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Digital Signal Processing
Lecture 9

In science one tries to tell people, in such a way as to be understood by everyone, something that no one ever knew before.

But in poetry, it's the exact opposite.

- Paul Dirac

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TIME!
teaching the 'Z-TRANSFORM'...
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z-Transform

- ✓ z-transform is a counterpart of the Laplace transform for discrete-time signals.

- ✓ z-transform is generalization of the Fourier Transform
  -Fourier Transform does not exist for all signals.

- ✓ For a sequence x[n], its z-transform is defined by:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Notations:

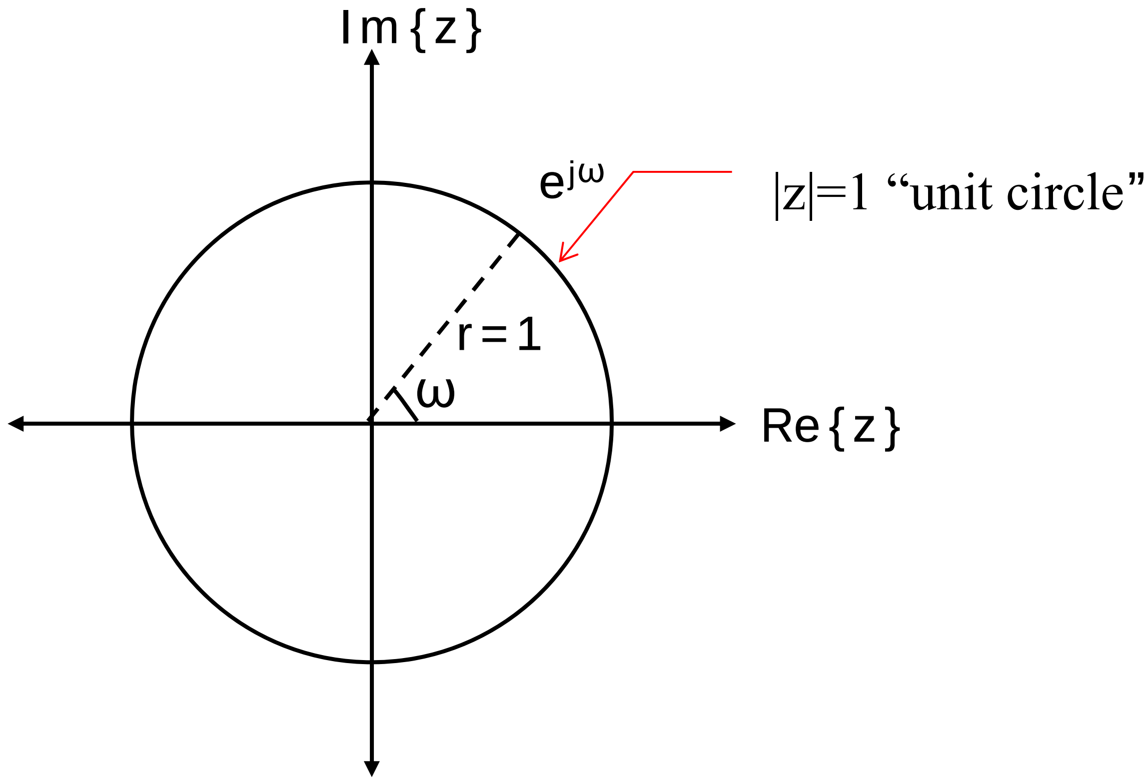
$$X(z) \equiv Z\{x[n]\}$$
$$x[n] \leftrightarrow X(z)$$

- ✓ Compare to DTFT: $X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$

- ✓ z is a complex variable that can be represented in polar form as $z = re^{j\theta}$
- ✓ Substituting $z=e^{j\omega}$, r=1 will reduce the z-transform to DTFT.

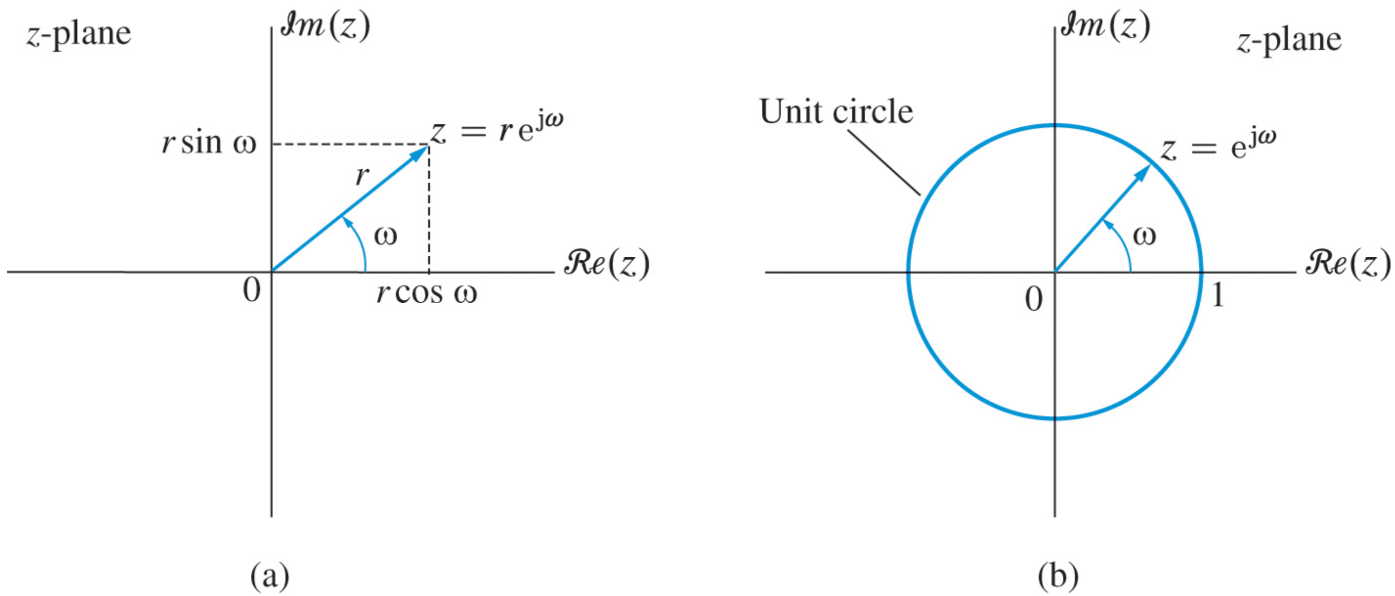
z-Transform

- ✓ The z-transform is a function of the complex z variable and it is convenient to describe on the complex z-plane.



- ✓ If we plot $z=e^{j\omega}$, we get the unit circle. As $-\pi < \omega < \pi$, $e^{j\omega}$ goes once around the unit circle. Thus, this confirms the periodicity of the DTFT.

z-Transform



a) a point $z=re^{j\omega}$ in the complex plane can be specified by the distance r from the origin and the angle ω with the positive real axis (polar coordinates) or the rectangular coordinates rcos(ω) and rsin(ω).

b) The unit circle in the complex plane.

Example

$$x[n] = \delta[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = \sum_{n=-\infty}^{\infty} \delta[n]\, z^{-n} = 1$$

$$x[n] = \delta[n - n_0]$$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n - n_0]\, z^{-n} = z^{-n_0}$$

z-Transform

Transfer function:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$$

Example:

$$h[n] = \begin{cases} 1 \;, & 0 \le n \le 4 \\ 0 \;, & \text{otherwise} \end{cases}$$

$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = \sum_{n=-\infty}^{\infty} (\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]) z^{-n}$$

$$H(z) = z^{-0} + z^{-1} + z^{-2} + z^{-3} + z^{-4} = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}$$

Convergence of z-transform

- Given a sequence, the set of values of z for which the z-transform converges, i.e., $|X(z)|<\infty$, is called the region of convergence, i.e.,

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| \le \sum_{n=-\infty}^{\infty} |x[n]| \, |z|^{-n} < \infty$$

$$z = re^{j\theta} \rightarrow \sum_{n=-\infty}^{\infty} |x[n]| \, |z|^{-n} = \sum_{n=-\infty}^{\infty} |x[n]| \, |r|^{-n} < \infty$$

- In other words, Region of Convergence (ROC) of X(z) is the set of all values of z for which X(z) attains a finite value.

- The z transform is therefore uniquely characterized by 1) expression of X(z) and 2) ROC of X(z).

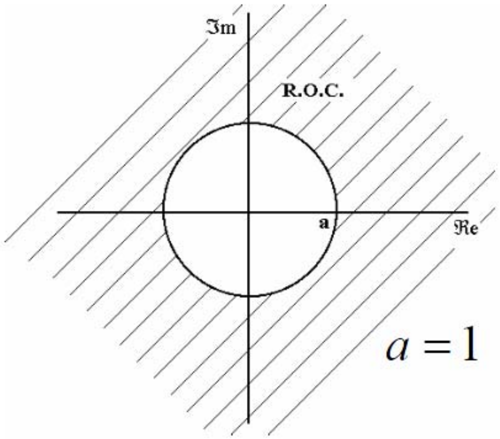
- ROC is centered on origin and consists of a set of rings.

Example-1

$$x[n]=u[n] \qquad X(z)=?$$

$$X(z)=\sum_{n=0}^{\infty} z^{-n}=\lim_{N\to\infty}\sum_{n=0}^{N} z^{-n}=\frac{\left(z^{-1}\right)^{N+1}-1}{z^{-1}-1},\left|z^{-1}\right|<1$$

$$=\frac{1}{1-z^{-1}},|z|>1$$



$a=1$

Example-2

$$x[n] = a^n u[n] \quad \Rightarrow \quad X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

- For Convergence we require

$$\sum_{n=0}^{\infty} \left|az^{-1}\right|^n < \infty$$

- Hence the ROC is defined as

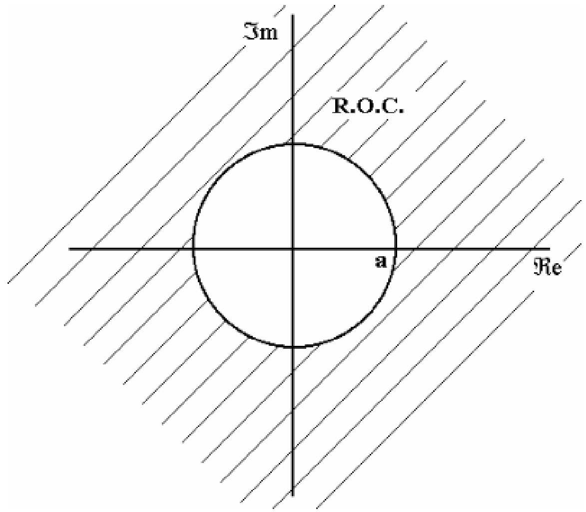
$$\left|az^{-1}\right|^n < 1 \Rightarrow |z| > |a|$$

- Inside the ROC series converges to

$$X(z) = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

- Geometric series formula

$$\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1 - a}$$



- Region outside the circle of radius a is the ROC
- Right-sided sequence ROCs extend outside a circle

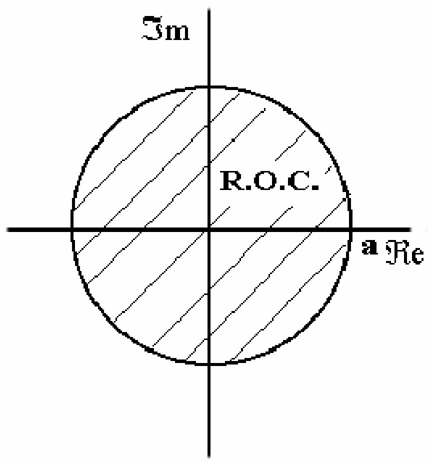
Example-3

$$x[n] = -a^n u[-n-1] \quad \Rightarrow \quad X(z) = \sum_{n=-\infty}^{\infty} -a^n u[n-1] z^{-n} = -\sum_{n=-\infty}^{-1} \left(az^{-1}\right)^n$$

$$X(z) = -\sum_{n=-\infty}^{-1} \left(az^{-1}\right)^n = -\sum_{m=1}^{\infty} \left(a^{-1}z\right)^m = -\sum_{m=0}^{\infty} \left(a^{-1}z\right)^m + 1$$

m=-n

$$= -\frac{1}{1-a^{-1}z} + 1 = \frac{1}{1-az^{-1}}$$



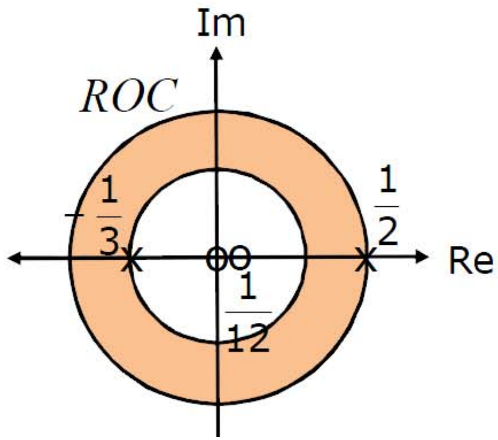
$$\left|a^{-1}z\right|^n < 1 \Rightarrow |z| < |a|$$

Example-4

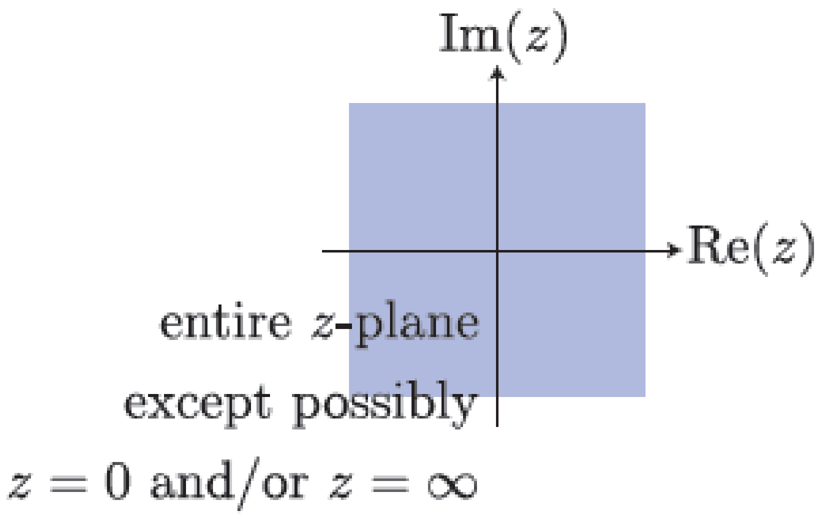
$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

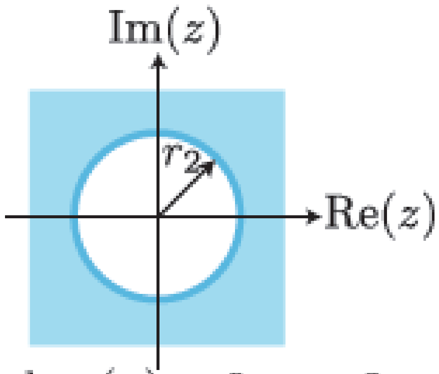
$$ROC: \frac{1}{3} < |z| < \frac{1}{2}$$



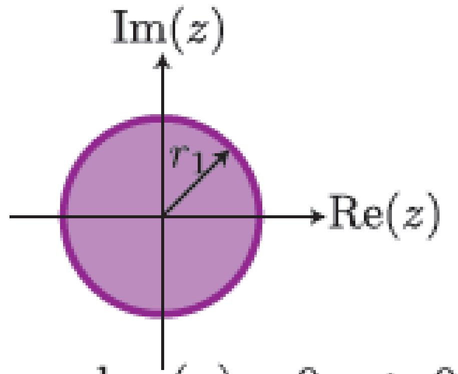
ROCs-Finite Duration Signals



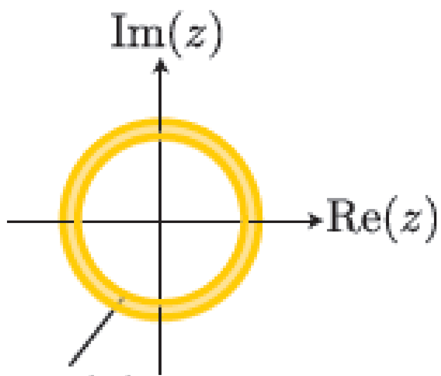
ROCs-Infinite Duration Signals



Causal: $x(n) = 0 \quad n < 0$



Anticausal: $x(n) = 0 \quad n > 0$



ROC: $r_1 < |z| < r_2$

Properties of z-transform-Linearity

$$x[n] \xleftrightarrow{Z} X(z) \qquad ROC = R_x$$

$$\boxed{\begin{aligned} ax_1[n] + bx_2[n] \xleftrightarrow{Z} aX_1(z) + bX_2(z) \\ ROC = R_{x_1} \cap R_{x_2} \end{aligned}}$$

Example:

$$x[n] = (0.5^n + 2)u[n]$$
$$X(z) = ?$$

Properties of z-transform-Time Shifting

$$x[n] \xleftrightarrow{Z} X(z) \qquad ROC = R_x$$

$$\boxed{x[n-n_o] \xleftrightarrow{Z} z^{-n_o} X(z) \qquad ROC = R_x}$$

Except z= 0 (n_0>0)
or z=∞(n_0 <0)

Example:

$$x[n] = a^{n-1}u[n-1]$$
$$X(z) = ?$$

Example

$$X(z) = z^{-1}\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right), \text{ROC: } |z| > \frac{1}{4}$$

$$x[n] = \left(\frac{1}{4}\right)^{n\text{-}1} u[n\text{ - }1]$$

Properties of z-transform-Multiplication by an Exponential Sequence

$$z_o^n x[n] \xleftrightarrow{Z} X(z/z_o) \qquad ROC = |z_o| R_x$$

Example:

$$u[n] \xleftrightarrow{Z} \frac{1}{1 - z^{-1}} \qquad ROC : |z| > 1$$

$$x[n] = r^n \cos(\omega_o n) u[n] = \frac{1}{2}\left(re^{j\omega_o}\right)^n u[n] + \frac{1}{2}\left(re^{-j\omega_o}\right)^n u[n]$$

$$X(z) = \frac{1/2}{1 - re^{j\omega_o} z^{-1}} + \frac{1/2}{1 - re^{-j\omega_o} z^{-1}} \qquad |z| > r$$

Properties of z-transform-Multiplication by an Exponential Sequence

Proof:

$$y[n] = a^n x[n]$$

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n] z^{-n} = \sum_{n=-\infty}^{\infty} a^n x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n] \left(z / a \right)^{-n} = X(z/a)$$

Properties of z-transform-Differentiation

$$\boxed{nx[n] \overset{Z}{\longleftrightarrow} -z\frac{dX(z)}{dz} \qquad ROC = R_x}$$

Excluding possibly the point z= 0 or z=∞

Example:

$$na^n u[n] \overset{Z}{\longleftrightarrow} ?$$

$$a^n u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1-az^{-1}} \quad , \ \mathrm{ROC:}\ |z| > |a|$$

$$X(z) = -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) = \frac{az^{-1}}{\left(1-az^{-1}\right)^2} \quad , \ \mathrm{ROC:}\ |z| > |a|$$

Properties of z-transform-Differentiation

Proof:

$$\frac{dX(z)}{dz} = \frac{d}{dz}\left(\sum_{n=-\infty}^{\infty} x[n]z^{-n}\right) = \sum_{n=-\infty}^{\infty} x[n]\left(\frac{d}{dz}z^{-n}\right) = \sum_{n=-\infty}^{\infty} x[n](-nz^{-n-1})$$

$$\frac{dX(z)}{dz} = -z^{-1}\sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

$$-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

Properties of z-transform-Time Reversal

$$x[-n] \xleftrightarrow{Z} X(1/z) \qquad ROC = \frac{1}{R_x}$$

Example:

$$x[n] = a^{-n}u[-n]$$

Time reversed version of $a^n u[n]$. Therefore;

$$X(z) = \frac{1}{1-az} = \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \qquad |z| < |a^{-1}|$$

Properties of z-transform-Convolution

$$x_1[n] * x_2[n] \xleftrightarrow{Z} X_1(z) X_2(z) \qquad ROC: R_{x1} \cap R_{x_2}$$

$$x_1[n] = a^n u[n],\ x_2[n] = u[n]$$

$$y[n] = x_1[n] * x_2[n] = ?$$

$$X_1(z) = \frac{1}{1 - az^{-1}} \quad \text{ROC:} |z| > |a|$$

$$X_2(z) = \frac{1}{1 - z^{-1}} \quad \text{ROC:} |z| > 1$$

Properties of z-transform-Convolution

$$Y(z) = X_1(z) X_2(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})}$$

$$\text{ROC} = \begin{cases} |z| > 1 \ , & |a| < 1 \\ |z| > |a| \ , & |a| > 1 \end{cases}$$

- Assuming $\text{ROC: } |z| > 1$:

$$Y(z) = \frac{1}{1-a}\left(\frac{1}{1 - z^{-1}} - \frac{a}{1 - az^{-1}}\right) \ , \text{ROC: } |z| > 1$$

$$y[n] = \frac{1}{1-a}\left(u[n] - a^{n+1} u[n]\right)$$

Properties of z-transform-Convolution

Proof:

$$x_3[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$

$$X_3(z) = \sum_{n=-\infty}^{\infty} x_3[n]z^{-n} = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]z^{-n}$$

Interchanging the order of summation on the right hand side we have:

$$= \sum_{k=-\infty}^{\infty} x_1[k]\sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n}$$

Substituting $r = n-k$, we arrive at:

$$X_3(z) = \sum_{k=-\infty}^{\infty} x_1[k]\sum_{r=-\infty}^{\infty} x_2[r]z^{-r}z^{-k}$$

$$X_3(z) = X_1(z)X_2(z)$$

Properties of z-transform-Convolution

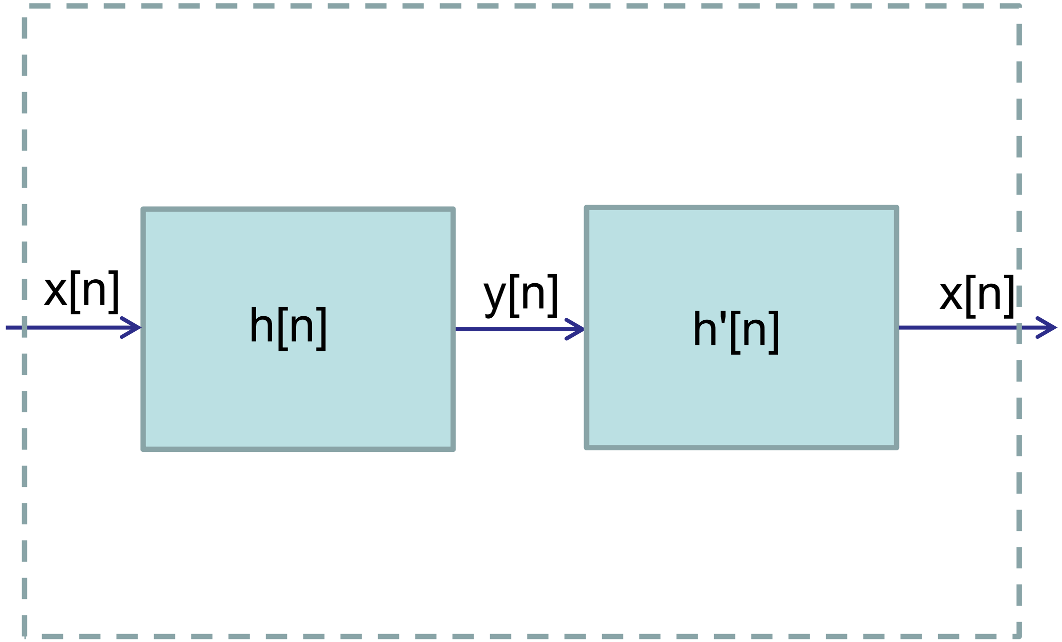
Example:

$$x[n] = 0.9^n u[n]$$

$$h[n] = \delta[n-2]$$

$$y[n] = ?$$

Inverse Systems



$$h[n] * h'[n] = \delta[n]$$

$$H(z)H'(z) = 1$$

$$H(z) = \frac{1}{H'(z)}$$

z-Transform-Poles and Zeros

- ✓ z-Transform expressions that are a fraction of polynomials in z^{-1} (or z) are called rational.

- z-Transforms that are rational represent an important class of signals and systems.

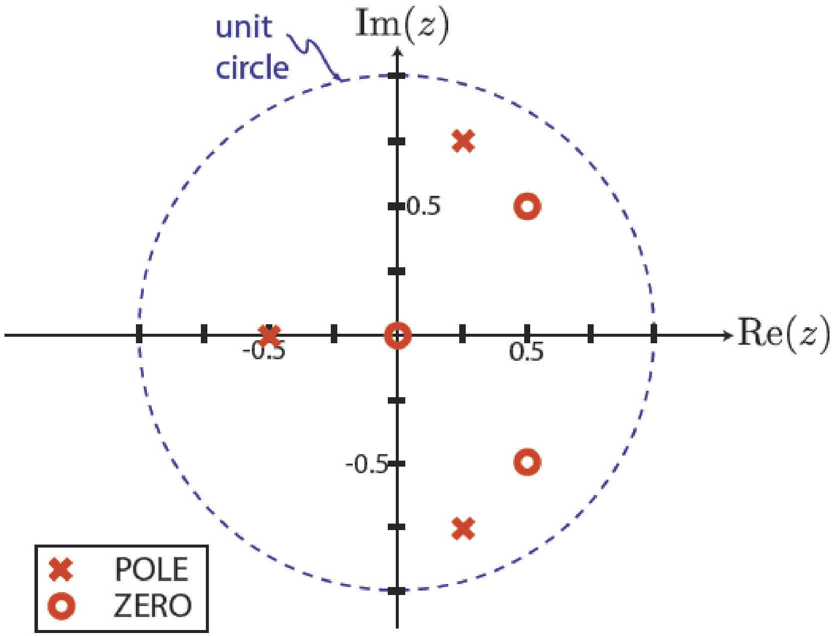
- X(z) is a rational function if it can be represented as the ratio of two polynomials in z^{-1}(or z):

$$X(z) = \frac{P(z)}{Q(z)}$$

  - Roots of P(z):zeros "o"
  - Roots of Q(z)=poles "x"
  - zeros of X(z): values of z for which X(z) =0
  - poles of X(z): values of z for which X(z)=∞
  - we may have poles/zeros at z=0,∞ in the case order of Q(z) ≠order of P(z).

z-Transform-Poles and Zeros

- zeros of $X(z)$: values of z for which $X(z) = 0$
- poles of $X(z)$: values of z for which $X(z) = \infty$



complex poles and zeros must occur in conjugate pairs

note: real poles and zeros do not have to be paired up!

$$\boxed{\text{Total number of zeros} = \text{Total number of poles}}$$

z-Transform-Poles and Zeros

- If X(z)=P(z)/Q(z) and order of P(z) is M and order of Q(z) is N

1) if N>M, there are zeros @z=∞ and/or @z=0

2) if N≤M, there are poles @z=∞ and/or @z=0

$$X(z) = \frac{z+1}{(z+2)(z-1)}$$

$zero@z=-1$

$poles@z=-2,1$

BUT

$$\lim_{z\to\infty} X(z) \approx \lim_{z\to\infty} \frac{1}{z} = 0$$

$z=\infty$ is also a zero

$$X(z) = \frac{(z+2)(z-1)}{(z+1)}$$

$zero@z=-2,1$

$poles@z=-1$

BUT

$$\lim_{z\to\infty} X(z) \approx \lim_{z\to\infty} z = \infty$$

$z=\infty$ is also a pole

Inverse z-transform

- Inverse z-transform (Cauchy integral):

$$x[n] = Z^{-1}\{X(z)\} = \frac{1}{2\pi j}\oint_c X(z) z^{n-1} dz$$

- Alternatively we use:
  - Inspection
  - Partial fraction expansion
  - Power series expansion

Inverse z-transform

- Inspection: Make use of known z-transform pairs such as

$$a^n u[n] \xleftrightarrow{z} \frac{1}{1-az^{-1}} \quad , \quad \text{ROC}: |z| > |a|$$

- Example:

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \quad , \text{ROC}: |z| > \frac{1}{2} \quad \rightarrow \quad \text{x}[\text{n}] = \left(\frac{1}{2}\right)^n u[n]$$

Power Series Expansion

- z-Transform is a power series

$$X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
$$= \ldots + x[2]z^{2} + x[-1]z + x[0]z^{-1} + x[2]z^{-2} + \ldots$$

- Power series expansion approach aims to

  1-Write the function to be inverted as a power series.

  2-Identify x[n] as coefficient of z^{-n}

- This approach can also be used to invert rational X(z) with long division.

- For a rational X(z), a convenient way to determine the power series is to express the numerator and denominator as polynomials in z^{-1} and then obtain the power series expansion by long division.

Power Series Expansion-Example

$$X(z) = z^2\left(1-\frac{1}{2}z^{-1}\right)\left(1+z^{-1}\right)\left(1-z^{-1}\right)$$

$$= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

$$x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$$

Power Series Expansion-Examples

$$X(z) = (1-2z^{-1})(2-5z^{-1})(3-z^{-1})$$
$x[n]$?

$$X(z) = \frac{2z^{-1} - z^{-2}}{1 - 1.6z^{-1} - 0.8z^{-2}} \qquad R.O.C: |z| > 2$$
$x[n]$?

$$X(z) = \frac{2z^{-1} - z^{-2}}{1 - 1.6z^{-1} - 0.8z^{-2}} \qquad R.O.C: |z| < 0.4$$
$x[n]$?

Partial Fraction Expansion

- Assume that a given z-transform can be expressed as

$$X(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- Apply partial fractional expansion:

$$X(z)=\sum_{r=0}^{M-N} B_r z^{-r}+\sum_{k=1,k\neq i}^{N}\frac{A_k}{1-d_k z^{-1}}+\sum_{m=1}^{s}\frac{C_m}{\left(1-d_i z^{-1}\right)^m}$$

- First term exist only if M≥N
  - B_r is obtained by long division.
- Second term represents all first order poles.
- Third term represents an order s pole.
  - There will be a similar term for every high-order pole.
- Each term can be inverse transformed by inspection.

Partial Fraction Expansion-Example1

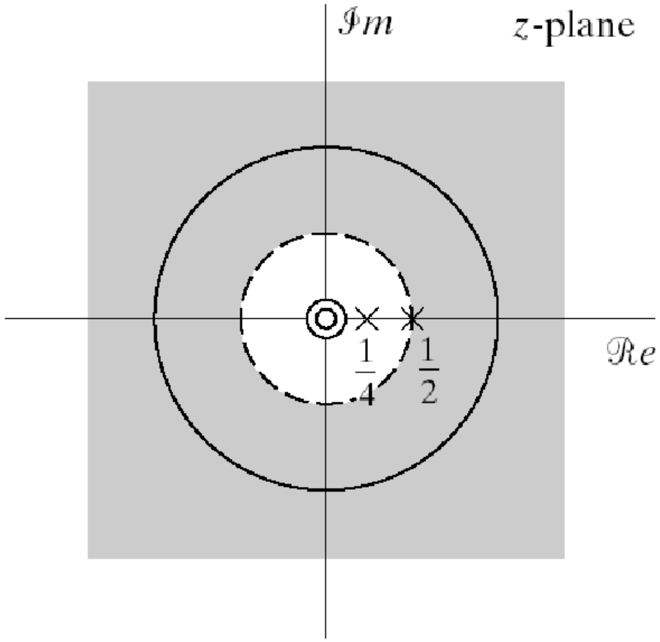
$$X(z)=\frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} \quad , \quad \text{ROC: } |z|>\frac{1}{2}$$

$$X(z)=\frac{A_1}{\left(1-\frac{1}{4}z^{-1}\right)}+\frac{A_2}{\left(1-\frac{1}{2}z^{-1}\right)}$$

$$A_1=\left(1-\frac{1}{4}z^{-1}\right)X(z)\Big|_{z=\frac{1}{4}}=\frac{1}{\left(1-\frac{1}{2}\left(\frac{1}{4}\right)^{-1}\right)}=-1 \qquad A_2=\left(1-\frac{1}{2}z^{-1}\right)X(z)\Big|_{z=\frac{1}{2}}=\frac{1}{\left(1-\frac{1}{4}\left(\frac{1}{2}\right)^{-1}\right)}=2$$

Partial Fraction Expansion-Example1-Cont

$$X(z) = \frac{-1}{\left(1-\frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1-\frac{1}{2}z^{-1}\right)} \quad , \ \text{ROC:}|z| > \frac{1}{2}$$



$$x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$$

Partial Fraction Expansion-Example2

$$X(z)=\frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}=\frac{\left(1+z^{-1}\right)^{2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)} \quad , \ \mathrm{ROC}:|z|>1$$

$$\begin{array}{l|l}
z^{-2}+2z^{-1}+1 & \frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1 \\ \hline
\underline{z^{-2}-3z^{-1}+2} & \qquad 2 \\
5z^{-1}-1 &
\end{array}$$

$$X(z)=2+\frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}$$

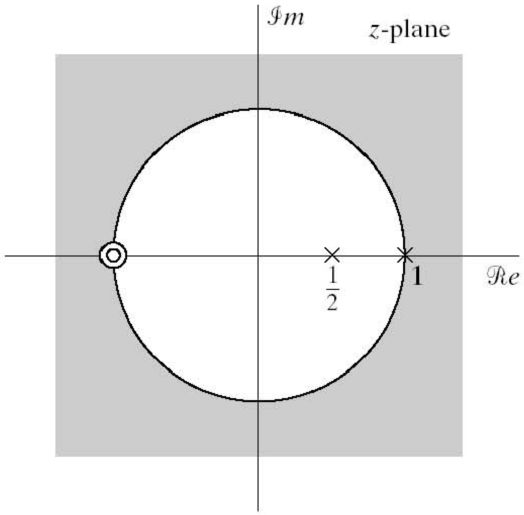
$$X(z)=2+\frac{A_1}{1-\frac{1}{2}z^{-1}}+\frac{A_2}{1-z^{-1}}$$

Partial Fraction Expansion-Example2-Cont

$$A_1 = \left(1 - \frac{1}{2} z^{-1}\right) X(z)\Big|_{z=\frac{1}{2}} = -9$$

$$A_2 = \left(1 - z^{-1}\right) X(z)\Big|_{z=1} = 8$$

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2} z^{-1}} + \frac{8}{1 - z^{-1}} \text{ , ROC:} |z| > 1$$



$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] \text{-} 8u[n]$$

Partial Fraction Expansion-Examples

$$X(z)=\frac{1}{1-3z^{-1}+2z^{-2}} \qquad R.O.C:|z|<1$$

$x[n]$?

$$X(z)=\frac{1}{1-3z^{-1}+2z^{-2}} \qquad R.O.C:1<|z|<2$$

$x[n]$?

$$X(z)=\frac{1}{1-3z^{-1}+2z^{-2}} \qquad R.O.C:|z|>2$$

$x[n]$?

$$X(z)=\frac{1-0.64z^{-2}}{1-0.2z^{-1}+0.08z^{-2}} \qquad R.O.C:|z|>0.4$$

$x[n]$?