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Partition Analysis via Polyhedral Geometry

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Linear Diophantine systems appear in many different areas of mathematics (from number theory and combinatorics to optimization) and computer science (i.e. compiler theory). Mathematicians are concerned with their solution for more than 2 millenia, since the time of Diophantus. Partition analysis is a general methodology for the treatment of linear Diophantine systems. The $\Omega_{\geq}$ operator, the central tool of partition analysis introduced by MacMahon 100 years ago, has received attention in recent years. There have been various theoretical improvements of algorithms implementing the $\Omega_{\geq}$ operator. Andrews, Paule and Riese, with the Omega package[2], gave a completely algorithmic implementation of the $\Omega_{\geq}$ operator powered by symbolic computation. At the same time, significant progress have been made in the geometric theory of lattice point enumeration. In this work, we connect these two branches of research: We give a new algorithm for computing the $\Omega_{\geq}$ operator using geometric methods.

The problem under consideration is "Given $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$ find all $x \in \mathbb{N}^n$ satisfying $Ax \geq b$". We will illustrate through an example how partition analysis works. The example of a linear inhomogeneous Diophantine inequality is essentially the building block of the algorithm.

Example 1. *Let* $A = \left[\begin{array}{ccc} 2 & 3 & -5 \end{array}\right]$ *and* $b = \left[\begin{array}{c} 4 \end{array}\right]$*. Find all* $x \in \mathbb{N}^3$ *satisfying* $Ax \geq b$*.*

Our goal is to obtain the generating function $g(S) = \sum_{\mathbf{s}\in S} \mathbf{z}^{\mathbf{s}}$ (multi-index notation) of the solution set S. A reformulation of the problem using formal power series is $g(S) = \sum_{\substack{x,\,y,\,z\,\in\,\mathbb{N} \\ 2x\,+\,3y\,-\,5z\,\geq\,4}} z_1^x z_2^y z_3^z$. Introducing an extra variable to encode the inequality we have $\sum_{x,y,z\in\mathbb{N}} \lambda^{2x+3y-5z-4} z_1^x z_2^y z_3^z$ requiring the exponent of λ to be non-negative in the monomials that appear in the generating function of the solution set. We take the definition of the $\Omega_{\geq}$ operator from [2]:

Definition 1. *The* $\Omega_{\geq}$ *operator is defined on functions with absolutely convergent multisum expansions in an open neighborhood of the complex circles* $|\lambda_i| = 1$ *and the action is given by*

$$\Omega_{\geq} \sum_{s_1=-\infty}^{\infty} \sum_{s_2=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,s_2,\ldots,s_r} \lambda_1^{s_1} \lambda_2^{s_2} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \sum_{s_2=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,s_2,\ldots,s_r}$$

Based on the geometric series expansion formula $(1-z)^{-1} = \sum_{x\geq 0} z^x$ and the definition of $\Omega_{\geq}$ we transform the series into a rational function: $g(S) = \Omega_{\geq} \sum_{x,y,z\in\mathbb{N}} \lambda^{2x+3y-5z-4} z_1^x z_2^y z_3^z = \Omega_{\geq} \frac{\lambda^{-4}}{(1-z_1\lambda^2)(1-z_2\lambda^3)(1-z_3\lambda^{-5})}$

The last expression is the crude generating function for Example 1. If we can evaluate $\Omega_{\geq}$ on such an expression then we recursively eliminate λ's in order to solve linear Diophantine systems.

In order to employ polyhedral geometry, we need to translate the problem to a problem about cones. Let A_i denote the i-th column of A. Let $v_i = [e_i : A_i]^t \in \mathbb{Z}^n$ and $V = (v_1, \ldots, v_n) \in \mathbb{Z}^{(n+1)\times n}$, using multi-index notation and denoting concatenation by colon. Although there is a geometric way to eliminate multiple λ at once, for simplicity and presentation clarity, in what follows we present an algorithm to eliminate a single λ assuming $b > 0$. This is the building block of all traditional Partition Analysis implementations. Define the cone $C := \mathbb{R}_+(v_1, \ldots, v_n)$ spanned by the v_i, and the set $H_b = \{(x_1, \ldots, x_n, x_{n+1}) \mid b \leq x_{n+1}\}$ of vectors with last component greater than b. Since we have one λ, the column A_i is an integer denoted by a_i. Then the rational function $\frac{\lambda^{-b}}{(1-z_1\lambda^{a_1})\cdot\ldots\cdot(1-z_n\lambda^{a_n})}$ is the generating function of the set of integer points in the cone $C - b$. Computing $\Omega_{\geq} \frac{\lambda^{-b}}{(1-z_1\lambda^{a_1})\cdot\ldots\cdot(1-z_n\lambda^{a_n})}$ now amounts to computing the generating function $g(\pi(C \cap H_b); z_1, \ldots, z_{n+1})$, where π denotes projection wrt the last coordinate, as follows:

1. Compute a vertex description of the tangent cones K_v at the vertices of $C \cap H_b$.
2. Compute the generating functions $g(K_v, z_1, z_2, \ldots, z_{n+1})$, either using Barvinok's algorithm [1] or by explicit formulas using modular arithmetic (similar to the expressions in [2]).
3. Substitute $z_{n+1} \mapsto 1$ o obtain λ-free generating functions (projection of the polyhedron).
4. Sum all the projected generating functions for the tanget cones. By Brion's theorem[1], this yields the desired generating function $\Omega_{\geq} \frac{\lambda^{-b}}{(1-z_1\lambda^{a_1})\cdot\ldots\cdot(1-z_n\lambda^{a_n})}$.

To illustrate the algorithm we present it with an example. Given an inequality $2x_1 + 3x_2 - 5x_3 \geq 4$, we want to compute $\Omega_{\geq} \frac{\lambda^{-4}}{(1-z_1\lambda^2)(1-z_2\lambda^3)(1-z_3\lambda^{-5})}$.

Our goal is to apply Brion's theorem to the polyhedron $P := C \cap H_b$. Any polyhedron is the Minkowski sum of a cone and a polytope. In our case $P = \mathrm{conv}(0, u_i : i \in \{1, \ldots, l\}) + \mathbb{R}_+\{w_{i,j} : i \in \{1, \ldots, l\}, j \in \{l+1, \ldots, n\}\}$ where $u_i = \frac{b}{a_i} v_i = (\ldots \frac{b}{a_i} \ldots b)$ and $w_{i,j} = -a_j v_i + a_i v_j = (\ldots -a_j \ldots a_i \ldots 0)$.

The vertices of P, denoted by u_i, are the intersection points of the hyperplane at height $x_{n+1} = b$ with the generators of P that are pointing "up" ($a_i \geq 0$). The $w_{i,j}$ are positive linear combinations of the generators v_i that are pointing "up" and the generators v_j that are pointing "down" ($a_j < 0$) such that $w_{i,j}$ has last coordinate equal to zero.

$$v_1 = (1, 0, 0, 2) \qquad v_2 = (0, 1, 0, 3) \qquad v_3 = (0, 0, 1, -5) \qquad V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & -5 \end{bmatrix}$$

$$C = \mathbb{R}_+\{v_1, v_2, v_3\}$$
$$H_4 = \{(z_1, z_2, z_3, z_4) \in \mathbb{R}^4 : z_4 \geq 4\}$$

$$u_1 = \tfrac{4}{2} v_1 = (2, 0, 0, 4) \qquad w_{1,3} = (-5, 0, 2, 0)$$
$$u_1 = \tfrac{4}{3} v_2 = (0, \tfrac{4}{3}, 0, 4) \qquad w_{2,3} = (0, -5, 3, 0)$$

$$K_{u_1} = \mathbb{R}_+\{(-2, 0, 0, -4), (-2, \tfrac{4}{3}, 0, 0), (-5, 0, 2, 0)\}$$
$$K_{u_2} = \mathbb{R}_+\{(0, -\tfrac{4}{3}, 0, -4), (2, -\tfrac{4}{3}, 0, 0), (0, -5, 3, 0)\}$$

The generators of the tangent cone K_{u_i} are $-u_i$, $u_{i'} - u_i$ for $i' \in \{1, \ldots, \ell\}, i' \neq i$ and $w_{i,j}$ for $j \in \{\ell+1, \ldots, n\}$. These n generators are linearly independent. Thus, K_{u_i} is a simplicial cone. We denote by U_i the matrix that has as columns the vectors $c_i u_i$, $\mathrm{lcm}(c_i, c_{i'}) \cdot (u_{i'} - u_i)$ for $i' \in \{1, \ldots, \ell\} \setminus \{i\}$ and $w_{i,j}$ for $j \in \{\ell+1, \ldots, n\}$. U_i is an $(n+1) \times n$ integer matrix that has a full-rank diagonal submatrix. The generating functions of the lattice points in the projected cones generated by the columns of U_1 and U_2 are $\frac{z_1^4\left(z_1+z_2+z_1^4z_3+z_1^2z_2z_3\right)}{(1-z_1)\left(z_1^3-z_2^2\right)\left(1-z_1^5z_3^2\right)}$ and $-\frac{z_2^2\left(z_1^2+z_1z_2+z_2^2+z_1^2z_2^2z_3+z_2^3z_3+z_1z_2^3z_3+z_1z_2^4z_3^2+z_1^2z_2^4z_3^2+z_2^5z_3^2\right)}{(1-z_2)\left(z_1^3-z_2^2\right)\left(1-z_2^5z_3^3\right)}$.
By Brion their sum is the generating function for $\pi(C \cap H_4)$, which is equal to $\Omega_{\geq} \frac{\lambda^{-4}}{(1-z_1\lambda^2)(1-z_2\lambda^3)(1-z_3\lambda^{-5})}$.

References

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