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Digital Signal Processing
Lecture 5

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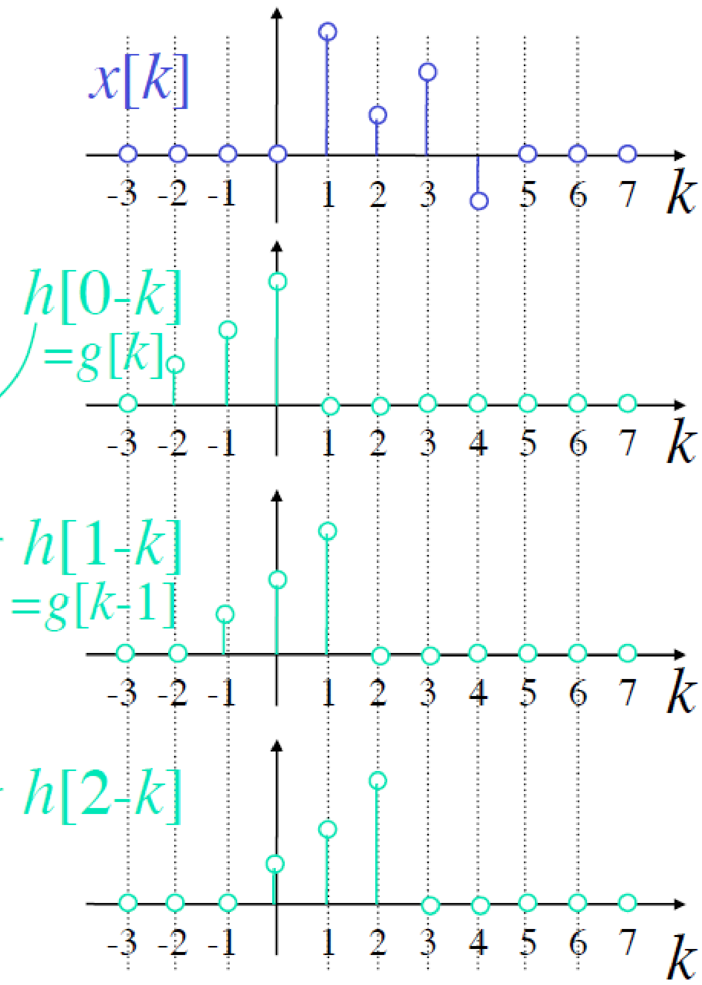
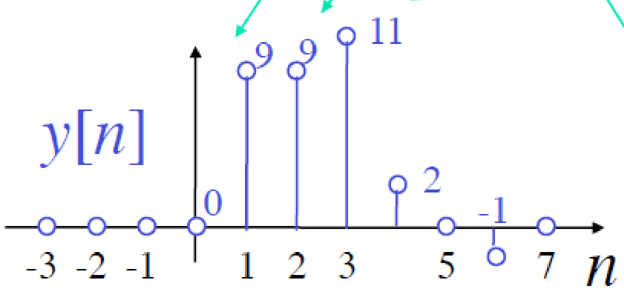
Convolution

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

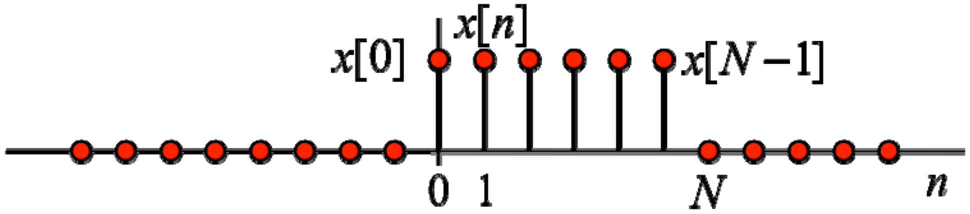
call $h[-n] = g[n]$

- Time-reverse h, shift by n, take inner product against fixed x

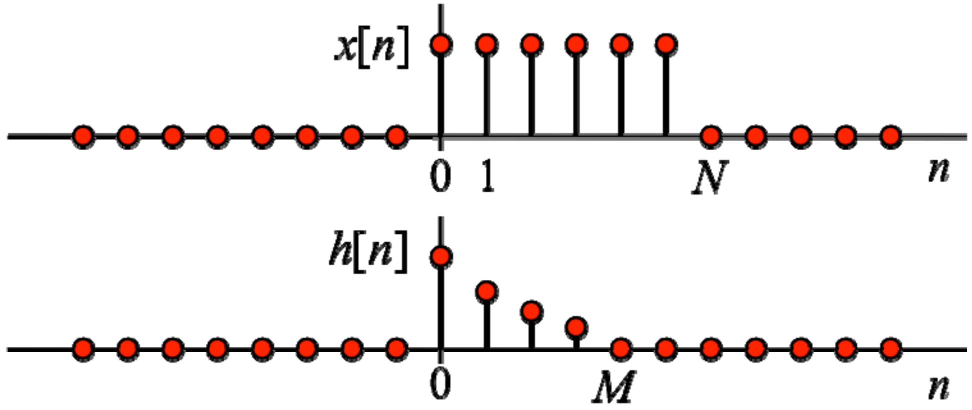




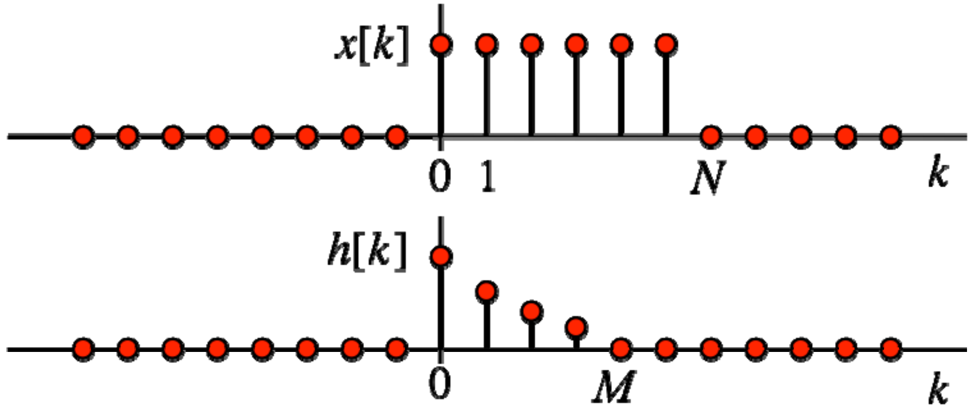
Convolution



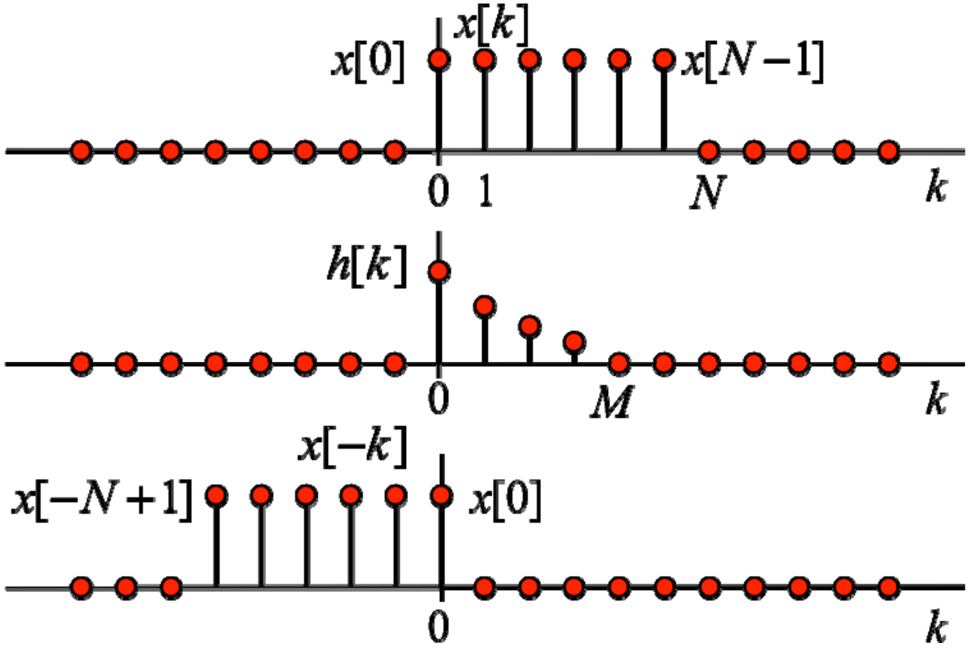
Convolution



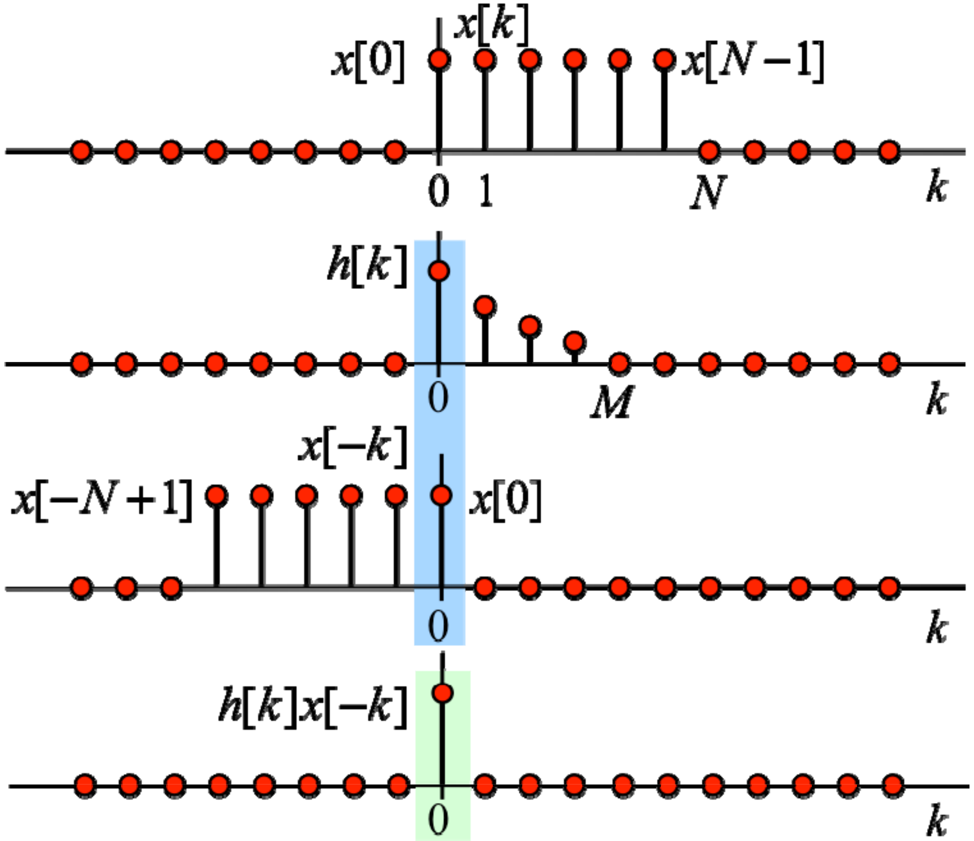
Convolution



Convolution

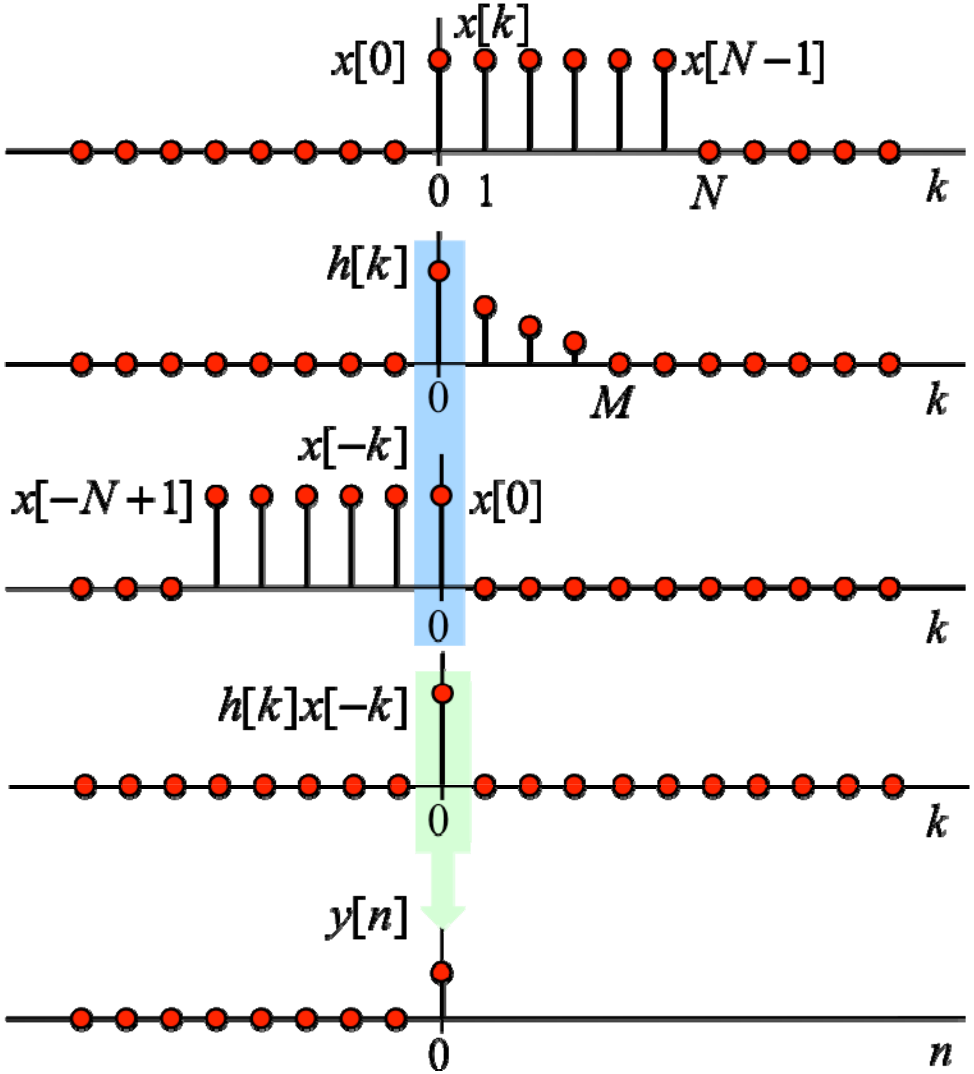


Convolution



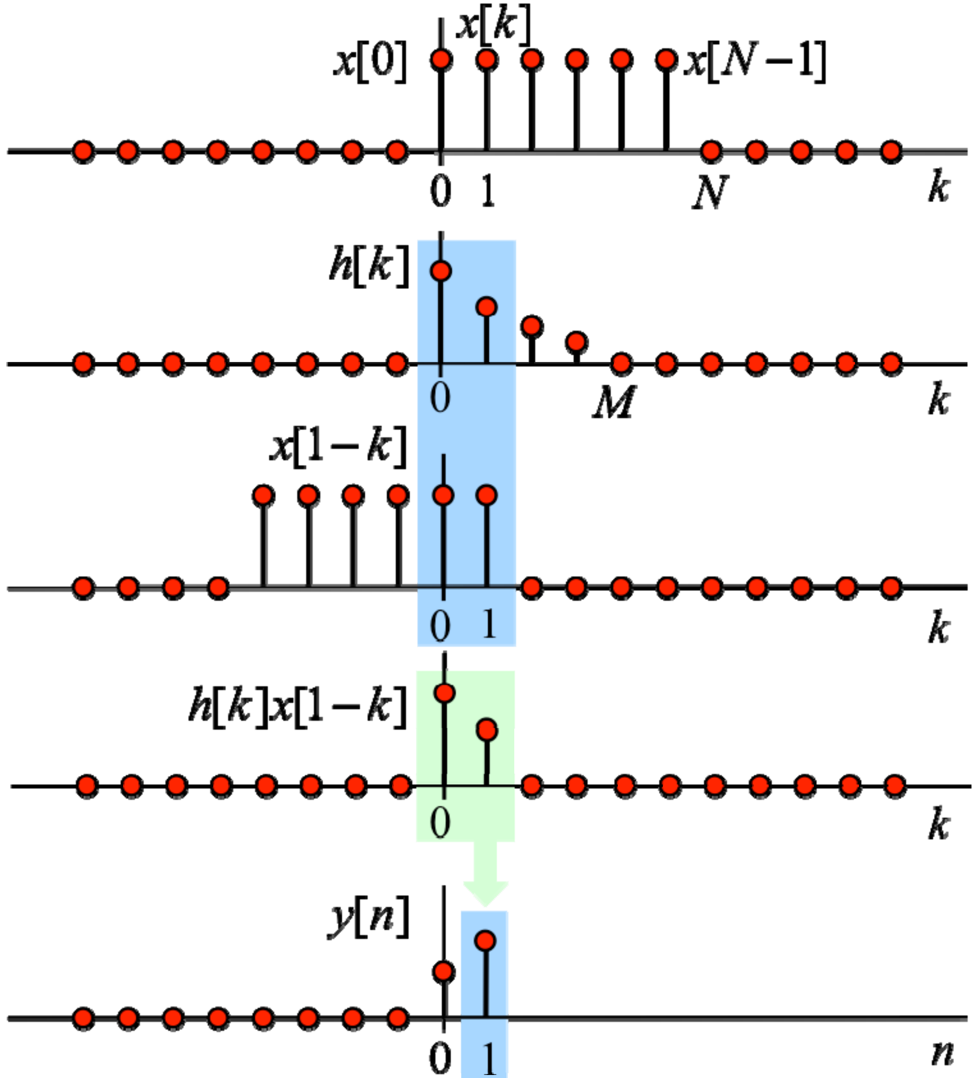
No Overlap for $n < 0$

Convolution



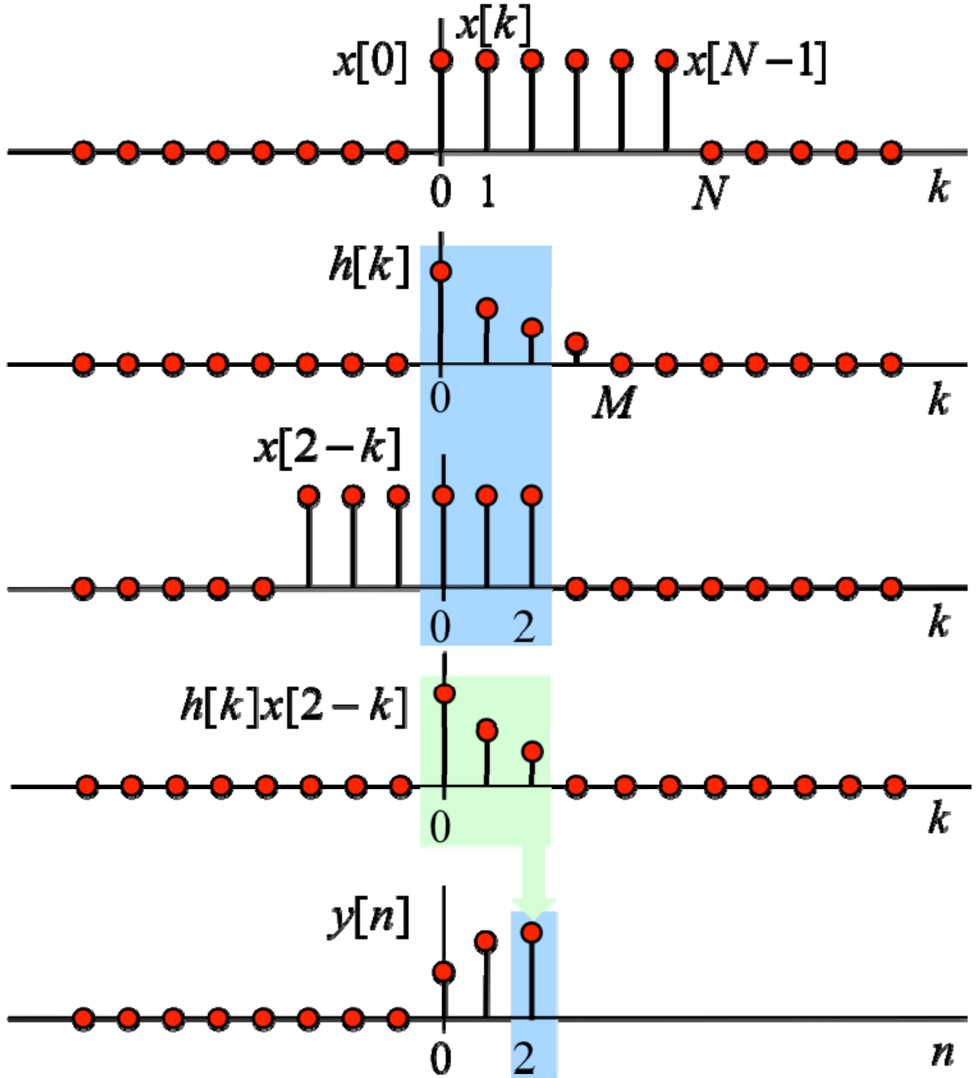
Partial Overlap

Convolution



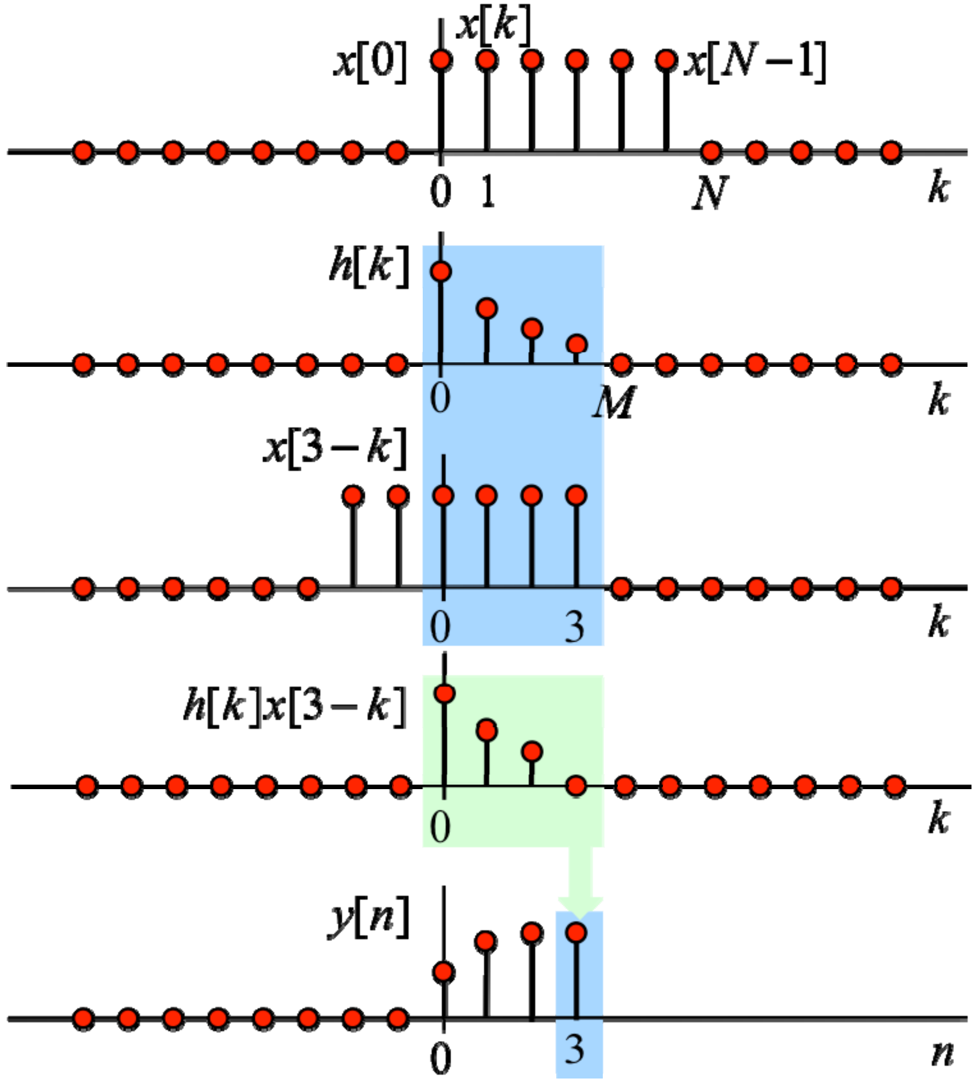
Partial Overlap

Convolution



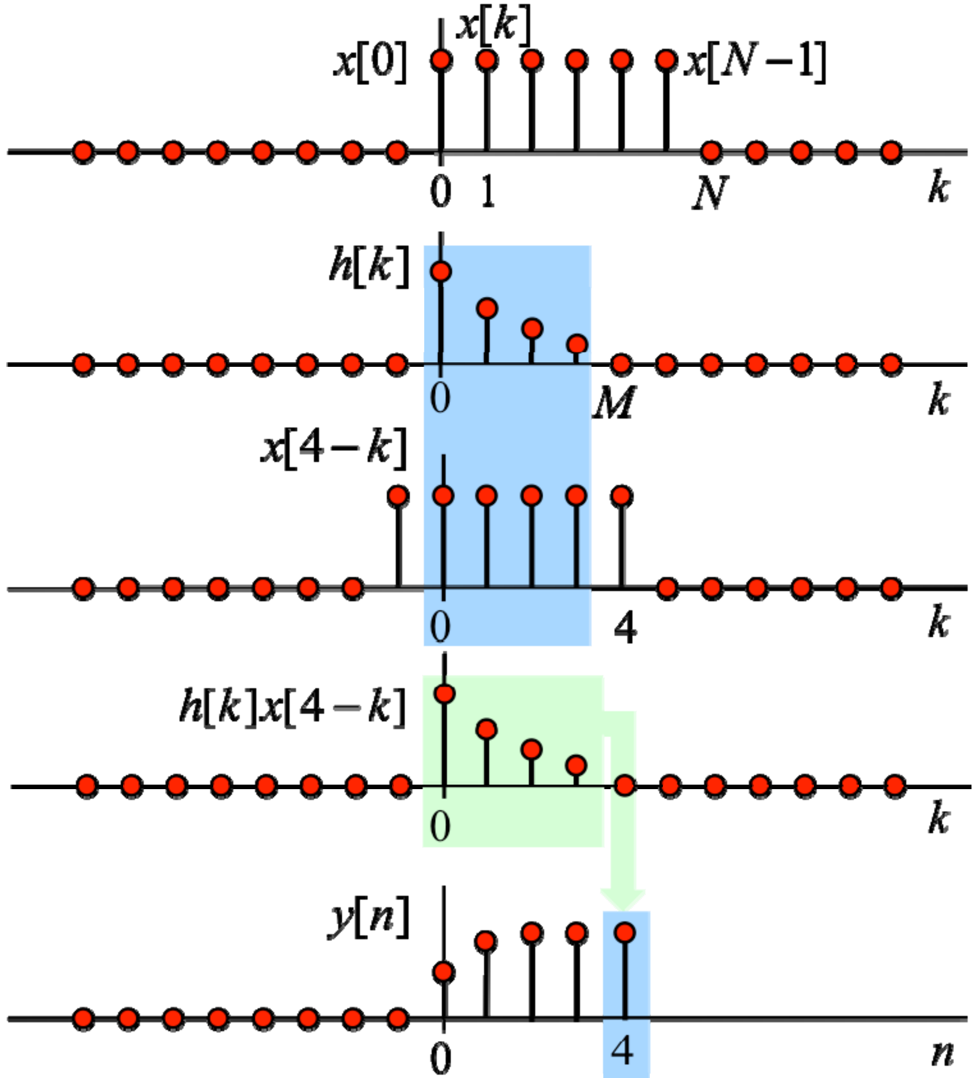
Partial Overlap

Convolution



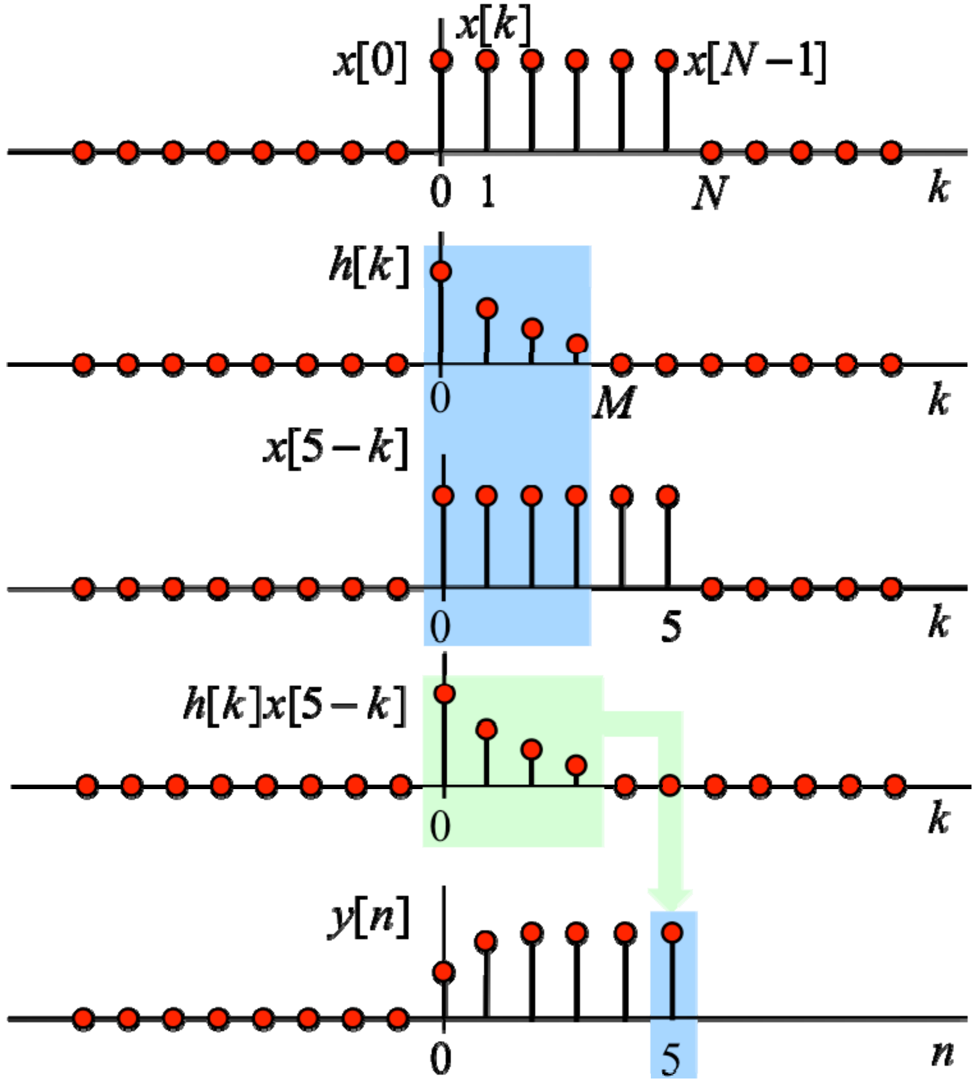
Full Overlap

Convolution



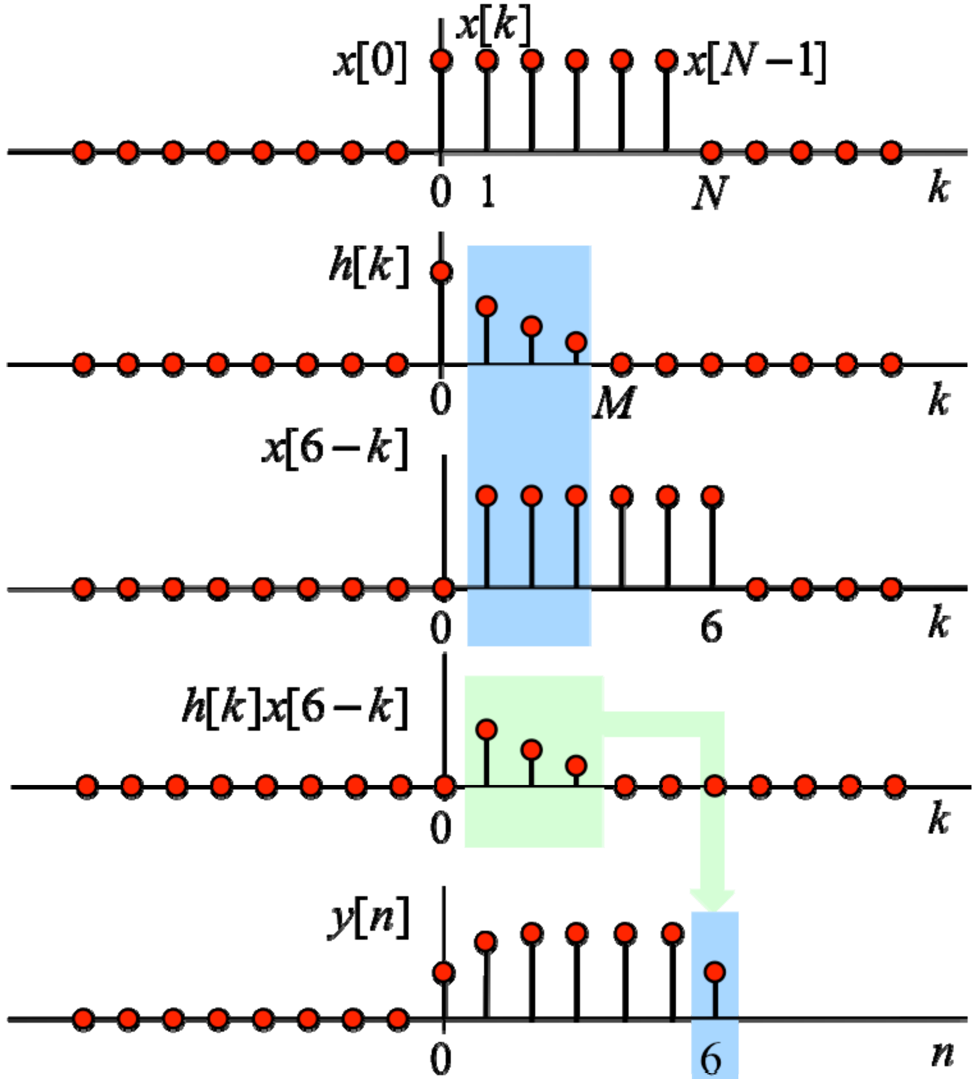
Full Overlap

Convolution



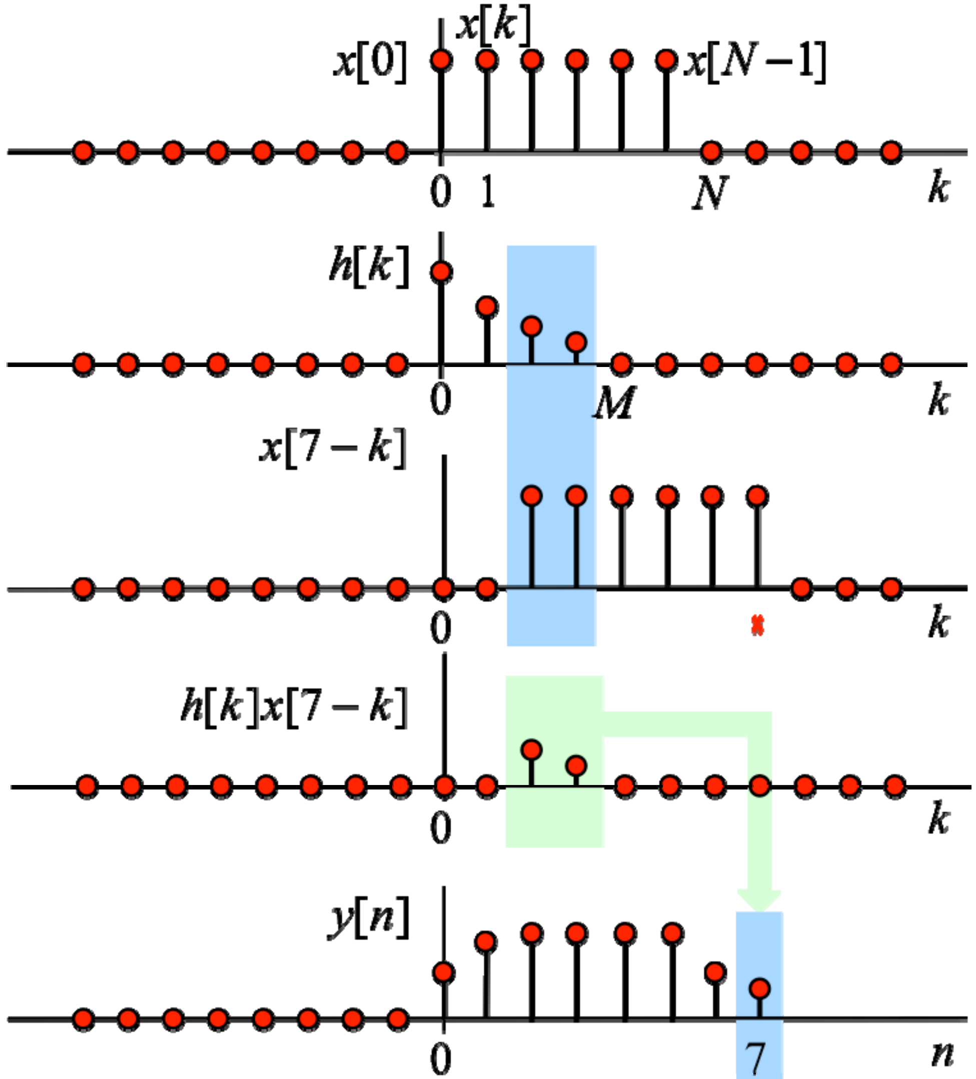
Full Overlap

Convolution



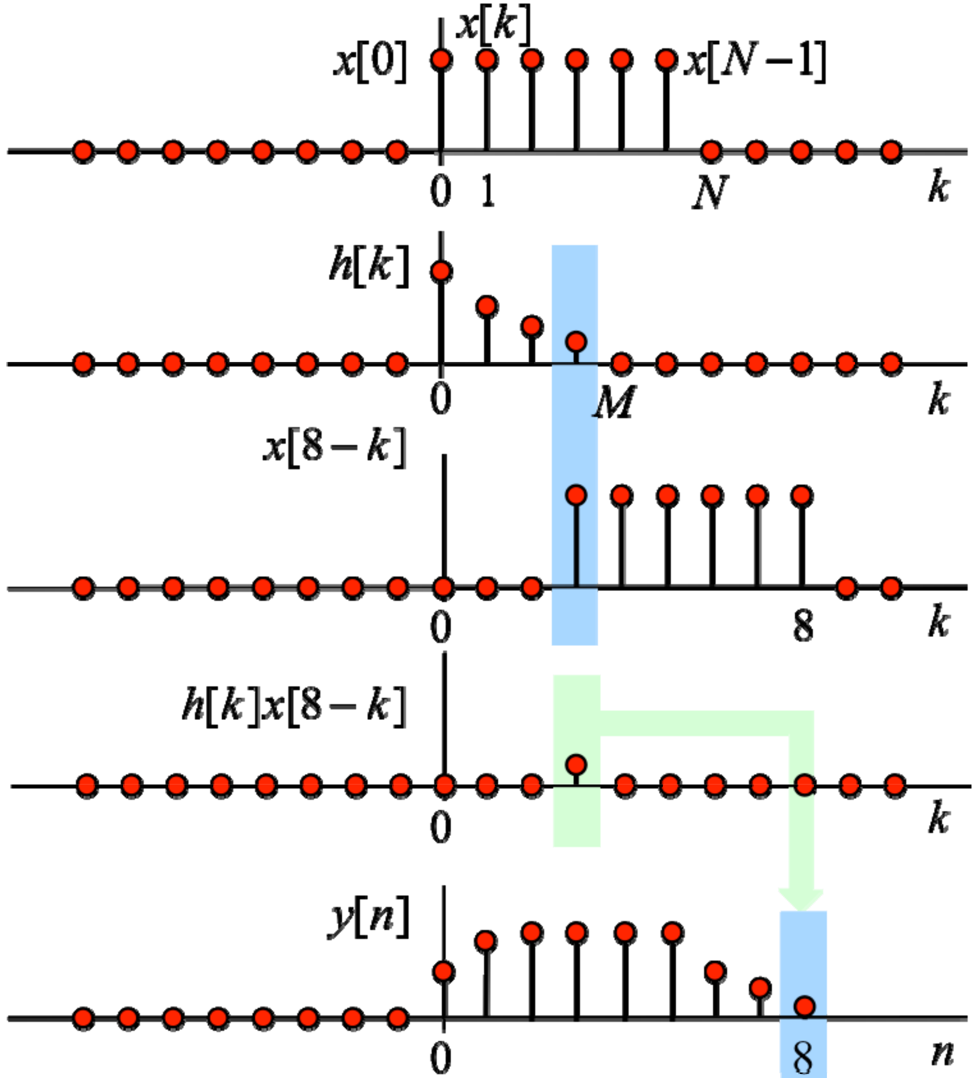
Partial Overlap

Convolution



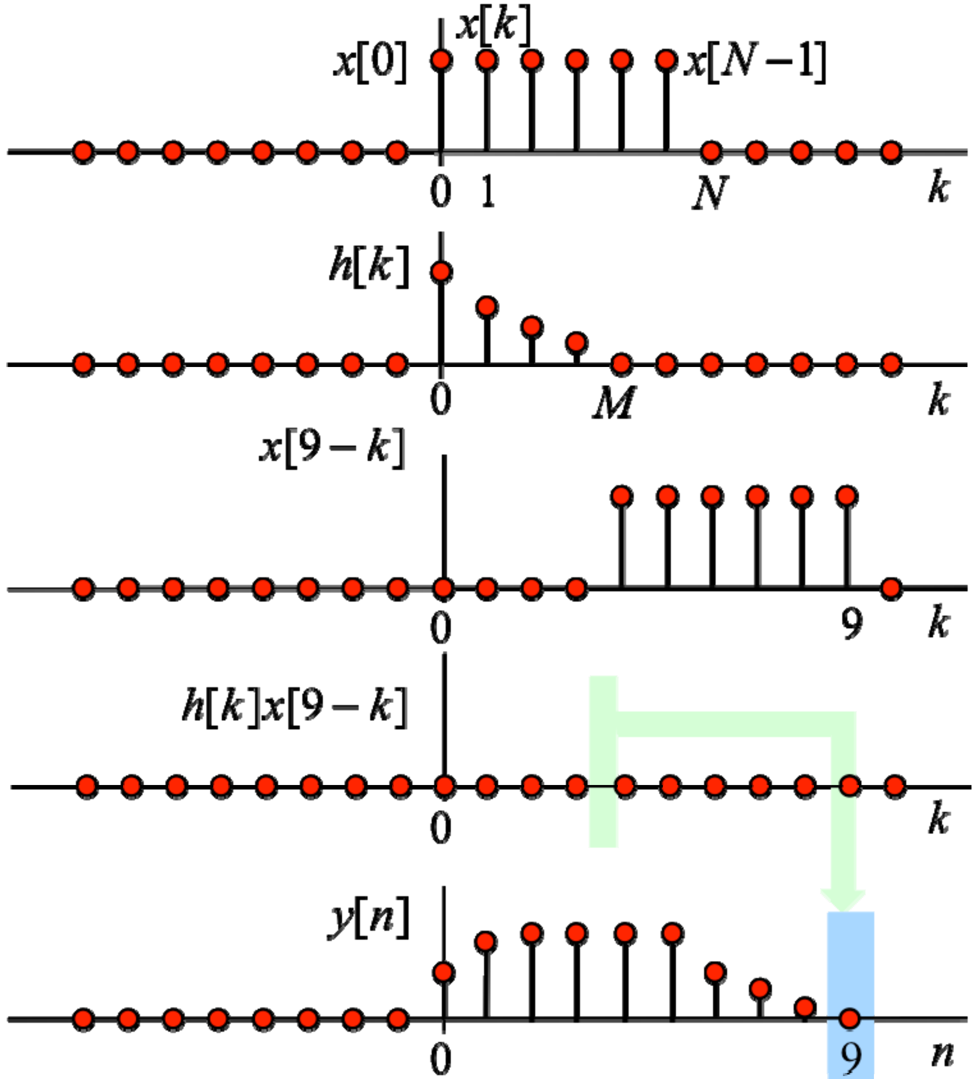
Partial Overlap

Convolution



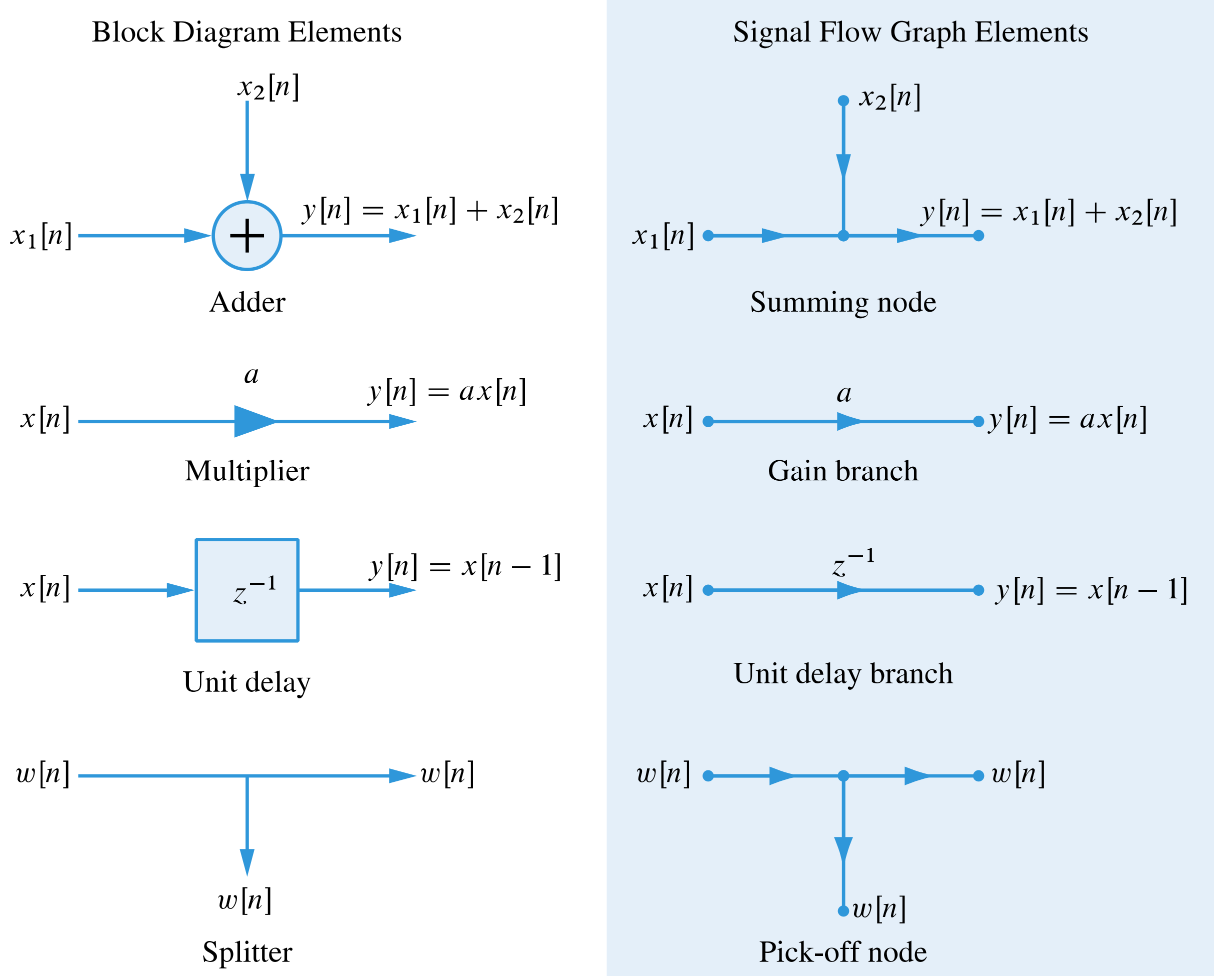
Partial Overlap

Convolution



No Overlap for $n > 0$

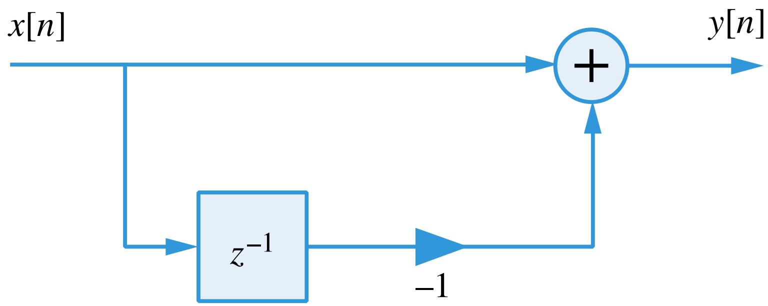
Building Blocks for DT Systems

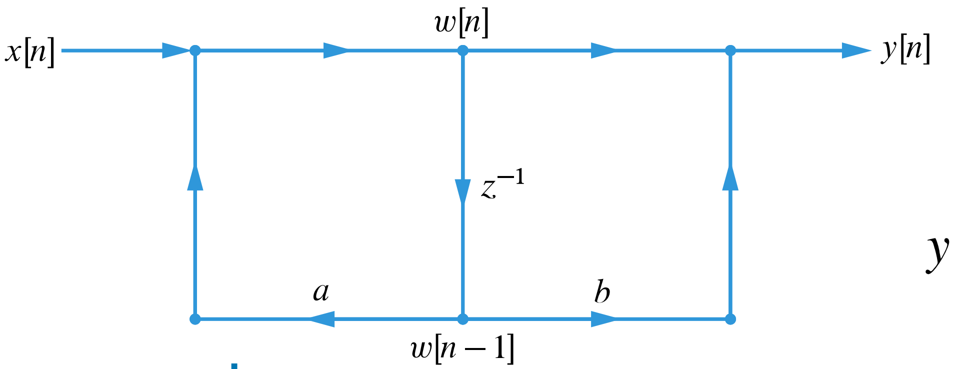


Unit delay = Memory => store at one sampling interval and read at the next one

Examples

$$y[n] = x[n] - x[n-1]$$





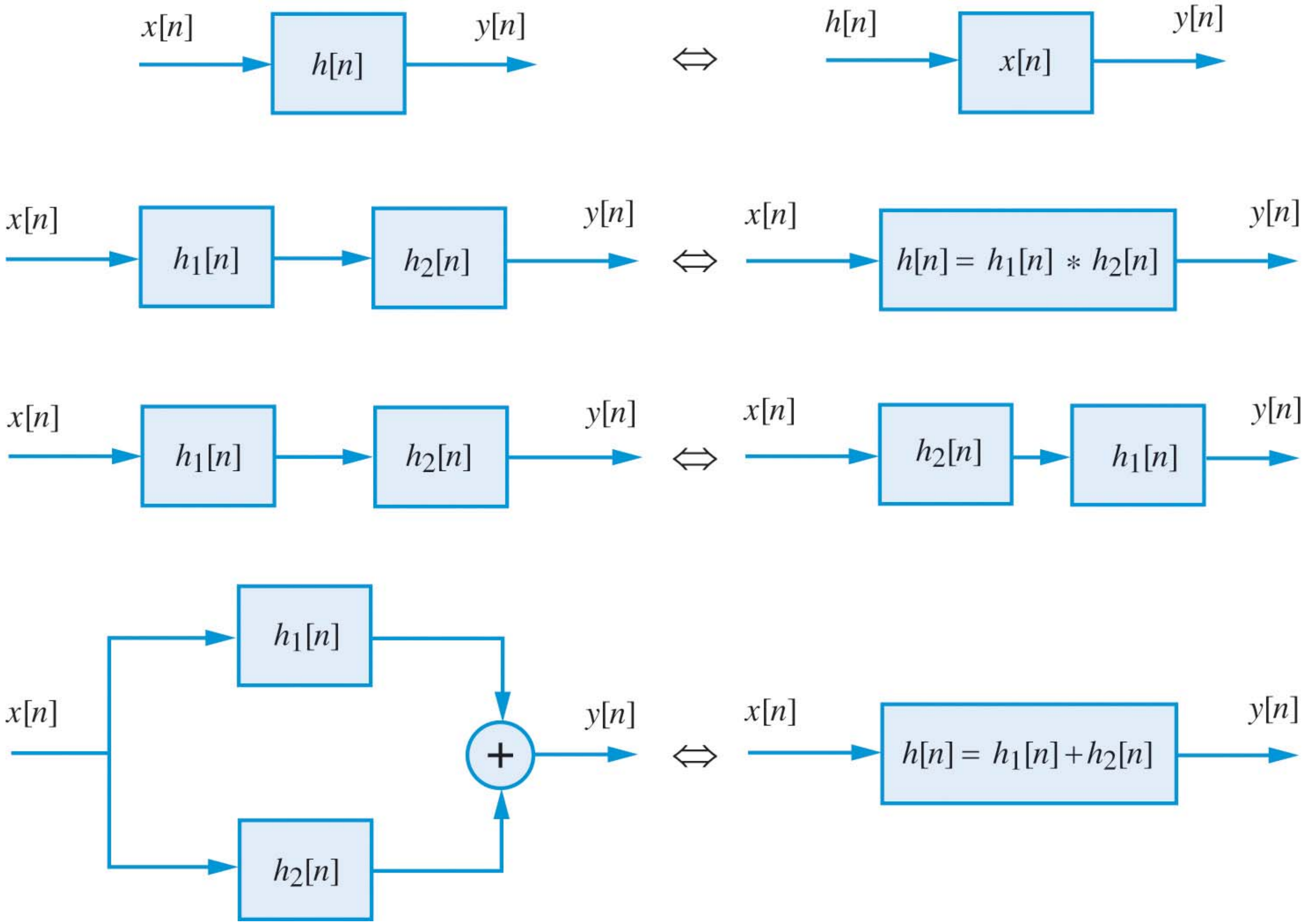
$$y[n] = x[n] + bx[n-1] + ay[n-1]$$

$$w[n] = x[n] + aw[n-1],$$

$$y[n] = w[n] + bw[n-1].$$

Signal flow graphs provide compact representation

Interconnection of LTI Systems



System Realization

For causal LTI systems, h[n] = 0 for n < 0.

Finite impulse response (FIR): $y[n] = \sum_{k=0}^{M-1} h[k]x[n-k]$

Infinite impulse response (IIR): $y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$

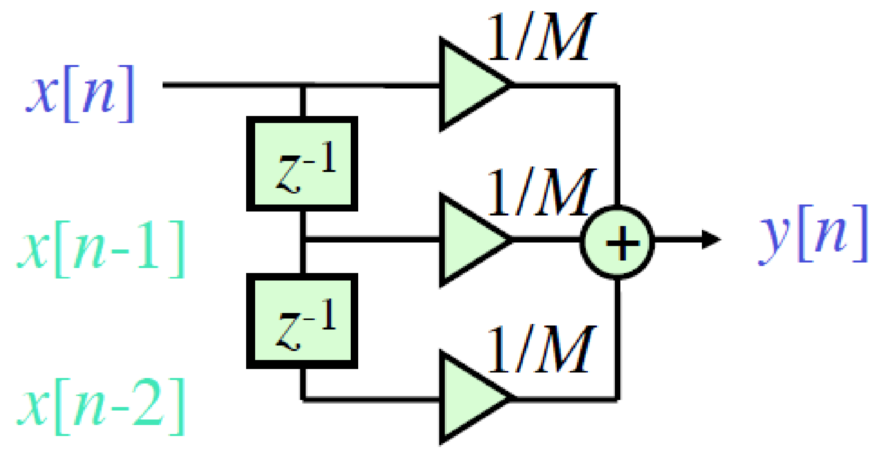
The convolution summation formula expresses the output of the linear time-invariant system explicitly and only in terms of the input signal. When n is increasing, memory requirements also increases with time.

How would one realize these systems?

System Realization

Moving Average

$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$



if y[n] depends only on the present and past inputs, such a system is called nonrecursive.

System Realization

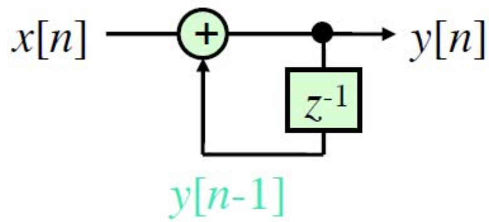
Accumulator

Output accumulates all past inputs:

$$y[n] = \sum_{\ell=-\infty}^{n} x[\ell]$$

$$= \sum_{\ell=-\infty}^{n-1} x[\ell] + x[n]$$

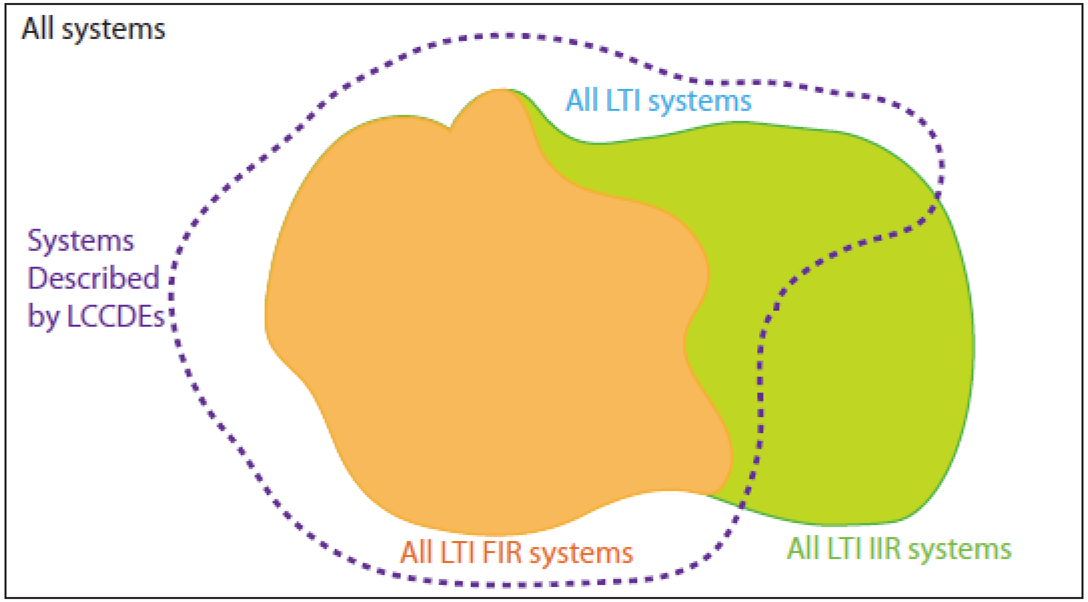
$$= y[n-1] + x[n]$$



- ✓ This is an example of a recursive system. In the recursive systems y[n] depends not only on the present and past inputs, but also available past output values.

There is a practical and computationally efficient means of implementing all FIR and a family of IIR systems that makes use of . . .

. . . difference equations.



Linear Constant-Coefficient Difference Equations (LCCDE)

- ✓ Discrete-time systems described by difference equations express the output of the system not only in terms of the present and past values of the input, but also in terms of the already available past output values:

$$a_0 y[n] + a_1 y[n-1] + \ldots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M]$$

⇩

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

⇩

$$y[n] = \frac{1}{\alpha_0}\left(\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k] \right)$$

Linear Constant-Coefficient Difference Equations

- ✓ If the output signal does not depend on the past values of output (N=0), it is defined as:

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\Downarrow$$

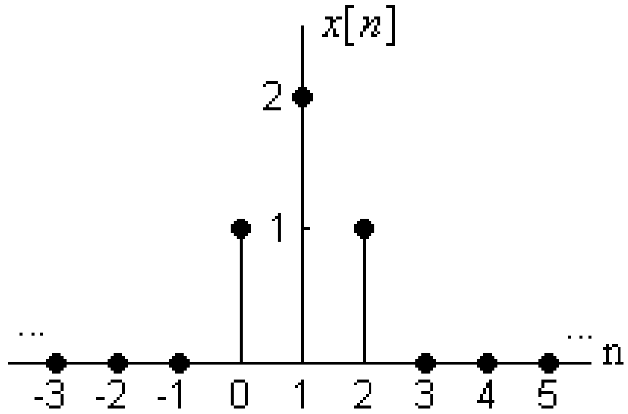
$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_k \quad , & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

- ✓ The length of impulse response is M+1.

Example-1

$$y[n] = y[n-1] + x[n]$$

$$y[-1] = 0$$



$$n = 0 \quad y[0] = y[-1] + x[0] = 1$$

$$n = 1 \quad y[1] = y[0] + x[1] = 3$$

$$n = 2 \quad y[2] = y[1] + x[2] = 4$$

$$n = 3 \quad y[3] = y[2] + x[3] = 4$$

$$n \geq 4 \quad x[n] = 0, y[n] = 4$$

$$n < 0 \quad x[n] = 0, y[-1] = 0, y[n] = 0$$

Example-2

$$y[n] = ay[n-1] + x[n]$$

$$x[n] = b\delta[n] \ , \ y[-1] = 1$$

$$y[0] = a + b \qquad y[-2] = a^{-1}$$

$$y[1] = a^2 + ab \qquad y[-3] = a^{-2}$$

$$y[2] = a^3 + a^2 b \qquad y[-4] = a^{-3}$$

$$y[3] = a^4 + a^3 b \qquad y[-5] = a^{-4}$$

$$\vdots \qquad \vdots$$

$$y[n] = a^{n+1} + a^n b \ , \ n \geq 0 \qquad y[n] = a^{n+1} \ , \ n < 0$$

Example-2-Cont

$$y[n] = a^{n+1} + a^n b \quad , \quad n \geq 0 \qquad\qquad y[n] = a^{n+1} \quad , \quad n < 0$$

$$\boxed{y[n] = a^{n+1} + a^n b u[n]}$$

- If b=0 → x[n]=0, but y[n]=a^{n+1} is not equal to zero.
- Therefore, scaling the input with zero is not gives zero output (the system is not linear).

Example-2-Cont

- For the shifted input

$$x_1[n] = x[n-n_d] = b\delta[n-n_d]$$

$$y_1[n] = a^{n+1} + a^{n-n_0} b u[n-n_0]$$

$$y_1[n] \neq y[n-n_0]$$

- the system is time variant.

Example-2-Cont

- If y[-1]=0 is given

$$y[n] = a^n b u[n]$$

- The system is linear and time invariant in this case (evaluate with $x[n] = b\delta[n-1]$).
- NOTE: Initial conditions of a LCCDE for systems affect the characteristics directly!
- In general, initial conditions and x[n]=0, n<0 are chosen as zero for causal LTI systems.

Linear Constant-Coefficient Difference Equations

- ✓ Given LCCDE as the I/O relationship describing LTI system, the objective is to determine an explicit expression for the output y[n].

- ✓ Basically, the goal is to determine y[n], n≥0, of the system given a specific input x[n], n≥0, and set of initial conditions.

- ✓ The direct solution method assumes that the total solution is the sum of two parts:

$$y[n] = y_h[n] + y_p[n]$$

$y_h[n]$: homogeneous /complementary solution

$y_p[n]$: particular solution

Linear Constant-Coefficient Difference Equations

Homogeneous Solution:

$$\sum_{k=0}^{N} a_k y[n-k] = 0$$

It is assumed that the solution of this eq. is in the form of

$$y[n] = \lambda^n$$

and the eq. is described as polynomial eq.

$$\lambda^n + a_1 \lambda^{n-1} + \ldots + a_N \lambda^{n-N} = 0$$

$$\underbrace{\lambda^N + a_1 \lambda^{N-1} + \ldots + a_N}_{characteristic \quad poly.} = 0$$

Linear Constant-Coefficient Difference Equations

- ✓ The polynomial has N roots ($\lambda_1, \lambda_2, \ldots, \lambda_N$).
- ✓ The roots can be real or complex valued.
- ✓ Complex-valued roots occur as complex conjugate pairs.
- ✓ Some of N roots may be identical.
- ✓ If the roots are distinct:

$$y_h[n] = C_1\lambda_1^{\,n} + C_2\lambda_2^{\,n} + \cdots + C_N\lambda_N^{\,n}$$

where $C_1, C_2, \ldots, C_N$ are weighting coefficients. These coefficients are determined from the initial conditions.

- ✓ If λ_1 is a root of multiplicity *m*, then eq. becomes

$$y_h[n] = C_1\lambda_1^{\,n} + C_2 n\lambda_1^{\,n} + C_3 n^2\lambda_1^{\,n} + C_4 n^3\lambda_1^{\,n} \cdots$$
$$+ C_m n^{m-1}\lambda_1^{\,n} + C_{m+1}\lambda_{m+1}^{\,n} + \cdots + C_N\lambda_N^{\,n}$$

Linear Constant-Coefficient Difference Equations

Particular Solution:

$$\sum_{k=0}^{N} a_k y_p[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- ✓ To solve this eq., it is assumed for $y_p[n]$, a form that depends on the form of the input x[n].
- ✓ If x[n] is given as an exponential, it is assumed that the particular solution is also exponential.
- ✓ If x[n] is sinusoidal, the particular solution is also sinusoidal.
- ✓ Thus, the assumed form for the particular solution takes the basic form of the signal x[n].

Example-1

$$y[n]+y[n-1]-6y[n-2]=x[n]$$

- Complementary solution:

$$y[n]+y[n-1]-6y[n-2]=0; \quad y[n]=\lambda^n$$

$$\Rightarrow \lambda^{n-2}\left(\lambda^2+\lambda-6\right)=0$$

$$\Rightarrow (\lambda+3)(\lambda-2)=0 \rightarrow \text{roots } \lambda_1=-3,\ \lambda_2=2$$

$$\Rightarrow y_c[n]=\alpha_1(-3)^n+\alpha_2(2)^n$$

- α_1, α_2 are unknown at this point

Example-1

- Particular solution:
- Input $x[n]$ is constant =8u[n]

  assume $y_p[n] = \beta$, substitute in:

  $y[n]+y[n-1]-6y[n-2]=x[n]$ ('large' n)

  $\Rightarrow \beta+\beta-6\beta=8\mu[n]$

  $\Rightarrow -4\beta=8 \Rightarrow \beta=-2$

Example-1 Cont

- Total solution $y[n] = y_c[n] + y_p[n]$
$$= \alpha_1(-3)^n + \alpha_2(2)^n + \beta$$
- Solve for unknown α_is by substituting *initial conditions* into DE at $n = 0, 1, \ldots$
$$y[n] + y[n-1] - 6y[n-2] = x[n]$$

from ICs

- $n = 0$ $\quad y[0] + y[-1] - 6y[-2] = x[0]$
$$\Rightarrow \alpha_1 + \alpha_2 + \beta + 1 + 6 = 8$$
$$\Rightarrow \alpha_1 + \alpha_2 = 3$$

Example-1 Cont

- $\underline{n=1}$ $\quad y[1]+y[0]-6y[-1]=x[1]$

  $\Rightarrow \alpha_1(-3)+\alpha_2(2)+\beta+\alpha_1+\alpha_2+\beta-6=8$

  $\Rightarrow -2\alpha_1+3\alpha_2=18$
- solve: $\alpha_1 = -1.8$, $\alpha_2 = 4.8$
- Hence, system output:

  $y[n]=-1.8(-3)^n+4.8(2)^n-2 \quad n\geq 0$
- *Don't* find α_is by solving with ICs at $n = -1,-2$

Linear Constant-Coefficient Difference Equations

- ✓ The total solution can be also defined as the sum of two parts:

$$y[n] = y_{zi}[n] + y_{zs}[n]$$

$y_{zi}[n]$= zero input response

$$\sum_{k=0}^{N} a_k y_{zi}[n-k] = 0$$

$$y_{zi}[n] = \sum_{j=1}^{N} C_j \lambda_j^n$$

y_{zs}=zero state response

$$y_{zs}[n] = y_h[\mathrm{n}] + y_p[\mathrm{n}]$$

Assume that all the initial conditions are zero

Example

$$y[n]-3y[n-1]-4y[n-2]=0$$

y[-1]=5 ve y[-2]=0

$$y_{zi}[n]=(-1)^{n}+(4)^{n+2}$$

Linear Constant-Coefficient Difference Equations

✓ h[n] is the zero state response of LTI causal systems since h[n]=0 when n<0.

- Impulse response: $\delta[n] \longrightarrow \boxed{\text{LCCDE}} \longrightarrow h[n]$

  i.e. solve with $x[n] = \delta[n] \rightarrow y[n] = h[n]$
  (zero ICs)

- With $x[n] = \delta[n]$, 'form' of $y_p[n] = \beta\, \delta[n]$

  → solve $y[n]$ for $n = 0,1,2...$

Example

- e.g. $y[n]+y[n-1]-6y[n-2]=x[n]$
  (from before); $x[n] = \delta[n]; y[n] = 0$ for $n<0$
- $y_c[n]=\alpha_1(-3)^n+\alpha_2(2)^n \qquad y_p[n] = \beta\delta[n]$
- $\underline{n=0:}$ $y[0]+\cancel{y[-1]}-6\cancel{y[-2]}=\cancel{x[0]}^{\,1}$
  $\Rightarrow \alpha_1 + \alpha_2 + \beta = 1$
- $\underline{n=1:}$ $\alpha_1(-3) + \alpha_2(2) + 1 = 0$
- $\underline{n=2:}$ $\alpha_1(9) + \alpha_2(4) - 1 - 6 = 0$
  $\Rightarrow \alpha_1 = 0.6,\ \alpha_2 = 0.4,\ \beta = 0$
- thus $h[n]=0.6(-3)^n+0.4(2)^n \quad n \geq 0$