# Data Structures – Week #8

## Heaps (Priority Queues)

# Outline

- Motivation for Heaps
- Implementation Alternatives of PQs
- Binary Heaps
- Basic Heap Operations (Insert, Delete*Min*)
- Other Heap Operation
  - BuildHeap, DecreaseKey, IncreaseKey, Delete
- d-Heaps
- Leftist Heaps
- Binomial Heaps

# Motivation for Heaps

- Priority queues are queues in which jobs with *different priorities* are enqueued and handled accordingly.
- Heaps are data structures that are used to implement priority queues.
- Heaps can be represented in an array since a complete binary tree is very regular.
- Two basic operations
  - Insert (average O(1), worst case O(log n)), and
  - Delete*Min* (O(log n)).

# Implementation Issues

| Implementation | Insertion | DeleteMin |
|---|---|---|
| Array (Cmplt BT) | O(1) | O(log(n)) |
| Linked List | O(1) | O(n) |
| Linked List (sorted) | O(n) | O(1) |
| BST | O(log(n)) | O(log(n)) |

# Keys in Heaps

- In the scope of this class (CSE 225),
  - a *key* in a heap is the priority value of the corresponding node which determines the position of its node in the heap;
  - i.e., in a ***min-heap*** the node with the minimum key is the node with the highest priority, hence it is the root.

# Binary Heaps

- A binary heap is
  - a completely filled binary tree with the possible exception of the bottom level, (known as a *complete binary tree*)
  - filled from *left to right*
  - with two properties:
    - *Structure property*
    - *Heap order property*

# Structure Property

If a complete binary tree is represented in an array, then for *any element* in array position *i*, the *left child* is in position *2i* and the *right child* in *2i+1* iff they exist (i.e., $2i < n$ and $2i+1 < n$, respectively).

# (Min)Heap Order Property

In a heap, *for every node X, the key in the parent of X is smaller than (or equal to) the key in X*, with the exception of the root since it has no parent. (Depending on the application, the opposite of this may be considered as the heap order property, too!!!)

# Insertion

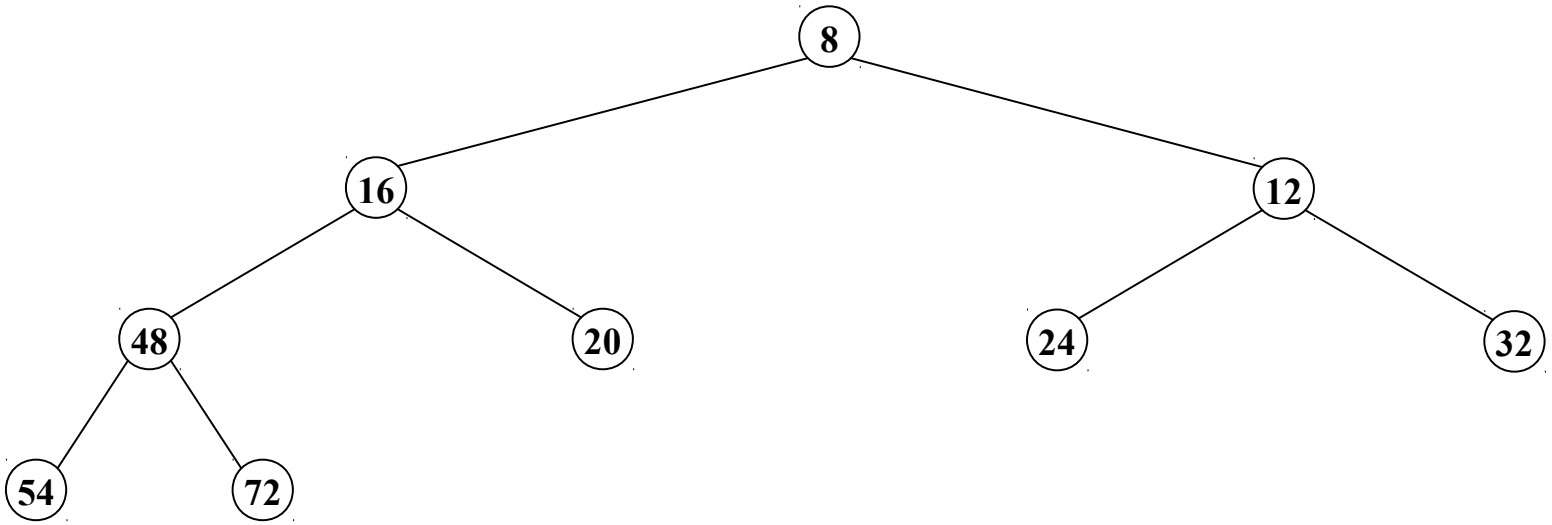
- **Steps of Insertion operation**
  - Create a hole in the next available location;
  - If heap order property is not violated
    - Then we are done;
    - Else
      - loop
        - » exchange the hole with the parent node
      - until the heap property is restored (i.e., *percolate the hole up*)
- Worst time best upper bound: O(log(n))

# Insert Function

```
void insert(ElmntType x, PrQ h)
{   // array starts at cell #1 not #0
    int i;
    if (isFull(h)){
       display("queue full: unsuccessful insertion");
       return;
    }
    for (i = ++h->Size; h->elements[i/2] > x; i/=2 )
       h->elements[i] = h->elements[i/2];
    h->elements[i] = x;
}
```

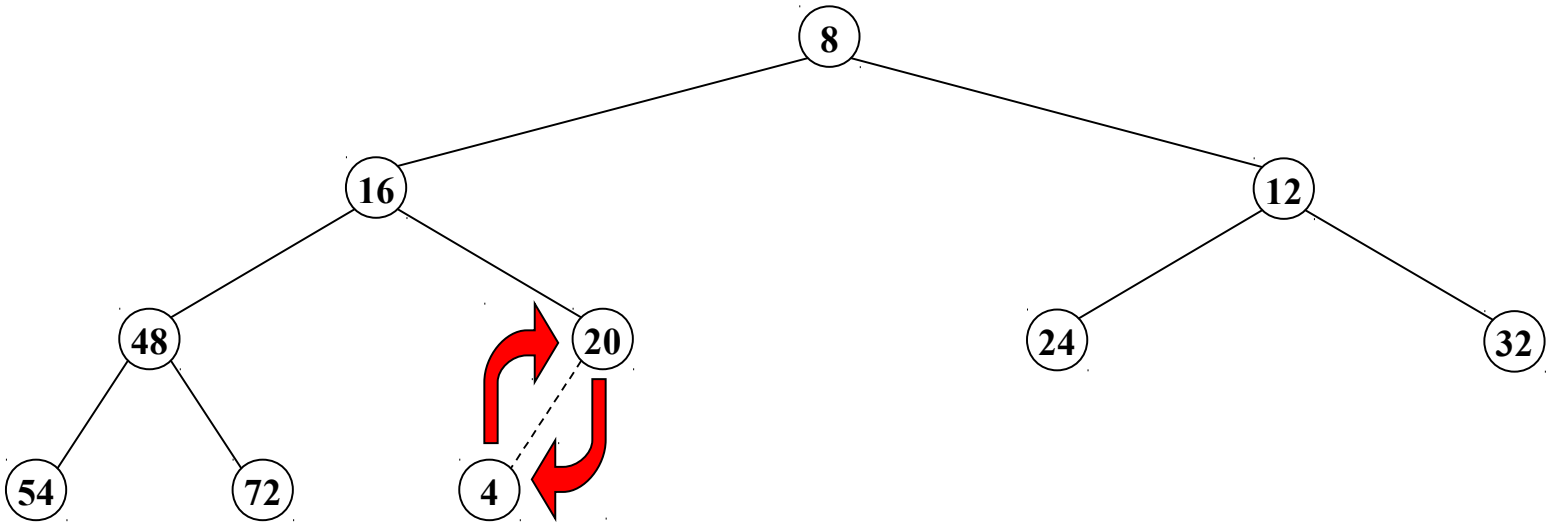
# Animated Insertion Example – Current Heap

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 |
|---|---|---|---|---|---|---|---|---|



# Inserting 4…

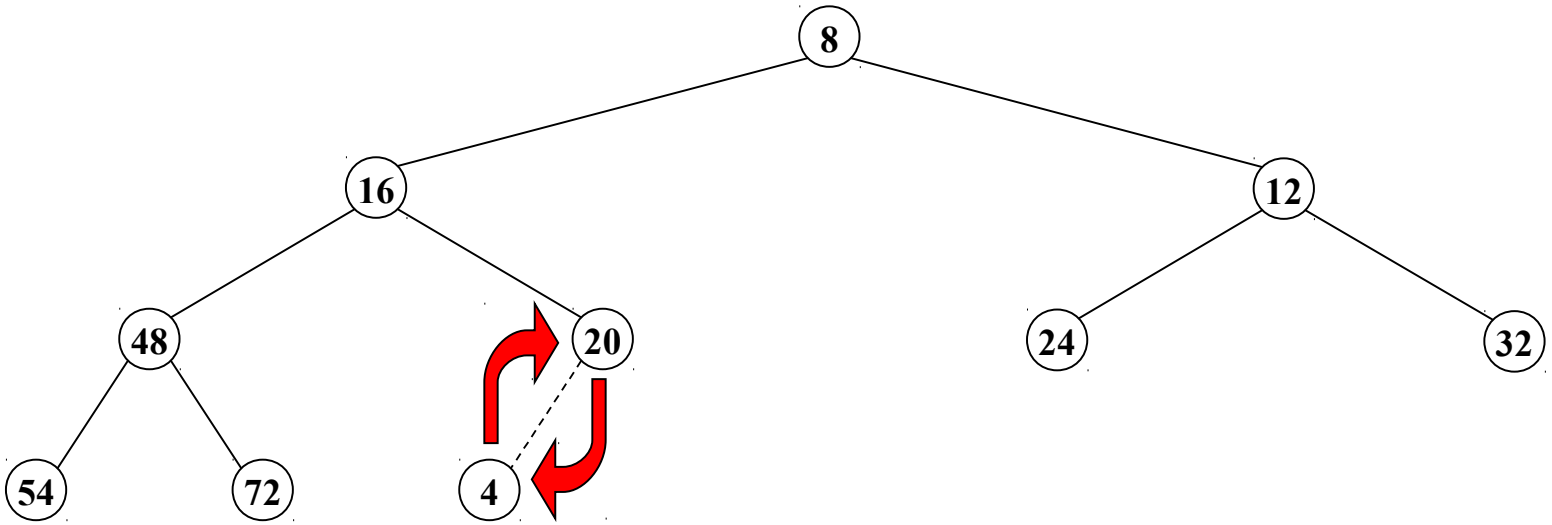
| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 |
|---|---|---|---|---|---|---|---|---|---|




**4 < 20**

# Inserting 4…

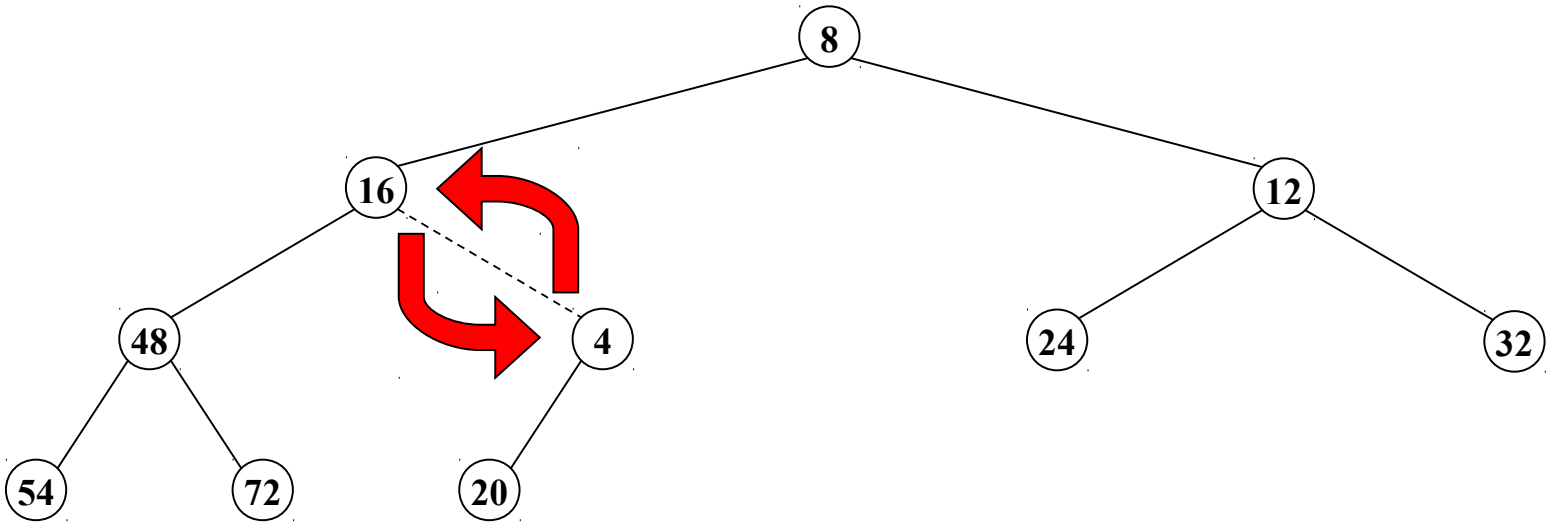
| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 |
|---|---|---|---|---|---|---|---|---|---|




**4 < 20**

# Inserting 4…

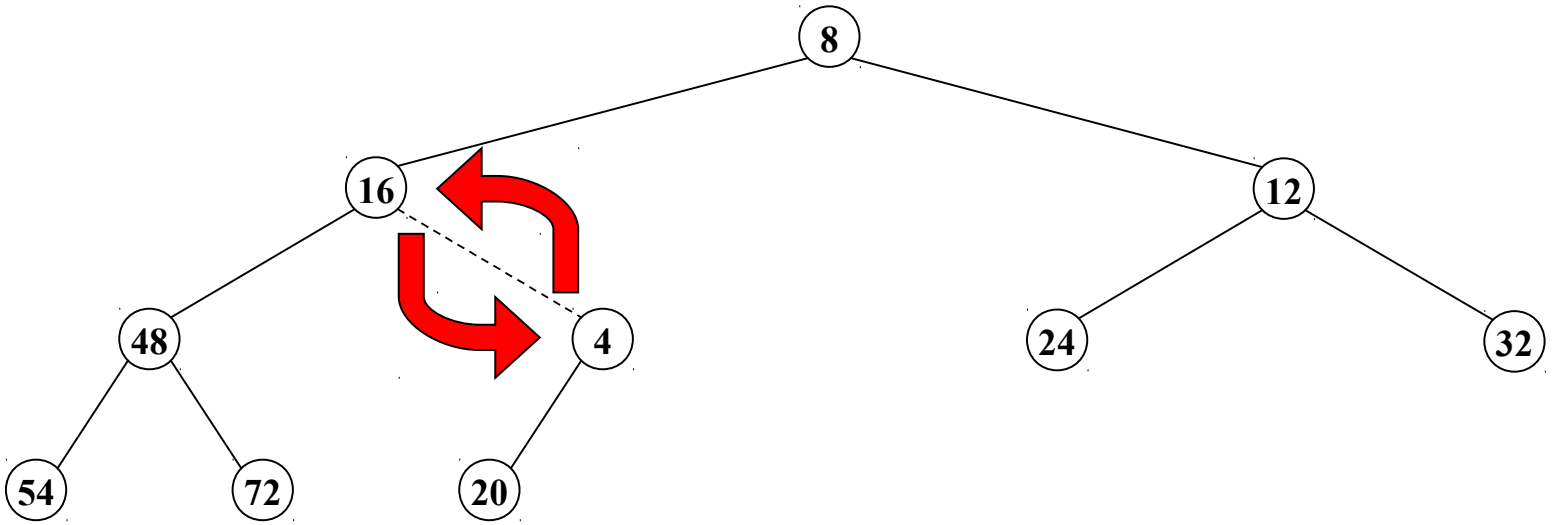
| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 |
|---|---|---|---|---|---|---|---|---|---|




**4 < 16**

# Inserting 4…

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 |
|---|---|---|---|---|---|---|---|---|---|




**4 < 16**

# Inserting 4…

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 |
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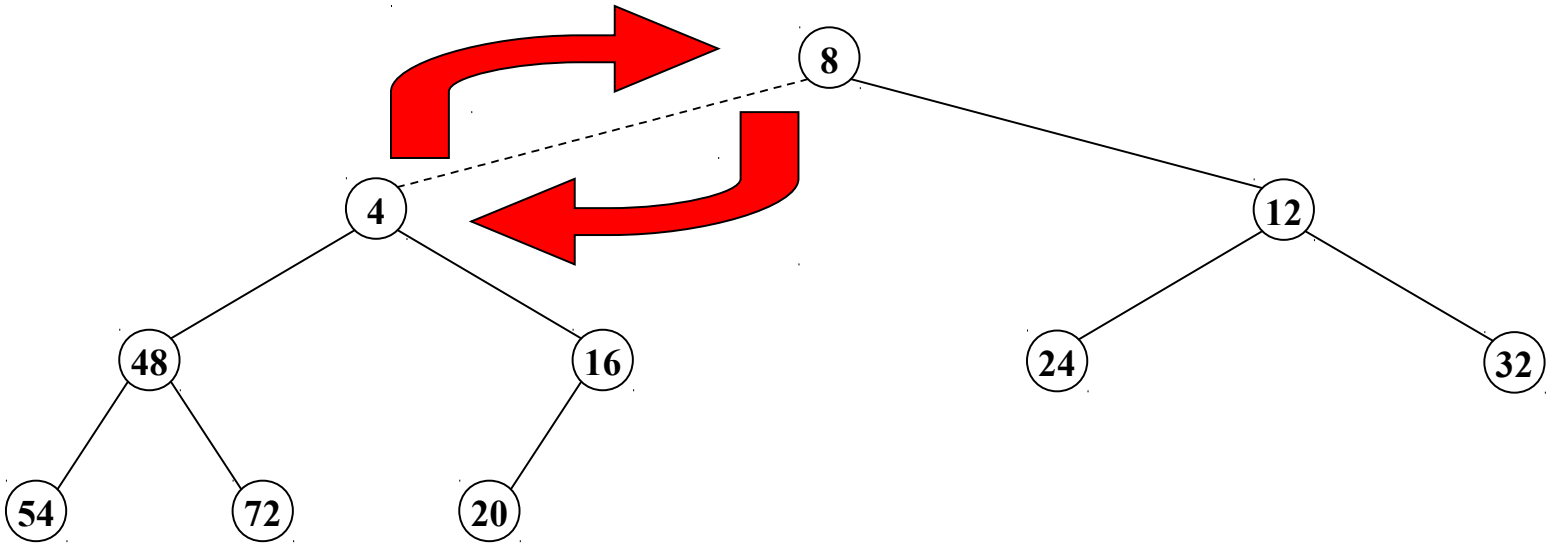
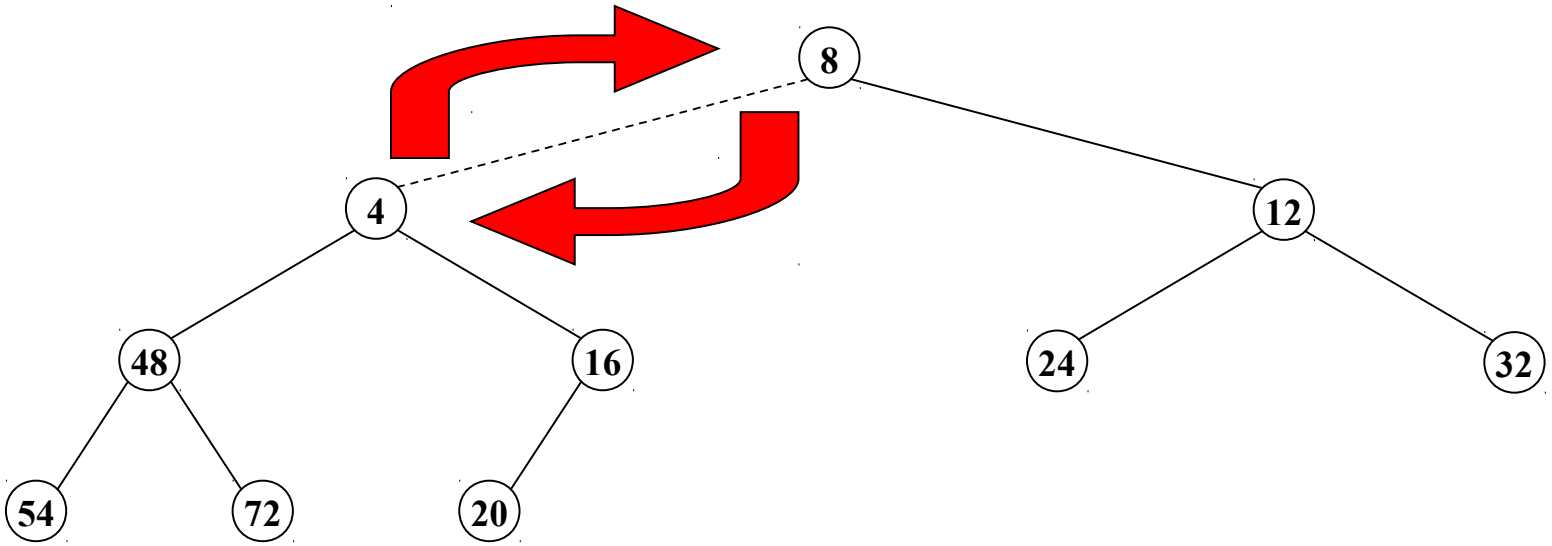



**4 < 8**

# Inserting 4…

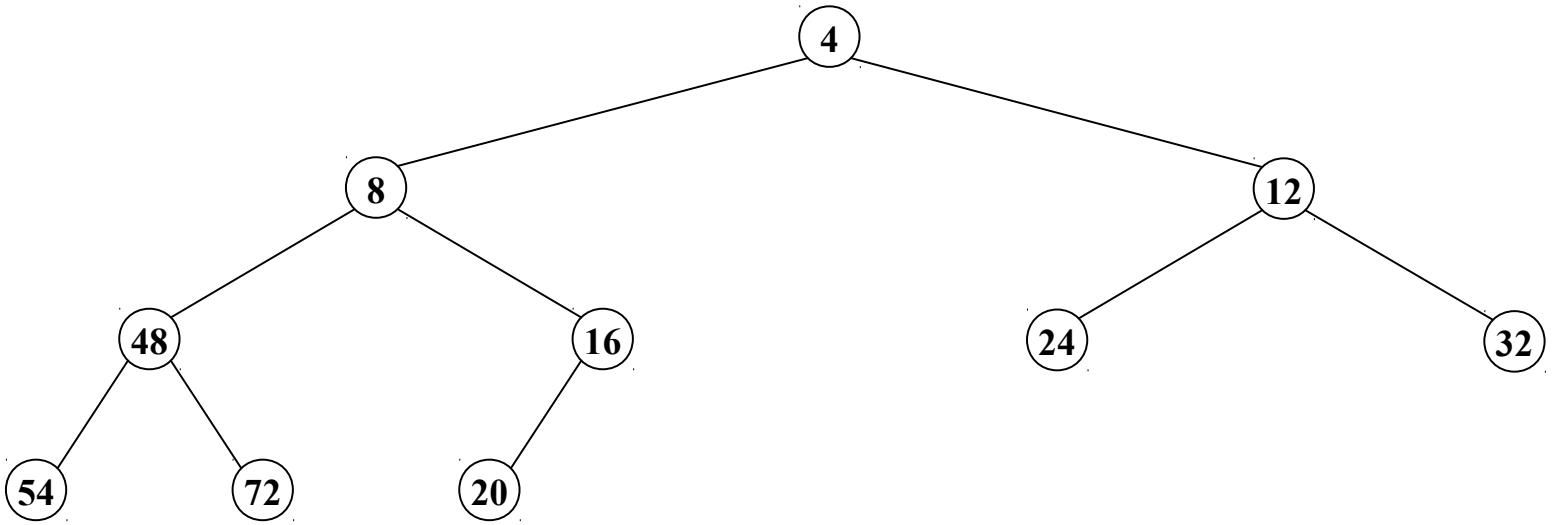
| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 |
|---|---|---|---|---|---|---|---|---|---|




**4 < 8**

# Inserting 4…

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 |
|---|---|---|---|---|---|---|---|---|---|



# Delete*Min Operation*

- **Steps of DeleteMin operation**
  - Remove the minimum element (at the root) from the heap;
  - If the last element can be placed in the hole
    - Then we are done;
    - Else
      - Loop
        - » exchange the hole with the smaller child node
      - until the last element moves in the heap (i.e., *percolate the hole down*).

# Delete*Min Operation*

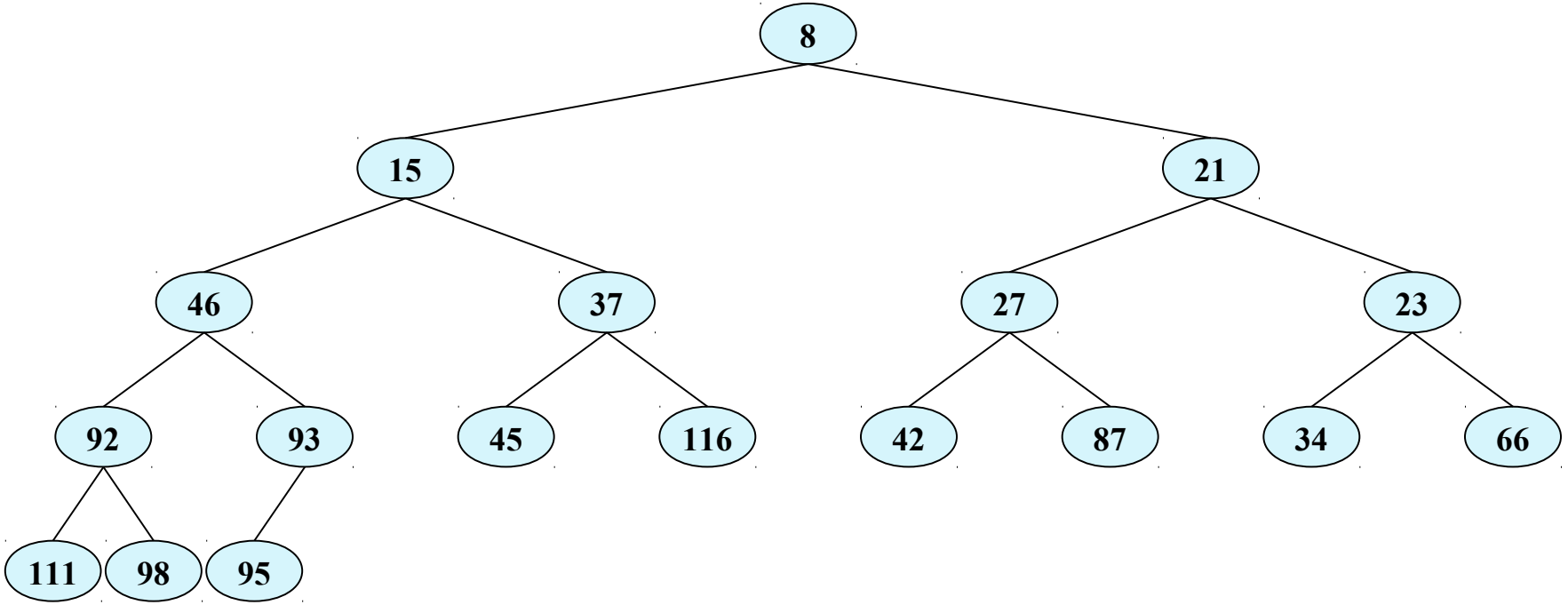
```
ElmntType DeleteMin(PrQ h)
{
    int i, chld;
    ElmntType minelm, lastelm;
    if isEmpty(h){
            display("queue empty")
            return (h->elements[0]);
    }
    minelm=h->elements[1];
    lastelm=h->elements[h->size--];
```

*... Cont'd at the next page!*

# DeleteMin Operation... (cont'd)

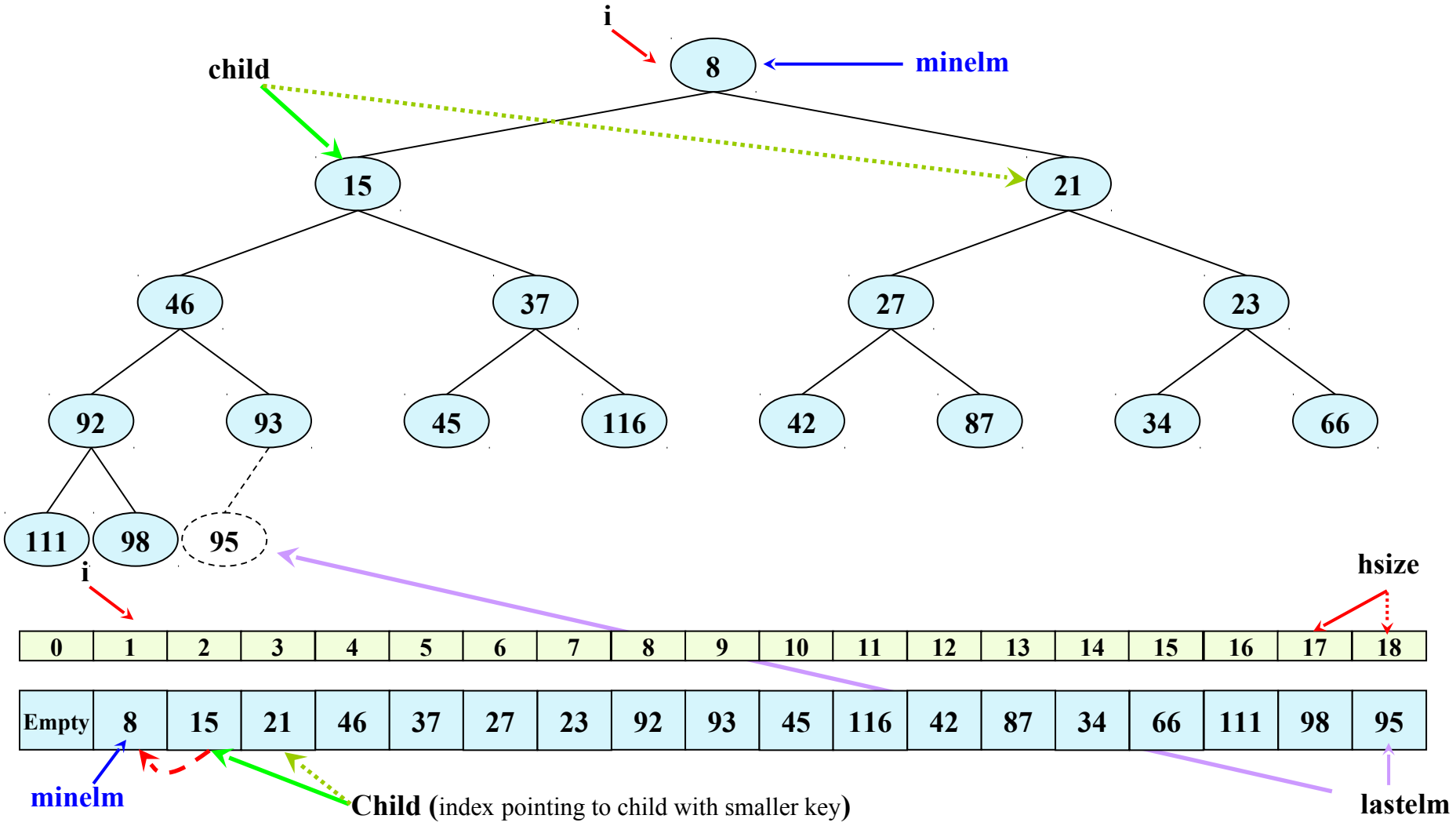
```
for (i=1; i * 2 <= h->size; i=chld) {
   // find smaller child
   chld=i*2;
   if (chld != h->size && h->elements[chld+1] < h->elements[chld]) chld++;
   // percolate one level
   if ( lastelm > h->elements[chld] )
      h->elements[i] = h->elements[chld];
   else break;
}
h->elements[i] =lastelm;
//restore min-heap property in case it is violated by placing lastelm to heap's ith node
for (j = i; h->elements[j/2] > lastelm; j/=2 )
   h->elements[j] = h->elements[j/2];
h->elements[j] = lastelm;
return minelm;
}
```

# DeleteMin Operation




| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Empty | 8 | 15 | 21 | 46 | 37 | 27 | 23 | 92 | 106 | 45 | 116 | 42 | 87 | 34 | 66 | 111 | 98 | 95 |

# *Removing 8 ...*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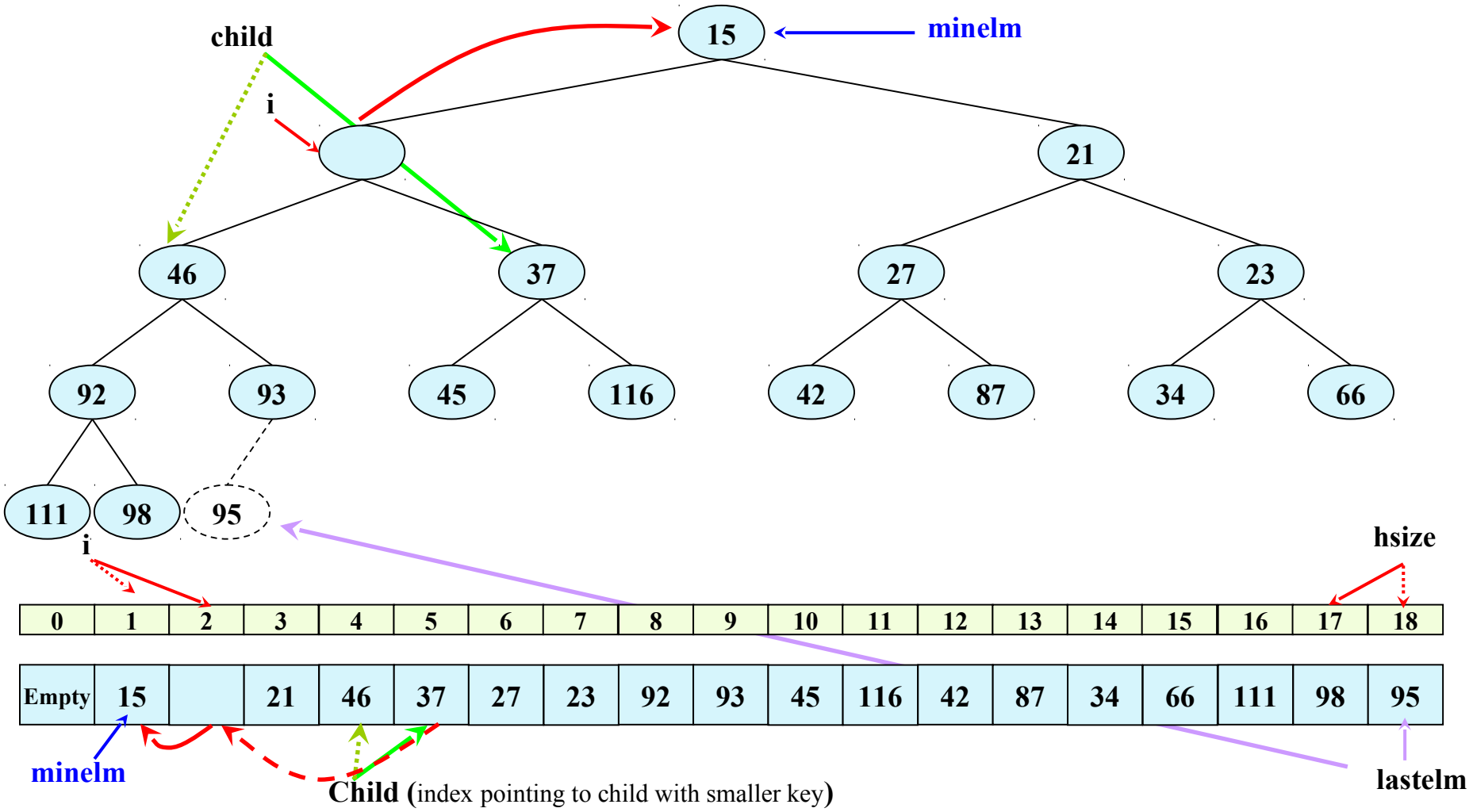
Empty 8 15 21 46 37 27 23 92 93 45 116 42 87 34 66 111 98 95
minelm
Child (index pointing to child with smaller key)
lastelm

# *Removing 8 ...*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minelm
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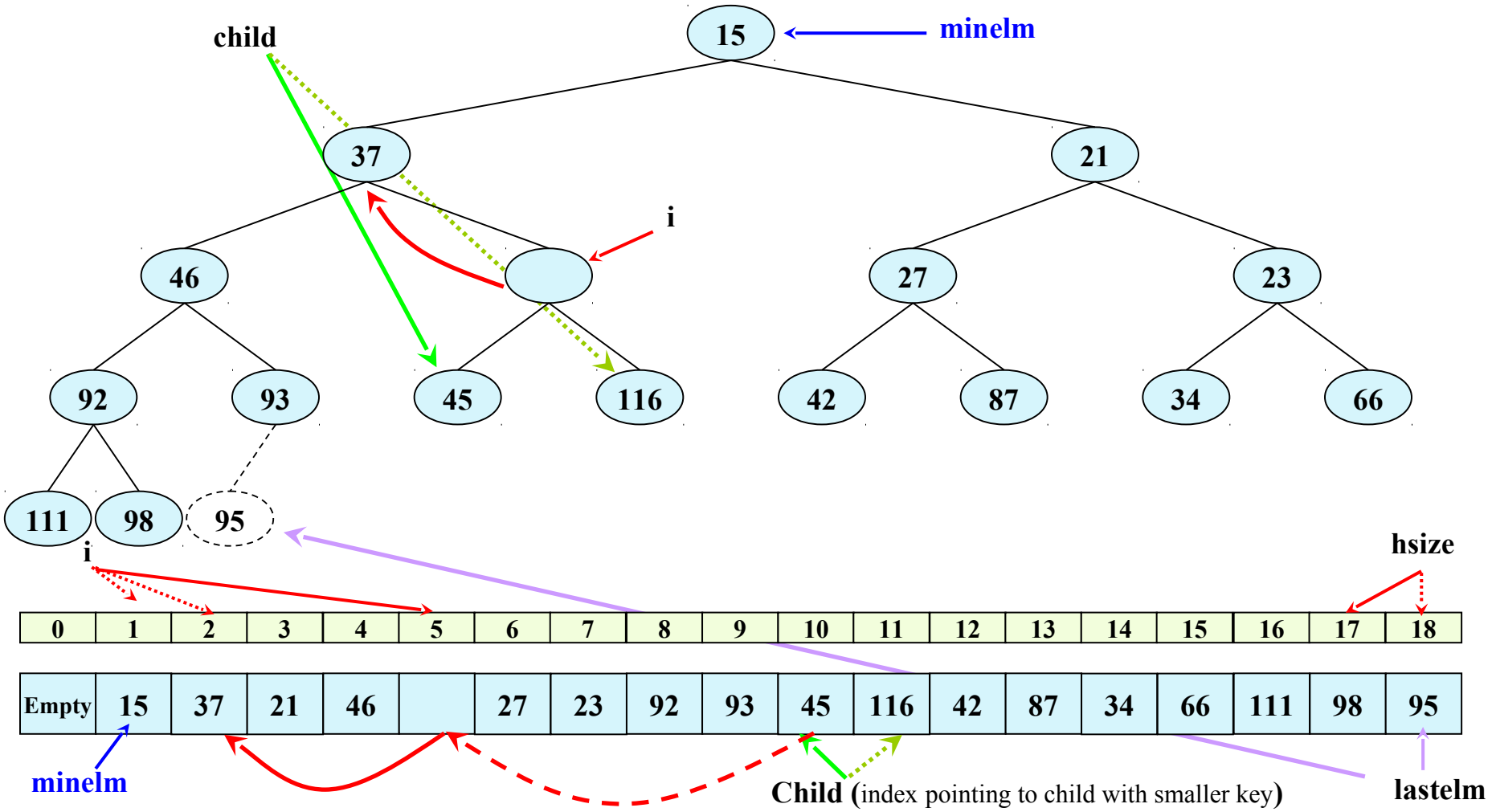
# *Removing 8 ...*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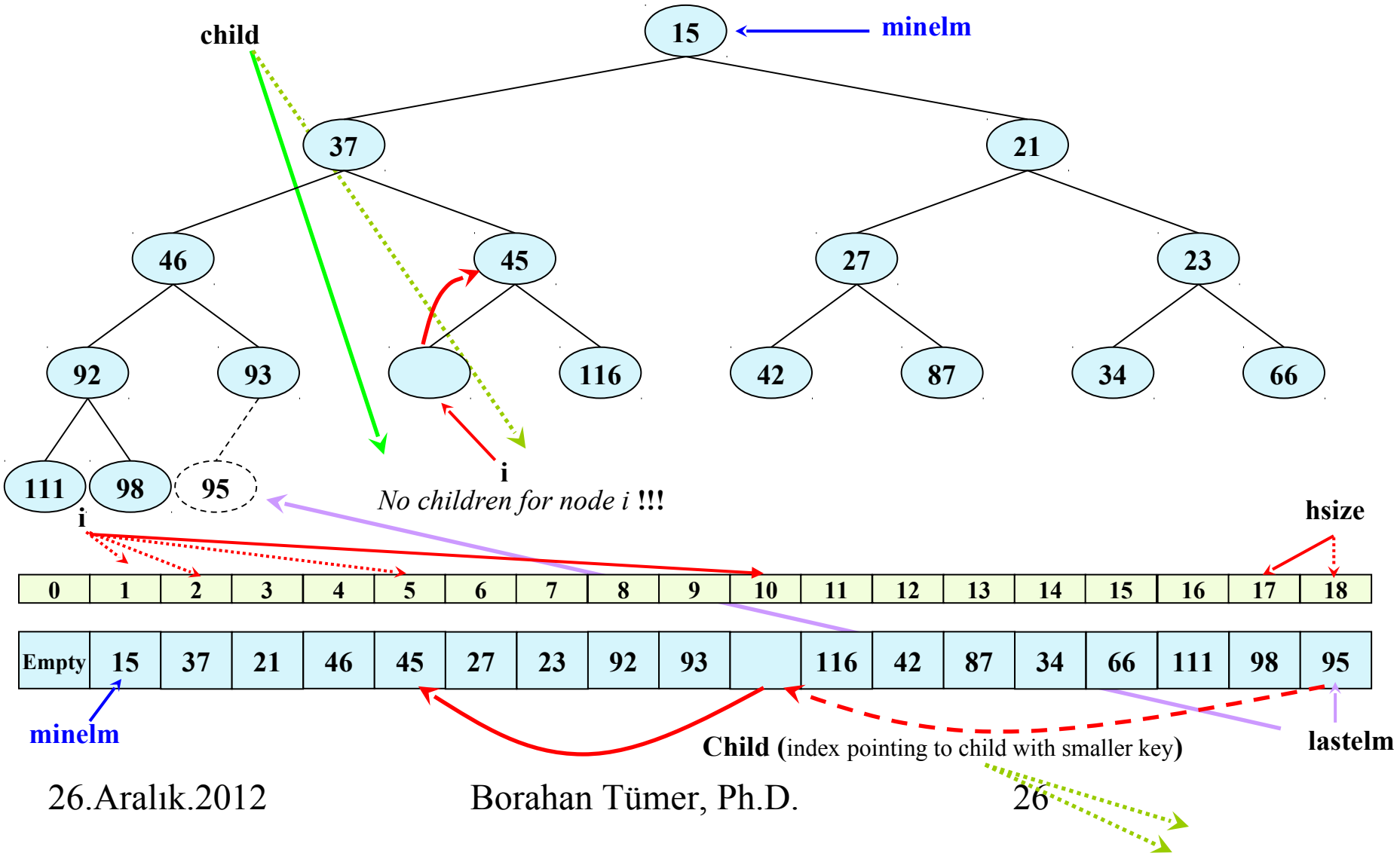
minelm
Child (index pointing to child with smaller key)
lastelm

# *Removing 8 ...*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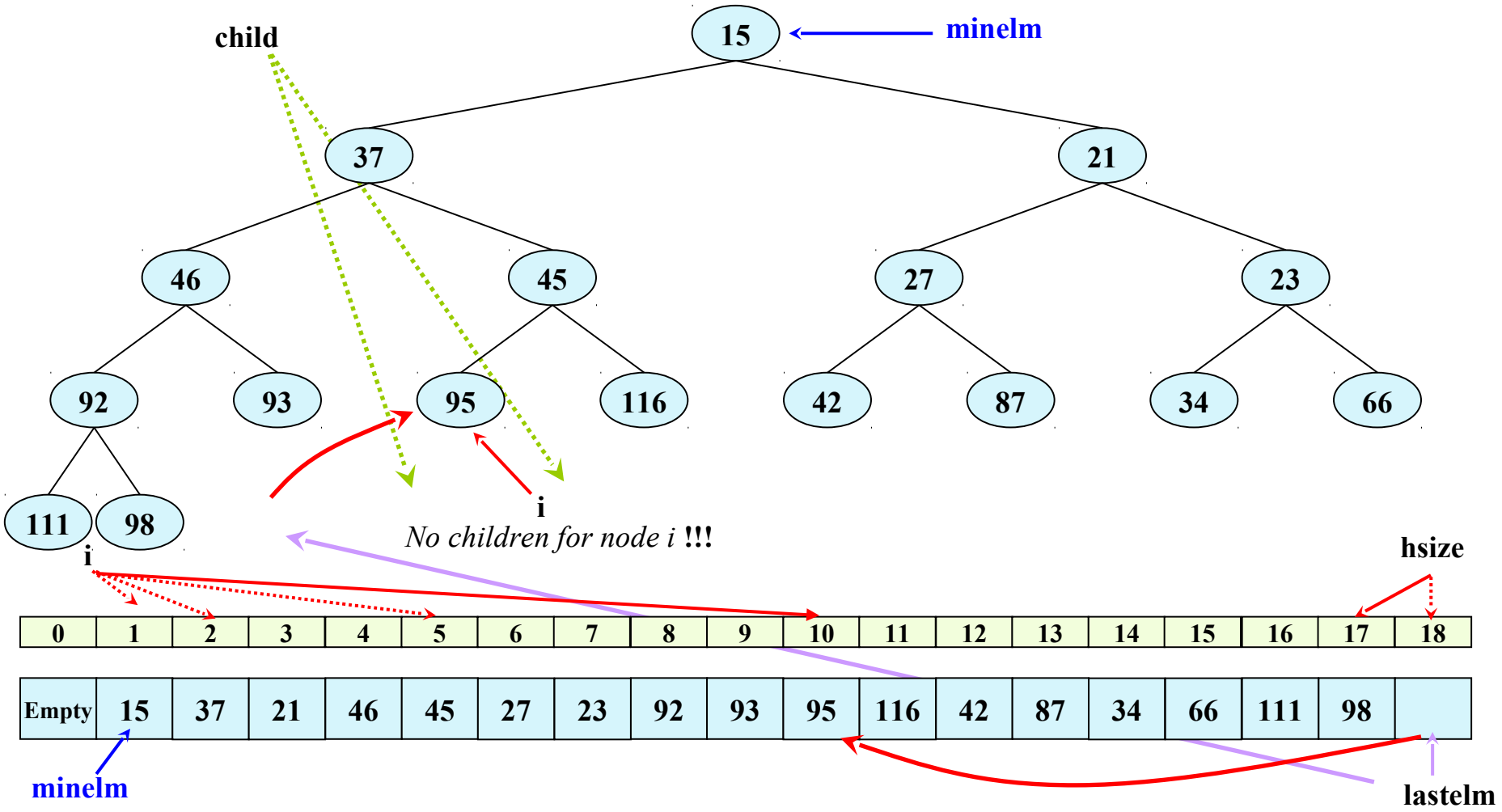
No children for node i !!!
i
hsize
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
Empty 15 37 21 46 45 27 23 92 93 116 42 87 34 66 111 98 95
minelm
Child (index pointing to child with smaller key)
lastelm

# *Removing 8 ...*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No children for node i !!!
i
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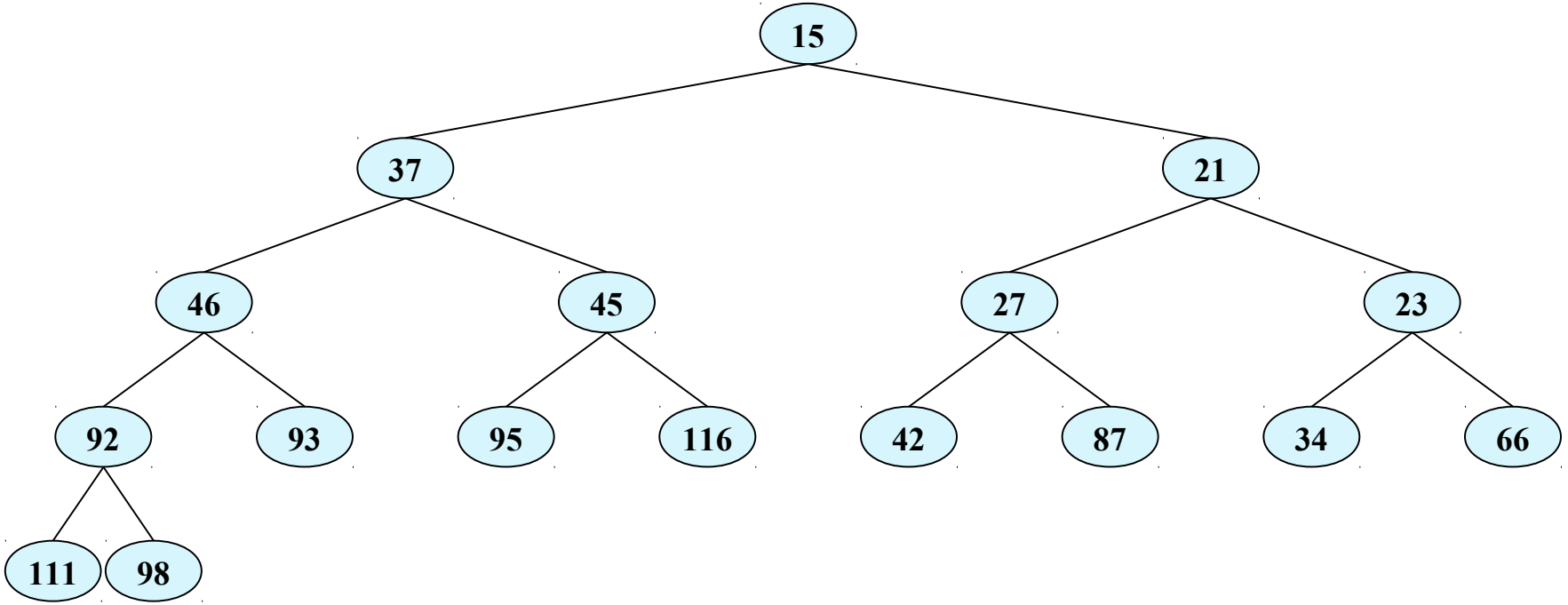
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
Empty 15 37 21 46 45 27 23 92 93 95 116 42 87 34 66 111 98
minelm
lastelm
null

# *8 Removed!*




| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Empty | 15 | 37 | 21 | 46 | 45 | 27 | 23 | 92 | 93 | 95 | 116 | 42 | 87 | 34 | 66 | 111 | 98 | |

# Other Heap Operations

- **Other Heap Operations**
  - DecreaseKey(I,D,HeapHeader);
  - IncreaseKey(I,D,HeapHeader);
  - Delete(I,HeapHeader);
  - BuildHeap(HeapHeader);

# DecreaseKey and IncreaseKey

- **DecreaseKey (I,D,HeapHeader) & IncreaseKey (I,D,HeapHeader)**
- These two operations decrease or increase the key at position i of the heap the root of which is pointed to by the HeapHeader by the amount D, respectively.
- Any of these two operations may violate the heap order. By percolating the modified key up or down after the DecreaseKey operation or IncreaseKey operation, respectively, the heap order property may be restored.

# Removal of any Key

- **Delete (I,HeapHeader)**
- This operation removes any key in the heap structure.

- Again, after the removal of the element from the heap, the heap order property may be violated.
- In this case, we may consider
  - the node of the key removed as the root, and
  - the corresponding subtree as the tree we perform a Delete*Min* on.
- Then using the Delete*Min* operation, the heap order property is restored.

# BuildHeap

- **BuildHeap(HeapHeader)**
- This operation is used to build a heap from a set of input data (e.g., numbers).
- Assuming that a set of numbers are arbitrarily (i.e., with no consideration of the heap order property) placed in a complete binary tree, we build a binary heap in
- For *n* numbers, this operation can be performed in *n* successive inserts. Since an insert takes *O(1)* in average and *O(log(n))* worst case, *Buildheap* takes an average time of *n*O(1)=O(n)* and *n*O(log(n)) = O(n*log(n))* in the worst case.

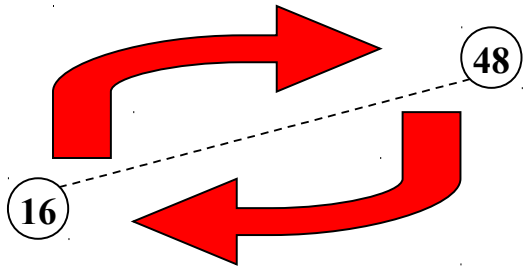
# BuildHeap Algorithm

- Starting from
  - the rightmost subtree with a height of 1,
  - Loop
    - compare the children and find the smaller child
    - compare the smaller child with the parent
    - exchange the smaller child with the parent.
  - until all nodes in the tree are processed.
- Subtrees with roots of height greater than 1,
  - the parent must be percolated down until the heap order property is restored.
- An example follows.

# Constructing a MinHeap – Animation


# Constructing a MinHeap – Animation

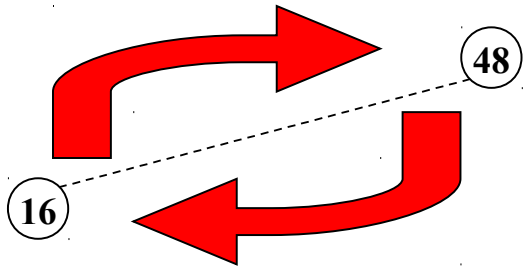
| 48 | 16 |
|---|---|




**48 > 16**

# Constructing a MinHeap – Animation

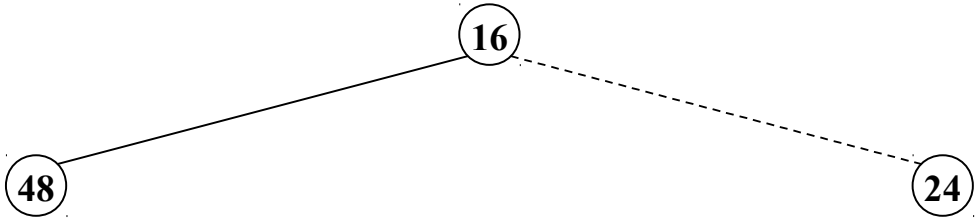
| 48 | 16 |
|---|---|




**48 > 16**

# Constructing a MinHeap – Animation

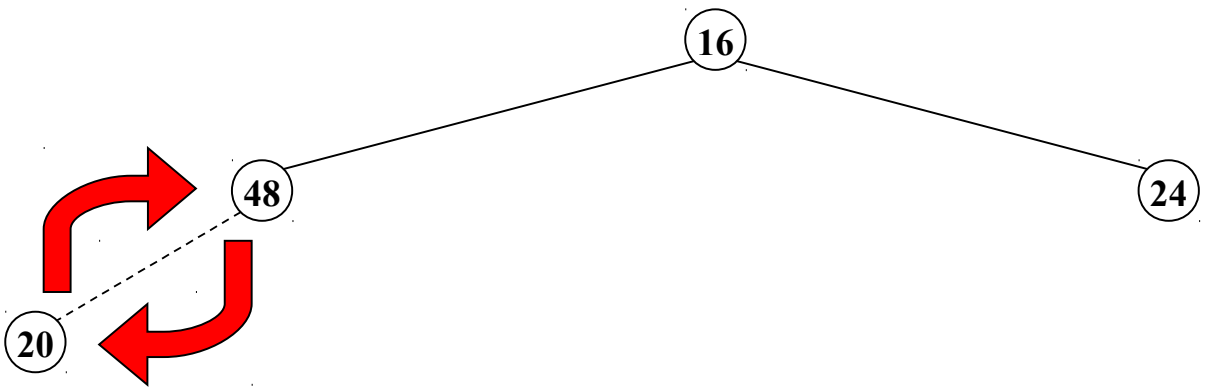
| 48 | 16 | 24 |
|---|---|---|




**OK!**

# Constructing a MinHeap – Animation

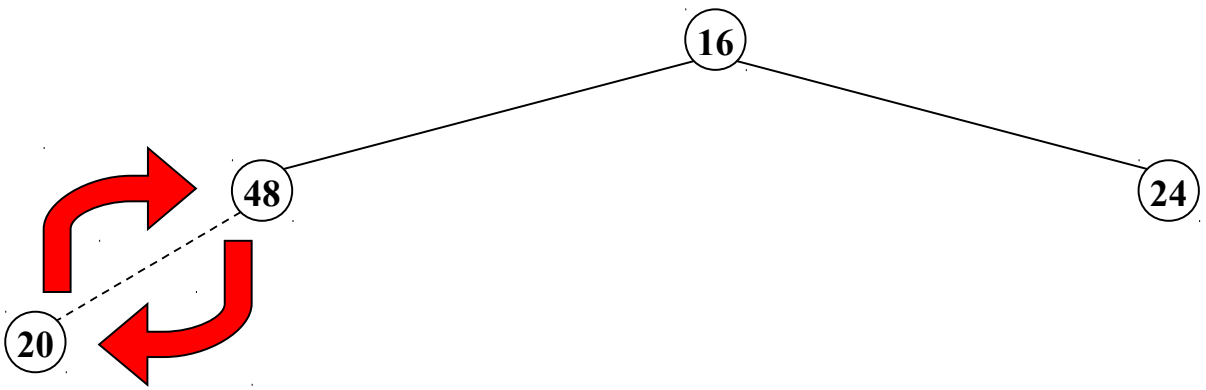
| 48 | 16 | 24 | 20 |
|---|---|---|---|




**48 > 20**

# Constructing a MinHeap – Animation

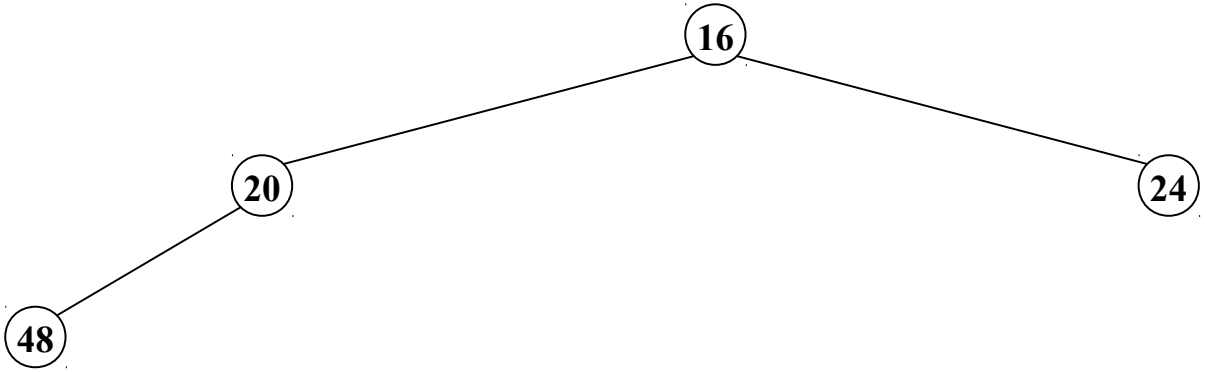
| 48 | 16 | 24 | 20 |
|---|---|---|---|




**48 > 20**

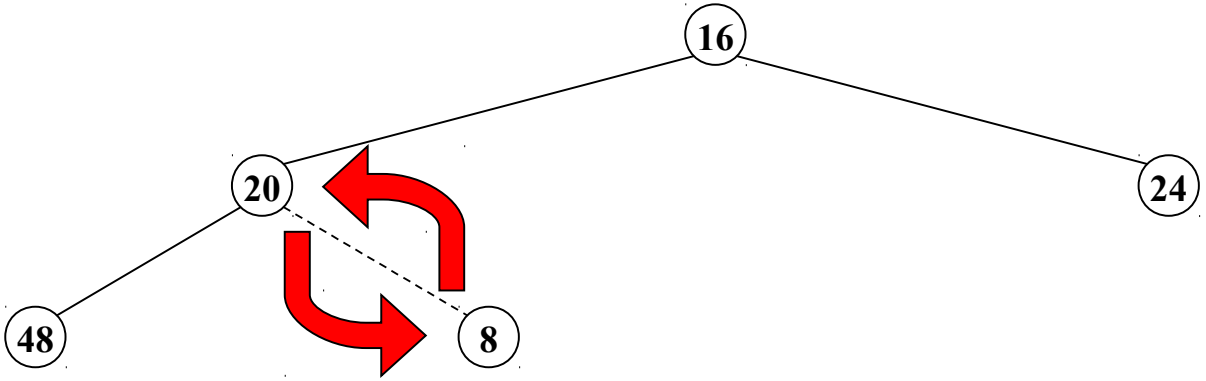
# Constructing a MinHeap – Animation

| 48 | 16 | 24 | 20 |
|---|---|---|---|



# Constructing a MinHeap – Animation

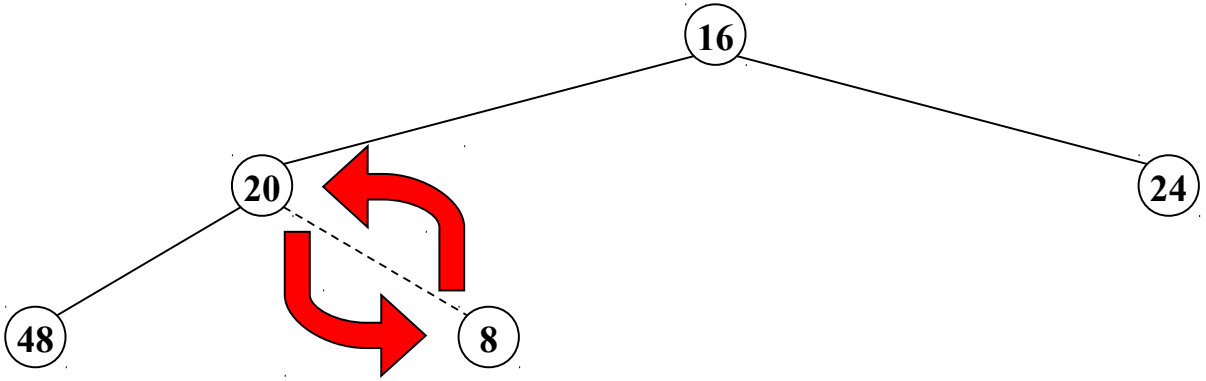
| 48 | 16 | 24 | 20 | 8 |
|---|---|---|---|---|




**20 > 8**

# Constructing a MinHeap – Animation

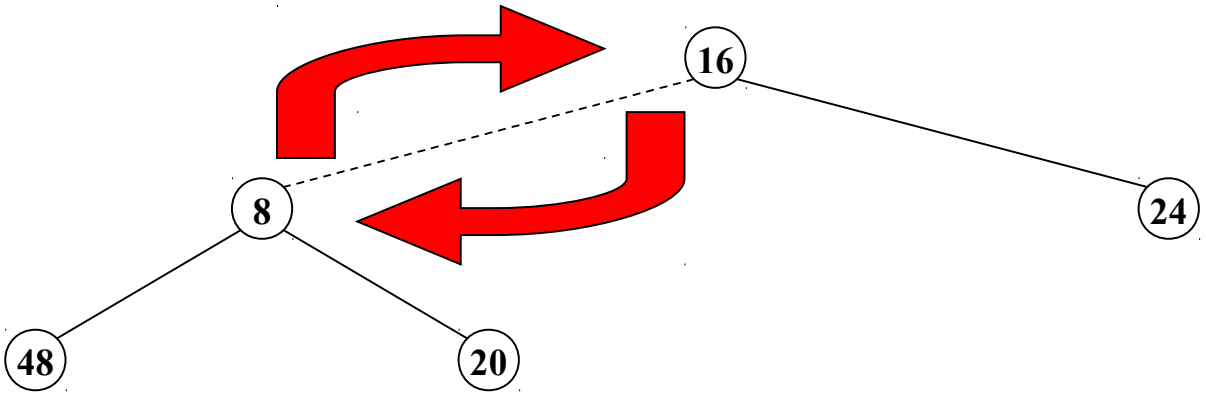
| 48 | 16 | 24 | 20 | 8 |
|---|---|---|---|---|




**20 > 8**

# Constructing a MinHeap – Animation

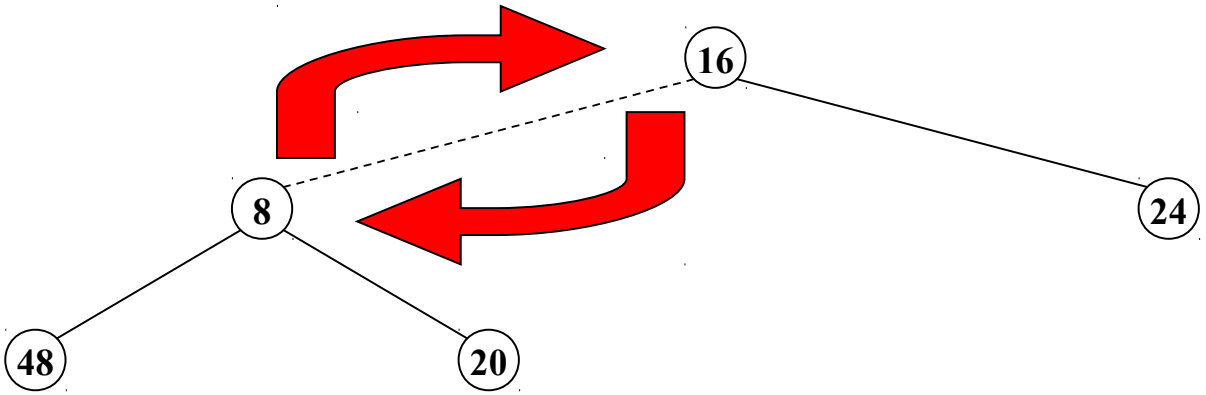
| 48 | 16 | 24 | 20 | 8 |
|---|---|---|---|---|




**16 > 8**

# Constructing a MinHeap – Animation

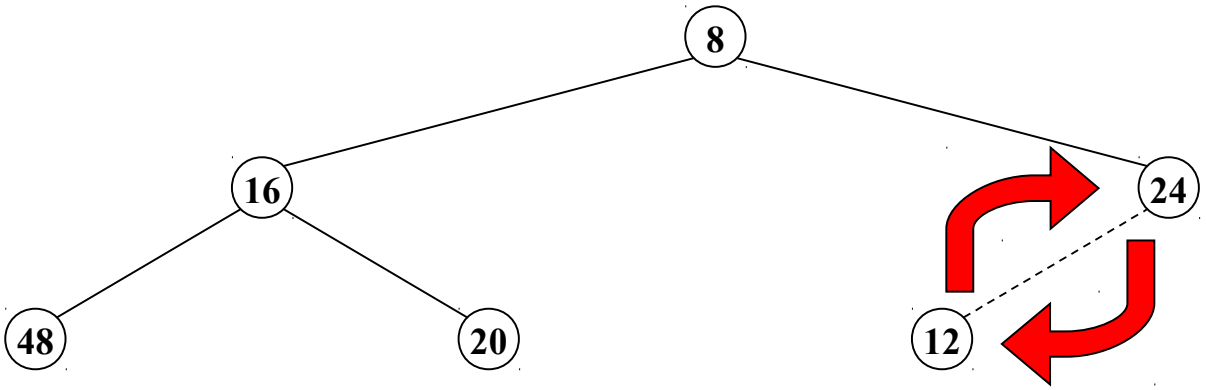
| 48 | 16 | 24 | 20 | 8 |
|---|---|---|---|---|




**16 > 8**

# Constructing a MinHeap – Animation

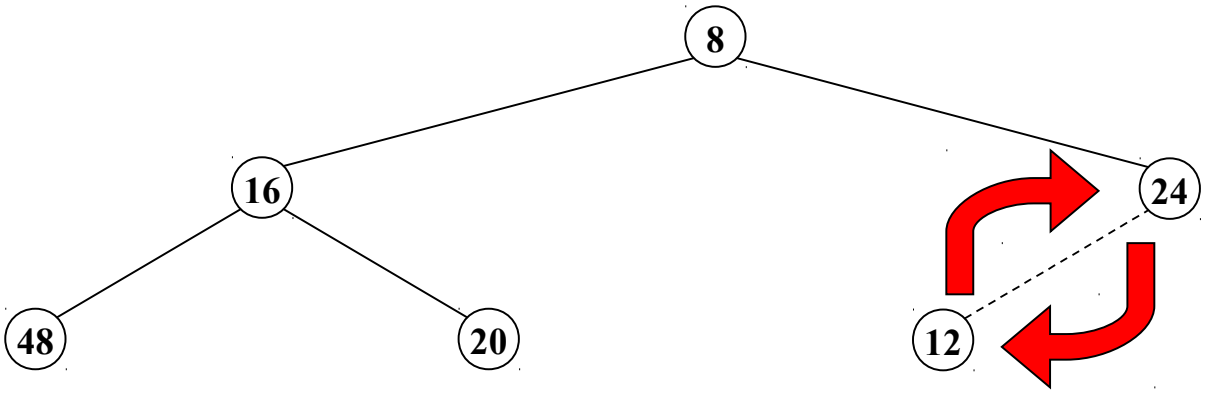
| 48 | 16 | 24 | 20 | 8 | 12 |
|---|---|---|---|---|---|




**24 > 12**

# Constructing a MinHeap – Animation

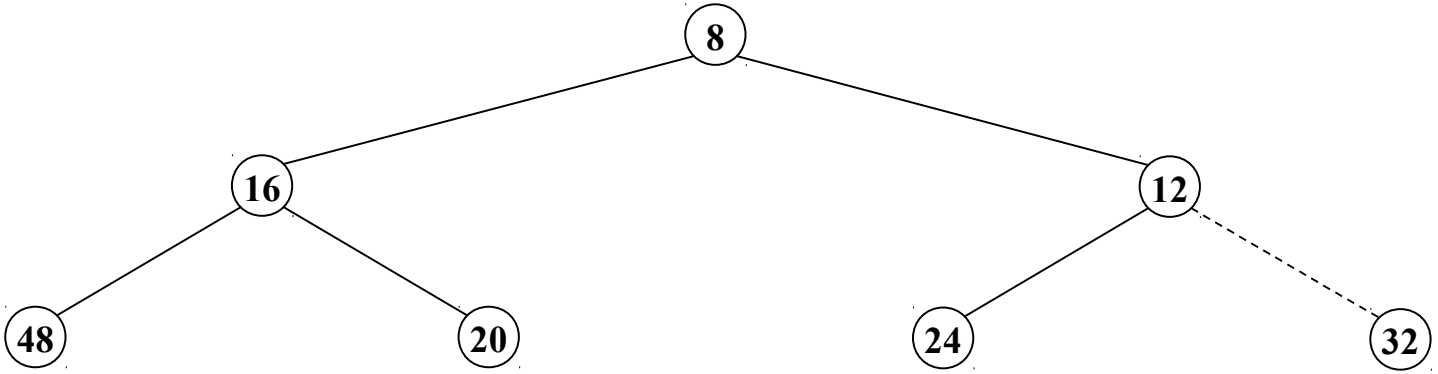
| 48 | 16 | 24 | 20 | 8 | 12 |
|---|---|---|---|---|---|




**24 > 12**

# Constructing a MinHeap – Animation

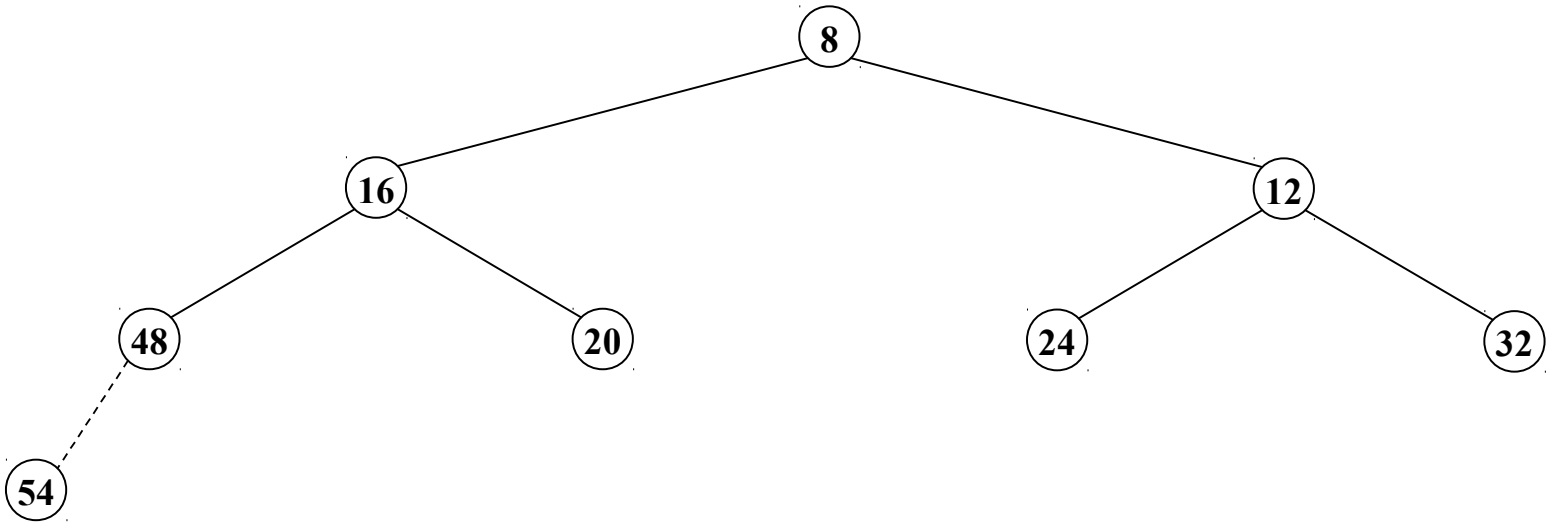
| 48 | 16 | 24 | 20 | 8 | 12 | 32 |
|---|---|---|---|---|---|---|




**OK!**

# Constructing a MinHeap – Animation

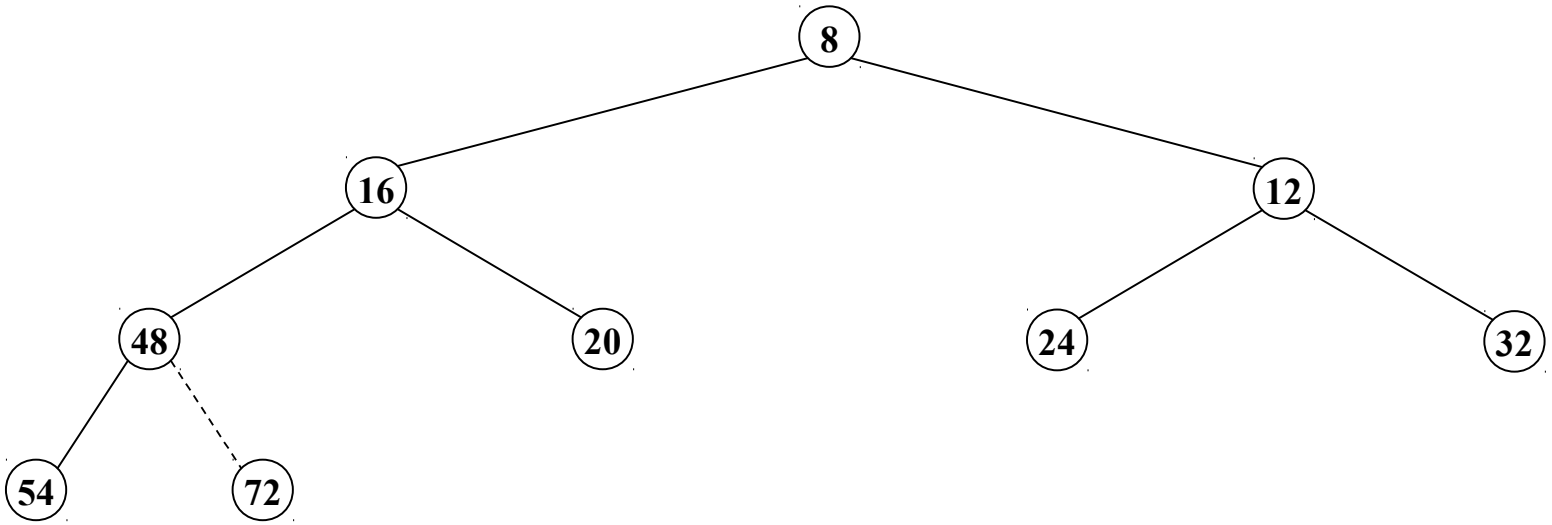
| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 |
|---|---|---|---|---|---|---|---|




**OK!**

# Constructing a MinHeap – Animation

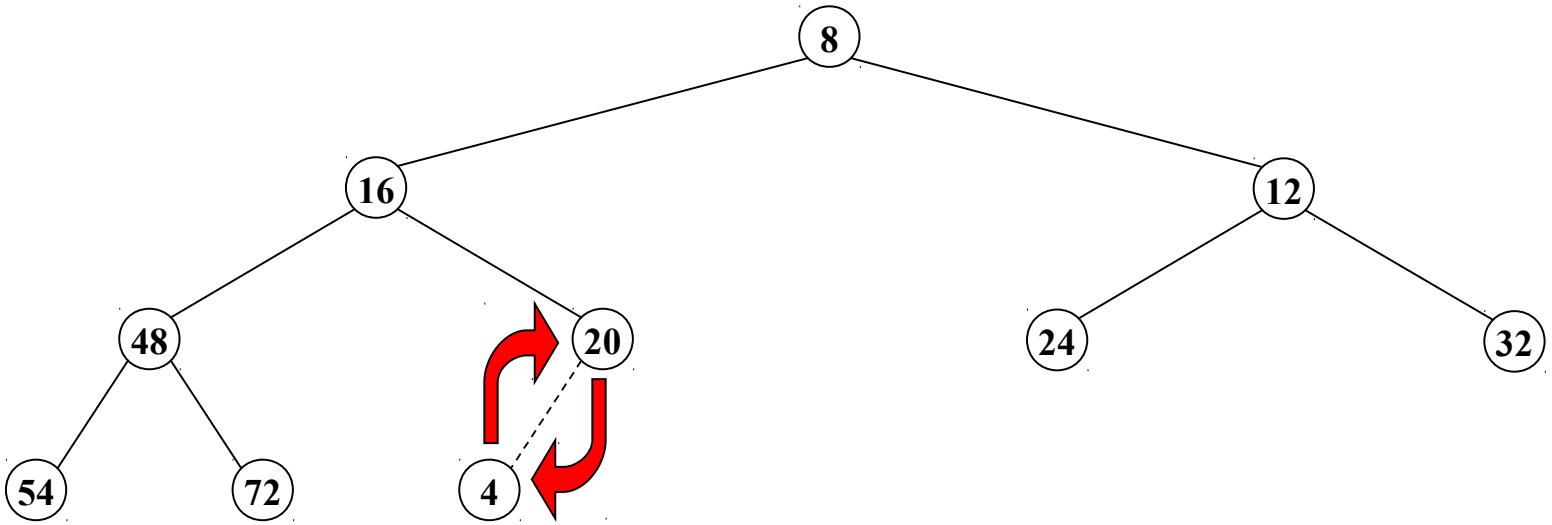
| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 |
|---|---|---|---|---|---|---|---|---|




**OK!**

# Constructing a MinHeap – Animation

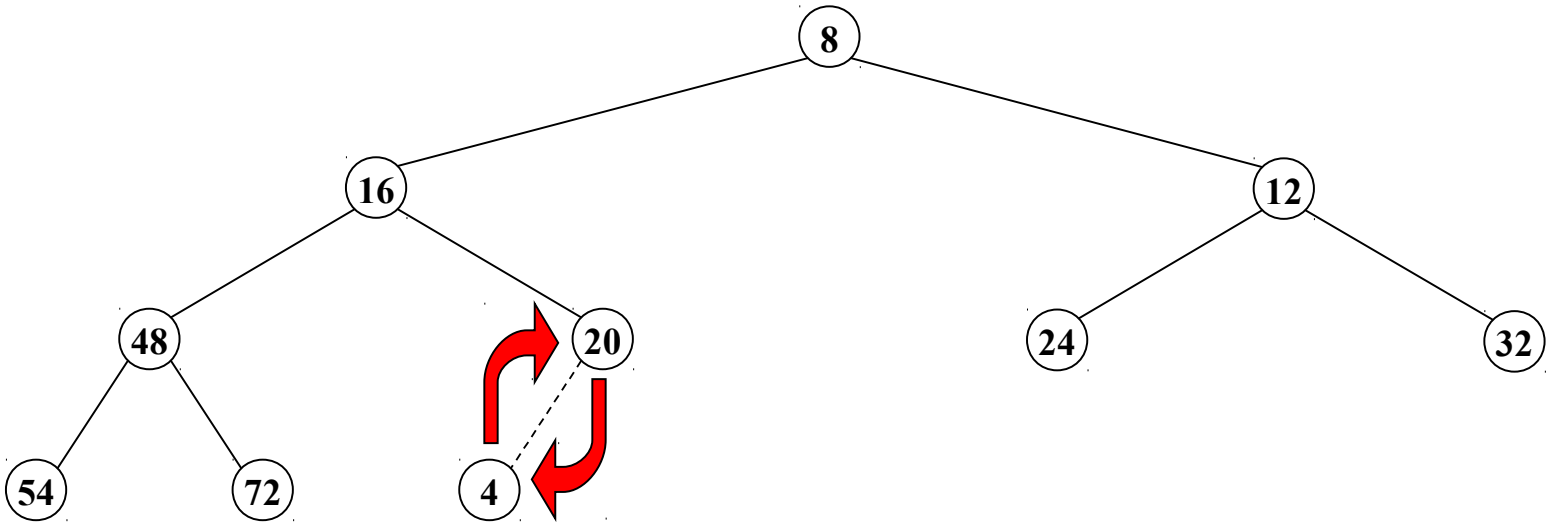
| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 |
|---|---|---|---|---|---|---|---|---|---|




**4 < 20**

# Constructing a MinHeap – Animation

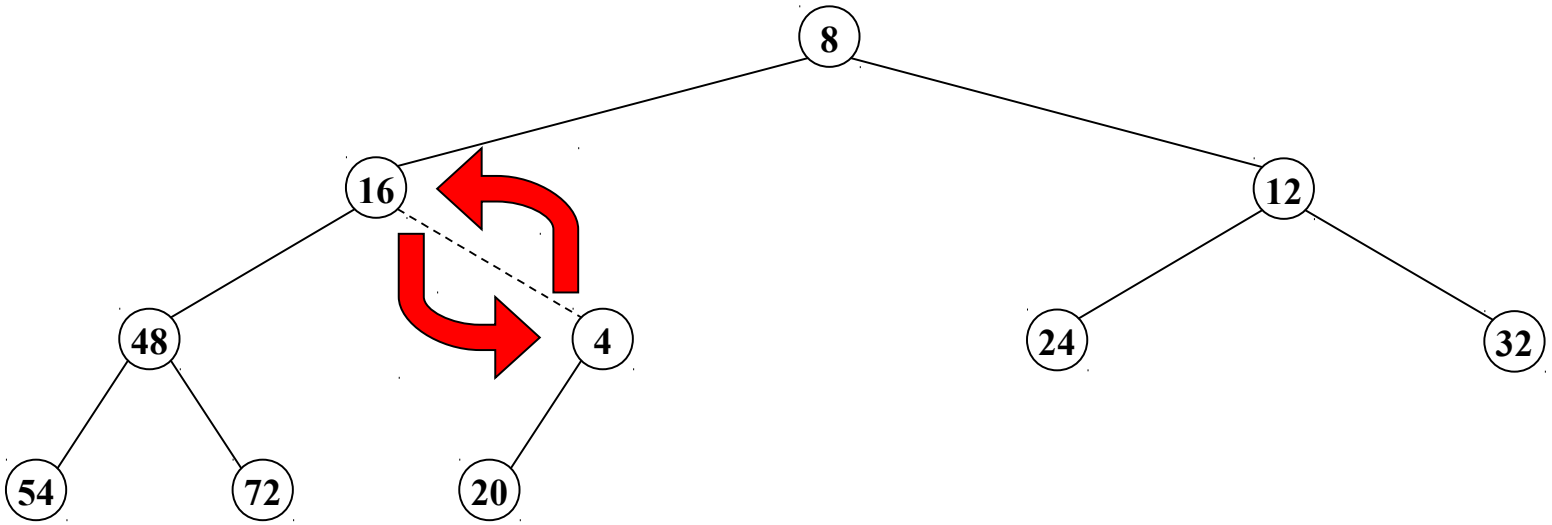
| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 |
|---|---|---|---|---|---|---|---|---|---|




**4 < 20**

# Constructing a MinHeap – Animation

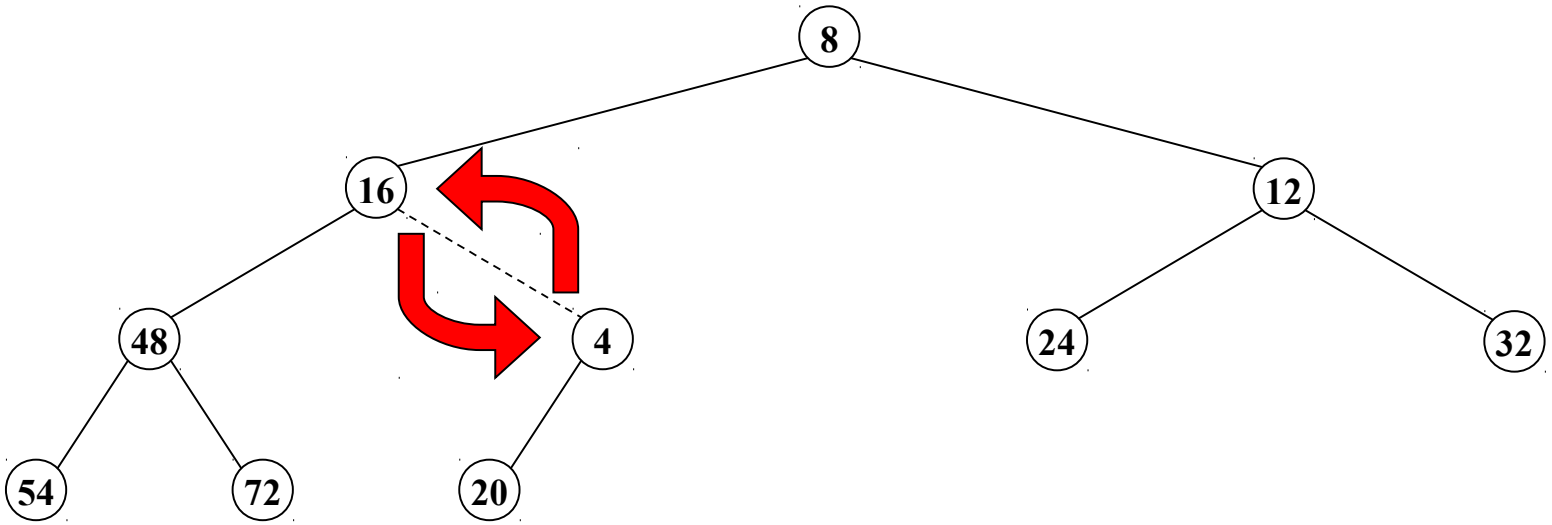
| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 |
|---|---|---|---|---|---|---|---|---|---|




**4 < 16**

# Constructing a MinHeap – Animation

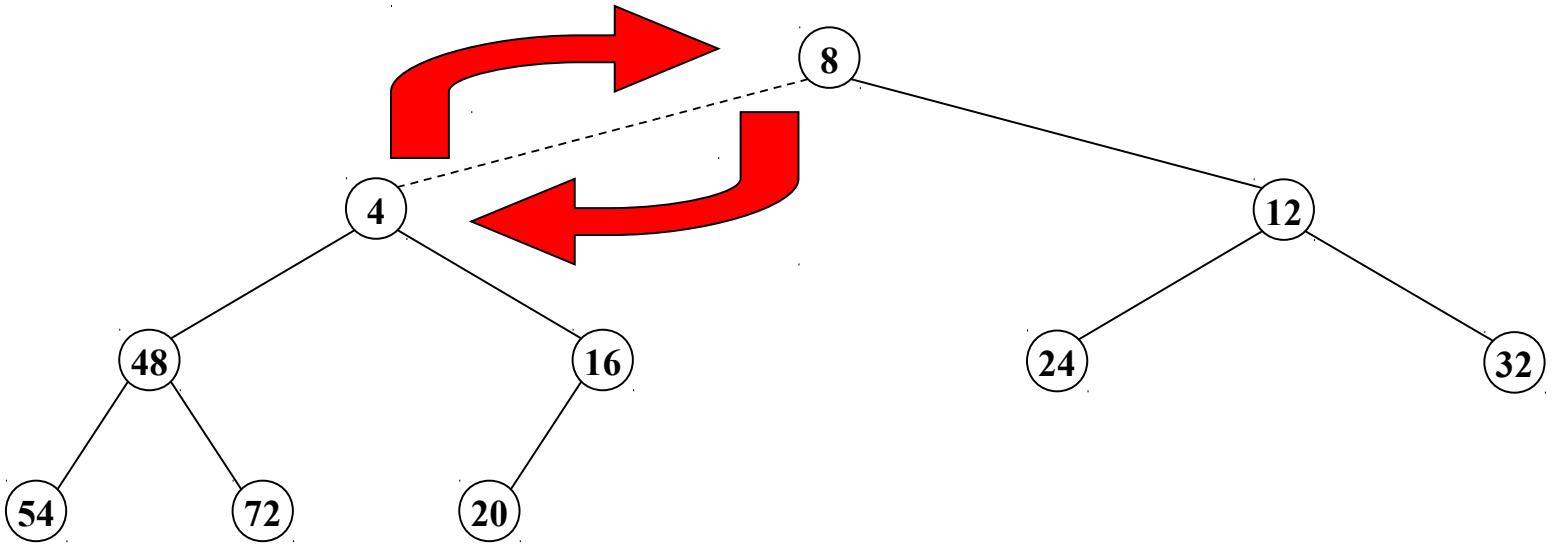
| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 |
|---|---|---|---|---|---|---|---|---|---|




**4 < 16**

# Constructing a MinHeap – Animation

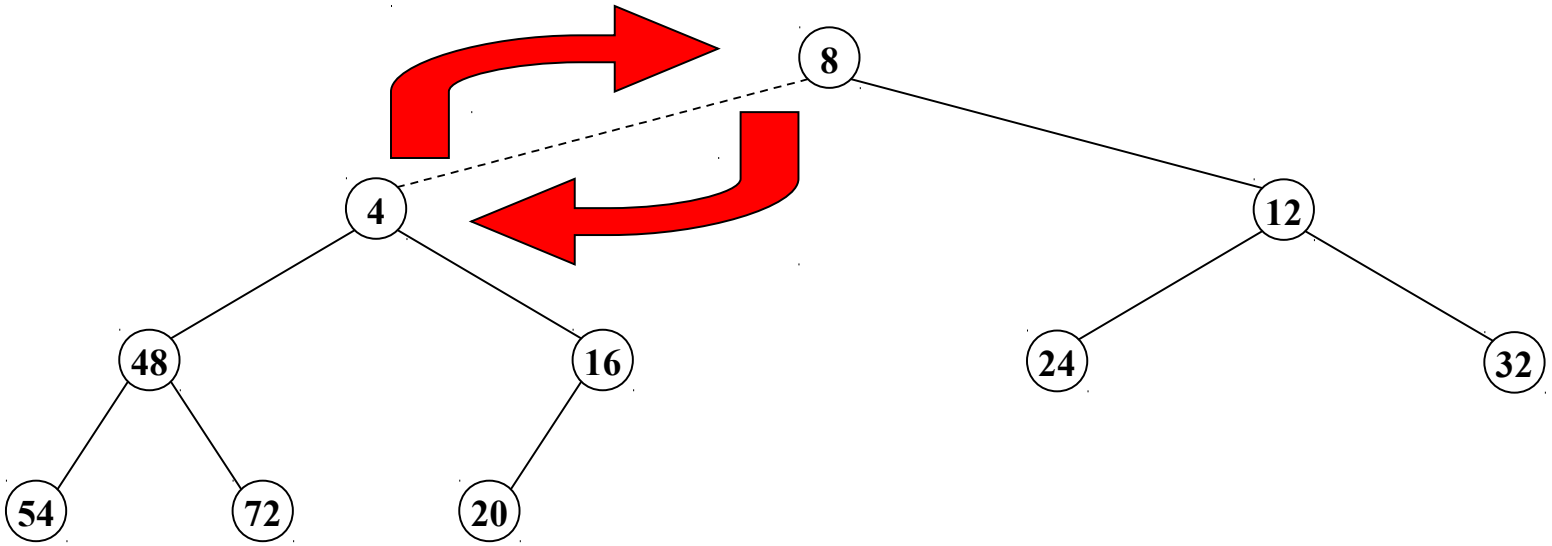
| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 |
|---|---|---|---|---|---|---|---|---|---|




**4 < 8**

# Constructing a MinHeap – Animation

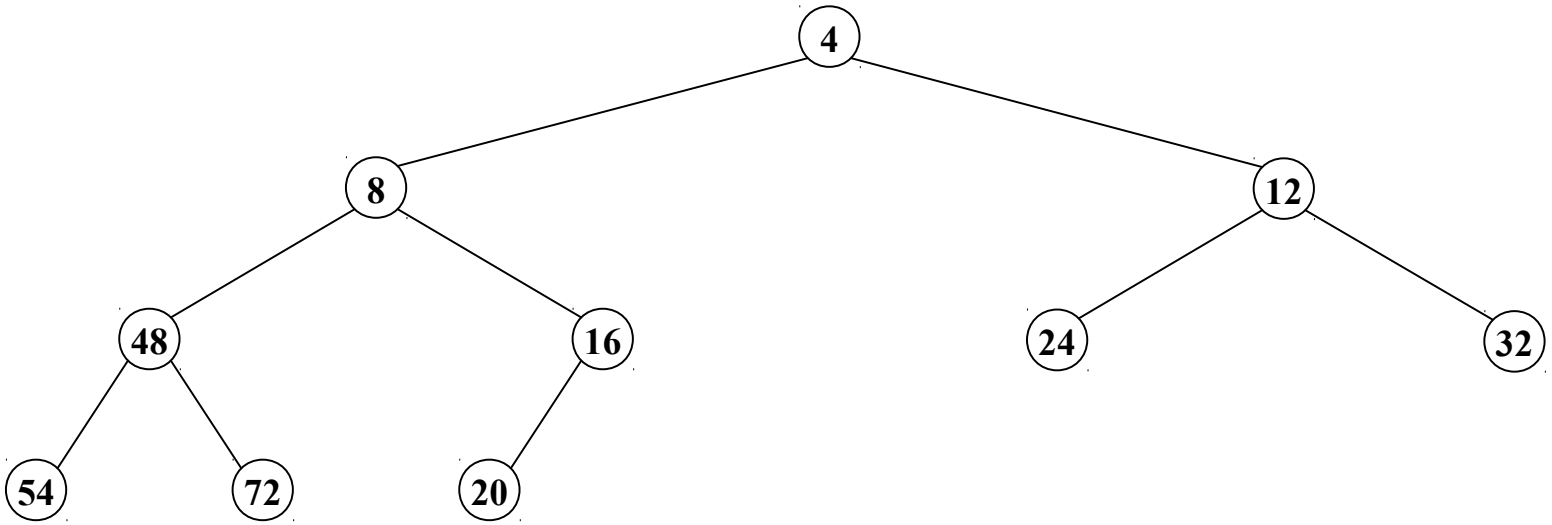
| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 |
|---|---|---|---|---|---|---|---|---|---|



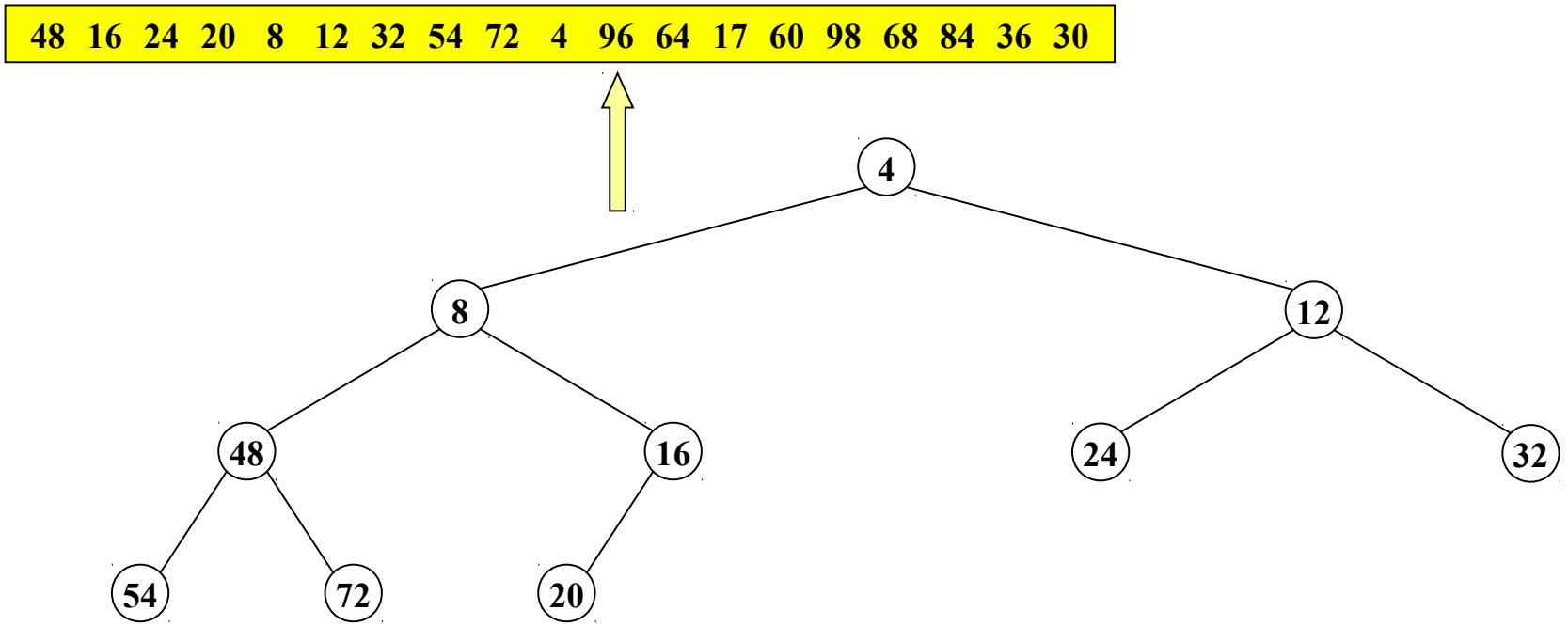

**4 < 8**

# Constructing a MinHeap – Animation

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 |
|---|---|---|---|---|---|---|---|---|---|

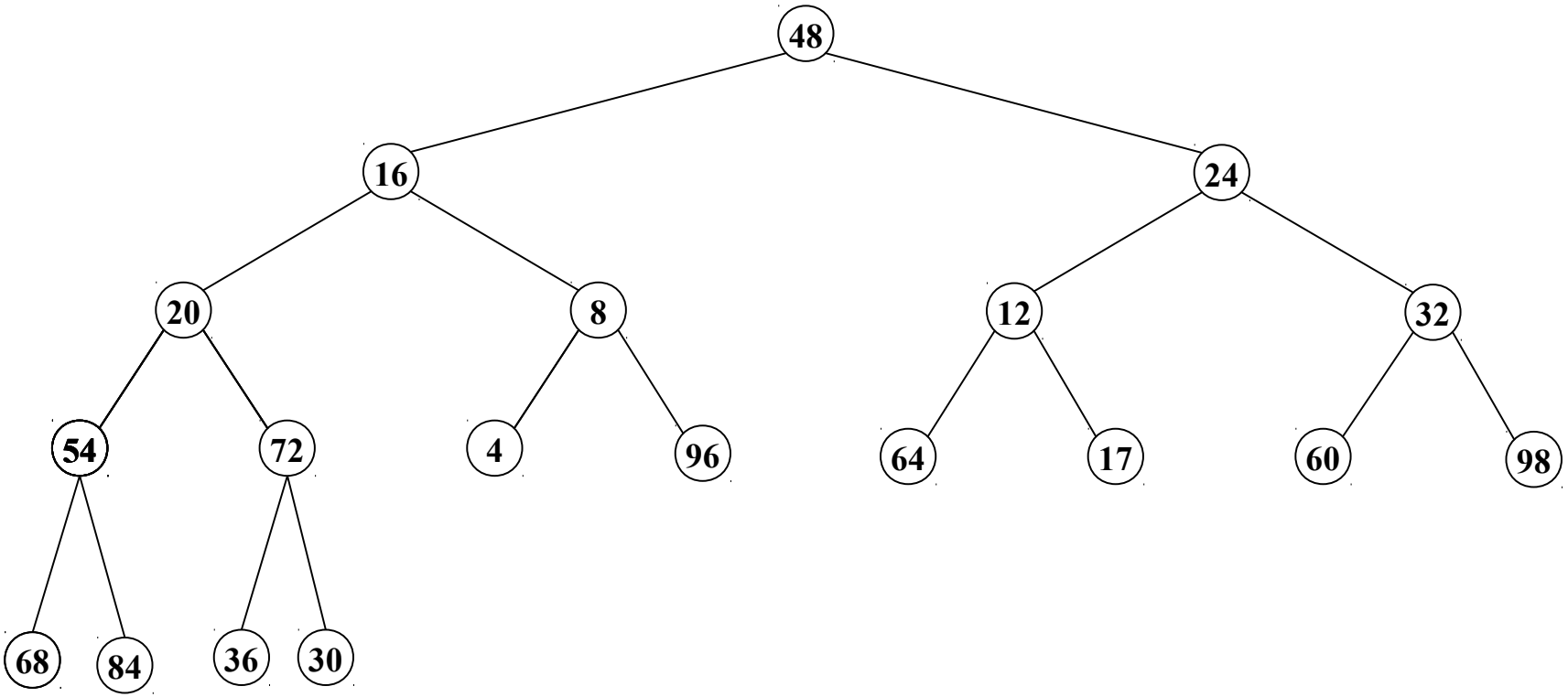


# Constructing a MinHeap – Classwork



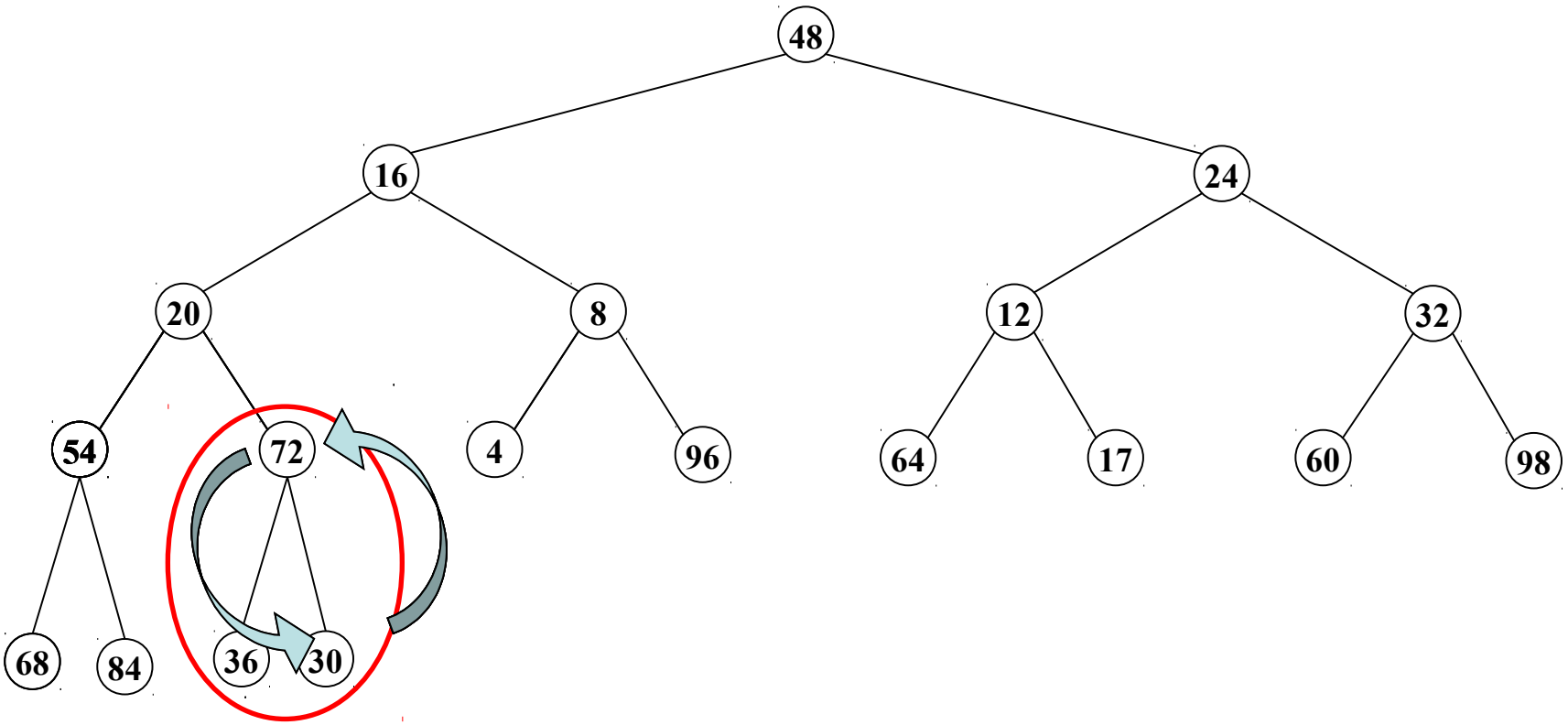
# Constructing a MinHeap

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 | 96 | 64 | 17 | 60 | 98 | 68 | 84 | 36 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|



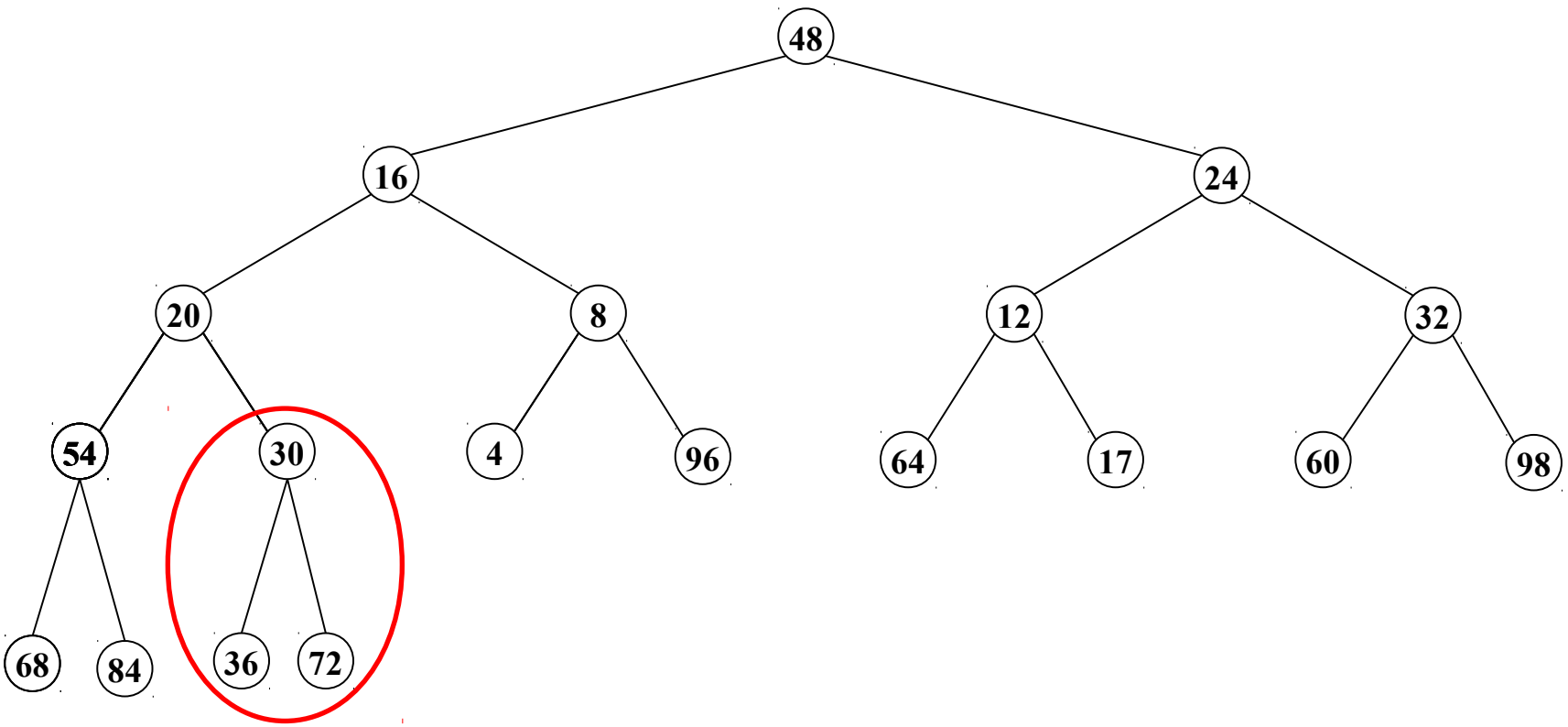
# Constructing a MinHeap

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 | 96 | 64 | 17 | 60 | 98 | 68 | 84 | 36 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|



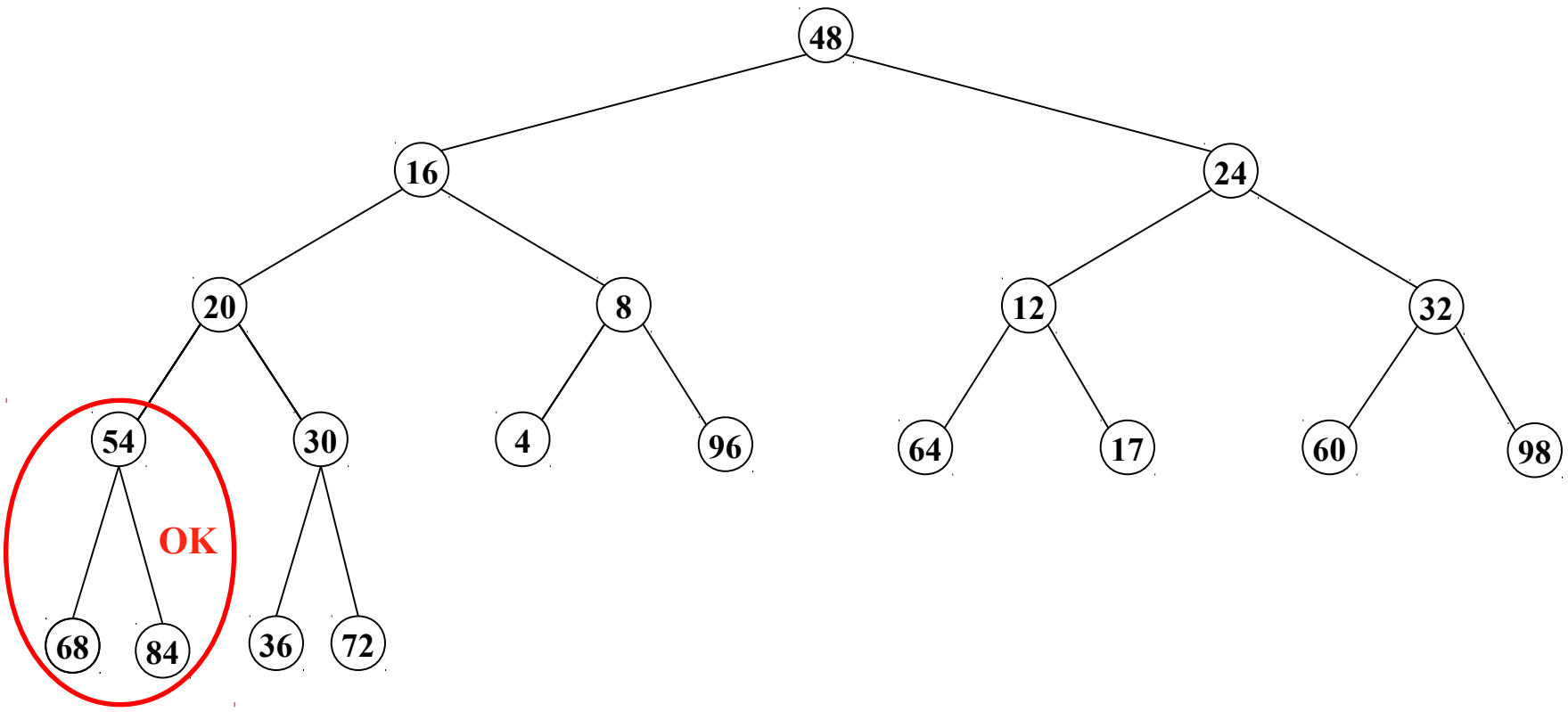
# Constructing a MinHeap -

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 | 96 | 64 | 17 | 60 | 98 | 68 | 84 | 36 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|



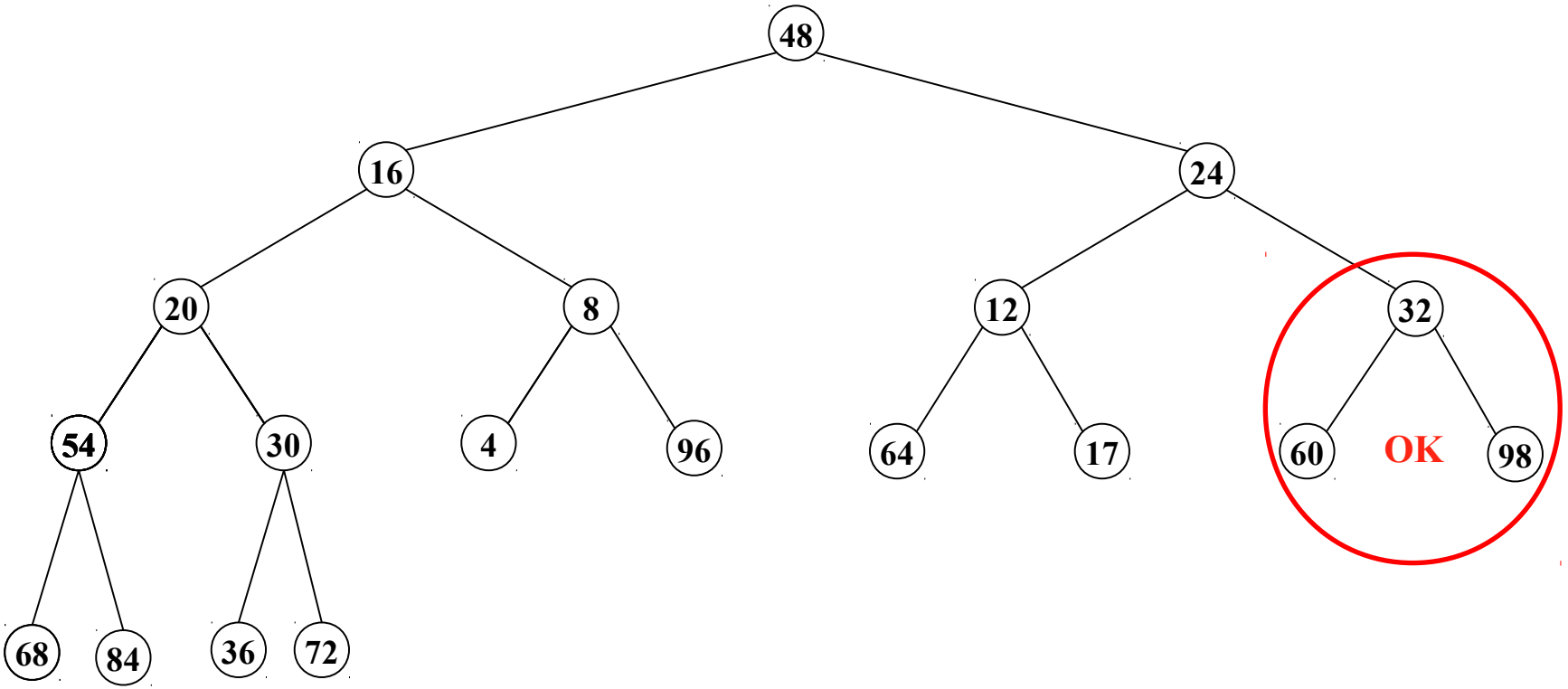
# Constructing a MinHeap -

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 | 96 | 64 | 17 | 60 | 98 | 68 | 84 | 36 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|



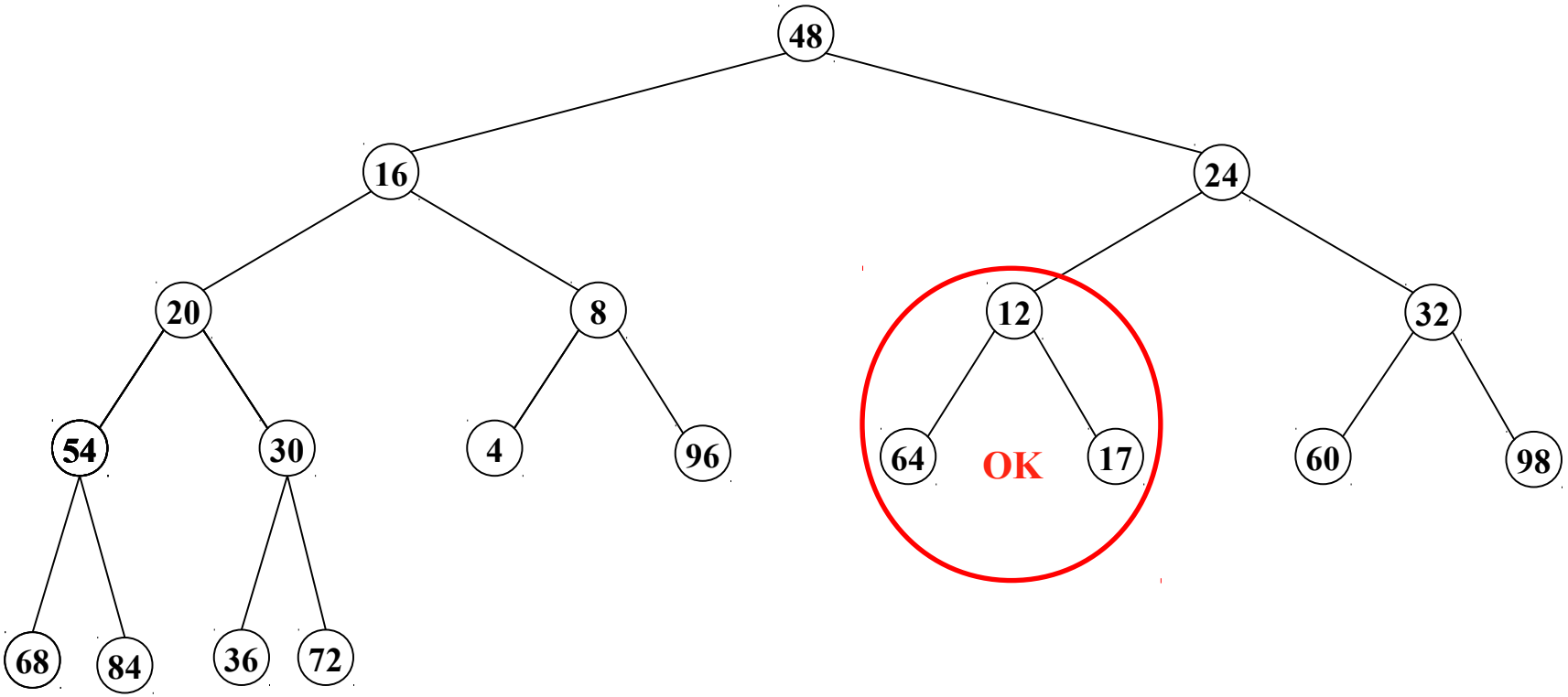
# Constructing a MinHeap -

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 | 96 | 64 | 17 | 60 | 98 | 68 | 84 | 36 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|



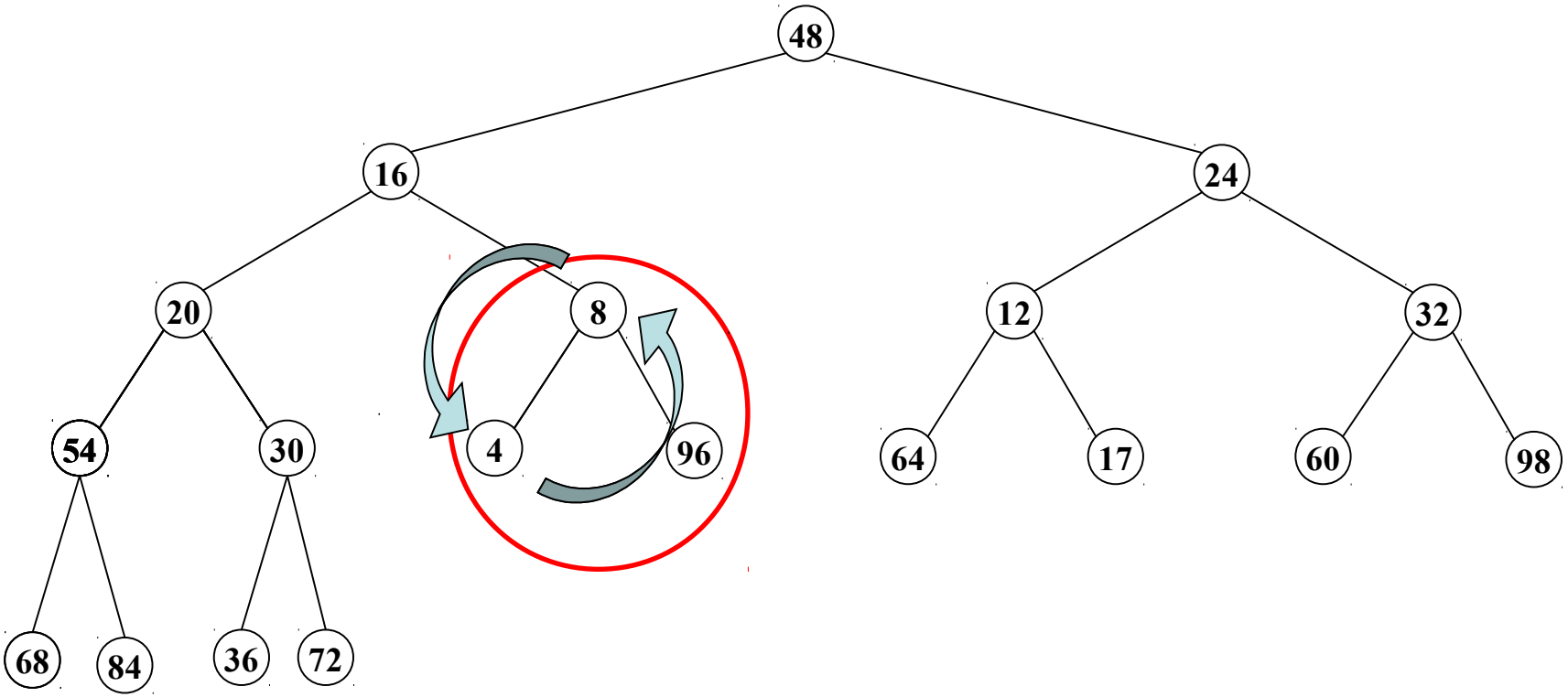
# Constructing a MinHeap -

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 | 96 | 64 | 17 | 60 | 98 | 68 | 84 | 36 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|



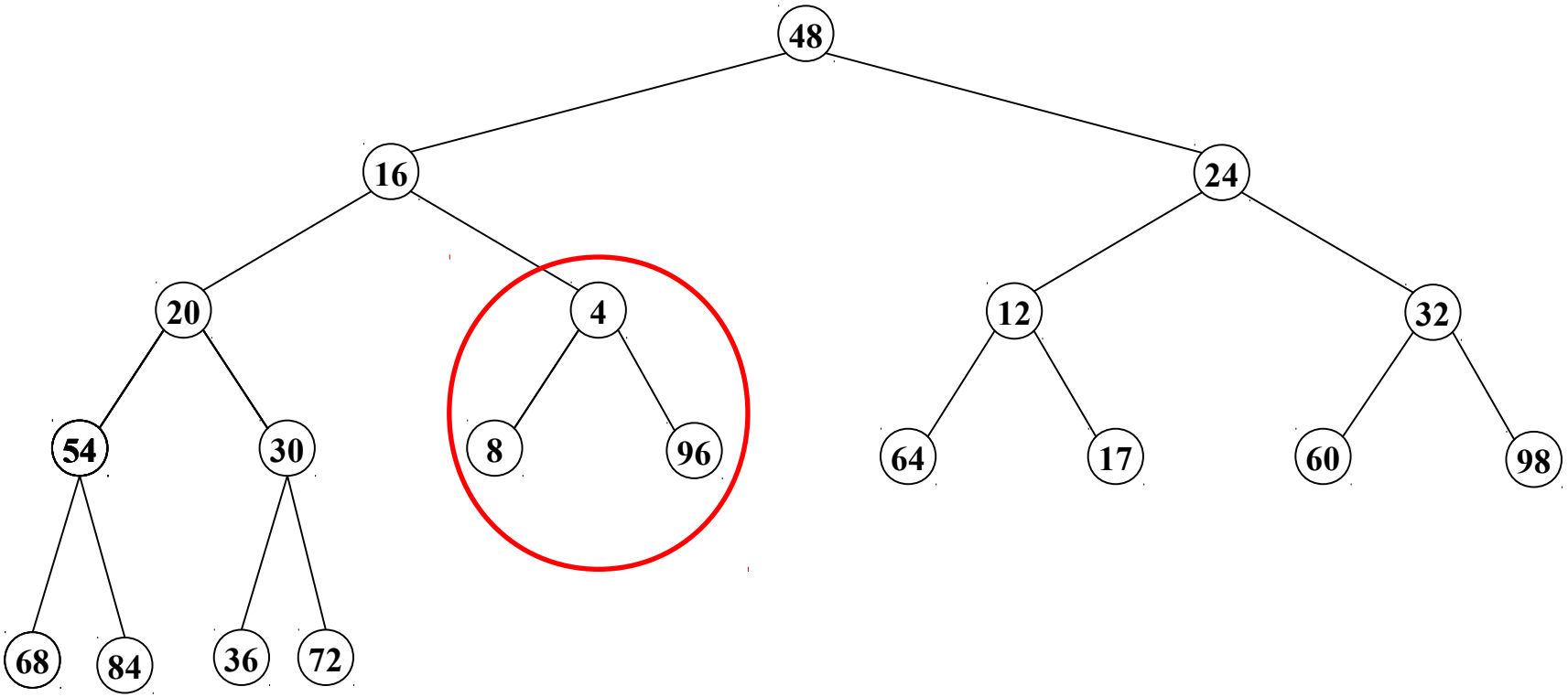
# Constructing a MinHeap -

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 | 96 | 64 | 17 | 60 | 98 | 68 | 84 | 36 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|



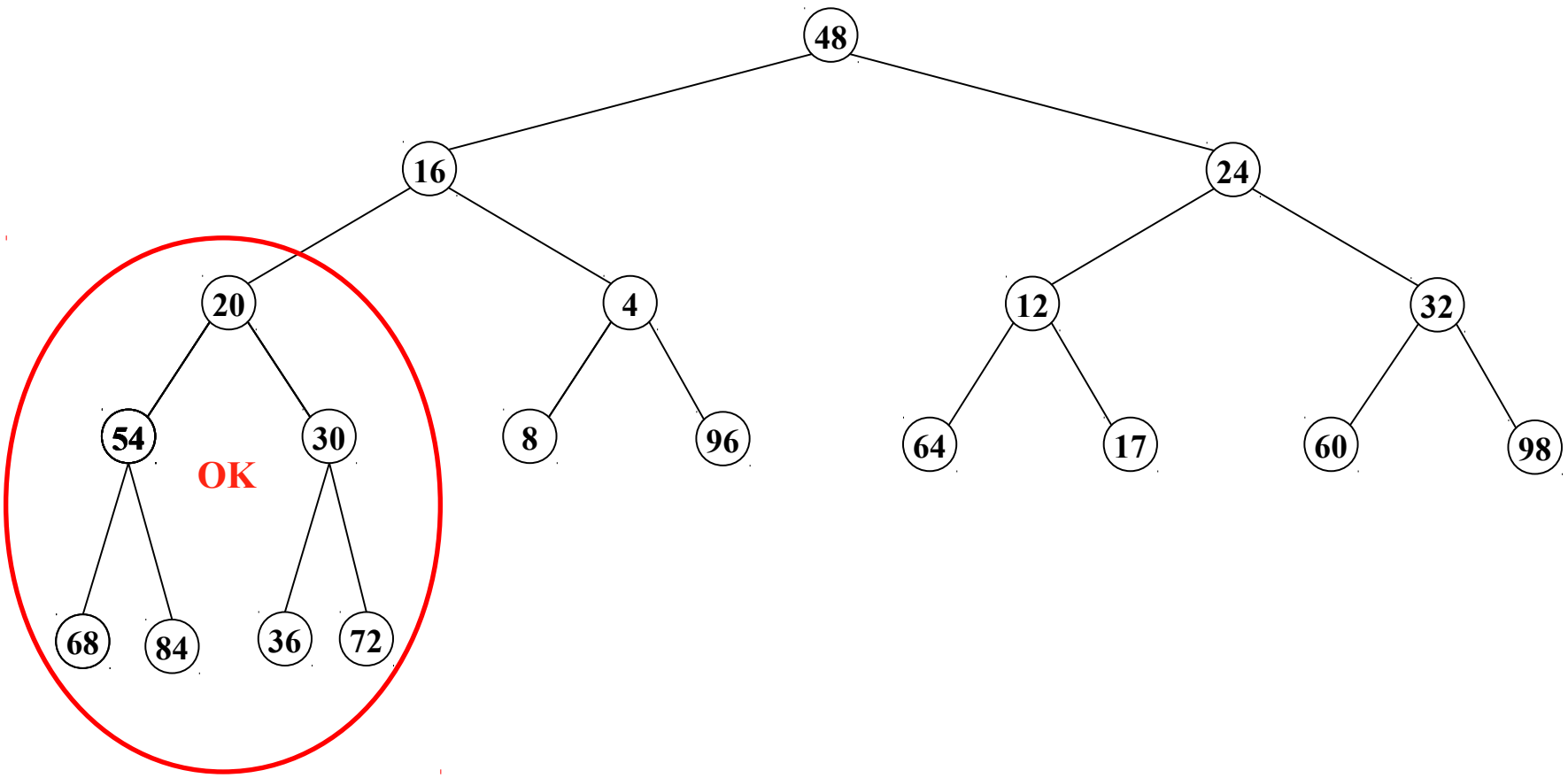
# Constructing a MinHeap -

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 | 96 | 64 | 17 | 60 | 98 | 68 | 84 | 36 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|



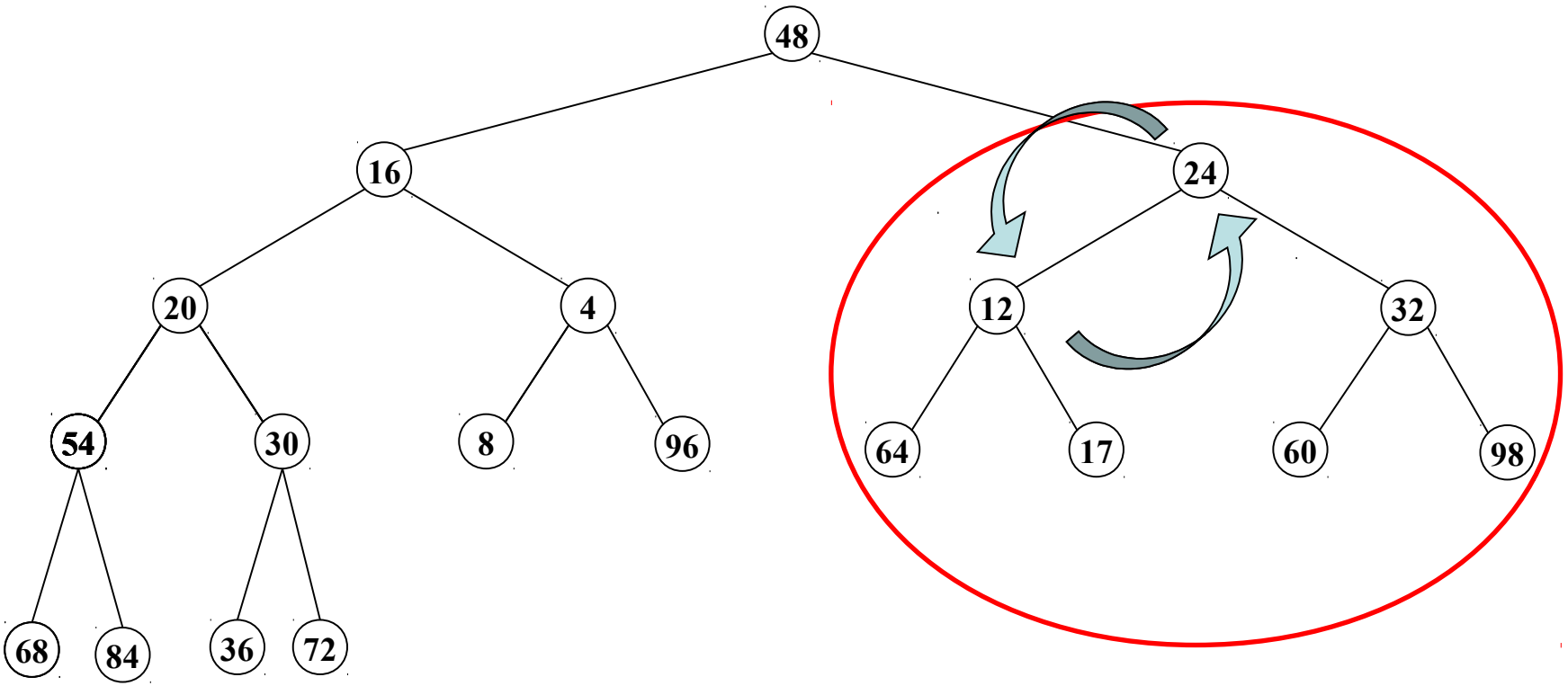
# Constructing a MinHeap -

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 | 96 | 64 | 17 | 60 | 98 | 68 | 84 | 36 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|



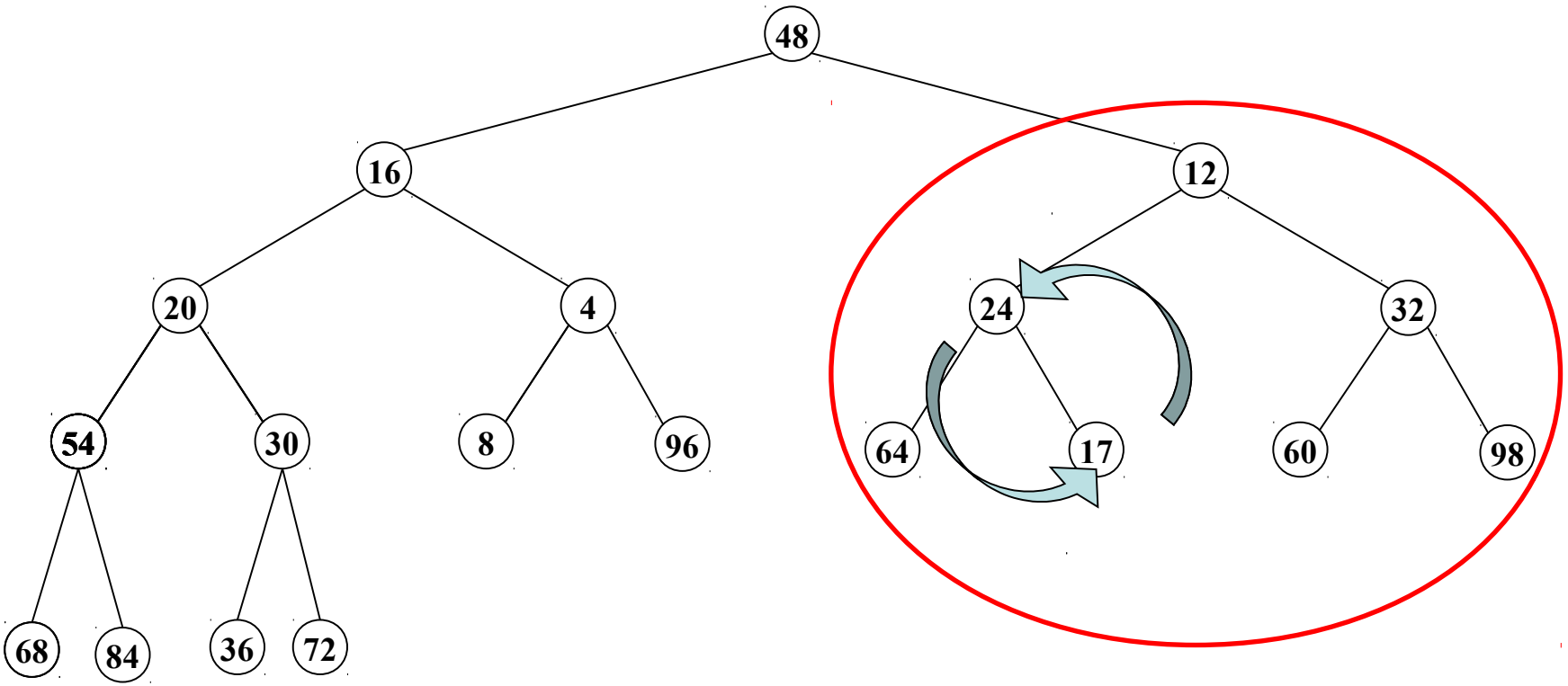
# Constructing a MinHeap -

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 | 96 | 64 | 17 | 60 | 98 | 68 | 84 | 36 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|



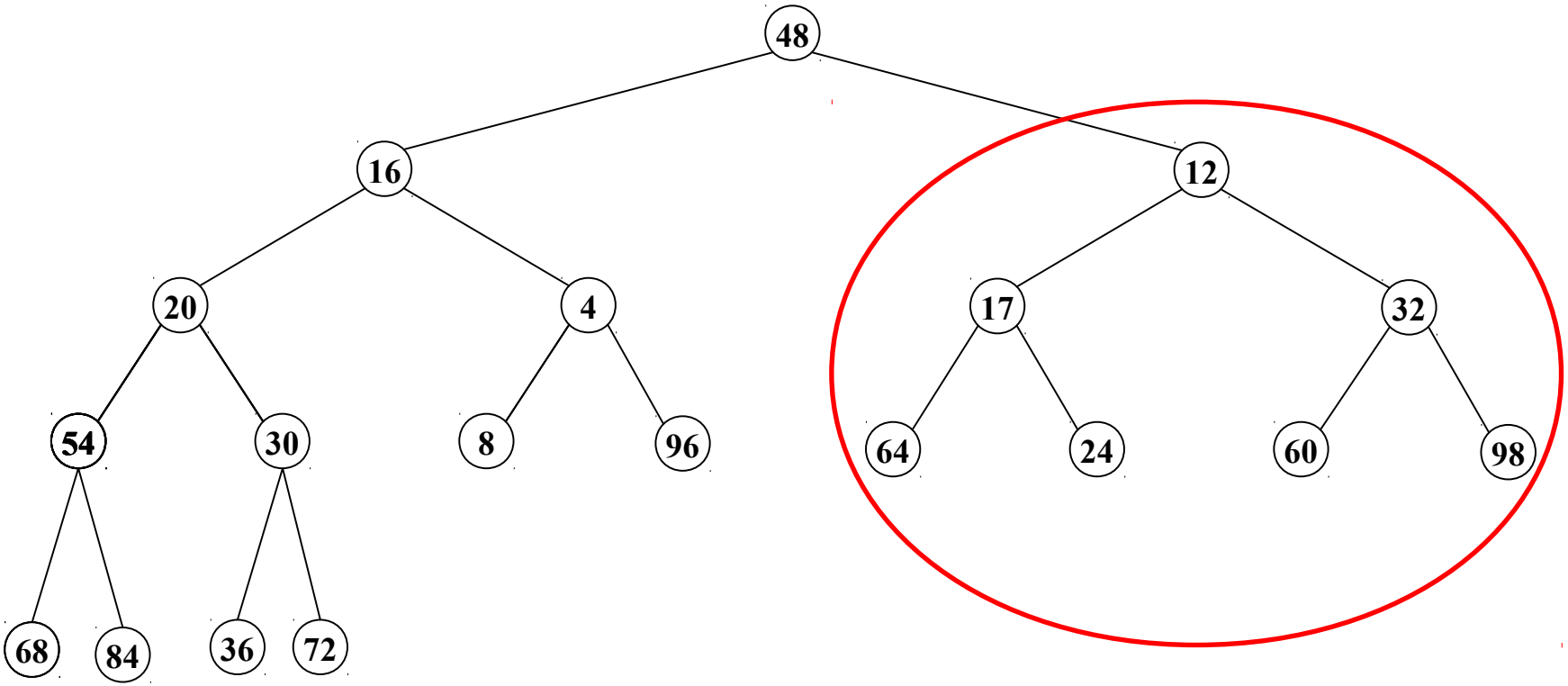
# Constructing a MinHeap -

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 | 96 | 64 | 17 | 60 | 98 | 68 | 84 | 36 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|



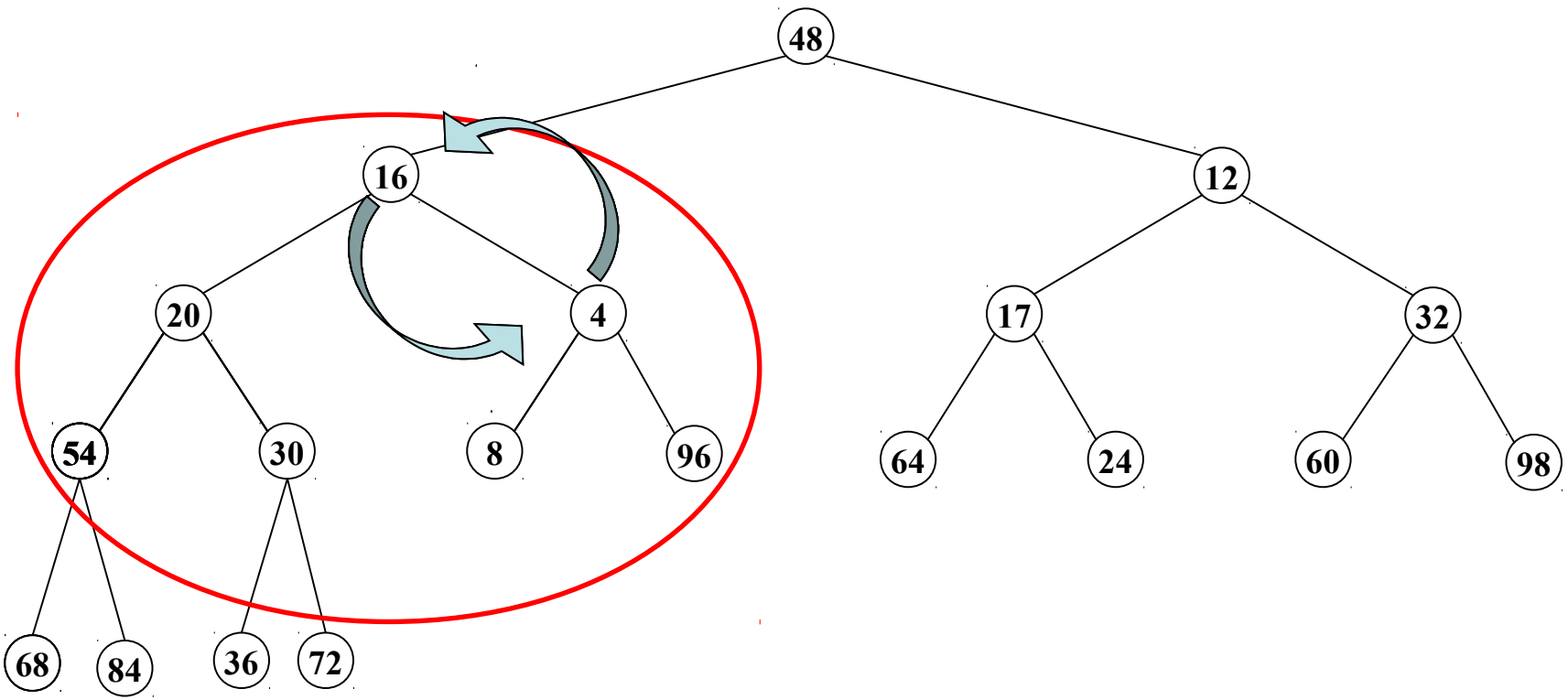
# Constructing a MinHeap -

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 | 96 | 64 | 17 | 60 | 98 | 68 | 84 | 36 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|



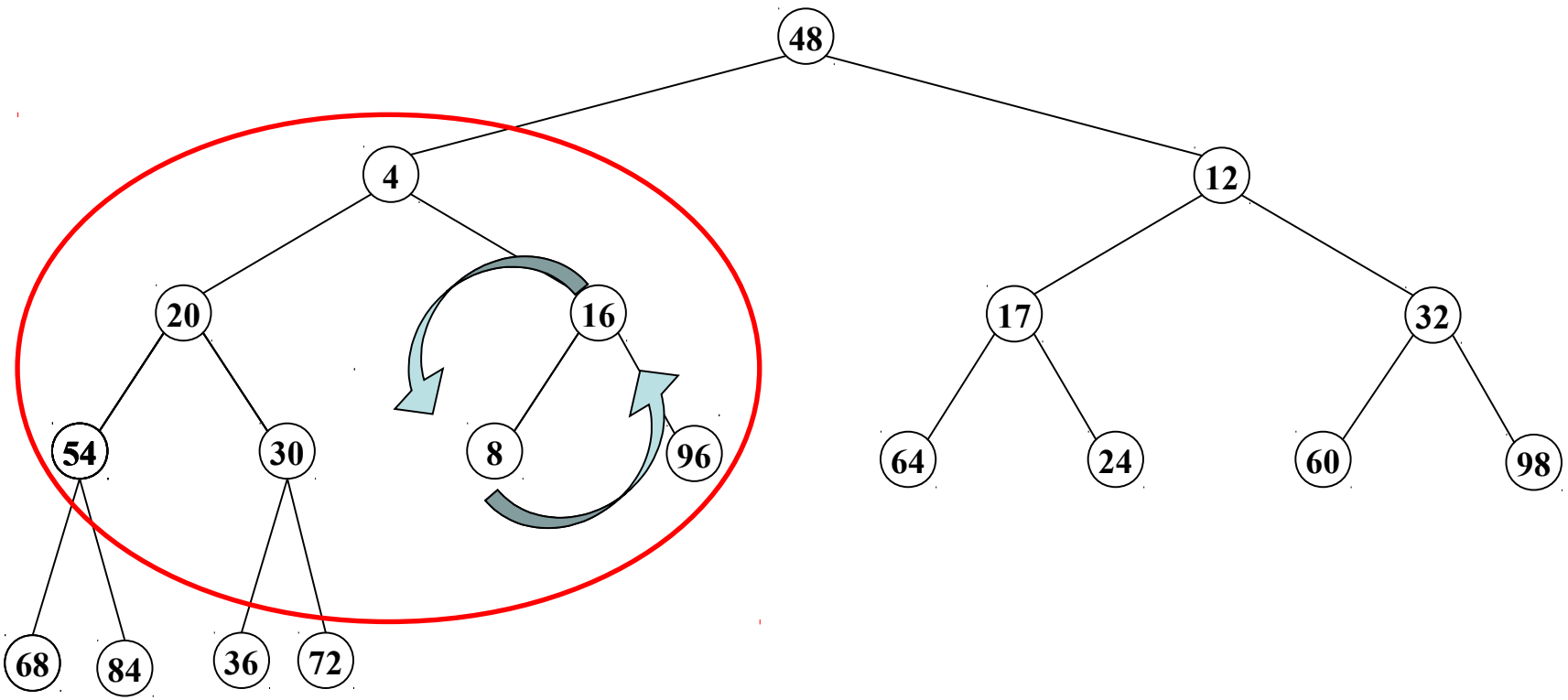
# Constructing a MinHeap -

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 | 96 | 64 | 17 | 60 | 98 | 68 | 84 | 36 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|



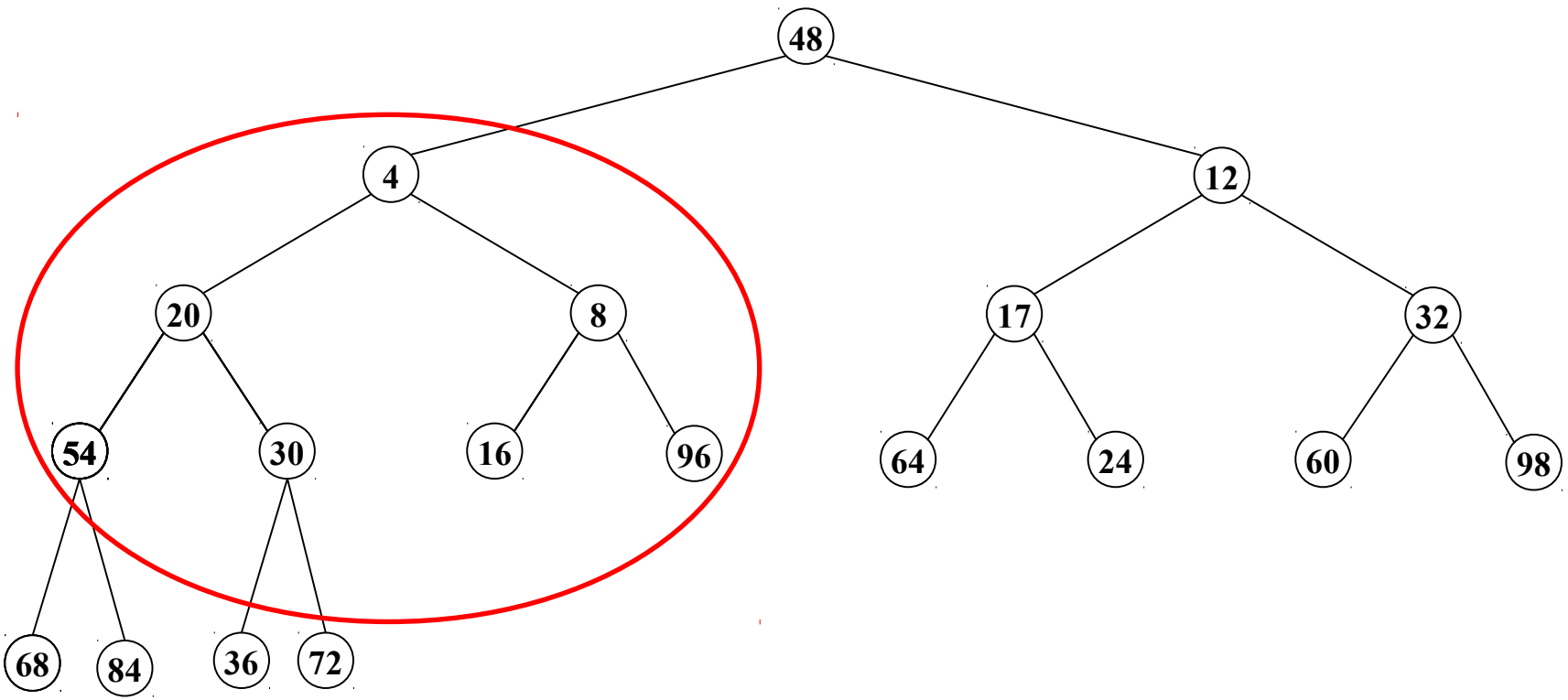
# Constructing a MinHeap -

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 | 96 | 64 | 17 | 60 | 98 | 68 | 84 | 36 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|

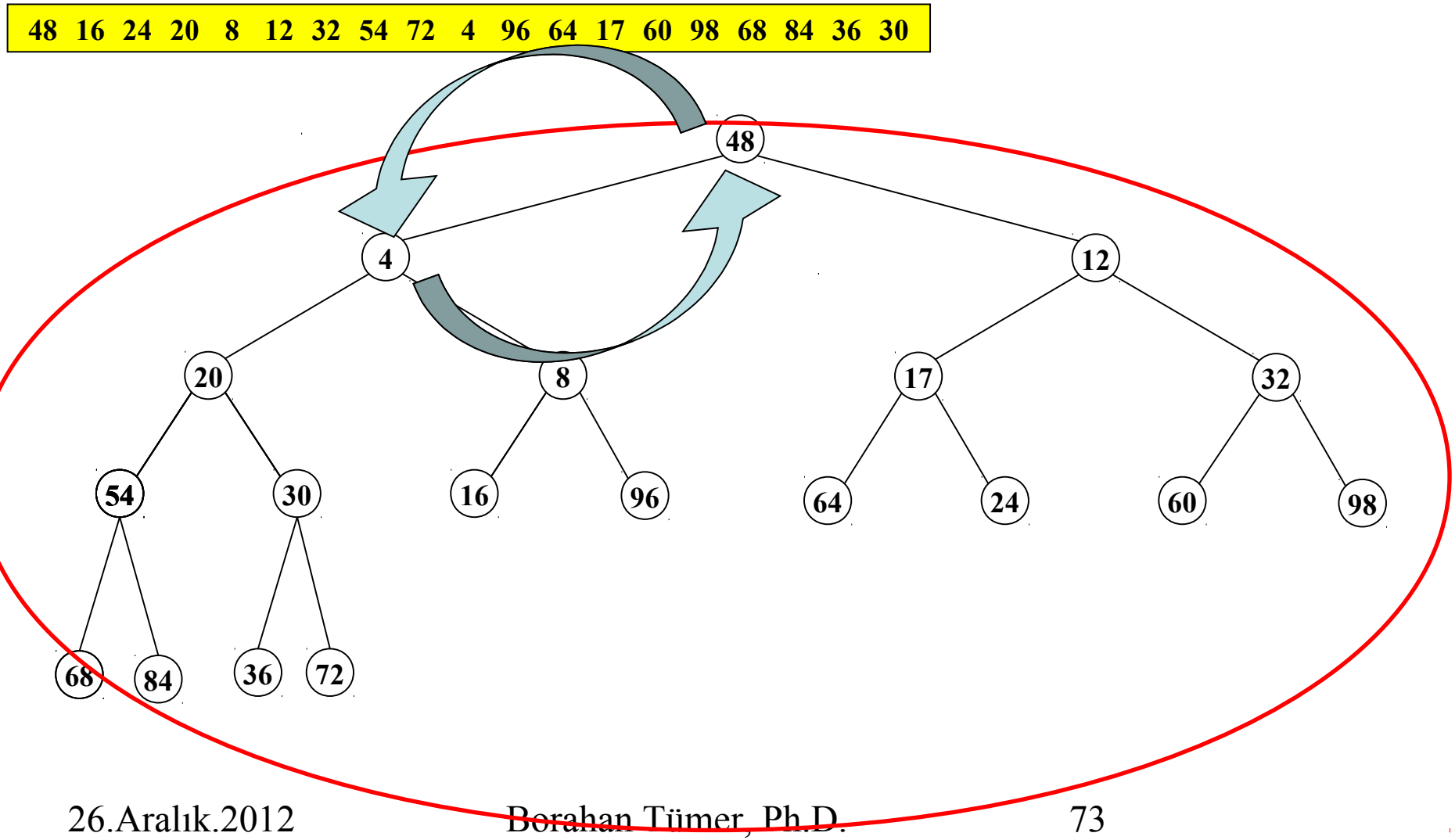


# Constructing a MinHeap -

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 | 96 | 64 | 17 | 60 | 98 | 68 | 84 | 36 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|

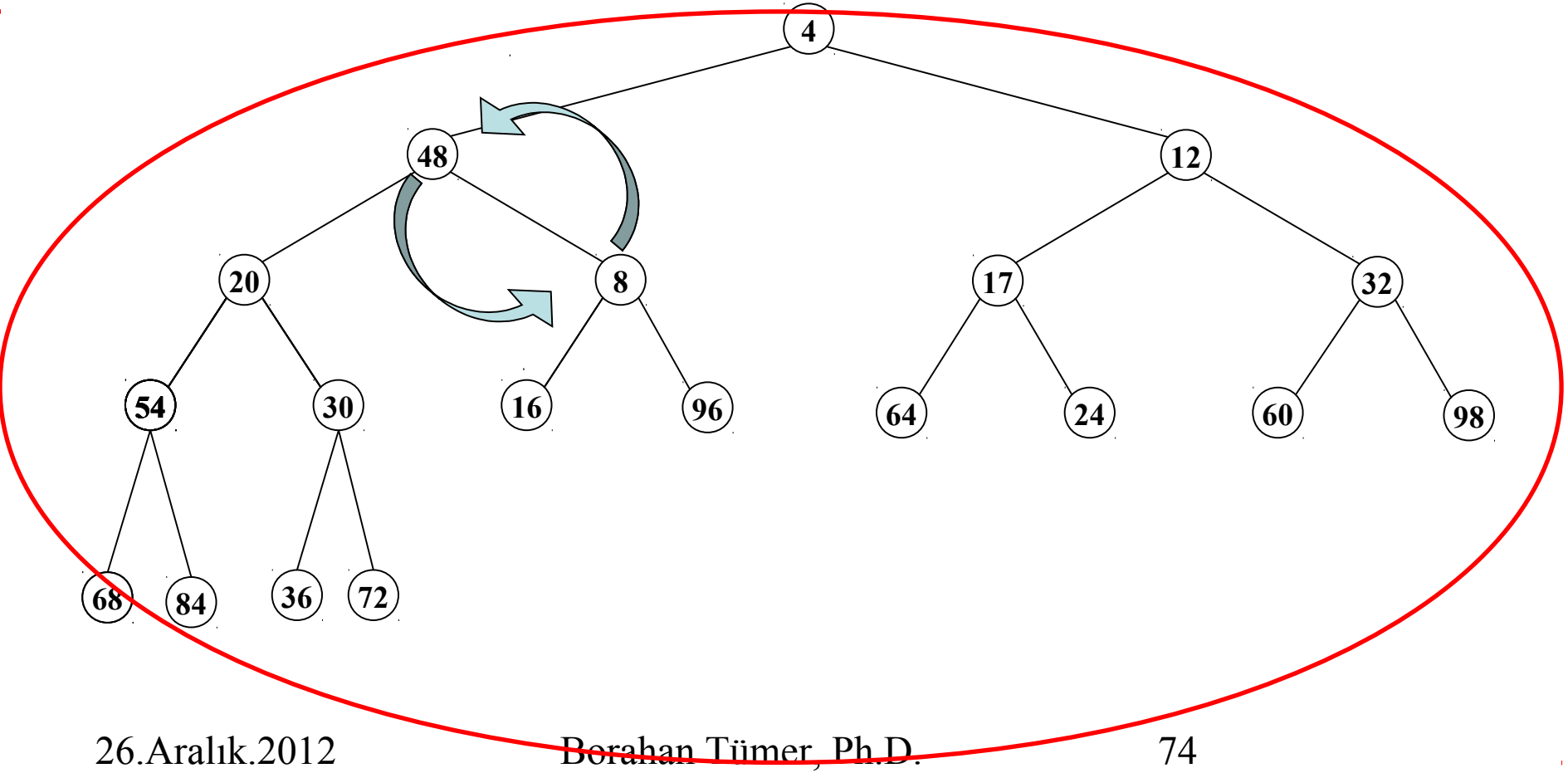


# Constructing a MinHeap -



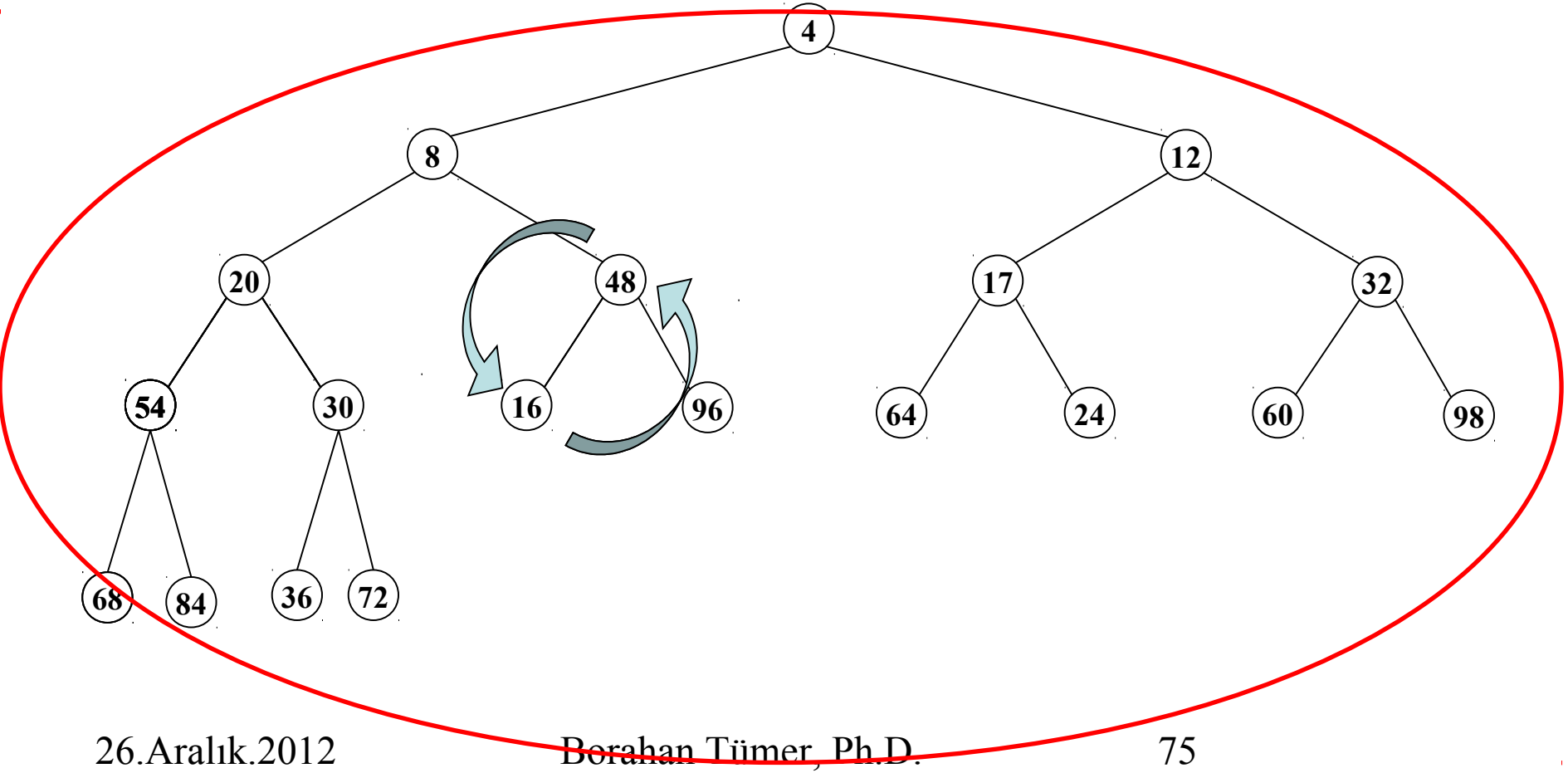
# Constructing a MinHeap -

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 | 96 | 64 | 17 | 60 | 98 | 68 | 84 | 36 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|



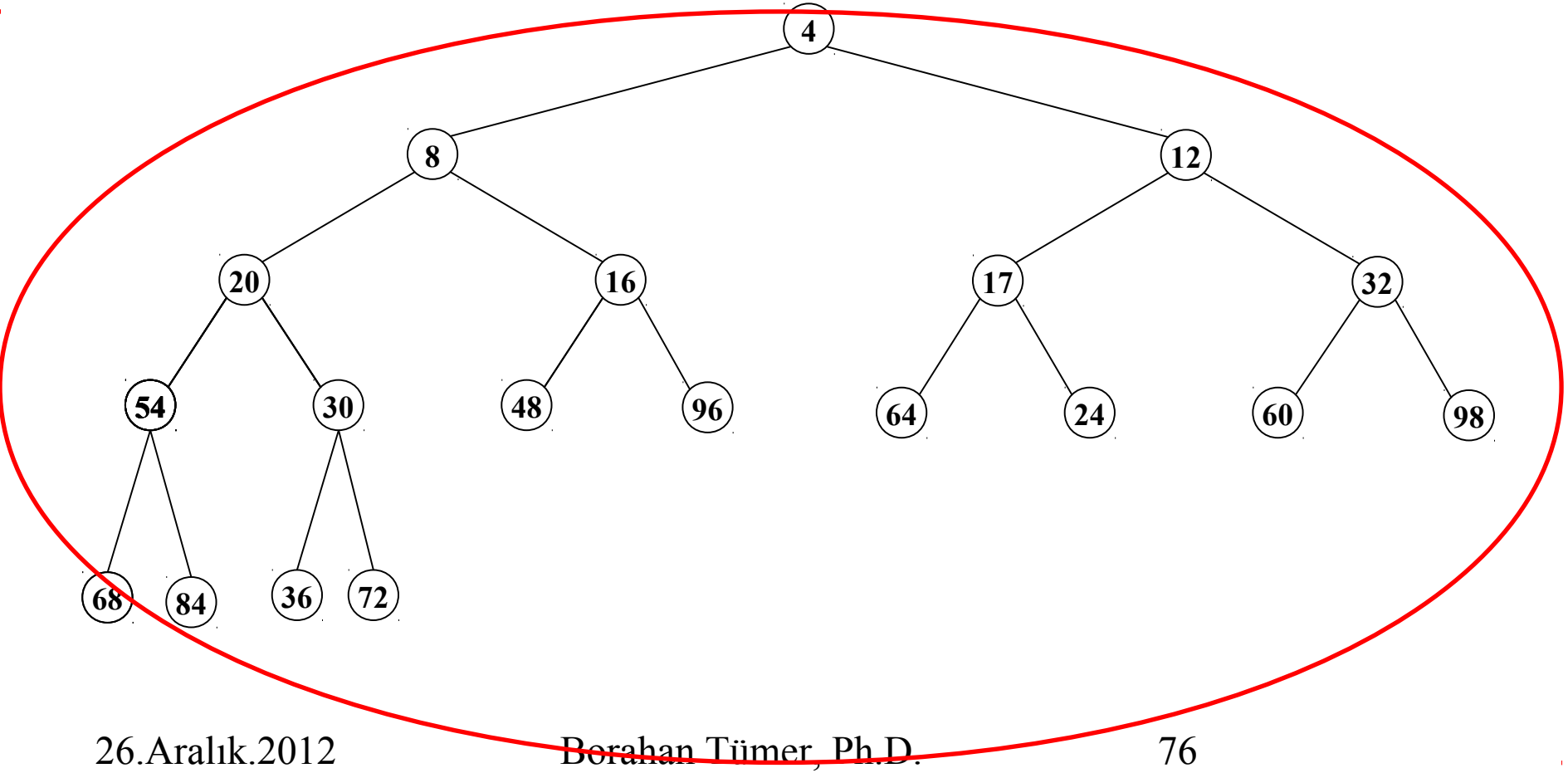
# Constructing a MinHeap -

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 | 96 | 64 | 17 | 60 | 98 | 68 | 84 | 36 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|



# Constructing a MinHeap -

| 48 | 16 | 24 | 20 | 8 | 12 | 32 | 54 | 72 | 4 | 96 | 64 | 17 | 60 | 98 | 68 | 84 | 36 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|



# d-Heaps

- **d-Heaps**
- A simple generalization to binary heaps is a d-heap,
- which is exactly like a binary heap except that all nodes have *d* children (i.e., a binary heap is a 2-heap).

# Leftist Heaps

# Motivation for Leftist Heaps

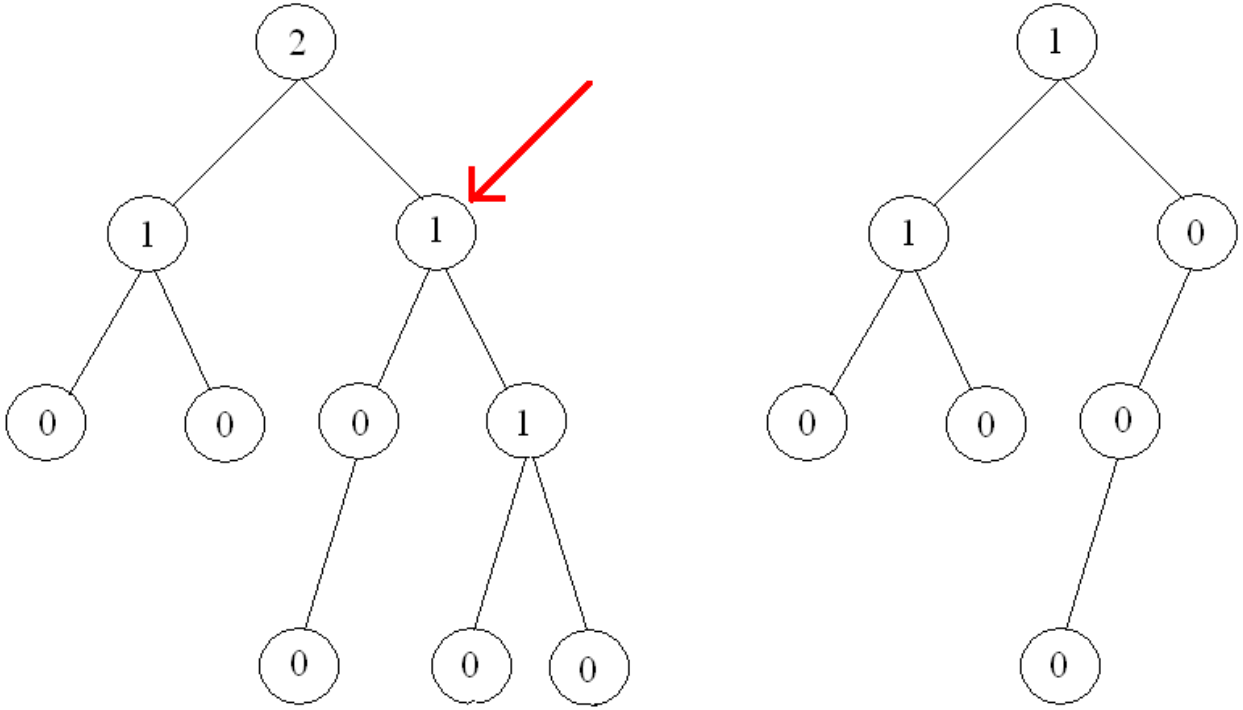
- Leftist heaps make *merging* possible in *O(log n)* time (*log n* insertions each with *O(1)* average time) using only an array as in binary heaps.
- LHs have both
  - a *structural* property, and
  - an *ordering* property.
- A LH has the *same heap order property*.
- A LH is a *binary tree*.
- Difference between a LH and a binary heap is
  - a *LH is not perfectly balanced*.

# Leftist Heap Property

- Definition:
  - *Null path length* of a node X, *Npl*(X), is defined as the *length of the shortest path from X to a node without two children*.
- By definition, Npl(NULL)=-1.
- Npl(leaf)=0.
- LH property is that for every node X in the heap,
  - *Npl(LCX) ≥ Npl(RCX)*
  - where LCX and RCX denote the left child and the right child of node X, respectively.

# Two binary trees

**Both are leftist heaps?**



# Leftist Heap Operations (*Merging*)

- The fundamental operation is *merging*.
- Two solutions
  - *recursive* version
  - *non-recursive* version
- Check and make sure that both binary trees are actually LHs!

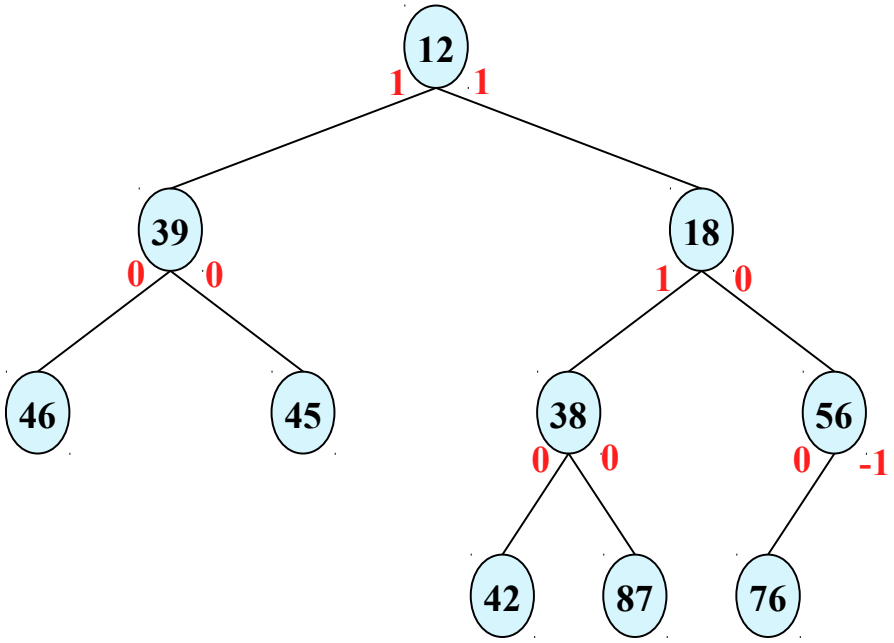
# *Recursive Merging Algorithm*

1. Input: two LHs.
2. First check that both binary trees are LHs.
3. If either heap is empty, then the result of the merge is the other heap.
4. If not, then compare the roots
5. Recursively merge the heap with the larger root with right subheap of the heap with the smaller root.

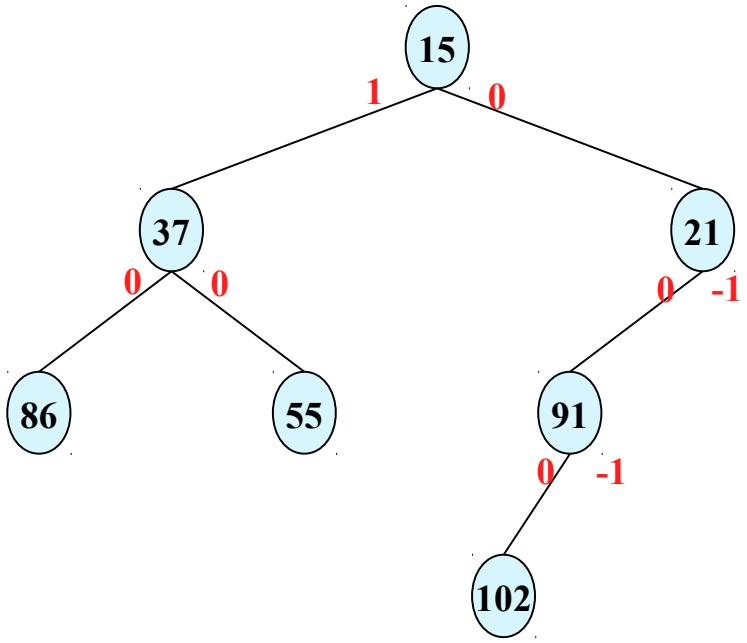
# *Recursive Merging Algorithm*

6. The recursive function at Step 5 will invoke itself until the base condition at Step 3 (i.e., that one LH is empty) will be attained. At this point the execution will start returning step by step to the original function call while building up the merged heap starting from the bottom level.
7. At each step, check if the LH property is violated. If so, swap the right child and the left child.
8. After each swap, compute the new Npl(LH) by adding 1 to the Npl(new RC)
9. End of the recursive algorithm

# Merging Example




H1

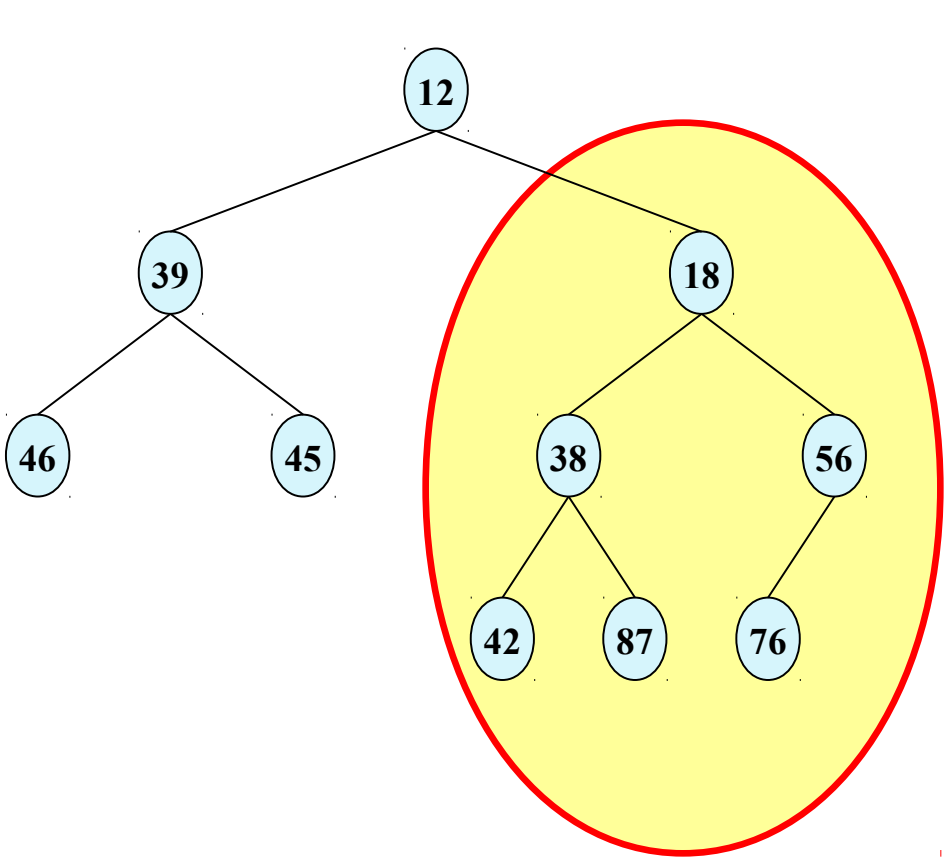



H2

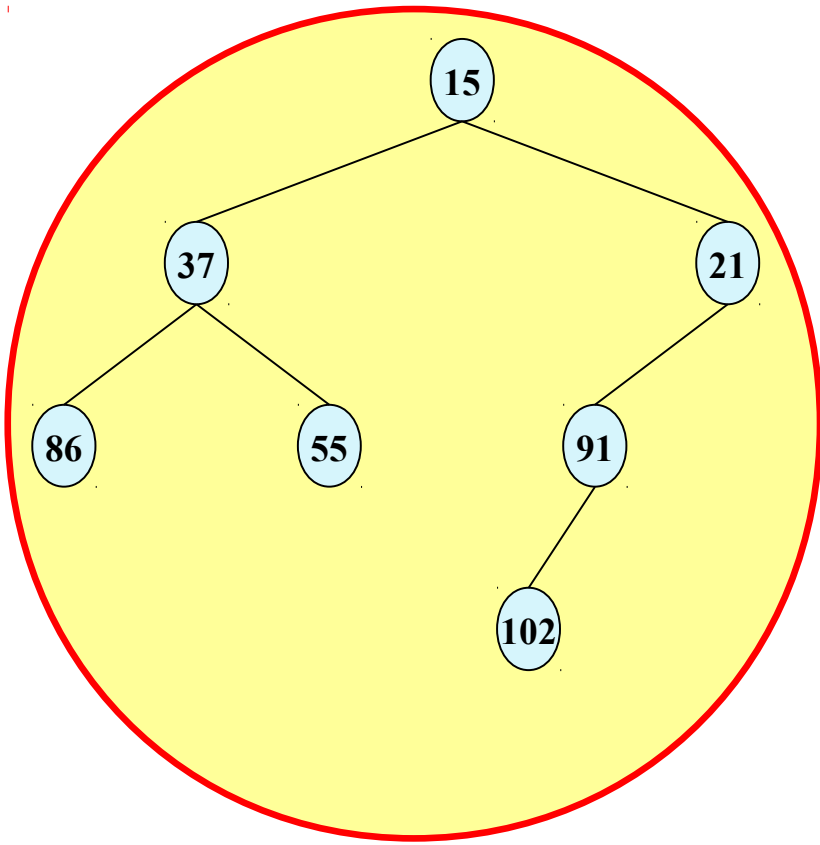
# *Recursive Merging Algorithm*

1. Input: two LHs H1 and H2 ( on slide 65 ).
2. Both binary trees are LHs.
3. No heap is empty.
4. 12 < 15
5. Recursively merge the heap with 15 with right subheap of the heap with 12.

# Merging Example




H1

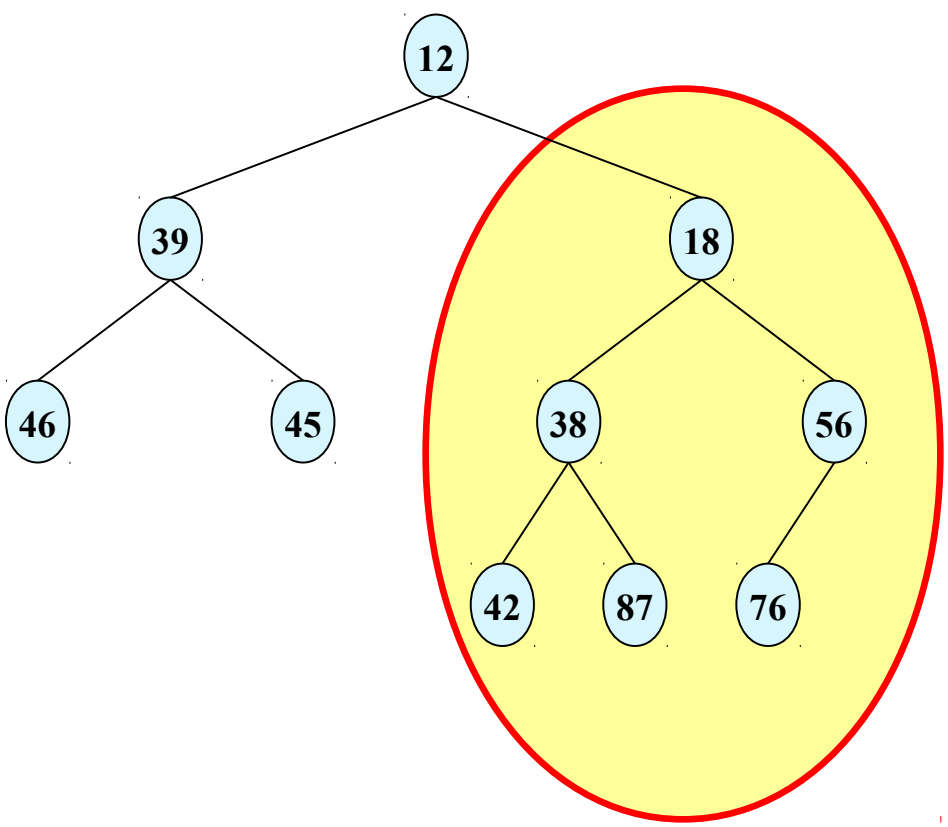



H2

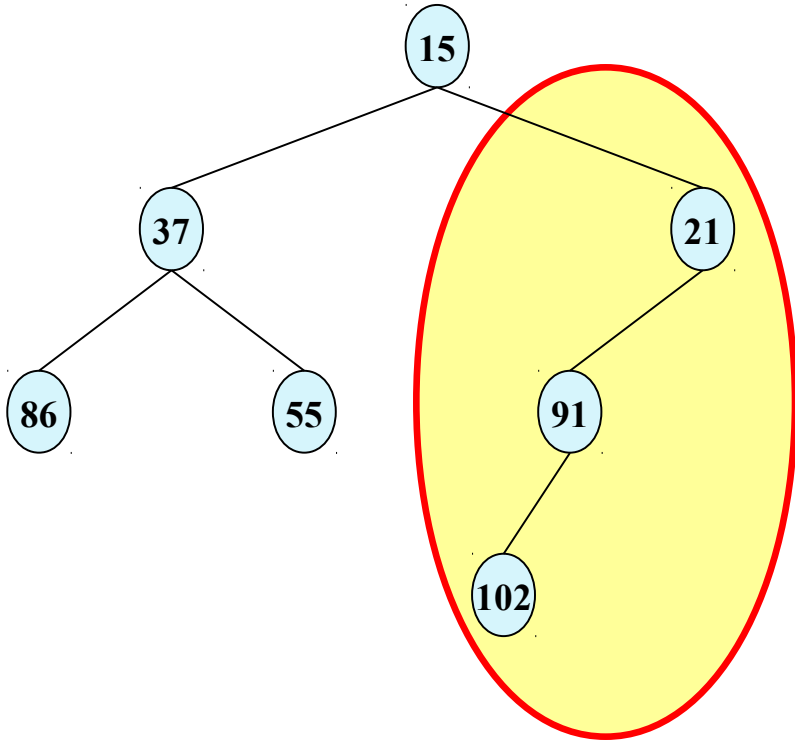
# *Recursive Merging Algorithm*

1. Input: two LHs H1 (within the red ellipse) and H2 (see slide 67).
2. Both binary trees are LHs.
3. No heap is empty.
4. 15 < 18
5. Recursively merge the heap with 18 with right subheap of the heap with 15.

# Merging Example




H1

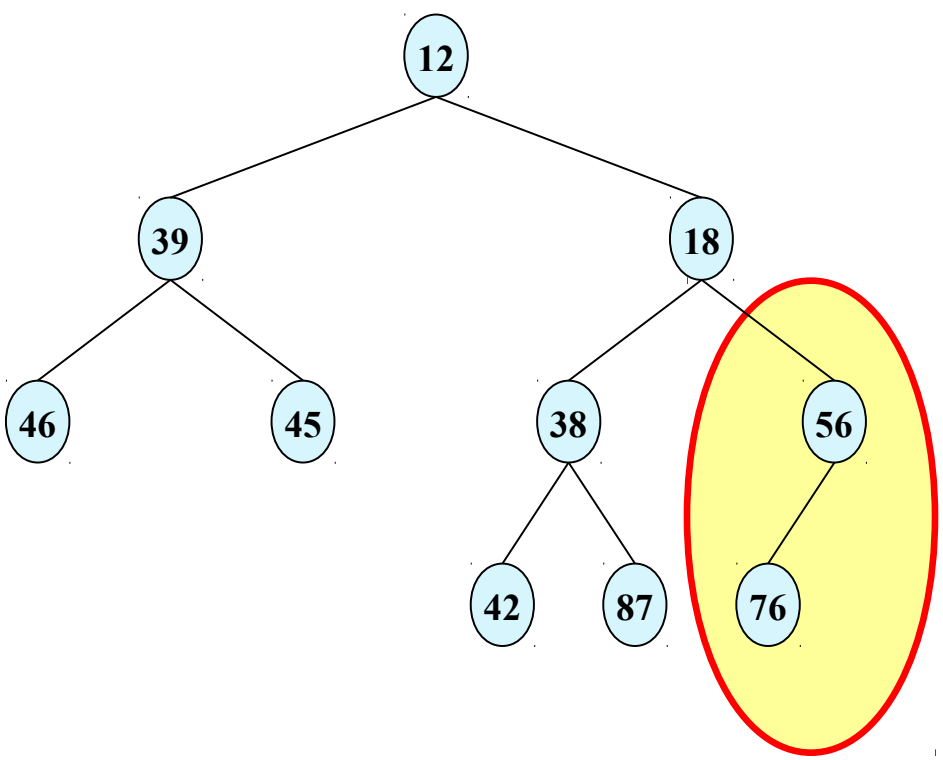


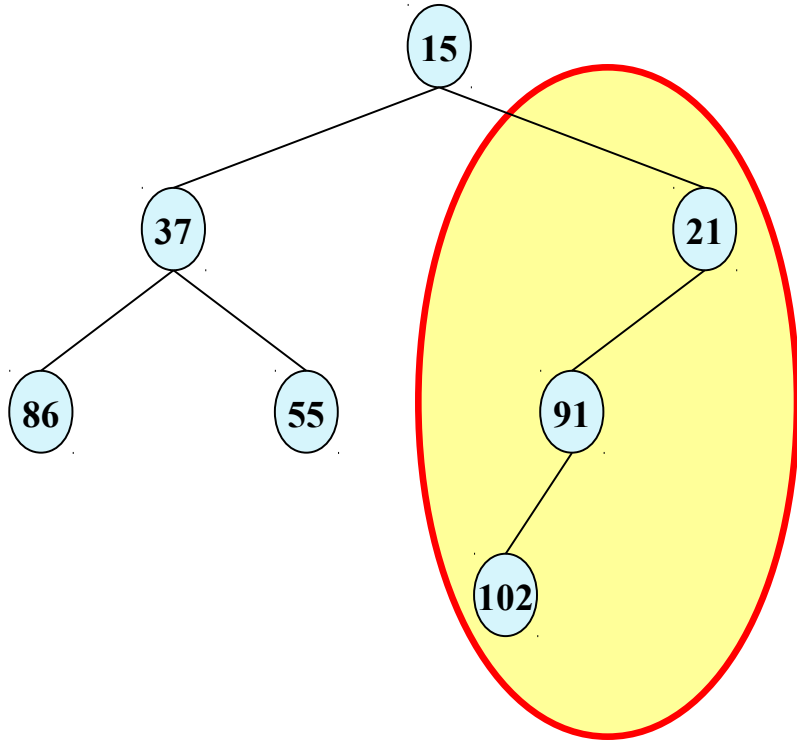

H2

# *Recursive Merging Algorithm*

1. Input: two LHs H1 and H2 ( both within the red ellipses) (see slide 69).
2. Both binary trees are LHs.
3. No heap is empty.
4. 18 < 21
5. Recursively merge the heap with 21 with right subheap of the heap with 18.

# Merging Example



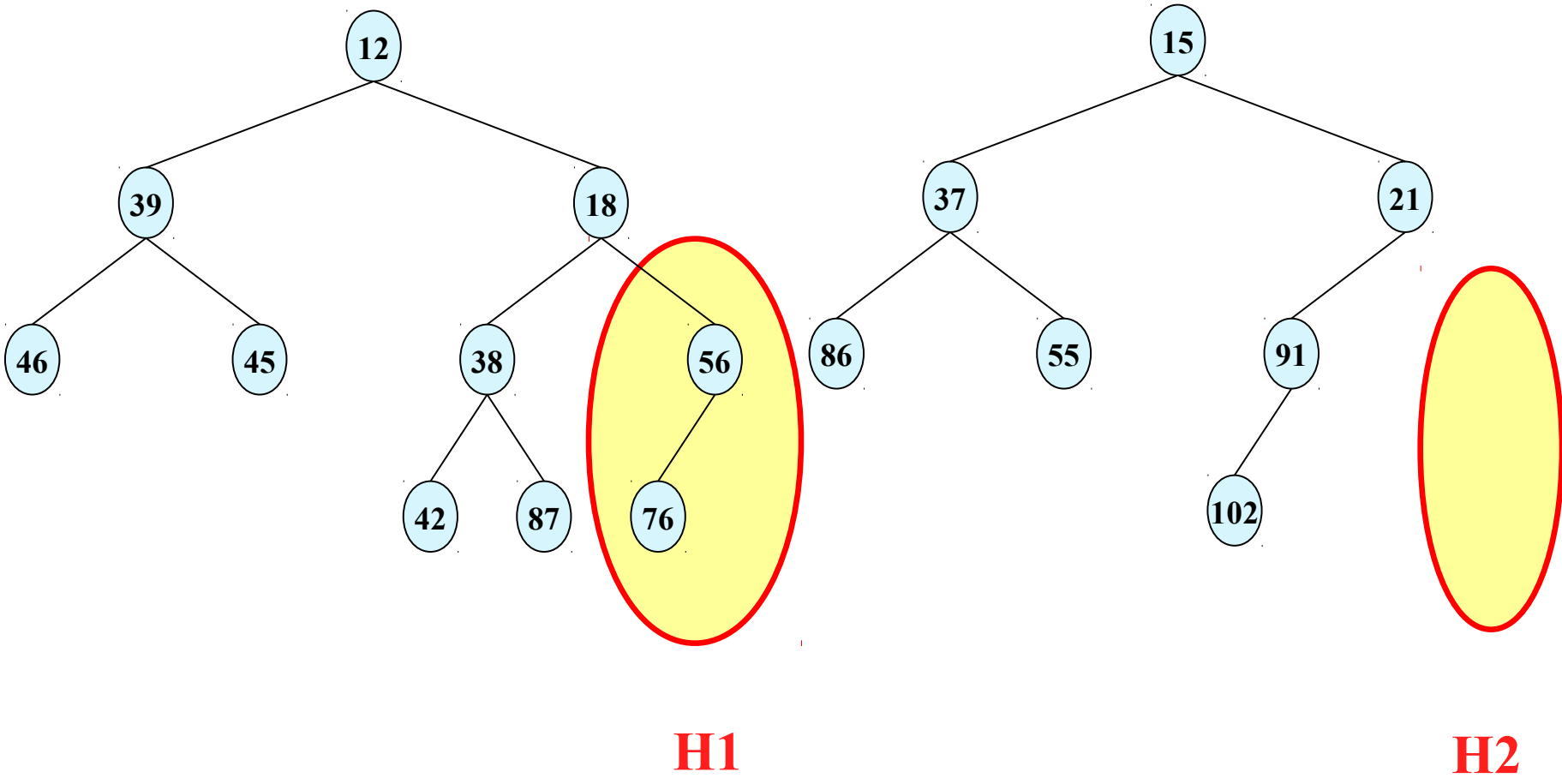




**H1** **H2**

# *Recursive Merging Algorithm*

1. Input: two LHs H1 and H2 ( both within the red ellipses) (see slide 71).
2. Both binary trees are LHs.
3. No heap is empty.
4. 21 < 56
5. Recursively merge the heap with 56 with right subheap of the heap with 21.

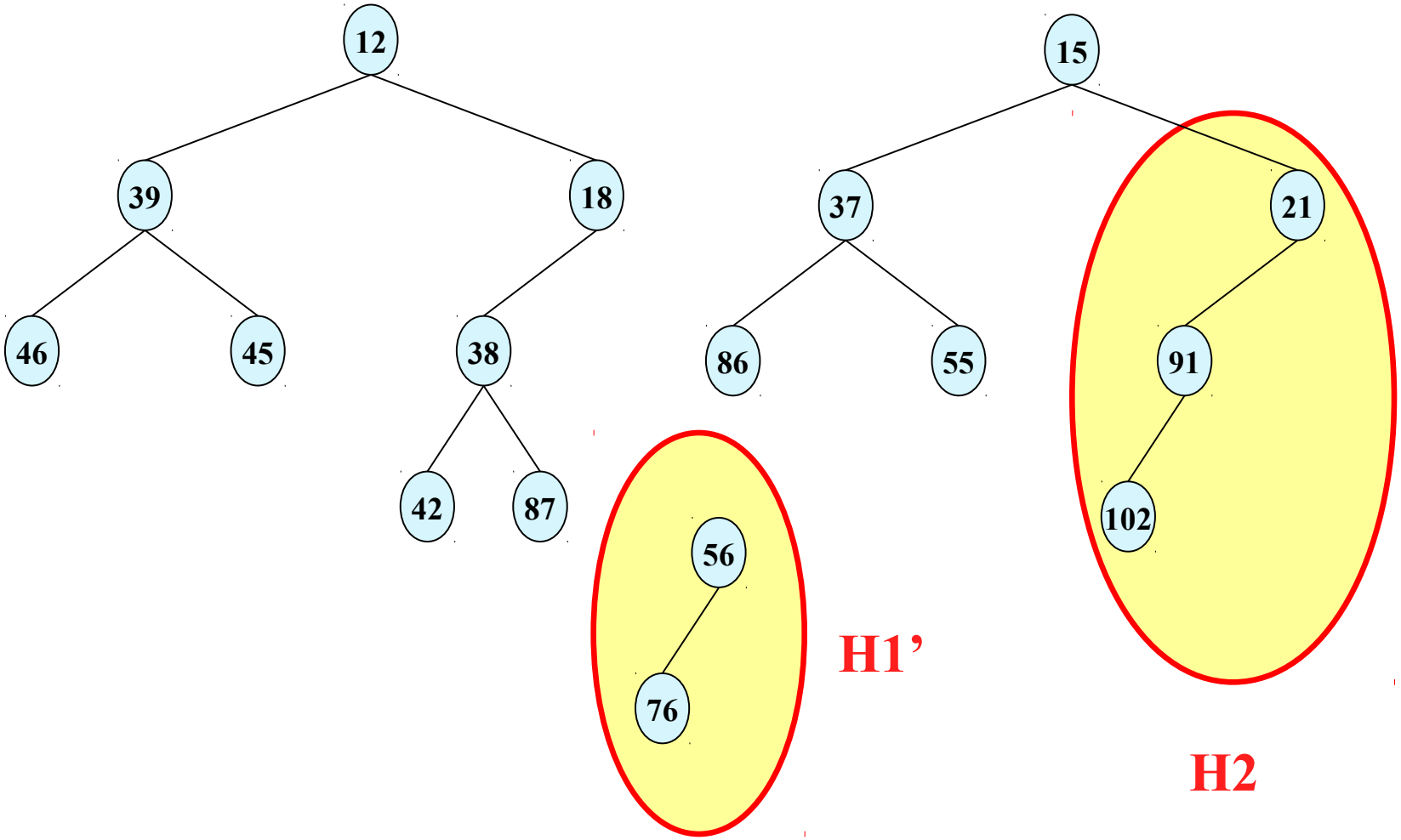
# Merging Example



# *Recursive Merging Algorithm*

1. Input: two LHs H1 and H2 ( both within the red ellipses) (see slide 73).
2. Both binary trees are LHs.
3. The right child of 21 is null. Hence, the resulting heap of merging the heap with root 56 (H1) and the right subheap of heap with root 21 (i.e., an empty heap) is H1 (the heap with root 56).
7. H1' is still a LH. No swaps necessary!
8. Skip;
9. The execution returns with H1' (see slide 75).

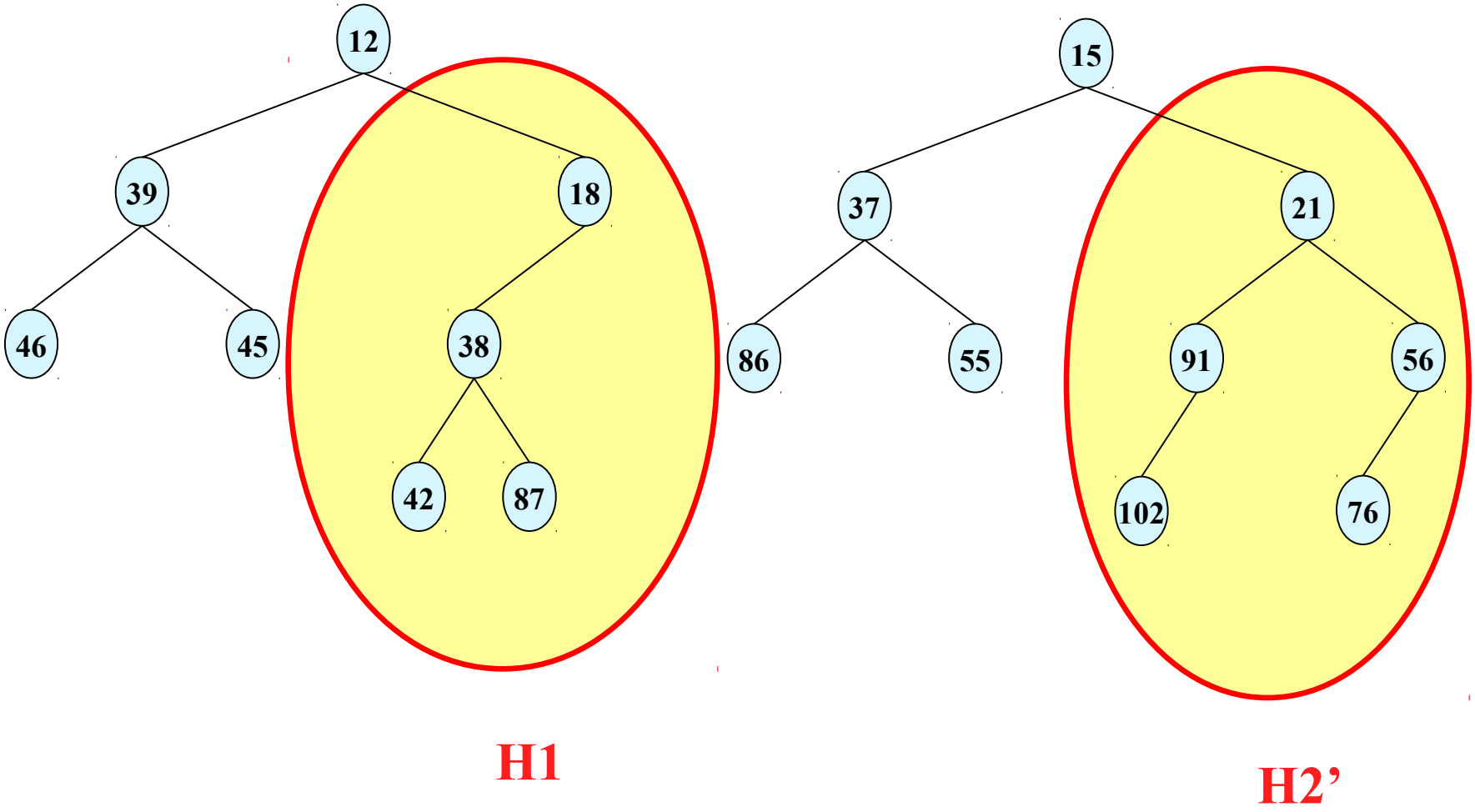
# Merging Example



# *Recursive Merging Algorithm*

6. The return address of execution is step 6 of the instance of merge algorithm at which H1 was the heap with root 56 and H2 was the heap with 21 (see slide 71!). Now, the new H1 (H1') is the same heap; but it is the result of the last recursive merge. At this step, H1' is merged with the current H2 (see slide 75!)
7. H2' is still a LH. No swaps necessary!
8. Skip;
9. The execution returns with H2' (see slide 77).

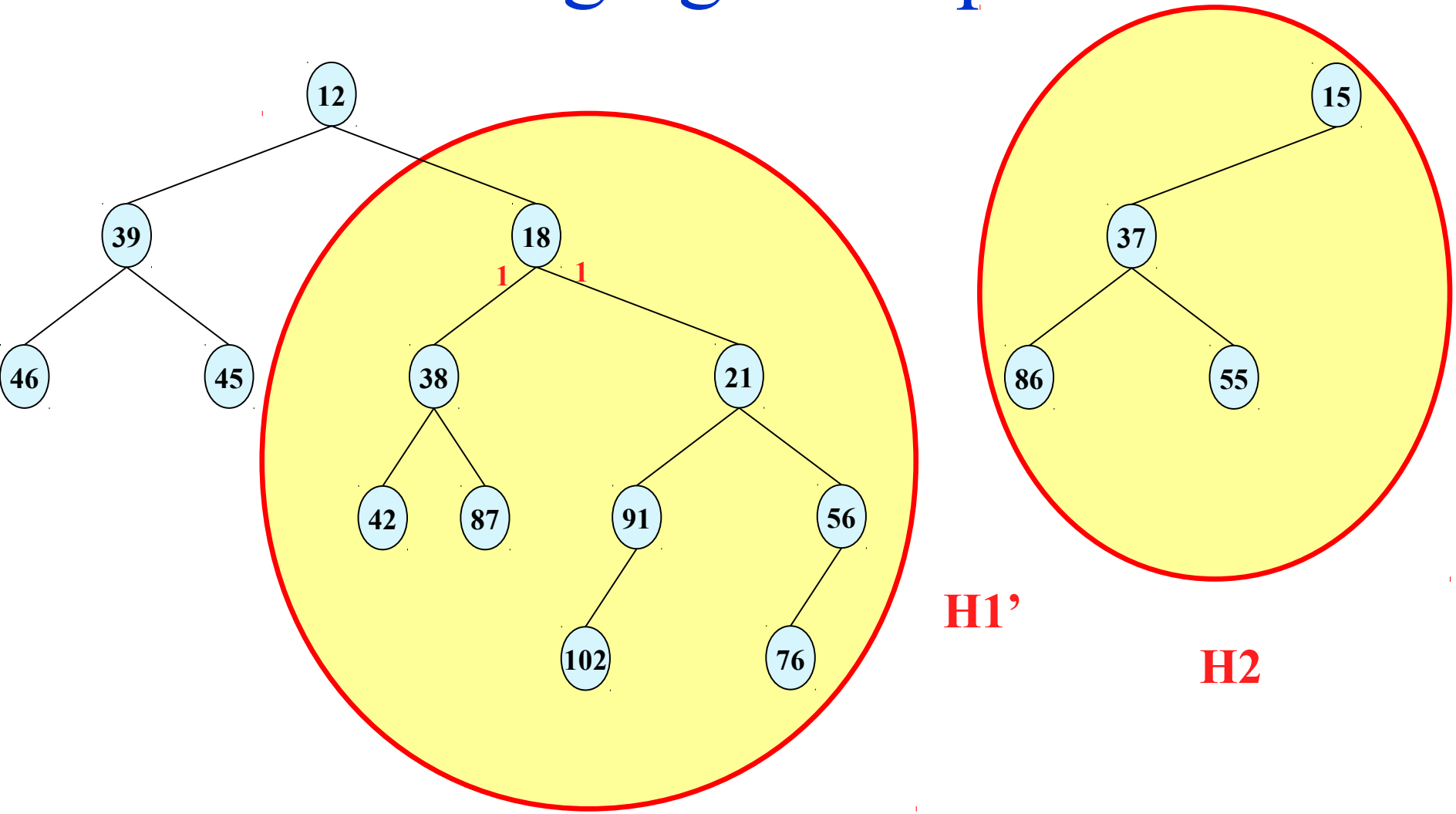
# Merging Example



# *Recursive Merging Algorithm*

6. The return address of execution is step 6 of the instance of merge algorithm at which H1 was the heap with root 18 and H2 was the heap with 21 (see slide 69!). Now, H2' is merged with the current H1 (see slide 77!)
7. H1' is still a LH. No swaps necessary!
8. Skip;
9. The execution returns with H1' (see slide 79).

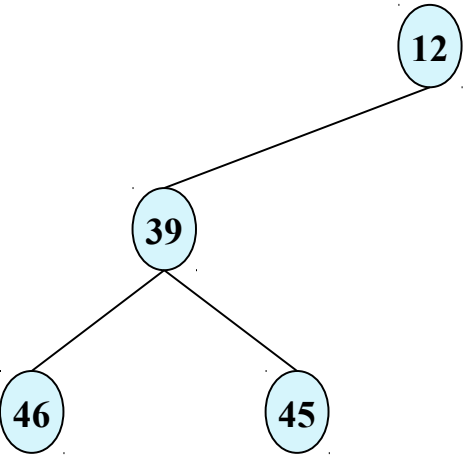
# Merging Example



# *Recursive Merging Algorithm*

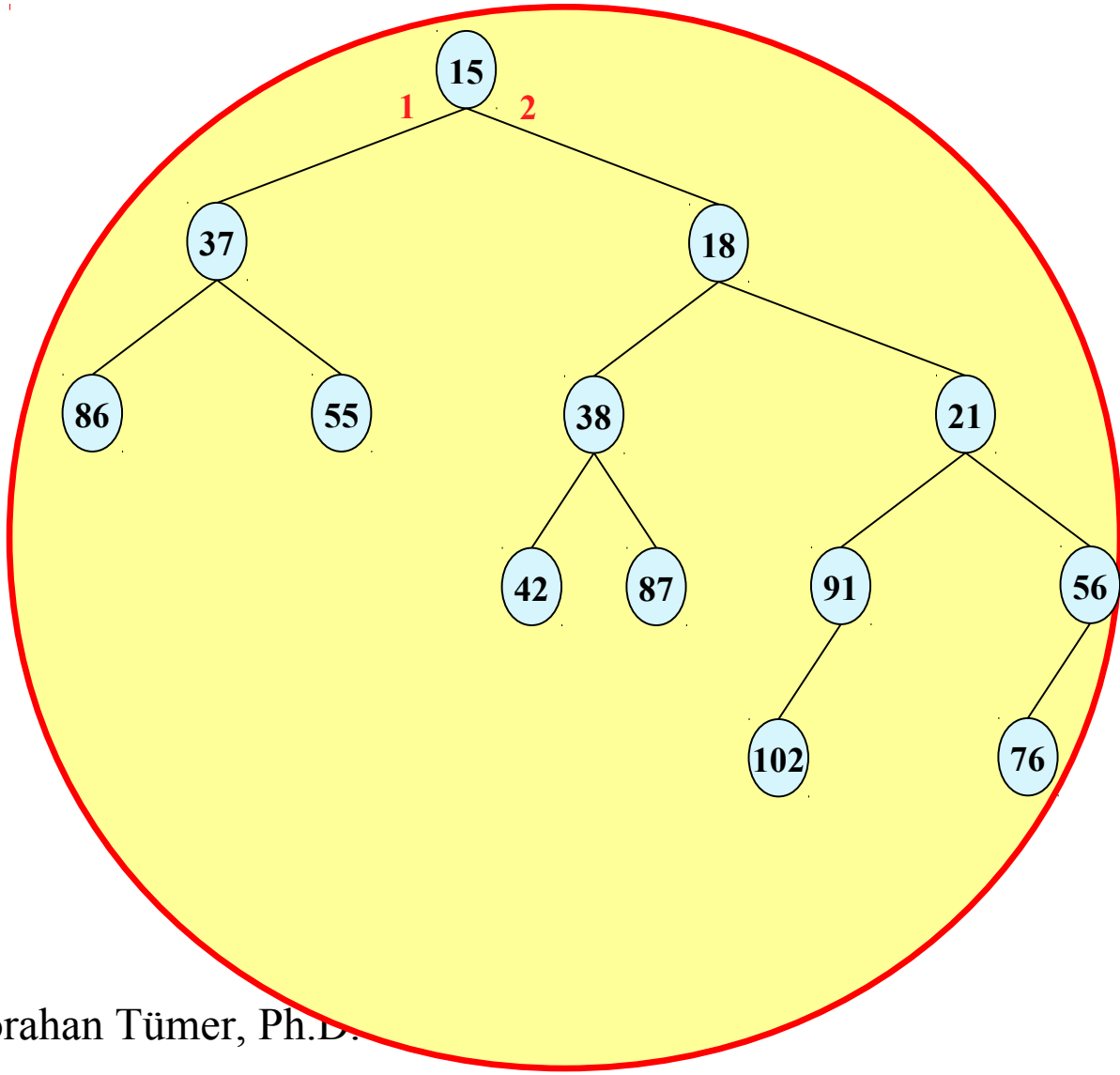
6. The return address of execution is step 6 of the instance of merge algorithm at which H1 was the heap with root 18 and H2 was the heap with 15 (see slide 67!). Now, H1' is merged with the current H2 (see slide 79!)
7. H2' is not a LH (*Npl(LC)<Npl(RC)*). (see slide 81!)
8. We swap LC and RC of 15!
9. Execution returns with H2'' (see slide 82).

# Merging Example

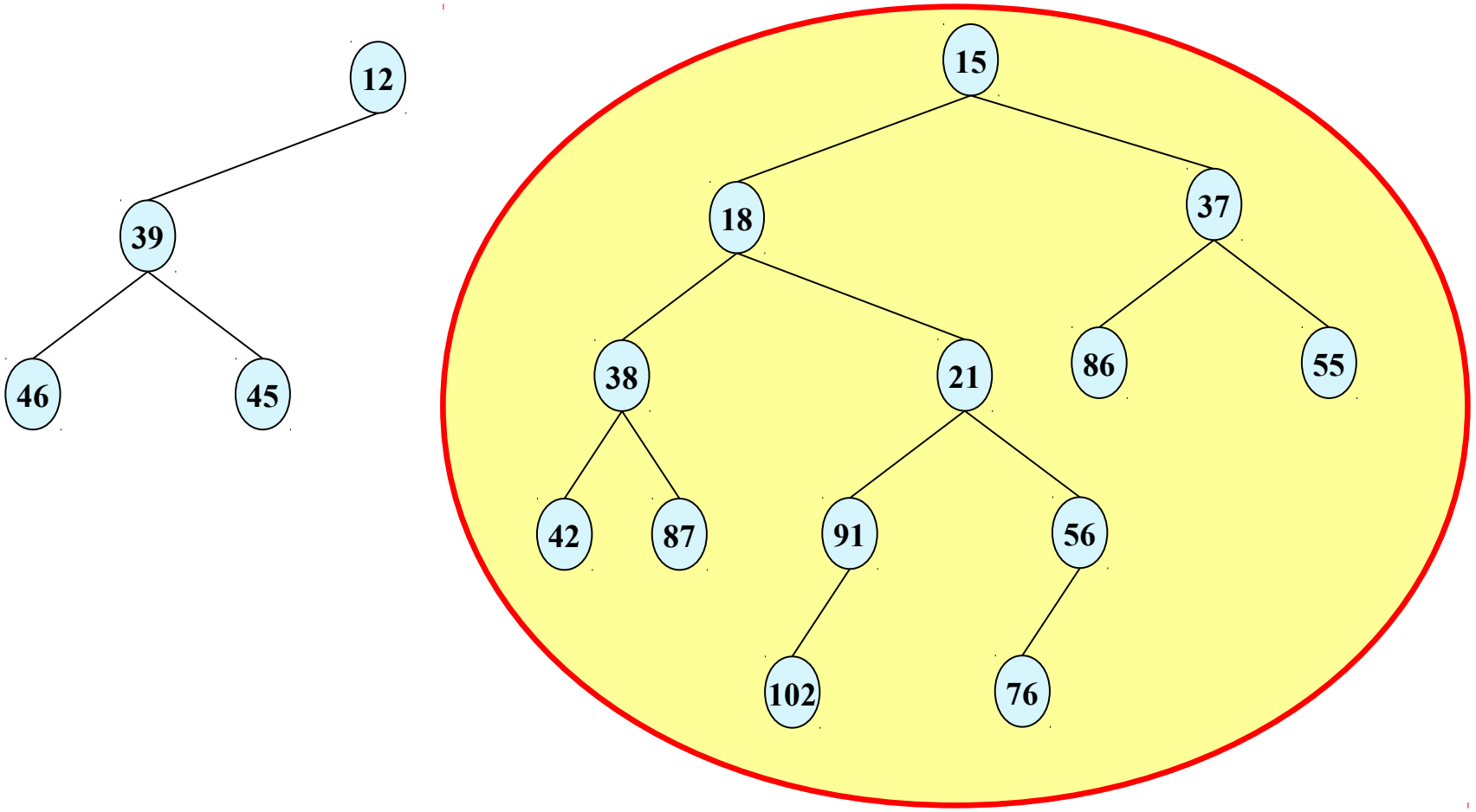



H1

H2’



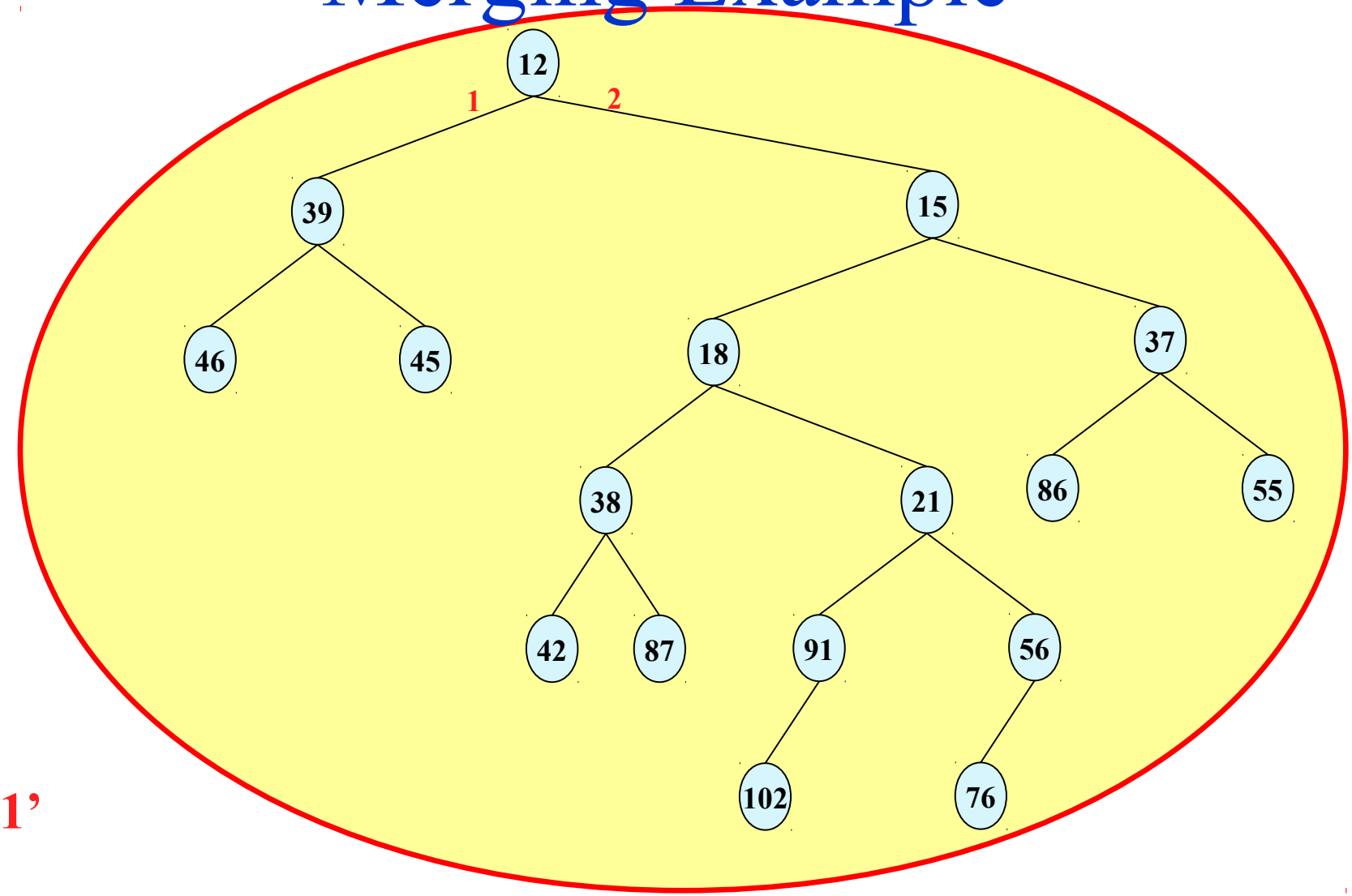
# Merging Example



# *Recursive Merging Algorithm*

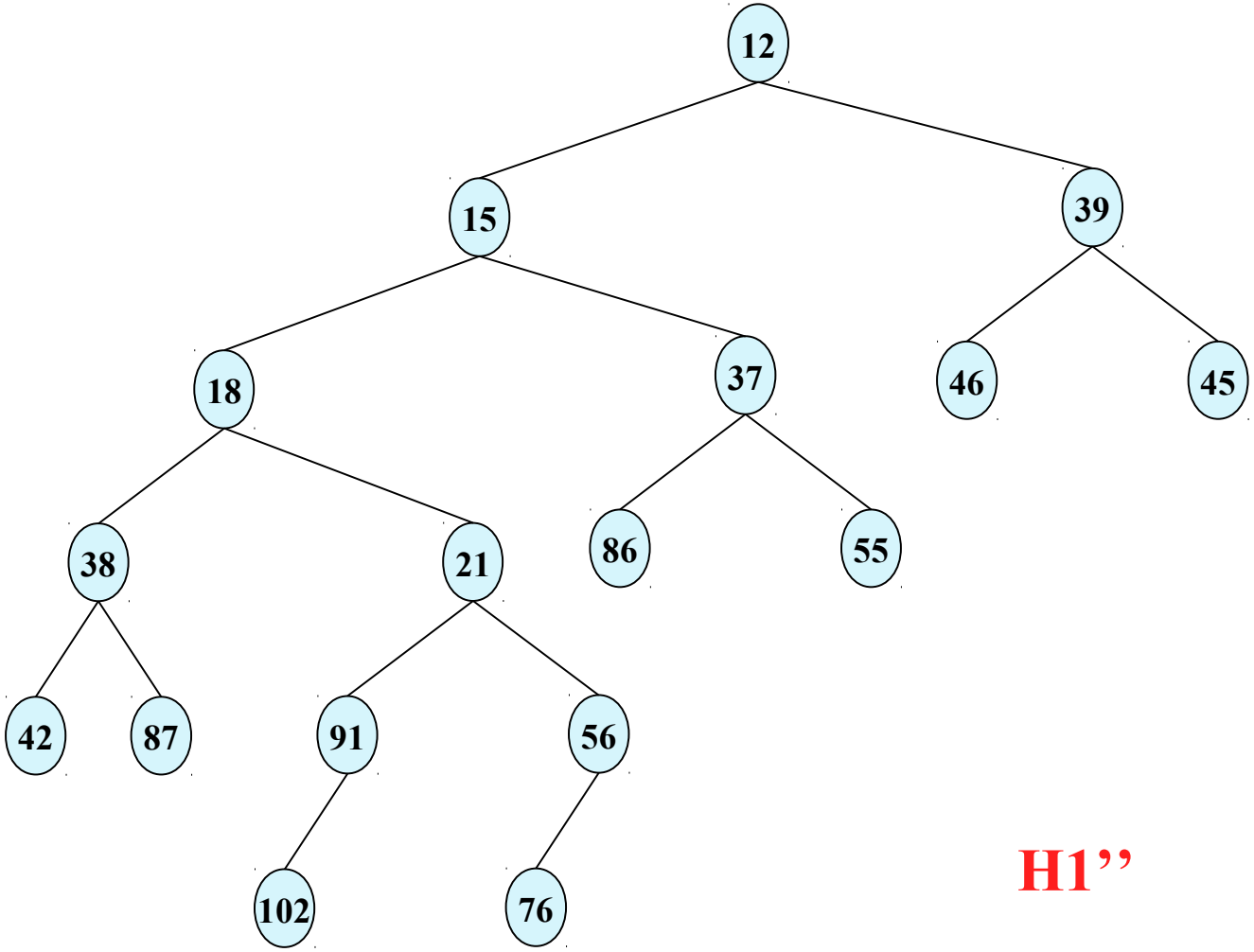
6. The return address of execution is step 6 of the instance of merge algorithm at which H1 was the heap with root 12 and H2 was the heap with 15 (see slide 65!). Now, H2'' is merged with the current H1 (see slide 84!)
7. H1' is not a LH (*Npl(LC)<Npl(RC)*).
8. We swap LC and RC of 12!
9. Execution returns with H1'' (see slide 85).

# Merging Example




**H1'**

# Merging Example




**H1''**

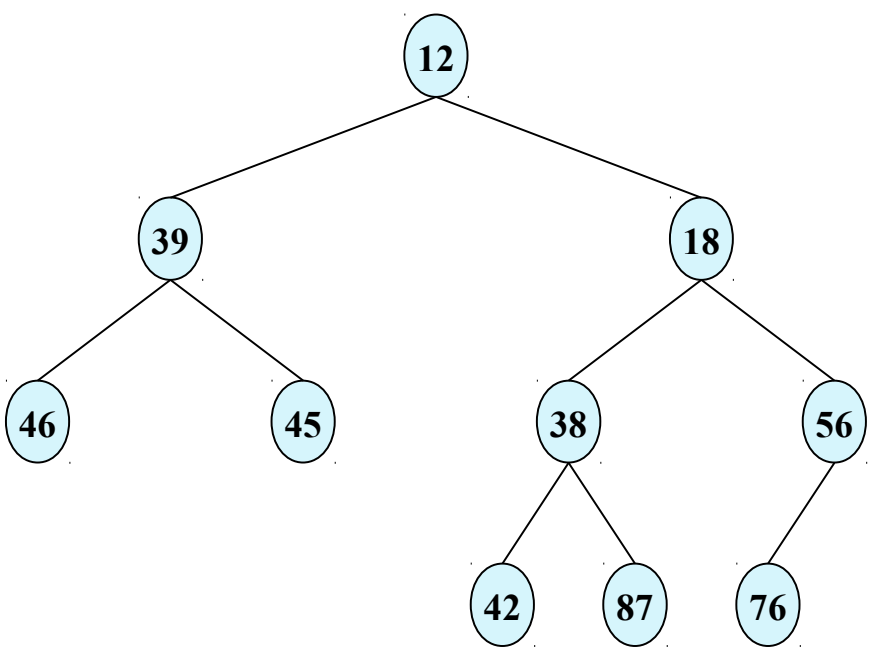
# Non-Recursive Merging Algorithm: First Pass

- *Two passes*
- *First pass*:
  - arrange the nodes on the *right-most path* of both LHs in *sorted order*, keeping their respective left children;
  - create a new tree from the two LHs;
    - sort and arrange the nodes above,
    - make them the right path of the new tree.

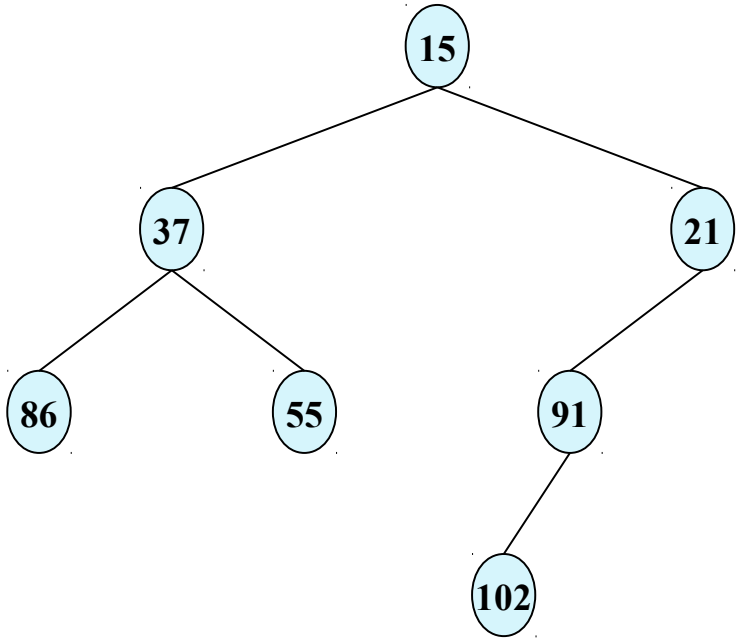
# Non-Recursive Merging Algorithm: Second Pass

- Start a *bottom-up analysis* to
  - check and determine the nodes at which the leftist heap property is violated, and
  - perform a swap at these nodes.

# Merging Example



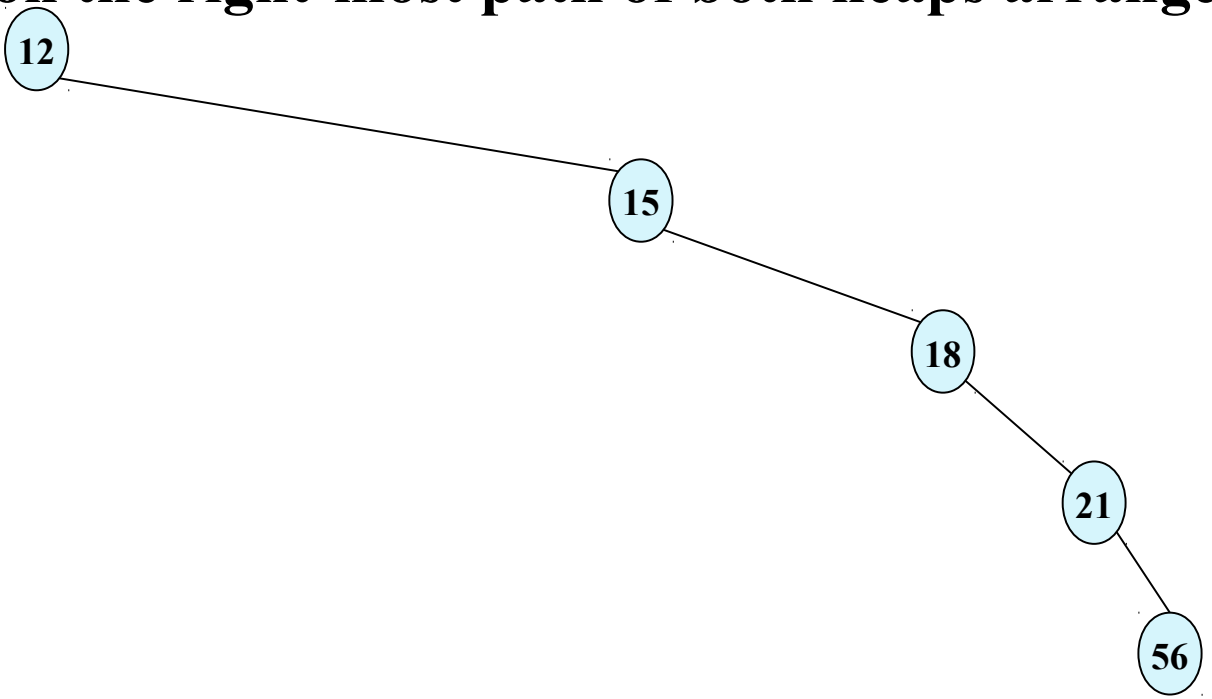

H1




H2

# Merging Example: First Pass

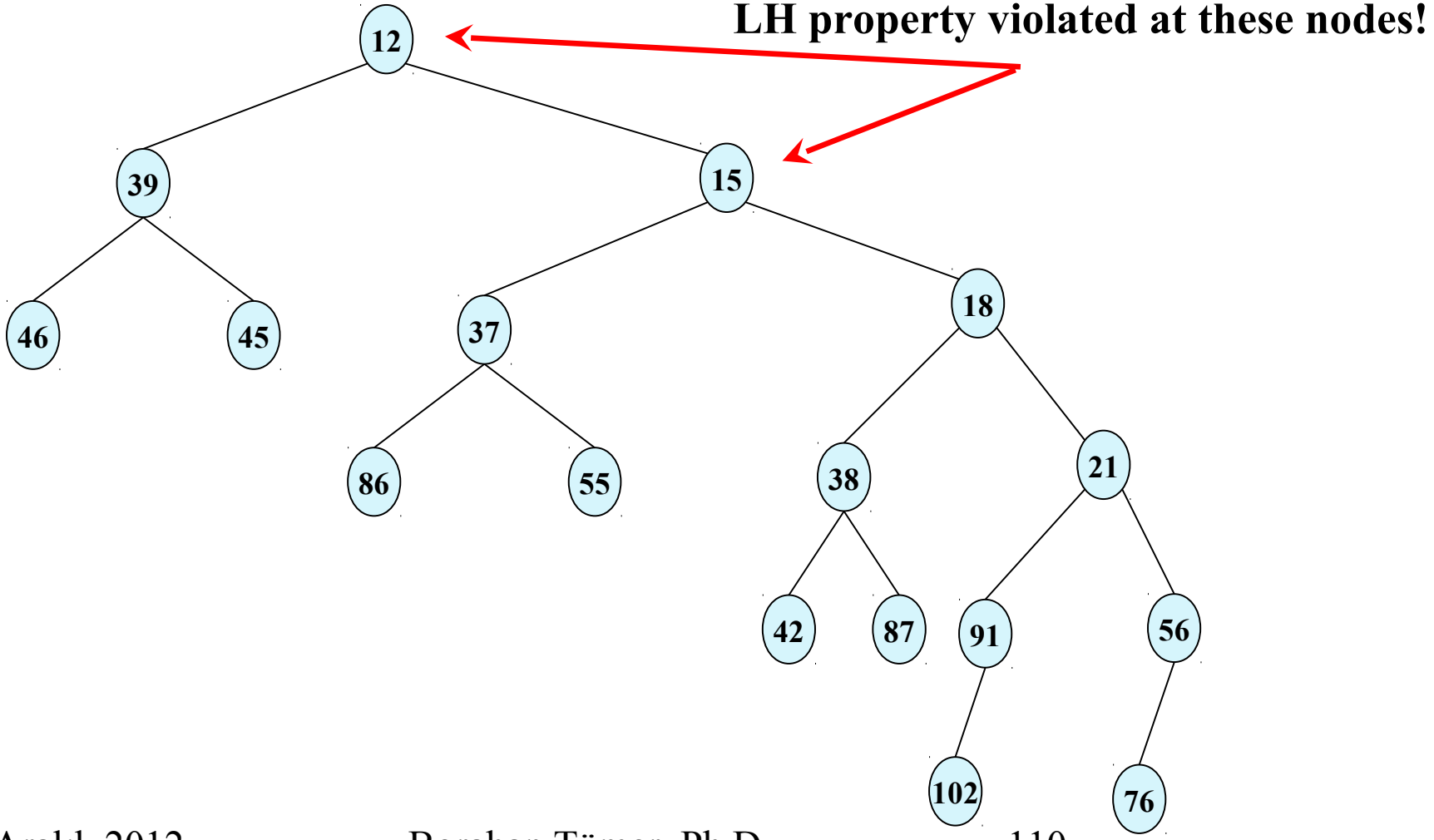

- **Nodes on the right-most path of both heaps arranged!**



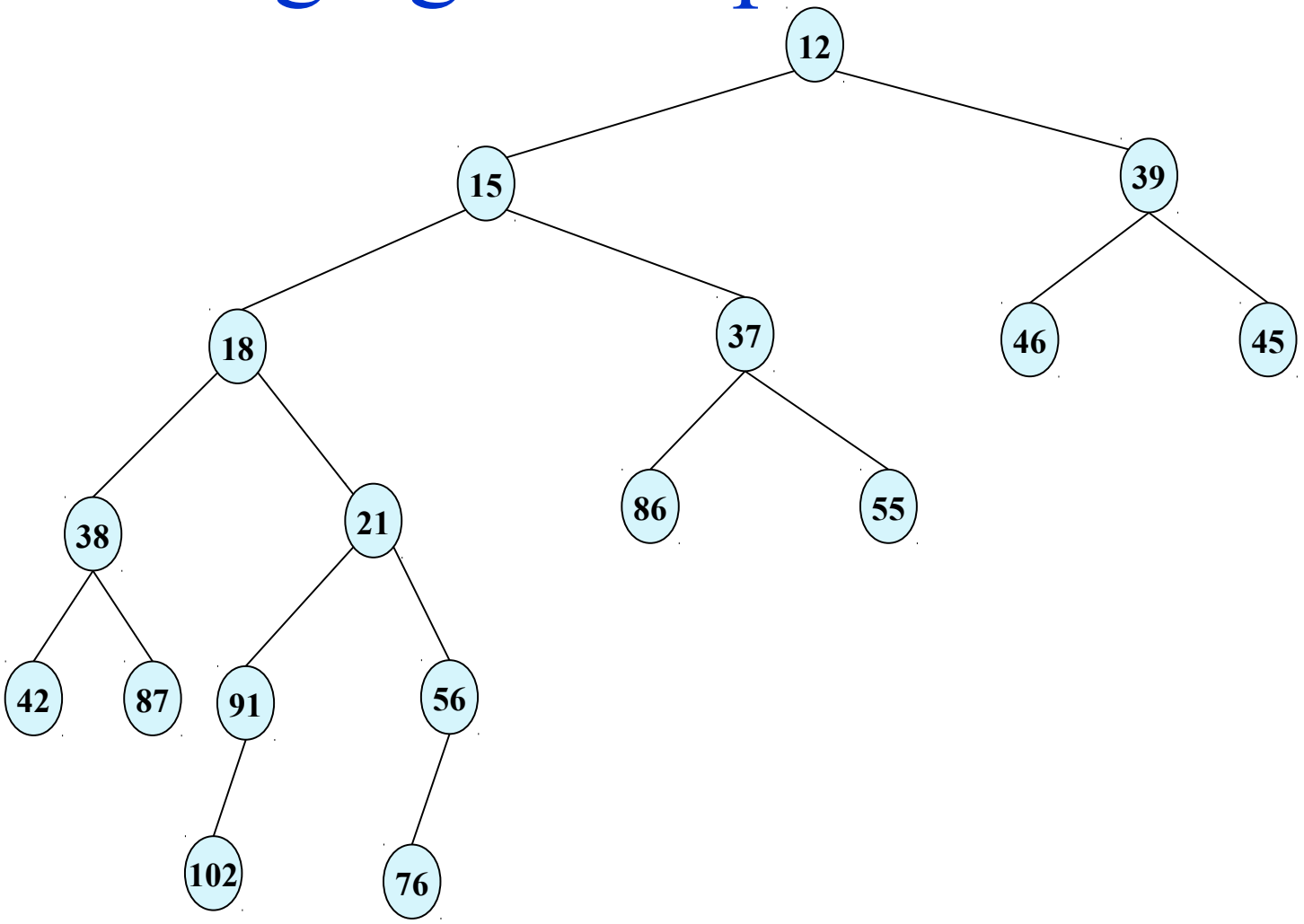

- **A new tree made with the above as its right-most path!**

# Merging Example: First Pass

- **Left children maintained!**



# Merging Example: Second Pass



# Time Analysis of Merging using LHs

Performing the recursive solution is proportional to the sum of the length of the right paths. The work done at each node visited on the right path is constant.

We have *O(log(n))* nodes on the right path in a LH with *n* nodes.

# Binomial Heaps

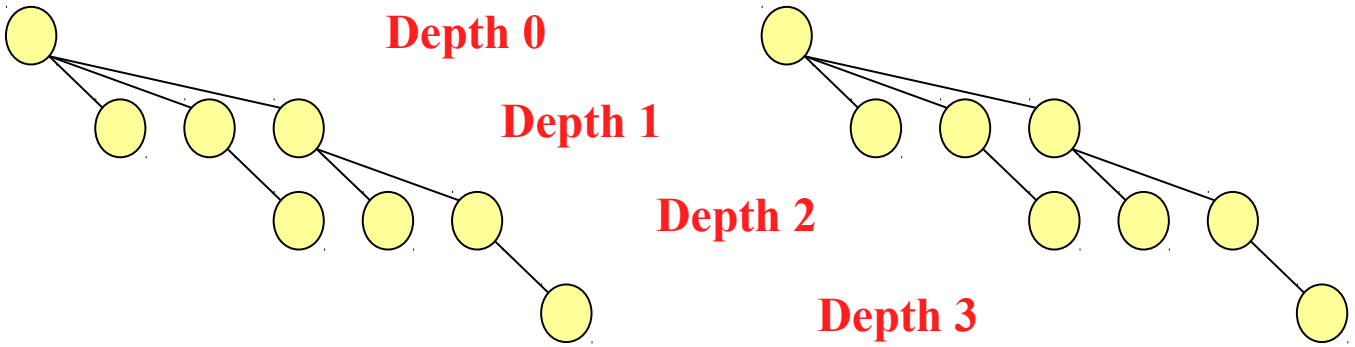
# Motivation for Binomial Heaps

- Leftist Heaps support
  - *merging, insertion, removal, and deleteMin*
- in *O(log(n))* time per operation.
- We know *binary heaps* have a constant (i.e., *O(1)*) insertion time.
- Question: May there be a data structure providing
- *O(1) time for insertion*, and
- *O(log(n)) time for each other operation*.
- This data structure is the so-called *binomial heaps (BHs) or binomial queues*.
- To study BHs we first need to discuss *binomial trees*.

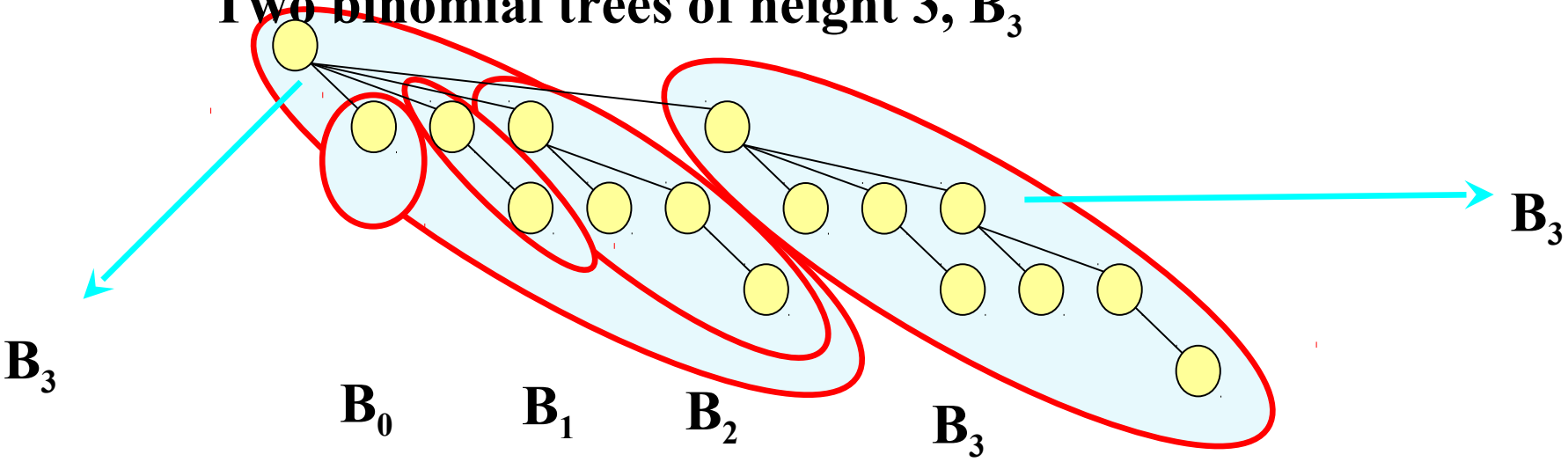
# Binomial Trees

- A *binomial tree* $B_k$ is *an ordered tree* (i.e., a rooted tree in which the children of each node are ordered; the order of the children matters) *defined recursively*.
- The binomial tree $B_0$ consists of a single node. A binomial tree $B_k$ of *height* $k$ is formed by attaching a $B_{k-1}$ to the root of another $B_{k-1}$.
- In the next slide, we see two $B_3$s combined to form a $B_4$.

# Binomial Trees




**Two binomial trees of height 3, $B_3$**




**A binomial tree of height 4, $B_4$**

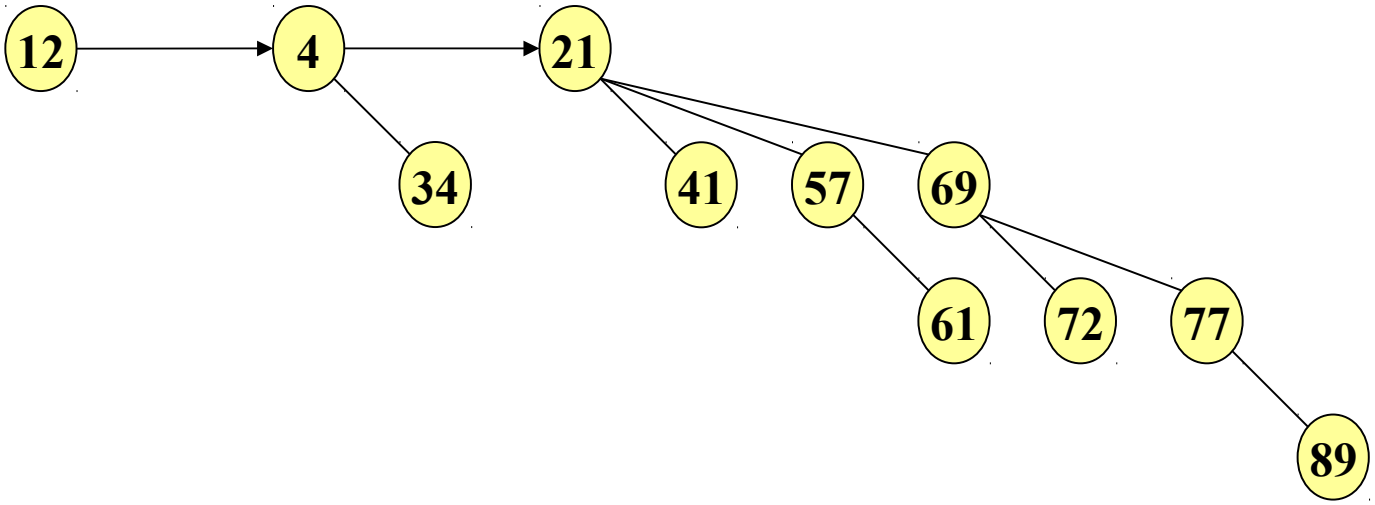
# Binomial Trees

- A *binomial tree* $B_k$ has
  - a height of $k$;
  - **$n = 2^k$ nodes (!!!)**;
  - $k+1$ depth levels ranging within $0,\ldots,k$.
  - $\binom{k}{d} = \frac{k!}{d!\ (k-d)!}$ nodes at *depth level* $d$.
  - a root and a $B_0$, $B_1$, $B_2$, $B_3$,..., $B_{k-1}$ connected to it in respective order (see slide 96!).

# Binomial Heaps

- *BHs* differ from other heap structures in that
  - a BH is *not a heap-ordered tree* but rather *a collection of heap ordered trees*, ***a forest***.
  - each heap-ordered tree is of a constrained form known as a *binomial tree*.
- Each binomial tree in a BH obeys *min-heap* or *heap order* property.
- There is *at most one* $B_k$ of *each height* $k$ in a *BH*.
- In the next slide, we see an 11-node BH.

# An 11-element Binomial Heap




$B_0$ $B_1$ $B_3$

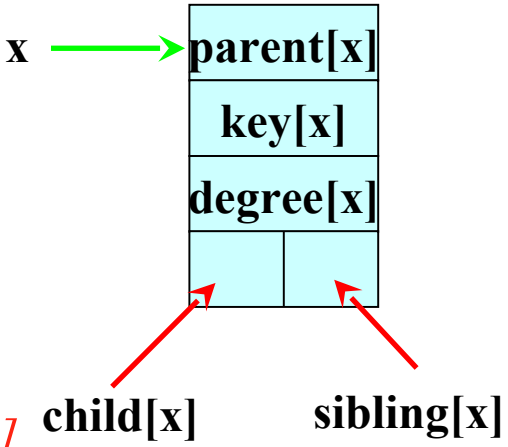
# Operations on Binomial Heaps

- *BH_Create()*
  - Creates the special header node for the BH.
- *BH_Find_Min()*
  - Finds the minimum key in the BH.
- *BH_Merge($H_1$,$H_2$)*
  - Merges two BHs $H_1$ and $H_2$.
- *BH_Insert($H_1$,x)*
  - *Inserts x into H.*
- *BH_Delete-Min(H)*
  - *Deletes the minimum key in H.*

- *In preparing algorithms/pseudocode for these operations, [1] has been the main reference.*

# Operations on Binomial Heaps

- *Assumptions*
  - Each node *x* in a BH contains
    - a key field *key[x]*,
    - a parent pointer, *p[x]*,
    - a child pointer to its left-most child, *child[x]*,
    - a pointer to its immediate right sibling, *sibling[x]*,
    - a field holding the number of children, *degree[x]*.



# Creating an Empty Tree

```
BH_Header * BH_Create()
{ // creates a special header node for a BH.
   BH_Header *BH_hdr;
   BH_hdr=(BH_Header *)
   malloc(sizeof(BH_Header));
   ... // here, proper values are assigned to special
   header     fields.
   BH_hdr->first=NULL;
   return BH_hdr;
}
```

Running time: $\Theta(1)$

# Finding Minimum Key in a BH

```
BH_Find_Min(BH_hdr)
{ // finds minimum key in the BH.
   y=NULL;  x=BH_hdr->first;  min=∞;
   while (x != NULL)
        if (key[x]<min) { min=key[x];  y=x; }
        x = sibling[x];
   return y;
}
```

Running time: *O(lg(n)) Why?*

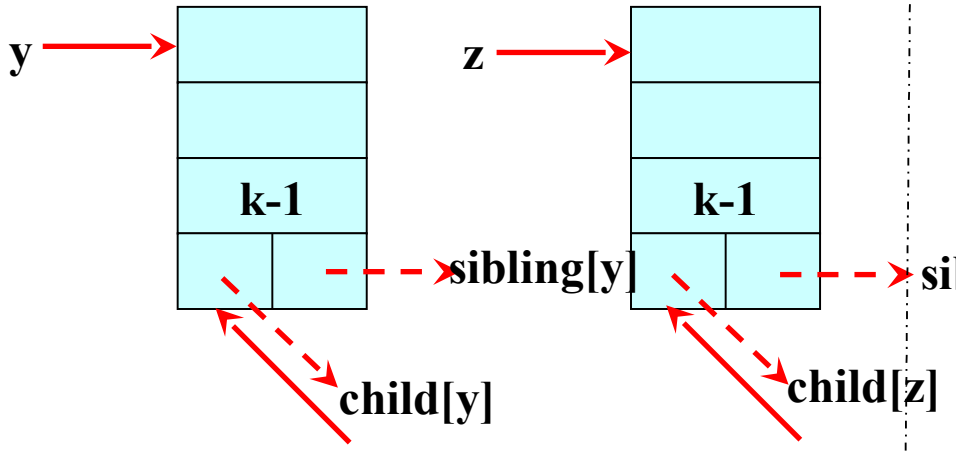
# Obtaining a $B_k$ from two $B_{k-1}$s

```
Get_a_BT_k(y,z)
{ // obtains a B_k from two B_k-1s.  Root of new B is z.
   p[y]=z;
   sibling[y]=child[z];
   child[z]=y;
   degree[z]++;
}
```

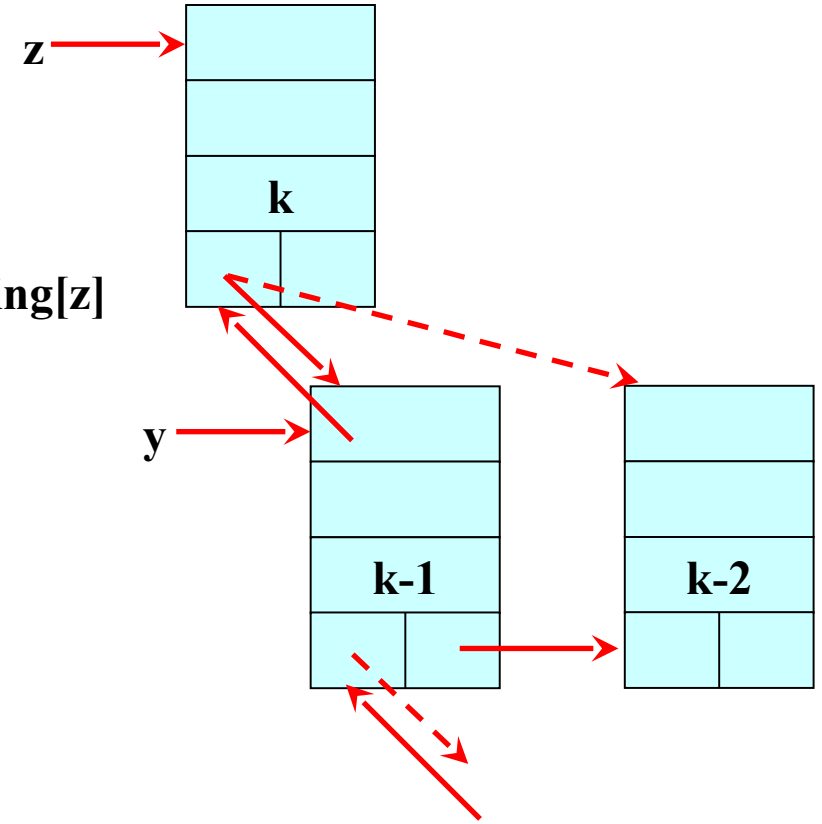
Running time: $\Theta(1)$

# Get_a_BT_k(y,z) illustrated...

**Before** Get_a_BT_k(y,z)




**After** Get_a_BT_k(y,z)




```
Get_a_BT_k(y,z)
{     p[y]=z;
      sibling[y]=child[z];
      child[z]=y;
      degree[z]++;
}
```

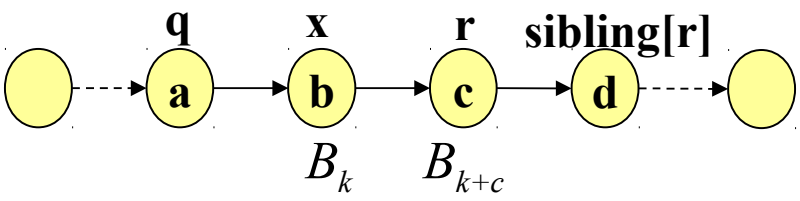
# Merging two BHs

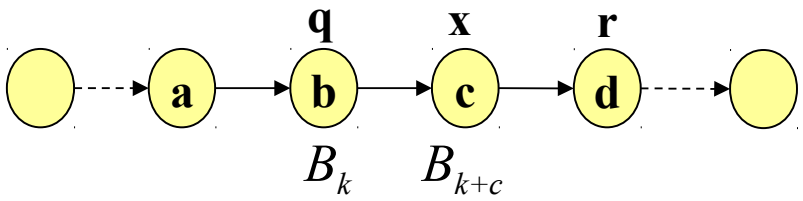
```
BH_Merge(BH1_hdr,BH2_hdr)
{ // merges two BH_1 and BH_2.
   BH_hdr= BH_Create();
   BH_hdr->first=Merge_Root_Lists(); // Merges root lists of BH_1 and BH_2 into one sorted
   by                                                    // ascending degree.
   if (BH_hdr->first == NULL) return BH_hdr->first;
   q=NULL; x=BH_hdr->first; r=sibling[x];
   while (r !=NULL)
         if (degree[r]!=degree[x] || sibling[r]!=NULL &&
            degree[sibling[r]]==degree[x]) {q=x; x=r;}
         else if (key[x]<=key[r]){sibling[x]=sibling[r];Get_a_BT_k(r,x);}
              else { if (q == NULL) BH_hdr->first=r; else sibling[q]=r;
               Get_a_BT_k(x,r); x=r;
              }
         r=sibling[x];
   return BH_hdr;
}
```

Running time: *O(lg(n))*

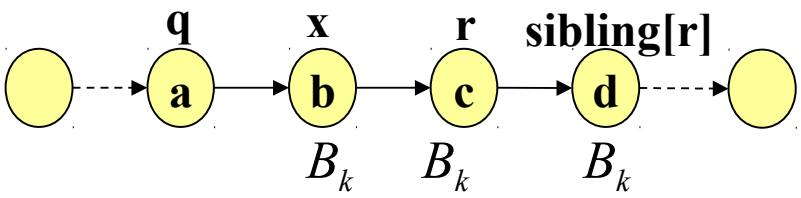
# Illustration of Various Cases in Merging

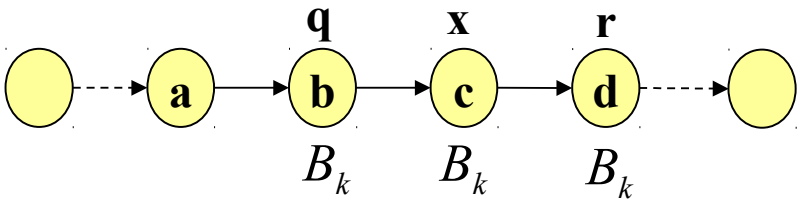
***degree[r] ≠ degree[x]***



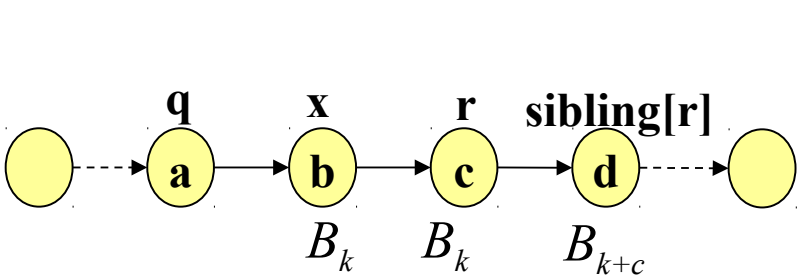

***sibling[r] ≠ NULL && degree[sibling[r]]=degree[x])***

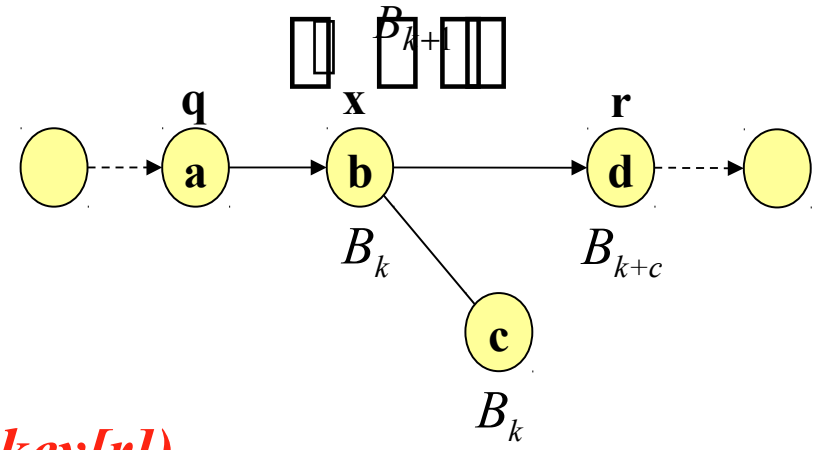


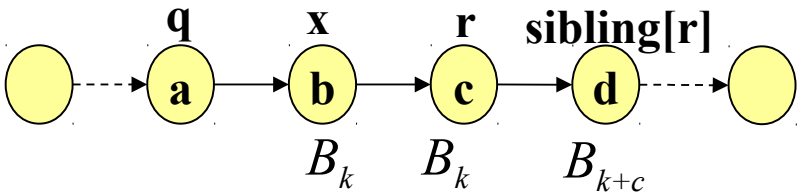
# Illustration of Various Cases in Merging

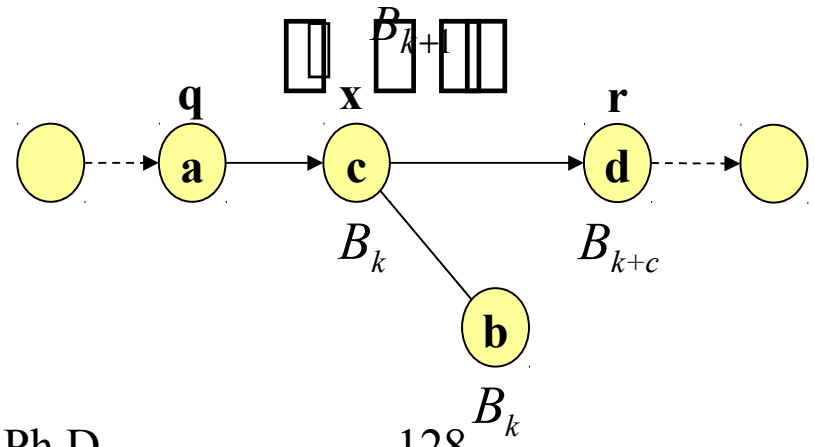
***key[x]≤key[r])***



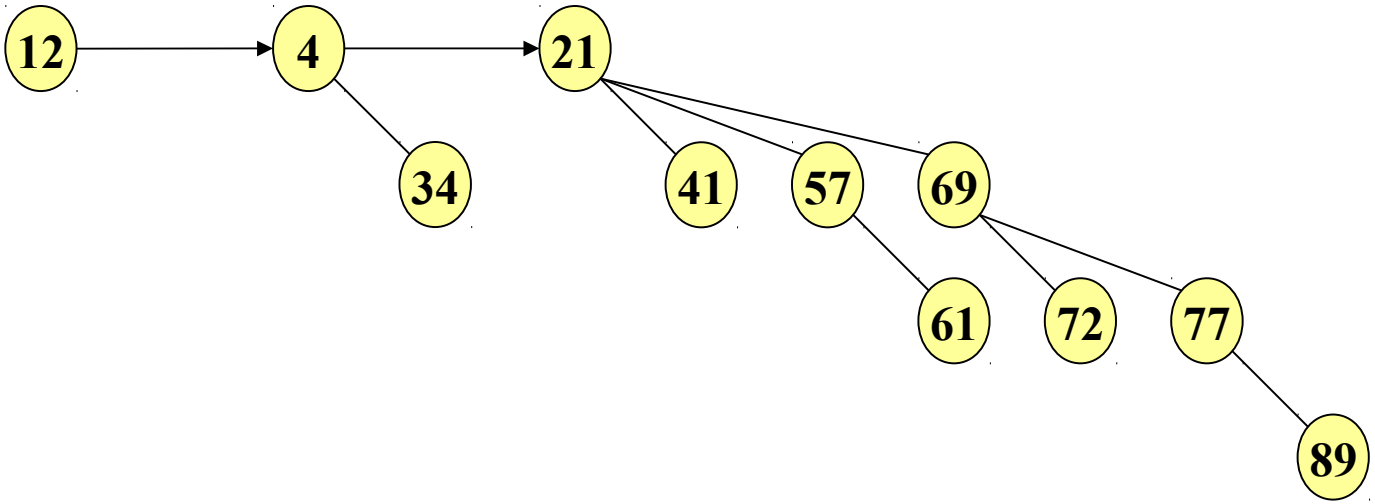




***key[x]>key[r])***



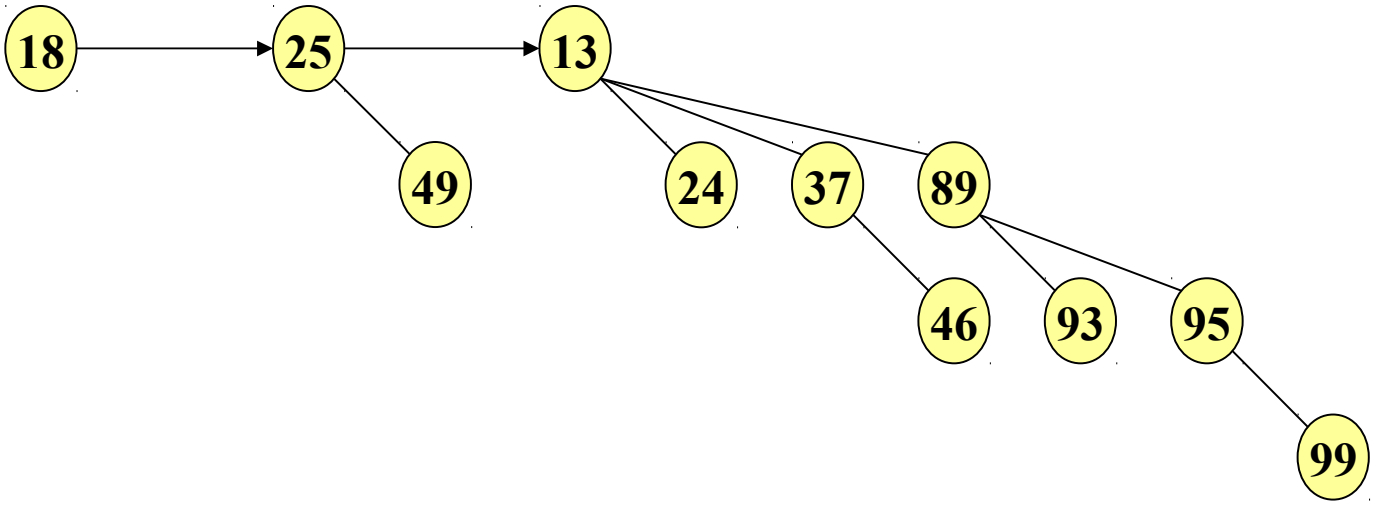



# An 11-element BH



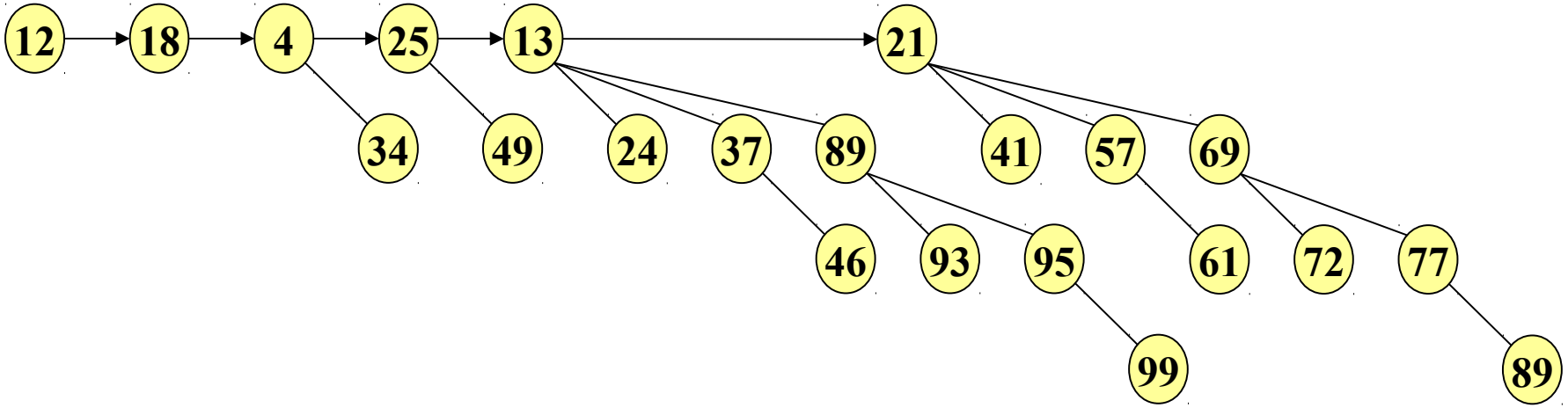

$B_0$ $B_1$ $B_3$

# Another 11-element BH

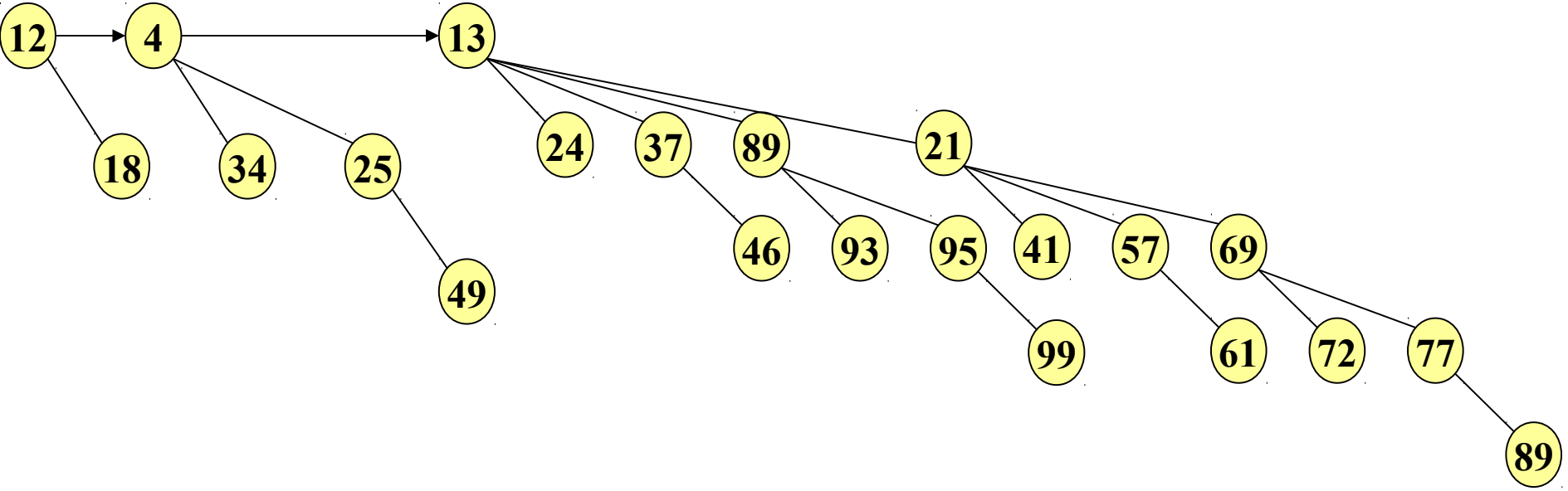



$B_0$ $B_1$ $B_3$

# Two BHs merged...



# Two BHs merged... into a 22-element BH



# Inserting a node into a BH

- Insertion of a node into a BH is the same as merging a single-node BH with another BH.

```
BH_Insert(BH_hdr,x)
{ //inserts x into BH.
   BH1_hdr=BH_Create();
   // makes a single-node (degree-0) BH out of x.
   p[x]=child[x]=sibling[x]=NULL; degree[x]=0;
   BH1_hdr->first=x;
   BH_hdr=BH_Merge(BH1_hdr,BH_hdr);
}
```

Running time: worst case: $O(lg(n))$

# Delete-Min in a BH

BH_Delete-Min(BH_hdr)

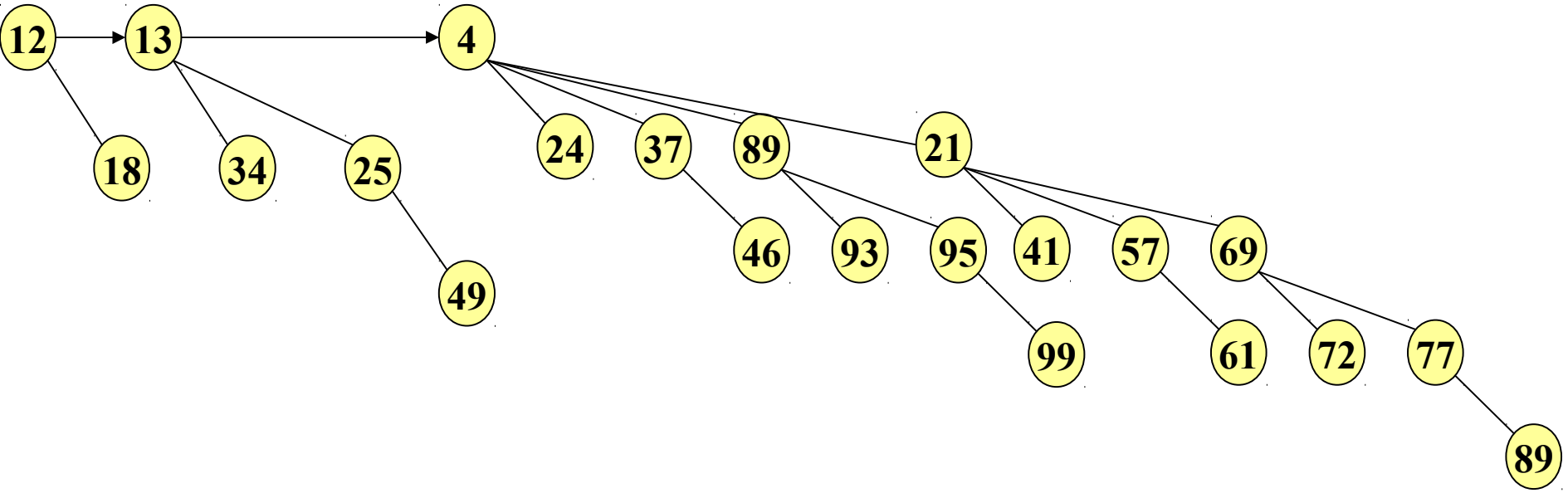
{ //*deletes the minimum key in the BH*.

- Find the root x with minimum key //(wc: $O(lg(n))$ )
- Remove x
- BH1_hdr=BH_Create();
- Establish a LL of x's children //(wc: $O(lg(n))$
- BH1_hdr->first=pointer to node with degree=0
- BH_hdr=BH_Merge(BH1_hdr,BH_hdr); //(wc: $O(lg(n))$

}

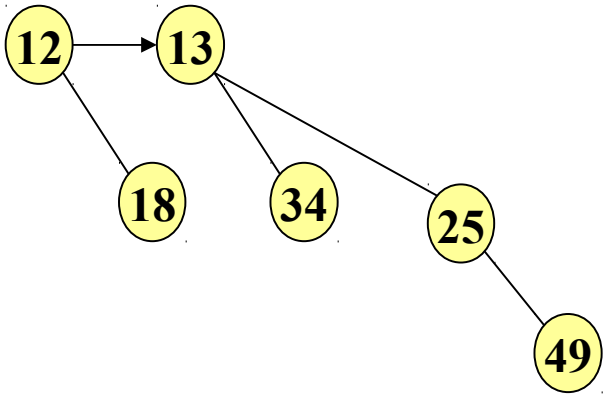
Running time: $O(lg(n))$

# A 22-element BH... Delete-min

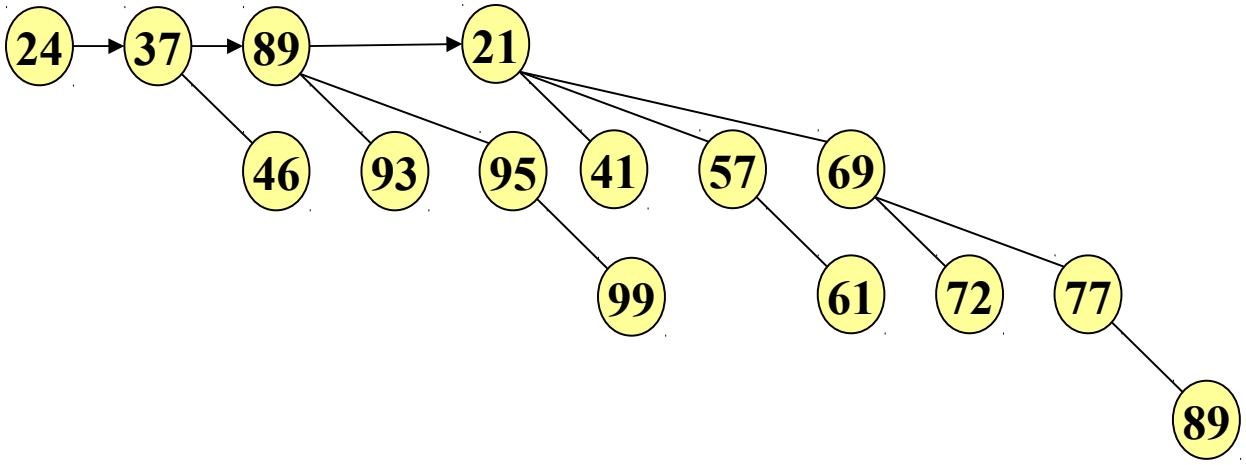


# 4 removed...

**BH**

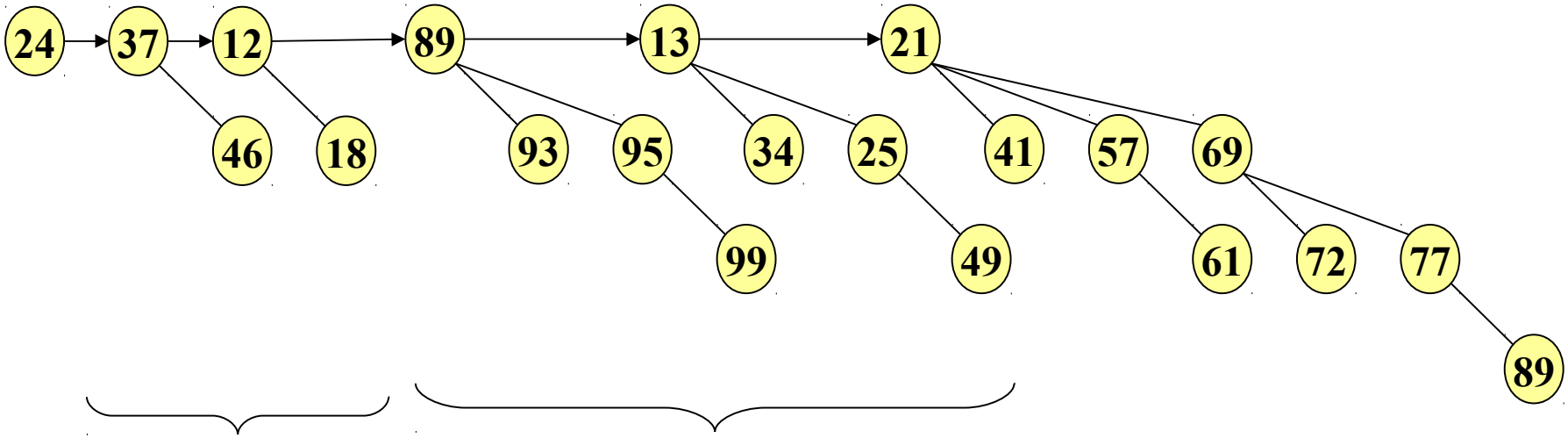



**BH1**



# BH merged with BH1... 2 $B_1$s, 2$B_2$s

**BH...**

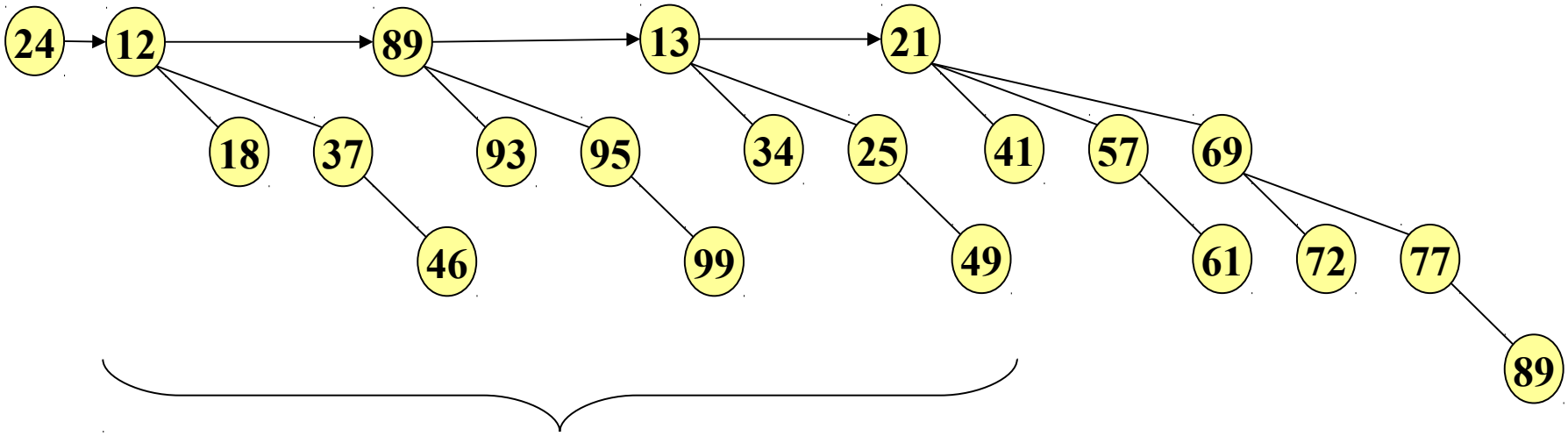



**2 $B_1$s to merge toa $B_2$**

**2 $B_2$s to merge to a $B_3$**

# BH merged with BH1... $3B_2$s

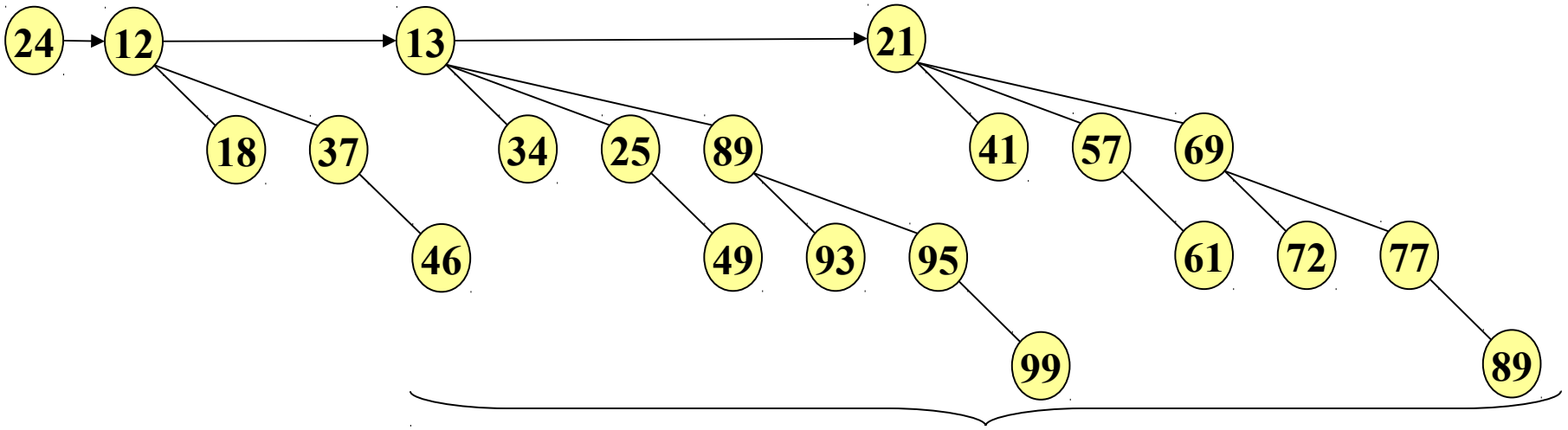
**BH...**




**$3B_2$s... Last two to merge to a $B_3$**

# BH merged with BH1... 2 $B_3$s

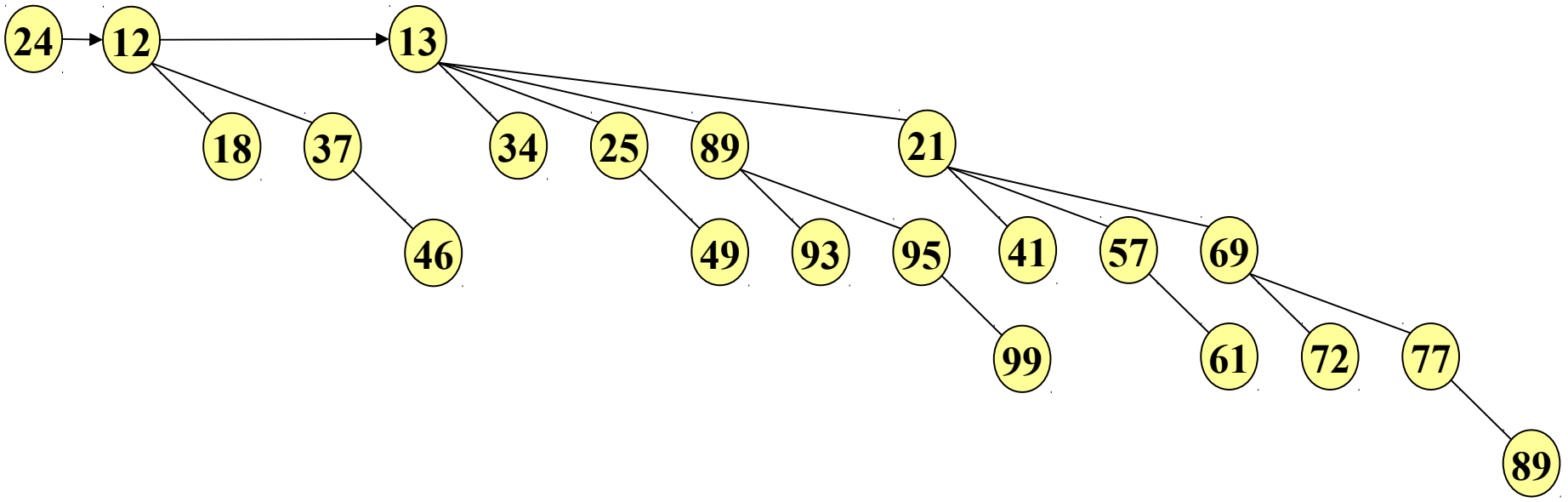
**BH...**




**2 $B_3$s to merge to a $B_4$**

# BH merged with BH1… Final appearance

**BH...**



# Reference...

[1] T.H. Cormen, C.E. Leiserson, R.L. Rivest, C. Stein,

"*Introduction to Algorithms*," 2nd edition, MIT Press, 2003