Quantum Reservoir Computing

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Outline

Quantum Reservoir Computing

Leveraging **Quantum Mechanics**

"Until recently, the fields of quantum computing and neuromorphic computing were evolving in parallel. However, cross-fertilization between the two domains has started and is likely to bring remarkable results." [5]

Why use quantum systems?

"Quantum systems have an exponentially large degree of freedom in the number of particles and hence provide a rich dynamics that could not be simulated on conventional computers." [2]

Quantum Reservoir Computing

Quantum reservoir computing is an intersection of two different paradigms of **natural computing**[1]:

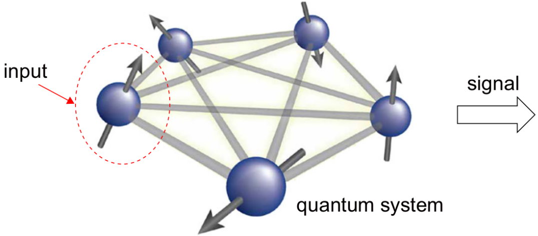
- Quantum computing
- Reservoir computing

Quantum reservoir computing allows to leverage the **complex dynamics** of quantum systems.

Quantum systems offer a unique venue for reservoir computing, given the presence of interactions unavailable in classical systems and a potentially **exponentially-larger computational space** [3].

[1] natural computing seeks to exploit natural physical or biological systems as computational resource

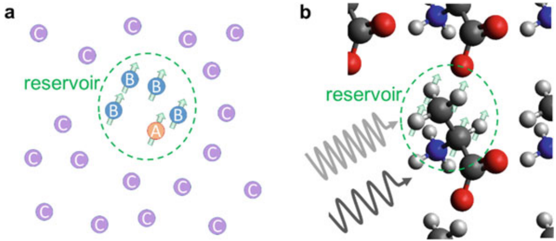
The QRC **Framework**



1. **Input**: inject the input signal into the quantum sistem.
2. **System evolution**: let the quantum system evolve following its dynamics.
3. **Readout**: Extract an output from the quantum sistem.

Physical Implementations of the Quantum System

- **Many different implementations**: quantum dots, exciton-polaritons, supercondtucting qubits, trapped ions...
- A promising approach is **NMR quantum reservoir computing** [9]:
  - Qubits are **nuclear spins** in a molecule, controlled by applying oscillating **magnetic fields** at radiofrequency.
  - An organic molecule, which can have many nuclei with spin-1/2, forms a many-qubit system
  - A usual NMR sample has a macroscopic ensemble of identical copies of such a **many-qubit system**



QRC System Definition [7]

- The QRC system in the *density matrix formalism*[2] is described by two equations:
  - the state-space equation, given by a complete positive trace-preserving (CPTP) map:

$$T : \mathcal{S}(\mathcal{H}) \times D_n \to \mathcal{S}(\mathcal{H})$$

  - the readout equation, given by the map:

$$h : \mathcal{S}(\mathcal{H}) \to \mathbb{R}^m$$

[2]Formalism used to describe ensembles of quantum states, using the density matrix ρ.
$\mathcal{S}(\mathcal{H})$ is the space of quantum *density matrices*. $\mathcal{S}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the set of all bounded operators that act on $\mathcal{H}$.
$D_n \subset \mathbb{R}^n$ is the subspace of inputs.

QRC System Definition

- A QRC system is determined by the state-space transformations:

$$\begin{cases} \rho_t = T(\rho_{t-1}, \mathbf{z}_t) \\ y_t = h(\rho_t) \end{cases} \tag{1}$$

The ESP in the context of QRC

- Consider the QRC system defined in (1)
- Given an input sequence $\mathbf{z} \in (D_n)^{\mathbb{R}}$, we say that $\boldsymbol{\rho} \in (\mathcal{S}(\mathcal{H}))^{\mathbb{Z}}$ is a *solution* of (1) for the input $\mathbf{z}$ if the components of the sequences $\mathbf{z}$ and $\boldsymbol{\rho}$ satisfy the first relation in (1) for any $t \in \mathbb{Z}$.
- The QRC system has the ESP when it has a *unique solution* for each input $\mathbf{z} \in (D_n)^{\mathbb{R}}$. More explicitly, for each $\mathbf{z} \in (D_n)^{\mathbb{R}}$, there exist a unique sequence $\boldsymbol{\rho} \in (\mathcal{S}(\mathcal{H}))^{\mathbb{Z}}$ such that

$$\rho_t = T(\rho_{t-1}, \mathbf{z}_t), \text{for all } t \in \mathbb{Z} \tag{2}$$

Condition for the ESP

Proposition:

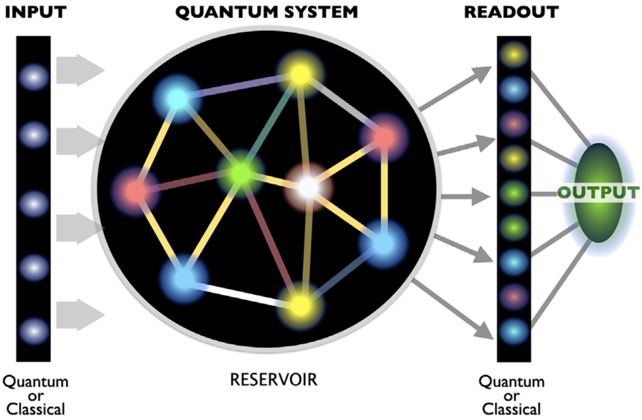
- Let $D_n \subset \mathbb{R}^n$ be a *compact subset* of $\mathbb{R}^n$. Let $T : \mathcal{S}(\mathcal{H}) \times D_n \to \mathcal{S}(\mathcal{H})$ be a continuous QRC system such that the CPTP maps $T(\cdot, \mathbf{z}) : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ are ***strictly contractive***[3] for all $\mathbf{z} \in D_n$, with a common contraction constant $0 \le r < 1$ associated to some norm in $\mathcal{B}(\mathcal{H})$.
- Then, the QRC system induced by T has the **ESP** (Echo State Property) and the **FMP** (Fading Memory Property).

[3] $||T(\rho_1) - T(\rho_2)|| \le r||\rho_1 - \rho_2||$ for all $\rho_1, \rho_2 \in \mathcal{S}(\mathcal{H})$, where $0 \le r < 1$.

Complex quantum dynamics for RC

The Framework

In the following part, we will focus on the framework developed by Fujii and Nakajima [1], summarizing their approach and results.



The Quantum System

- 1 qubit $\rightarrow$ a state $|\psi\rangle$ in a two-dimensional complex vector space spanned by $\{|0\rangle, |1\rangle\}$.
- System of N qubits $\rightarrow$ a state $|\psi\rangle$ in a 2^N dimensional complex vector space.
- To describe a **statistical mixture of states** $|\psi_j\rangle$ (each prepared with probability p_j) we introduce the **density matrix** ρ, a $2^N \times 2^N$ Hermitian matrix, defined as:

$$\rho = \sum_j p_j |\psi_j\rangle \langle\psi_j|$$

We consider an **ensemble quantum system**, consisting of a huge number of copies of ρ, i.e. $\rho^{\otimes m}$ (cold atomic ensembles and liquid- or solid-state molecules are natural candidates of such an ensemble quantum system).

The Quantum System as a Reservoir

- The density matrix ρ lives in a $(\mathbf{2^N} \times \mathbf{2^N})$**-dimensional operator space**:

$$\rho \in \mathcal{S}(\mathcal{H})$$

- As nodes of the network of the QR, we don't use individual qubits.
- Nodes of the network are defined by elements of an **orthogonal basis of the density matrices operator space**.

QR and the density matrix formalism

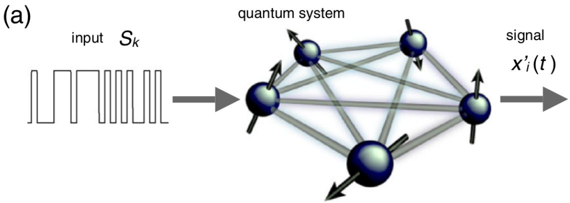
- The **density matrix** can be represented as a vector **x** on a 4^N-dimensional operator space.
  - The ith coefficient x_i of **x** is defined by $x_i = Tr[B_i \rho]$
  - Set of N-qubit products of Pauli operators[4]: $\{B_i\}_{i=1}^{4^N} = \{I, X, Y, Z\}^{\otimes N}$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

[4] Specifically, we choose the first N elements such that $B_i = Z_i$ for convenience in the definition of the observables later.

Introducing the Input

- We want to introduce into the system the input sequence $\{s_k\}_{k=1}^{M}$



- To do so, at each time $t = k\tau$, the input signal s_k is injected into a qubit, say, the first qubit, by replacing it with the state $\rho_{s_k} = |\psi_{s_k}\rangle \langle\psi_{s_k}|$, where:

$$|\psi_{s_k}\rangle = \sqrt{1 - s_k}\,|0\rangle + \sqrt{s_k}\,|1\rangle \quad (3)$$

Introducing the Input

- The action of the kth input on the density matrix of the system is given by the following CPTP map:

$$\rho \to \rho_{s_k} \otimes Tr_1[\rho] \tag{4}$$

- In terms of the state $\mathbf{x}$(t), the above action can be represented by a matrix[5] S_k:

$$\mathbf{x} \to S_k \mathbf{x} \tag{5}$$

[5] The action of a CPTP map $\mathcal{D}$, in terms of the state $\mathbf{x}$, can be expressed by a linear map: $\mathbf{x} \to W\mathbf{x}$, where $W_{ji} = Tr[B_j \mathcal{D}(B_i)]$

System Evolution

- After the injection, **the system evolves under the Hamiltonian** H for a time interval τ.
- Thus, the time evolution of the state for a unit time step is given by:

$$\mathbf{x}(k\tau) = U_\tau S_k \mathbf{x}[(k-1)\tau] \tag{6}$$

$$(U_\tau)_{ji} = Tr[B_j e^{-iH\tau} B_i e^{iH\tau}]$$

The time interval τ should be chosen within a physically allowed time scale for the input injections, which is determined by both the time scale for the initialization of the qubit and the operation for the input.

Measurements in Quantum Systems

- An **Observable** is a physical quantity that can be experimentally measured.
- Observables are represented by **Hermitian operators**.
- Measurements in a quantum system are described by the expectation value of an **Observable** O.

$$\langle O \rangle = Tr[O\rho] \tag{7}$$

Readout

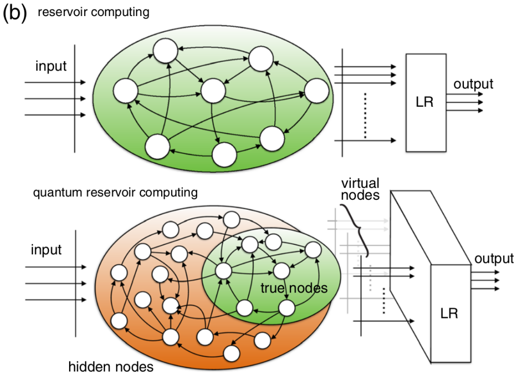
- The output signal is defined as an average value of a local observable on each qubit.
- Observable: **Pauli operator** Z_i acting on each ith qubit.
- For an appropriately ordered basis $\{B_i\}$, the observed signals are related with the first N elements of the state **x**(t):

$$x_i(t) = Tr[Z_i \rho(t)] \qquad (i = 1, ..., N) \tag{8}$$

We do not consider the backaction of the measurements to obtain the average values $\{x_i(t)\}$ by considering an ensemble quantum system.

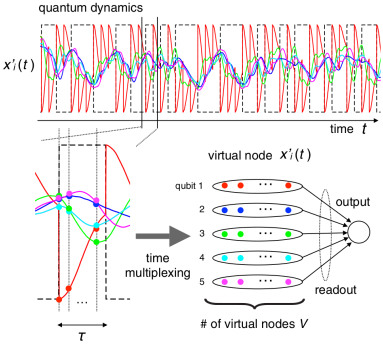
Readout

- The directly observed signals $\{x_i(t)\}_{i=1}^N$ are the **true nodes**.
- The remaining $(4^N - N)$ nodes of $\mathbf{x}(t)$ are called **hidden nodes**, as they are not employed as the signals for learning.



Time Multiplexing

- Divide the unit time interval τ into V subintervals to construct V virtual nodes.
- The signals are sampled from the QR not only at the time $k\tau$, but also at each of the subdivided V time stesps during the unitary evolution U_τ.
- This effectively increases the total number of computational nodes employed in the learning process from N true nodes to NV computational nodes.

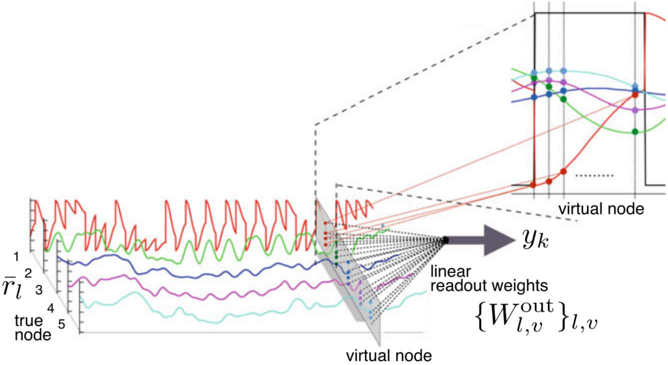


Another complementary approach that has been proposed is **Spatial Multiplexing** [8]

Time Multiplexing

- After each input s_k, the signals from the hidden nodes (via the true nodes) are measured for each subdivided intervals after the time evolution by $U_{v\tau/V}$, where $(v = 1, 2, ..., V)$:

$$\mathbf{x}(k\tau + (v/V)\tau) = U_{v\tau/V} S_k \mathbf{x}(k\tau)$$



Training Readout Weights

- The **output** at each time step is defined as the linear combination of the NV computational nodes.

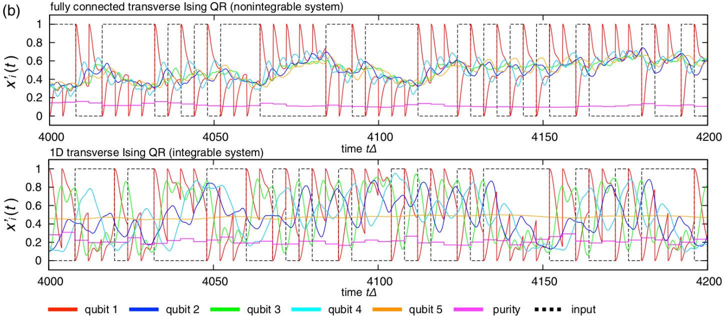
$$y_k = \sum_{l=1}^{N} \sum_{v=1}^{V} W_{j,v}^{out} x_l(k\tau + (v/V)\tau) \tag{9}$$

- Let $\{\bar{y}_k\}_{k=1}^{L}$ be the target sequence for learning.
- The linear readout weights $W_{j,v}^{out}$ can be determined using the pseudo inverse.

Numerical Simulations

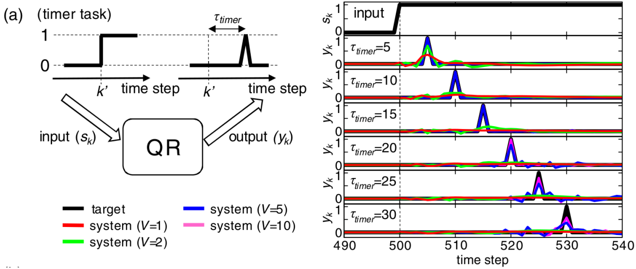
- In the following **numerical experiments**, we employ, as an example, the simplest quantum system, a fully connected transverse-field **Ising model**:

$$H = \sum_{ij} J_{ij} X_i X_j + h Z_i$$



Timer Task

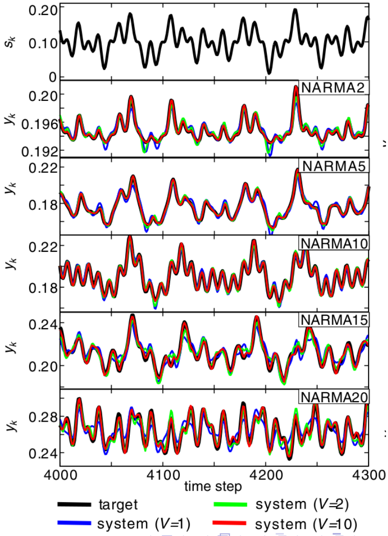
- Test whether the system contains **memory** or not:
  - The input is flipped from 0 to 1 at time step k'.
  - The system should output 1 after τ_{timer} time steps, otherwise it should output 0.



- The QR system is capable of embedding a timer.
- By increasing V the performance improves.
- Increasing τ_{timer}, the performance declines.

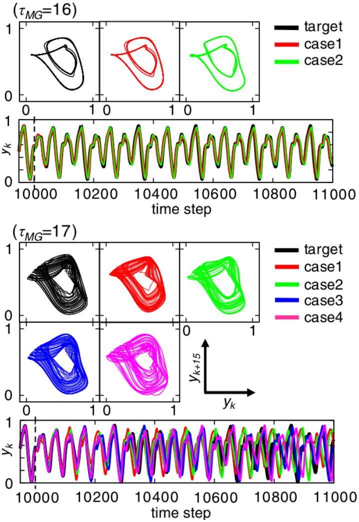
NARMA Task

- Emulation of **nonlinear** input-driven dynamical systems (NARMA systems).
- The input s_k is expressed as a product of three sinusoidal functions with different frequencies.
- Increasing V, the performance improves.



Mackey-Glass prediction task

- **Time series prediction** task.
- The system output is fed back as the input for the next time step.
- The MG system has a delay term τ_{MG}, and when $\tau_{MG} > 16.8$ it exhibits a chaotic attractor.
- When $\tau_{MG} = 16$, the system outputs overlap the target outputs.
- When $\tau_{MG} = 17$, after the fist time steps, the outputs start to deviate perceptibly large.



Conclusions

Pros

- Thanks to the **high dimensionality of the Hilbert space**, QRC approaches exhibit great computational potential.
- The QR framework can emulate nonlinear dynamical systems, and exhibit **robust information processing** against noise [1].
- A number of numerical experiments show that quantum systems consisting of 5–7 qubits possess computational capabilities comparable to conventional recurrent neural networks of 100–500 nodes [1].
- Aside excelling at classical tasks, reservoir computers based on quantum systems may be coupled to **quantum states in their input or their output**. QR systems can have several applications, such as the characterization of quantum states, and quantum state preparation [3].
- Many different approaches are being tested and research is being carried out to improve computational capabilities of QRC.

Cons

- There are **physical constraints**, related to quantum dynamics, for implementing the QRC framework.
- The performance of a system to be used as a quantum reservoir computer crucially depends on its **operation regime** [6].

Conclusions

- Both quantum computing and reservoir computing are flourishing research fields.
- QRC approaches have been shown to have **great computational potential**.
- More research is needed in the field of QRC, to develop new frameworks with better performances.
- The future prospects are promising...

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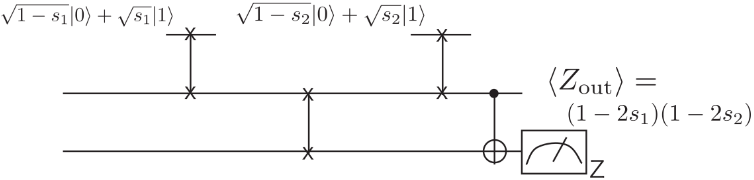
Emerging nonlinearity from a linear system [1]

- The evolution of the quantum system is **linear**
- Then how can we employ it for a **nonlinear learning task**?
- *Nonlinearity* defined for the learning task and *linearity* of the dynamics on the quantum system are different concepts!

Example

- Consider the following **quantum circuit**:



- The output has a **second-order nonlinearity** w.r.t. s_1 and s_2.

Unitary real-time evolution is essential for nonlinearity

- Whereas dynamics is described as a linear map, information with respect to any kind of correlation exists in exponentially many degrees of freedom.
- In the QRC, such higher-order correlations or nonlinear terms are mixed by the linear but quantum-chaotic (nonintegrable) dynamics U_τ.

Unitary real-time evolution is essential for nonlinearity

- There exists a state corresponding to an observable $B_l = Z_i Z_j$, i.e., $x_l(t) = Tr[Z_i Z_j \rho(t)]$ storing the **correlation** between $x_i(t) = Tr[Z_i \rho(t)]$ and $x_j(t) = Tr[Z_j \rho(t)]$, which can be **monitored from another true node** via U_τ.
- This mechanism allows us to find a **nonlinear dynamics** with respect to the input sequence $\{s_k\}$ from the dynamics of the true nodes $\{x_i(t)\}_{i=1}^N$.