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Complex numbers

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Complex numbers and coordinate systems

Two common types:

1. rectangular: real and imaginary components

2. polar: magnitude and phase components

$$
\begin{aligned}
w_1 \pm w_2 &= (x_1 + j\ y_1) \pm (x_2 + j\ y_2) \\
&= \underbrace{(x_1 \pm x_2)}_{\text{real part}} + j\ \underbrace{(y_1 \pm y_2)}_{\text{imag part}} \\
w_1 \cdot w_2 &= (x_1 + j\ y_1) \cdot (x_2 + j\ y_2) \\
&= x_1 x_2 + x_1 (j\ y_2) + (j\ y_1) x_2 + (j\ y_1)(j\ y_2) \\
&= x_1 x_2 + j x_1 y_2 + j x_2 y_1 + j^2 y_1 y_2 \\
&= \underbrace{(x_1 x_2 - y_1 y_2)}_{\text{real part}} + j\ \underbrace{x_1 y_2 + x_2 y_1)}_{\text{imag part}} \\
\frac{w_1}{w_2} &= \frac{x_1 + j\ y_1}{x_2 + j\ y_2} \cdot \left(\frac{x_2 - j\ y_2}{x_2 - j\ y_2} \right) = \cdots \\
&= \underbrace{\frac{x_1 x_2 + y_1 y2}{x_2^2 + y_2^2}}_{\text{real part}} + j\ \underbrace{\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}}_{\text{imag part}}
\end{aligned}
$$

Complex numbers and coordinate systems

Note:

$$w_1^* = x_1 - j\, y_1$$

$$|w_1| = \sqrt{x_1^2 + y_1^2}$$

and

$$w_1 \cdot w_1^* = (x_1 + j\, y_1)(x_1 - j\, y_1) = x_1^2 + y_1^2 = |w_1|^2 \quad \text{REAL}$$

$$w = re^{j\theta} \qquad r \in \mathbb{R}^+, \theta \in \mathbb{R}, w \in \mathbb{C}$$

Note: $r \equiv$ magnitude and $\theta \equiv$ phase

Let $w_1 = r_1 e^{j\theta_1}$ and $w_2 = r_2 e^{j\theta_2}$.

Complex numbers and coordinate systems

$$w_1 \cdot w_2 = r_1 e^{j\theta_1} \cdot r_2 e^{j\theta_2} = (r_1\ r_2) e^{j(\theta_1\ +\ \theta_2)}$$

$$\text{magnitude} \equiv r_1\ r_2$$

$$\text{phase} \equiv \theta_1 + \theta_2$$

$$\frac{w_1}{w_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1\ -\ \theta_2)}$$

$$\text{magnitude} \equiv \frac{r_1}{r_2}$$

$$\text{phase} \equiv \theta_1 - \theta_2$$

$$w_1 \pm w_2 = r_1 e^{j\theta_1} \pm r_2 e^{j\theta_2} = \cdots$$

$$\text{magnitude} \equiv \sqrt{(r_1 \cos\theta_1 \pm r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 \pm r_2 \sin\theta_2)^2}$$

$$\text{phase} \equiv \arctan\left(\frac{r_1 \sin\theta_1 \pm r_2 \sin\theta_2}{r_1 \cos\theta_1 \pm r_2 \cos\theta_2}\right) \quad ☹$$

<u>Note</u>:

$$w_1^* = r_1 e^{-j\ \theta_1}$$

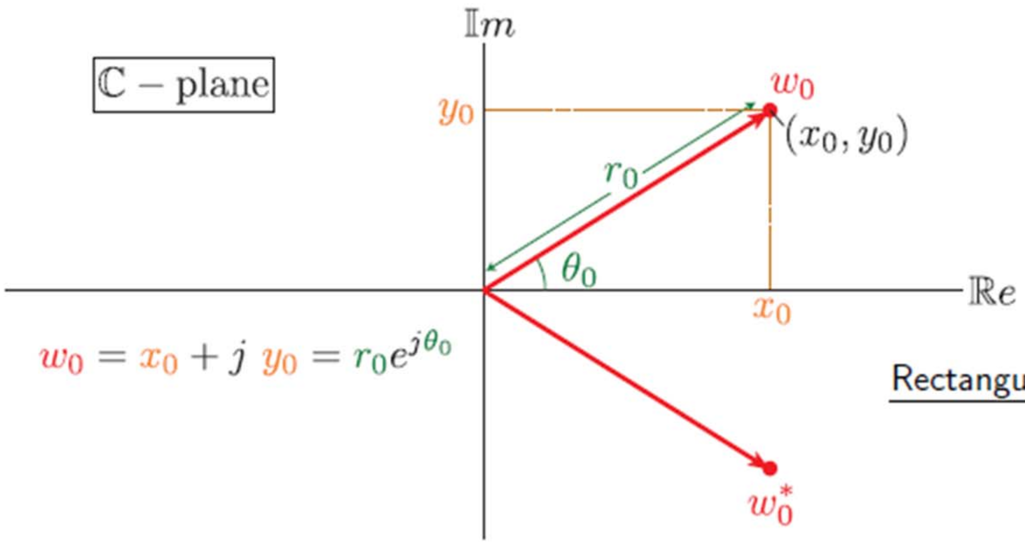
$$|w_1| = |r_1 e^{j\theta_1}| = |r_1| \cdot |e^{j\theta_1}| = r_1 \cdot 1 = r_1$$

and

$$w_1 \cdot w_1^* = (r_1 e^{+j\ \theta_1})(r_1 e^{-j\ \theta_1}) = r_1^2 = |w_1|^2 \quad \text{REAL}$$

Complex numbers and coordinate systems

Complex Plane Relations



Rectangular to Polar Conversion:

$$r_0 = \sqrt{x_0^2 + y_0^2}$$

$$\theta_0 = \arctan\left(\frac{y_0}{x_0}\right)$$

Polar to Rectangular Conversion:

$$x_0 = r_0 \cos\theta_0$$

$$y_0 = r_0 \sin\theta_0$$

Note:

$$z_0 = x_0 + j\ y_0 = r_0 \cos\theta_0 + j r_0 \sin\theta_0 = r_0 \underbrace{(\cos\theta_0 + j \sin\theta_0)}_{= e^{j\theta_0}\ \text{(EULER)}}$$

$$= r_0 e^{j\theta_0}$$

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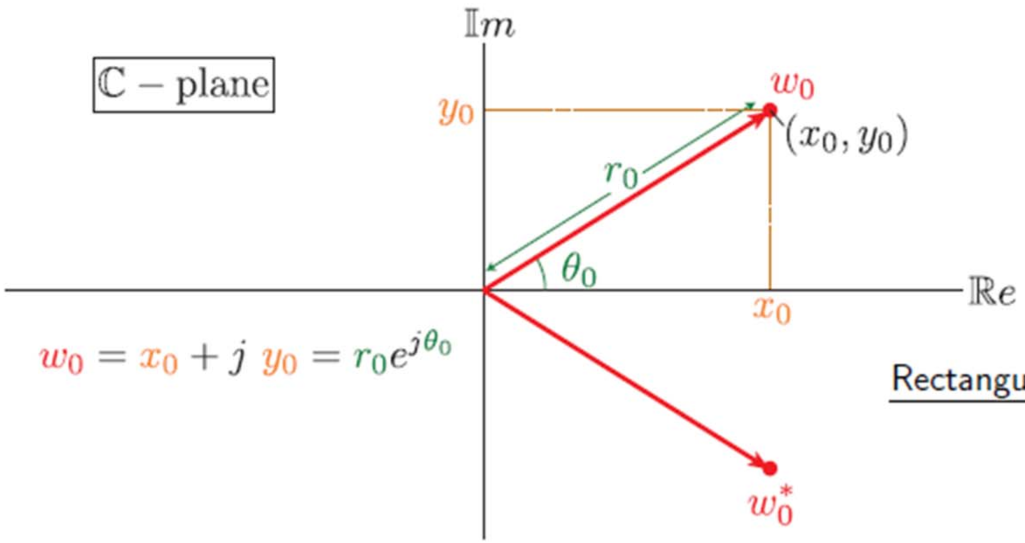

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$$= r_0 e^{j\theta_0}$$