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Digital Signal Processing
Lecture 7

Quote of the Day

Whoever despises the high wisdom of mathematics nourishes himself on delusion.

- Leonardo da Vinci

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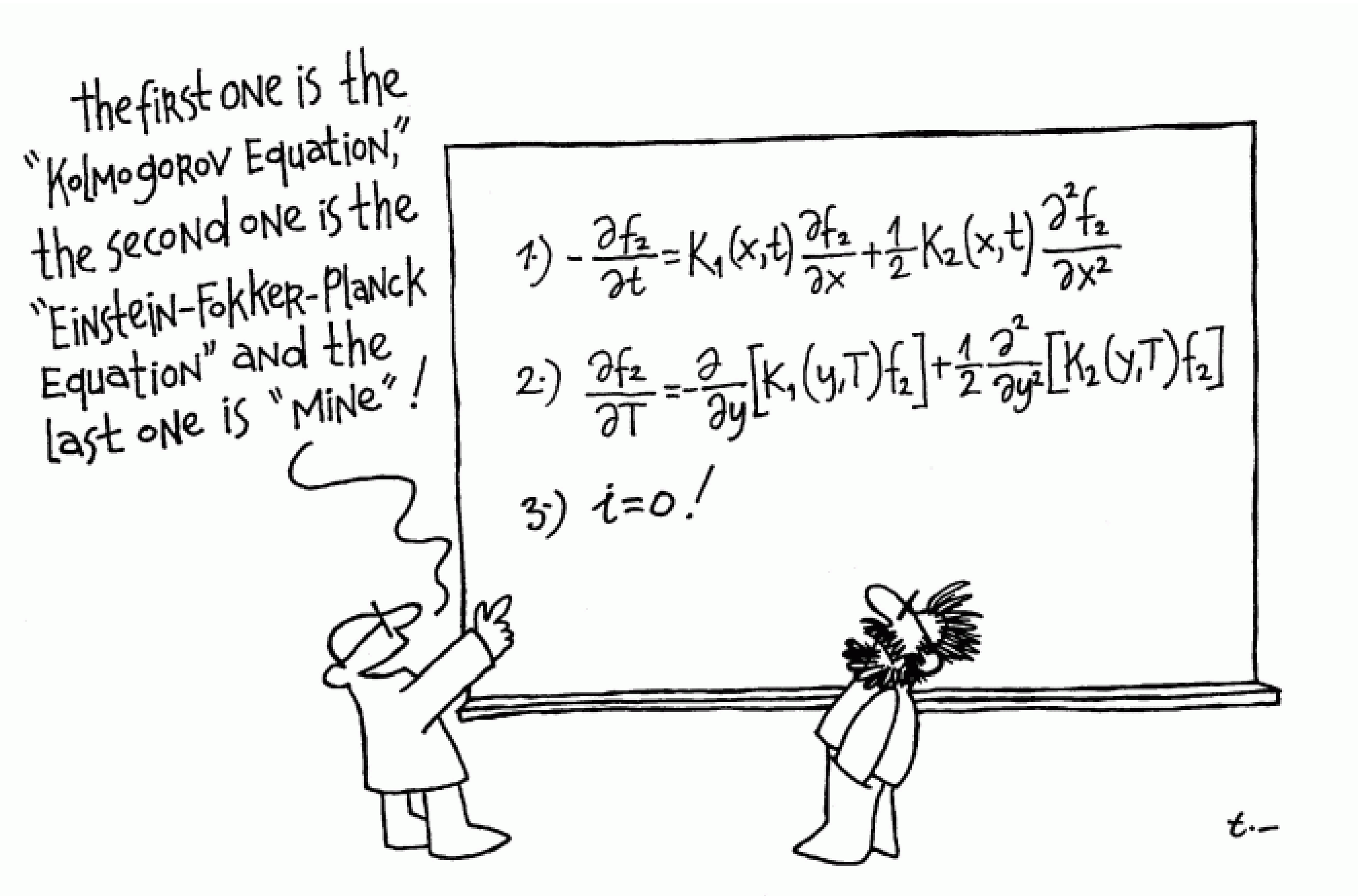


the first one is the "Kolmogorov Equation," the second one is the "Einstein-Fokker-Planck Equation" and the last one is "Mine"!
1:) $-\frac{\partial f_2}{\partial t} = K_1(x,t)\frac{\partial f_2}{\partial x} + \frac{1}{2}K_2(x,t)\frac{\partial^2 f_2}{\partial x^2}$
2:) $\frac{\partial f_2}{\partial T} = -\frac{\partial}{\partial y}[K_1(y,T)f_2] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[K_2(y,T)f_2]$
3:) $i = 0$!

Discrete Time Fourier Transform

- ✓ Frequency analysis of discrete-time signals must conveniently be performed on a computer or DSP.

- ✓ $X(e^{j\omega})$ can not be computed for the entire set of ω; for practically it must be computed for a discrete and finite set of values.

- ✓ Strategy to compute $X(e^{j\omega})$
  - Compute $X(e^{j\omega})$ for equally spaced samples.
  - Compute $X(e^{j\omega})$ samples for one period only (recall, $X(e^{j\omega})$ is periodic with period 2π.

- ✓ Therefore, we define the Discrete Fourier Transform (DFT) as being a computable transform that approximates the DTFT.

Discrete Fourier Transform

- Consider a sequence with a Fourier transform

$$x[n] \xleftrightarrow{DTFT} X\left(e^{j\omega}\right)$$

- Assume that a sequence is obtained by sampling the DTFT

$$\tilde{X}[k] = X\left(e^{j\omega}\right)\Big|_{\omega=(2\pi/N)k} = X\left(e^{j(2\pi/N)k}\right)$$

- Since the DTFT is periodic resulting sequence is also periodic.
- $\tilde{X}[k]$ is the Discrete Fourier Series of a sequence.
- Write the corresponding sequence:

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j(2\pi/N)kn}$$

Discrete Fourier Transform

- The only assumption made on the sequence is that DTFT exist.

$$X\left(e^{j\omega}\right)=\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}$$

$$\tilde{X}[k]=X\left(e^{j(2\pi/N)k}\right)$$

$$\tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j(2\pi/N)kn}$$

- Combine equation to get:

$$\tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=-\infty}^{\infty} x[m]e^{-j(2\pi/N)km}\right]e^{j(2\pi/N)kn}$$

$$=\sum_{m=-\infty}^{\infty} x[m]\left[\frac{1}{N}\sum_{k=0}^{N-1}e^{j(2\pi/N)k(n-m)}\right]=\sum_{m=-\infty}^{\infty} x[m]\tilde{p}[n-m]$$

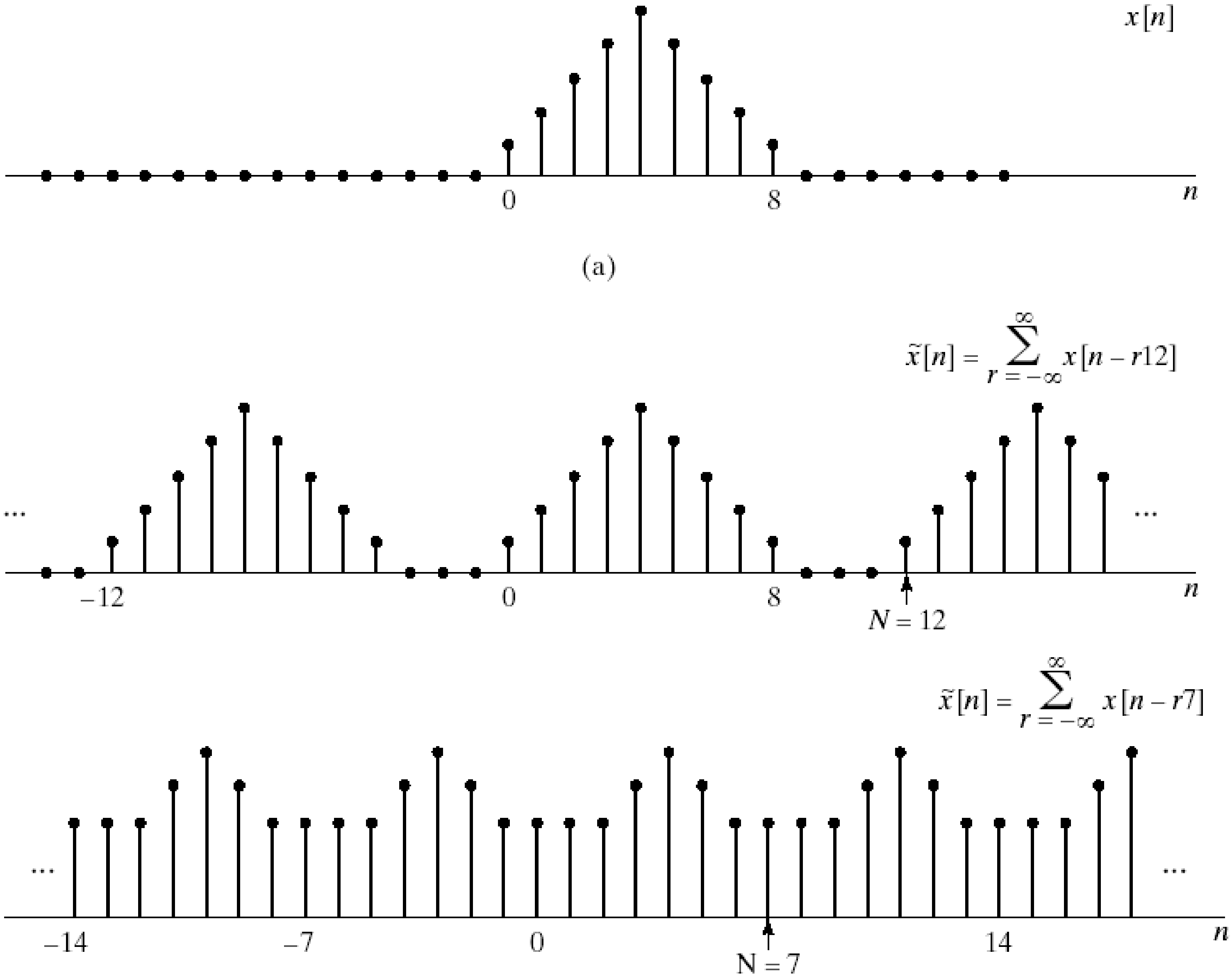
- Term in the parenthesis is

$$\tilde{p}[n-m]=\frac{1}{N}\sum_{k=0}^{N-1}e^{j(2\pi/N)k(n-m)}=\sum_{r=-\infty}^{\infty}\delta[n-m-rN]$$

- So we get

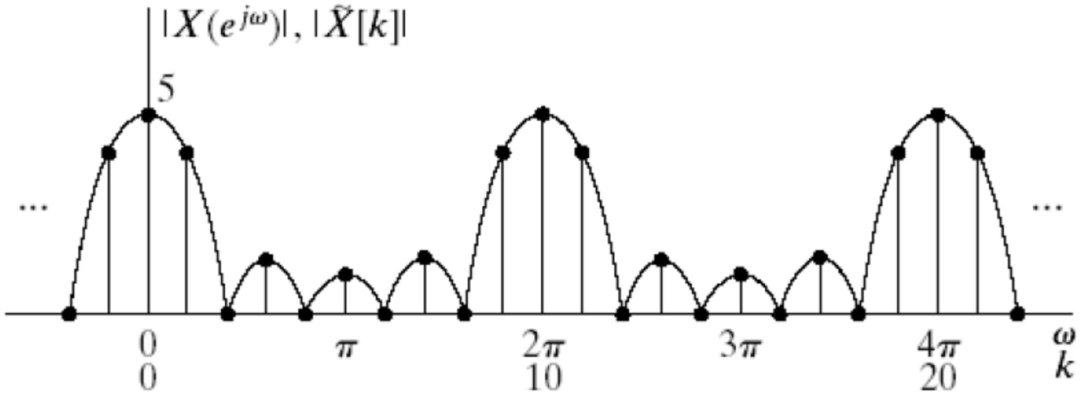
$$\tilde{x}[n]=x[n]*\sum_{r=-\infty}^{\infty}\delta[n-rN]=\sum_{r=-\infty}^{\infty}x[n-rN]$$

Discrete Fourier Transform

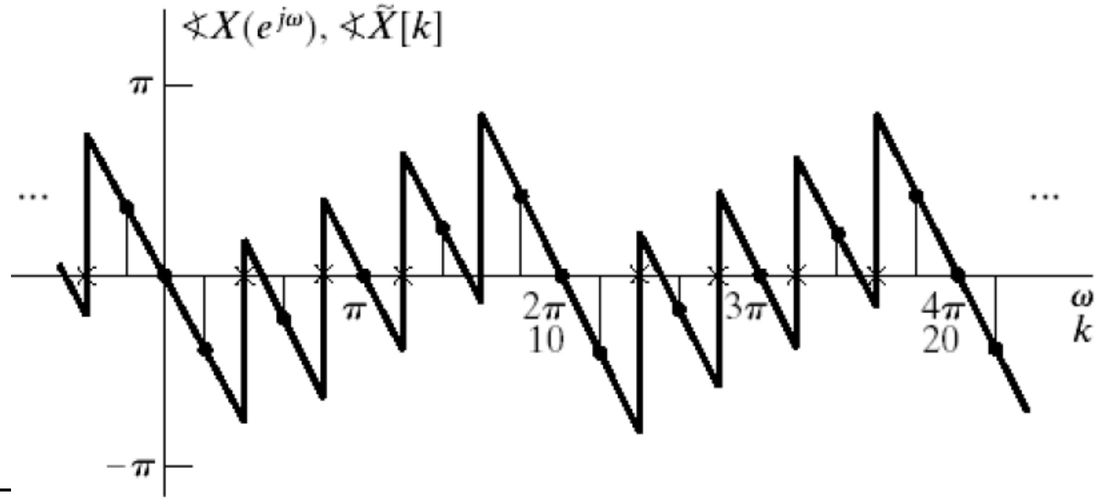


Discrete Fourier Transform

Sampling in Frequency Domain-Example



(a)



Discrete Fourier Transform

If the original sequence

- is of finite length
- and we take sufficient number of samples of its DTFT
- the original sequence can be recovered by

$$x[n] = \begin{cases} \tilde{x}[n] & 0 \le n \le N-1 \\ 0 & else \end{cases}$$

Discrete Fourier Transform

- Representing a finite length sequence by samples of DTFT

Discrete Fourier Transform

- Consider a finite length sequence x[n] of length N

$$x[n] = 0 \quad \text{outside of} \quad 0 \le n \le N-1$$

- For given length-N sequence associate a periodic sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN]$$

- The Discrete Fourier Series (DFS) coefficients of the periodic sequence are samples of the DTFT of x[n]. Since x[n] is of length N there is no overlap between terms of x[n-rN] and we can write the periodic sequence as:

$$\tilde{x}[n] = x[(n \mod \mathrm{N})] = x[((n))_N]$$

- To maintain duality between time and frequency
  - We choose one period of $\tilde{X}[k]$ as the Fourier transform of x[n]

$$X[k] = \begin{cases} \tilde{X}[k] & 0 \le k \le N-1 \\ 0 & else \end{cases}$$

$$\tilde{X}[k] = X[(k \mod \mathrm{N})] = X[((k))_N]$$

Discrete Fourier Transform

- The DFS pair: $\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j(2\pi/N)kn}$ $\quad \tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j(2\pi/N)kn}$

- The equations involve only on period so we can write

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j(2\pi/N)kn} & 0 \le k \le N-1 \\ 0 & else \end{cases}$$

$$x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j(2\pi/N)kn} & 0 \le k \le N-1 \\ 0 & else \end{cases}$$

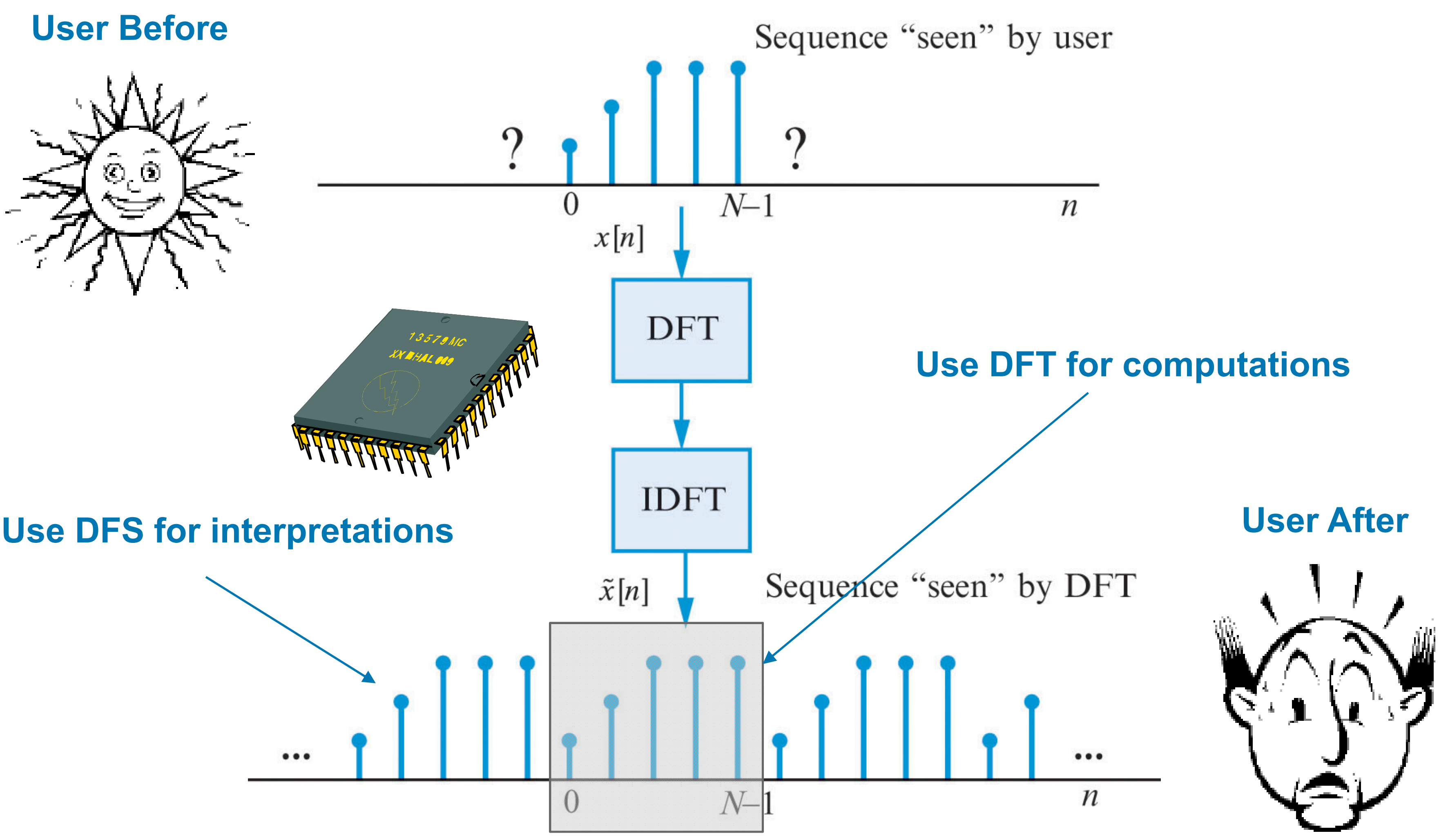
- The Discrete Fourier Transform (DFT):

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn} \qquad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)kn}$$

- The DFT pair can also be written as

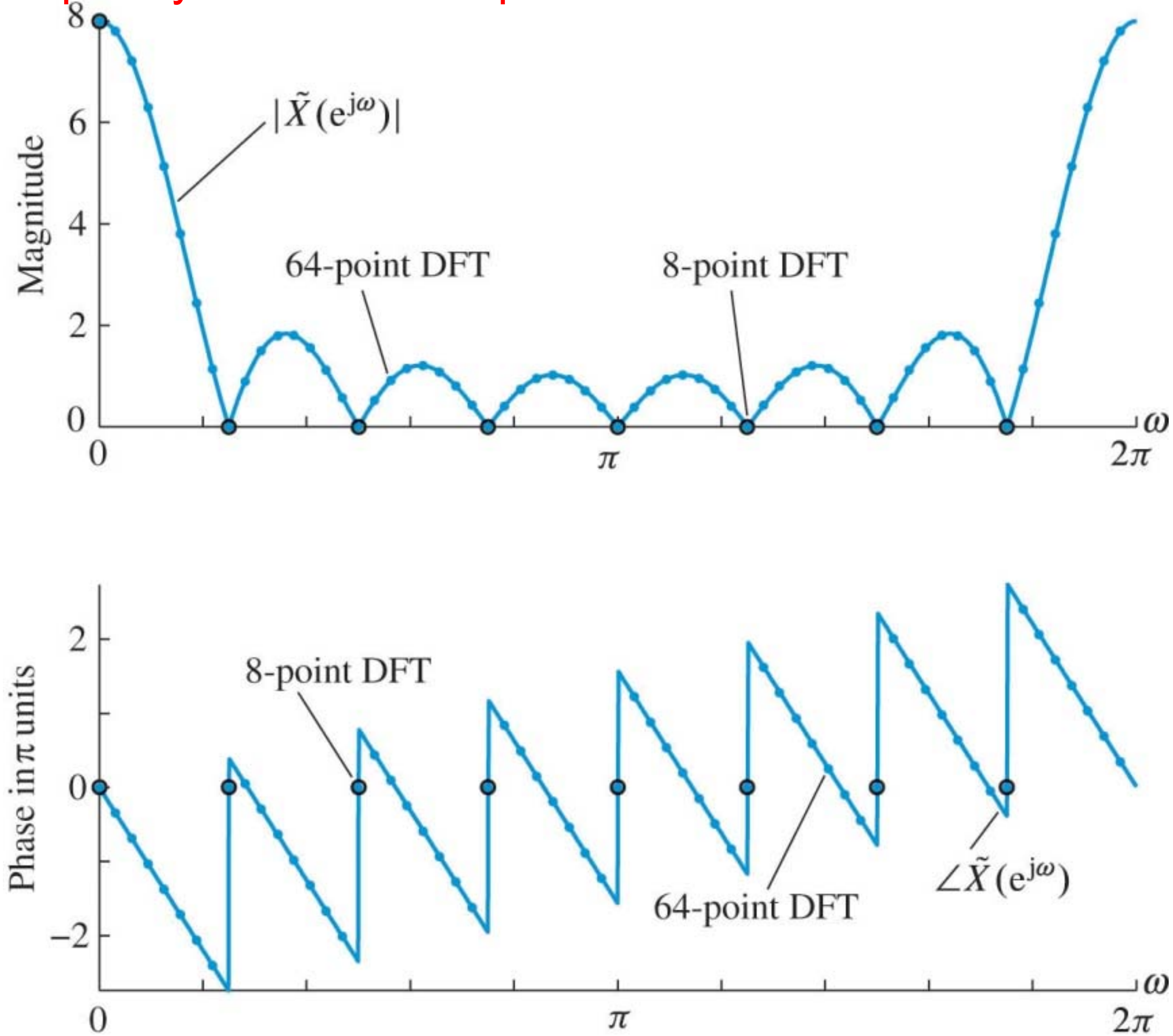
$$x[n] \xleftrightarrow{DFT} X[k]$$

Implications of Inherent DFT Periodicity



Discrete Fourier Transform

Sampling in Frequency Domain-Example



Discrete Fourier Transform

- Alternative representation of DFT and IDFT:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad , \quad k = 0,1,\ldots,N-1$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \quad , \quad n = 0,1,\ldots,N-1$$

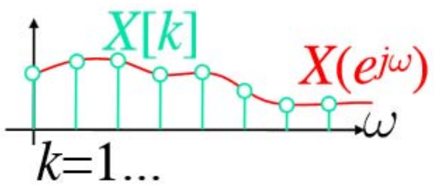
$$W_N = e^{-j(2\pi/N)}$$

DFT versus DTFT

DTFT	$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$	• *continuous freq* • *infinite $x[n]$, $-\infty<n<\infty$*
DFT	$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$	• *discrete freq* • *finite $x[n]$, $0 \le n < N$*

- DFT 'samples' DTFT at discrete freqs:

$$X[k] = X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}}$$



Discrete Fourier Transform

- Inverse DFT: **IDFT** $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-nk}$
- Check:

$$x[n] = \frac{1}{N}\sum_k \left(\sum_l x[l] W_N^{kl}\right) W_N^{-nk}$$

$$= \frac{1}{N}\sum_{l=0}^{N-1} x[l] \boxed{\sum_{k=0}^{N-1} W_N^{k(l-n)}} \quad = 0 \text{ if } l \neq n; = N \text{ if } l = n$$

$$= x[n] \quad \checkmark$$

$0 \leq n < N$

Discrete Fourier Transform

- *Finite* impulse $x[n] = \begin{cases} 1 & n = 0 \\ 0 & n = 1 \ldots N-1 \end{cases}$

$$\Rightarrow X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = W_N^0 = 1 \quad \forall k$$

- Periodic sinusoid:

$$x[n] = cos\left(\frac{2\pi rn}{N}\right) \quad (r \in \mathbb{Z}) \quad = \frac{1}{2}\left(W_N^{-rn} + W_N^{rn}\right)$$

$$\Rightarrow X[k] \quad = \tfrac{1}{2}\sum_{n=0}^{N-1}(W_N^{-rn} + W_N^{rn})W_N^{kn}$$

$(0 \le k < N)$

$$= \begin{cases} \frac{N}{2} & k = r, k = N - r \\ 0 & \text{otherwise} \end{cases}$$

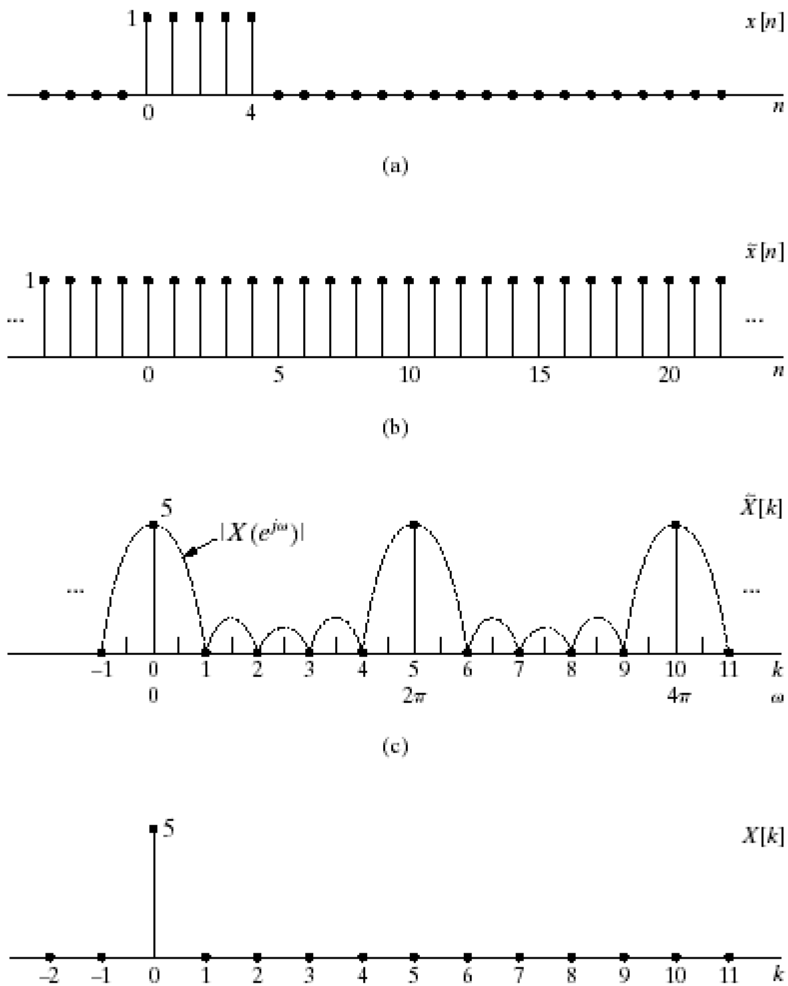
Example

- The DFT of a rectangular pulse
- x[n] is of length 5
- We can consider x[n] of any length greater than 5
- Let's pick N=5
- Calculate the DFS of the periodic form of x[n]

$$\tilde{X}[k] = \sum_{n=0}^{4} e^{-j(2\pi k/5)n}$$

$$= \frac{1-e^{-j2\pi k}}{1-e^{-j(2\pi k/5)}}$$

$$= \begin{cases} 5 & k = 0, \pm 5, \pm 10, \ldots \\ 0 & else \end{cases}$$



Discrete Fourier Transform

✓ How we can obtain other samples of DTFT?

- We should increase N. This we can achieve by treating x[n] as an N point sequence by appending zeros.

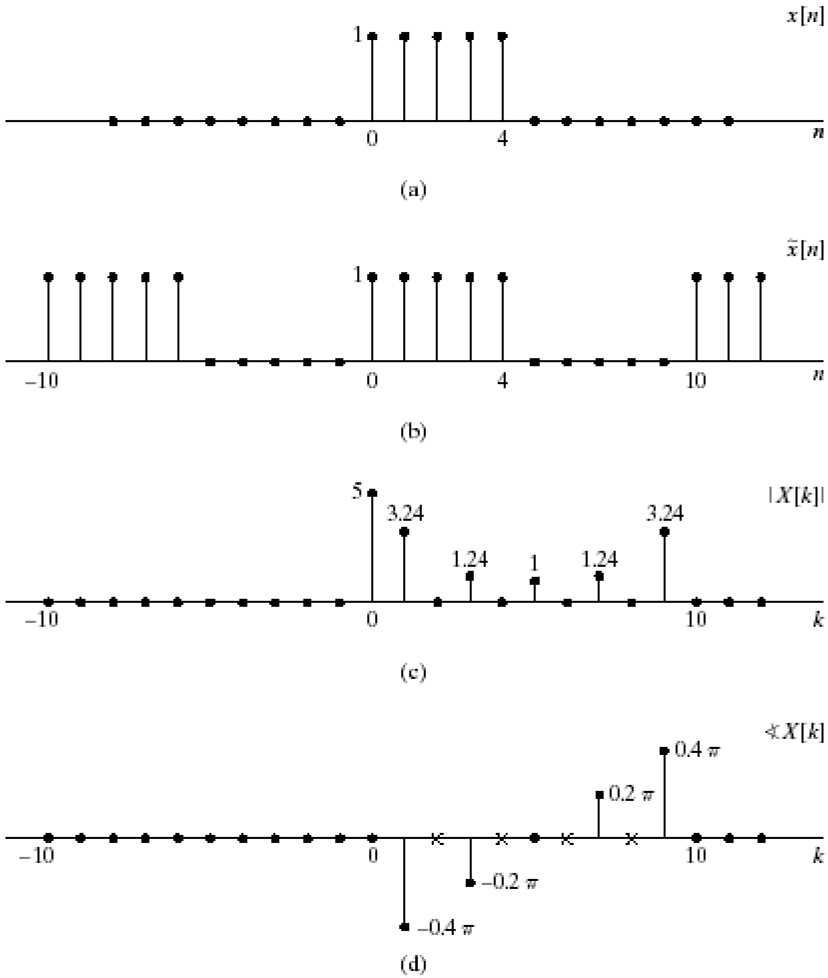
- This is a very important operation called zero padding operation.

- This operation is necessary in practice to obtain a dense spectrum of signals.

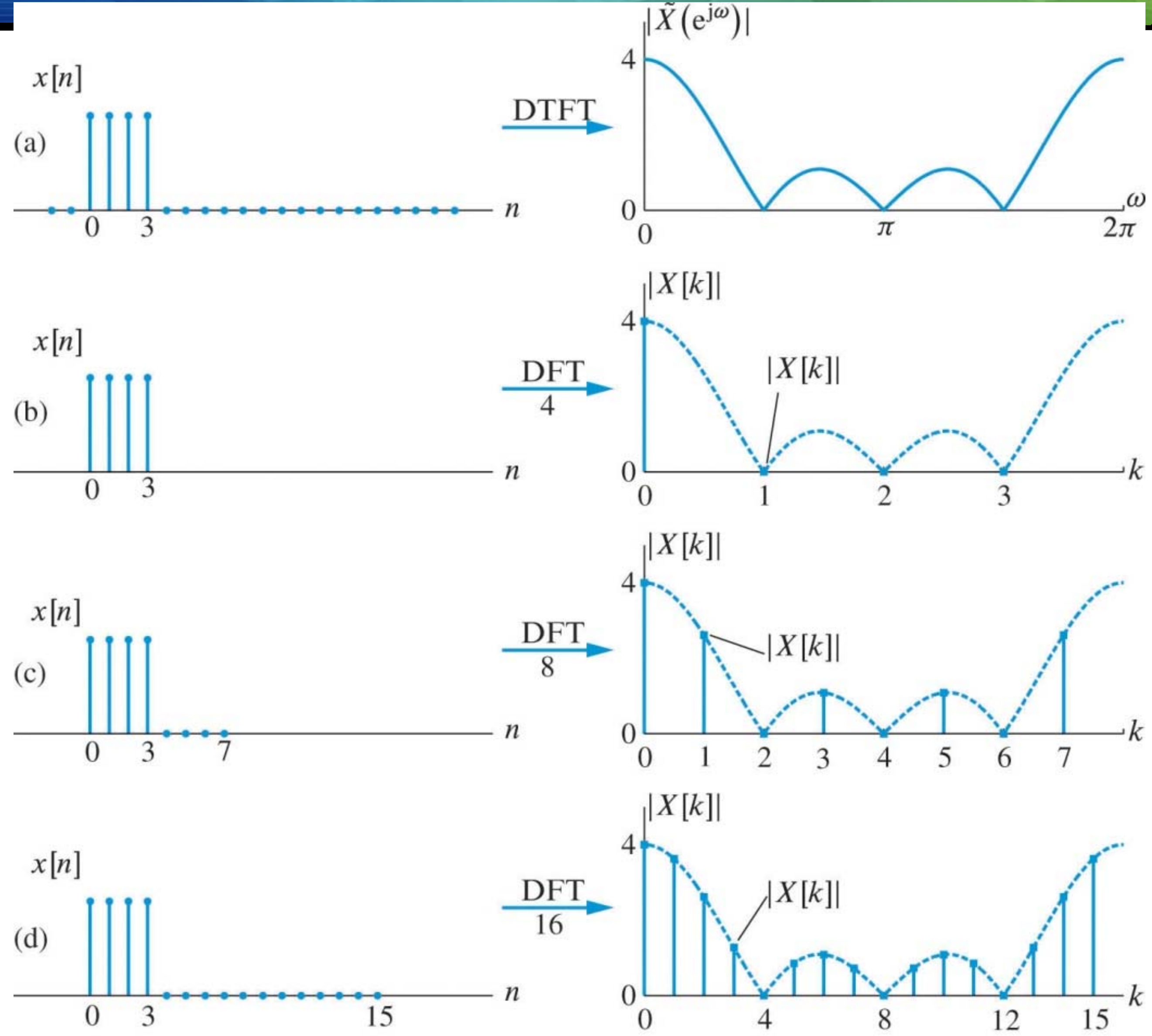
- Zero padding is an operation in which more zeros are appended to the original sequence. The resulting longer DFT provides closely spaced samples of the DTFT of the original sequence (i.e., a high density spectrum).

Example-Cont

- If we consider x[n] of length 10
- We get a different set of DFT coefficients
- Still samples of the DTFT but in different places



Discrete Fourier Transform



Time-domain aliasing

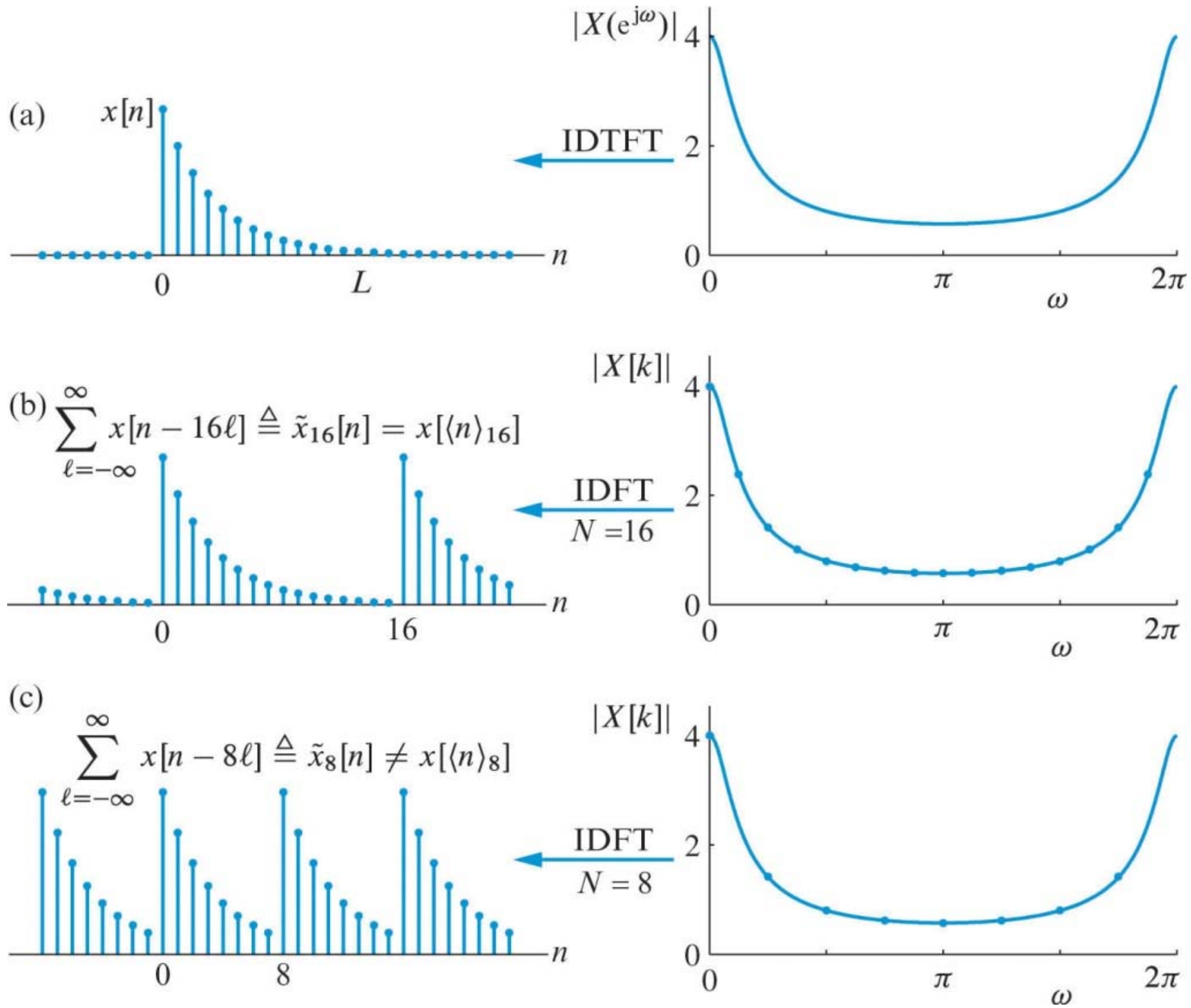


Illustration of Time-Domain Aliasing

- A causal exponential sequence and its Fourier transform are given by

$$x[n] = a^n u[n], \quad 0 < a < 1 \xleftrightarrow{\text{DTFT}} \tilde{X}\left(e^{j\omega}\right) = \frac{1}{1 - a e^{-j\omega}}$$

- Sampling the DTFT creates periodicity in the time domain

$$\tilde{x}[n] = \sum_{\ell=-\infty}^{\infty} x[n - \ell N] \xleftrightarrow[N]{\text{DFS}} \tilde{X}(e^{j\frac{2\pi}{N}k})$$

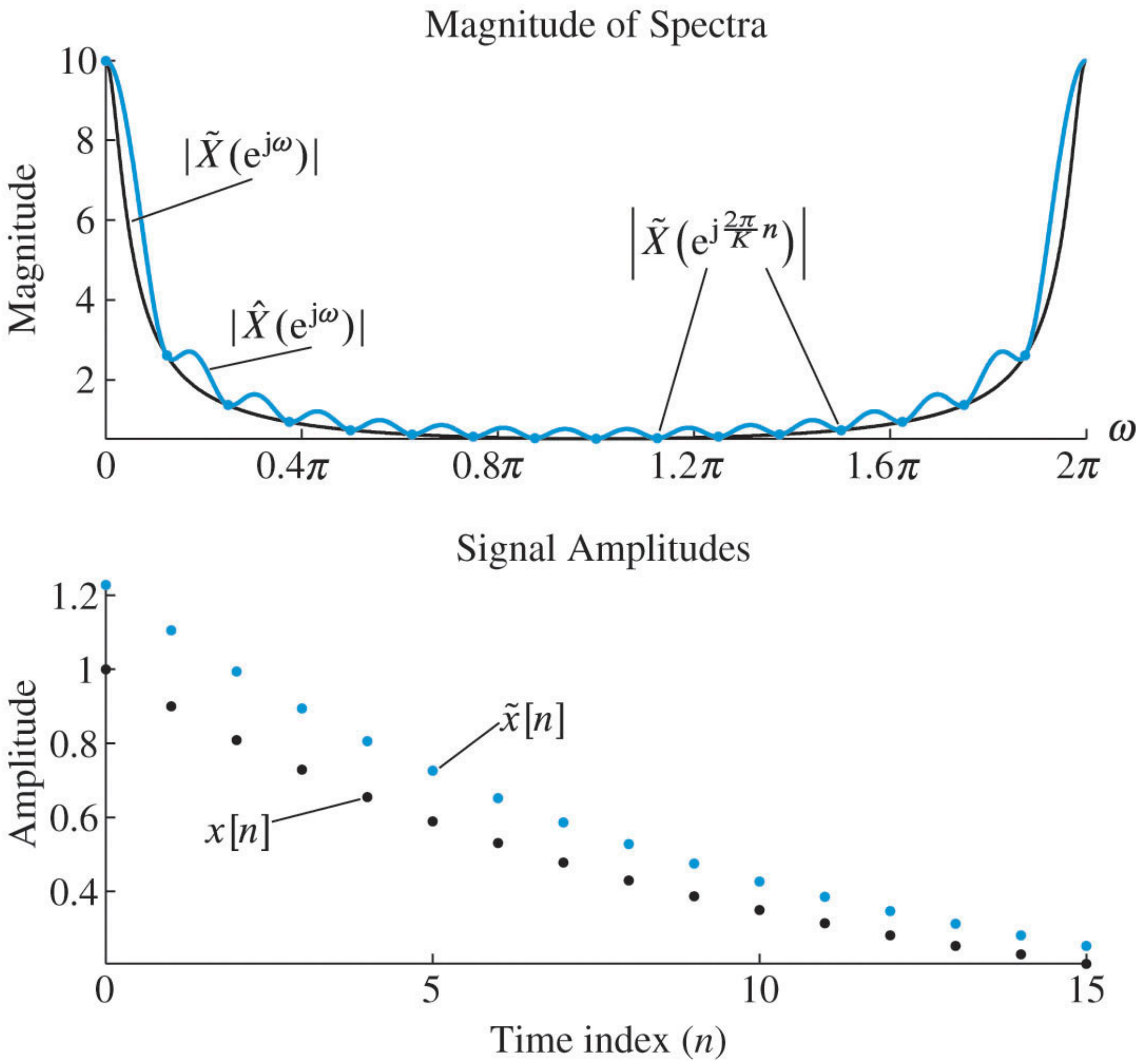
- Since we know $x[n]$ we can compute $\tilde{x}[n]$ directly

$$\tilde{x}[n] = \sum_{\ell=-\infty}^{0} a^{n-\ell N} = a^n \sum_{\ell=0}^{\infty} (a^N)^\ell = \frac{a^n}{1 - a^N} = \frac{x[n]}{1 - a^N}, \quad 0 \le n \le N-1$$

- The reconstructed DTFT is given by

$$\hat{\tilde{X}}(e^{j\omega}) = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\omega n} = \frac{1}{1 - a^N} \frac{1 - a^N e^{-j\omega N}}{1 - a e^{-j\omega}} \xrightarrow[N \to \infty]{} \tilde{X}(e^{j\omega})$$

Illustration of Time-Domain Aliasing



DFT: Matrix form

$X[k]=\sum_{n=0}^{N-1} x[n]\cdot W_N^{kn}$ as a matrix multiply:

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$$\mathbf{X}=\mathbf{D}_N\cdot\mathbf{x}$$

IDFT: Matrix form

If $\mathbf{X} = \mathbf{D}_N \cdot \mathbf{x}$

then $\mathbf{x} = \mathbf{D}_N^{-1} \cdot \mathbf{X}$

i.e. inverse DFT is also just a matrix,

$$\mathbf{D}_N^{-1} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix}$$

$= {}^{1}/_{N} D_N^{*}$

Example

- Compute the 4-point DFT of $x[n] = \{0, 1, 2, 3\}$
- We first compute the entries of the DFT matrix using its periodicity

$$\boldsymbol{W}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathrm{j} & -1 & \mathrm{j} \\ 1 & -1 & 1 & -1 \\ 1 & \mathrm{j} & -1 & -\mathrm{j} \end{bmatrix}$$

- The DFT coefficients are evaluated by the matrix-by-vector multiplication

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathrm{j} & -1 & \mathrm{j} \\ 1 & -1 & 1 & -1 \\ 1 & \mathrm{j} & -1 & -\mathrm{j} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 + \mathrm{j}2 \\ -2 \\ -2 - \mathrm{j}2 \end{bmatrix}$$

- MATLAB code: `x = [0 1 2 3]'; W = dftmtx(4); X = W*x;`
- Efficient MATLAB functions: `X=fft(x)` and `x=ifft(X)` (see FFT lectures)

Properties of Discrete Fourier Transform

- Linearity

$$x_1[n] \xleftrightarrow{DFT} X_1[k]$$

$$x_2[n] \xleftrightarrow{DFT} X_2[k]$$

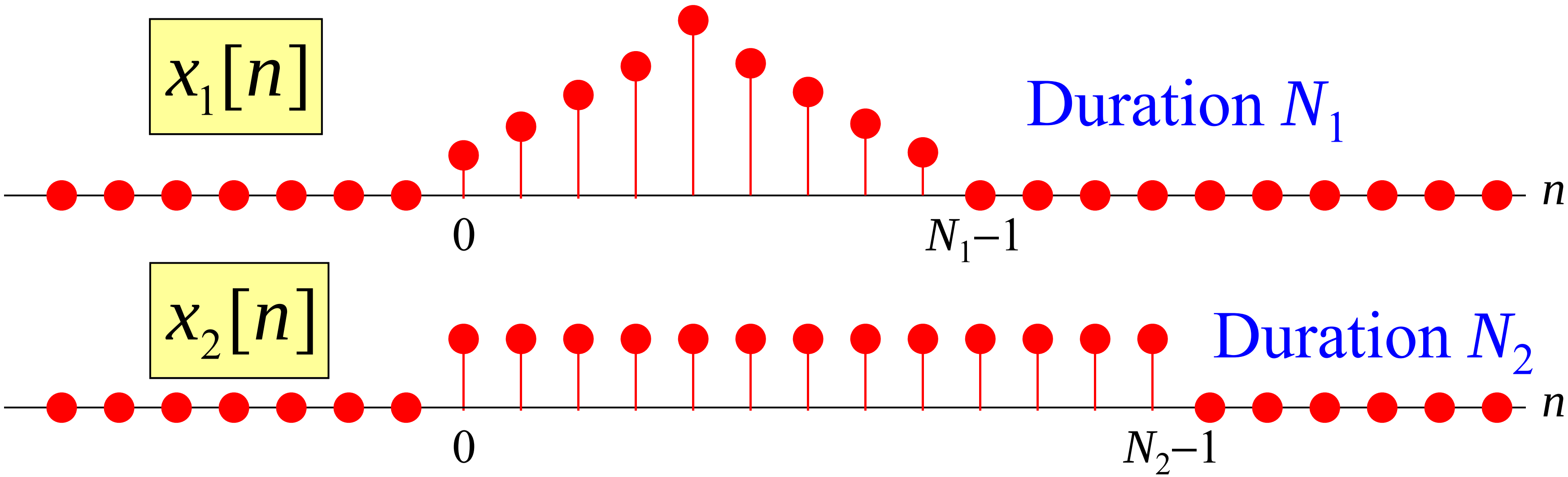
$$x_3[n] = ax_1[n] + bx_2[n] \xleftrightarrow{DFT} X_3[k] = aX_1[k] + bX_2[k]$$

✓If $x_1[n]$ and $x_2[n]$ have different durations-that is, they are N_1 point and N_2 point sequences, respectively-then the duration N_3 of $x_3[n]$ should be greater or equal to $\max(N_1, N_2)$.

✓Then $N_3 \geq \max(N_1, N_2)$ point DFTs should be estimated both for $x_1[n]$ and $x_2[n]$.

✓If the duration (i.e., length) of any of the signals is smaller than N_3, zero padding should be applied.

Properties of Discrete Fourier Transform: Linearity

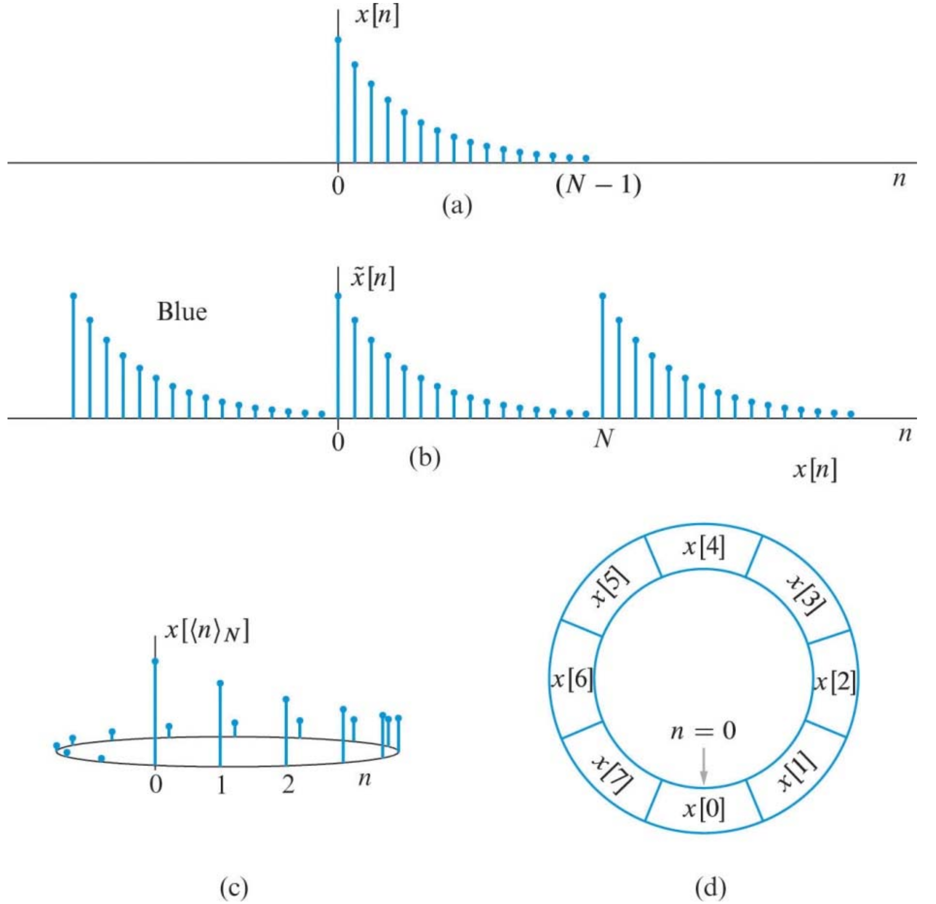


$$x_1[n] \xleftrightarrow{\mathcal{DFT}} X_1[k]$$

$$x_2[n] \xleftrightarrow{\mathcal{DFT}} X_2[k]$$

$$ax_1[n]+bx_2[n] \xleftrightarrow{\mathcal{DFT}} aX_1[k]+bX_2[k]$$

Periodic, circular and modulo-N operations

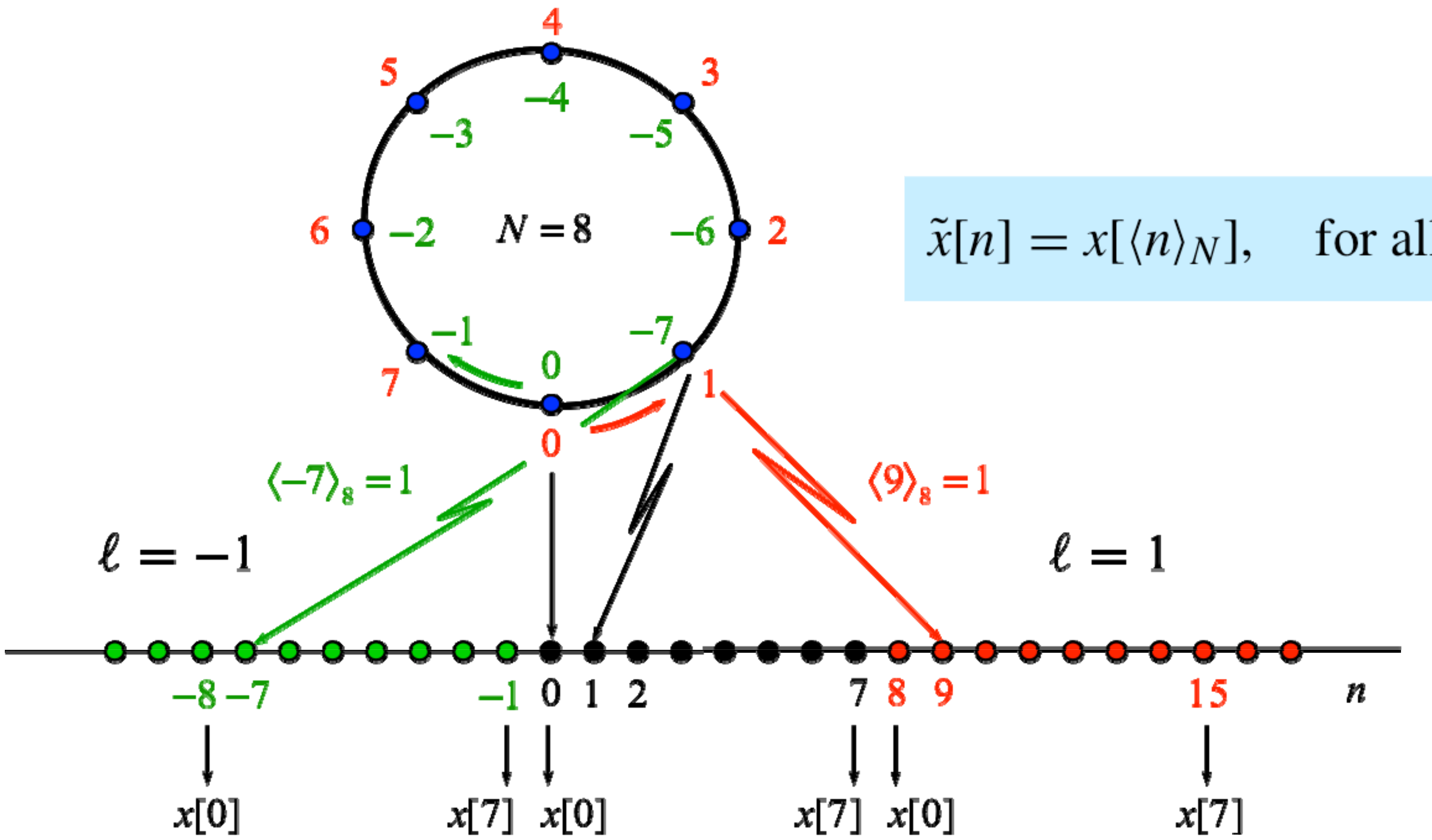


a) finite length sequence x[n] b) periodic extension sequence formed by replicated x[n] c) wrapping the sequence around a cylinder with circumference N and using modulo-N d) representation of a circular buffer with modulo-N indexing.

The n Modulo N Operation

$$n = \ell N + r,\ 0 \le r \le N-1 \Rightarrow \langle n \rangle_N \triangleq n \text{ modulo } N = r$$

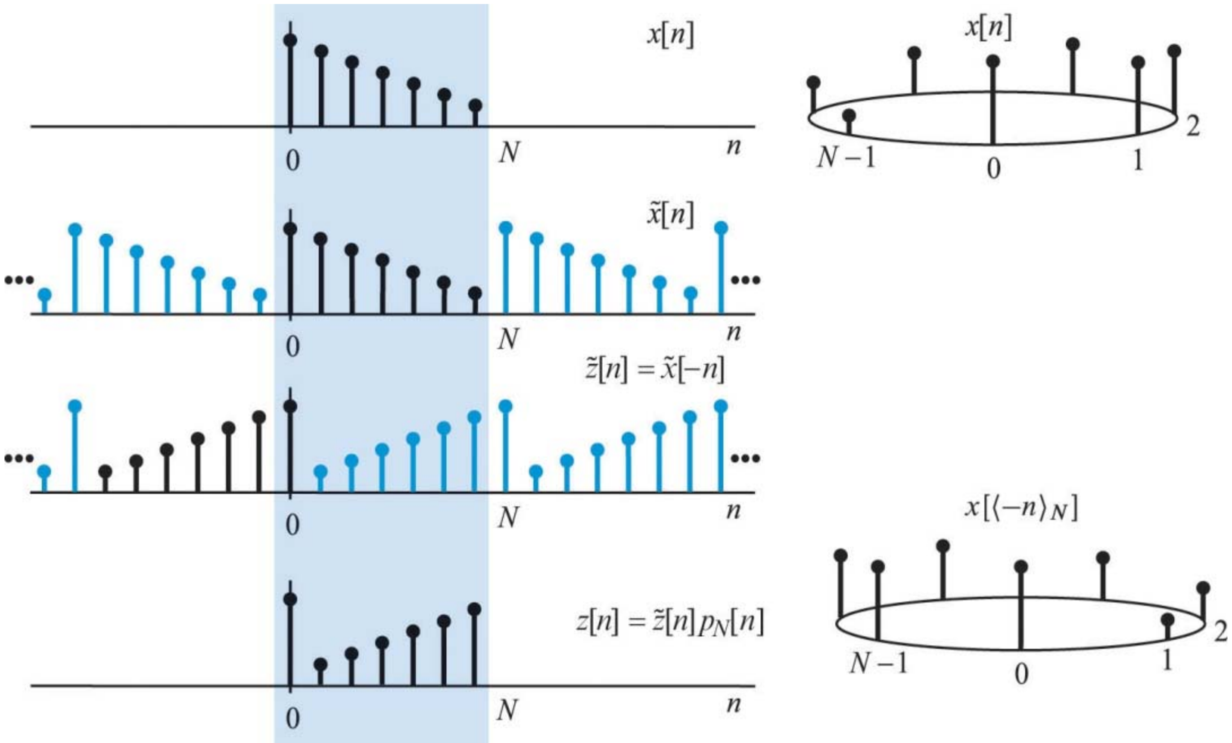
MATLAB function: `m = mod(n,N)`



$$\tilde{x}[n] = x[\langle n \rangle_N], \quad \text{for all } n$$

Circular folding (or reversal)

$$z[n] = x\left[\langle -n \rangle_N\right] = \begin{cases} x[0] & n=0 \\ x[N-n] & 1 \le n \le N-1 \end{cases}$$



Circular folding (or reversal)

$$x[n] \xleftrightarrow{DFT} X[k]$$

$$\Downarrow$$

$$x[\langle -n \rangle_N] \xleftrightarrow{DFT} X[\langle -k \rangle_N]$$

$$X\left[\langle -k \rangle_N\right] = \begin{cases} X[0] & \text{n=0} \\ X[N-k] & 1 \leq k \leq N\text{ - }1 \end{cases}$$

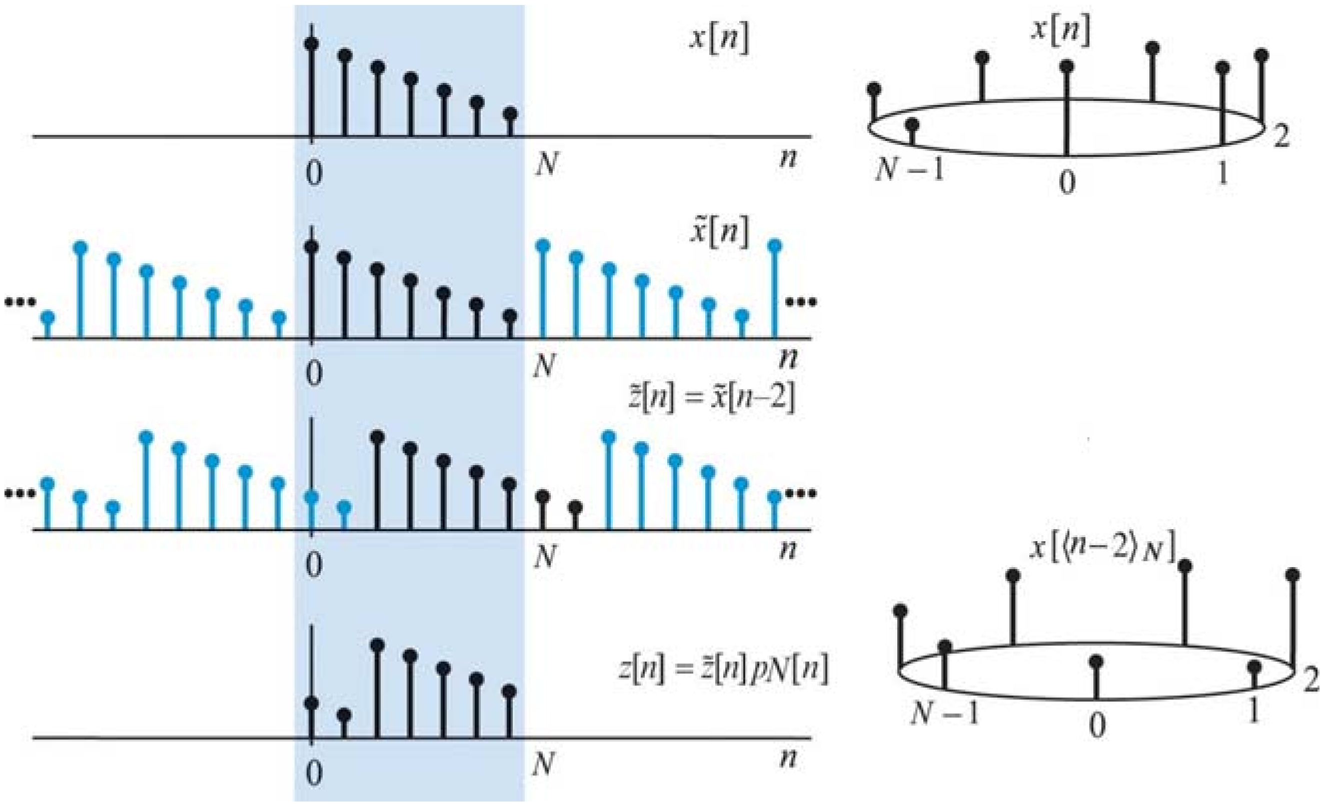
Circular Shift of a Sequence

Circular Time Shifting of a Sequence

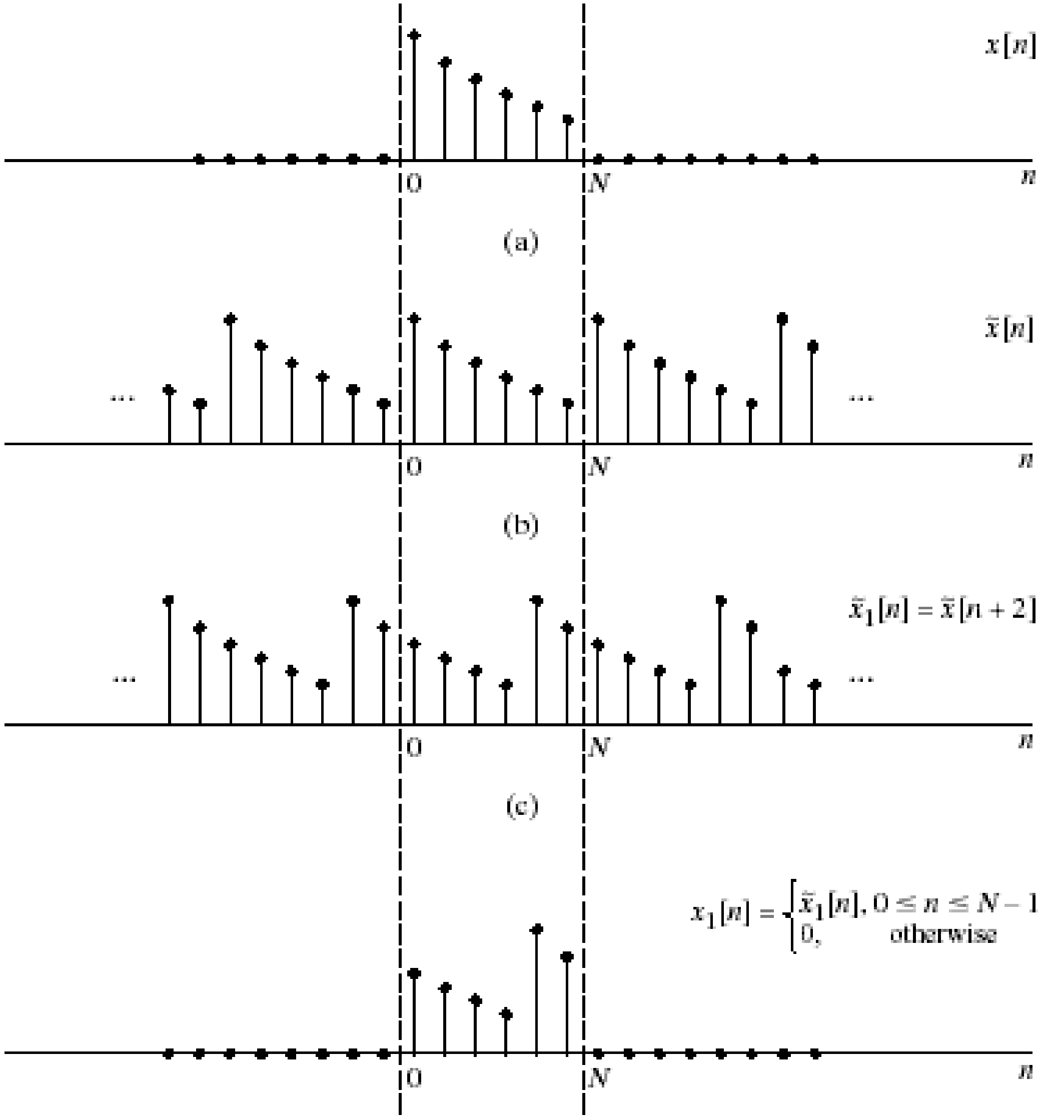
$$x[n] \xleftrightarrow{DFT} X[k]$$

$$x\left[\langle n-m\rangle_N\right] \quad 0 \le n \le N-1 \xleftrightarrow{DFT} X[k]e^{-j(2\pi k/N)m}$$

Circular Shift of a Sequence

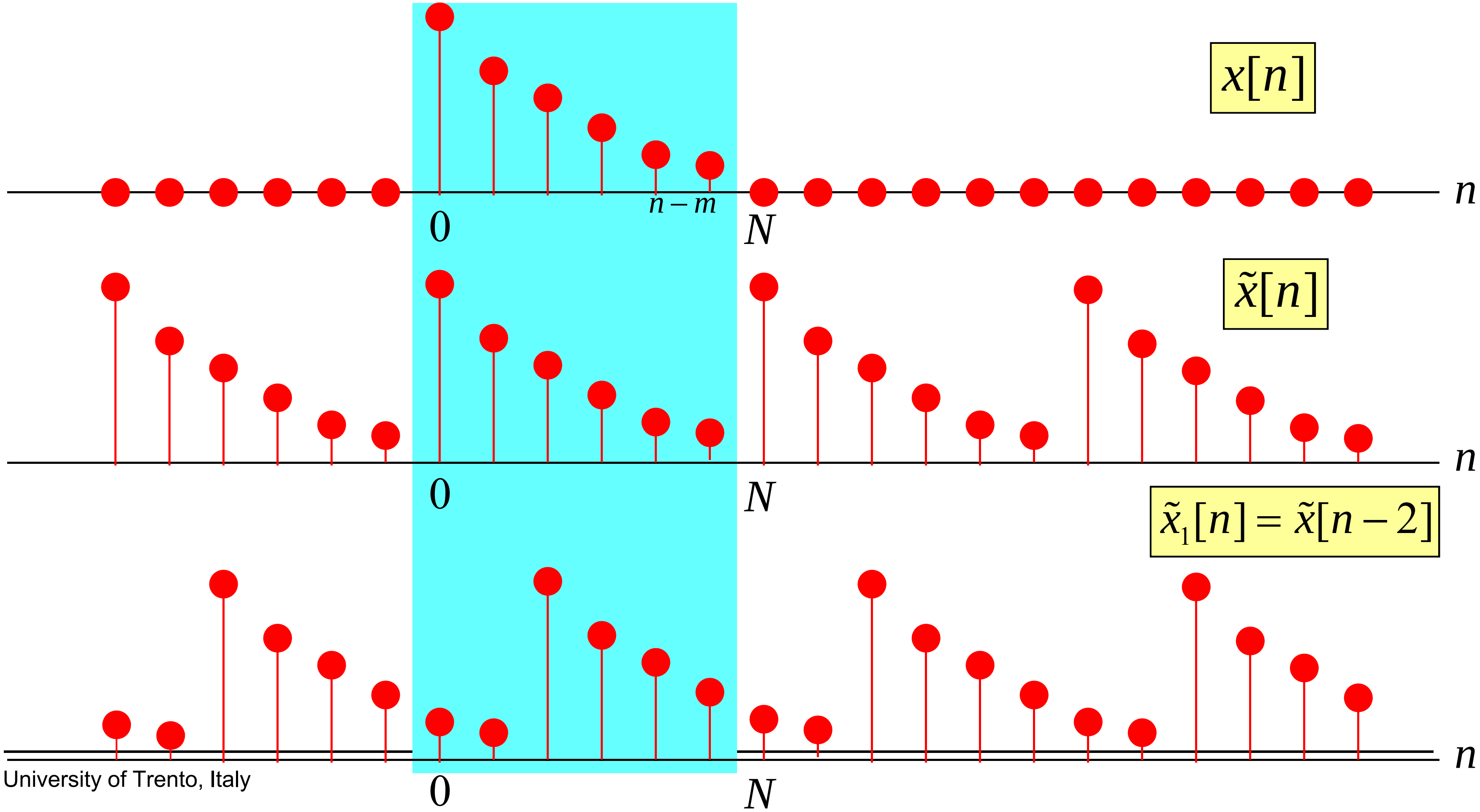


Properties of Discrete Fourier Transform-Shifting

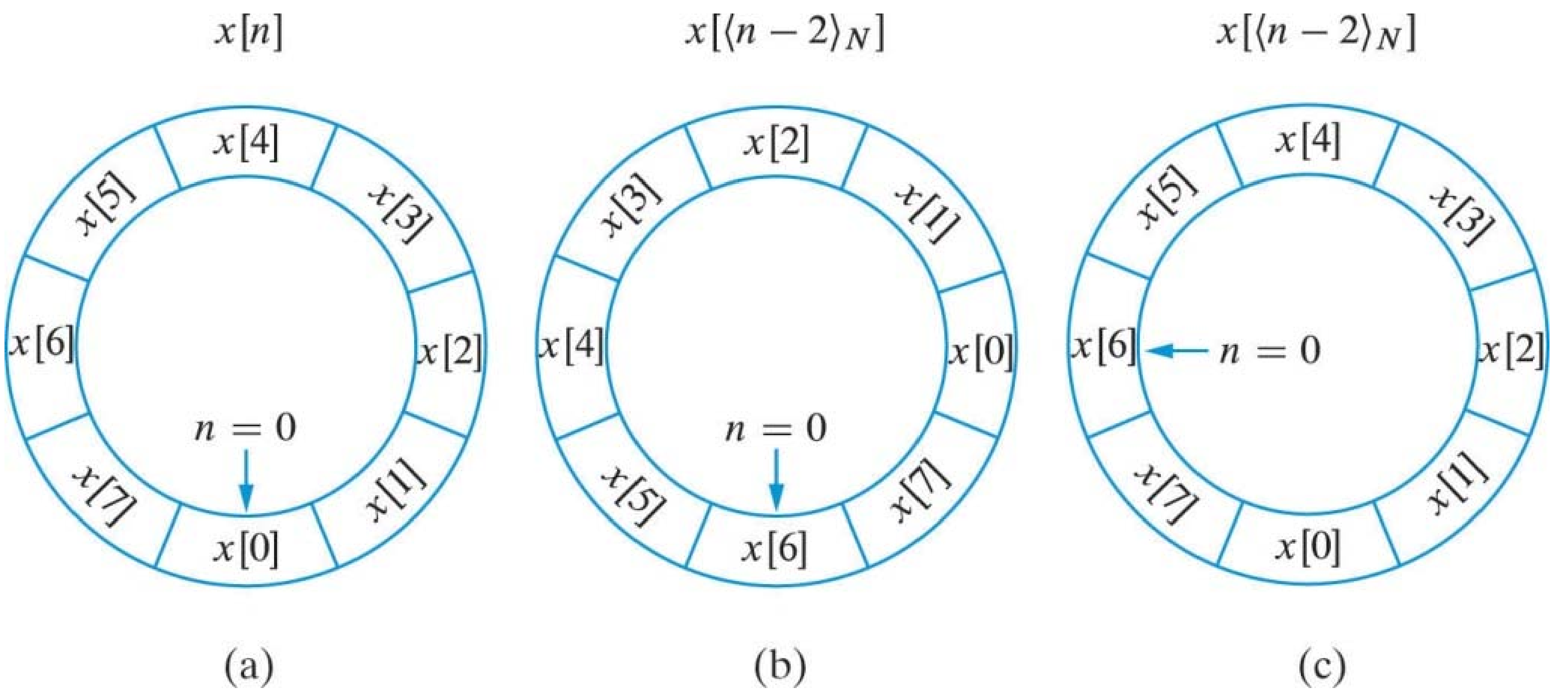


Circular Time Shifting of a Sequence

$$x_1[n] = \begin{cases} \tilde{x}_1[n] = x[\langle n-m \rangle_N] & 0 \le n \le N-1 \\ 0 & otherwise \end{cases}$$



Circular Shift of a Sequence



a) sequence x[n] ; b) sequence x[< $n-2$ >$_N$] ;
c) sequence x[< $n-2$ >$_N$] using circular addressing.

Circular Frequency Shifting Teorem:

Circular Frequency Shifting of a Sequence

$$x[n] \xleftrightarrow{DFT} X[k]$$

$$x[n]e^{j(2\pi k_0/N)n} \xleftrightarrow{DFT} X\left[\langle k-k_0\rangle_N\right] 0 \leq k \leq N-1$$

Circular Convolution

$$x_1[n] \xleftrightarrow{\mathcal{DFT}} X_1[k] \qquad x_2[n] \xleftrightarrow{\mathcal{DFT}} X_2[k]$$

both of length N

$$x_3[n] = \sum_{m=0}^{N-1} x_1[m] x_2[\langle n-m \rangle_N]$$

$$x_3[n] = \sum_{m=0}^{N-1} x_1[m] x_2[\langle n-m \rangle_N] \xleftrightarrow{\mathcal{DFT}} X_3[k] = X_1[k] X_2[k]$$

Multiplication of Two N-Point DFTs: Circular Convolution

$$X[k] = \sum_{m=0}^{N-1} x[m] W_N^{km}$$

$$H[k] = \sum_{\ell=0}^{N-1} h[\ell] W_N^{k\ell}$$

$$Y[k] = H[k]X[k] \xrightarrow[N]{\text{IDFT}} y[n] = ?$$

N-point sequence

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} H[k]X[k]W_N^{-kn} = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{l=0}^{N-1} h[\ell]W_N^{k\ell}\sum_{m=0}^{N-1} x[m]W_N^{km}\right)W_N^{-kn}$$

$$y[n] = \sum_{\ell=0}^{N-1} h[\ell]\sum_{m=0}^{N-1} x[m]\underbrace{\left(\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-\ell-m)}\right)}_{\substack{1 \text{ if } n-\ell-m=rN \\ 0 \text{ if } n-\ell-m\neq rN}}$$

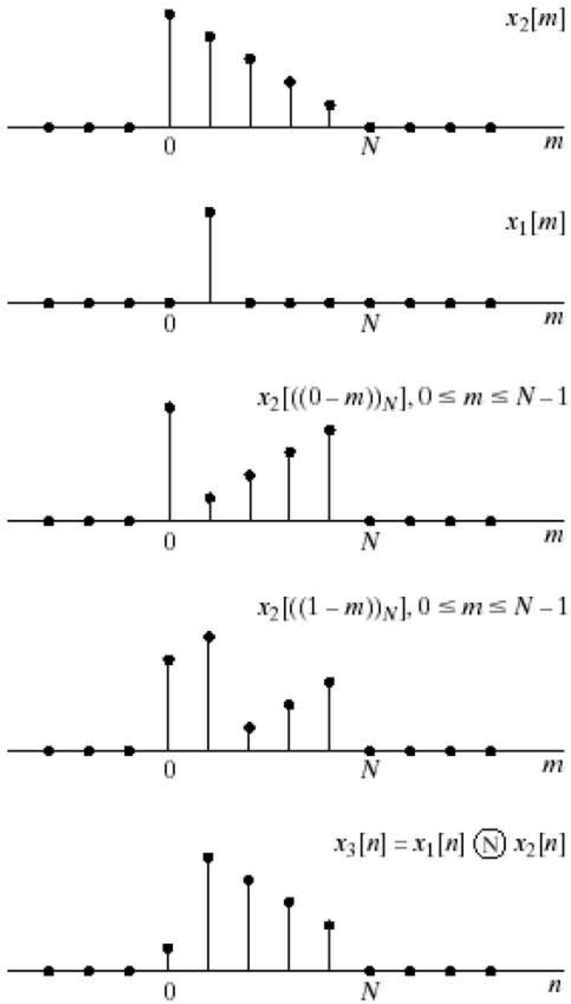
$$n-\ell-m = rN \Rightarrow m = n-\ell-rN$$
$$\Rightarrow m = (n-\ell) \text{ modulo } N = \langle n-\ell\rangle_N$$

$$y[n] = \sum_{\ell=0}^{N-1} h[\ell]x\left[(n-\ell) \text{ modulo } N\right] = \sum_{\ell=0}^{N-1} h[\ell]x\left[\langle n-\ell\rangle_N\right]$$

Circular Convolution

$$x_3[n] = \sum_{m=0}^{N-1} x_1[m]\, x_2\left[((n-m))_N\right]$$

$$x_3[n] = \sum_{m=0}^{N-1} x_2[m]\, x_1\left[((n-m))_N\right]$$



Circular Convolution-Example

$$x_1[n] = x_2[n] = \begin{cases} 1 & 0 \le n \le L-1 \\ 0 & else \end{cases}$$

- DFT of each sequence

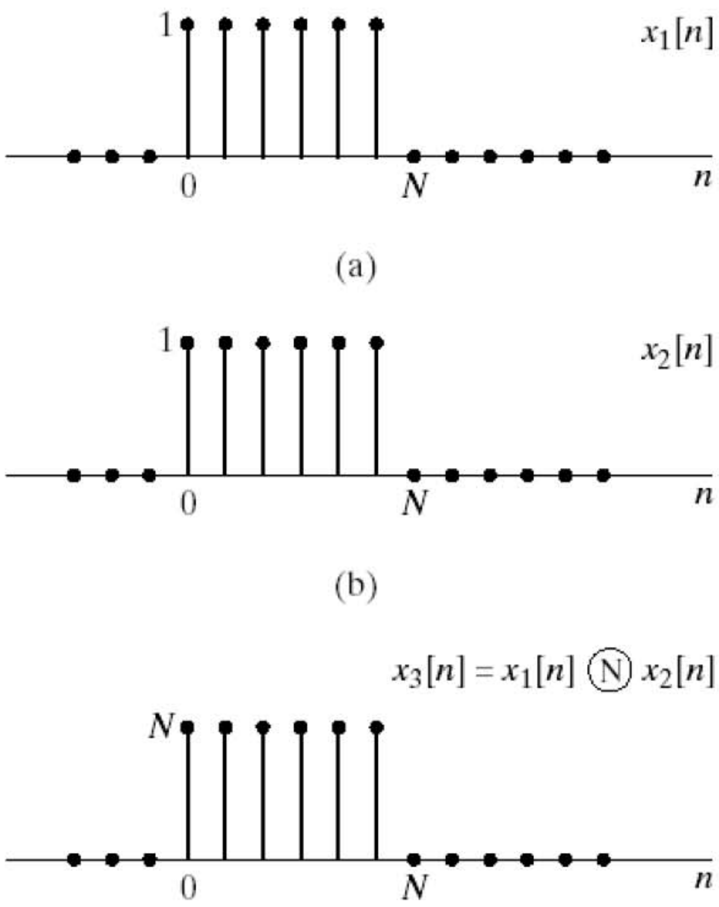
$$X_1[k] = X_2[k] = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn} = \begin{cases} N & k = 0 \\ 0 & else \end{cases}$$

- Multiplication of DFTs

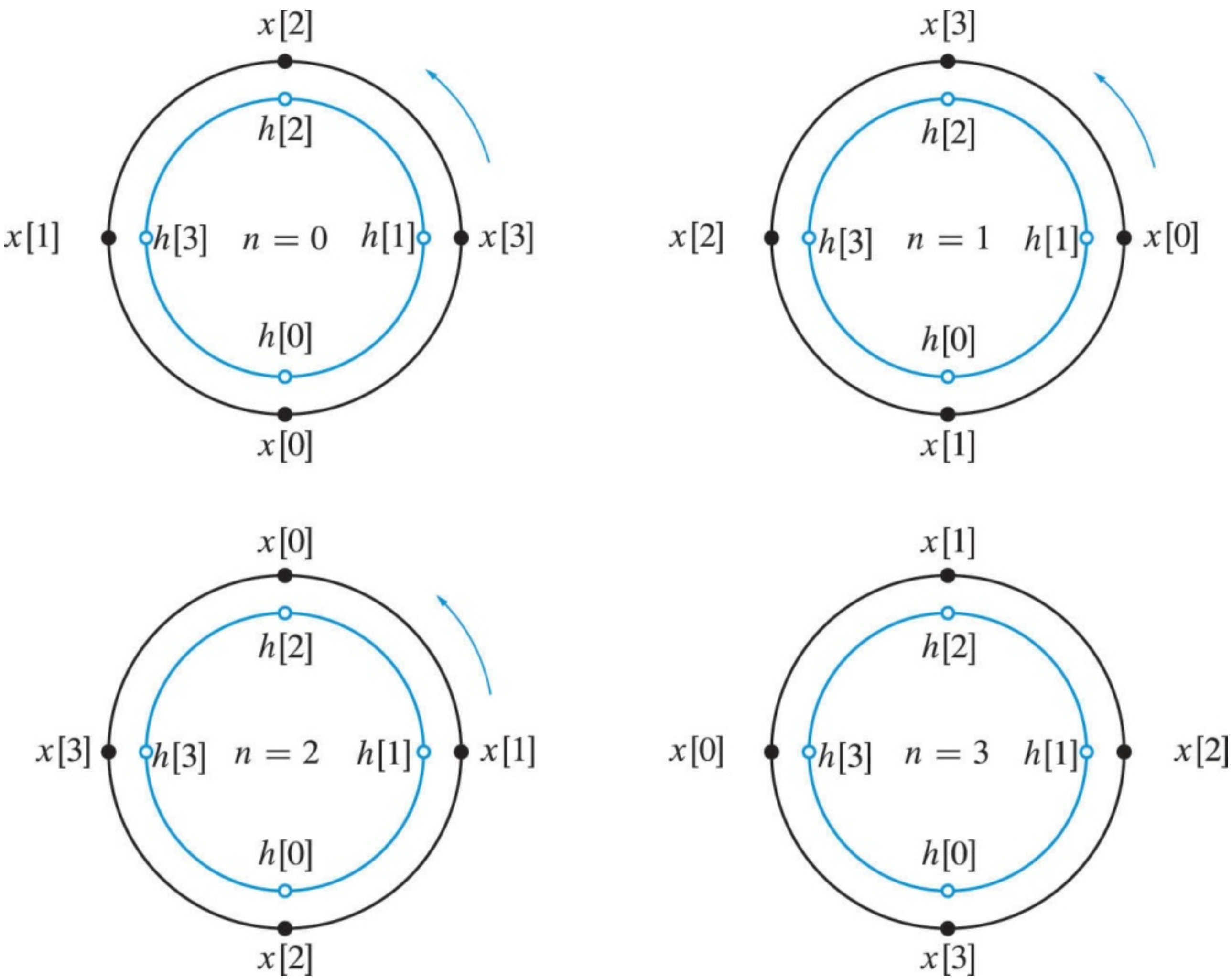
$$X_3[k] = X_1[k]X_2[k] = \begin{cases} N^2 & k = 0 \\ 0 & else \end{cases}$$

- the inverse DFT

$$x_3[n] = \begin{cases} N & 0 \le n \le N-1 \\ 0 & else \end{cases}$$



Circular Convolution-Graphical Interpretation



Circular Convolution-Matrix Form

Let's compute $y[n]$ for the case of $N = 4$. We have

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix} = \underbrace{\begin{bmatrix} x[0] & x[3] & x[2] & x[1] \\ x[1] & x[0] & x[3] & x[2] \\ x[2] & x[1] & x[0] & x[3] \\ x[3] & x[2] & x[1] & x[0] \end{bmatrix}}_{\mathbf{X}_N} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \end{bmatrix}$$

We note that the column of $\mathbf{X}_N$ are generated by circularly shifting $x[n]$. A matrix generated by this process is called a *circulant matrix*.

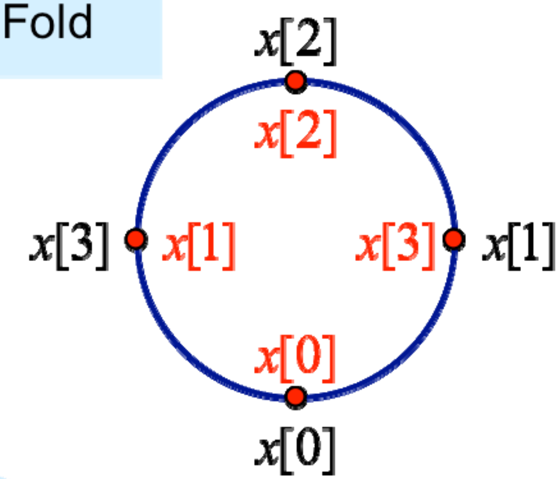
Circular Convolution-Matrix Form

$$y[n] = \sum_{m=0}^{N-1} h[m]\, x\left[\langle n-m \rangle_N\right]$$

For $N = 4$, using $x[-n] = x[N-n] \Rightarrow$

Circular Fold

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix} = \underbrace{\begin{bmatrix} x[0] & x[3] & x[2] & x[1] \\ x[1] & x[0] & x[3] & x[2] \\ x[2] & x[1] & x[0] & x[3] \\ x[3] & x[2] & x[1] & x[0] \end{bmatrix}}_{\text{Circulant Matrix}} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \end{bmatrix}$$



Circular Shift

A circulant matrix is Toeplitz

A Toeplitz matrix is not circulant!

$$y[n] = \sum_{m=0}^{N-1} h[m]\, x\left[\langle n-m \rangle_N\right] \triangleq h[n] \text{ Ⓝ } x[n] = x[n] \text{ Ⓝ } h[n] \xleftrightarrow[N]{\text{DFT}} Y[k] = H[k]X[k]$$

Example

$$x[n] = \{0 \quad 1 \quad 2 \quad 3\}, \quad \{[1 \quad 0 \quad 1 \quad 1]\} \quad \Rightarrow \quad y[n] = x[n] \, N \, h[n] = ?$$

Time Domain

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{h} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix} = \begin{bmatrix} 0 & 3 & 2 & 1 \\ 1 & 0 & 3 & 2 \\ 2 & 1 & 0 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 5 \\ 4 \end{bmatrix}$$

Using DFT

$$X[k] = x[0] + x[1]e^{-j\frac{2\pi}{4}k} + x[2]e^{-j\frac{2\pi}{4}2k} + x[3]e^{-j\frac{2\pi}{4}3k}, \quad k = 0,1,2,3$$

$$X[k] = \left\{ 6 \quad -2+j2 \quad -2 \quad -2-j2 \right\}$$

$$H[k] = \left\{ 3 \quad j \quad 1 \quad -j \right\}$$

$$Y[k] = X[k]H[k] = \left\{ 18 \quad -2-j2 \quad -2 \quad -2+j2 \right\}$$

$$y[n] = \{3 \quad 6 \quad 5 \quad 4\}$$

Linear Convolution (Review)

$$\boxed{x_3[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m] = x_1[n] * x_2[n]}$$

$$\boxed{x_1[n] \xleftrightarrow{DTFT} X_1(e^{j\omega})} \quad \boxed{x_2[n] \xleftrightarrow{DTFT} X_2(e^{j\omega})}$$

$$\Downarrow$$

$$\boxed{x_3[n] = x_1[n] * x_2[n] \xleftrightarrow{DTFT} X_3(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega})}$$

Linear Convolution Using DFT

- ✓ Let $x_1[n]$ be an N_1 point sequence and let $x_2[n]$ be an N_2 point sequence. Define the linear convolution of $x_1[n]$ and $x_2[n]$ by $x_3[n]$, that is:

$$\boxed{\begin{aligned} x_3[n] &= x_1[n] * x_2[n] \\ &= \sum_{m=-\infty}^{\infty} x_1[m] x_2[n-m] \end{aligned}}$$

- ✓ Then $x_3[n]$ is a N_1+N_2-1 point sequence.
- ✓ If we choose $N=\max(N_1, N_2)$ and compute an N point circular convolution we get an N point sequence which is obviously different from $x_3[n]$.

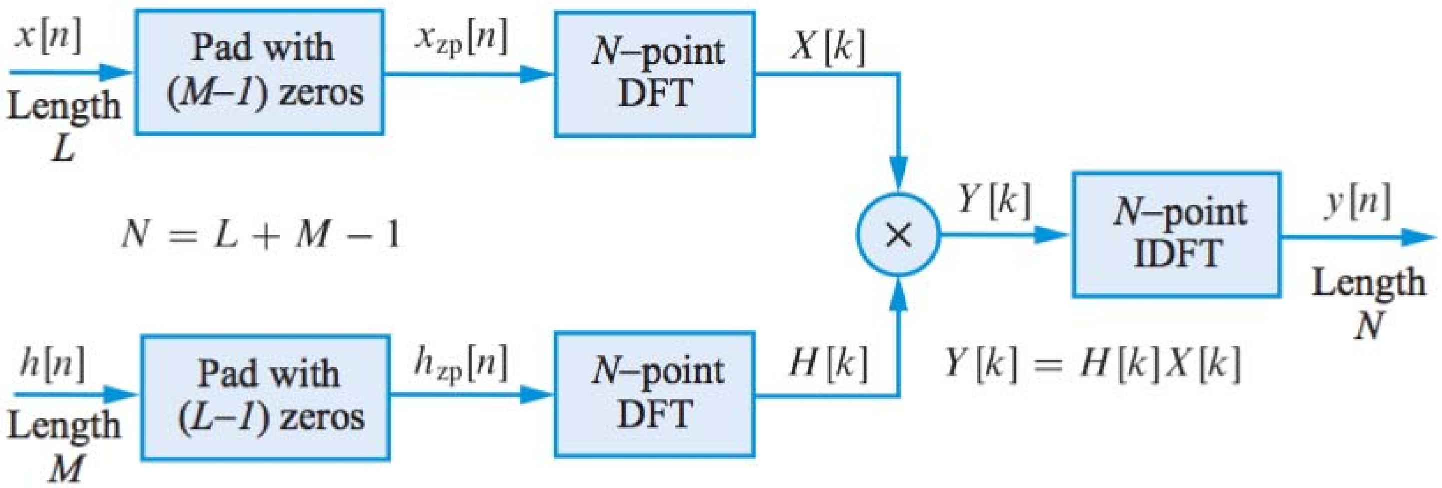
Linear Convolution Using DFT

- ✓ To obtain the same results with linear convolution, we should make both $x_1[n]$ and $x_2[n]$ N_1+N_2-1 point sequences by padding an appropriate number of zeros.

- ✓ By this way the circular convolution is identical to the linear convolution.

Linear Convolution Using DFT

linear convolution can be implemented by means of the DFT as shown below.



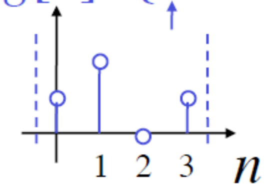
The length M of the impulse response at which the DFT based approach is more efficient than direct computation of convolution depends on the hardware and software available to implement the computations.

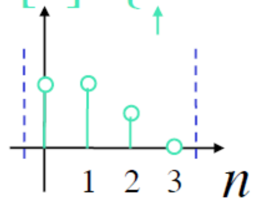
Example

- 4 pt sequences:

$$\sum_{m=0}^{N-1} g[m]h[\langle n-m\rangle_N]$$

$g[n]=\{1\ 2\ 0\ 1\}$ $h[n]=\{2\ 2\ 1\ 0\}$



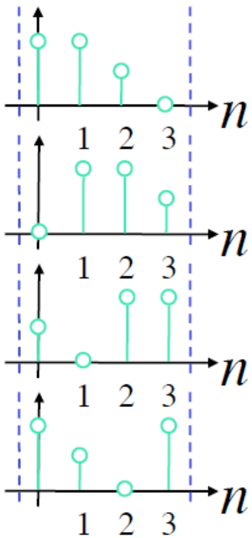


$h[<n-0>_4]$ $\cdot\ 1$

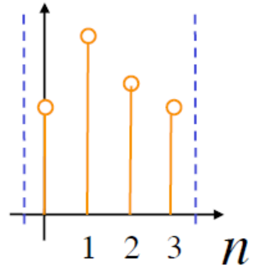
$h[<n-1>_4]$ $\cdot\ 2$

$h[<n-2>_4]$ $\cdot\ 0$

$h[<n-3>_4]$ $\cdot\ 1$



$g[n]\,④\,h[n]=\{4\ 7\ 5\ 4\}$



check: $g[n] \circledast h[n]$
$=\{2\ 6\ 5\ 4\ 2\ 1\ 0\}$

Windowing Property:

$$x_1[n] \xleftrightarrow{\mathcal{DFT}} X_1[k] \qquad x_2[n] \xleftrightarrow{\mathcal{DFT}} X_2[k]$$

both of length N

$$x_3[n] = x_1[n]x_2[n]$$

$$x_3[n] = x_1[n]x_2[n] \xleftrightarrow{\mathcal{DFT}} X_3[k] = \frac{1}{N}\sum_{l=0}^{N-1} X_1[l]X_2[\langle k-l \rangle_N]$$

Duality Property:

$$x[n] \xleftrightarrow{DFT} X[k]$$

$$X[n] \xleftrightarrow{DFT} Nx[\langle -k \rangle_N], \quad 0 \le k \le N-1$$

Properties of DFT

Proof for the Duality:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)kn}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

$$Nx[-n] = \sum_{k=0}^{N-1} X[k] e^{-j(2\pi/N)kn}$$

$$Nx[-k] = \sum_{n=0}^{N-1} X[n] e^{-j(2\pi/N)kn}$$

$$x[n] \xleftrightarrow{DFT} X[k]$$

$$X[n] \xleftrightarrow{DFT} Nx[-k]$$

Parseval's Relation (Related to DFT):

- ✓ This theorem expresses the sum of sample by sample product of two complex sequences in terms of sum of the product of their discrete Fourier transforms. Specifically, the most general form of this theorem is:

$$\sum_{n=0}^{N-1} g[n]h^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} G[k]H^*[k]$$

- ✓ The total energy of a length N sequence g[n] can be computed by summing the square of the absolute values of the DFT samples G[k] divided by N:

$$E = \sum_{n=0}^{N-1} \left|g[n]\right|^2 = \frac{1}{N}\sum_{k=0}^{N-1} \left|G[k]\right|^2$$

Symmetry properties of the DFT

✓ x[n] is a real sequence:

An N-point sequence $x[n]$ is called *circularly even* if

$$x[n] = x[\langle -n \rangle_N]$$

An N-point sequence $x[n]$ is called *circularly odd* if

$$x[n] = -x[\langle -n \rangle_N]$$

Symmetry properties of the DFT

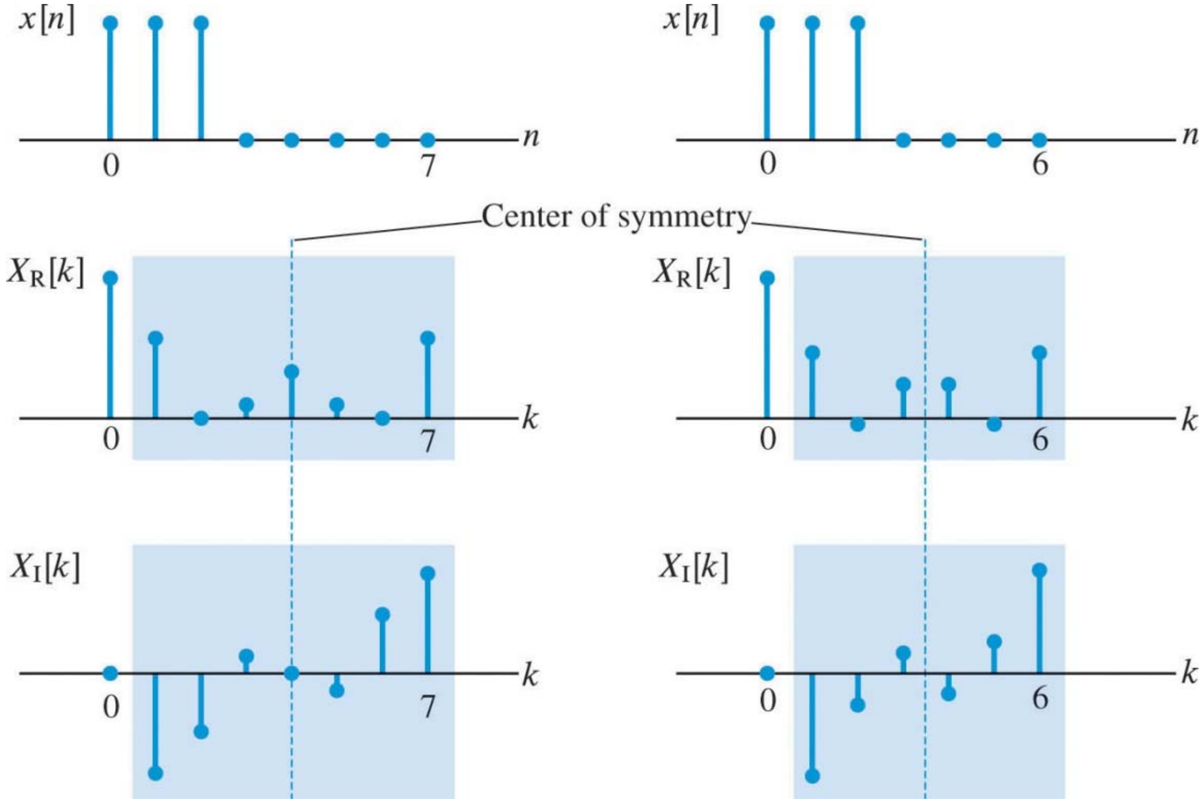
✓ x[n] is a real sequence:

$$x[n] = x^*[n] \xrightarrow{DFT} X[k] = X^*\left[\left(-k\right)_N\right]$$

Symmetry relation of the discrete Fourier transform for a real sequence

Sequence x[n]	Discrete Fourier Transform X[k]
Any real x[n]	$X[k]=X^*[(-k)_N]$
Any real x[n]	$X_{Re}[k]=X_{Re}[(-k)_N]$
Any real x[n]	$X_{Im}[k]=-X_{Im}[(-k)_N]$
Any real x[n]	$\lvert X[k]\rvert=\lvert X[(-k)_N]\rvert$
Any real x[n]	$\angle X[k]=-\angle X[(-k)_N]$
$x_e[n]$	$X_{Re}[k]$
$x_o[n]$	$jX_{Im}[k]$

Symmetry properties of the DFT



Symmetries of real and imaginary parts of the DFT of a real valued sequence with even and odd number of samples. Note that we check the values for k=1,2,…,N-1 for even and odd symmetry about the point N/2; for odd symmetry the value at k=0 should be zero.

DFT Symmetry Properties-Conjugate Symmetry

Let's consider a complex sequence x[n]:

Circular conjugate symmetric

Circular conjugate antisymmetric

$$x[n] = x_{cs}[n] + x_{ca}[n], \ 0 \le n \le N-1$$

$$x_{cs}[n] = \frac{1}{2}\left(x[n] + x^*\left[(-n)_N\right]\right), \ 0 \le n \le N-1$$

$$x_{ca}[n] = \frac{1}{2}\left(x[n] - x^*\left[(-n)_N\right]\right), \ 0 \le n \le N-1$$

x[n] is said to be circular conjugate symmetric if $x[n] = x^*\left[(-n)_N\right]$

x[n] is said to be circular conjugate antisymmetric if $x[n] = -x^*\left[(-n)_N\right]$

DFT Symmetry Properties-Conjugate Symmetry

Similarly, symmetry properties can be extended to DFT of a signal:

Circular conjugate symmetric Circular conjugate antisymmetric

$$X[k] = X_{cs}[k] + X_{ca}[k], \ 0 \le k \le N-1$$

$$X_{cs}[k] = \frac{1}{2}\left(X[k] + X^*\left[(-k)_N\right]\right), \ 0 \le k \le N-1$$

$$X_{ca}[k] = \frac{1}{2}\left(X[k] - X^*\left[(-k)_N\right]\right), \ 0 \le k \le N-1$$

X[k] is said to be circular conjugate symmetric if $X[k] = X^*\left[(-k)_N\right]$

X[k] is said to be circular conjugate antisymmetric if $X[k] = -X^*\left[(-k)_N\right]$

DFT Symmetry Properties-Conjugate Symmetry

Symmetry relation of the discrete Fourier transform for a complex sequence

Sequence $x[n]$	Discrete Fourier Transform $X[k]$
$x^*[n]$	$X^*[(-k)_N]$
$x^*[(-n)_N]$	$X^*[k]$
$Re\{x[n]\}$	$X_{cs}[k]$ (circular conjugate-symmetric part)
$jIm\{x[n]\}$	$X_{ca}[k]$ (circular conjugate-antisymmetric part)
$x_{cs}[n]$	$X_{Re}[k] = Re\{X[k]\}$
$x_{ca}[n]$	$jX_{Im}[k] = jIm\{X[k]\}$