begin{aligned}
X(e^{j\omega}) &= e^{j\omega} + 1 + e^{-j\omega} = 1 + 2\cos(\omega) \\
H(e^{j\omega}) &= -e^{j\omega} + 1 - e^{-j\omega} = 1 - 2\cos(\omega) \\
Y(e^{j\omega}) &= X(e^{j\omega})H(e^{j\omega}) = 1 - 4\cos^2(\omega) \\
&= 1 - 4\big(1 + \cos(2\omega)\big)/2 = -1 - 2\cos(2\omega) \\
&= -e^{j2\omega} - 1 - e^{-j2\omega}
\end{aligned}
$$

or

$$y[n] = \{-1, 0, \underset{\uparrow}{-1}, 0, -1\}$$

Graphical Illustration



DTFT Teorems-Windowing:

$$x[n] \overset{DTFT}{\Leftrightarrow} X\left(e^{j\omega}\right) \qquad w[n] \overset{DTFT}{\Leftrightarrow} W\left(e^{j\omega}\right)$$

$$y[n] = x[n]w[n] \overset{DTFT}{\Leftrightarrow} Y\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W\left(e^{j(\omega-\theta)}\right)d\theta$$

Parseval's Relation:

✓ This theorem expresses the sum of sample by sample product of two complex sequences in terms of an integral of the product of their Fourier transforms. Specifically, the most general form of this theorem is:

$$\sum_{n=-\infty}^{\infty} g[n]h^{*}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})H^{*}(e^{j\omega})d\omega$$

✓ One important application of Parseval’s relation is in the computation of the energy of a inifite energy sequence.

✓ The total energy of a finite energy g[n] is given by:

$$E = \sum_{n=-\infty}^{\infty} \left|g[n]\right|^{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|G\left(e^{j\omega}\right)\right|^{2} d\omega$$

Energy density spectrum

Example:

$$x[n] = 0.2^n u[n-4] + 0.4^{n-1} u[n] \rightarrow X(e^{j\omega}) = ?$$

Example:

$$y[n]-0.2y[n-1]=0.5x[n]-0.3x[n-1]+0.1x[n-2]$$

$$h[n]=?$$