SIAM J. DISCRETE MATH.
Vol. 30, No. 3, pp. 1470–1479

GENERATING FUNCTIONS AND TRIANGULATIONS FOR LECTURE HALL CONES*

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Abstract. We investigate the arithmetic-geometric structure of the lecture hall cone $L_n := \left\{\lambda \in \mathbb{R}^n : 0 \le \frac{\lambda_1}{1} \le \frac{\lambda_2}{2} \le \frac{\lambda_3}{3} \le \cdots \le \frac{\lambda_n}{n}\right\}$. We show that L_n is isomorphic to the cone over the lattice pyramid of a reflexive simplex whose Ehrhart h^*-polynomial is given by the $(n-1)$st Eulerian polynomial and prove that lecture hall cones admit regular, flag, unimodular triangulations. After explicitly describing the Hilbert basis for L_n, we conclude with observations and a conjecture regarding the structure of unimodular triangulations of L_n, including connections between enumerative and algebraic properties of L_n and cones over unit cubes.

Key words. lecture hall, triangulations, generating functions, Eulerian

AMS subject classifications. 05A17, 05A19, 52B11, 13A02, 13H10

DOI. 10.1137/15M1036907

1. Introduction.

For any subset $K \subset \mathbb{R}^n$, we define the *integer point transform* of K to be the formal power series

$$\sigma_K(z_1, \ldots, z_n) := \sum_{m \in K \cap \mathbb{Z}^n} z_1^{m_1} \cdots z_n^{m_n}.$$

Given a pointed rational cone $C := \{x \in \mathbb{R}^n : A\,x \ge 0\}$, it is well known that $\sigma_C(z_1, \ldots, z_n)$ is a rational function [3]. (Here *pointed* requires that C not contain any nontrivial linear subspace of $\mathbb{R}^n$, and *rational* requires that A have rational—or, equivalently, integral—entries.) For $n \in \mathbb{Z}_{\ge 1}$, the *lecture hall cone* is

$$L_n := \left\{\lambda \in \mathbb{R}^n : 0 \le \frac{\lambda_1}{1} \le \frac{\lambda_2}{2} \le \frac{\lambda_3}{3} \le \cdots \le \frac{\lambda_n}{n}\right\}.$$

The elements of $L_n \cap \mathbb{Z}^n$ are called *lecture hall partitions*, and the generating functions $\sigma_{L_n}(z_1, z_2, \ldots, z_n)$ and $\sigma_{L_n}(q, q, \ldots, q)$ have been the subject of active research [2, 4, 5, 6, 7, 12, 13] since the discovery of the following surprising result.

*Received by the editors August 26, 2015; accepted for publication (in revised form) May 13, 2016; published electronically August 2, 2016.
http://www.siam.org/journals/sidma/30-3/M103690.html

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THEOREM 1.1 (see Bousquet-Mélou and Eriksson [4]). *For* $n \geq 1$,

$$\sigma_{L_n}(q, q, \ldots, q) \;=\; \frac{1}{\prod_{j=1}^{n}(1 - q^{2j-1})}\,.$$

Our focus in this note is on the arithmetic-geometric structure of the lecture hall cone, continuing our work in [2]. After giving the necessary background and terminology in section 2, we prove in section 3 that L_n is isomorphic to the cone over the lattice pyramid of a reflexive simplex whose Ehrhart h^*-polynomial is given by the $(n-1)$st Eulerian polynomial. In section 4 we prove that the lecture hall cones admit regular, flag, unimodular triangulations. In section 5 we explicitly describe the Hilbert basis for L_n. We conclude in section 6 with observations and a conjecture regarding the structure of unimodular triangulations of L_n, including connections between enumerative and algebraic properties of L_n and cones over unit cubes.

2. Background. This section contains the necessary terminology and background literature to understand our results; it can be safely skipped by the experts.

2.1. Eulerian polynomials. We recall the *descent statistic* for $\pi \in S_n$,

$$\operatorname{des}(\pi) \;:=\; \left|\{i : 1 \leq i \leq n-1,\ \pi_i > \pi_{i+1}\}\right|,$$

which is encoded in the *Eulerian polynomial*

$$A_n(z) \;:=\; \sum_{\pi \in S_n} z^{\operatorname{des}(\pi)}.$$

An alternative definition of the Eulerian polynomial is via

$$\sum_{t \geq 0}(1+t)^n z^t \;=\; \frac{A_n(z)}{(1-z)^{n+1}}\,.$$

2.2. Lattice polytopes. A *lattice polytope* is the convex hull of finitely many integer vectors in $\mathbb{R}^n$. The *Ehrhart polynomial* of a lattice polytope P is

$$i(P,t) \;:=\; |tP \cap \mathbb{Z}^n|\,,$$

and the *Ehrhart series* of P is $1 + \sum_{t \geq 1} i(P,t)\, z^t$. Ehrhart's theorem [8] asserts that $i(P,t)$ is indeed a polynomial; equivalently, the Ehrhart series evaluates to a rational function of the form

$$1 + \sum_{t \geq 1} i(P,t)\, z^t \;=\; \frac{h^*(z)}{(1-z)^{\dim(P)+1}},$$

where we write $h^*(z) = \sum_{j=0}^{\dim(P)} h_j^*(P)\, z^j$. By a theorem of Stanley [14], the $h_j^*(P)$ are nonnegative integers. The polynomial $h^*(z)$ is called the *Ehrhart h^*-polynomial* of P. Ehrhart polynomials and series have far-reaching applications in combinatorics, number theory, and beyond (see, e.g., [3] for more).

A lattice polytope is *reflexive* if its polar dual is also a lattice polytope. A theorem of Hibi [10] says that P is a lattice translate of a reflexive polytope if and only if the vector $(h_0^*(P), \ldots, h_{\dim(P)}^*(P))$ is symmetric. Reflexive polytopes are of considerable interest in combinatorics, algebraic geometry, and theoretical physics [1].

2.3. Triangulations. A *triangulation* of a lattice polytope P is a collection of simplices that only meet in faces and whose union is P. A triangulation is *unimodular* if all of its maximal simplices are. (A *unimodular* simplex is the convex hull of some $v_0, v_1, \ldots, v_n \in \mathbb{Z}^n$ such that $\{v_1 - v_0, v_2 - v_0, \ldots, v_n - v_0\}$ is a lattice basis of $\mathbb{Z}^n$.) A triangulation of P is *regular* if there is a convex function $P \to \mathbb{R}$ whose domains of linearity are exactly the maximal simplices in the triangulation. A triangulation is *flag* if, viewed as a simplicial complex, its minimal nonfaces are pairs of vertices. (See, e.g., [9, section 1] for more details.)

2.4. Gradings of cones. Let $C \subset \mathbb{R}^n$ be a pointed, rational, n-dimensional cone. A *grading* of C is a vector $a \in \mathbb{Z}^n$ such that $a \cdot p > 0$ for all $p \in C \setminus \{0\}$. With such a grading $a \in \mathbb{Z}^n$, we associate its *Hilbert function*

$$h_C^a(t) \;:=\; |\{m \in C \cap \mathbb{Z}^n : \, a \cdot m = t\}| \,.$$

Since C is pointed, $h_C^a(t) < \infty$ for all $t \in \mathbb{Z}_{\geq 1}$, and thus we can define the *Hilbert series*

$$H_C^a(z) \;:=\; 1 + \sum_{t \geq 1} h_C^a(t)\, z^t.$$

It is naturally connected to the integer transform of C via

$$H_C^a(z) \;=\; \sigma_C(z^{a_1}, \ldots, z^{a_d})\,.$$

For example, $\sigma_{L_n}(q, q, \ldots, q)$ is the Hilbert series of the lecture hall cone L_n with respect to the grading $(1, 1, \ldots, 1)$, and the Ehrhart series of P is the Hilbert series of the *cone over* P,

$$\mathrm{cone}(P) \;:=\; \mathrm{span}_{\mathbb{R}_{\geq 0}}\{(1, p) : \, p \in P\}\,,$$

with respect to the grading $(1, 0, \ldots, 0)$.

3. Eulerian gradings and reflexive simplices. The *lecture hall polytope* was defined in [13] as

$$P_n \;:=\; \{\lambda \in L_n : \, \lambda_n \leq n\}\,.$$

From [7, Corollary 2] we know that $|tP_n \cap \mathbb{Z}^n| = (t+1)^n$, which means that the Ehrhart series of P_n is $\sum_{t \geq 0}(t+1)^n z^t$; so the Eulerian polynomial $A_n(z)$ is the h^*-polynomial of P_n.

We are interested in associating a reflexive polytope with the lecture hall cone L_n. The coefficients of an Eulerian polynomial are symmetric. Note, however, that P_n has dimension n whereas A_n has degree $n-1$, and so P_n is not reflexive. On the other hand, Savage and Schuster [13, Corollary 2(b) and Lemma 1] proved that

$$\sum_{\lambda \in L_n \cap \mathbb{Z}^n} z^{\lceil \frac{\lambda_n}{n} \rceil} \;=\; \frac{A_n(z)}{(1-z)^n}\,,$$

which is almost a Hilbert series (the exponent vector on the left-hand side does not correspond to a grading). Instead we consider the grading $(0, \ldots, 0, -1, 1)$, which defines $\deg(\lambda) := \lambda_n - \lambda_{n-1}$ for $\lambda = (\lambda_1, \ldots, \lambda_n) \in L_n$.

Theorem 3.1. *For all $n \geq 1$,*

$$\sigma_{L_n}(1, 1, \ldots, 1, z^{-1}, z) \;=\; \sum_{\lambda \in L_n \cap \mathbb{Z}^n} z^{\deg(\lambda)} \;=\; \frac{A_{n-1}(z)}{(1-z)^n}\,.$$

Proof. Fix $t \geq 0$. We show that the map $(\lambda_1, \ldots, \lambda_n) \mapsto (\lambda_1, \ldots, \lambda_{n-1})$ is a bijection from the set of elements $\lambda \in L_n$ satisfying $\deg(\lambda) = t$ to $tP_{n-1} \cap \mathbb{Z}^{n-1}$. Observe first that if λ is a lecture hall partition, then $\lambda_n - \lambda_{n-1} = t$ implies

$$\frac{\lambda_{n-1}}{n-1} \;\leq\; \frac{\lambda_n}{n} \;=\; \frac{\lambda_{n-1}+t}{n},$$

which in turn simplifies to

$$\frac{\lambda_{n-1}}{n-1} \;\leq\; t\,.$$

It is immediate that our map is injective, since the preimage of a point in $tP_{n-1} \cap \mathbb{Z}^{n-1}$ is uniquely determined by adding t to λ_{n-1}. To see that the map is surjective, suppose that $(\lambda_1, \ldots, \lambda_{n-1}) \in tP_{n-1} \cap \mathbb{Z}^{n-1}$ and consider the partition

$$(\lambda_1, \ldots, \lambda_{n-1}, \lambda_{n-1} + t)\,.$$

This is an element of L_n of degree t, which is verified by observing that

$$\frac{\lambda_{n-1}+t}{n} \;\geq\; \frac{\lambda_{n-1}}{n} + \frac{1}{n}\frac{\lambda_{n-1}}{(n-1)} \;=\; \frac{(n-1)\lambda_{n-1}+\lambda_{n-1}}{n(n-1)} \;=\; \frac{n\lambda_{n-1}}{n(n-1)} \;=\; \frac{\lambda_{n-1}}{n-1}\,,$$

and hence our map is surjective.

Thus,

$$\sum_{\lambda \in L_n \cap \mathbb{Z}^n} z^{\deg(\lambda)} \;=\; \sum_{t \geq 0} i(P_{n-1}, t)\, z^t \;=\; \frac{A_{n-1}(z)}{(1-z)^n}\,.$$

This completes the proof. ∎

Throughout the remainder of this note we represent a lecture hall partition λ as either the row vector $(\lambda_1, \ldots, \lambda_n)$ or as a column vector

$$\begin{bmatrix} \lambda_n \\ \lambda_{n-1} \\ \vdots \\ \lambda_1 \end{bmatrix}.$$

Theorem 3.1 says that $A_{n-1}(z)$ is the h^*-polynomial of the $(n-1)$-dimensional simplex obtained as the intersection of L_n with the hyperplane $\lambda_n - \lambda_{n-1} = 1$. Because the columns of the matrix below are the minimal ray generators for L_n and they each lie on the hyperplane $\lambda_n - \lambda_{n-1} = 1$, these columns must form the vertices of the intersection, which we denote

$$Q_n \;:=\; \operatorname{conv} \begin{bmatrix} 1 & n & n & \cdots & n \\ 0 & n-1 & n-1 & \cdots & n-1 \\ 0 & 0 & n-2 & \cdots & n-2 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

We will establish two geometric properties of Q_n.

Applying to Q_n the unimodular transformation that takes consecutive row differences produces the polytope

$$R_n := \operatorname{conv}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & n-1 & 1 & \cdots & 1 \\ 0 & 0 & n-2 & \cdots & 1 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Note that R_n is an $(n-1)$-dimensional simplex embedded at height 1 in $\mathbb{R}^n$. The unimodular transformation above shows that L_n is unimodularly equivalent to $\operatorname{cone}(R_n)$—this means that $\operatorname{cone}(R_n)$ is the image of L_n under some mapping in $\mathrm{SL}_n(\mathbb{Z})$—and hence we can freely use either presentation of this cone with respect to lattice-point enumeration.

Among the vertices for R_n, the rightmost column in the defining matrix is the apex of a height-1 lattice pyramid over the convex hull of the remaining $n-1$ columns, which we denote by $\tilde{R}_n$. Theorem 3.1 can be used to show that $\tilde{R}_n$ satisfies the following condition.

THEOREM 3.2. *The polytope $\tilde{R}_n$ is a lattice translate of a reflexive polytope.*

Proof. Theorem 3.1 implies that the h^*-vector of R_n is given by $A_{n-1}(z)$. It is not hard to see that for a height-1 lattice pyramid over a polytope P, the h^*-polynomial of the pyramid is the same as that of P [3, Theorem 2.4]. By the previous theorem of Hibi, since $A_{n-1}(z)$ has symmetric coefficients and is of degree $n-2$, and R_n is a lattice pyramid over the $(n-2)$-dimensional polytope $\tilde{R}_n$, the result follows. ∎

4. Triangulating the lecture hall cone. In this section we prove that the polytopes R_n admit regular, flag, unimodular triangulations. We can represent R_n as the convex hull of

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 0 & n-2 & \cdots & 1 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ n-1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Thus, we see that the intersection of R_n with the hyperplane $\lambda_{n-1} = 1$, which arises as the convex hull of the matrix on the left above, is a subpolytope of R_n that is an embedded copy of R_{n-1}. Thus, R_n is a union of two pyramids, with different heights, over R_{n-1}, pictured in Figure 1.

Given a lattice polytope $S \subset \mathbb{R}^n$, consider two integral linear functionals $\mathbf{l}$ and $\mathbf{u}$, where $\mathbf{l}$ and $\mathbf{u}$ have integer coefficients, such that $\mathbf{l} \le \mathbf{u}$ on S. Define the *chimney polytope*

$$\operatorname{Chim}(S, \mathbf{l}, \mathbf{u}) := \{(y, x) \in \mathbb{R} \times \mathbb{R}^n : x \in S,\ \mathbf{l}(x) \le y \le \mathbf{u}(x)\}.$$

THEOREM 4.1 (see Haase et al., [9]). *If S admits a regular, flag, unimodular triangulation, then so does* $\operatorname{Chim}(S, \mathbf{l}, \mathbf{u})$.

THEOREM 4.2. *For all $n \ge 1$, the polytope R_n admits a regular, flag, unimodular triangulation.*

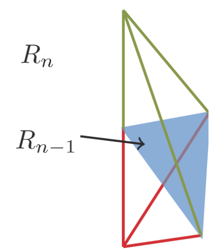

FIG. 1. *R_n is a union of two pyramids over R_{n-1}.*

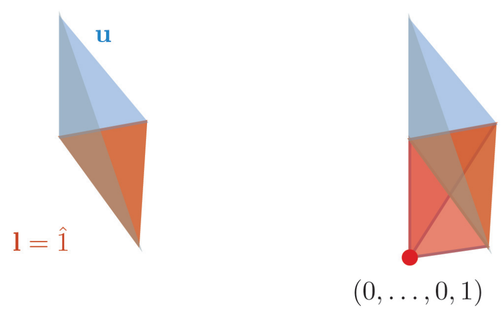

FIG. 2. *The triangulation of the chimney polytope restricts to a regular, flag, unimodular triangulation of the face defined by $\hat{1}$. Taking a unimodular lattice pyramid over this face extends this triangulation to all of R_n.*

Proof. Let $\hat{1}$ denote the constant function with value 1. We first observe that R_n is the union of

$$\operatorname{Chim}\left(R_{n-1}, \hat{1}, (n-1)\lambda_{n-1} - \sum_{i=1}^{n-2} \lambda_i\right)$$

with the height-1 lattice pyramid formed from the convex hull of $(0, \ldots, 0, 1)$ and the face of the chimney polytope above obtained from the lower linear functional $\hat{1}$. To see this, recall from our previous discussion that R_{n-1} arises as the $(\lambda_{n-1} = 1)$-slice of R_n in $\mathbb{R}^n$. The above chimney polytope is the convex hull of the embedded copy of R_{n-1} and the height-$(n-1)$ pyramid point, as is verified by observing that the upper linear functional on R_{n-1} agrees with the λ_{n-1}-value of the vertices of R_n with $(n-1)$st coordinate strictly greater than 0. That the remainder of R_n is contained, as claimed, in the height-1 lattice pyramid is clear from our description of R_n relative to the embedded copy of R_{n-1}. See Figure 2.

Second, we use the fact that R_1 is a unit lattice segment as a base for induction to establish the theorem. Assuming that R_{n-1} admits a regular, flag, unimodular triangulation, Theorem 4.1 implies the same for $\operatorname{Chim}(R_{n-1}, \hat{1}, (n-1)\lambda_{n-1} - \sum_{i=1}^{n-2} \lambda_i)$. This triangulation restricts to a regular, flag, unimodular triangulation of the face defined by the lower linear functional $\hat{1}$, and taking a unimodular lattice pyramid over this face extends this triangulation to all of R_n. ∎

5. Hilbert bases. Given a pointed rational cone $C \subset \mathbb{R}^n$, the integer points $C \cap \mathbb{Z}^n$ form a semigroup. It is known that such a semigroup has a unique minimal generating set called the *Hilbert basis* of C (see, e.g., [11]). In this section we show that the cone L_n has a (we think) interesting Hilbert basis.

To an element

$$A \ = \ \{i_1 < i_2 < \cdots < i_k\} \ \in \ 2^{[n-1]},$$

i.e., a subset $A \subseteq [n-1] := \{1, 2, \ldots, n-1\}$, we associate the vector $v_A \in \mathbb{Z}^n$ defined by

$$v_A \ := \ (0, 0, \ldots, 0, i_1, i_2, \ldots, i_k, i_k + 1),$$

where $v_\emptyset := (0, 0, \ldots, 0, 1)$. The following lemmas will be useful.

LEMMA 5.1. *For integers $0 < i$ and $0 \le k \le i$,*

$$\frac{k}{i} \ \le \ \frac{k+1}{i+1}.$$

Proof. From $0 < i$ and $0 \le k \le i$, we have that

$$\frac{k+1}{i+1} - \frac{k}{i} \ = \ \frac{i(k+1) - k(i+1)}{i(i+1)} \ = \ \frac{i-k}{i(i+1)} \ \ge \ 0.$$

This completes the proof. ∎

LEMMA 5.2. *For positive integers k, ℓ, i, if*

$$\frac{k}{i} \ \le \ \frac{\ell}{i+1},$$

then $k < \ell$.

Proof. Positivity of the variables implies $\ell i \ \ge \ k(i+1) \ = \ ki + k \ > \ ki$. So $\ell > k$. ∎

THEOREM 5.3. *The Hilbert basis for L_n is*

$$\mathcal{H}_n \ := \ \left\{ v_A : \, A \in 2^{[n-1]} \right\}.$$

Remark 5.4. There is a standard bijection between subsets of $n-1$ and compositions (i.e., ordered partitions) of n given by sending $\{i_1 < i_2 < \cdots < i_k\}$ to $(n - i_k) + (i_k - i_{k-1}) + \cdots + (i_2 - i_1) + i_1$. This is precisely the unimodular transformation taking consecutive row differences previously used to convert between Q_n and R_n. For example, the subset $\{2, 4, 5\} \subset [6]$ bijects to the composition $(1, 1, 2, 2)$, which is equivalent to our consecutive row difference transformation sending

$$\begin{bmatrix} 6 \\ 5 \\ 4 \\ 2 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

Proof of Theorem 5.3. $\mathcal{H}_n$ can be characterized as the set of $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ such that $\lambda_n - \lambda_{n-1} = 1$ and for some j, $1 \le j \le n$,

$$0 = \lambda_1 = \cdots = \lambda_{j-1} < \lambda_j < \cdots \lambda_{n-1} < \lambda_n \le n.$$

We first show that $\mathcal{H}_n \subset L_n$. Let $\lambda \in \mathcal{H}_n$. We need to check that $\frac{\lambda_i}{i} \le \frac{\lambda_{i+1}}{i+1}$ for $1 \le i < n$. If $\lambda_i = 0$, the condition holds. If $\lambda_i > 0$, then by Lemma 5.1 and the characterization of $\mathcal{H}_n$,

$$\frac{\lambda_i}{i} \ \le \ \frac{\lambda_i + 1}{i+1} \ \le \ \frac{\lambda_{i+1}}{i+1}.$$

We next show that $\mathcal{H}_n$ contains all degree-1 elements in L_n with respect to the grading $\lambda_n - \lambda_{n-1} = 1$. Let λ be a degree-1 element in L_n. Since $\lambda \in L_n$,

$$-n\,\lambda_{n-1} + (n-1)\lambda_n \;\geq\; 0\,,$$

and substituting $\lambda_n - 1 = \lambda_{n-1}$ gives

$$-n\,(\lambda_n - 1) + (n-1)\lambda_n \;\geq\; 0\,,$$

from which it follows that

$$n \;\geq\; \lambda_n\,.$$

To check that the other coordinates of λ satisfy the constraints of $\mathcal{H}_n$, note that if $\lambda_j > 0$ for $1 \leq j < n-1$, then since, as a lecture hall partition, $\frac{\lambda_j}{j} \leq \frac{\lambda_{j+1}}{j+1}$, we conclude with Lemma 5.2 that $\lambda_j < \lambda_{j+1}$. It is immediate that, since every v_A has degree 1, no v_A is a nonnegative integer combination of other elements in $\mathcal{H}_n$. Hence, our Hilbert basis must contain $\mathcal{H}_n$.

To finish the proof, we show by induction on the degree that all elements in L_n are nonnegative integer combinations of elements of $\mathcal{H}_n$. We have just shown that degree-1 elements of L_n are in $\mathcal{H}_n$. Let $a \in L_n$ have degree $t > 1$. Let j be the largest index such that $a_j < j$. If $j = n-1$ or $j = n$, write $a = b + c$ where $b = (0, \ldots, 0, 1)$ and $c = (a_1, \ldots, a_{n-1}, a_n - 1)$. Then $b \in \mathcal{H}_n$, and c has degree $t-1$, and, by Lemma 5.1, $c \in L_n$.

Otherwise, $j \leq n-2$ and $a = b + c$ where

$$b = (a_1, \ldots, a_j, j+1, \ldots, n)$$

and

$$c = (0, 0, \ldots, 0, a_{j+1} - (j+1), a_{j+2} - (j+2), \ldots, a_n - n).$$

Then $b \in \mathcal{H}_n$ and c has degree $t-1$. It remains to show $c \in L_n$. As $a \in L_n$, we have for $1 \leq i < n$

$$(i-1)a_i - i\,a_{i-1} \;\geq\; 0\,.$$

It follows that $c \in L_n$, since for all $i \geq j$,

$$(i-1)(a_i - i) - i(a_{i-1} - (i-1)) \;=\; (i-1)a_i - i\,a_{i-1} \;\geq\; 0\,. \qquad \square$$

Remark 5.5. We give here an alternative proof that all elements in L_n are nonnegative integer combinations of elements of $\mathcal{H}_n$—we chose to highlight the proof above because we feel it is independently interesting, as it explicitly relies on the number-theoretic structure of L_n. The existence of the unimodular triangulation given by Theorem 4.2 implies that the Hilbert basis for $L_n \cap \mathbb{Z}^n$ consists of precisely the elements of degree 1. From the bijection in the proof of Theorem 3.1, we have that the elements of degree t in L_n are in bijection with the elements of $tP_{n-1} \cap \mathbb{Z}^{n-1}$. Since $|tP_n \cap \mathbb{Z}^n| = (t+1)^n$, we have $|\{\lambda \in L_n : \deg(\lambda) = t\}| = (t+1)^{n-1}$. Thus, the Hilbert basis $\mathcal{H}_n$ contains (at most) 2^{n-1} elements, since $|\{\lambda \in L_n : \deg(\lambda) = 1\}| = 2^{n-1}$. As we have identified 2^{n-1} such elements, these must constitute all of $\mathcal{H}_n$.

6. Triangulations and unit cubes. In this section we briefly discuss consequences of the existence of regular, flag, unimodular triangulations of Q_n. It is known [14] that for a lattice polytope P with a unimodular triangulation, the triangulation has $\sum_{i=0}^{\dim(P)} h_i^*(P)$ maximal unimodular simplices. Hence, Theorem 3.1

implies that any unimodular triangulation of Q_n has $(n-1)!$ maximal unimodular simplices.

There is another well-known polytope having h^*-polynomial given by $A_{n-1}(z)$ and admitting regular, flag, unimodular triangulations, namely the cube $[0,1]^{n-1}$. An interesting connection between L_n and $\operatorname{cone}([0,1]^{n-1})$ is that the integer points in both of them are generated over $\mathbb{Z}_{\geq 0}$ by degree-1 elements in bijection with subsets of $[n-1]$. Namely, we can encode each subset

$$A \;=\; \{i_1 < i_2 < \cdots < i_k\} \;\subseteq\; [n-1]$$

as a vector in $\mathbb{Z}^n$ in two ways. First, let $(1, m^A) := (1, m_1^A, \ldots, m_{n-1}^A)$ be defined by $m_i^A = 1$ if $i \in A$, and $m_i^A = 0$ otherwise. Second, let $v_A \in \mathbb{Z}^n$ be defined as before. Informally, for $(1, m^A)$ we encode A by a characteristic vector, while for v_A we encode A directly by placing the elements of A in increasing order, making the last entry one more than $\max(A)$, and padding the leftmost entries with zeros as needed. Note that, by definition, $(1, m^A) \in \operatorname{cone}([0,1]^{n-1})$ and $v_A \in L_n$ each have degree 1 in their respective gradings. Thus, both L_n and $\operatorname{cone}([0,1]^{n-1})$ can be viewed as cones generated by the subsets of $[n-1]$. While their geometric and arithmetic structure are quite different, our results and Conjecture 6.1 below indicate that these cones share surprising properties.

An important regular, flag, unimodular triangulation of $[0,1]^{n-1}$ is obtained by intersecting $[0,1]^{n-1}$ with the real braid arrangement, i.e., the set of hyperplanes defined by $x_i = x_j$ for all $i \neq j$ between 1 and $n-1$. Recall that a *Sperner* 2*-pair* of $[n-1]$ is a pair of subsets $A, B \subset [n-1]$ such that neither A nor B is contained in each other. One important property of the braid triangulation of $[0,1]^{n-1}$ is that the number of minimal nonedges equals the number of Sperner 2-pairs of $[n-1]$. (All of these properties are most easily seen by observing that $[0,1]^{n-1}$ is the order polytope [15] for an $(n-1)$-element antichain; hence the maximal unimodular simplices in this triangulation correspond to linear extensions of the antichain, i.e., chains in the Boolean algebra $2^{[n-1]}$.)

Based on experimental evidence for $n \leq 7$, and the established connections between L_n and $\operatorname{cone}([0,1]^{n-1})$, the following conjecture seems plausible.

Conjecture 6.1. *There exists a regular, flag, unimodular triangulation of R_n that admits a shelling order such that the maximal simplices of the triangulation are indexed by $\pi \in S_{n-1}$ and each such simplex is attached along $\operatorname{des}(\pi)$ many of its facets. Further, the number of minimal nonedges in the triangulation is the number of Sperner* 2*-pairs of* $[n-1]$.

Conjecture 6.1 has thus far proven to be a real challenge. One might hope that a "deformed" version of the braid triangulation for $[0,1]^{n-1}$ could be imposed on R_n, though so far such a triangulation has proven to be elusive.

The second part of Conjecture 6.1 can be reformulated in the language of toric algebra using Sturmfels' correspondence between regular, flag, unimodular triangulations and quadratic, squarefree Gröbner bases for toric ideals [11, 16]. Letting I_n denote the toric ideal defining the semigroup algebra $\mathbb{C}[L_n \cap \mathbb{Z}^n]$, Theorem 4.2 implies that there exists a term order for which I_n has a quadratic, squarefree Gröbner basis. The second part of Conjecture 6.1 is equivalent to the statement that there exists a reduced Gröbner basis for I_n containing the same number of elements as Sperner 2-pairs of $[n-1]$. When considering randomly tested term orders, with surprising frequency we have observed squarefree quadratic initial ideals with the desired number of Gröbner basis elements.

Acknowledgments. Thanks to Christian Haase for directing our attention to the existence of chimney polytopes and their triangulations, and to the anonymous referees for their helpful suggestions. The authors thank the American Institute of Mathematics for their support of our SQuaRE working group *Polyhedral Geometry and Partition Theory*.

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