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Digital Signal Processing
Lecture 10

Quote of the Day

When you look at yourself from a universal standpoint, something inside always reminds or informs you that there are bigger and better things to worry about.

- Albert Einstein

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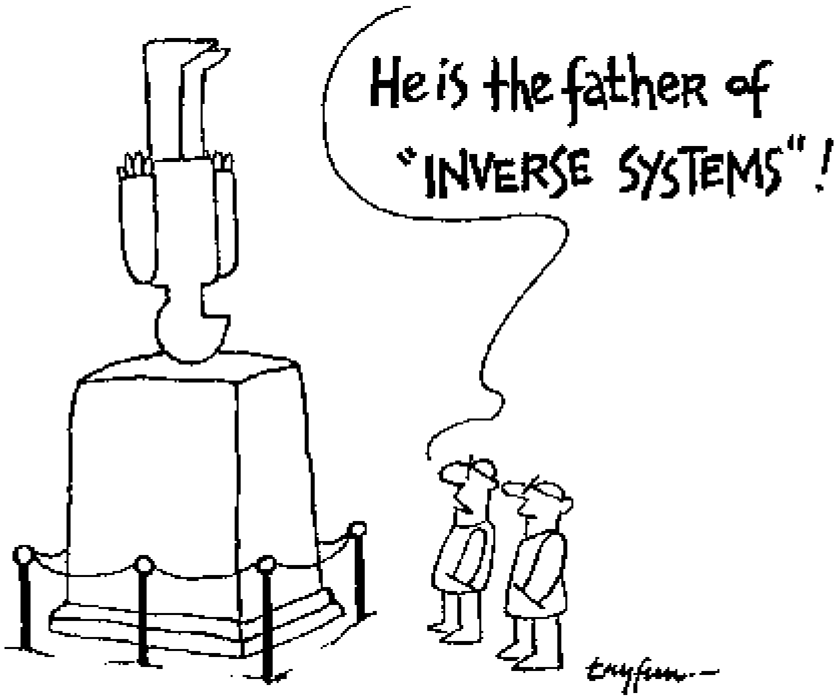
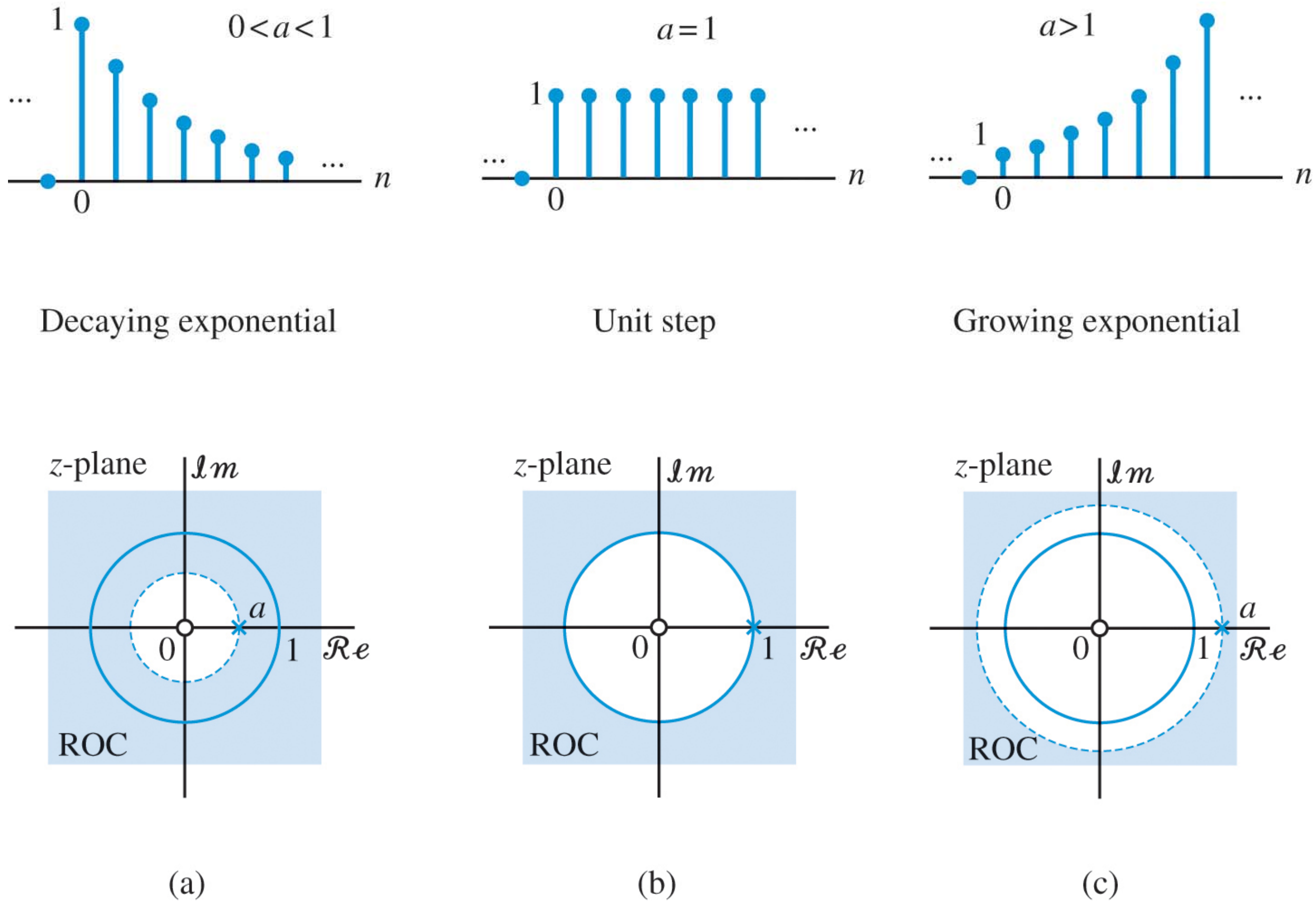
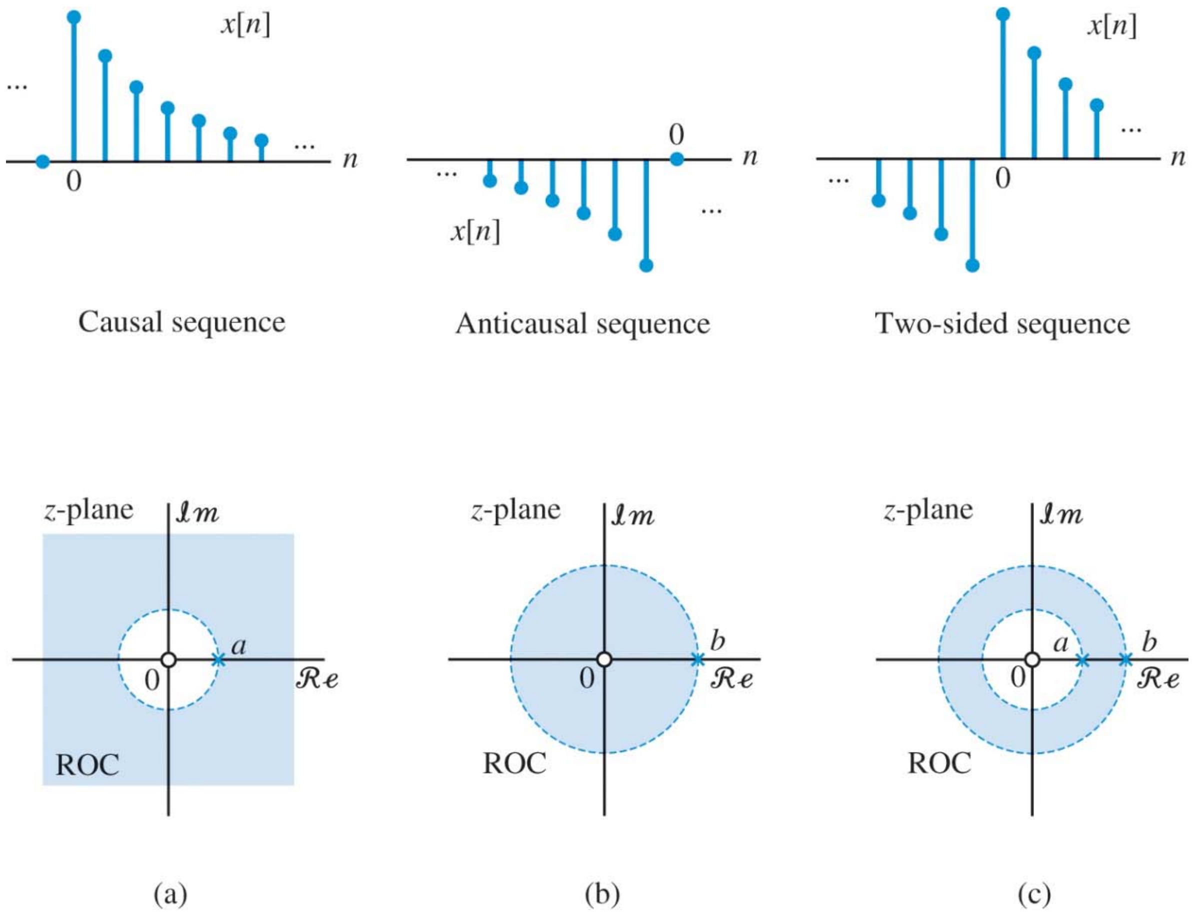

He is the father of
"INVERSE SYSTEMS"!

Reminder



Pole-zero plot and region of convergence of a causal exponential sequence $x[n]=a^n u[n]$ with a) decaying exponential, b) unit step sequence and c) growing sequence

Reminder

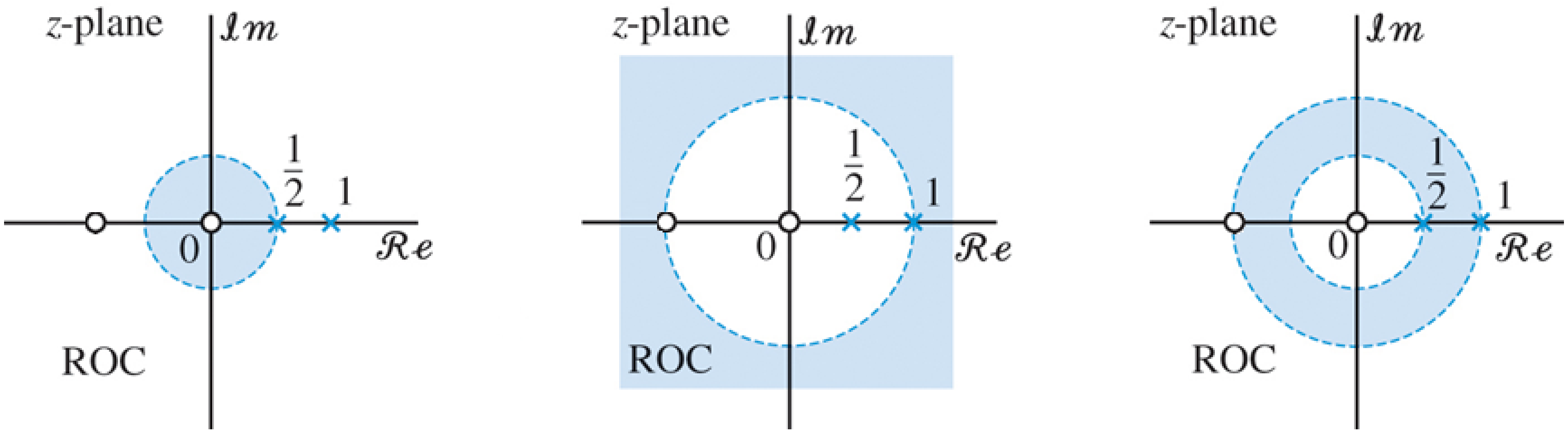


Pole-zero plot and region of convergence for the a) causal, b) anticausal and c) two sided exponential sequences.

Example

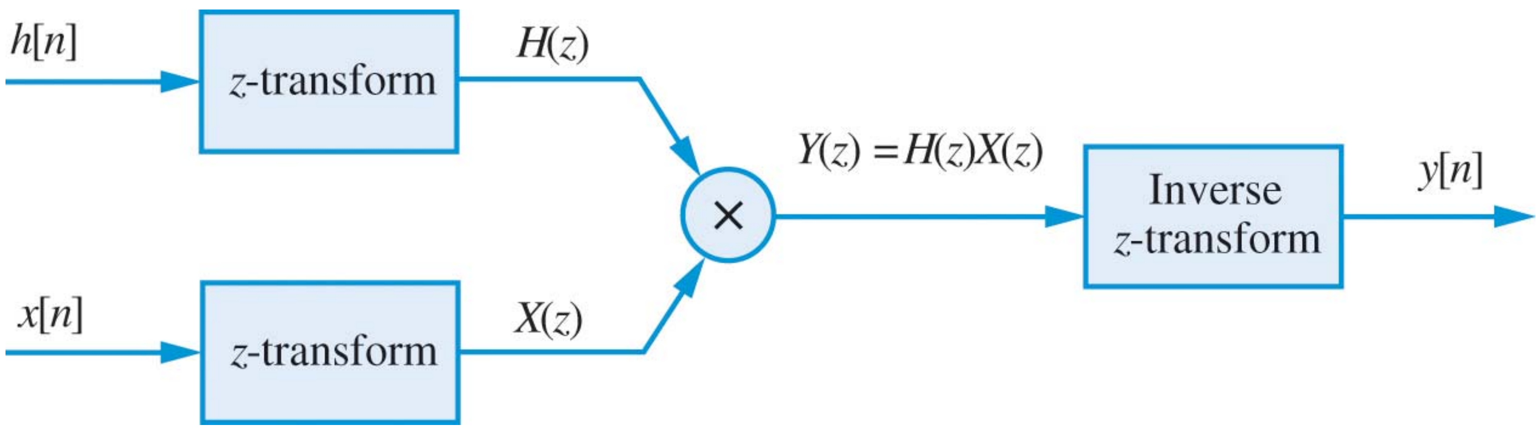
Pole-zero plot and region of possible convergence for the z transform in

$$X(z) = \frac{1+z^{-1}}{\left(1-z^{-1}\right)\left(1-0.5z^{-1}\right)}$$

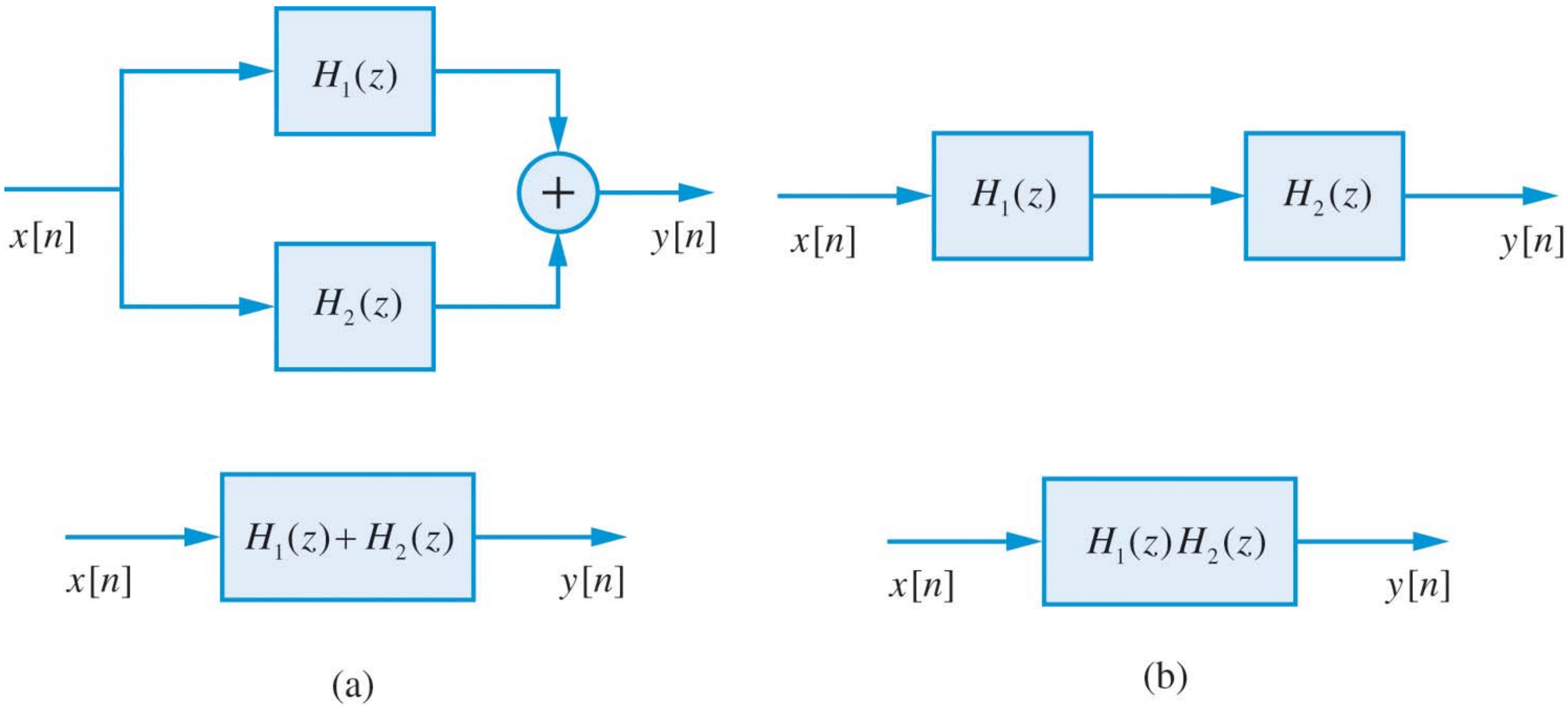


System function of LTI systems

Procedure for the analytical computation of the output of an LTI system using the convolution theorem of the z-transform:



System function of LTI systems



Equivalent system function of linear time-invariant systems combined in a) parallel connection and b) cascade connection

Stability-Causality

- Stability & Causality:
  - Causal systems must be right sided
    - ROC is outside the outermost pole
  - Stable system requires absolute summable impulse response $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$
    - Absolute summability implies existence of DTFT
    - DTFT exists if unit circle is in the ROC
    - Therefore, stability implies that the ROC includes the unit circle

Stability-Causality

Stability & Causality:

– Causal & stable systems have all poles inside unit circle
  - Causal hence the ROC is outside outermost pole
  - Stable hence unit circle included in ROC
  - This means outermost pole is inside unit circle
  - Hence all poles are inside unit circle

Linear constant-coefficient difference equations

A class of LTI systems whose input and output sequences satisfy a linear constant-coefficient difference equation is written as:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Using the properties of linearity and shifting we obtain:

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

Then the system function (transfer function) is obtained as:

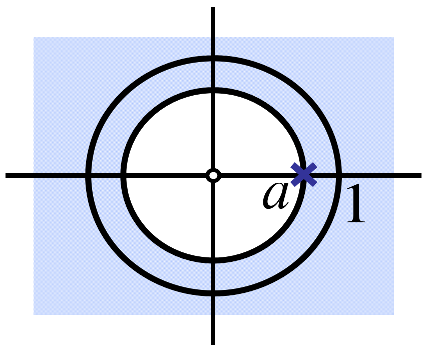
$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Example

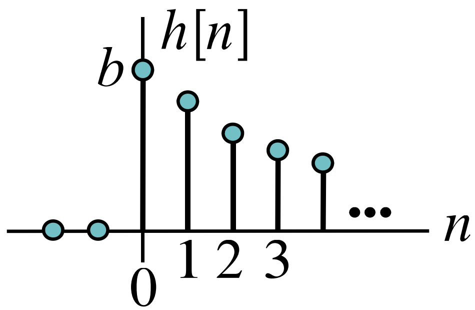
- LCCDE (causal): $y[n] = ay[n-1] + bx[n], \quad -1 < a < 1$
- System function

$$Y(z) = az^{-1}Y(z) + bX(z) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{b}{1 - az^{-1}} = \frac{bz}{z-a}$$

- Since system is causal $\Rightarrow h[n] = ba^n u[n]$
- The system is stable because $p = a$ and $|a| < 1$



$\Rightarrow$



Example

$$H(z)=\frac{\left(1+z^{-1}\right)^2}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{3}{4}z^{-1}\right)}=\frac{1+2z^{-1}+z^{-2}}{1+\frac{1}{4}z^{-1}+\frac{3}{8}z^{-2}}=\frac{Y(z)}{X(z)}$$

$$\left(1+\frac{1}{4}z^{-1}+\frac{3}{8}z^{-2}\right)Y(z)=\left(1+2z^{-1}+z^{-2}\right)X(z)$$

$$y[n]+\frac{1}{4}y[n-1]+\frac{3}{8}y[n-2]=x[n]+2x[n-1]+x[n-2]$$

Example-Cont

If we apply $x[n] = u[n]$ (initial cond. are zero)

$$Y(z) = \frac{\left(1+z^{-1}\right)^2}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{3}{4}z^{-1}\right)\left(1-z^{-1}\right)}$$

Then, we can find y[n] using partial fraction expansion.

Linear constant-coefficient difference equations

H(z) is a rational function, that is, the ratio of two polynomials in z^{-1}.

$$H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}=\left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}\left(1-z_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1-p_k z^{-1}\right)}=\left(\frac{b_0}{a_0}\right) z^{N-M}\frac{\prod_{k=1}^{M}\left(z-z_k\right)}{\prod_{k=1}^{N}\left(z-p_k\right)}$$

We recall that any rational function of z^{-1} can be expressed in the form of:

$$H(z)=\sum_{k=0}^{M-N} C_k z^{-k}+\sum_{k=1}^{N}\frac{A_k}{\left(1-p_k z^{-1}\right)}$$

Where the first summation is included only if M≥N.

Linear constant-coefficient difference equations

If we assume that the system is causal, then the ROC is the exterior of a circle starting at the outermost pole, and the impulse response is:

$$h[n] = \sum_{k=0}^{M-N} C_k \delta[n-k] + \sum_{k=1}^{N} A_k (p_k)^n u[n]$$

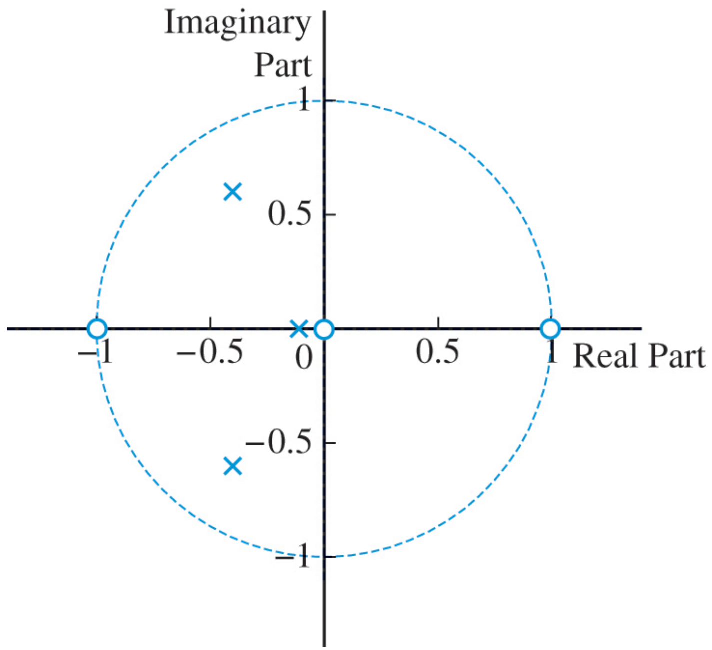
For a causal system to be stable, the impulse response must be absolutely summable.

$$\sum_{n=-\infty}^{\infty} |h[n]| < \sum_{k=0}^{M-N} |C_k| + \sum_{k=1}^{N} |A_k| \sum_{n=0}^{\infty} |p_k|^n < \infty$$

This is possible if and only if $|p_k|<1$. Hence a causal LTI system with a rational system function is stable if and only if all poles of H(z) are inside the unit circle in the z plane. The zeros can be anywhere.

Example

Pole-zero plot for the system function given by: $H(z)=\dfrac{1-z^{-2}}{\left(1+0.9z^{-1}+0.6z^{-2}+0.05z^{-3}\right)}$

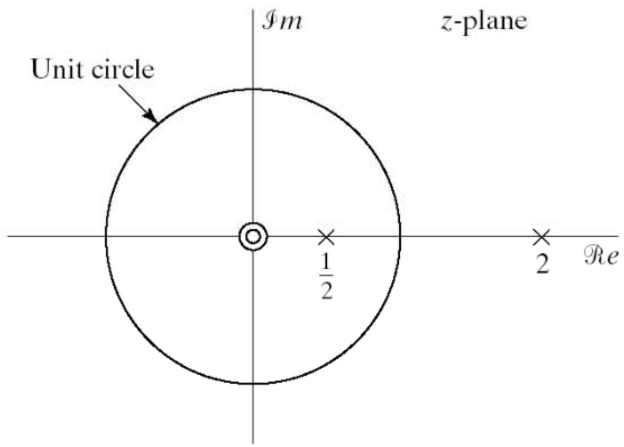


This causal system is stable because its poles are inside the unit circle.

Example

$$y[n]-\frac{5}{2}y[n-1]+y[n-2]=x[n]$$

$$H(z)=\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)}$$



Causal & not stable ← $ROC_1 : |z| > 2$

Not causal & stable ← $ROC_2 : \frac{1}{2} < |z| < 2$

Not causal & not stable ← $ROC_3 : |z| < \frac{1}{2}$

One-sided z-Transform

- To solve LCCDEs with initial conditions or to obtain output when an input is stepped into a system, we need the one-sided z-transform.

- **Definition and Notation:**

$$\boxed{\mathcal{Z}^{+}\{x[n]\} \triangleq \mathcal{Z}\{x[n]u[n]\} \triangleq X^{+}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}}$$

- **One Important Property:**

$$\begin{aligned}
\mathcal{Z}^{+}\{x[n-k]\} &= \mathcal{Z}\{x[n-k]u[n]\}, \quad k > 0 \\
&= \sum\nolimits_{n=0}^{\infty} x[n-k]z^{-n} = \sum\nolimits_{m=-k}^{\infty} x[m]z^{-(m+k)} \\
&= \sum\nolimits_{m=-k}^{-1} x[m]z^{-(m+k)} + \left[\sum\nolimits_{m=0}^{\infty} x[m]z^{-m}\right] z^{-k} \\
&= x[-1]z^{1-k} + \cdots + x[-k] + z^{-k}X^{+}(z)
\end{aligned}$$

Linear constant-coefficient difference equations

$$\mathcal{Z}^+\{x[n-k]\} = \underbrace{x[-1]z^{1-k} + \cdots + x[-k]}_{\text{initial condition effect}} + x^{-k}X(z), \quad k > 0$$

- Now we can solve the difference equation

$$y[n] = \sum_{m=0}^{M} b_m x[n-m] - \sum_{k=1}^{N} a_k y[n-k], \qquad n > 0$$

subject to the initial conditions:

$$\{y[i],\ i = -1, -2, \ldots, -N\}; \quad \{x[i],\ i = -1, -2, \ldots, -M\}$$

after obtaining an algebraic equation for $Y(z)$ and then taking its inverse z-transform.

Example

$$y[n]-\frac{1}{5}y[n-1]-\frac{3}{100}y[n-2]=\frac{1}{5}x[n]-\frac{3}{10}x[n-1]$$

$$y[-1]=1 \text{ and } y[-2]=2 \qquad \text{find } y[n] \text{ for } x[n]=u[n]$$

$$Y(z)+\frac{1}{5}\left(y[-1]+z^{-1}Y(z)\right)+\frac{3}{100}\left(y[-2]+z^{-1}y[-1]+z^{-2}Y(z)\right)=\frac{1}{5}X(z)-\frac{3}{10}\left(x[-1]+z^{-1}X(z)\right)$$

$$\left(1-0.2z^{-1}-0.03z^{-2}\right)Y(z)-0.26-0.03z^{-1}=\left(0.2-0.3z^{-1}\right)X(z)$$

$$X(z)=\frac{1}{1-z^{-1}}\ ,\ \text{ROC:}|z|>1$$

$$\left(1-0.2z^{-1}-0.03z^{-2}\right)Y(z)=\frac{0.2-0.3z^{-1}}{1-z^{-1}}+0.26+0.03z^{-1}$$

Example-Cont

$$Y(z) = \frac{\dfrac{0.2 - 0.3z^{-1}}{1 - z^{-1}} + 0.26 + 0.03z^{-1}}{1 - 0.2z^{-1} - 0.03z^{-2}}$$

$$= \frac{-0.1299}{1 - z^{-1}} + \frac{0.5271}{1 - 0.3z^{-1}} + \frac{0.0627}{1 + 0.1z^{-1}}$$

$$y[n] = -0.1299u[n] + 0.5271(0.3)^n u[n] + 0.0627(-0.1)^n u[n]$$

Ideal and Practical filters

- ✓ Systems that are designed to pass some frequency components without significant distortion while severely or completely eliminating others are known as frequency selective filters.

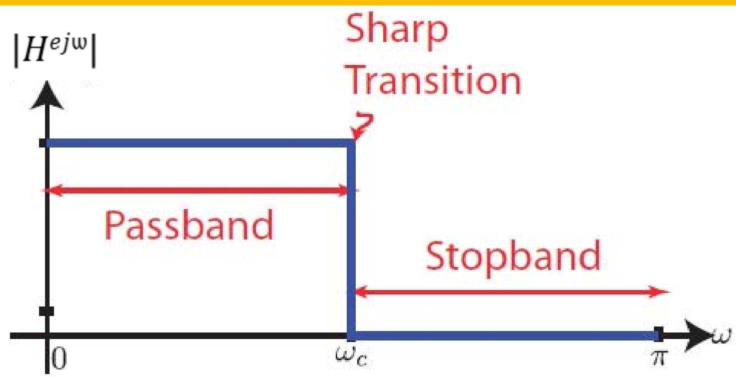
- ✓ By definition, an ideal frequency selective filter satisfies the requirements for distortionless response over one or more frequency bands.

- ✓ Ideal filters are used in the early stages of a design process to specify modules in a signal processing systems. However they are not realizable in practice, they must be approximated by practical and non-ideal filters.

- ✓ This is usually done by minimizing some approximation error between the non-ideal filter and a prototype ideal filter.

Filters-Reminder

Ideal frequency selective filters either pass or eliminate a region of input spectrum perfectly and they have abrupt transitions between passbands and stopbands.

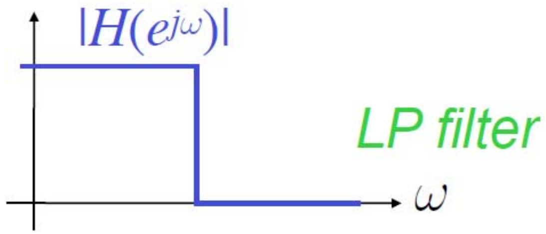


However ideal filters can not be implemented in practice; therefore they have to be approximated by practically realizable filters. Practical filters differ from ideal filters in several respect:

- ✓ The bandpass responses are not perfectly flat;
- ✓ The stopband responses can not completely eliminate bands of frequencies;
- ✓ The transition between passband and stopband regions takes place over a finite transition band.

Ideal filters

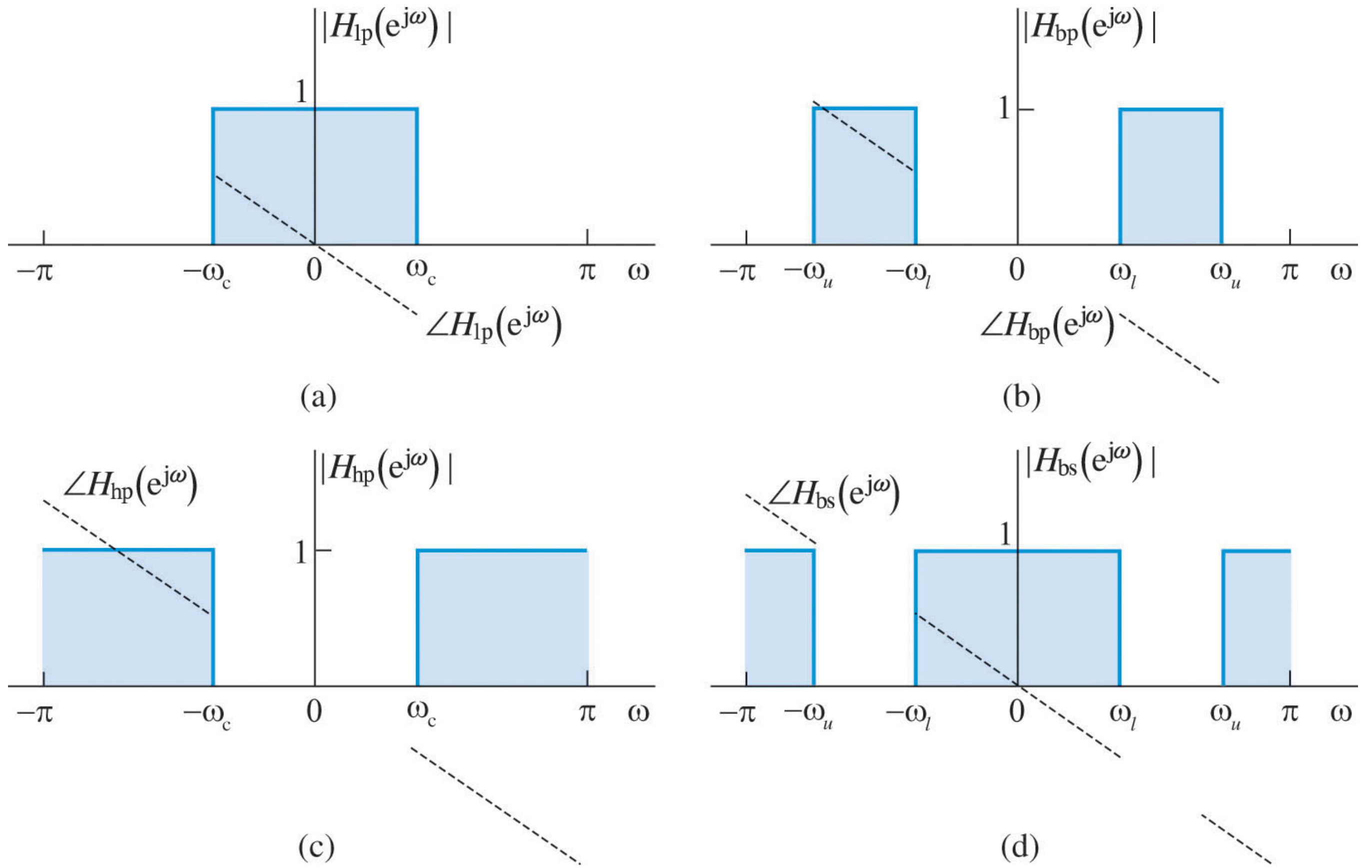
- Typical filter requirements:
  - gain = 1 for wanted parts (pass band)
  - gain = 0 for unwanted parts (stop band)
- "Ideal" characteristics would be like:



- What is this filter?
  - can calculate IR $h[n]$ as IDTFT of ideal response...

Ideal and Practical filters

Frequency responses of four types of ideal filter



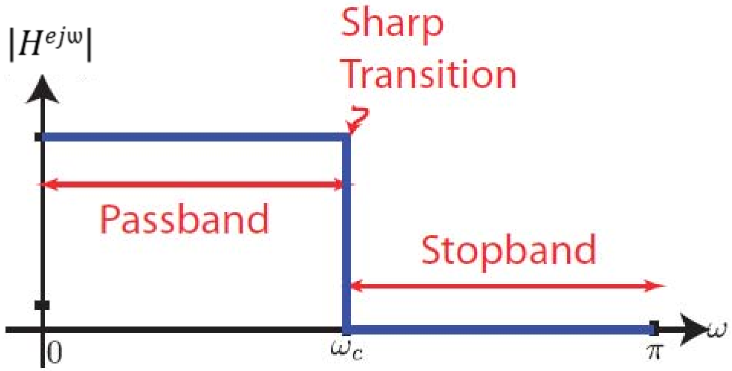
a) Lowpass filter; b) bandpass filter; c) highpass filter; bandstop filter.

Ideal and Practical filters

- ✓ As the impulse response corresponding to each of these ideal filters is noncausal and of infinite length, these filters are not realizable.
- ✓ In practice, the magnitude response specifications of a digital filter in the passband and in the stopband are given with some acceptable tolerances.
- ✓ In addition, a transition band is specified between the passband and stopband.

Filter design

Ideal frequency selective filters either pass or eliminate a region of input spectrum perfectly and they have abrupt transitions between passbands and stopbands.



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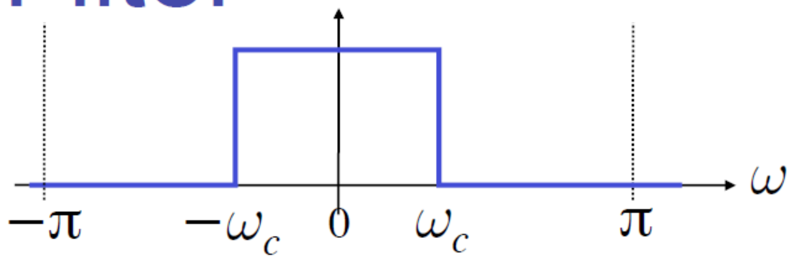
Ideal Lowpass Filter



- Given ideal $H(e^{j\omega})$:
  (assume $\theta(\omega) = 0$)

$$\Rightarrow \quad h[n] = IDTFT\left\{H\left(e^{j\omega}\right)\right\}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{j\omega}\right)e^{j\omega n}\,d\omega$$

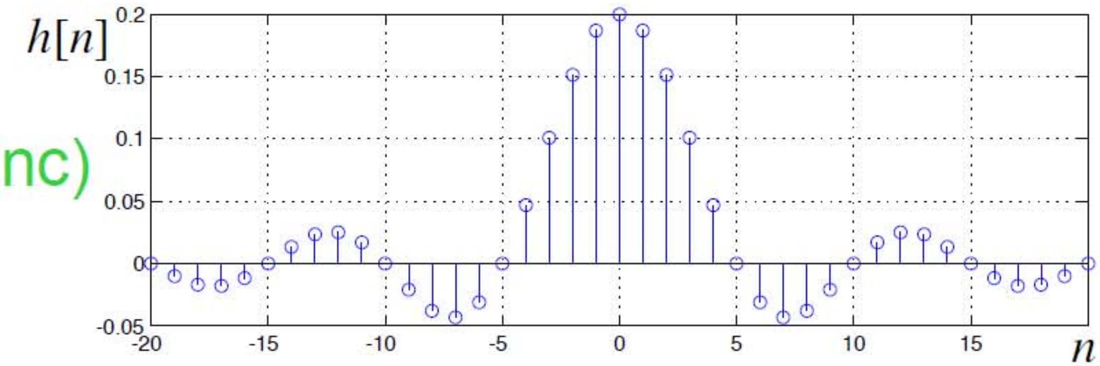
$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\,d\omega$$

$$\Rightarrow h[n] = \frac{\sin \omega_c n}{\pi n}$$

Ideal lowpass filter

Ideal Lowpass Filter

$$h[n] = \frac{\sin \omega_c n}{\pi n}$$ (sinc)



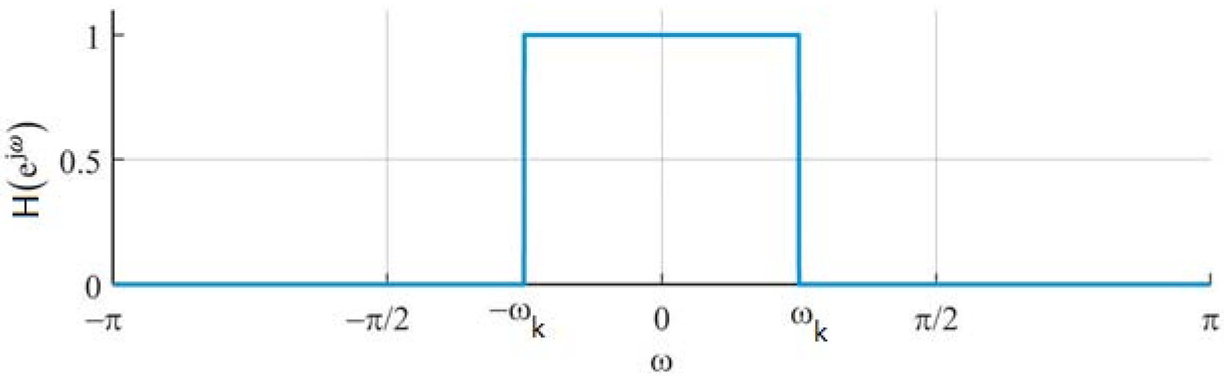
- Problems!
  - doubly infinite ($n = -\infty..\infty$)

Ex Example from DTFT:

Ideal Low Pass Filter:

$$H\left(e^{j\omega}\right)=\begin{cases}1 \ , & |\omega|<\omega_k \\ 0 \ , & \omega_k<|\omega|<\pi\end{cases}$$

$$h[n]=?$$



Example-Cont:

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{j\omega}\right)e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_k}^{\omega_k} e^{j\omega n}d\omega = \frac{1}{2\pi}\left(\frac{e^{j\omega n}}{jn}\right)\Bigg|_{-\omega_k}^{\omega_k}$$

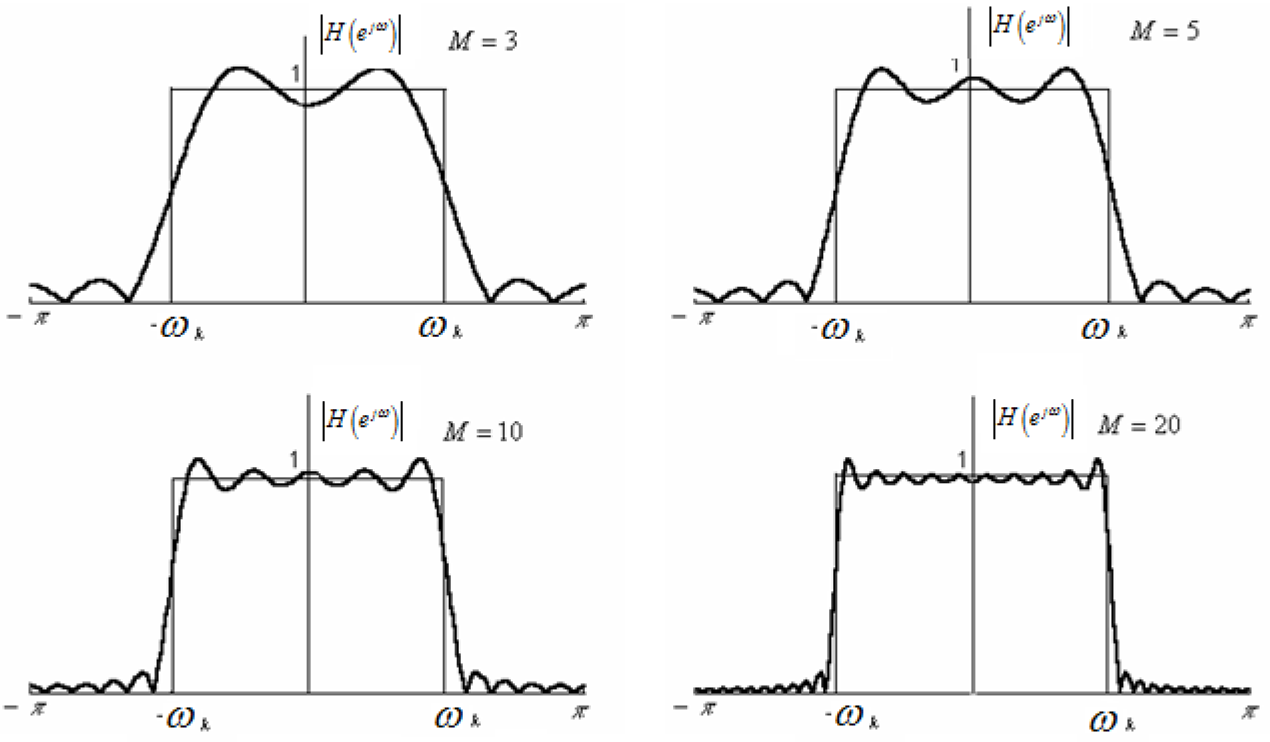
$$h[n] = \frac{1}{2\pi}\left(\frac{e^{j\omega n}}{jn}\right)\Bigg|_{-\omega_k}^{\omega_k} = \frac{1}{2\pi}\left(\frac{e^{j\omega_k n} - e^{-j\omega_k n}}{jn}\right)$$

$$h[n] = \frac{1}{2\pi}\left(\frac{e^{j\omega_k n} - e^{-j\omega_k n}}{jn}\right) = \frac{1}{2\pi}\frac{2j\sin\omega_k n}{jn} = \frac{\sin\omega_k n}{\pi\ n} \quad ,-\infty < n < \infty$$

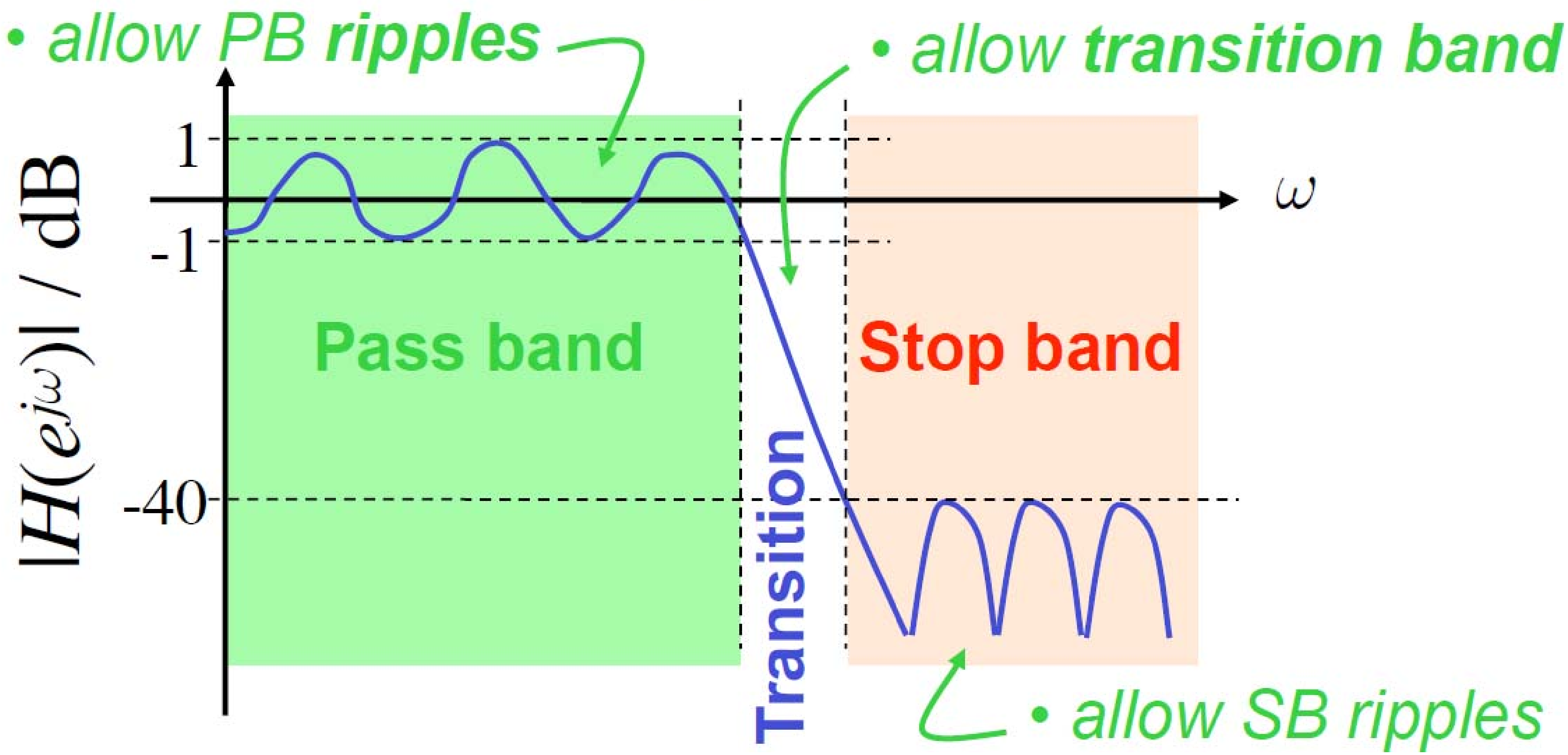
$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty}\frac{\sin\omega_k n}{\pi\ n}e^{-j\omega n}$$

Example-Cont:

$$H_M\left(e^{j\omega}\right) = \sum_{n=-M}^{M} \frac{\sin \omega_k n}{\pi\, n} e^{-j\omega n}$$

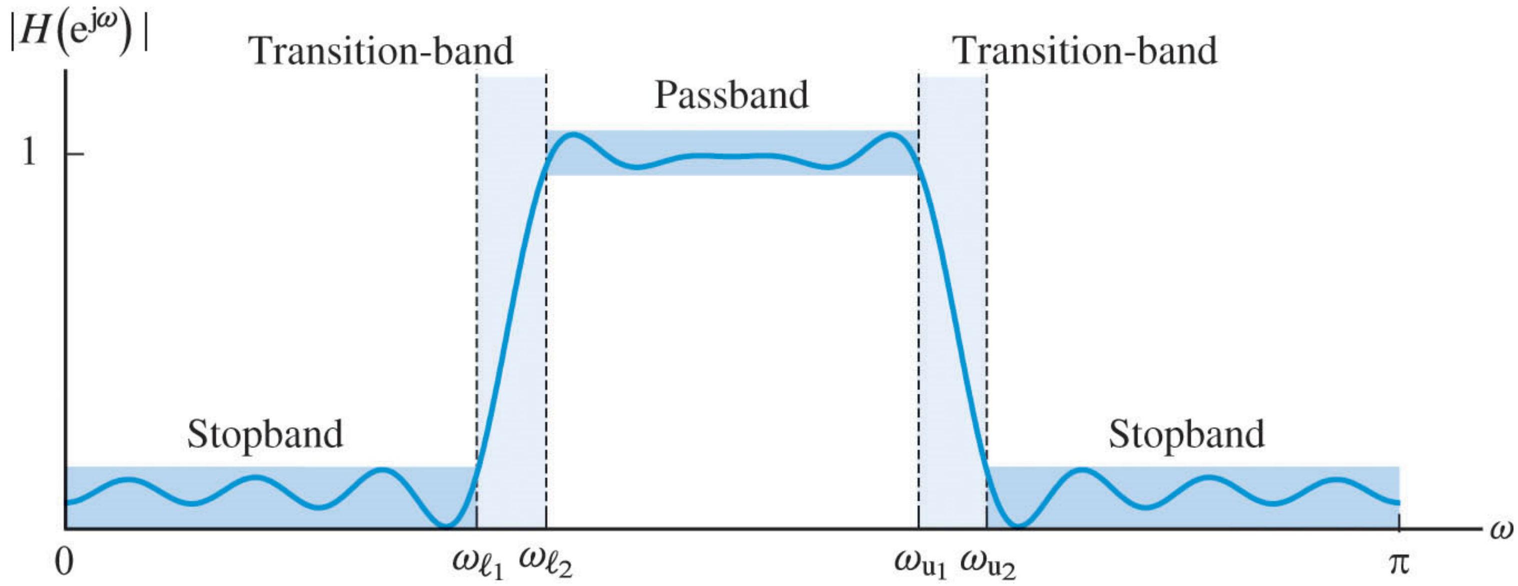


Practical filter specifications



Ideal and Practical filters

Typical characteristics of a practical bandpass filter



A good filter should have only a small ripple in the passband, high attenuation in the stopband, and very narrow transition bands.

Frequency Response of Rational System Functions

- All LTI systems of practical interest are described by a difference equation of the form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

and have a rational system function:

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- For a stable system the system function convergences on the unit circle. Therefore, we obtain:

$$H\left(e^{j\omega}\right) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

which expresses $H(e^{j\omega})$ as a ratio of two polynomials in the variable $e^{j\omega}$

Frequency Response of Rational System Functions

- DTFT of a stable and LTI rational system function:

$$H\left(e^{j\omega}\right)=\frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}=\left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}\left(1-z_k e^{-j\omega}\right)}{\prod_{k=1}^{N}\left(1-p_k e^{-j\omega}\right)}$$

- Magnitude Response:

$$\left|H\left(e^{j\omega}\right)\right|=\left|\frac{b_0}{a_0}\right|\frac{\prod_{k=1}^{M}\left|1-z_k e^{-j\omega}\right|}{\prod_{k=1}^{N}\left|1-p_k e^{-j\omega}\right|}$$

- Phase response:

$$\left|H\left(e^{j\omega}\right)\right|=\angle\left(\frac{b_0}{a_0}\right)+\prod_{k=1}^{M}\angle\left(1-z_k e^{-j\omega}\right)-\prod_{k=1}^{N}\angle\left(1-p_k e^{-j\omega}\right)$$

Dependence of frequency response on poles and zeros

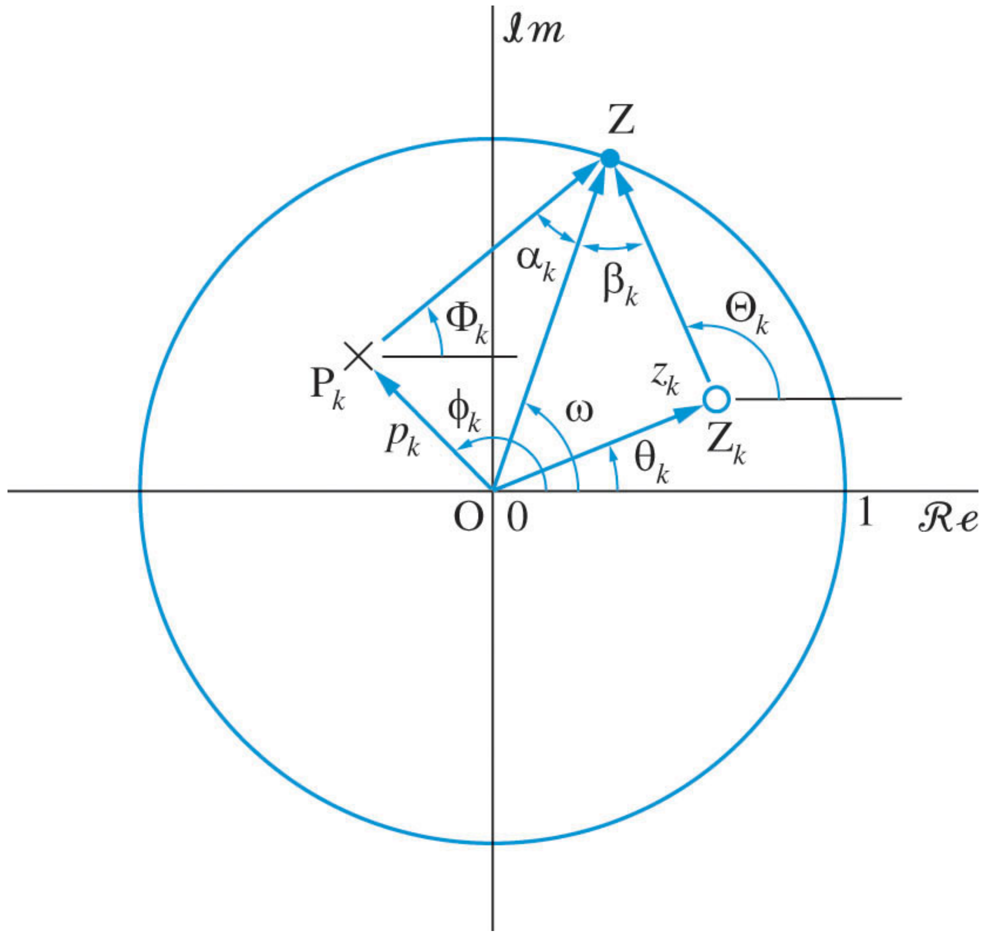
$$H\left(e^{j\omega}\right)=\left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}\left(1-z_k e^{-j\omega}\right)}{\prod_{k=1}^{N}\left(1-p_k e^{-j\omega}\right)}=\left(\frac{b_0}{a_0}\right)e^{j\omega(N-M)}\frac{\prod_{k=1}^{M}\left(e^{j\omega}-z_k\right)}{\prod_{k=1}^{N}\left(e^{j\omega}-p_k\right)}$$

- This equation consists of factors of the form $\left(e^{j\omega}-z_k\right)$ and $\left(e^{j\omega}-p_k\right)$
- The factor $\left(e^{j\omega}-z_k\right)$ is a complex number represented by a vector drawn from the point z_k (zero) to the point $z=e^{j\omega}$ in the complex plane and can be written as:

$$\left(e^{j\omega}-z_k\right)=Q_k e^{j\Theta_k}$$

- Q_k is the distance from the z_k to the point $z=e^{j\omega}$, and Θ_k is the angle of the vector with the positive real axis.

Dependence of frequency response on poles and zeros



Dependence of frequency response on poles and zeros

- The factor $\left(e^{j\omega}-p_k\right)$ is a complex number represented by a vector drawn from the point p_k (pole) to the point $z=e^{j\omega}$ in the complex plane.

$$\left(e^{j\omega}-p_k\right)=R_k e^{j\Phi_k}$$

- R_k is the distance from the pole to the point, and Φ_k is the angle of the vector with the positive real axis.

- Then, we can write:

$$H\left(e^{j\omega}\right)=\left|\frac{b_0}{a_0}\right|\frac{\prod_{k=1}^{M}Q_k(\omega)}{\prod_{k=1}^{N}R_k(\omega)}\times\exp\left[\angle\left(\frac{b_0}{a_0}\right)+\omega(N-M)+\sum_{k=1}^{M}\Theta_i(\omega)-\sum_{k=1}^{M}\Phi_k(\omega)\right]$$

Dependence of frequency response on poles and zeros

$$H\left(e^{j\omega}\right)=\left|\frac{b_0}{a_0}\right|\frac{\prod_{k=1}^{M}Q_k(\omega)}{\prod_{k=1}^{N}R_k(\omega)}\times\exp\left[\angle\left(\frac{b_0}{a_0}\right)+\omega(N-M)+\sum_{k=1}^{M}\Theta_i(\omega)-\sum_{k=1}^{M}\Phi_k(\omega)\right]$$

- We note that $\angle\left(\frac{b_0}{a_0}\right)=\pi$ if $\left(\frac{b_0}{a_0}\right)<0$ because $e^{j\pi}=-1$, and

  $\angle\left(\frac{b_0}{a_0}\right)=0$ if $\left(\frac{b_0}{a_0}\right)\geq 0$ because $e^{j0}=1$

$$\left|H\left(e^{j\omega}\right)\right|=\left|\frac{b_0}{a_0}\right|\frac{\prod_{k=1}^{M}Q_k(\omega)}{\prod_{k=1}^{N}R_k(\omega)}$$

$$\angle H\left(e^{j\omega}\right)=\left[\angle\left(\frac{b_0}{a_0}\right)+\omega(N-M)+\sum_{k=1}^{M}\Theta_i(\omega)-\sum_{k=1}^{M}\Phi_k(\omega)\right]$$

Dependence of frequency response on poles and zeros

$$\left|H\left(e^{j\omega}\right)\right| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M} Q_k\left(\omega\right)}{\prod_{k=1}^{N} R_k\left(\omega\right)}$$

$$\angle H\left(e^{j\omega}\right) = \left[\angle\left(\frac{b_0}{a_0}\right) + \omega(N-M) + \sum_{k=1}^{M}\Theta_i(\omega) - \sum_{k=1}^{M}\Phi_k(\omega)\right]$$

- ✓ where ω is the angle of the point $z=e^{j\omega}$ with the positive real axis and

$$Q_k\left(\omega\right) = \text{distance of the } k\text{-}th \text{ zero from } e^{j\omega}$$
$$R_k\left(\omega\right) = \text{distance of the } k\text{-}th \text{ pole from } e^{j\omega}$$
$$\Theta_k\left(\omega\right) = \text{angle of the } k\text{-}th \text{ zero with the real axis}$$
$$\Phi_k\left(\omega\right) = \text{angle of the } k\text{-}th \text{ pole with the real axis}$$

- ✓ Therefore the magnitude response at a certain frequency ω is given by the product of the lengths of the vectors from the zeros to $z=e^{j\omega}$ divided by the product of the lengths of vectors drawn from the poles to $z=e^{j\omega}$.
- ✓ Similarly the phase response is obtained by subtracting from the sum of angles of zeros the sum of angles of poles. All angles are determined with respect to the positive real axis. We can determine $H(e^{j\omega})$ for each value of ω or equivalently any location of the point $e^{j\omega}$ on the unit circle.

Dependence of frequency response on poles and zeros

- ✓ To suppress a frequency component at $\omega=\omega_0$, we should place a zero at angle ω_0 on the unit circle.

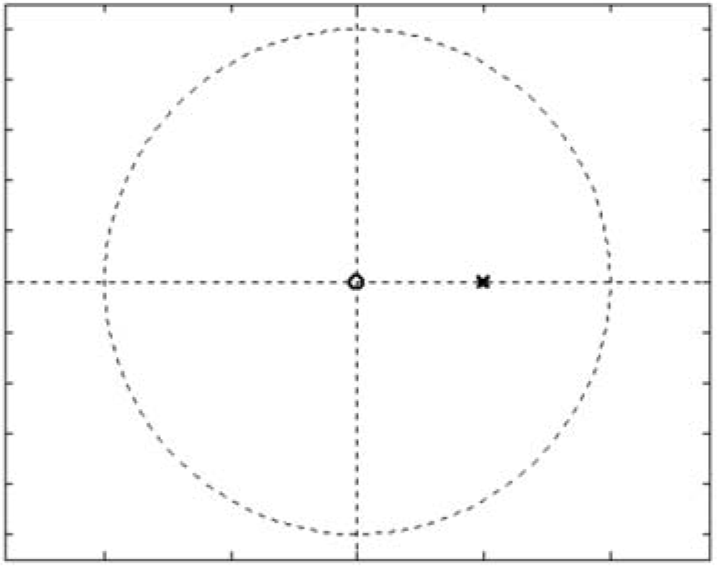
- ✓ To enhance or amplify a frequency component at $\omega=\omega_0$ we should place a pole at angle ω_0 close but inside the unit circle.

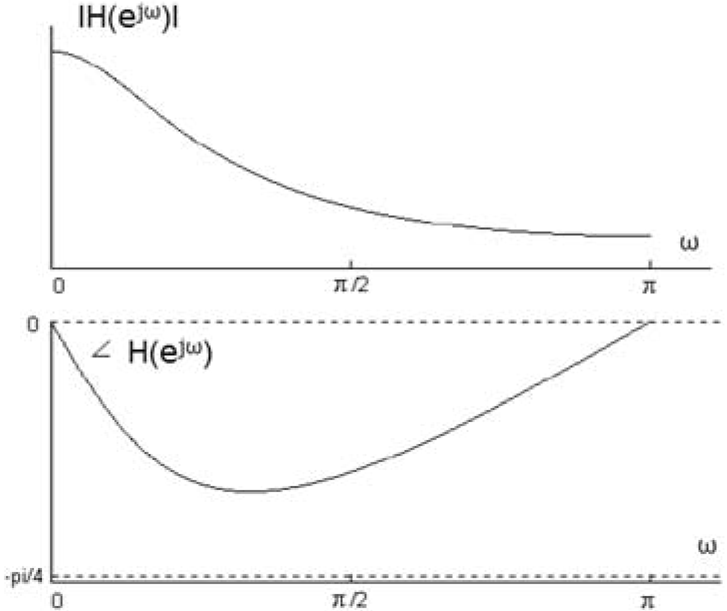
- ✓ Complex poles or zeros should appear in complex conjugate pairs to assure that the system has real coefficients. This stems from the fact that the frequency ω is defined as the angle with respect to positive real axis. Therefore all symmetries should be defined with respect to the real axis.

- ✓ Poles and zeros at the origin do not influence the magnitude response because their distance from any point on the unit circle is unity. However a pole at the origin adds (or subtracts) a linear phase of ω rads to the phase response. We often introduce poles and zeros at $z=0$ to assure that $N=M$

Significance of poles and zeros

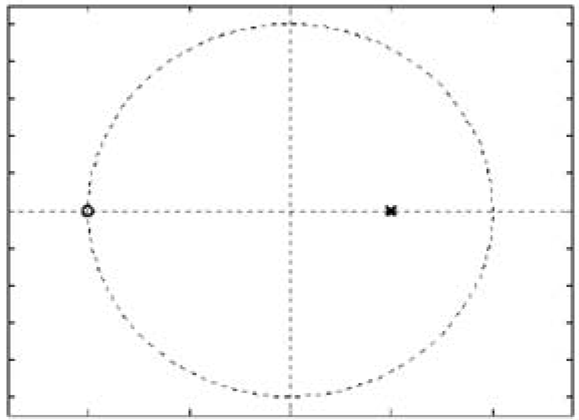
$$H(z) = \frac{1}{1 - 0.5z^{-1}}$$

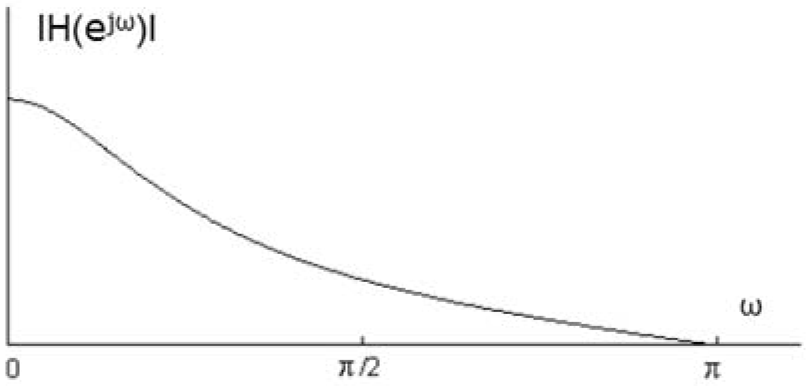


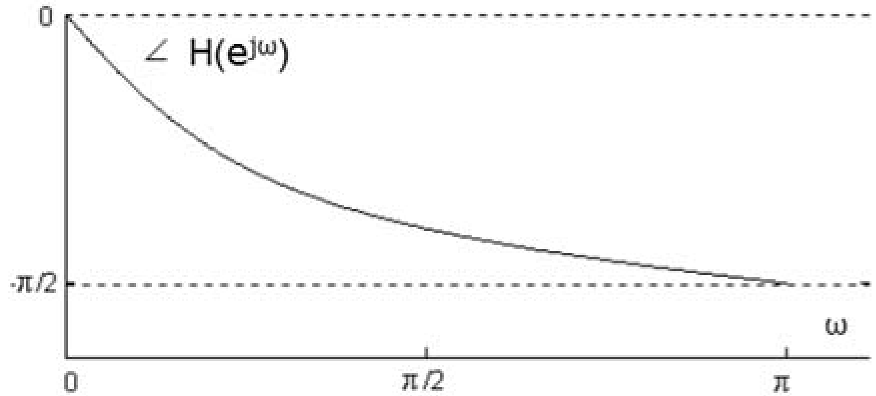


Significance of poles and zeros

$$H(z) = \frac{1 + z^{-1}}{1 - 0.5z^{-1}}$$



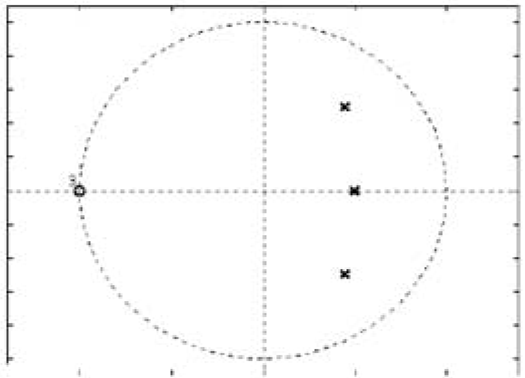


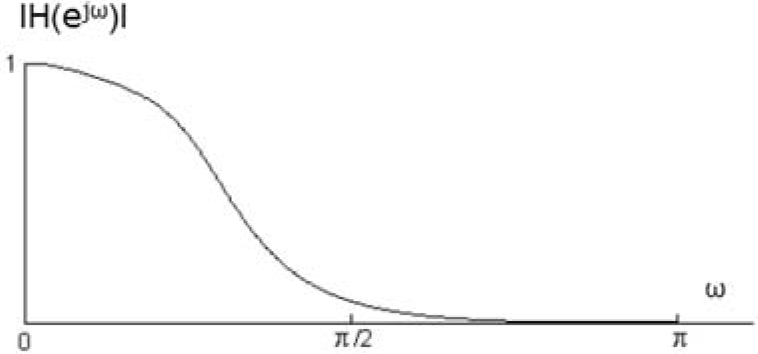


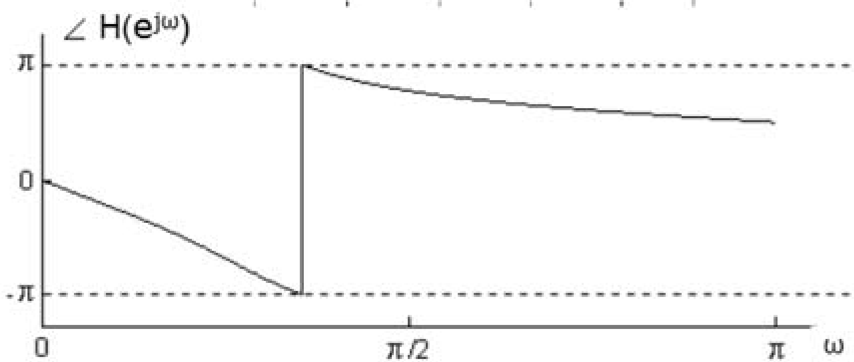
Significance of poles and zeros

$$H(z) = \frac{1}{29} \frac{\left(1+z^{-1}\right)^3}{\left(1-0.5z^{-1}\right)\left(1-0.9z^{-1}+0.4525z^{-2}\right)}$$

$$H\left(e^{j\omega}\right)\Big|_{\omega=0} = H(z)\big|_{z=1} = 1$$

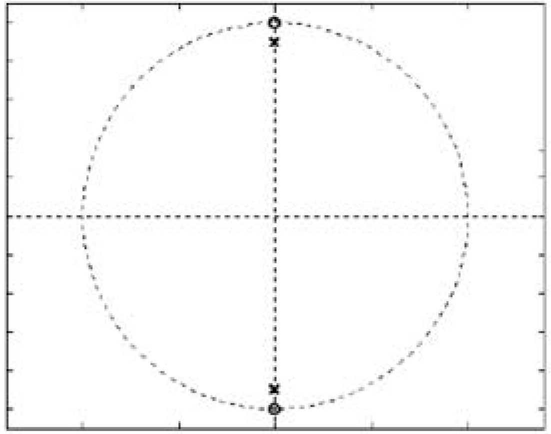


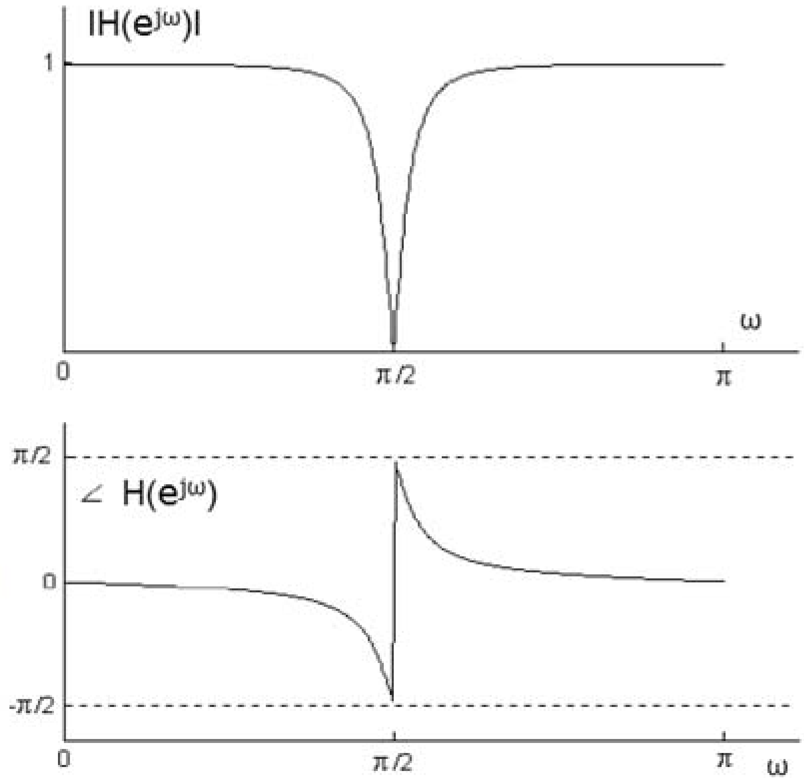




Significance of poles and zeros

- Example: Band-stop filter





Zeros: $z = e^{j\pi/2} = 0 + j \quad z = e^{-j\pi/2} = 0 - j$

Poles: $z = 0.9e^{j\pi/2} = 0 + 0.9j \quad z = 0.9e^{-j\pi/2} = 0 - 0.9j$

$$H(z) = \frac{1}{1.105} \frac{1 + z^{-2}}{1 + 0.81z^{-2}}$$

Allpass systems

- The frequency response of an allpass system has constant magnitude at all frequencies.

$$\left|H\left(e^{j\omega}\right)\right| = G$$

- If we assume that G=1, allpass systems preserve that the power and energy of their input signals.

- In this sense allpass systems are said to be lossless systems. The simplest allpass system simply scale and delay the input signal:

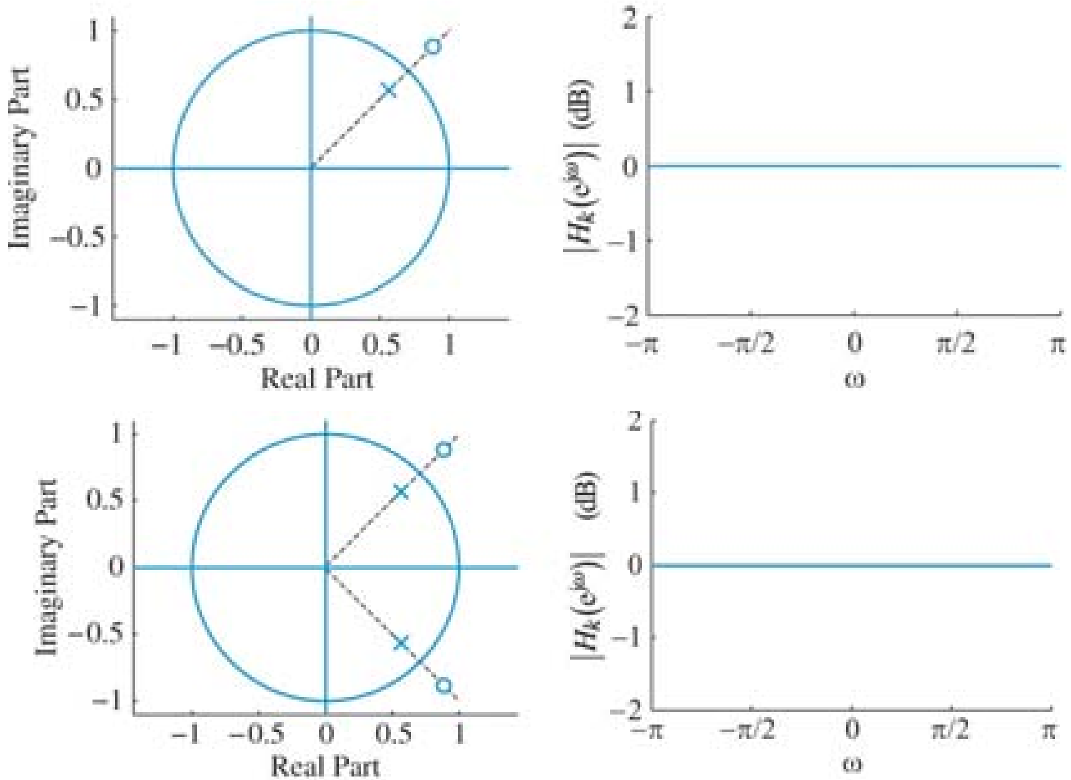
$$H_{ap}\left(z\right) = Gz^{-k}$$

- A more interesting all pass system is: $H_{ap}\left(z\right) = \dfrac{z^{-1} - p_k^*}{1 - p_k z^{-1}}$

Allpass systems

- ✓ We note that each pole p_k of an allpass system should be accompanied by a complex reciprocal zero $1/p_k^*$.

- ✓ Since causal and stable allpass systems must have all poles inside the unit circle, all zeros are outside the unit circle.



Invertibility and minumum-phase systems

$$h[n] * h_{inv}[n] = \delta[n]$$

$$H(z)H_{inv}(z) = 1 \qquad H_{inv}(z) = \frac{1}{H(z)}$$

- ✓ The zeros of H(z) become the poles of its inverse system and vice versa. A causal and stable H(z) should have its poles inside the unit circle; its zeros can be anywhere.
- ✓ In inverse filtering applications the system $H_{inv}(z)$ should be causal and stable as well.
- ✓ A causal and stable LTI system with a causal and stable inverse is known as a minumum-phase system.
- ✓ A LTI system is stable and causal and also has a stable and causal inverse if and only if both the poles and the zeros of H(z) are inside the unit circle.

Minimum phase and allpass decomposition

- ✓ Any system with a rational system function can be decomposed into a minimum phase system and an allpass system.
- ✓ Suppose that H(z) has one zero z=1/a* where |a|<1 outside the unit circle, and all the other poles and zeros are inside the unit circle. Then we can factor H(z) as:

$$H(z) = H_1(z)(z^{-1} - a^*)$$

minimum phase system

allpass system

$$H(z) = H_1(z)(1 - az^{-1})\frac{(z^{-1} - a^*)}{(1 - az^{-1})}$$

- ✓ If we repeat this process for every zero outside the unit circle, we obtain:

$$H(z) = H_{\min}(z)H_{ap}(z)$$

Relationship between time, frequency and z-transform domains