Introduction to Image Processing

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Topic 4
Filtering in the Frequency Domain

1. Background

• Jean Baptiste Joseph Fourier was a French mathematician born in 1768.

• The contribution for which he is most remembered was outlined in a memoir in 1807 and published in 1822 in his book, *La Théorie Analitique de la Chaleur* (The Analytic Theory of Heat).

• Basically, Fourier's contribution in this field states that any periodic function can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient (we now call this sum a **Fourier series**).

• Even functions that are not periodic (but whose area under the curve is finite) can be expressed as the integral of sines and/or cosines multiplied by a weighing function. The formulation in this case is the **Fourier transform**.

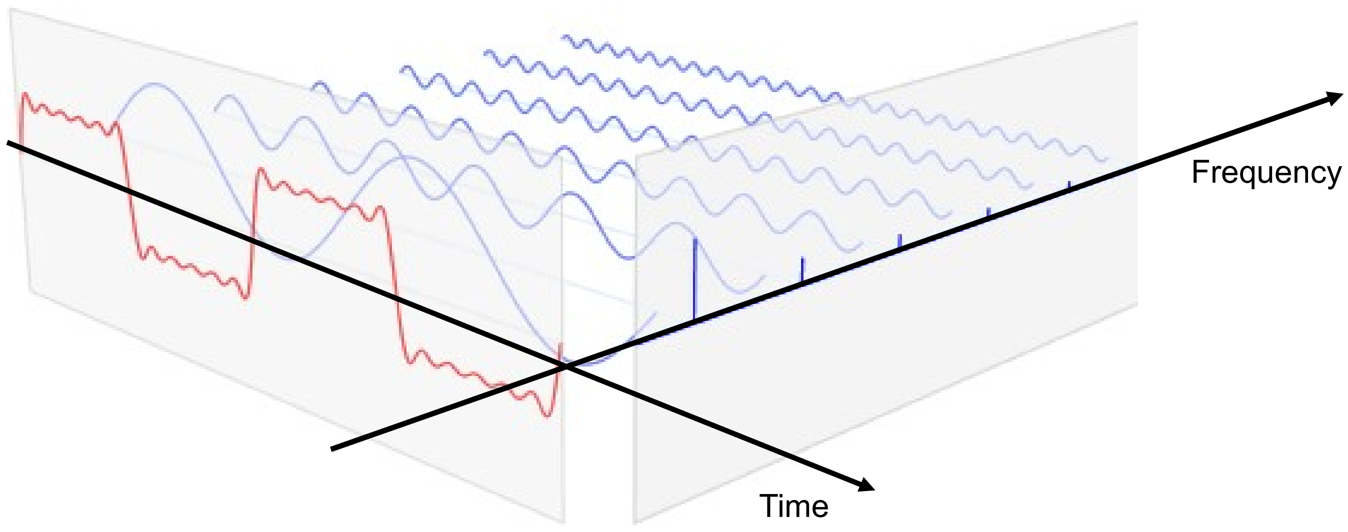
2. Preliminary Concepts

- Both representations share a important characteristic:

  - A function, expressed in either a Fourier series or transform, **can be reconstructed completely** via with no loss of information.

2. Preliminary Concepts

• The advent of digital **computers** and the “discovery” of a **fast Fourier transform** (FFT) algorithm in the early 1960s (more about this later) revolutionized the field of signal processing.



2. Preliminary Concepts

- Complex numbers

$$C = R + jI$$

- Conjugate

$$C^{*} = R - jI$$

- Polar coordinates

$$C = |C|(\cos\theta + j\sin\theta)$$

$$|C| = \sqrt{R^2 + I^2}$$

$$\theta = \arctan(I/R)$$

- Exponential form

$$e^{j\theta} = \cos\theta + j\sin\theta \longrightarrow C = |C|e^{j\theta}$$

2. Preliminary Concepts

- The preceding equations are applicable also to complex functions.

$$F(u) = R(u) + jI(u)$$

$$|F(u)| = \sqrt{R(u)^2 + I(u)^2}$$

$$\theta(u) = \arctan[I(u)/R(u)]$$

- A function *f(t)* of a continuous variable *t* that is periodic with period, T, can be expressed as the **sum of sines and cosines multiplied by appropriate coefficients**.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-j\frac{2\pi n}{T}t}\, dt \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots$$

2. Preliminary Concepts

- Fourier series (Trigonometric):

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n}{T}t\right) + b_n \sin\left(\frac{2\pi n}{T}t\right)\right]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nw_0 t) + b_n \sin(nw_0 t)\right] \qquad w_0 = \frac{2\pi}{T}$$

$$a_0 = \frac{2}{T}\int_0^T f(t)dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(nw_0 t)dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(nw_0 t)dt$$

2. Preliminary Concepts

- Fourier series (Exponential):

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(nw_0 t\right) + b_n \sin\left(nw_0 t\right)\right]$$

$$e^{jnw_0 t} = \cos(nw_0 t) + j\sin(nw_0 t)$$
$$e^{-jnw_0 t} = \cos(nw_0 t) - j\sin(nw_0 t)$$

$$\longrightarrow$$

$$\cos(nw_0 t) = \frac{1}{2}(e^{jnw_0 t} + e^{-jnw_0 t})$$
$$\sin(nw_0 t) = \frac{1}{2j}(e^{jnw_0 t} - e^{-jnw_0 t})$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{a_n}{2}(e^{jnw_0 t} + e^{-jnw_0 t}) + \frac{b_n}{2j}(e^{jnw_0 t} - e^{-jnw_0 t})\right]$$

2. Preliminary Concepts

- Fourier series (Exponential):

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{2}(e^{jnw_0t} + e^{-jnw_0t}) + \frac{b_n}{2j}(e^{jnw_0t} - e^{-jnw_0t}) \right]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - jb_n}{2} \right) e^{jnw_0t} + \left(\frac{a_n + jb_n}{2} \right) e^{-jnw_0t} \right]$$

2. Preliminary Concepts

- Fourier series (Exponential):

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{a_n - jb_n}{2}\right)e^{jnw_0 t} + \left(\frac{a_n + jb_n}{2}\right)e^{-jnw_0 t}\right]$$

- Taking

$$n \geq 1$$

$$c_0 = \frac{1}{2}a_0, \quad c_n = \frac{1}{2}(a_n - jb_n), \quad c_{-n} = \frac{1}{2}(a_n + jb_n)$$

- It follows that

$$f(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{jnw_0 t} + \sum_{n=1}^{\infty} c_{-n} e^{-jnw_0 t}$$

2. Preliminary Concepts

- Fourier series (Exponential):

$$f(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{jnw_0 t} + \sum_{n=1}^{\infty} c_{-n} e^{-jnw_0 t}$$

$$f(t) = \sum_{n=1}^{\infty} c_{-n} e^{-jnw_0 t} + c_0 e^{j0w_0 t} + \sum_{n=1}^{\infty} c_n e^{jnw_0 t}$$

$$f(t) = \sum_{n=-\infty}^{-1} c_n e^{-jnw_0 t} + c_0 e^{j0w_0 t} + \sum_{n=1}^{\infty} c_n e^{jnw_0 t}$$

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnw_0 t}}$$

2. Preliminary Concepts

- Fourier series (Exponential):

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(nw_0 t)dt$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(nw_0 t)dt$$

$$c_n = \frac{1}{2}\left[\frac{2}{T}\int_0^T f(t)\cos(nw_0 t)dt - j\frac{2}{T}\int_0^T f(t)\sin(nw_0 t)dt\right]$$

$$= \frac{1}{T}\int_0^T f(t)\left[\cos(nw_0 t) - j\sin(nw_0 t)\right]dt$$

2. Preliminary Concepts

- Fourier series (Exponential):

$$c_n = \frac{1}{T}\int_0^T f(t)\left[\cos(nw_0 t) - j\sin(nw_0 t)\right]dt$$

$$= \frac{1}{T}\int_0^T f(t)e^{-jnw_0 t}dt$$

2. Preliminary Concepts

- Fourier series (Exponential):
- In summary

$$f(t) = \sum_{n=-\infty}^{\infty} F(n) e^{jw_0 nt} \qquad w_0 = \frac{2\pi}{T}$$

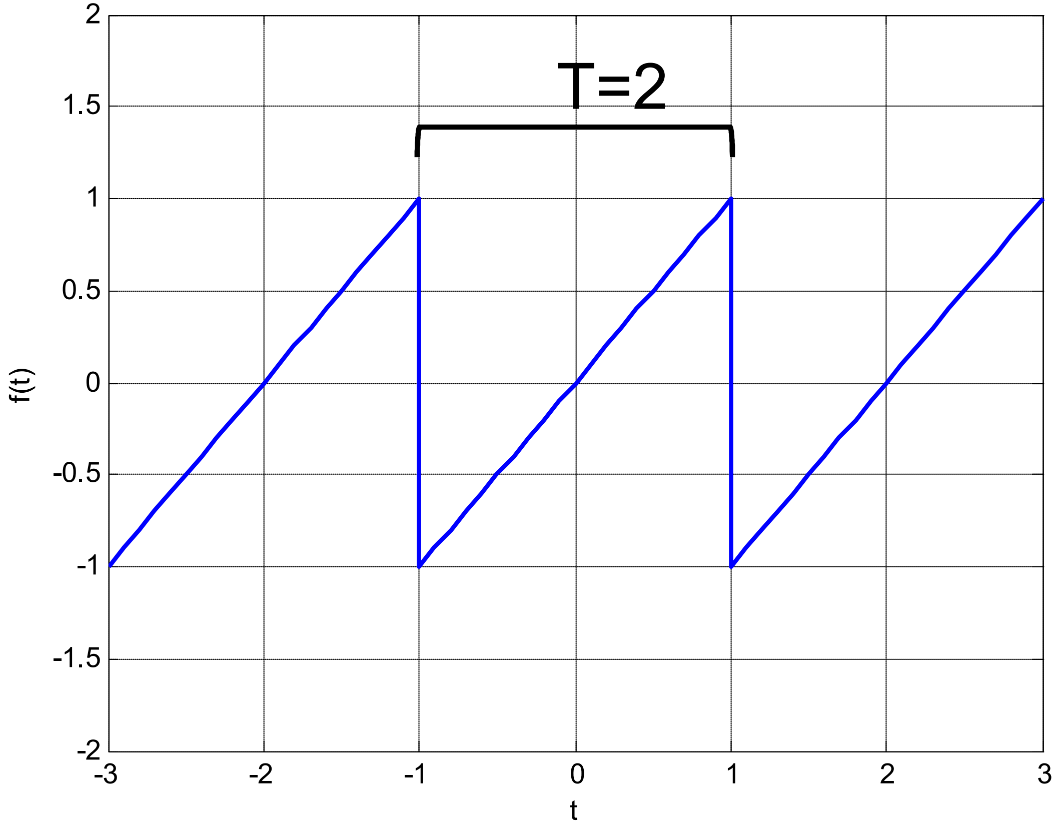
$$F(n) = \frac{1}{T}\int_0^T f(t) e^{-jw_0 nt} dt, \qquad n = 0, \pm 1, \pm 2, \ldots$$

f(t) is complex, continuous and periodic with period T

F(n) is complex, discrete and non-periodic

2. Preliminary Concepts

- Example:



2. Preliminary Concepts

- Example:

$$F(n) = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\frac{2\pi n}{T}t}dt = \frac{1}{2}\int_{-1}^{1} te^{-j\frac{2\pi n}{2}t}dt =$$

$$= \frac{1}{2}\int_{-1}^{1} t\left[\cos(\pi nt) - j\sin(\pi nt)\right]dt$$

$$= \frac{1}{2}\int_{-1}^{1} t\cos(\pi nt)dt - \frac{j}{2}\int_{-1}^{1} t\sin(\pi nt)dt$$

0

2. Preliminary Concepts

- Example:

$$F(n) = -\frac{j}{2}\int_{-1}^{1} t\sin(\pi n t)dt = -j\int_{0}^{1} t\sin(\pi n t)dt$$

$$= -j\left[\frac{\sin(\pi n t)}{(\pi n)^2} - t\frac{\cos(\pi n t)}{\pi n}\right]_0^1$$

$$= -j\left[\frac{\sin(\pi n) - \pi n\cos(\pi n)}{(\pi n)^2}\right]$$

$$\boxed{\int x\sin ax\,dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a} + C}$$

2. Preliminary Concepts

- Example:

$$= -j\left[\frac{\underbrace{\sin(\pi n)}_{0} - \pi n\cos(\pi n)}{(\pi n)^2}\right]$$

$$= -j\left[\frac{-\cos(\pi n)}{\pi n}\right] = -j\left[\frac{-(-1)^n}{\pi n}\right]$$

$$F(n) = j\frac{(-1)^n}{\pi n}, n \neq 0$$

$$F(0) = \frac{1}{T}\int_0^T f(t)e^{-jw_0 nt}dt = \int_{-1}^{1} t dt = 0, n = 0$$

2. Preliminary Concepts

- Example:

$$f(t) = \sum_{n=-\infty}^{\infty} F(n) e^{j\frac{2\pi n}{T}t} = \sum_{n=-\infty}^{\infty} j\frac{(-1)^n}{\pi n} e^{j\pi n t}$$

$$= \sum_{n=-\infty}^{\infty} j\frac{(-1)^n}{\pi n}\left[\cos(\pi n t) + j\sin(\pi n t)\right], \quad n \neq 0$$

$$= \underbrace{\sum_{n=-\infty}^{\infty} j\frac{(-1)^n}{\pi n}\cos(\pi n t)}_{0} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\pi n}\sin(\pi n t)$$

2. Preliminary Concepts

- MATLAB: s21FourierSeriesSawtooth.m

$$f(t) = -\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\pi n} \sin(\pi n t)$$

$$= -2\sum_{n=1}^{\infty} \frac{(-1)^n}{\pi n} \sin(\pi n t)$$

2. Preliminary Concepts

- Impulses and Their Sifting Property

  - A **unit impulse** of a continuous variable *t* is define as

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

  - An impulse may be viewed as a spike of infinity amplitude and zero duration, having unit area.

2. Preliminary Concepts

- Impulses and Their Sifting Property

  - An impulse has the so called **sifting property** with respect to integration

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$$

provided that $f(t)$ is continuous at $t = 0$.

  - A more general statement

$$\int_{-\infty}^{\infty} f(t)\,\delta(t - t_0)\,dt = f(t_0)$$

2. Preliminary Concepts

- Impulses and Their Sifting Property

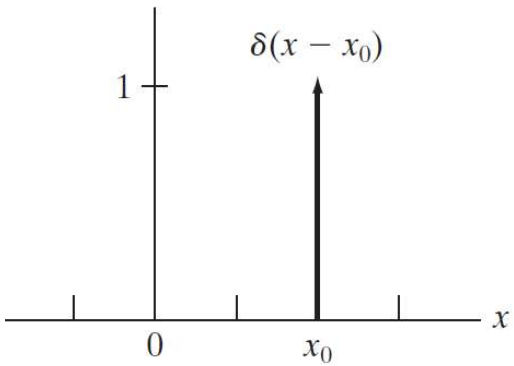
  - The **unit discrete impulse**

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases} \qquad \sum_{x=-\infty}^{\infty} \delta(x) = 1$$

  - *The* **sifting property**

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0)$$

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$$

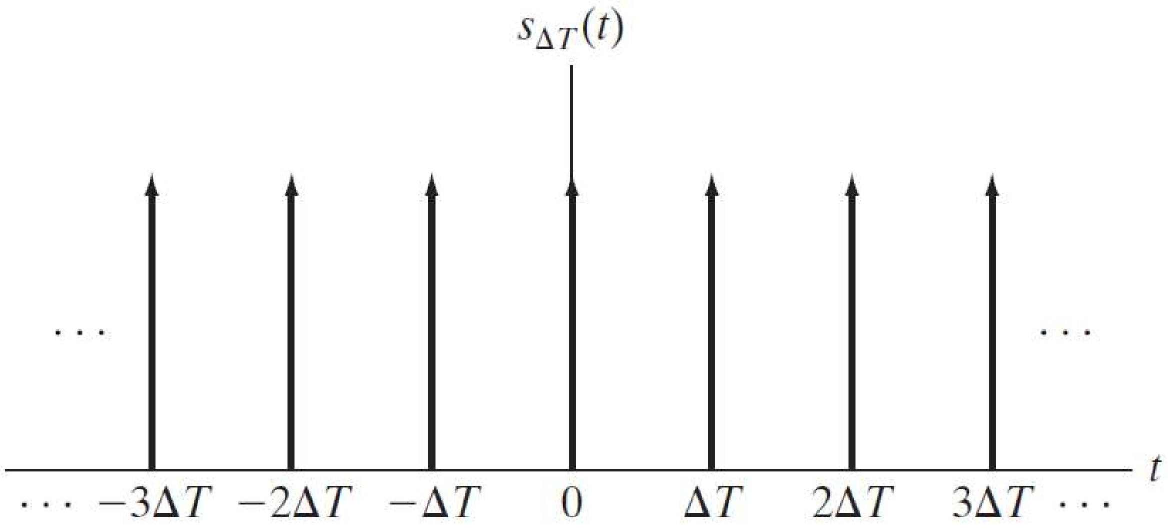


2. Preliminary Concepts

- Impulses and Their Sifting Property

  - **Impulse train**

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$



2. Preliminary Concepts

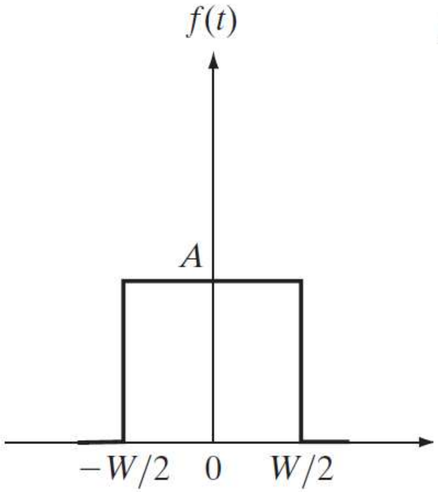
- The Fourier Transform of Functions of One Continuous Variable

$$\Im\{f(t)\} = F(\mu) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\, dt$$

$$\Im^{-1}\{F(\mu)\} = f(t) = \int_{-\infty}^{\infty} F(\mu)\, e^{j2\pi\mu t}\, d\mu$$

2. Preliminary Concepts

• The Fourier Transform of Functions of One Continuous Variable



$$F(\mu) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\, dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t}\, dt$$

$$= \frac{-A}{j2\pi\mu}\left[e^{-j2\pi\mu t}\right]_{-W/2}^{W/2} = \frac{-A}{j2\pi\mu}\left[e^{-j\pi\mu W} - e^{j\pi\mu W}\right]$$

$$= \frac{A}{j2\pi\mu}\left[e^{j\pi\mu W} - e^{-j\pi\mu W}\right]$$

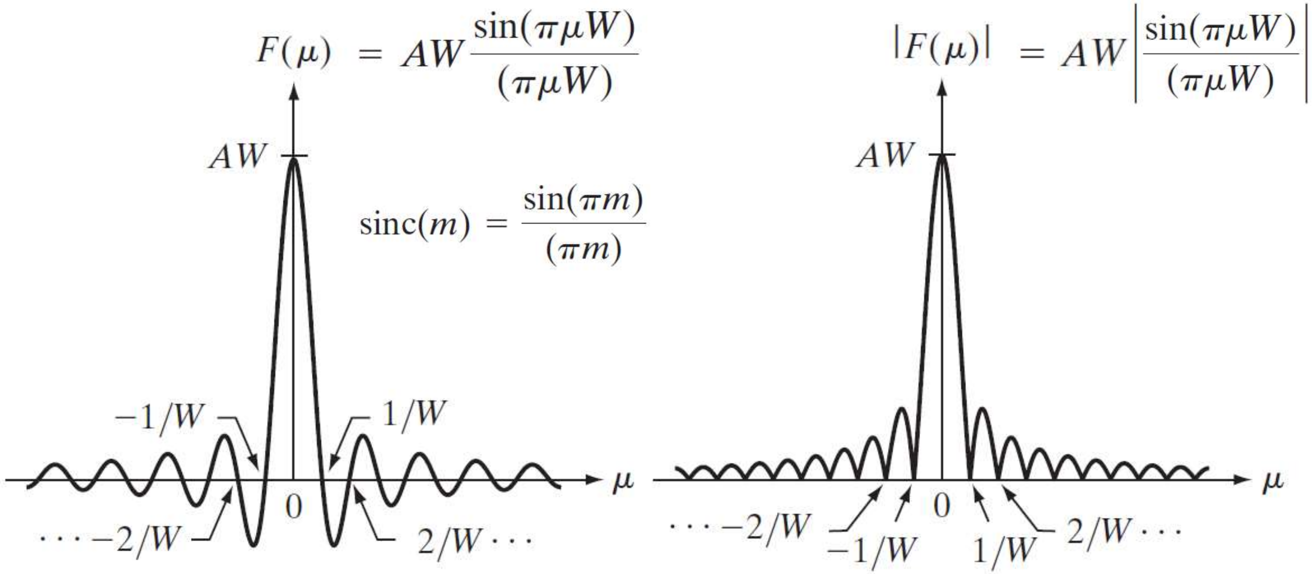
$$= AW\frac{\sin(\pi\mu W)}{(\pi\mu W)}$$

$$\boxed{\int e^{cx}\, dx = \frac{1}{c} e^{cx}}$$

2. Preliminary Concepts

• The Fourier Transform of Functions of One Continuous Variable

- In general, the Fourier transform contains complex terms, and it is customary for display purposes to work with the magnitude of the transform, which is called the **frequency spectrum**.



2. Preliminary Concepts

• The Fourier Transform of Functions of One Continuous Variable

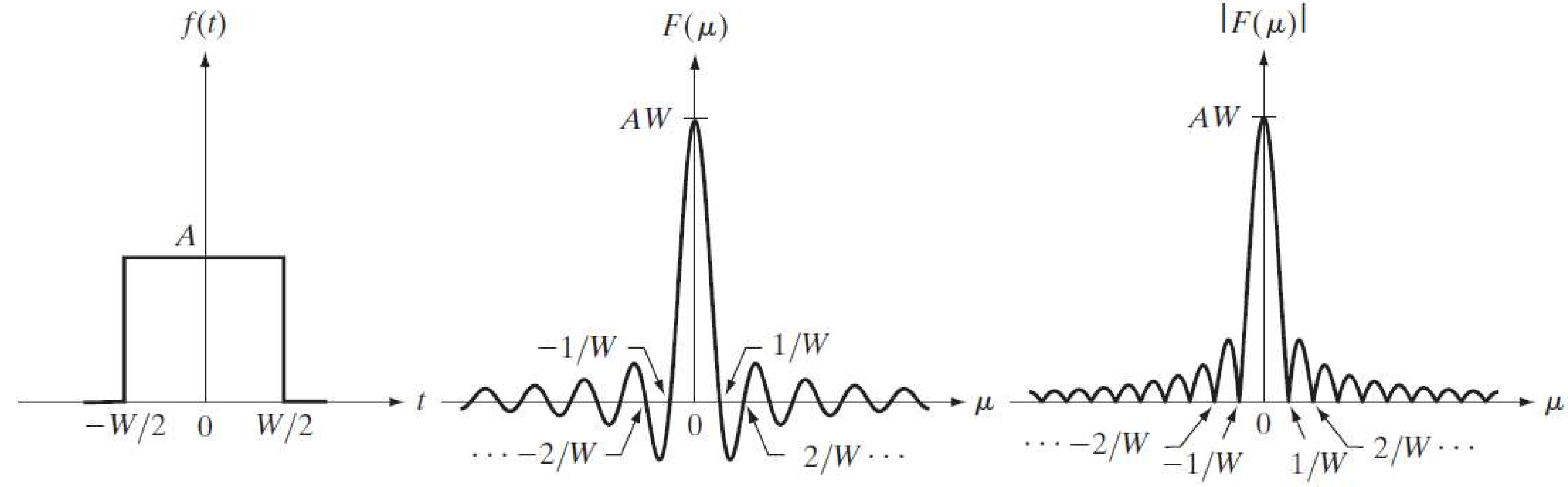


FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

2. Preliminary Concepts

• The Fourier Transform of Functions of One Continuous Variable

- The Fourier transform of a unit impulse located at the origin

$$
\begin{aligned}
F(\mu) &= \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi\mu t} dt \\
&= \int_{-\infty}^{\infty} e^{-j2\pi\mu t}\, \delta(t)\, dt \\
&= e^{-j2\pi\mu 0} = e^{0} \\
&= 1
\end{aligned}
$$

2. Preliminary Concepts

• The Fourier Transform of Functions of One Continuous Variable

➢ Similarly, the Fourier transform of an impulse located at $t = t_0$

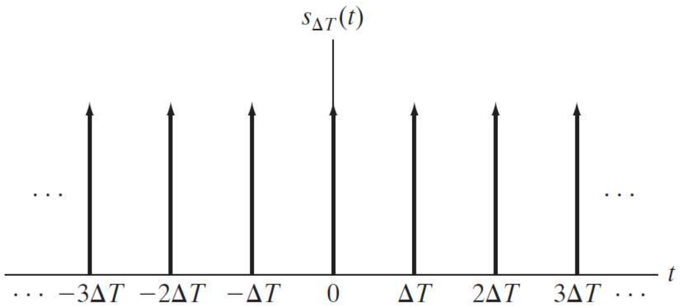
$$
\begin{aligned}
F(\mu) &= \int_{-\infty}^{\infty} \delta(t - t_0)\, e^{-j2\pi\mu t} dt \\
&= \int_{-\infty}^{\infty} e^{-j2\pi\mu t}\, \delta(t - t_0)\, dt \\
&= e^{-j2\pi\mu t_0} \\
&= \cos(2\pi\mu t_0) - j\sin(2\pi\mu t_0)
\end{aligned}
$$

2. Preliminary Concepts

• The Fourier Transform of Functions of One Continuous Variable

- The **impulse train** below is periodic so that it can be expressed as a Fourier series

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$



$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t}$$

$$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t)\, e^{-j\frac{2\pi n}{\Delta T}t}\, dt$$

2. Preliminary Concepts

•The Fourier Transform of Functions of One Continuous Variable

- The preceding equation becomes

$$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} \delta(t) e^{-j\frac{2\pi n}{\Delta T}t} dt$$

$$= \frac{1}{\Delta T} e^0$$

$$= \frac{1}{\Delta T}$$

- The Fourier series expansion then becomes

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t} \longrightarrow s_{\Delta T}(t) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}$$

2. Preliminary Concepts

• The Fourier Transform of Functions of One Continuous Variable

- Our objective is to obtain the Fourier transform of this expression

$$s_{\Delta T}(t) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}$$

- Because summation is a linear process, obtaining the **Fourier transform of a sum** is the same as obtaining the **sum of the transforms** of the individual components.

$$\Im\left\{e^{j\frac{2\pi n}{\Delta T}t}\right\}$$

2. Preliminary Concepts

- The Fourier Transform of Functions of One Continuous Variable

  - Suppose we have *f(t)* and we know the Fourier Transform of a function *g(t)*

$$f(t) = e^{j\frac{2\pi}{\Delta T}t} \qquad G(\mu) = \delta(\mu - 1/\Delta T)$$

then

$$g(t) = \int_{-\infty}^{+\infty} G(\mu) e^{j2\pi\mu t}\, d\mu$$

$$= \int_{-\infty}^{+\infty} \delta(\mu - 1/\Delta T) e^{j2\pi\mu}\, d\mu = e^{j\frac{2\pi}{\Delta T}t}$$

It follows that $f(t) = g(t)$

2. Preliminary Concepts

- The Fourier Transform of Functions of One Continuous Variable

  - If

$$f(t) = g(t)$$

then

$$\Im\{f(t)\} = \Im\{g(t)\}$$

and

$$f(t) = e^{j\frac{2\pi}{\Delta T}t} \longleftrightarrow F(\mu) = \delta(\mu - 1/\Delta T)$$

2. Preliminary Concepts

- The Fourier Transform of Functions of One Continuous Variable

  - Therefore, if

$$f(t) = e^{j\frac{2\pi}{\Delta T}t} \longleftrightarrow F(\mu) = \delta(\mu - 1/\Delta T)$$

and

$$s_{\Delta T}(t) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}$$

then

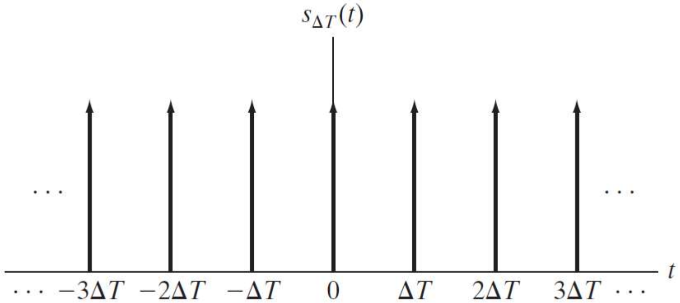
$$S(\mu) = \Im\{s_{\Delta T}(t)\} = \Im\left\{\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} = \frac{1}{\Delta T}\Im\left\{\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\}$$

$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$

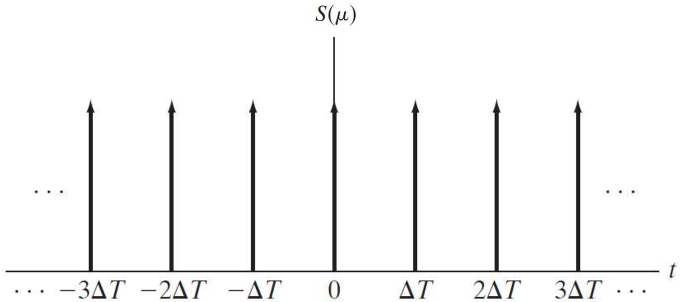
2. Preliminary Concepts

• The Fourier Transform of Functions of One Continuous Variable

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$



$$S(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$



2. Preliminary Concepts

- Convolution

$$f(t) \star h(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$$

➢ Fourier transform of the convolution

$$\Im\left\{f(t) \star h(t)\right\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau\right] e^{-j2\pi\mu t}\, dt$$

$$= \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j2\pi\mu t}\, dt\right] d\tau$$

2. Preliminary Concepts

- Convolution

  - It can be shown that (exercise),

$$\Im\{h(t-\tau)\} = H(\mu)\mathrm{e}^{-j2\pi\mu\tau}$$

  - From the previous equations

$$\Im\{f(t)\star h(t)\} = \int_{-\infty}^{\infty} f(\tau)\left[H(\mu)e^{-j2\pi\mu\tau}\right]d\tau$$

$$= H(\mu)\int_{-\infty}^{\infty} f(\tau)e^{-j2\pi\mu\tau}\,d\tau$$

$$= H(\mu)F(\mu)$$

2. Preliminary Concepts

- Convolution

  - This result is one-half of the **convolution theorem** and is written as

$$f(t) \star h(t) \Leftrightarrow H(\mu)F(\mu)$$

  - Following a similar development would result in the other half of the convolution theorem:

$$f(t)h(t) \Leftrightarrow H(\mu) \star F(\mu)$$

3. Sampling and Fourier Transform of Sampled Functions

- Sampling

  - One way to model sampling is to multiply *f(t)* by a **sampling function** equal to a train of impulses *ΔT* units apart

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\,\delta(t - n\Delta T)$$

  - The **value** of each sample is then given by the "strength" of the weighted impulse, which we obtain by integration. That is, the value, f_k, of an arbitrary sample in the sequence is given by

$$f_k = \int_{-\infty}^{\infty} f(t)\,\delta(t - k\Delta T)\,dt = f(k\Delta T)$$

$$k = \ldots, -2, -1, 0, 1, 2, \ldots$$

3. Sampling and Fourier Transform of Sampled Functions

- Sampling

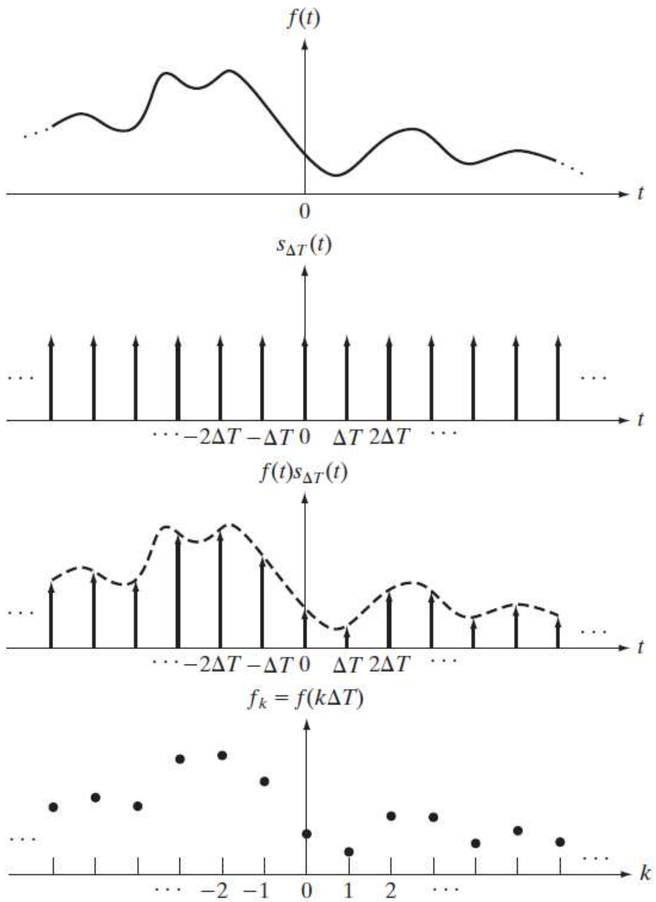
$$f(t)$$

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

$$f_k = \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)\, dt$$

$$= f(k\Delta T)$$



a
b
c
d

FIGURE 4.5
(a) A continuous function. (b) Train of impulses used to model the sampling process. (c) Sampled function formed as the product of (a) and (b). (d) Sample values obtained by integration and using the sifting property of the impulse. (The dashed line in (c) is shown for reference. It is not part of the data.)

3. Sampling and Fourier Transform of Sampled Functions

- Fourier Transform of Sampled Functions

  - From the convolution theorem

$$\begin{aligned}\tilde{F}(\mu) &= \Im\left\{\tilde{f}(t)\right\} \\ &= \Im\left\{f(t)s_{\Delta T}(t)\right\} \\ &= F(\mu) \star S(\mu)\end{aligned}$$

  - And we know that

$$S(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta\left(\mu - \frac{n}{\Delta T}\right)$$

3. Sampling and Fourier Transform of Sampled Functions

- Fourier Transform of Sampled Functions

  - But directly from the definition

$$\begin{aligned}
\tilde{F}(\mu) &= F(\mu) \star S(\mu) \\
&= \int_{-\infty}^{\infty} F(\tau) S(\mu - \tau)\, d\tau \\
&= \frac{1}{\Delta T} \int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{\infty} \delta\left(\mu - \tau - \frac{n}{\Delta T} \right) d\tau \\
&= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(\tau)\, \delta\left(\mu - \tau - \frac{n}{\Delta T} \right) d\tau \\
&= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T} \right)
\end{aligned}$$

$$S(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T} \right)$$

3. Sampling and Fourier Transform of Sampled Functions

- Fourier Transform of Sampled Functions

  - This summation shows that the Fourier transform of a sampled function is an **infinite**, **periodic** sequence of **copies** of the transform of the original continuous function.

$$\tilde{F}(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

3. Sampling and Fourier Transform of Sampled Functions

- The Sampling Theorem

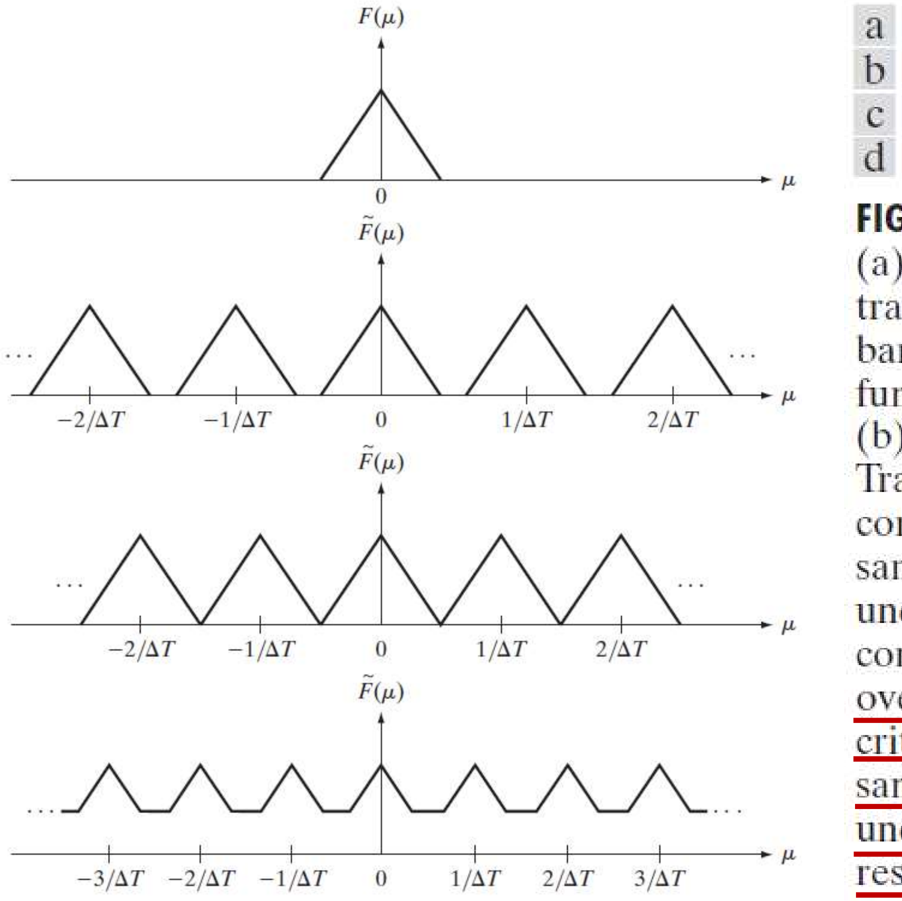
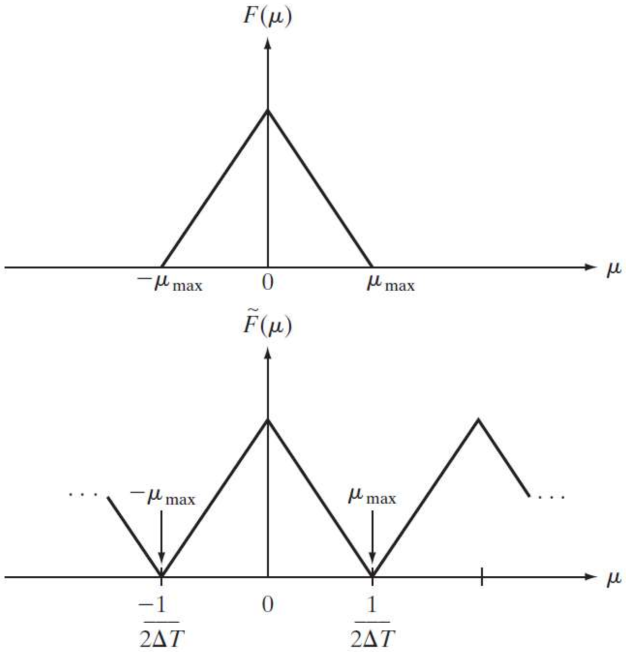


FIGURE 4.6 (a) Fourier transform of a band-limited function. (b)–(d) Transforms of the corresponding sampled function under the conditions of over-sampling, critically-sampling, and under-sampling, respectively.

3. Sampling and Fourier Transform of Sampled Functions

- The Sampling Theorem

  - A function *f(t)* whose Fourier transform is zero for values of frequencies outside a finite interval (band) about the origin is called a **band-limited** function.



a
b

FIGURE 4.7
(a) Transform of a band-limited function.
(b) Transform resulting from critically sampling the same function.

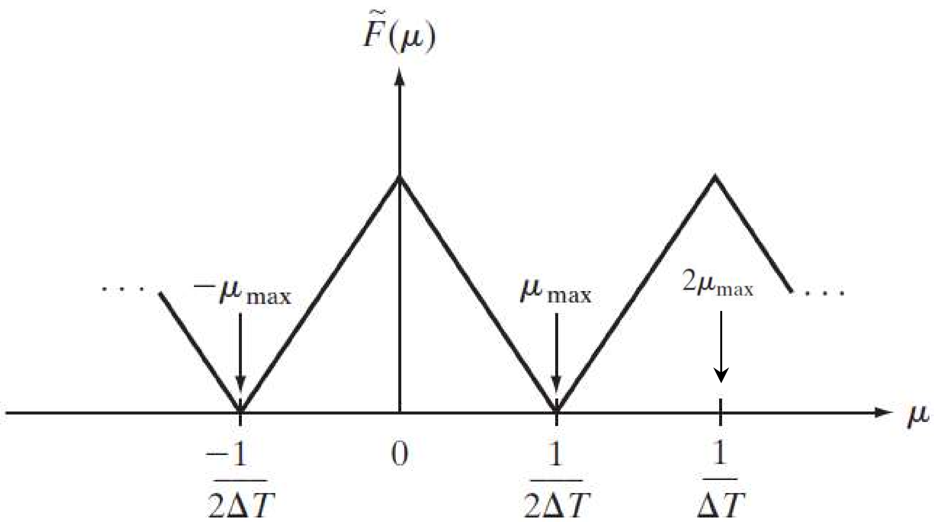
3. Sampling and Fourier Transform of Sampled Functions

- The Sampling Theorem

  - Extracting from $\tilde{F}(\mu)$ a single period that is equal to $F(\mu)$ is possible if the separation between copies is sufficient

  - Sufficient separation is guaranteed if

$$\frac{1}{\Delta T} > 2\mu_{\max}$$



3. Sampling and Fourier Transform of Sampled Functions

- The Sampling Theorem

  - This equation indicates that a continuous, band-limited function can be recovered completely from a set of its samples if the samples are acquired at a rate **exceeding twice the highest frequency content of the function**.

$$\mu_s > 2\mu_{\max}$$

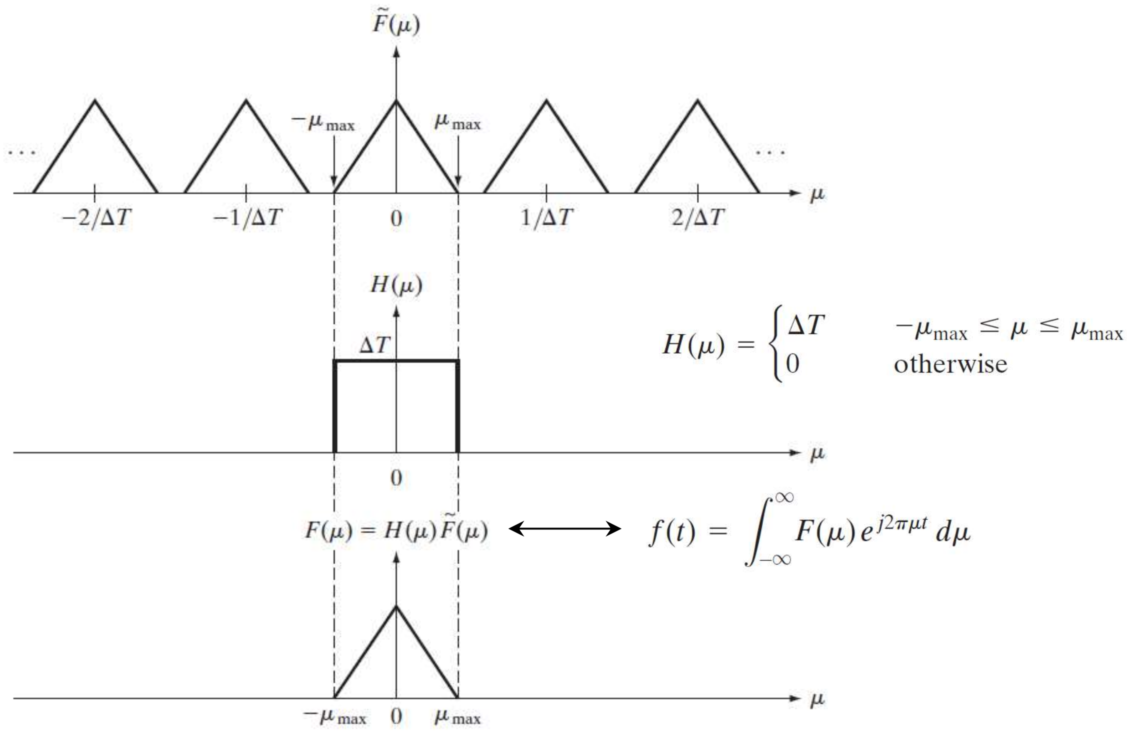
  - This result is known as the **sampling theorem**.

  - A sampling rate equal to exactly twice the highest frequency is called the **Nyquist rate**.

3. Sampling and Fourier Transform of Sampled Functions

- The Sampling Theorem

  - Extracting one period of the transform of a band-limited function using an **ideal lowpass filter**.



3. Sampling and Fourier Transform of Sampled Functions

- The Sampling Theorem

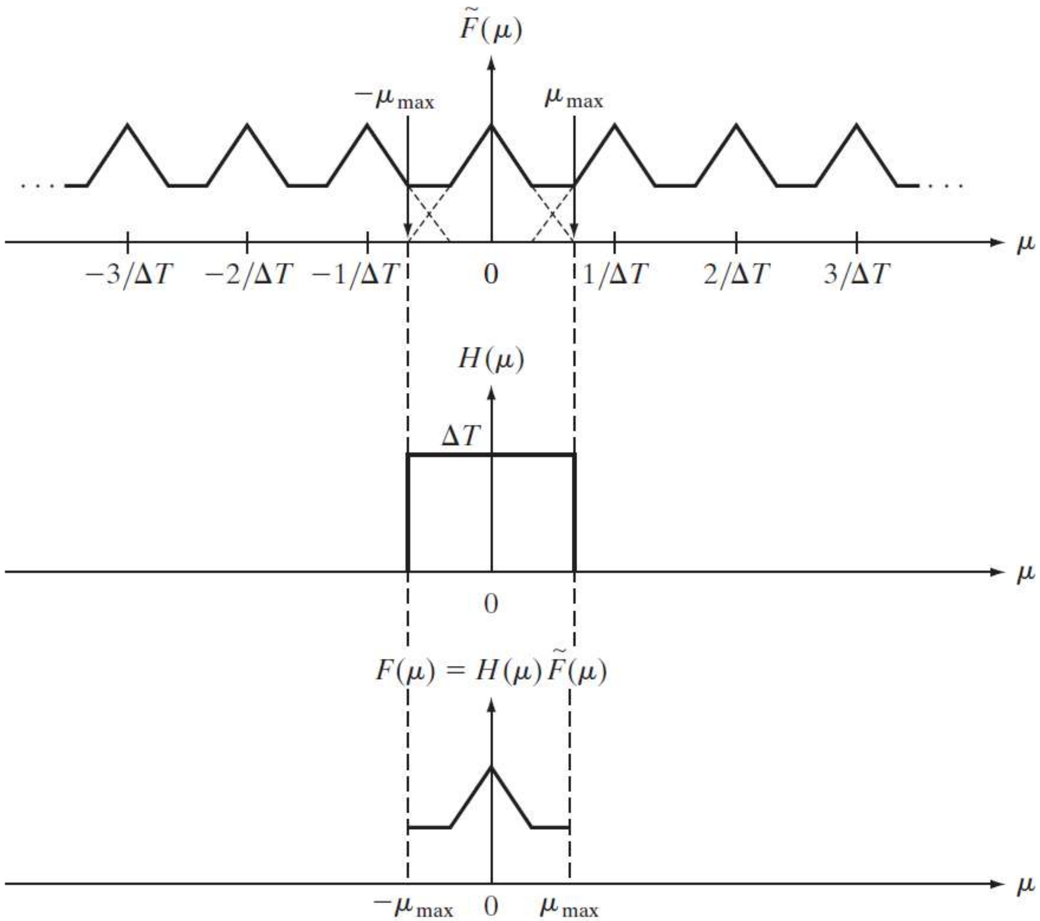
  - Function $H(\mu)$ is called a **lowpass filter** because it passes frequencies at the low end of the frequency range but it eliminates all higher frequencies.

  - It is called also an **ideal lowpass filter** because of its infinitely rapid transitions in amplitude.

  - We will have much more to say about filtering later.

3. Sampling and Fourier Transform of Sampled Functions

- Aliasing



3. Sampling and Fourier Transform of Sampled Functions

- Aliasing

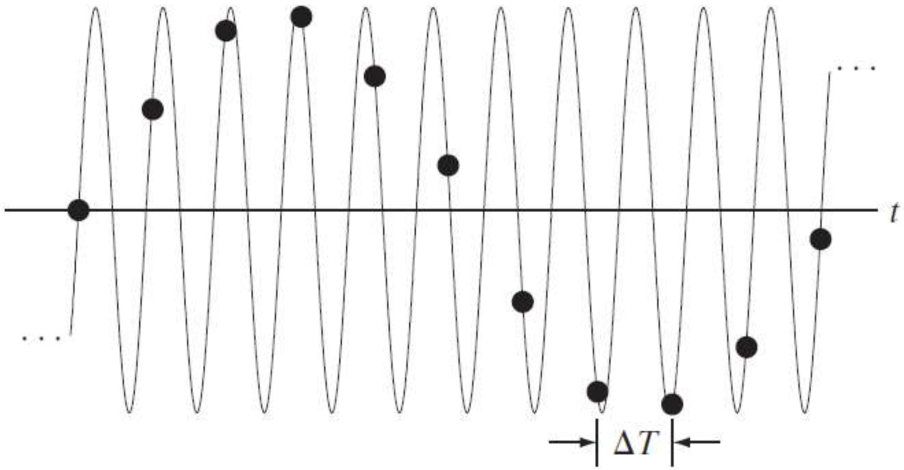


FIGURE 4.10 Illustration of aliasing. The under-sampled function (black dots) looks like a sine wave having a frequency much lower than the frequency of the continuous signal. The period of the sine wave is 2 s, so the zero crossings of the horizontal axis occur every second. ΔT is the separation between samples.

3. Sampling and Fourier Transform of Sampled Functions

- Function reconstruction

  - Using the convolution theorem

$$
\begin{aligned}
f(t) &= \Im^{-1}\{F(\mu)\} \\
&= \Im^{-1}\{H(\mu)\tilde{F}(\mu)\} \\
&= h(t) \star \tilde{f}(t) = \tilde{f}(t) \star h(t)
\end{aligned}
$$

3. Sampling and Fourier Transform of Sampled Functions

- Function reconstruction

  - Using the convolution theorem

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

$$h(t) = \Im^{-1}\{H(u)\} = \int_{-\mu_{\max}}^{\mu_{\max}} \Delta T e^{j2\pi ut} du = \frac{\Delta T}{\pi t}\sin(2\pi\mu_{\max}t)$$

$$= 2\Delta T\mu_{\max}\frac{\sin(\pi 2\mu_{\max}t)}{\pi 2\mu_{\max}t} = 2\Delta T\mu_{\max}\text{sinc}(2\mu_{\max}t)$$

$$= 2\Delta T\frac{1}{2\Delta T}\text{sinc}\left(\frac{t}{\Delta T}\right) = \text{sinc}\left(\frac{t}{\Delta T}\right)$$

$$\boxed{\int e^{cx}\,\mathrm{d}x = \frac{1}{c}e^{cx}}$$

$$\boxed{\text{sinc}(m) = \frac{\sin(\pi m)}{(\pi m)}}$$

3. Sampling and Fourier Transform of Sampled Functions

- Function reconstruction
  - Using the convolution theorem

$$f(t) = \tilde{f}(t) \star h(t) = \int_{-\infty}^{\infty} \tilde{f}(\tau)\, h(t-\tau)\, d\tau$$

$$f(t) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(\tau)\delta(\tau - n\Delta T)h(t-\tau)d\tau$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\delta(\tau - n\Delta T)d\tau$$

$$= \sum_{n=-\infty}^{\infty} f(n\Delta T)h(t - n\Delta T)$$

$$h(t) = \mathrm{sinc}\left(\frac{\mathrm{t}}{\Delta \mathrm{T}}\right) \rightarrow h(t - n\Delta T) = \mathrm{sinc}\left(\frac{t - n\Delta T}{\Delta \mathrm{T}}\right)$$

$$= \sum_{n=-\infty}^{\infty} f(n\Delta T)\mathrm{sinc}\left[(t - n\Delta T)/\Delta \mathrm{T}\right]$$

4. The Discrete Fourier Transform (DFT) of One Variable

- Obtaining the DFT from the Continuous Transform of a Sampled Function

  - In practice, we work with a finite number of samples.

  - First we find $\tilde{F}(u)$ in terms of the sampled function $\tilde{f}(t)$itself.

$$\begin{aligned}\tilde{F}(\mu) &= \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi\mu t}\, dt \\ &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t}\, dt \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t}\, dt \\ &= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}\end{aligned}$$

$$\boxed{\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)}$$

4. The Discrete Fourier Transform (DFT) of One Variable

• Obtaining the DFT from the Continuous Transform of a Sampled Function

- Although f_n is a discrete function, its Fourier $\tilde{F}(\mu)$ is continuous and infinitely periodic with period $1/\Delta T$.

- All we need to characterize $\tilde{F}(\mu)$ is one period, and sampling one period is the basis for the DFT.

- Suppose that we want to obtain M equally spaced samples of $\tilde{F}(\mu)$ taken from $\mu = 0$ to $\mu = 1/\Delta T$.

- This is accomplished by taking the samples at the following frequencies:

$$\mu = \frac{m}{M\Delta T} \qquad m = 0, 1, 2, \ldots, M-1$$

4. The Discrete Fourier Transform (DFT) of One Variable

• Obtaining the DFT from the Continuous Transform of a Sampled Function

➢ Substituting $\mu = \dfrac{m}{M\Delta T} \qquad m = 0, 1, 2, \ldots, M-1$

into

$$\tilde{F}(\mu) = \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}$$

and letting F_m denote the result, then

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad m = 0, 1, 2, \ldots, M-1$$

➢ This is the *discrete Fourier transform* we are seeking. A set $\{f_n\}$ consisting of M samples of *f(t)* yields a set $\{F_m\}$ of M complex discrete values.

4. The Discrete Fourier Transform (DFT) of One Variable

- Obtaining the DFT from the Continuous Transform of a Sampled Function

  - Conversely, given $\{F_m\}$ we can recover the sample set $\{f_n\}$ by using the IDFT.

$$f_n = \frac{1}{M}\sum_{m=0}^{M-1} F_m e^{j2\pi mn/M} \qquad n = 0, 1, 2, \ldots, M-1$$

  - Demonstration

$$f_n = \frac{1}{M}\sum_{m=0}^{M-1}\left\{\sum_{k=0}^{M-1} f_k e^{-j2\pi mk/M}\right\} e^{j2\pi mn/M}$$

$$= \sum_{k=0}^{M-1} f_k \left\{\frac{1}{M}\sum_{m=0}^{M-1} e^{j2\pi\frac{m(k-n)}{M}}\right\} =$$

$$= \sum_{k=0}^{M-1} f_k \delta(k-n) = f_n, \quad n = 0 \ldots M-1$$

$$\frac{1}{M}\sum_{m=0}^{M-1} e^{j2\pi\frac{m(k-n)}{M}} = \frac{1}{M}\sum_{m=0}^{M-1} e^{j2\pi\frac{m\alpha}{M}}, \quad \alpha = (k-n)$$

$$\left.\begin{aligned} \frac{1}{M}\sum_{m=0}^{M-1} e^{j2\pi\frac{m\alpha}{M}} &= 0, \text{ for } \alpha \neq 0 \\ \frac{1}{M}\sum_{m=0}^{M-1} e^{j2\pi\frac{m\alpha}{M}} &= 1, \text{ for } \alpha = 0 \end{aligned}\right\} \frac{1}{M}\sum_{m=0}^{M-1} e^{j2\pi\frac{m(k-n)}{M}} = \delta(k-n)$$

4. The Discrete Fourier Transform (DFT) of One Variable

• Obtaining the DFT from the Continuous Transform of a Sampled Function

- It is more intuitive, especially in two dimensions, to use the notation *x* and *y* for image coordinate variables and *u* and *v* for frequency variables.

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad u = 0, 1, 2, \dots, M-1$$

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \qquad x = 0, 1, 2, \dots, M-1$$

- It can be shown that both the forward and inverse discrete transforms are infinitely periodic, with period M.

$$f(x) = f(x + kM) \qquad F(u) = F(u + kM)$$

4. The Discrete Fourier Transform (DFT) of One Variable

• Obtaining the DFT from the Continuous Transform of a Sampled Function

- The discrete equivalent of the convolution

$$f(x) \star h(x) = \sum_{m=0}^{M-1} f(m)h(x-m) \qquad x = 0, 1, 2, \ldots, M-1$$

- Because in the preceding formulations the functions are periodic, their convolution is also periodic.

- For this reason, the process inherent in this equation often is referred to as **circular convolution**, and is a direct result of the periodicity of the DFT and its inverse.

- Finally, the convolution theorem is applicable also to discrete variables.

4. The Discrete Fourier Transform (DFT) of One Variable

- Relationship between the sampling and frequency intervals

  - If *f(x)* consists of *M* samples of a function *f(t)* taken *ΔT* units apart, the duration of the record comprising the set

$$\{f(x)\},\ x = 0, 1, 2, \ldots, M-1$$

is

$$T = M\Delta T$$

  - The corresponding spacing *Δμ*, in the discrete frequency domain follows from

$$\Delta u = \frac{1}{M\Delta T} = \frac{1}{T}$$

  - The entire frequency range spanned by the M components of the DFT is

$$\Omega = M\Delta u = \frac{1}{\Delta T}$$

4. The Discrete Fourier Transform (DFT) of One Variable

- Relationship between the sampling and frequency intervals

  - Thus, we see that the resolution in frequency, Δu, of the DFT depends on the duration T over which the continuous function, *f(t)*, is sampled.

$$\Delta u = \frac{1}{M\Delta T} = \frac{1}{T}$$

  - The range of frequencies spanned by the DFT depends on the sampling interval.

$$\Omega = M\Delta u = \frac{1}{\Delta T}$$

  - Observe that both expressions exhibit inverse relationships with respect to T and ΔT.

5. Extension to Functions of Two Variables

- The 2-D Impulse and Its Sifting Property

  - The impulse of two continuous variables is given by

$$\delta(t, z) = \begin{cases} \infty & \text{if } t = z = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t, z)\, dt\, dz = 1$$

  - The 2-D impulse exhibits the sifting property under integration

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t, z)\,\delta(t, z)\, dt\, dz = f(0, 0)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t, z)\,\delta(t - t_0, z - z_0)\, dt\, dz = f(t_0, z_0)$$

5. Extension to Functions of Two Variables

- The 2-D Impulse and Its Sifting Property

  - For discrete variables

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x, y)\,\delta(x, y) = f(0, 0)$$

  - For an impulse located at coordinates (x_0, y_0)

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x, y)\,\delta(x - x_0, y - y_0) = f(x_0, y_0)$$

5. Extension to Functions of Two Variables

- The 2-D Impulse and Its Sifting Property
  - For an impulse located at coordinates (x_0, y_0)

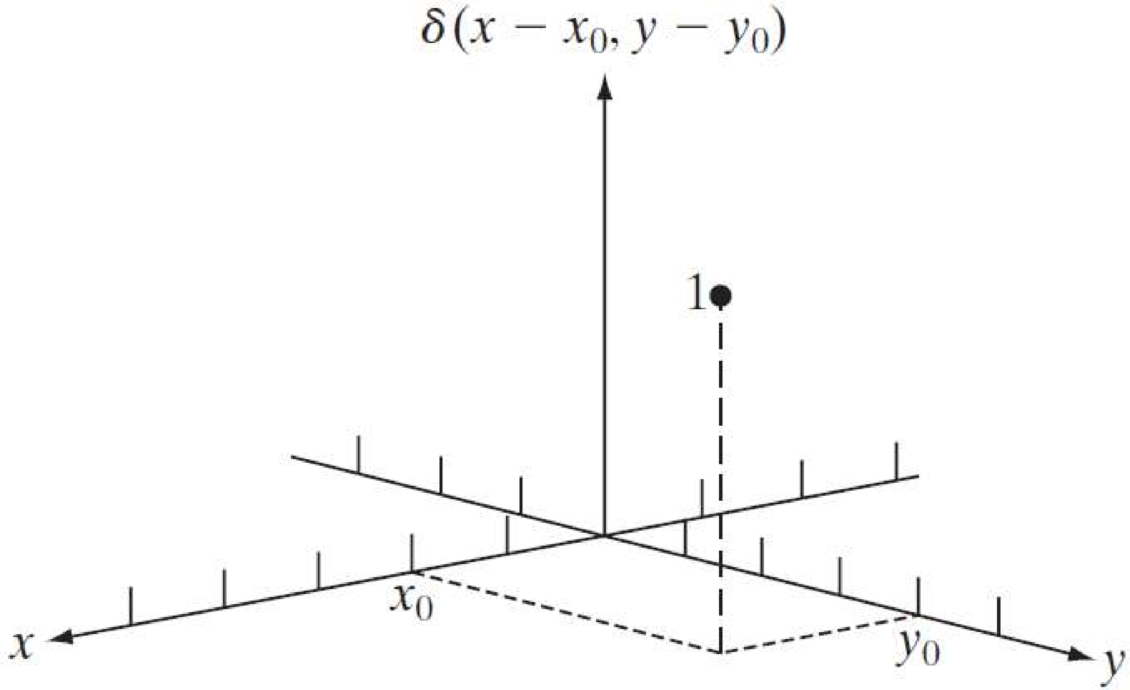


FIGURE 4.12 Two-dimensional unit discrete impulse. Variables x and y are discrete, and δ is zero everywhere except at coordinates (x_0, y_0).

5. Extension to Functions of Two Variables

- The 2-D Continuous Fourier Transform Pair

  - Let $f(t,z)$ be a continuous function of two continuous variables, t and z.

  - The two-dimensional, continuous Fourier transform pair is given by the expressions

$$F(\mu, \nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) e^{-j2\pi(\mu t + \nu z)} dt\, dz$$

$$f(t, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\mu, \nu) e^{j2\pi(\mu t + \nu z)} d\mu\, d\nu$$

5. Extension to Functions of Two Variables

- The 2-D Continuous Fourier Transform Pair

  - Example

$$F(\mu, \nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t, z)\, e^{-j2\pi(\mu t + \nu z)}\, dt\, dz$$

$$= \int_{-T/2}^{T/2}\int_{-Z/2}^{Z/2} A\, e^{-j2\pi(\mu t + \nu z)}\, dt\, dz$$

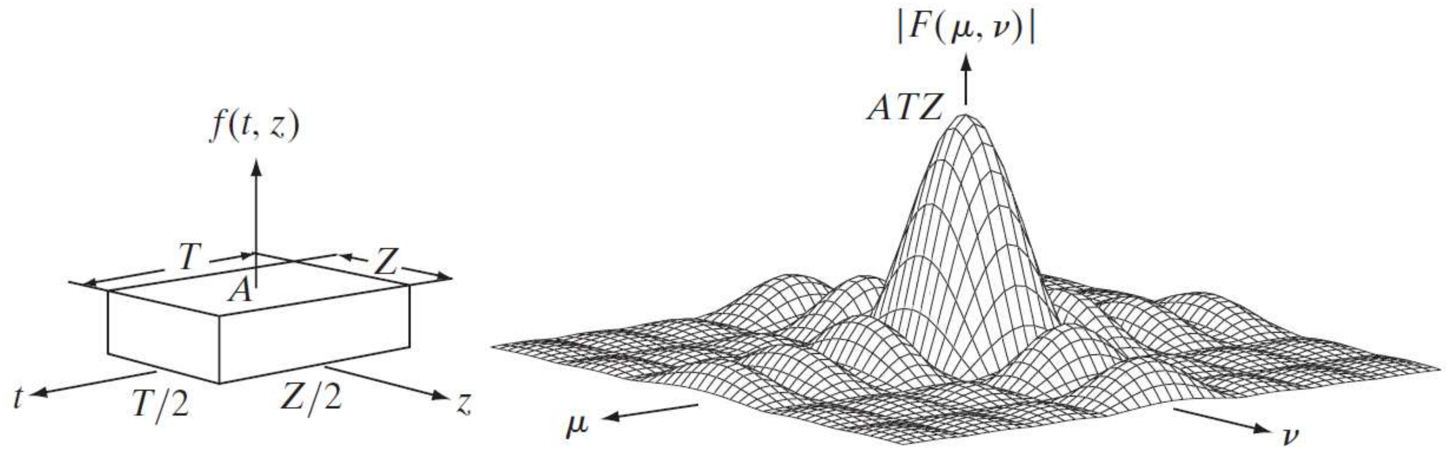
$$= ATZ\left[\frac{\sin(\pi\mu T)}{(\pi\mu T)}\right]\left[\frac{\sin(\pi\nu Z)}{(\pi\nu Z)}\right]$$

$$|F(\mu, \nu)| = ATZ\left|\frac{\sin(\pi\mu T)}{(\pi\mu T)}\right|\left|\frac{\sin(\pi\nu Z)}{(\pi\nu Z)}\right|$$

5. Extension to Functions of Two Variables

- The 2-D Continuous Fourier Transform Pair

  - Example



a b

FIGURE 4.13 (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the t-axis, so the spectrum is more "contracted" along the μ-axis. Compare with Fig. 4.4.

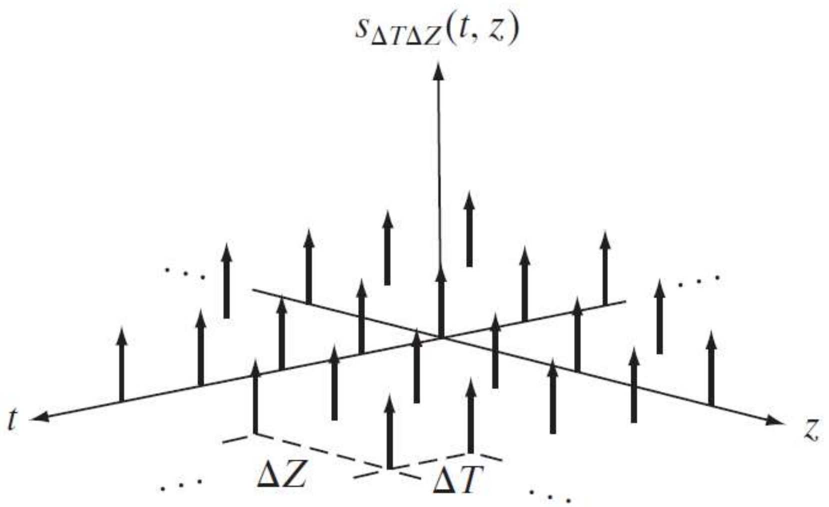
5. Extension to Functions of Two Variables

- Two-Dimensional Sampling and the 2-D Sampling Theorem

  - 2-D impulse train

$$s_{\Delta T \Delta Z}(t, z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$

$$F(\mu, \nu) = 0 \quad \text{for } |\mu| \geq \mu_{\max} \text{ and } |\nu| \geq \nu_{\max}$$



5. Extension to Functions of Two Variables

- Two-Dimensional Sampling and the 2-D Sampling Theorem

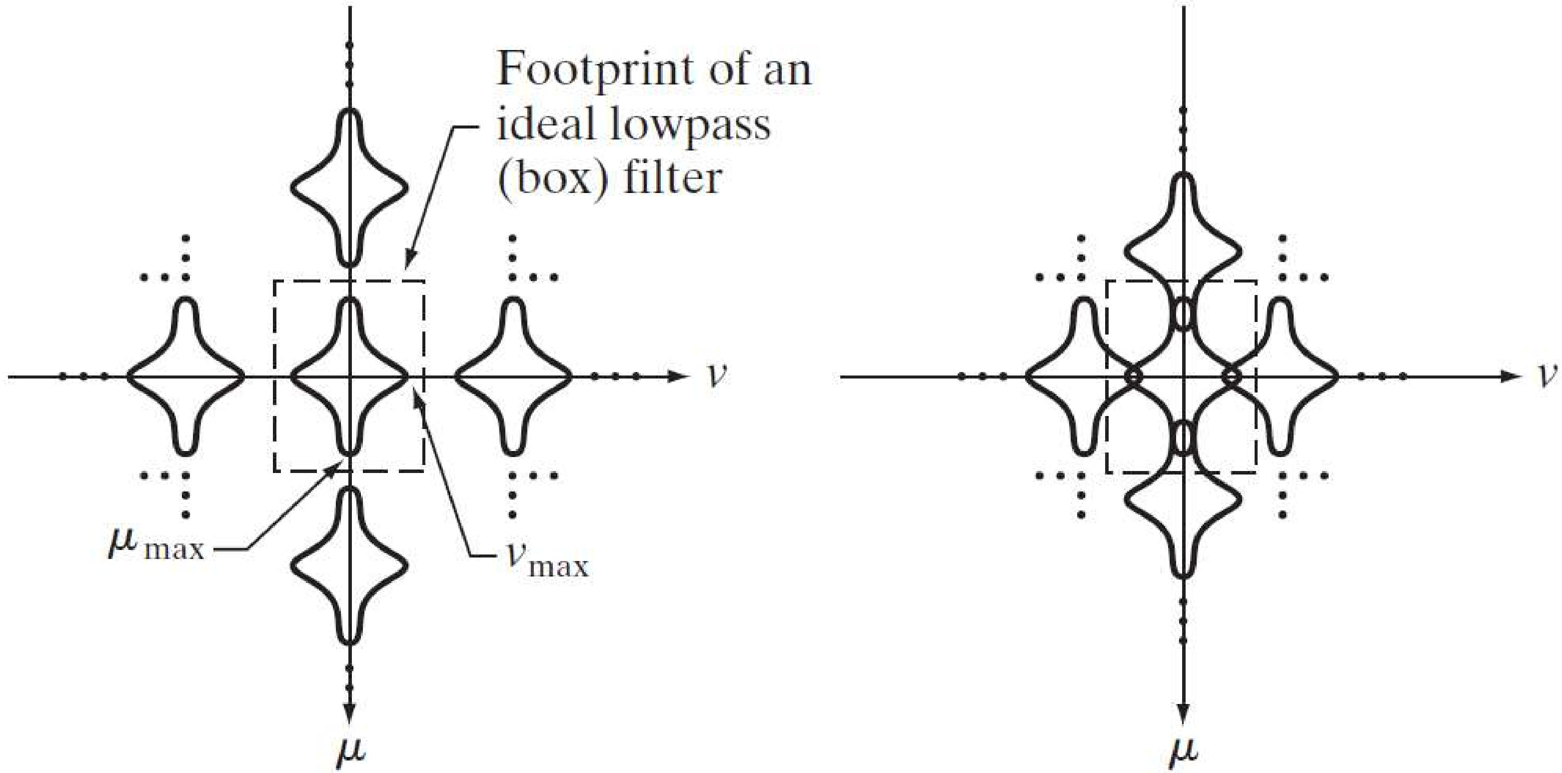
  - The two-dimensional sampling theorem states that a continuous, band-limited function can be recovered with no error from a set of its samples if the sampling intervals are

$$\frac{1}{\Delta T} > 2\mu_{\max} \qquad \frac{1}{\Delta Z} > 2\nu_{\max}$$

  - Stated another way, no information is lost if a 2-D, band-limited, continuous function is represented by samples acquired at rates greater than twice the highest frequency content of the function in both the μ- ν-directions.

5. Extension to Functions of Two Variables

- Aliasing in Images



5. Extension to Functions of Two Variables

- Aliasing in Images (sampling)

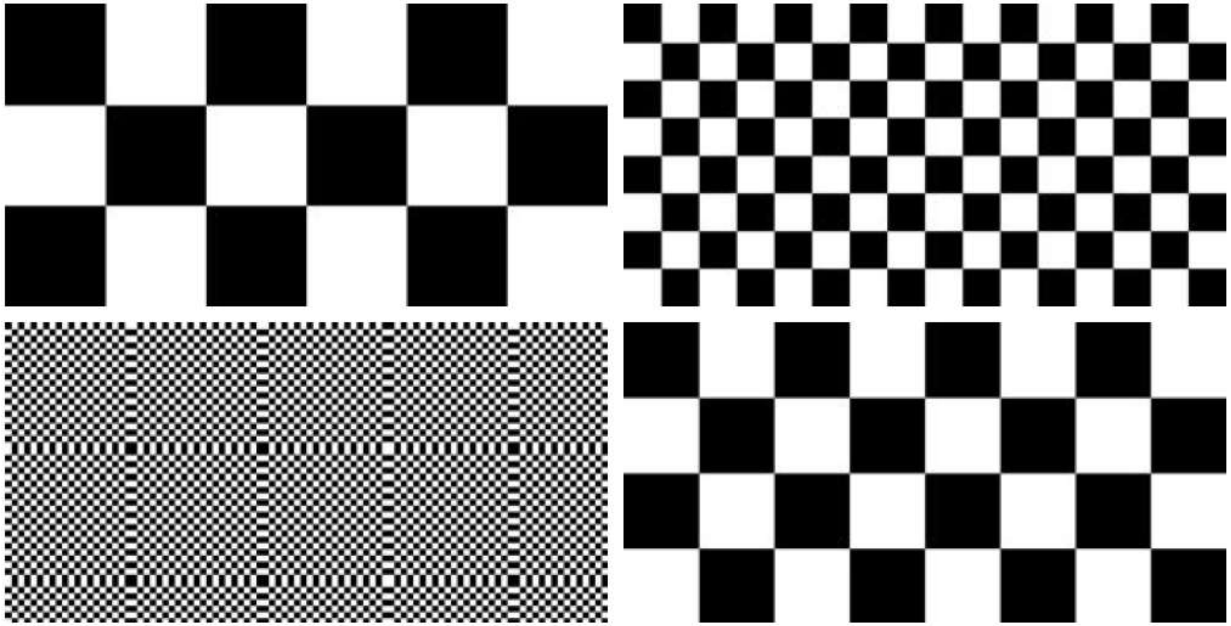
  - Suppose an imaging system whose number of samples it can take is fixed at 96 x 96 pixels.

  - If we use this system to digitize checkerboard patterns, it will be able to resolve patterns that are up to 96 x 96 pixels in which the size of each square is 1 x 1 pixels.

  - What happens when the detail (the size of the checkerboard squares) is less than one camera pixel?

5. Extension to Functions of Two Variables

- Aliasing in Images (sampling)



a b
c d

FIGURE 4.16 Aliasing in images. In (a) and (b), the lengths of the sides of the squares are 16 and 6 pixels, respectively, and aliasing is visually negligible. In (c) and (d), the sides of the squares are 0.9174 and 0.4798 pixels, respectively, and the results show significant aliasing. Note that (d) masquerades as a "normal" image.

5. Extension to Functions of Two Variables

- Aliasing in Images (interpolation and resampling)

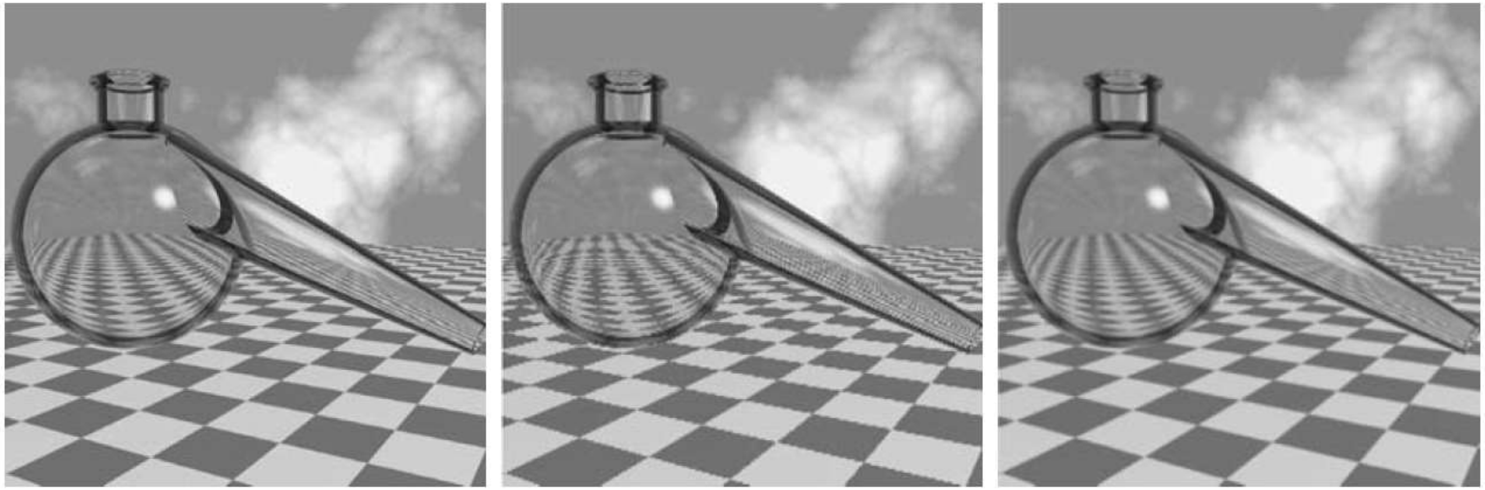
a b c

FIGURE 4.17 Illustration of aliasing on resampled images. (a) A digital image with negligible visual aliasing. (b) Result of resizing the image to 50% of its original size by pixel deletion. Aliasing is clearly visible. (c) Result of blurring the image in (a) with a 3×3 averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)

5. Extension to Functions of Two Variables

- Aliasing in Images (interpolation and resampling)

  - When you work with images that have strong edge content, the effects of aliasing are seen as block-like image components, called *jaggies*.



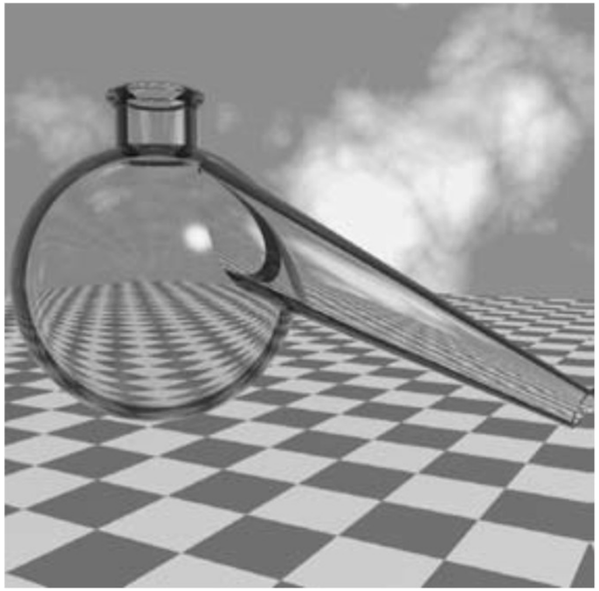
a b c

FIGURE 4.18 Illustration of jaggies. (a) A 1024 × 1024 digital image of a computer-generated scene with negligible visible aliasing. (b) Result of reducing (a) to 25% of its original size using bilinear interpolation. (c) Result of blurring the image in (a) with a 5 × 5 averaging filter prior to resizing it to 25% using bilinear interpolation. (Original image courtesy of D. P. Mitchell, Mental Landscape, LLC.)

5. Extension to Functions of Two Variables

- Aliasing in Images (interpolation and resampling)

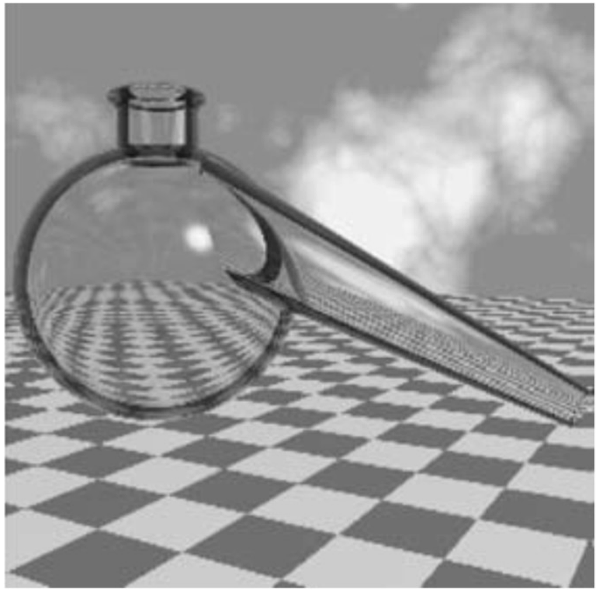
  - When you work with images that have strong edge content, the effects of aliasing are seen as block-like image components, called *jaggies*.



5. Extension to Functions of Two Variables

- Aliasing in Images (interpolation and resampling)

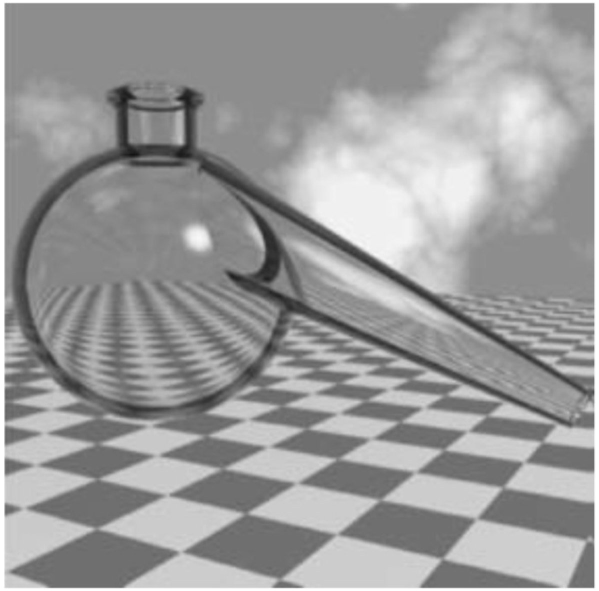
  - When you work with images that have strong edge content, the effects of aliasing are seen as block-like image components, called *jaggies*.



5. Extension to Functions of Two Variables

- Aliasing in Images (interpolation and resampling)

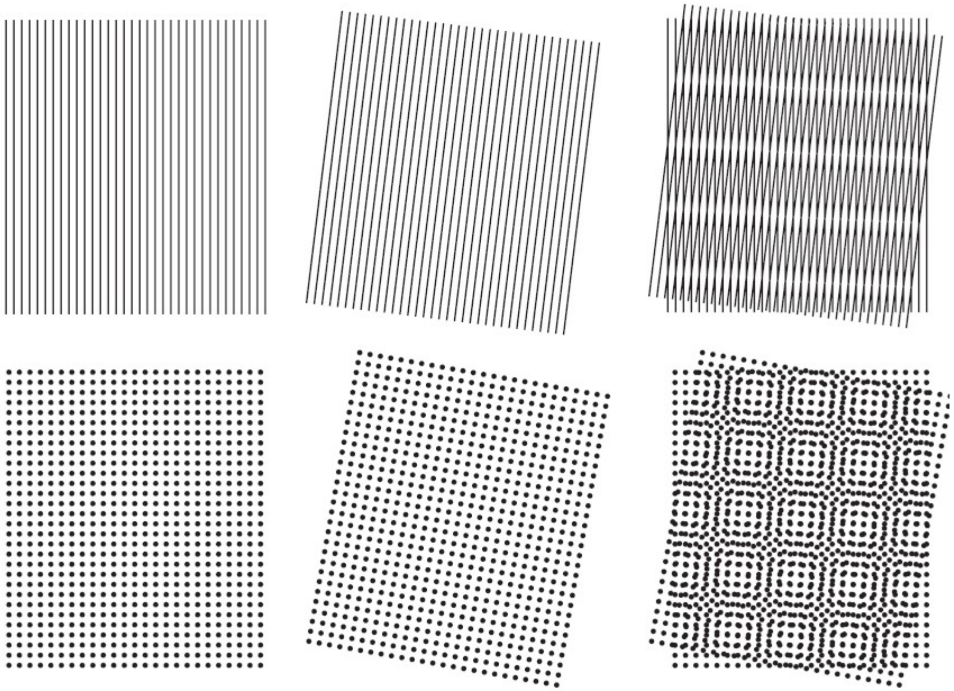
  - When you work with images that have strong edge content, the effects of aliasing are seen as block-like image components, called *jaggies*.



5. Extension to Functions of Two Variables

- Aliasing in Images (Moiré patterns)

    - The problem arises routinely when scanning media print, such as newspapers and magazines, or in images with periodic components.



a	b	c
d	e	f

FIGURE 4.20 Examples of the moiré effect. These are ink drawings, not digitized patterns. Superimposing one pattern on the other is equivalent mathematically to multiplying the patterns.

5. Extension to Functions of Two Variables

- Aliasing in Images (Moiré patterns)

  - Printed materials make use of halftone dots, which are black dots whose sizes and joining schemes are used to simulate gray tones.

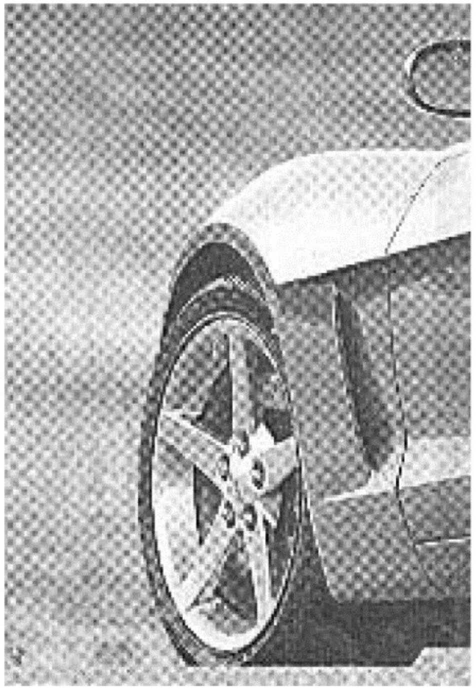


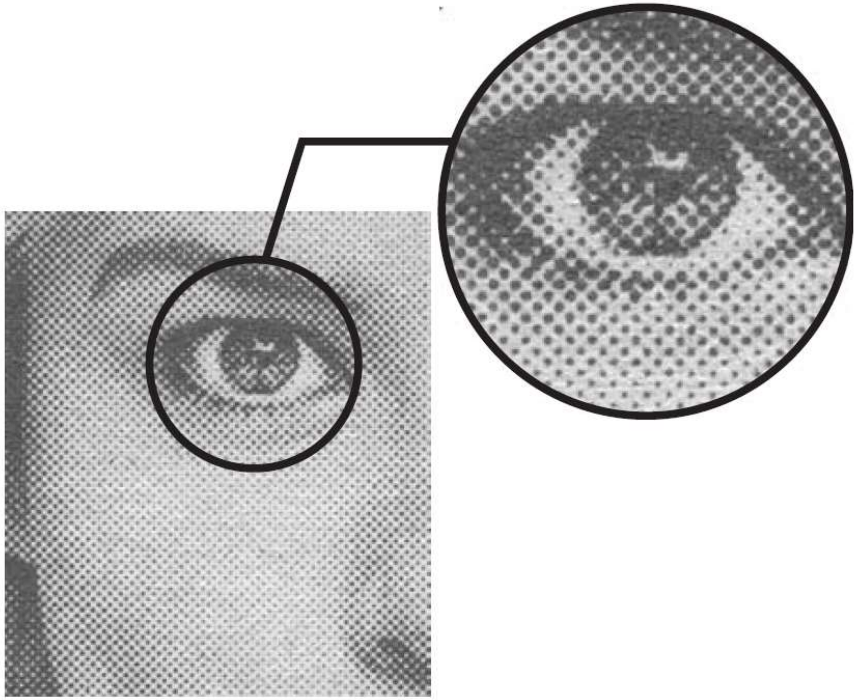
FIGURE 4.21 A newspaper image of size 246×168 pixels sampled at 75 dpi showing a moiré pattern. The moiré pattern in this image is the interference pattern created between the $\pm 45°$ orientation of the halftone dots and the north–south orientation of the sampling grid used to digitize the image.

5. Extension to Functions of Two Variables

- Aliasing in Images (Moiré patterns)

  - Printed materials make use of halftone dots, which are black dots whose sizes and joining schemes are used to simulate gray tones.

FIGURE 4.22 A newspaper image and an enlargement showing how halftone dots are arranged to render shades of gray.



5. Extension to Functions of Two Variables

- The 2-D Discrete Fourier Transform and Its Inverse

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

$u = 0, 1, 2, \dots, M-1$ and $v = 0, 1, 2, \dots, N-1$.

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$$

$x = 0, 1, 2, \dots, M-1$ and $y = 0, 1, 2, \dots, N-1$.

6. Some Properties of the 2-D DFT

- Relationships Between Spatial and Frequency Intervals

  - Let ΔT and ΔZ denote the separations between samples.

  - Then, the separations between the corresponding discrete, frequency domain variables are given by

$$\Delta u = \frac{1}{M\Delta T} \qquad \Delta v = \frac{1}{N\Delta Z}$$

6. Some Properties of the 2-D DFT

- Translation

$$f(x, y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u - u_0, v - v_0)$$

$$f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(x_0u/M+y_0v/N)}$$

- Rotation

$$x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$$

$$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$$

6. Some Properties of the 2-D DFT

- Periodicity

  - The 2-D Fourier transform and its inverse are infinitely periodic in the *u* and *v* directions; that is,

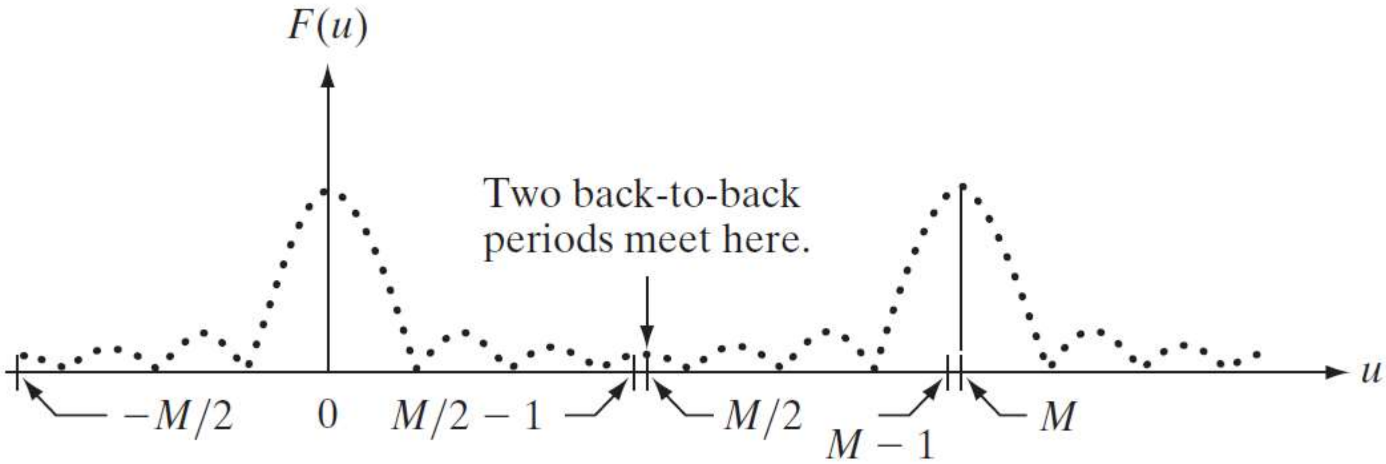
$$F(u, v) = F(u + k_1M, v) = F(u, v + k_2N) = F(u + k_1M, v + k_2N)$$

$$f(x, y) = f(x + k_1M, y) = f(x, y + k_2N) = f(x + k_1M, y + k_2N)$$

6. Some Properties of the 2-D DFT

- Periodicity

  - Consider the 1-D spectrum. The transform data in the interval from 0 to consists of two back-to-back half periods meeting at point M/2.

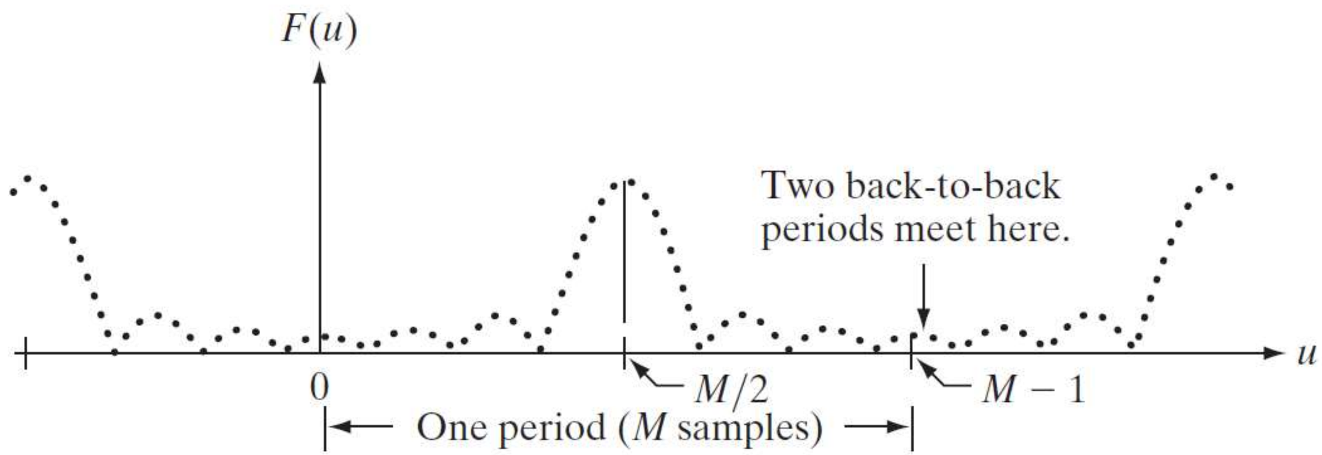


6. Some Properties of the 2-D DFT

- Periodicity

  - For display and filtering purposes, it is more convenient to have in this interval a complete period of the transform in which the data are contiguous.

$$f(x)e^{j2\pi(u_0x/M)} \Leftrightarrow F(u - u_0) \qquad f(x)(-1)^x \Leftrightarrow F(u - M/2)$$

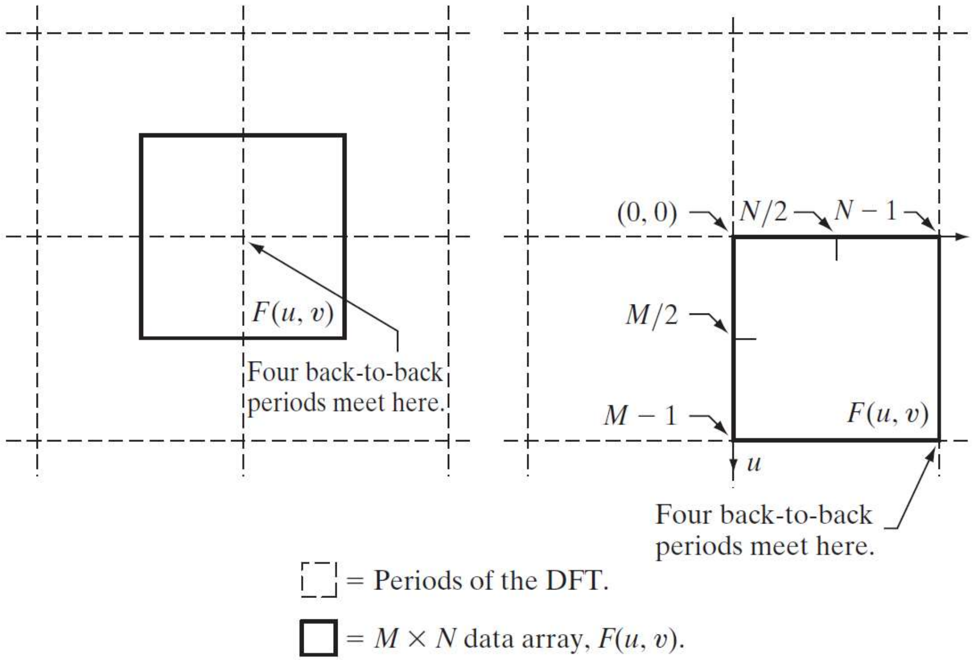


6. Some Properties of the 2-D DFT

- Periodicity

  - In 2-D the situation is more difficult to graph, but the principle is the same.

$$f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$$



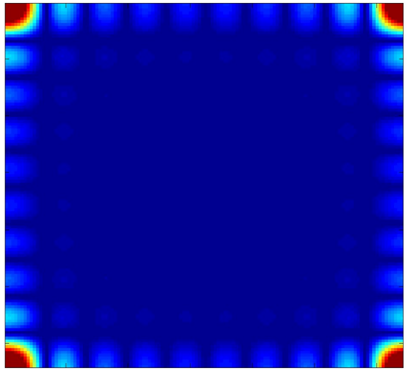
6. Some Properties of the 2-D DFT

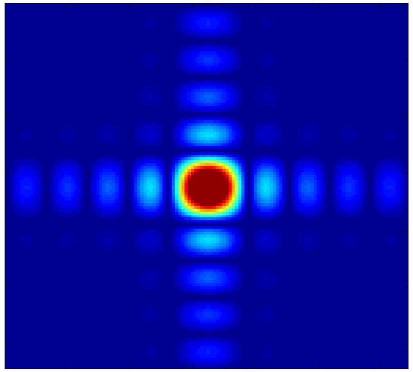
- Periodicity

  - In 2-D the situation is more difficult to graph, but the principle is the same.

$$f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$$

  - Using this equation shifts the data so that F(0,0) is at the center of the frequency rectangle defined by the intervals and [0 M-1], [0 N-1].





6. Some Properties of the 2-D DFT

- Symmetry

  - An important result from functional analysis is that any real or complex function, w(x, y), can be expressed as the sum of an even and an odd part (each of which can be real or complex):

$$w(x, y) = w_e(x, y) + w_o(x, y)$$

  - Where,

$$w_e(x, y) \triangleq \frac{w(x, y) + w(-x, -y)}{2}$$

$$w_o(x, y) \triangleq \frac{w(x, y) - w(-x, -y)}{2}$$

  - It follows that

$$w_e(x, y) = w_e(-x, -y) \qquad w_o(x, y) = -w_o(-x, -y)$$

6. Some Properties of the 2-D DFT

- Symmetry

  - Even functions are said to be *symmetric* and odd functions are *antisymmetric*.

  - When we talk about symmetry we are referring to symmetry about the center point of a sequence

  - It is more convenient to think only in terms of nonnegative indices, in which case the definitions of evenness and oddness become:

$$w_e(x, y) = w_e(M - x, N - y)$$

$$w_o(x, y) = -w_o(M - x, N - y)$$

6. Some Properties of the 2-D DFT

- Symmetry

  - Although evenness and oddness are visualized easily for continuous functions, these concepts are not as intuitive when dealing with discrete sequences.

    - ✓ Even sequences

      - $f(x) = f(M - x)$

      - $f(0)$ is immaterial in the test for evenness.

6. Some Properties of the 2-D DFT

- Symmetry

  - Although evenness and oddness are visualized easily for continuous functions, these concepts are not as intuitive when dealing with discrete sequences.

    - ✓ Odd sequences

      - $g(x) = -g(M - x)$

      - When M (length of the sequence) is an *even* number, a 1-D *odd sequence* has the property that the points at locations 0 and M/2 are always zero.

      - When M is *odd*, the first term still has to be 0, but the remaining terms form pairs with equal value but opposite sign.

6. Some Properties of the 2-D DFT

- Symmetry properties

  - Now, a number of important symmetry properties of the DFT and its inverse can be establish.

  - A property used frequently is that the Fourier transform of a *real* function, *f(x, y)*, is *conjugate symmetric,*

$$F^*(u, v) = F(-u, -v)$$

  - If *f(x, y)* is *imaginary*, its Fourier transform is *conjugate antisymmetric,*

$$F^*(-u, -v) = -F(u, v)$$

6. Some Properties of the 2-D DFT

- Symmetry properties

  - Now, a number of important symmetry properties of the DFT and its inverse can be establish.

	Spatial Domain†		Frequency Domain†
1)	$f(x, y)$ real	$\Leftrightarrow$	$F^*(u, v) = F(-u, -v)$
2)	$f(x, y)$ imaginary	$\Leftrightarrow$	$F^*(-u, -v) = -F(u, v)$
3)	$f(x, y)$ real	$\Leftrightarrow$	$R(u, v)$ even; $I(u, v)$ odd
4)	$f(x, y)$ imaginary	$\Leftrightarrow$	$R(u, v)$ odd; $I(u, v)$ even
5)	$f(-x, -y)$ real	$\Leftrightarrow$	$F^*(u, v)$ complex
6)	$f(-x, -y)$ complex	$\Leftrightarrow$	$F(-u, -v)$ complex
7)	$f^*(x, y)$ complex	$\Leftrightarrow$	$F^*(-u - v)$ complex
8)	$f(x, y)$ real and even	$\Leftrightarrow$	$F(u, v)$ real and even
9)	$f(x, y)$ real and odd	$\Leftrightarrow$	$F(u, v)$ imaginary and odd
10)	$f(x, y)$ imaginary and even	$\Leftrightarrow$	$F(u, v)$ imaginary and even
11)	$f(x, y)$ imaginary and odd	$\Leftrightarrow$	$F(u, v)$ real and odd
12)	$f(x, y)$ complex and even	$\Leftrightarrow$	$F(u, v)$ complex and even
13)	$f(x, y)$ complex and odd	$\Leftrightarrow$	$F(u, v)$ complex and odd

†Recall that x, y, u, and v are *discrete* (integer) variables, with x and u in the range $[0, M-1]$, and y, and v in the range $[0, N-1]$. To say that a complex function is *even* means that its real *and* imaginary parts are even, and similarly for an odd complex function.

TABLE 4.1 Some symmetry properties of the 2-D DFT and its inverse. $R(u, v)$ and $I(u, v)$ are the real and imaginary parts of $F(u, v)$, respectively. The term *complex* indicates that a function has nonzero real and imaginary parts.

6. Some Properties of the 2-D DFT

- Fourier Spectrum and Phase Angle.

  - Because the 2-D DFT is complex in general, it can be expressed in polar form:

$$F(u, v) = |F(u, v)|e^{j\phi(u,v)}$$

where

$$|F(u, v)| = \left[R^2(u, v) + I^2(u, v)\right]^{1/2}$$

is the frequency spectrum and

$$\phi(u, v) = \arctan\left[\frac{I(u, v)}{R(u, v)}\right]$$

is the phase angle.

6. Some Properties of the 2-D DFT

- Fourier Spectrum and Phase Angle.

  - The *power spectrum* is defined as

$$P(u,v) = |F(u,v)|^2$$
$$= R^2(u,v) + I^2(u,v)$$

  - The Fourier transform of a real function is conjugate symmetric, which implies that the spectrum has even symmetry about the origin:

$$|F(u,v)| = |F(-u,-v)|$$

  - The phase angle exhibits the following odd symmetry about the origin:

$$\phi(u,v) = -\phi(-u,-v)$$

6. Some Properties of the 2-D DFT

- Fourier Spectrum and Phase Angle.

  - Finally, it follows from

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

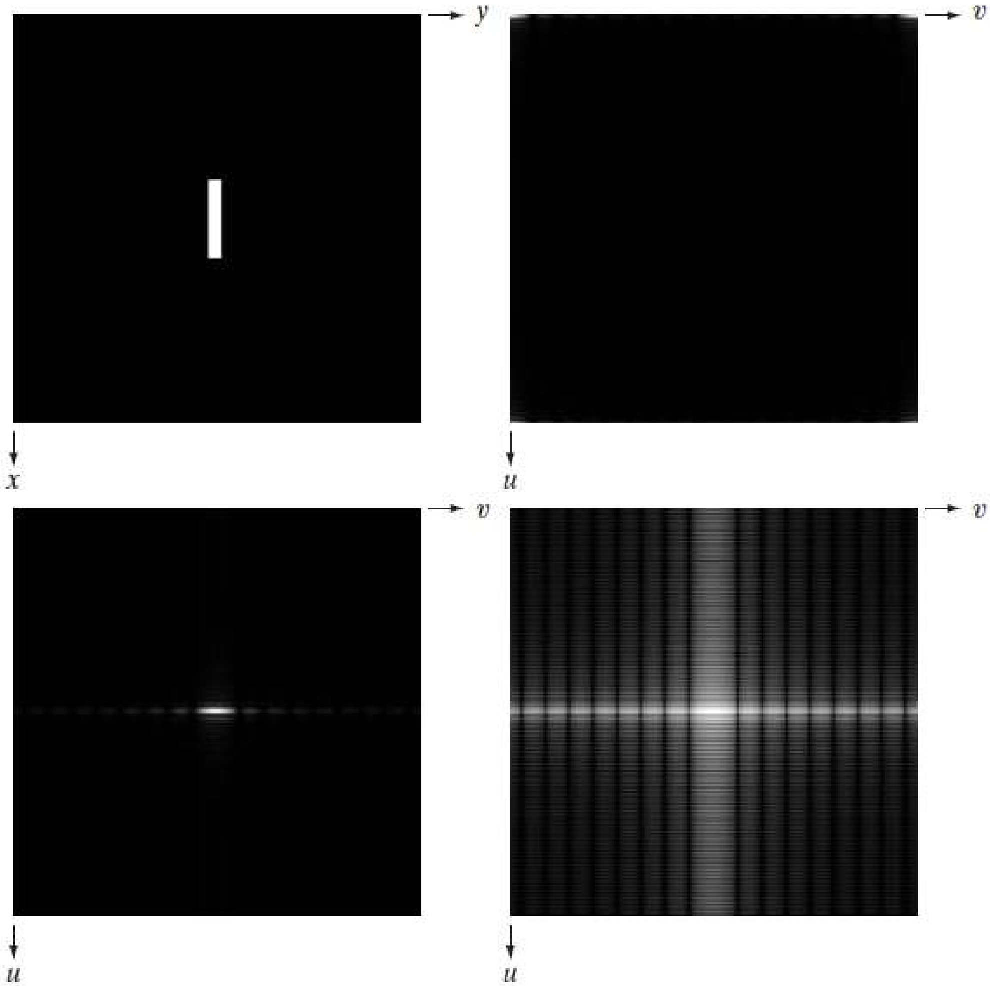
that

$$F(0, 0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)$$

which indicates that the zero-frequency term is proportional to the average value of *f(x, y)*.

$$\begin{aligned} F(0, 0) &= MN \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \\ &= MN\bar{f}(x, y) \end{aligned}$$

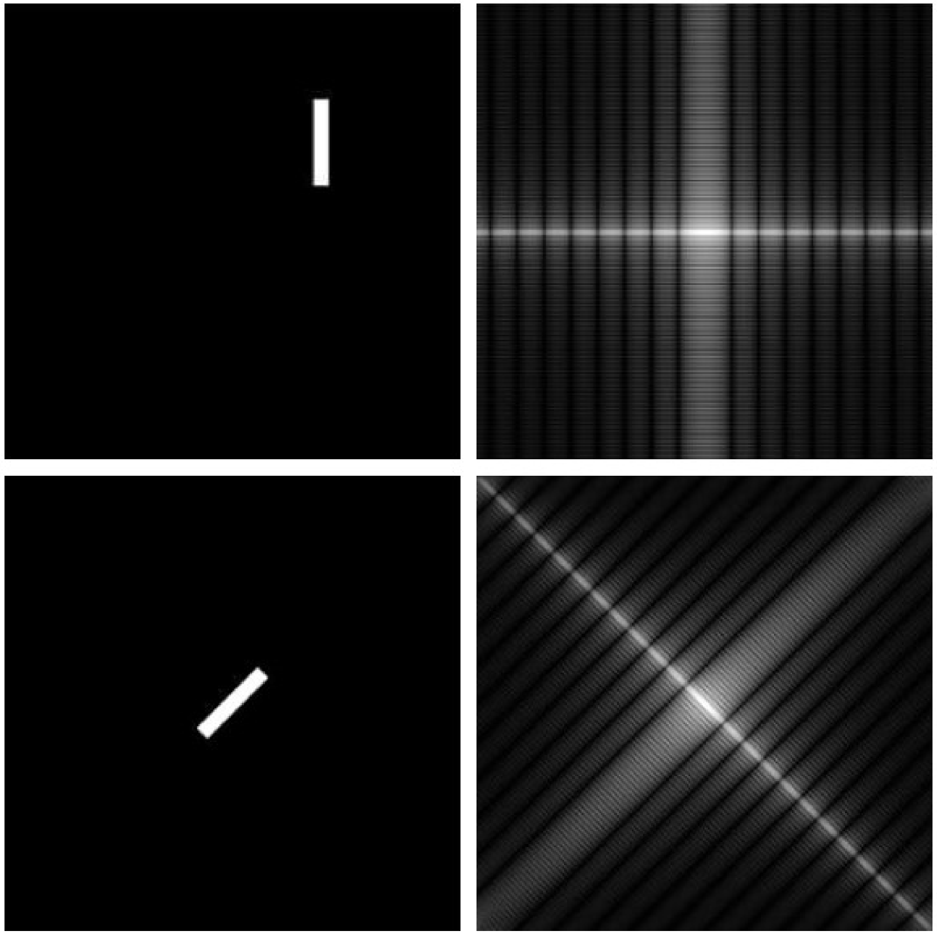
6. Some Properties of the 2-D DFT



a b
c d

FIGURE 4.24
(a) Image. (b) Spectrum showing bright spots in the four corners. (c) Centered spectrum. (d) Result showing increased detail after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The coordinate convention used throughout the book places the origin of the spatial and frequency domains at the top left.

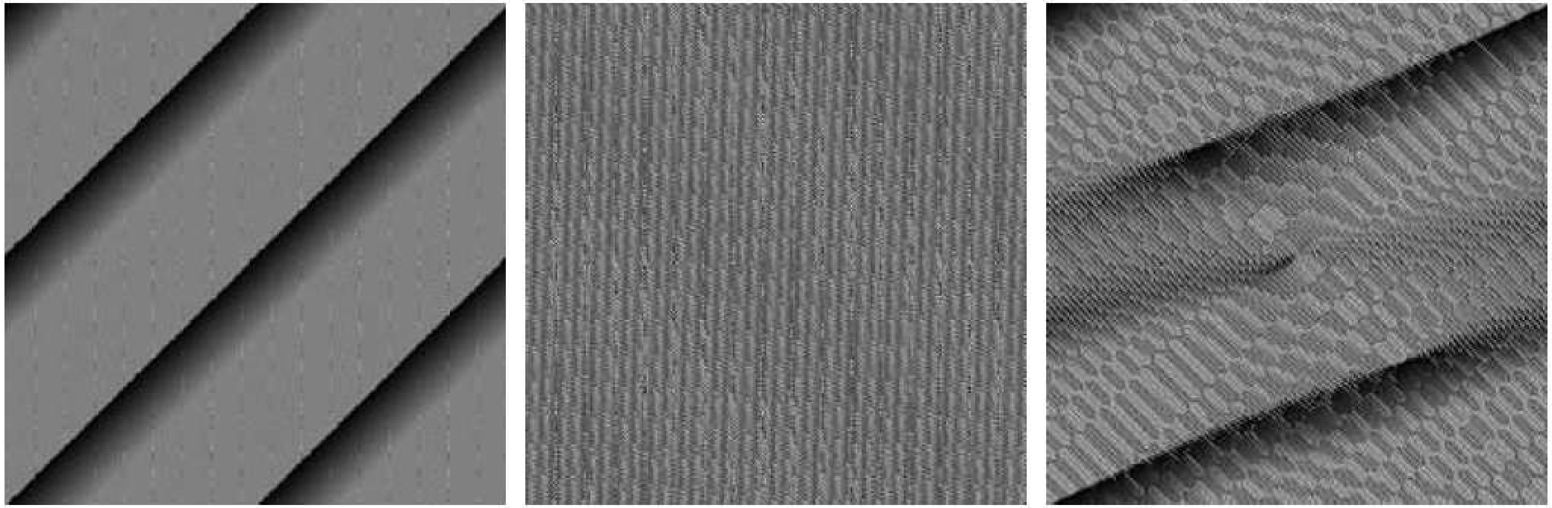
6. Some Properties of the 2-D DFT



a b
c d

FIGURE 4.25 (a) The rectangle in Fig. 4.24(a) translated, and (b) the corresponding spectrum. (c) Rotated rectangle, and (d) the corresponding spectrum. The spectrum corresponding to the translated rectangle is identical to the spectrum corresponding to the original image in Fig. 4.24(a).

6. Some Properties of the 2-D DFT



a b c

FIGURE 4.26 Phase angle array corresponding (a) to the image of the centered rectangle in Fig. 4.24(a), (b) to the translated image in Fig. 4.25(a), and (c) to the rotated image in Fig. 4.25(c).

6. Some Properties of the 2-D DFT

- The 2-D Convolution Theorem

$$f(x, y) \star h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) h(x-m, y-n)$$

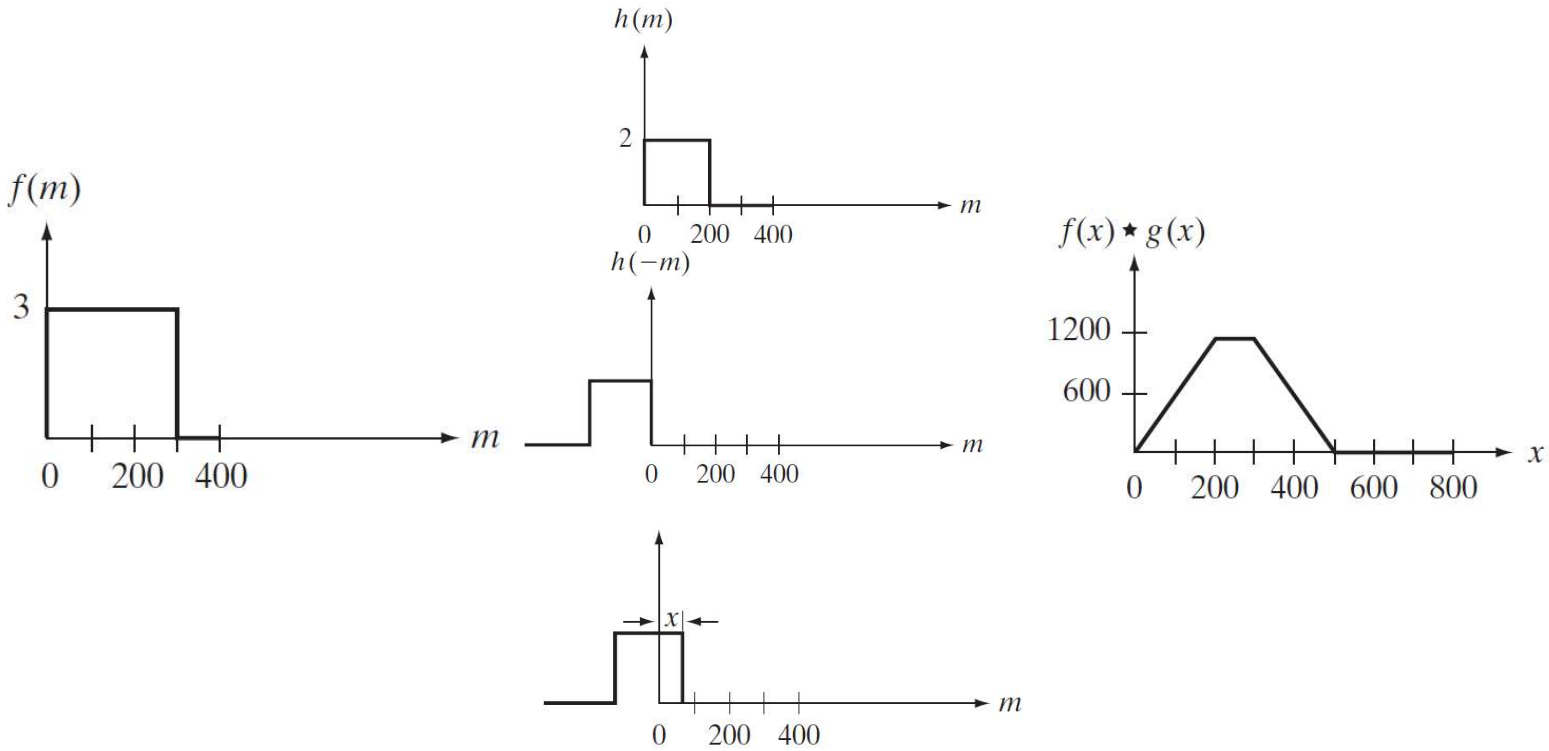
$$x = 0, 1, 2, \ldots, M-1 \quad \text{and} \quad y = 0, 1, 2, \ldots, N-1$$

$$f(x, y) \star h(x, y) \Leftrightarrow F(u, v) H(u, v)$$

$$f(x, y) h(x, y) \Leftrightarrow F(u, v) \star H(u, v)$$

6. Some Properties of the 2-D DFT

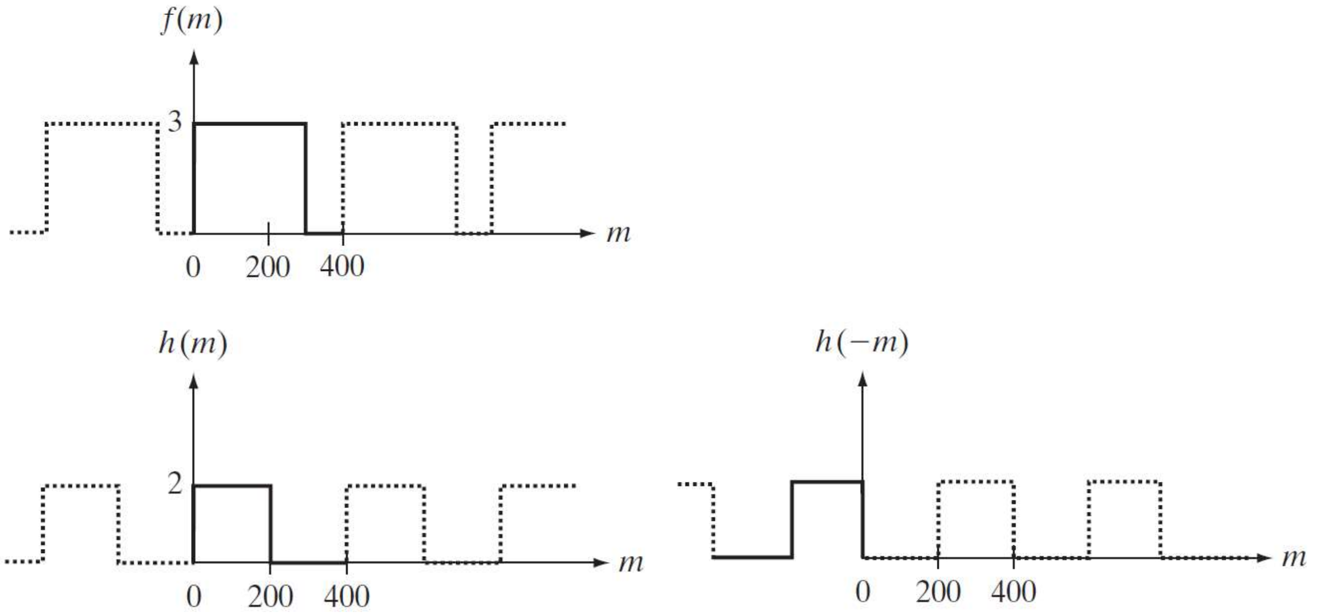
- The 2-D Convolution Theorem
  - 1-D example: convolution of two functions, f and h.



6. Some Properties of the 2-D DFT

- The 2-D Convolution Theorem

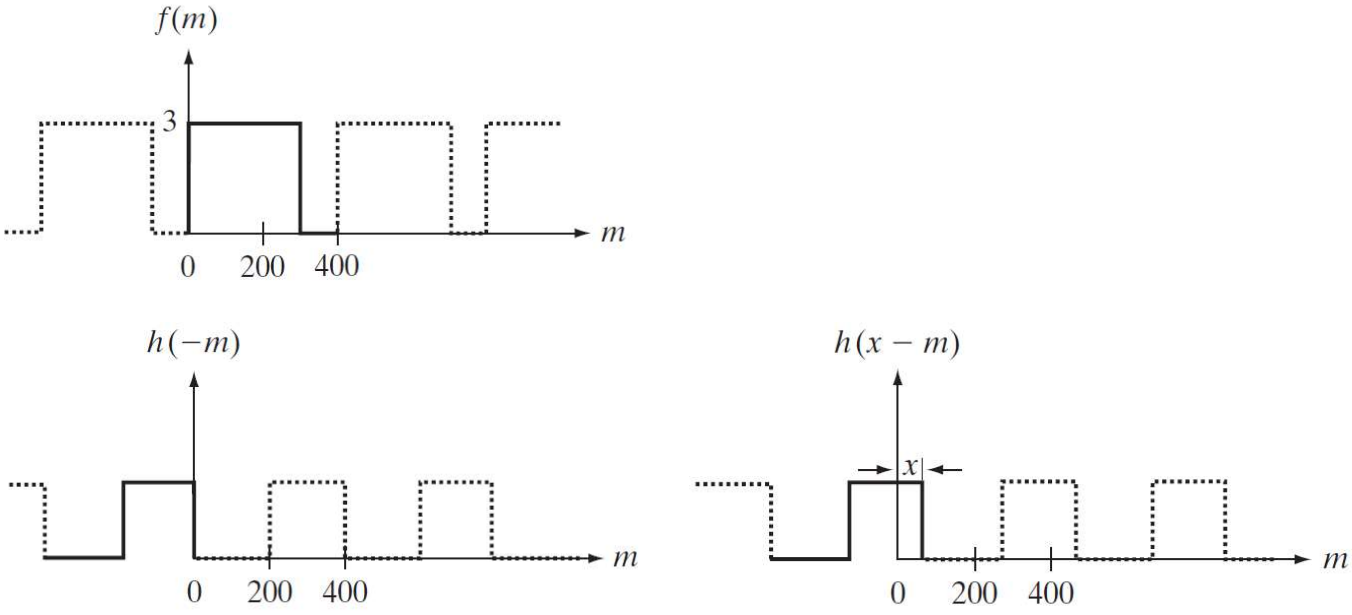
  - 1-D example: if we use the DFT and the convolution theorem to obtain the same result, we must take into account the periodicity inherent in the expression for the DFT.



6. Some Properties of the 2-D DFT

- The 2-D Convolution Theorem

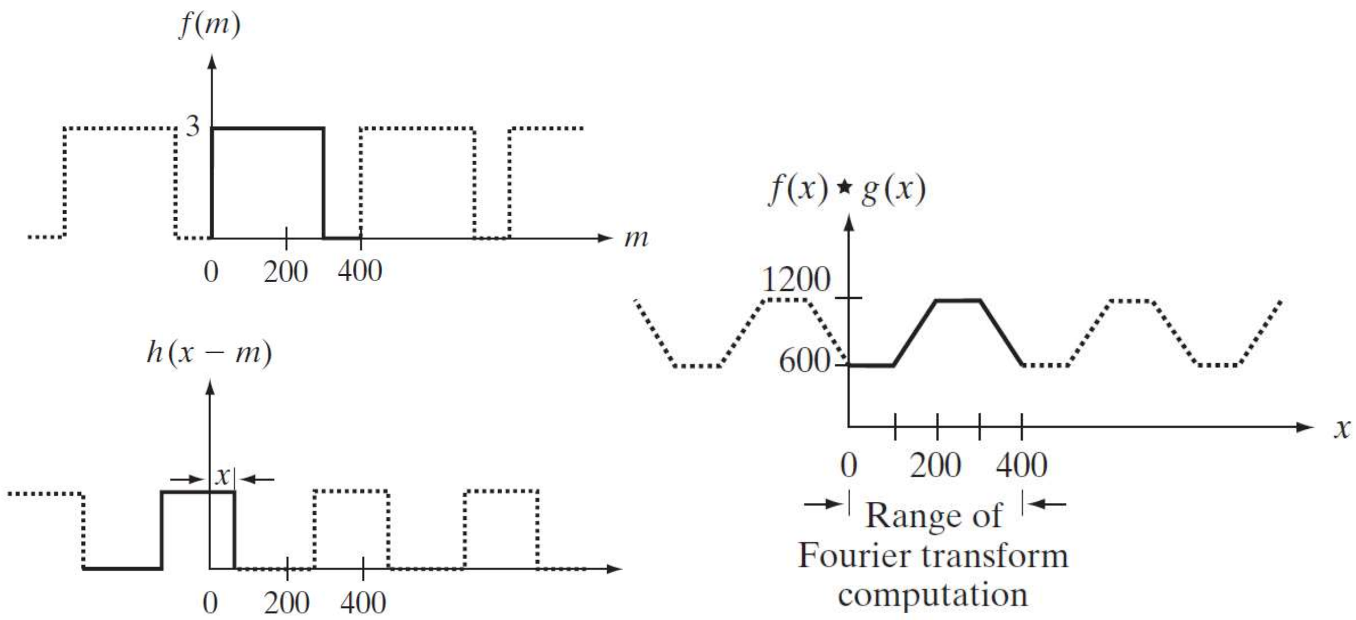
  - 1-D example: if we use the DFT and the convolution theorem to obtain the same result, we must take into account the periodicity inherent in the expression for the DFT.



6. Some Properties of the 2-D DFT

- The 2-D Convolution Theorem

  - 1-D example: if we use the DFT and the convolution theorem to obtain the same result, we must take into account the periodicity inherent in the expression for the DFT.



6. Some Properties of the 2-D DFT

- The 2-D Convolution Theorem

  - 1-D example: if we use the DFT and the convolution theorem to obtain the same result, we must take into account the periodicity inherent in the expression for the DFT.

  - The closeness of the periods is such that they interfere with each other to cause what is commonly referred to as *wraparound error*.

  - Fortunately, the solution to the wraparound error problem is simple. Consider two functions, f(x) and h(x) composed of A and B samples, respectively. We need to append P zeros to both functions,

$$P \geq A + B - 1$$

6. Some Properties of the 2-D DFT

- The 2-D Convolution Theorem

  - Visualizing a similar example in 2-D would be more difficult, but the conclusions are similar.

  - Let f(x, y) and h(x, y) be two image arrays of sizes A x B and C x D pixels, respectively.

  - Wraparound error in their circular convolution can be avoided by padding these functions with zeros, as follows,

$$f_p(x,y) = \begin{cases} f(x,y) & 0 \le x \le A-1 \quad \text{and} \quad 0 \le y \le B-1 \\ 0 & A \le x \le P \quad \text{or} \quad B \le y \le Q \end{cases}$$

$$h_p(x,y) = \begin{cases} h(x,y) & 0 \le x \le C-1 \quad \text{and} \quad 0 \le y \le D-1 \\ 0 & C \le x \le P \quad \text{or} \quad D \le y \le Q \end{cases}$$

$$P \ge A + C - 1 \qquad Q \ge B + D - 1$$

7. The Basics of Filtering in the Frequency Domain

- Additional Characteristics of the Frequency Domain

  - Filtering techniques in the frequency domain are based on modifying the Fourier transform of a signal to achieve a specific objective and then computing the inverse DFT to get us back to the image domain.

  - Frequency is directly related to spatial rates of change.

  - The slowest varying frequency component is proportional to the average intensity of an image.

  - As we move away from the origin of the transform, the low frequencies correspond to the slowly varying intensity components of an image.

7. The Basics of Filtering in the Frequency Domain

- Additional Characteristics of the Frequency Domain

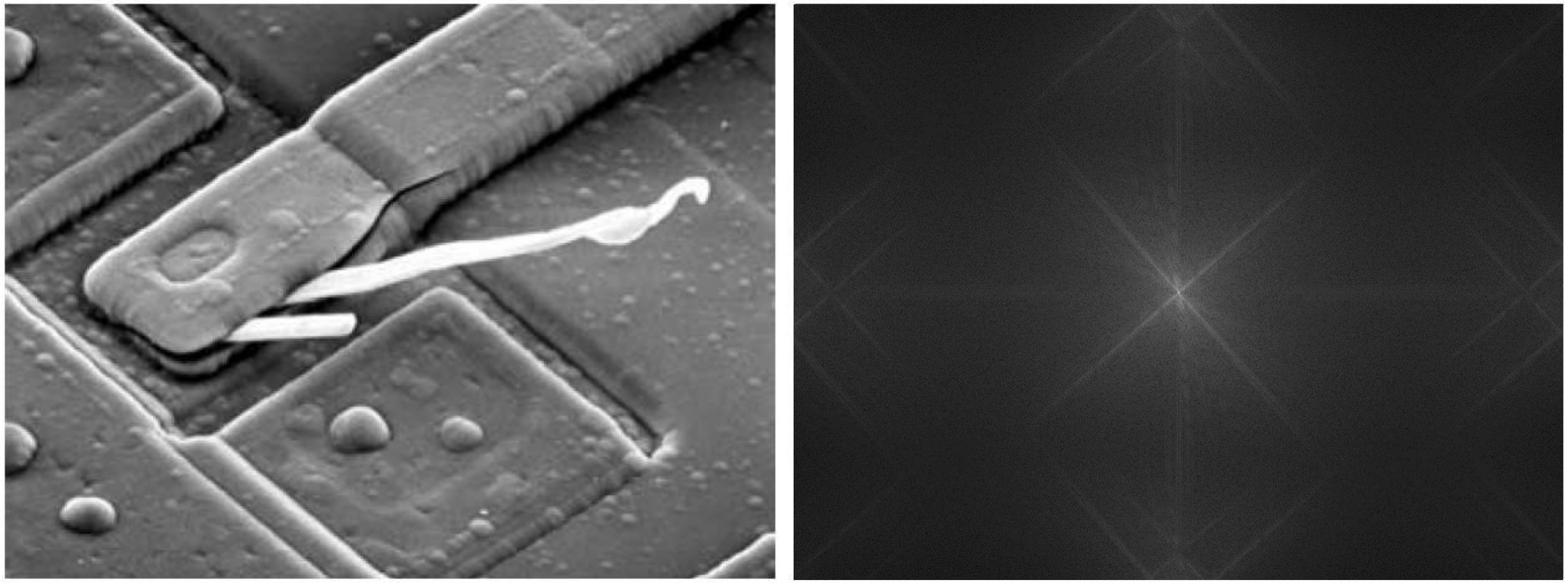
  - The two components of the transform to which we have access are the transform magnitude (spectrum) and the phase angle.

  - Visual analysis of the phase component generally is not very useful.

  - The spectrum, however, provides some useful visual information.

7. The Basics of Filtering in the Frequency Domain

- Additional Characteristics of the Frequency Domain



a b

FIGURE 4.29 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

7. The Basics of Filtering in the Frequency Domain

- Frequency Domain Filtering Fundamentals

  - Given a digital image *f(x,y)* of M x N the basic filtering equation in which we are interested has the form:

$$g(x, y) = \Im^{-1}[H(u, v)F(u, v)]$$

  - *F(u,v)* is the DFT of the input image, *f(x, y)*, and *H(u,v)* is the filter; *g(x,y)* is the filterd image.

  - We are now in a position to consider the filtering process in some detail.

  - One of the simplest filters we can construct is a filter *H(u, v)* that is 0 at the center of the transform and 1 elsewhere.

7. The Basics of Filtering in the Frequency Domain

- Frequency Domain Filtering Fundamentals

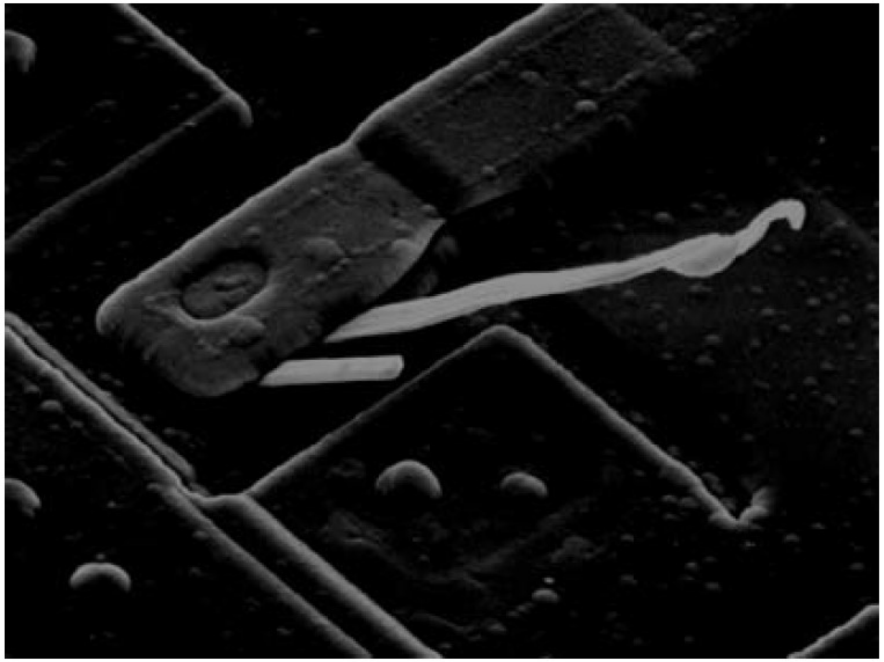
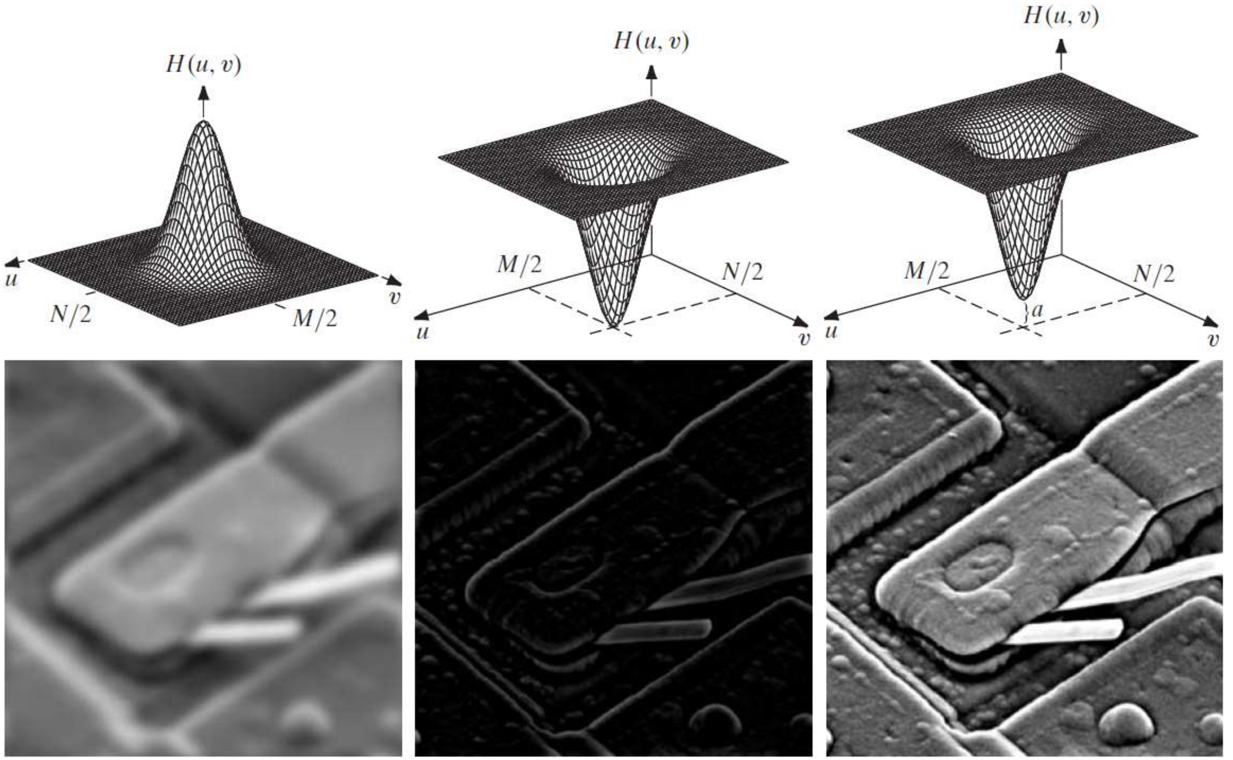


FIGURE 4.30 Result of filtering the image in Fig. 4.29(a) by setting to 0 the term $F(M/2, N/2)$ in the Fourier transform.

7. The Basics of Filtering in the Frequency Domain

- Frequency Domain Filtering Fundamentals



a b c
d e f

FIGURE 4.31 Top row: frequency domain filters. Bottom row: corresponding filtered images obtained using Eq. (4.7-1). We used $a = 0.85$ in (c) to obtain (f) (the height of the filter itself is 1). Compare (f) with Fig. 4.29(a).

7. The Basics of Filtering in the Frequency Domain

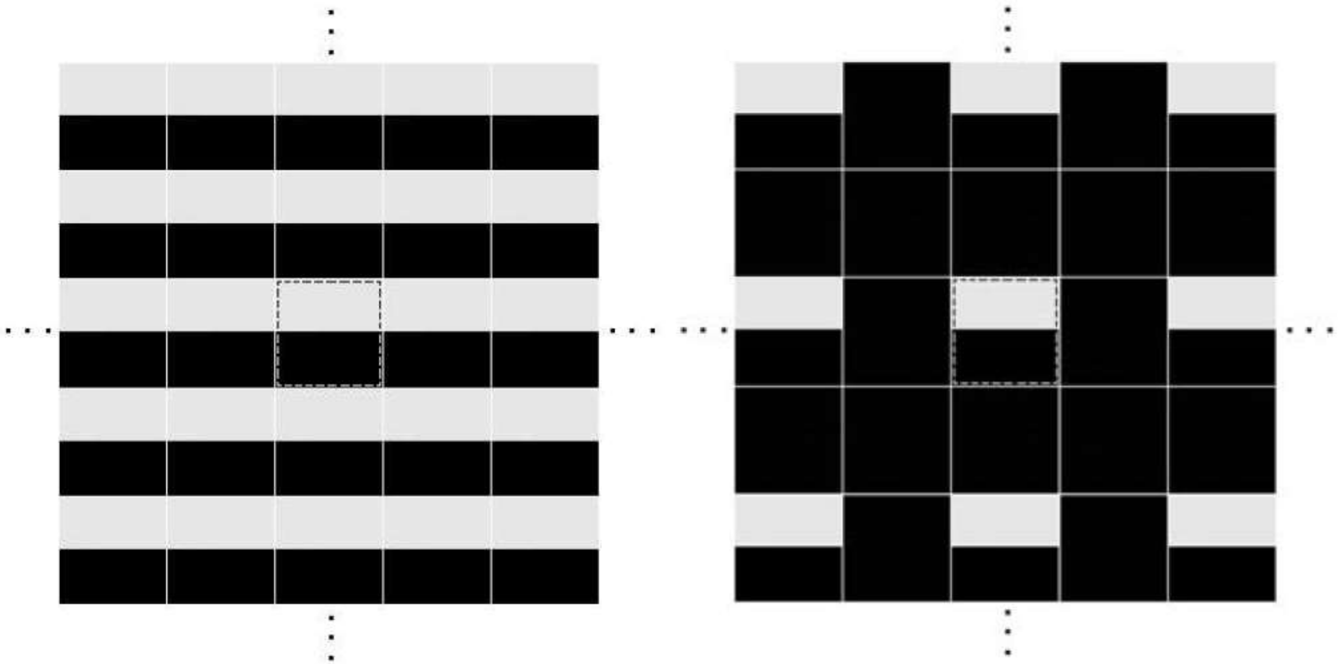
- Frequency Domain Filtering Fundamentals

a b c

FIGURE 4.32 (a) A simple image. (b) Result of blurring with a Gaussian lowpass filter without padding. (c) Result of lowpass filtering with padding. Compare the light area of the vertical edges in (b) and (c).

7. The Basics of Filtering in the Frequency Domain

- Frequency Domain Filtering Fundamentals



a b

FIGURE 4.33 2-D image periodicity inherent in using the DFT. (a) Periodicity without image padding. (b) Periodicity after padding with 0s (black). The dashed areas in the center correspond to the image in Fig. 4.32(a). (The thin white lines in both images are superimposed for clarity; they are not part of the data.)

7. The Basics of Filtering in the Frequency Domain

- Frequency Domain Filtering Fundamentals

  - Low frequencies in the transform are related to slowly varying intensity components in an image.

  - High frequencies are caused by sharp transitions in intensity

  - A filter that attenuates high frequencies while passing low frequencies (appropriately called a *lowpass filter*) would blur an image.

  - A filter with the opposite property (called a *highpass filter*) would enhance sharp detail.

7. The Basics of Filtering in the Frequency Domain

- Summary of Steps for Filtering in the Frequency Domain

I. Given an input image $f(x,y)$ of size M x N obtain the padding parameters P and Q from P = 2M e Q = 2N.

I. Form a padded image $f_p(x,y)$ of size P x Q.

II. Multiply $f_p(x,y)$ by $(-1)^{x+y}$ to center its transform ➔ $f_{ps}(x,y)$

III. Compute DFT of $f_{ps}(x,y)$ ➔ $F_{ps}(u,v)$.

IV. Generate a real, symmetric filter function $H(u,v)$ of size P x Q, with center at coordinates P/2, Q/2.

7. The Basics of Filtering in the Frequency Domain

- Summary of Steps for Filtering in the Frequency Domain

V. Calculate $G(u,v) = H(u,v).F_{ps}(u,v)$.

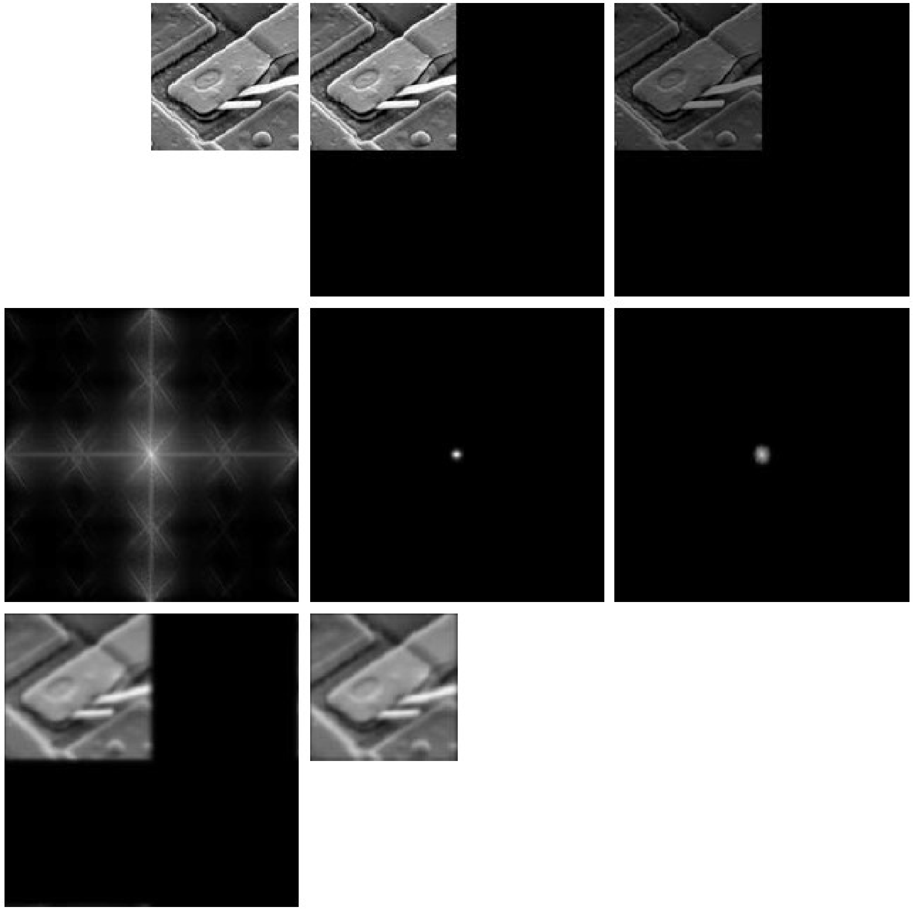
VI. Obtain the processed image:

$$g_p(x,y) = \left\{ real\left[\Im^{-1}[G(u,v)] \right] \right\} (-1)^{x+y}$$

VII. Obtain the final processed result *g(x, y)* by extracting the M x N region from the top, left quadrant $g_p(x, y)$.

7. The Basics of Filtering in the Frequency Domain

- Summary of Steps for Filtering in the Frequency Domain



a b c
d e f
g h

FIGURE 4.36
(a) An $M \times N$ image, f.
(b) Padded image, f_p of size $P \times Q$.
(c) Result of multiplying f_p by $(-1)^{x+y}$.
(d) Spectrum of F_p. (e) Centered Gaussian lowpass filter, H, of size $P \times Q$.
(f) Spectrum of the product HF_p.
(g) g_p, the product of $(-1)^{x+y}$ and the real part of the IDFT of HF_p.
(h) Final result, g, obtained by cropping the first M rows and N columns of g_p.

7. The Basics of Filtering in the Frequency Domain

- Correspondence Between Filtering in the Spatial and Frequency Domains

  - The link between filtering in the spatial and frequency domains is the convolution theorem.

  - Given a filter $H(u, v)$, suppose that we want to find its equivalent representation in the spatial domain.

  - If we let $f(x,y) = \delta(x, y)$, then $F(u,v) = 1$.

  - The filtered output is $\Im^{-1}\{F(u,v).H(u,v)\} = \Im^{-1}\{H(u,v)\}$, which is the corresponding filter in the spatial domain, $h(x, y)$.

7. The Basics of Filtering in the Frequency Domain

- Correspondence Between Filtering in the Spatial and Frequency Domains

  - Because this filter can be obtained from the response of a frequency domain filter to an impulse, $h(x, y)$ sometimes is referred to as the impulse response of $H(u, v)$.

  - And because the impulse response is of finite length, such filters are called *finite impulse response* (FIR) filters.

  - These are the only types of linear spatial filters considered in this course.

7. The Basics of Filtering in the Frequency Domain

- Correspondence Between Filtering in the Spatial and Frequency Domains

  - In practice, we prefer to implement convolution filtering using with small filter masks.

  - However, filtering concepts are more intuitive in the frequency domain.

  - One way to take advantage of the properties of both domains is to specify a filter in the frequency domain, compute its IDFT, and then use the resulting, full-size spatial filter as a guide for constructing smaller spatial filter masks.

7. The Basics of Filtering in the Frequency Domain

• Correspondence Between Filtering in the Spatial and Frequency Domains

➢ Example (lowpass filter):

✓ Let *H(u)* denote the 1-D frequency domain Gaussian filter:

$$H(u) = Ae^{-u^2/2\sigma^2}$$

✓ The corresponding filter in the spatial domain is obtained by taking the inverse Fourier transform of H(u):

$$h(x) = \sqrt{2\pi}\sigma A e^{-2\pi^2\sigma^2 x^2}$$

✓ When H(u) has a broad profile (large value of s), h(x) has a narrow profile, and vice versa.

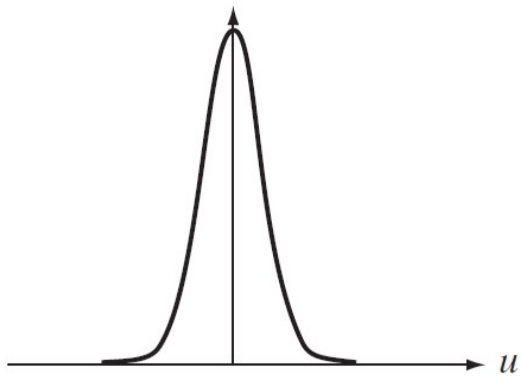
7. The Basics of Filtering in the Frequency Domain

• Correspondence Between Filtering in the Spatial and Frequency Domains

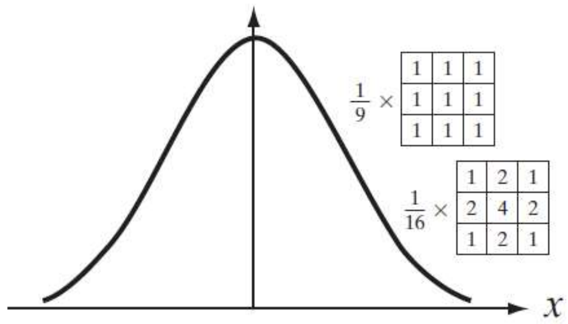
➢ Example (lowpass filter):

✓ Suppose that we want to use the shape of h(x) as a guide for specifying the coefficients of a small spatial mask

$$H(u) = Ae^{-u^2/2\sigma^2}$$



$$h(x) = \sqrt{2\pi}\sigma A e^{-2\pi^2\sigma^2 x^2}$$

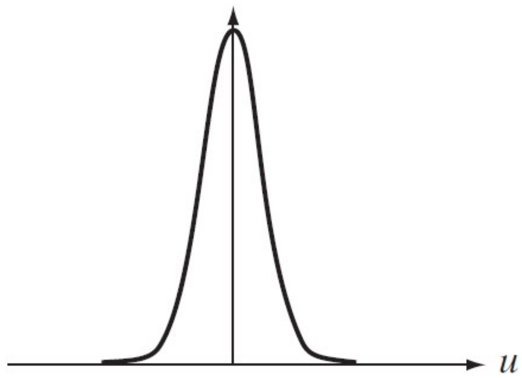


7. The Basics of Filtering in the Frequency Domain

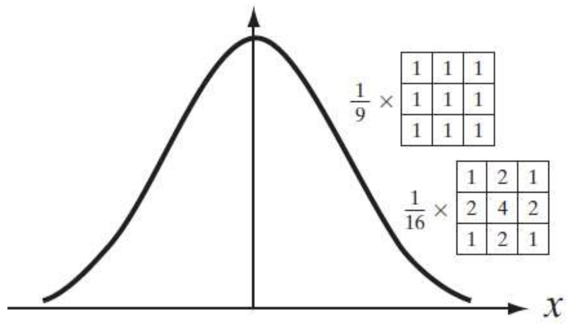
• Correspondence Between Filtering in the Spatial and Frequency Domains

- ➢ Example (lowpass filter):
  - ✓ All coefficients are positive.

$$H(u) = Ae^{-u^2/2\sigma^2}$$



$$h(x) = \sqrt{2\pi}\sigma A e^{-2\pi^2\sigma^2 x^2}$$



7. The Basics of Filtering in the Frequency Domain

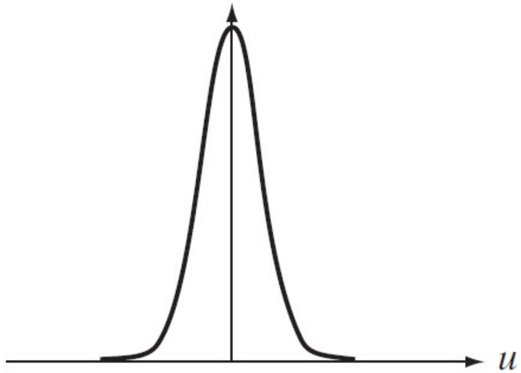
• Correspondence Between Filtering in the Spatial and Frequency Domains

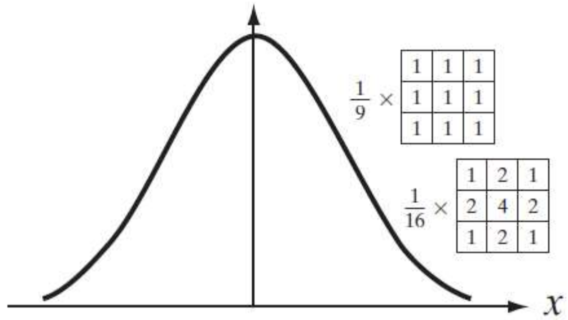
➢ Example (lowpass filter):

✓ The narrower the frequency domain filter, the more it will attenuate the low frequencies, resulting in increased blurring.

$$H(u) = Ae^{-u^2/2\sigma^2}$$

$$h(x) = \sqrt{2\pi}\sigma A e^{-2\pi^2\sigma^2 x^2}$$





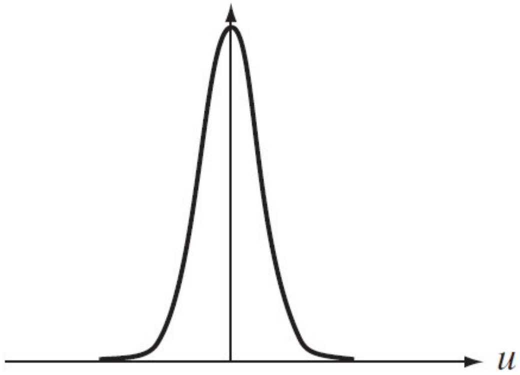
7. The Basics of Filtering in the Frequency Domain

- Correspondence Between Filtering in the Spatial and Frequency Domains

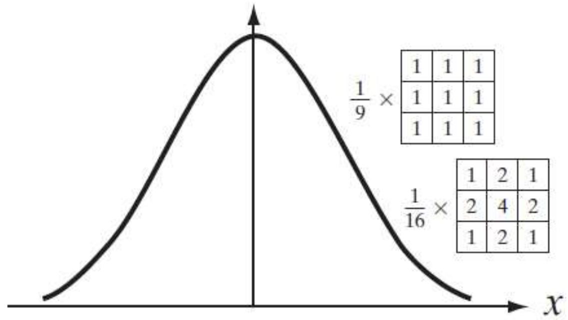
  - Example (lowpass filter):

    - ✓ In the spatial domain, this means that a larger mask must be used to increase blurring.

$$H(u) = Ae^{-u^2/2\sigma^2}$$



$$h(x) = \sqrt{2\pi}\sigma A e^{-2\pi^2\sigma^2 x^2}$$



7. The Basics of Filtering in the Frequency Domain

• Correspondence Between Filtering in the Spatial and Frequency Domains

➢ Example (highpass filter):

✓ Difference of Gaussians in the frequency domain:

$$H(u) = Ae^{-u^2/2\sigma_1^2} - Be^{-u^2/2\sigma_2^2}$$

$$A \geq B \text{ and } \sigma_1 > \sigma_2$$

✓ The corresponding filter in the spatial domain is

$$h(x) = \sqrt{2\pi}\sigma_1 A e^{-2\pi^2\sigma_1^2 x^2} - \sqrt{2\pi}\sigma_2 B e^{-2\pi^2\sigma_2^2 x^2}$$

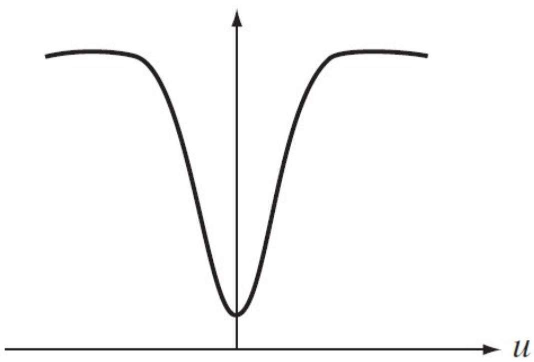
7. The Basics of Filtering in the Frequency Domain

- Correspondence Between Filtering in the Spatial and Frequency Domains

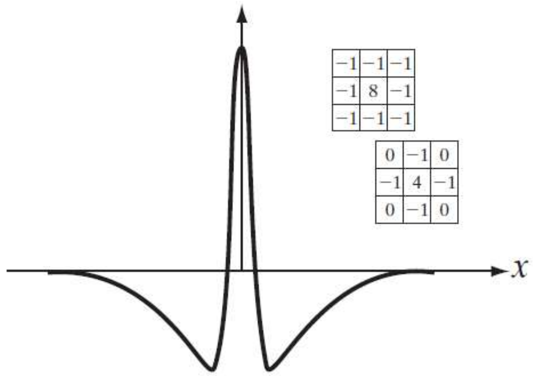
  - Example (highpass filter):

    - Has a positive center term with negative terms on either side.

$$H(u) = Ae^{-u^2/2\sigma_1^2} - Be^{-u^2/2\sigma_2^2}$$



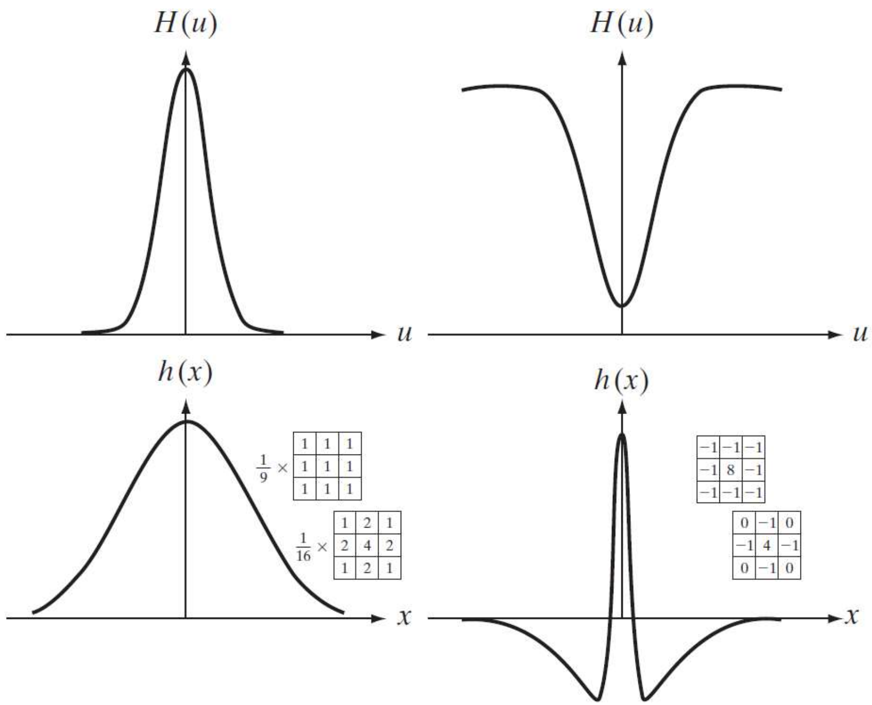
$$h(x) = \sqrt{2\pi}\sigma_1 A e^{-2\pi^2\sigma_1^2 x^2} - \sqrt{2\pi}\sigma_2 B e^{-2\pi^2\sigma_2^2 x^2}$$



7. The Basics of Filtering in the Frequency Domain

- Correspondence Between Filtering in the Spatial and Frequency Domains

  - ➢ Example (lowpass and highpass filters):



a	c
b	d

FIGURE 4.37
(a) A 1-D Gaussian lowpass filter in the frequency domain. (b) Spatial lowpass filter corresponding to (a). (c) Gaussian highpass filter in the frequency domain. (d) Spatial highpass filter corresponding to (c). The small 2-D masks shown are spatial filters we used in Chapter 3.

7. The Basics of Filtering in the Frequency Domain

- Correspondence Between Filtering in the Spatial and Frequency Domains

  - Frequency domain can be viewed as a "laboratory".

  - Some tasks that would be difficult to formulate in the spatial domain become almost trivial in the frequency domain.

  - Once we have selected a filter in the frequency domain, the implementation is usually done in the spatial domain.

  - One approach is to specify small spatial masks that attempt to capture the "essence" of the full filter function in the spatial domain.

  - There are more formal approaches to design 2-D digital filters.

7. The Basics of Filtering in the Frequency Domain

• Correspondence Between Filtering in the Spatial and Frequency Domains

- In this example, we start with a spatial mask and show how to generate the filter in the frequency domain.

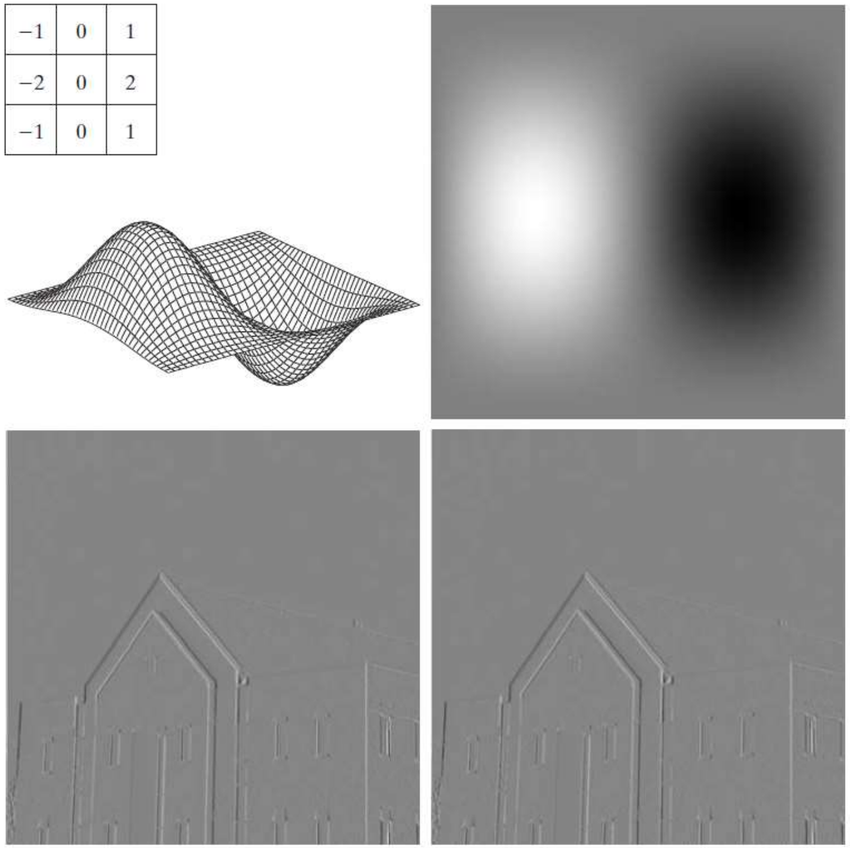
- To keep matters simple, we use the 3x3 Sobel vertical edge detector.

- The figure below shows a pixel image that we wish to filter.

FIGURE 4.38

7. The Basics of Filtering in the Frequency Domain

- Correspondence Between Filtering in the Spatial and Frequency Domains



−1	0	1
−2	0	2
−1	0	1

a b
c d

FIGURE 4.39 (a) A spatial mask and perspective plot of its corresponding frequency domain filter. (b) Filter shown as an image. (c) Result of filtering Fig. 4.38(a) in the frequency domain with the filter in (b). (d) Result of filtering the same image with the spatial filter in (a). The results are identical.

7. The Basics of Filtering in the Frequency Domain

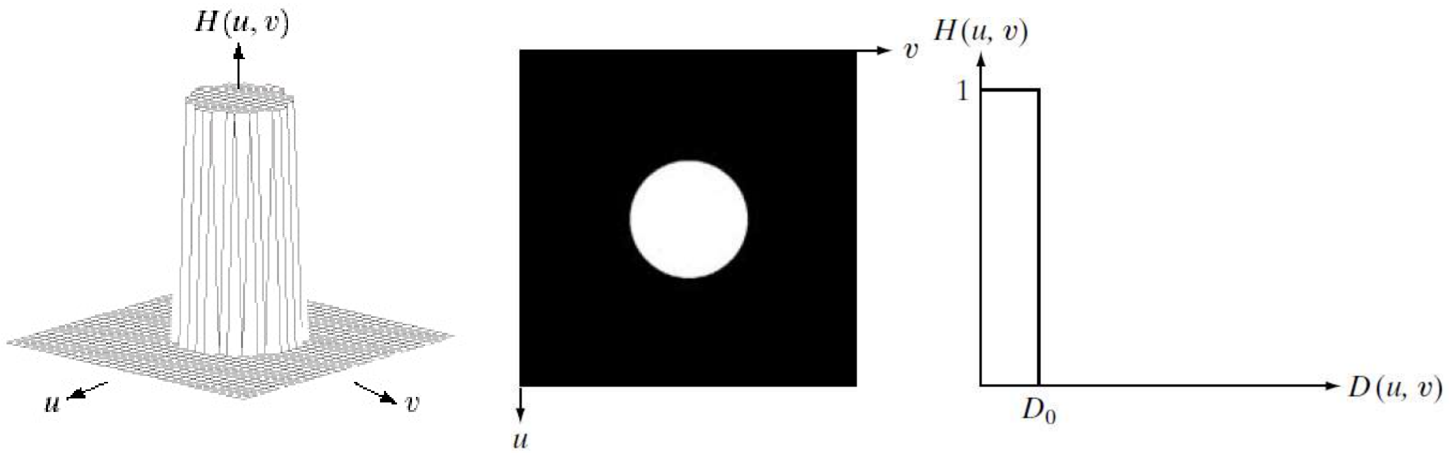
- MATLAB: s137FilterGen.m

8. Image Smoothing Using Frequency Domain Filters

- Ideal Lowpass Filters

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$$

$$D(u,v) = \left[(u - P/2)^2 + (v - Q/2)^2\right]^{1/2}$$



8. Image Smoothing Using Frequency Domain Filters

- Ideal Lowpass Filters

  - The name *ideal* indicates that all frequencies on or inside a circle of radius D_0 are passed, without attenuation.

  - For an ILPF cross section, the point D_0 of transition between 1 and 0 is called the *cutoff frequency*.

  - The sharp cutoff frequencies of an ILPF cannot be realized with electronic components, although they certainly can be simulated in a computer.

  - The lowpass filters introduced in this chapter are compared by studying their behavior as a function of the same cutoff frequencies.

8. Image Smoothing Using Frequency Domain Filters

- Ideal Lowpass Filters

  - One way to establish a set of standard cutoff frequency loci is to compute circles that enclose specified amounts of total image power P_T.

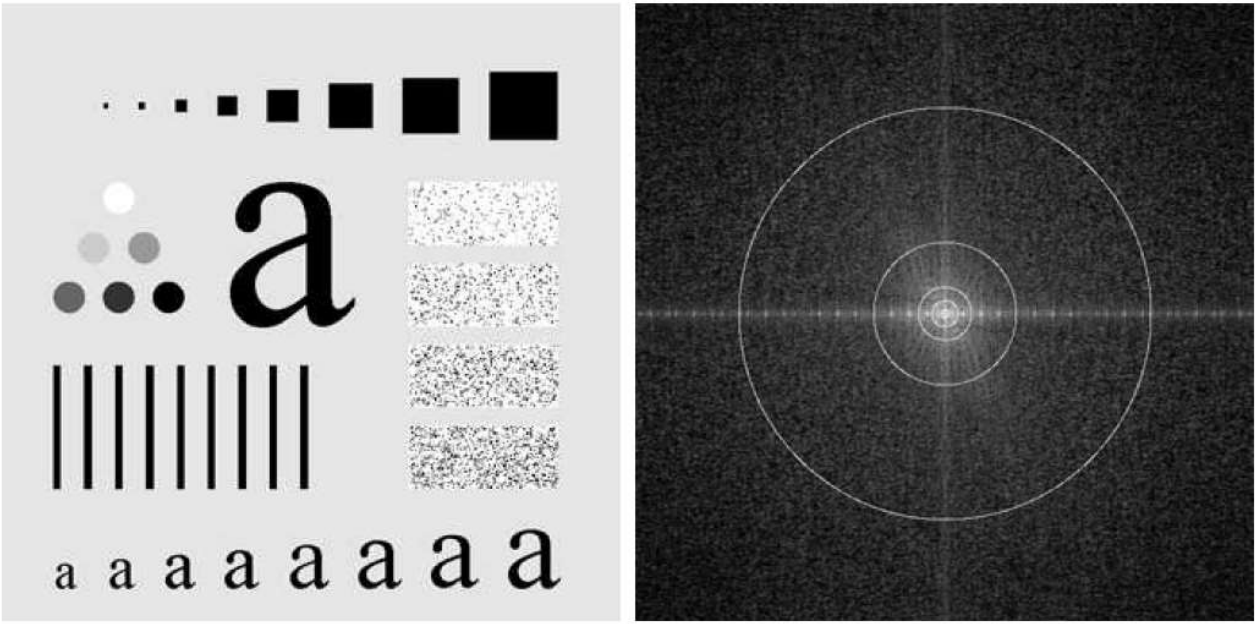
  - This quantity is obtained by summing the components of the power spectrum of the padded images at each point

$$P_T = \sum_{u=0}^{P-1} \sum_{v=0}^{Q-1} P(u, v) \qquad \alpha = 100\left[\sum_u \sum_v P(u, v)/P_T \right]$$

$$\begin{aligned} P(u, v) &= |F(u, v)|^2 \\ &= R^2(u, v) + I^2(u, v) \end{aligned}$$

8. Image Smoothing Using Frequency Domain Filters

- Ideal Lowpass Filters



a b

FIGURE 4.41 (a) Test pattern of size 688 × 688 pixels, and (b) its Fourier spectrum. The spectrum is double the image size due to padding but is shown in half size so that it fits in the page. The superimposed circles have radii equal to 10, 30, 60, 160, and 460 with respect to the full-size spectrum image. These radii enclose 87.0, 93.1, 95.7, 97.8, and 99.2% of the padded image power, respectively.

8. Image Smoothing Using Frequency Domain Filters

- Ideal Lowpass Filters

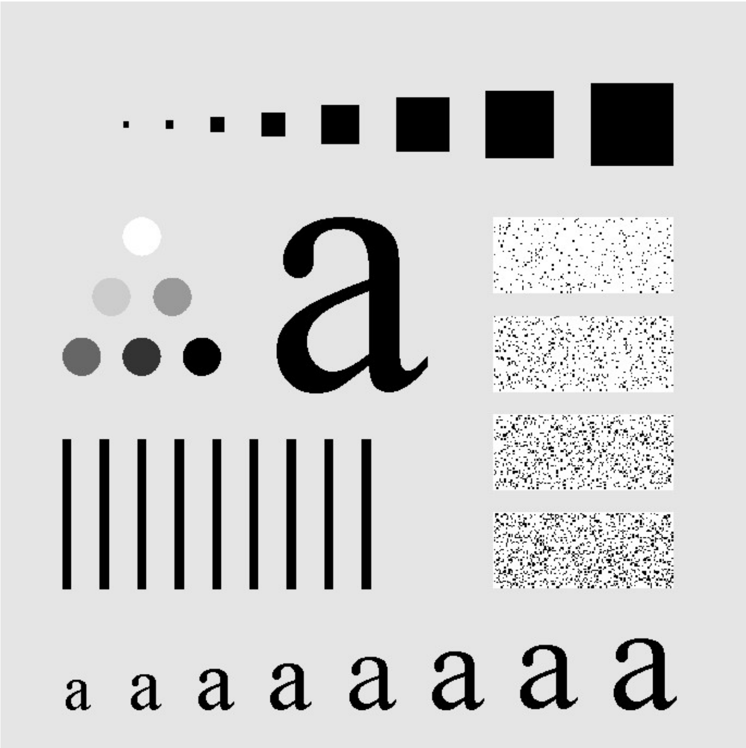


FIGURE 4.42 (a) Original image. (b)–(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460, as shown in Fig. 4.41(b). The power removed by these filters was 13, 6.9, 4.3, 2.2, and 0.8% of the total, respectively.

8. Image Smoothing Using Frequency Domain Filters

- Ideal Lowpass Filters

FIGURE 4.42 (a) Original image. (b)–(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460, as shown in Fig. 4.41(b). The power removed by these filters was 13, 6.9, 4.3, 2.2, and 0.8% of the total, respectively.

8. Image Smoothing Using Frequency Domain Filters

- Ideal Lowpass Filters

FIGURE 4.42 (a) Original image. (b)–(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460, as shown in Fig. 4.41(b). The power removed by these filters was 13, 6.9, 4.3, 2.2, and 0.8% of the total, respectively.

8. Image Smoothing Using Frequency Domain Filters

- Ideal Lowpass Filters

Ringing.

FIGURE 4.42 (a) Original image. (b)–(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460, as shown in Fig. 4.41(b). The power removed by these filters was 13, 6.9, 4.3, 2.2, and 0.8% of the total, respectively.

8. Image Smoothing Using Frequency Domain Filters

- Ideal Lowpass Filters

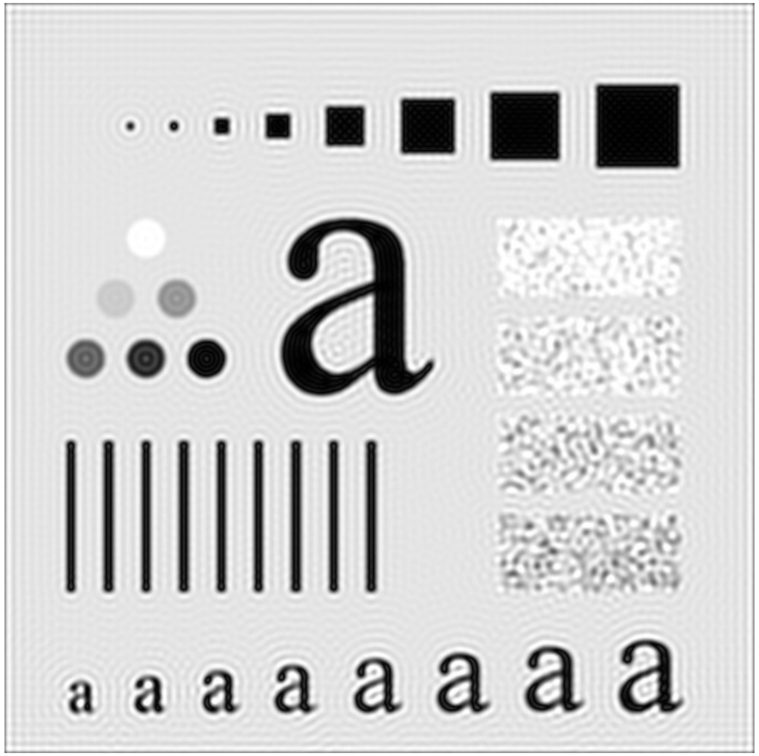


FIGURE 4.42 (a) Original image. (b)–(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460, as shown in Fig. 4.41(b). The power removed by these filters was 13, 6.9, 4.3, 2.2, and 0.8% of the total, respectively.

8. Image Smoothing Using Frequency Domain Filters

- Ideal Lowpass Filters

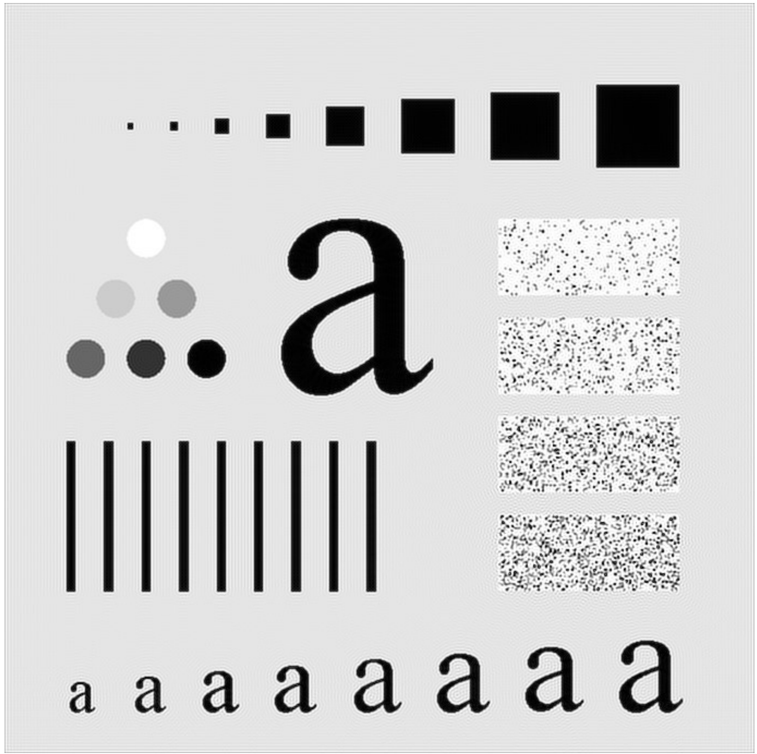
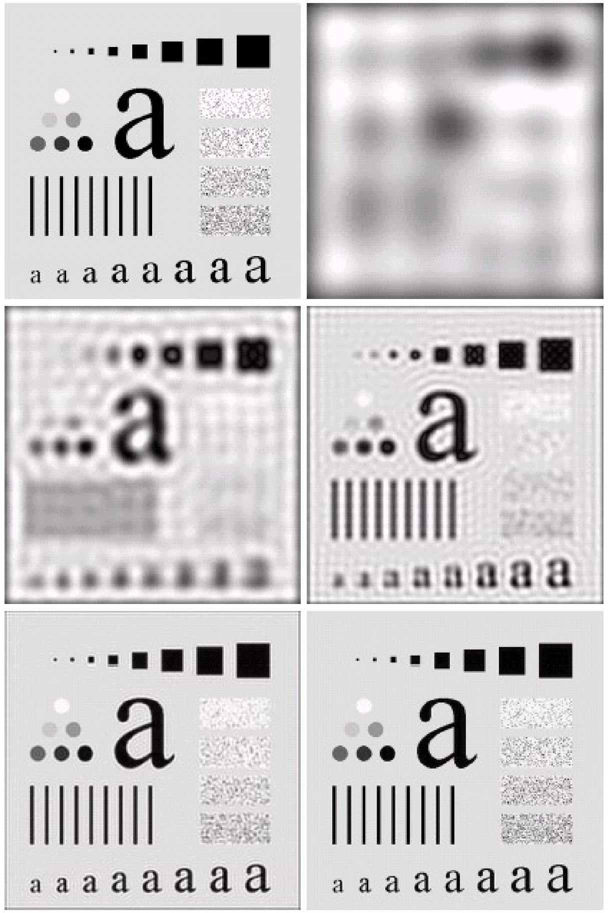


FIGURE 4.42 (a) Original image. (b)–(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460, as shown in Fig. 4.41(b). The power removed by these filters was 13, 6.9, 4.3, 2.2, and 0.8% of the total, respectively.

8. Image Smoothing Using Frequency Domain Filters

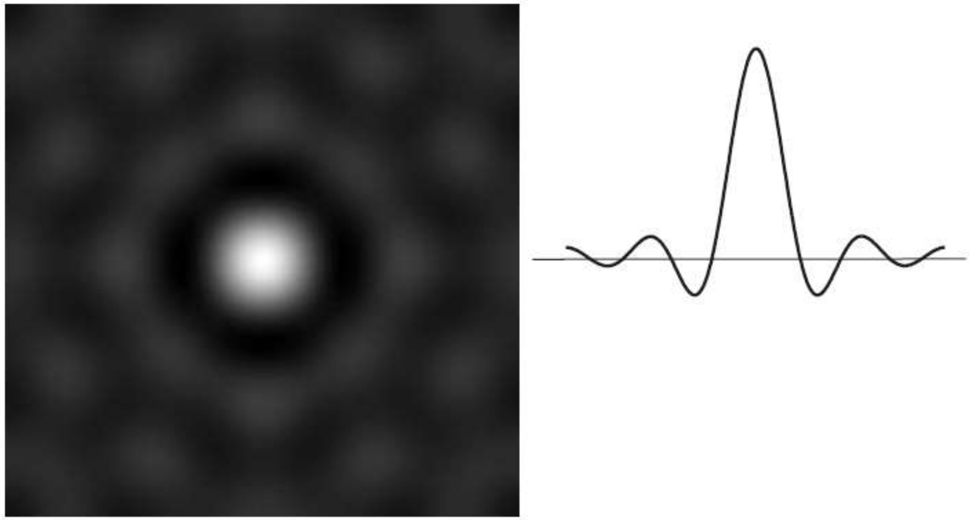
- Ideal Lowpass Filters



8. Image Smoothing Using Frequency Domain Filters

- Ideal Lowpass Filters

  - The blurring and ringing properties of ILPFs can be explained using the convolution theorem.



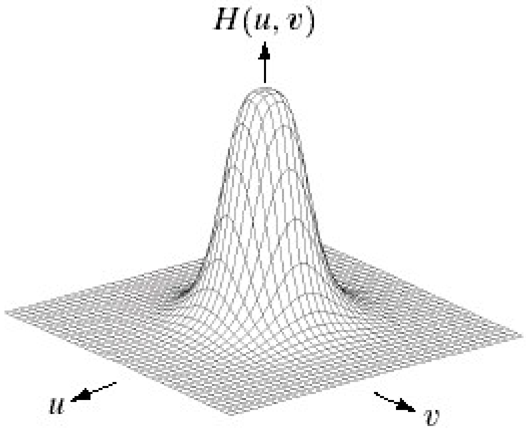
a b

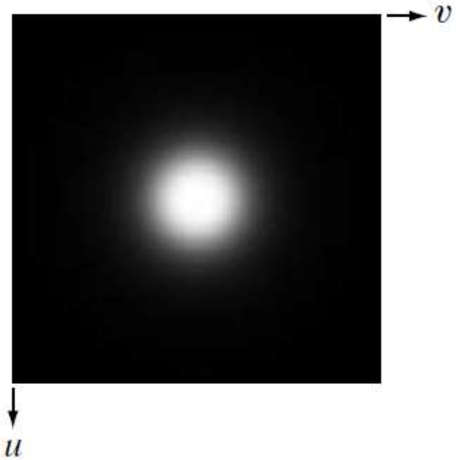
FIGURE 4.43 (a) Representation in the spatial domain of an ILPF of radius 5 and size 1000 × 1000. (b) Intensity profile of a horizontal line passing through the center of the image.

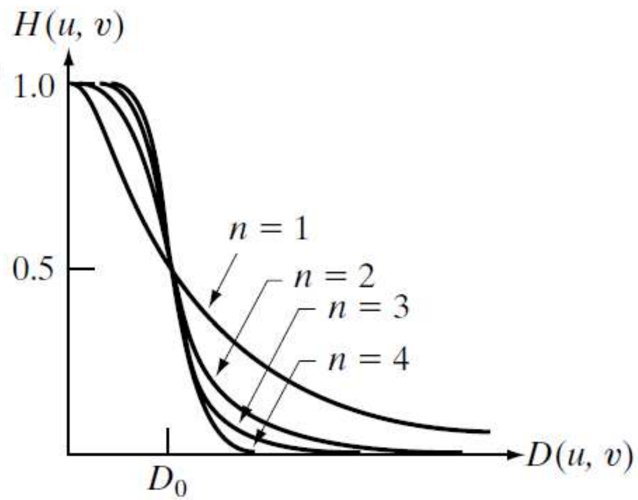
8. Image Smoothing Using Frequency Domain Filters

- Butterworth Lowpass Filters

$$H(u, v) = \frac{1}{1 + [D(u, v)/D_0]^{2n}}$$







8. Image Smoothing Using Frequency Domain Filters

- Butterworth Lowpass Filters

  - Unlike the ILPF, the BLPF transfer function does not have a sharp discontinuity that gives a clear cutoff between passed and filtered frequencies.

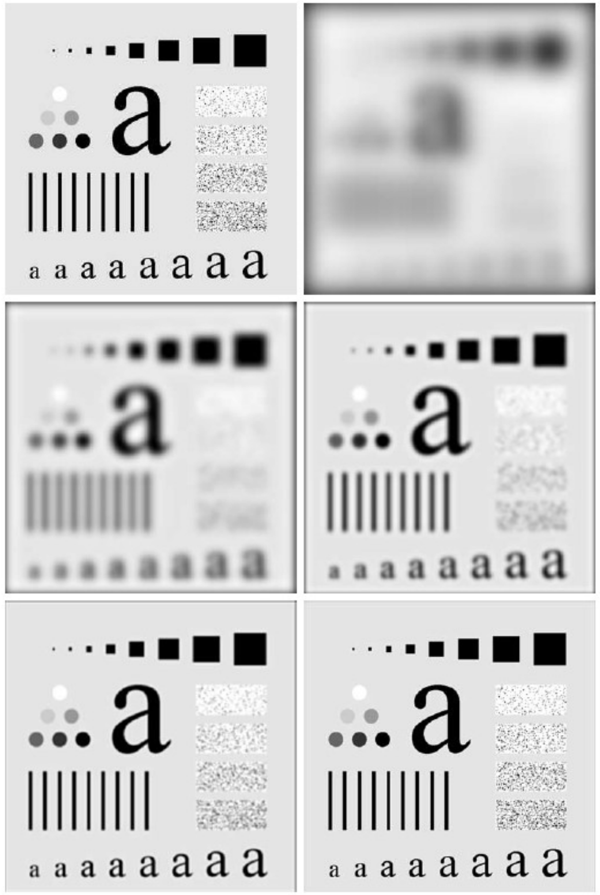
  - For filters with smooth transfer functions, cutoff frequency defined as those for which $H(u, v)$ is down to a certain fraction of its maximum value (eg. 0.707).

  - A BLPF of order 1 has no ringing in the spatial domain. Ringing generally is imperceptible in filters of order 2, but can become significant in filters of higher order.

8. Image Smoothing Using Frequency Domain Filters

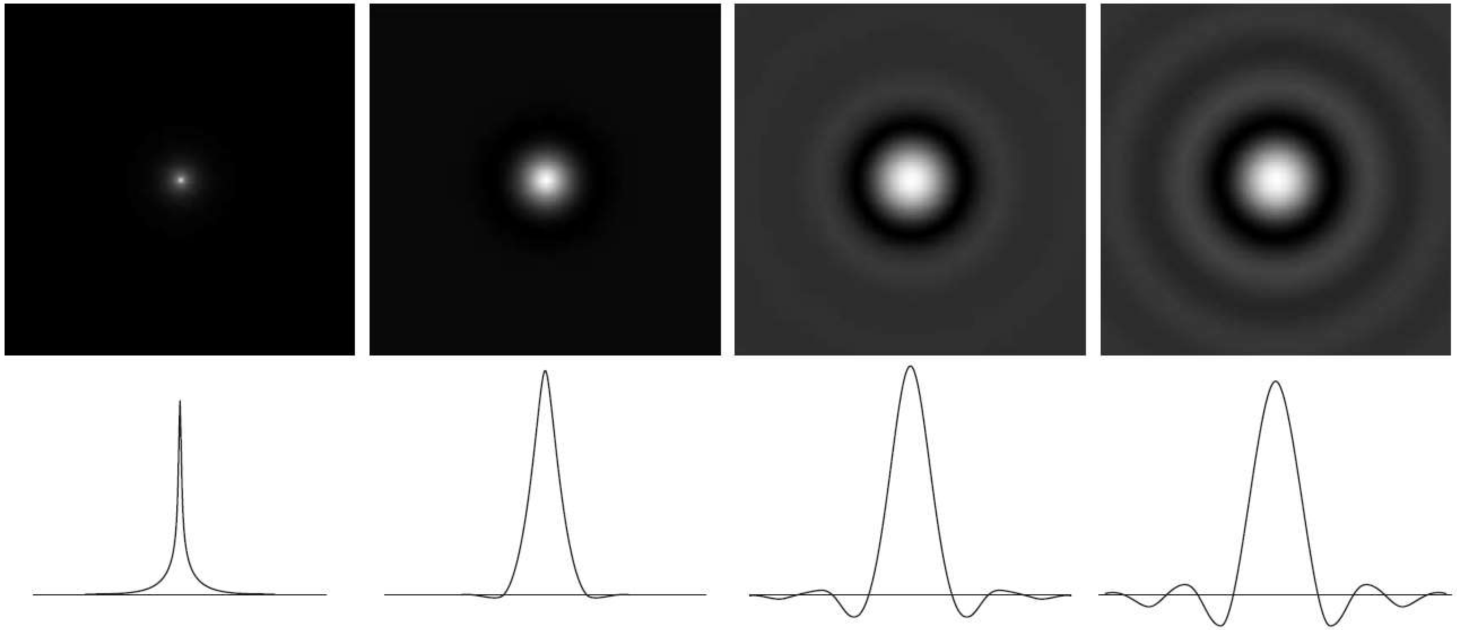
- Butterworth Lowpass Filters

  - BLPFs of order 2
  - Cutoff frequencies same as before



8. Image Smoothing Using Frequency Domain Filters

- Butterworth Lowpass Filters



a b c d

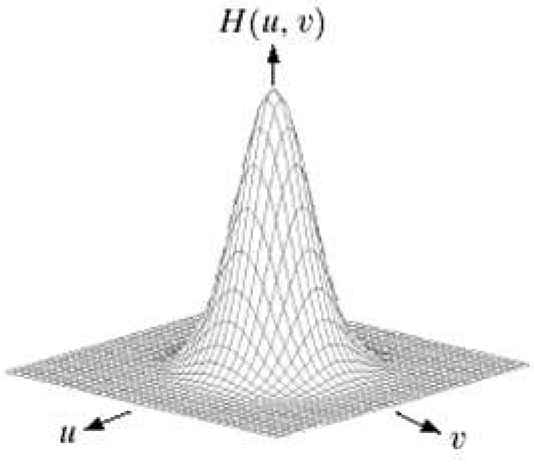
FIGURE 4.46 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding intensity profiles through the center of the filters (the size in all cases is 1000 × 1000 and the cutoff frequency is 5). Observe how ringing increases as a function of filter order.

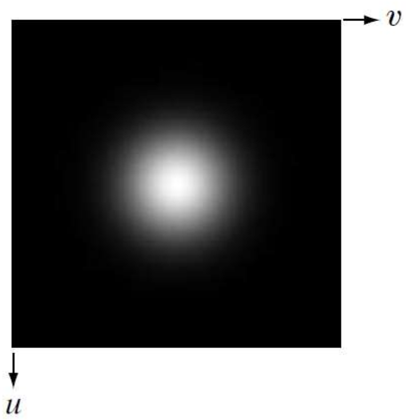
8. Image Smoothing Using Frequency Domain Filters

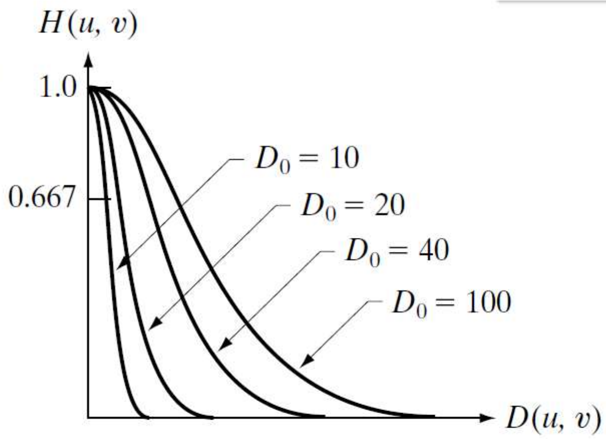
- Gaussian Lowpass Filters

$$H(u, v) = e^{-D^2(u,v)/2D_0^2}$$

- D_0 is the cutoff frequency. When $D(u,v) = D_0$, the GLPF is down to 0.607 of its maximum value.





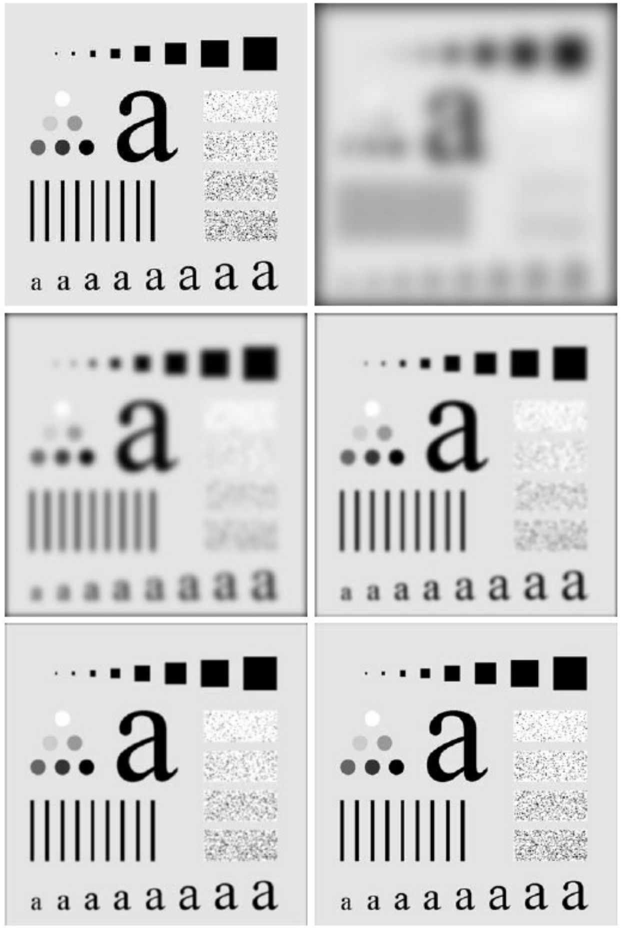


8. Image Smoothing Using Frequency Domain Filters

- Gaussian Lowpass Filters

  - Cutoff frequencies same as before.

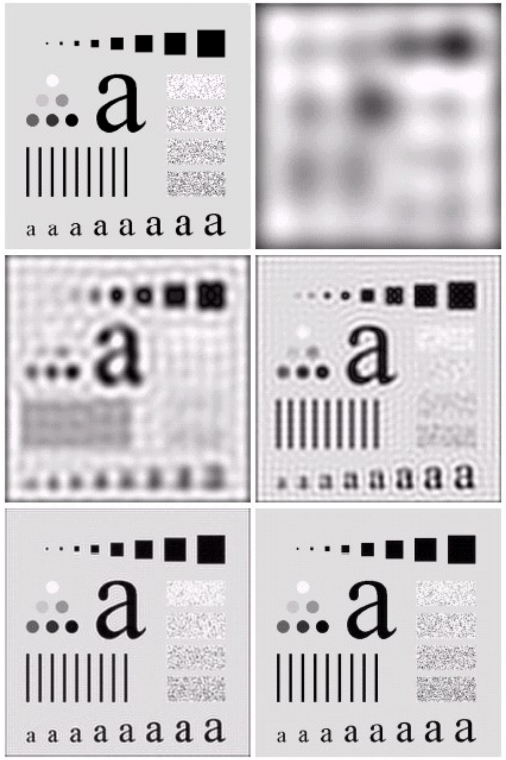
  - Spatial Gaussian filters will have no ringing.



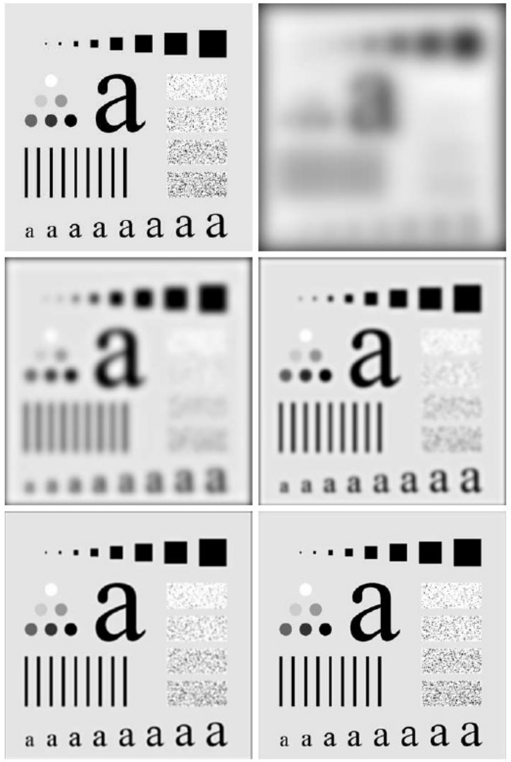
8. Image Smoothing Using Frequency Domain Filters

- Comparison

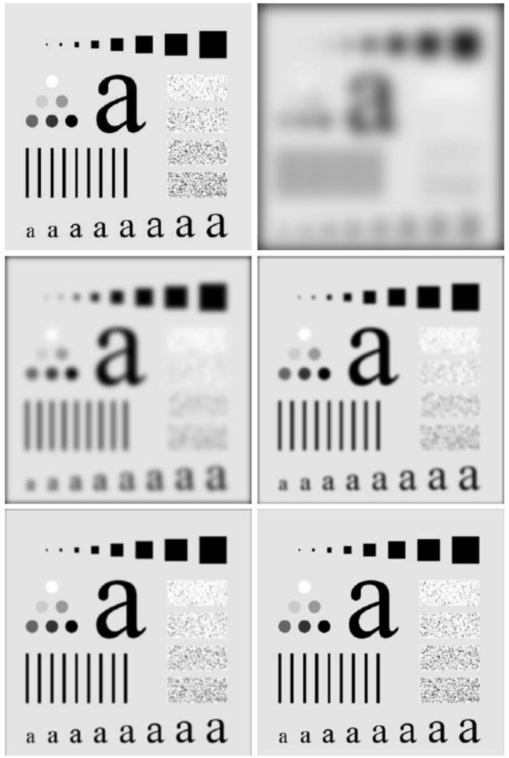
Ideal



Butterworth

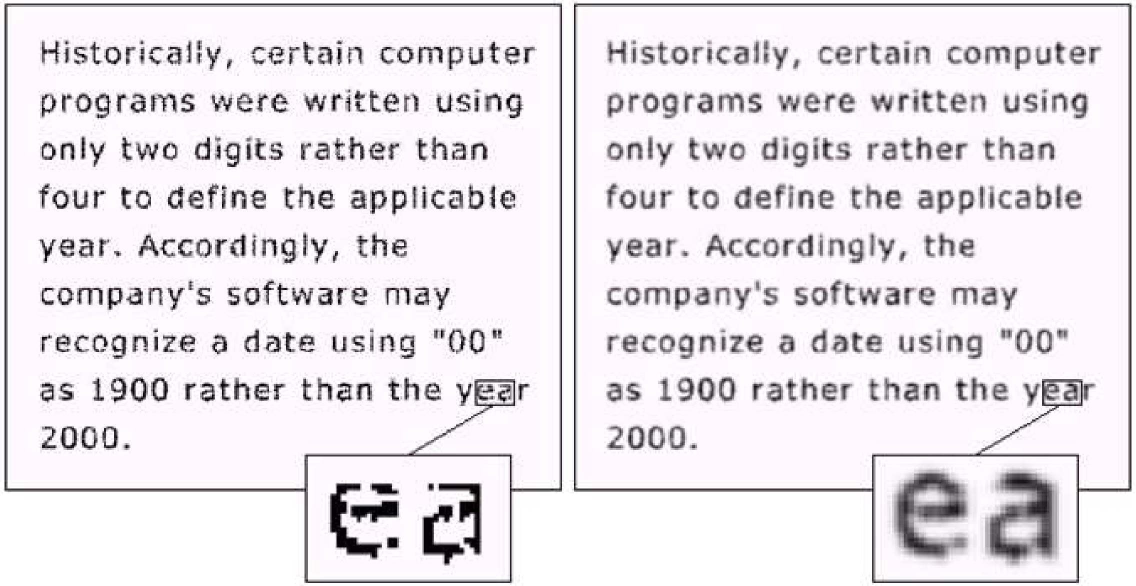


Gaussian



8. Image Smoothing Using Frequency Domain Filters

- Additional Examples of Lowpass Filtering



a b

FIGURE 4.49 (a) Sample text of low resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).

8. Image Smoothing Using Frequency Domain Filters

- Additional Examples of Lowpass Filtering

a b c

FIGURE 4.50 (a) Original image (784 × 732 pixels). (b) Result of filtering using a GLPF with $D_0 = 100$. (c) Result of filtering using a GLPF with $D_0 = 80$. Note the reduction in fine skin lines in the magnified sections in (b) and (c).

9. Image Sharpening Using Frequency Domain Filters

- Additional Examples of Lowpass Filtering

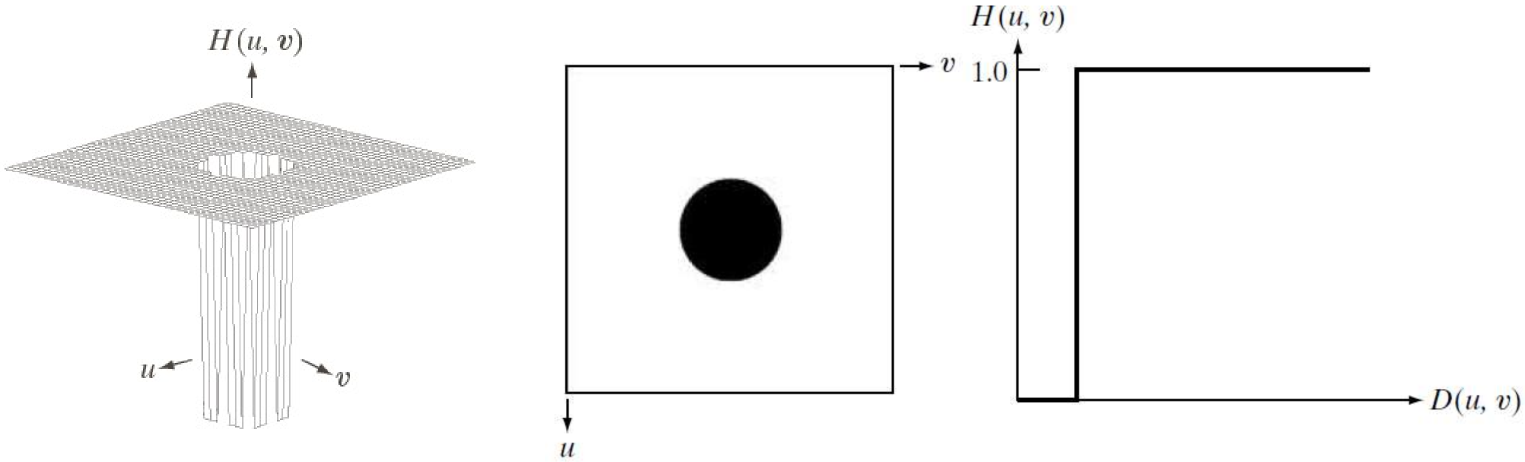
  - A highpass filter is obtained from a given lowpass filter using the equation

$$H_{\mathrm{HP}}(u, v) = 1 - H_{\mathrm{LP}}(u, v)$$

9. Image Sharpening Using Frequency Domain Filters

- Ideal Highpass Filters

$$H(u, v) = \begin{cases} 0 & \text{if } D(u, v) \leq D_0 \\ 1 & \text{if } D(u, v) > D_0 \end{cases}$$



9. Image Sharpening Using Frequency Domain Filters

- Ideal Highpass Filters

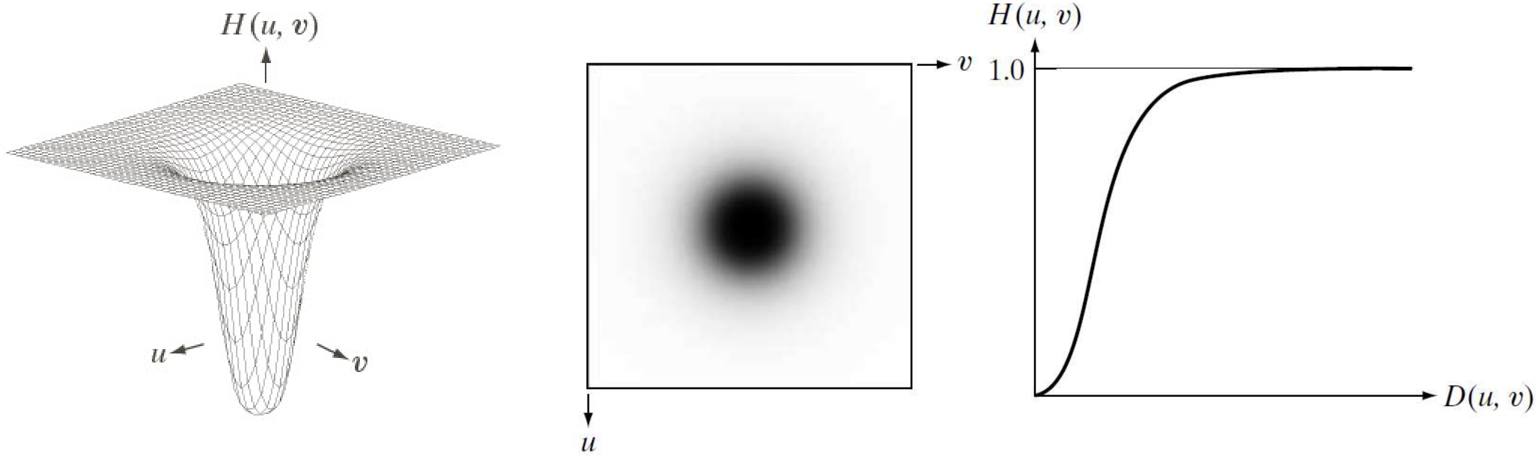
a b c

FIGURE 4.54 Results of highpass filtering the image in Fig. 4.41(a) using an IHPF with $D_0 = 30, 60,$ and 160.

9. Image Sharpening Using Frequency Domain Filters

- Butterworth Highpass Filters

$$H(u, v) = \frac{1}{1 + [D_0/D(u, v)]^{2n}}$$



9. Image Sharpening Using Frequency Domain Filters

- Butterworth Highpass Filters

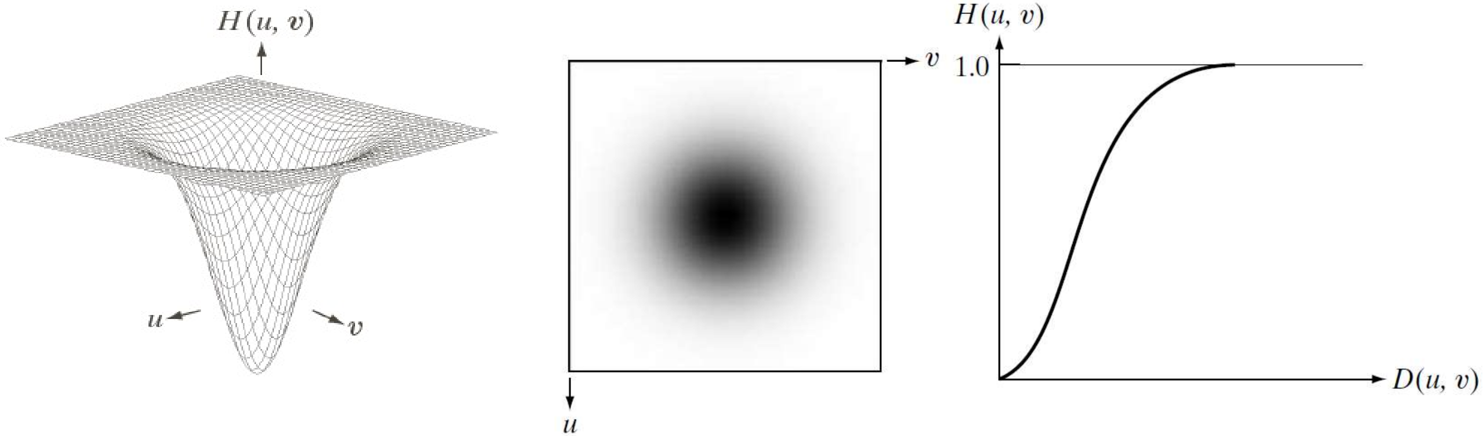
a b c

FIGURE 4.55 Results of highpass filtering the image in Fig. 4.41(a) using a BHPF of order 2 with $D_0 = 30, 60,$ and 160, corresponding to the circles in Fig. 4.41(b). These results are much smoother than those obtained with an IHPF.

9. Image Sharpening Using Frequency Domain Filters

- Gaussian Highpass Filters

$$H(u, v) = 1 - e^{-D^2(u,v)/2D_0^2}$$



9. Image Sharpening Using Frequency Domain Filters

- Gaussian Highpass Filters

a b c

FIGURE 4.56 Results of highpass filtering the image in Fig. 4.41(a) using a GHPF with $D_0 = 30$, 60, and 160, corresponding to the circles in Fig. 4.41(b). Compare with Figs. 4.54 and 4.55.

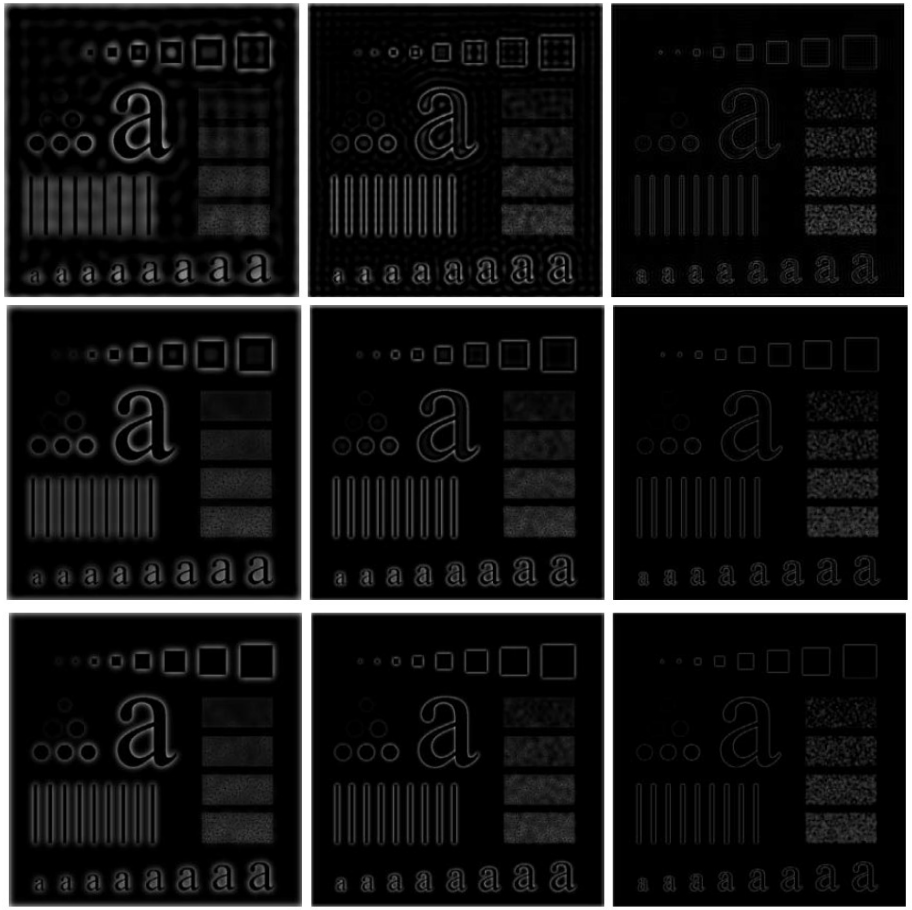
9. Image Sharpening Using Frequency Domain Filters

- Comparison

  - Ideal

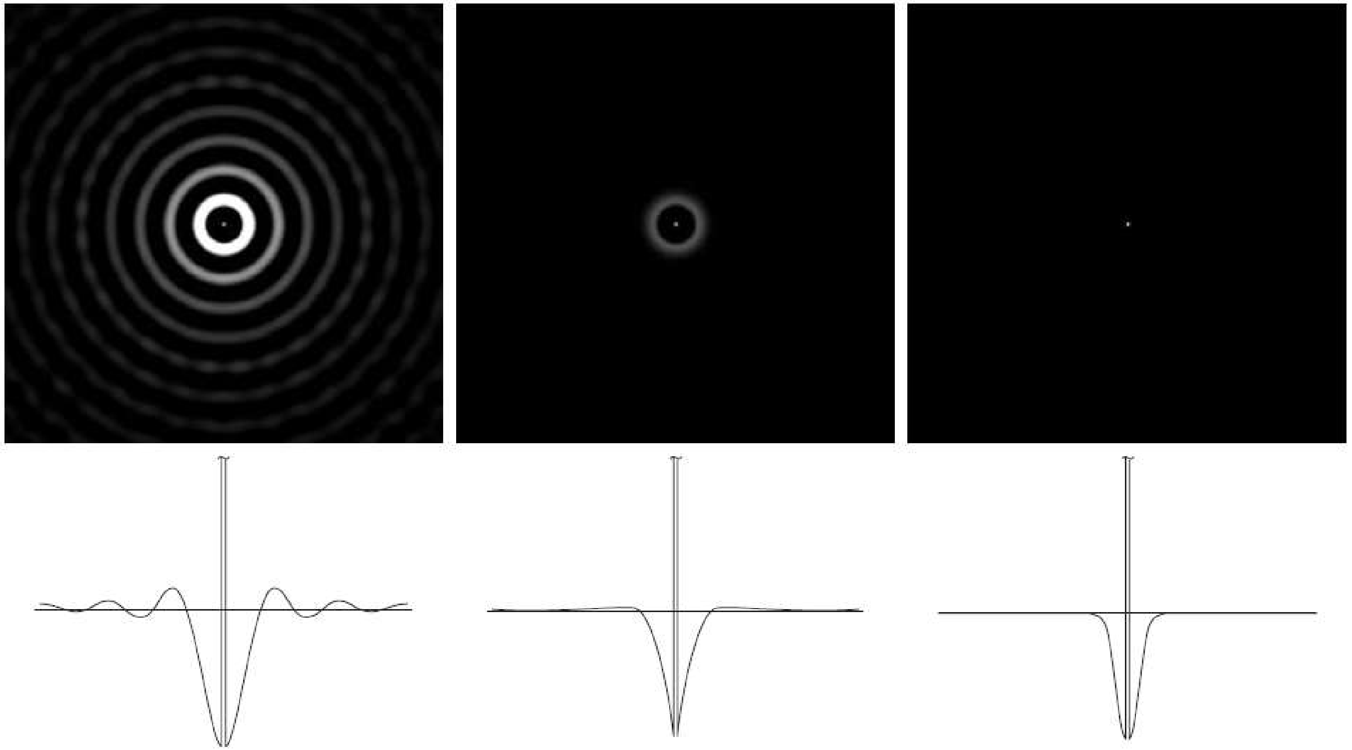
  - Butterworth

  - Gaussian



9. Image Sharpening Using Frequency Domain Filters

- Spatial representation



a b c

FIGURE 4.53 Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.

9. Image Sharpening Using Frequency Domain Filters

- Example

a b c

FIGURE 4.57 (a) Thumb print. (b) Result of highpass filtering (a). (c) Result of thresholding (b). (Original image courtesy of the U.S. National Institute of Standards and Technology.)

9. Image Sharpening Using Frequency Domain Filters

- Low/highpass filters equations

Lowpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \leq D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = \dfrac{1}{1 + [D(u,v)/D_0]^{2n}}$	$H(u,v) = e^{-D^2(u,v)/2D_0^2}$

Highpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

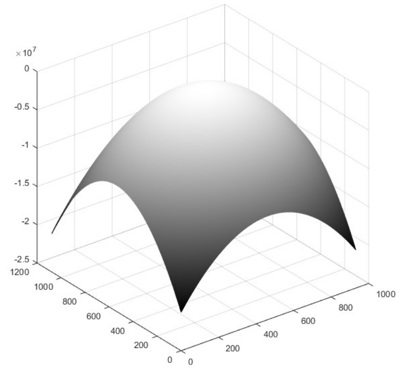
Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \leq D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = \dfrac{1}{1 + [D_0/D(u,v)]^{2n}}$	$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$

9. Image Sharpening Using Frequency Domain Filters

- The Laplacian in the Frequency Domain

  - It can be shown that the Laplacian can be implemented in the frequency domain using the filter

$$H(u, v) = -4\pi^2(u^2 + v^2)$$



or, with respect to the center of the frequency rectangle, using the filter

$$H(u, v) = -4\pi^2\left[(u - P/2)^2 + (v - Q/2)^2\right]$$
$$= -4\pi^2 D^2(u, v)$$

9. Image Sharpening Using Frequency Domain Filters

- The Laplacian in the Frequency Domain

  - Then, the Laplacian image is obtained as:

$$\nabla^2 f(x, y) = \Im^{-1}\{H(u, v)F(u, v)\}$$

  - As explained before, enhancement is achieved using the equation:

$$g(x, y) = f(x, y) + c\nabla^2 f(x, y)$$

  - Here, c = -1 because is negative. In the frequency domain, is written as

$$\begin{aligned} g(x, y) &= \Im^{-1}\{F(u, v) - H(u, v)F(u, v)\} \\ &= \Im^{-1}\{[1 - H(u, v)]F(u, v)\} \\ &= \Im^{-1}\{[1 + 4\pi^2 D^2(u, v)]F(u, v)\} \end{aligned}$$

9. Image Sharpening Using Frequency Domain Filters

- MATLAB: s172LaplacianFreq.m

a b

FIGURE 4.58 (a) Original, blurry image. (b) Image enhanced using the Laplacian in the frequency domain. Compare with Fig. 3.38(e).

9. Image Sharpening Using Frequency Domain Filters

- Unsharp Masking, Highboost Filtering, and High-Frequency-Emphasis Filtering

✓ First a mask is defined

$$g_{\text{mask}}(x, y) = f(x, y) - f_{\text{LP}}(x, y)$$

with $$f_{\text{LP}}(x, y) = \Im^{-1}\left[H_{\text{LP}}(u, v)F(u, v)\right]$$

where $H_{LP}(u,v)$ is a lowpass filter $F(u,v)$ and is the Fourier transform of $f(x, y)$.

✓ Then $$g(x, y) = f(x, y) + k * g_{\text{mask}}(x, y)$$

✓ This expression defines unsharp masking when and k = 1 and highboost filtering when k > 1.

9. Image Sharpening Using Frequency Domain Filters

• Unsharp Masking, Highboost Filtering, and High-Frequency-Emphasis Filtering

✓ We can express the preceding result entirely in terms of frequency domain computations involving a lowpass filter:

$$g(x, y) = \Im^{-1}\Big\{\big[1 + k * [1 - H_{\mathrm{LP}}(u, v)]\big]F(u, v)\Big\}$$

✓ We can express this result in terms of a highpass filter:

$$g(x, y) = \Im^{-1}\Big\{[1 + k * H_{\mathrm{HP}}(u, v)]F(u, v)\Big\}$$

✓ The expression contained within the square brackets is called a *high-frequency emphasis* filter.

✓ The constant, k, gives control over the proportion of high frequencies that influence the final result.

9. Image Sharpening Using Frequency Domain Filters

• Unsharp Masking, Highboost Filtering, and High-Frequency-Emphasis Filtering

✓ A slightly more general formulation of high-frequency-emphasis filtering is the expression

$$g(x, y) = \Im^{-1}\Big\{[k_1 + k_2 * H_{\mathrm{HP}}(u, v)]F(u, v)\Big\}$$

with $k_1 \geq 1$ (offset from the origin) and $k_2 \geq 0$ (contribution of high frequencies).

9. Image Sharpening Using Frequency Domain Filters

- Unsharp Masking, Highboost Filtering, and High-Frequency-Emphasis Filtering

$D_0 = 40$

$k_1 = 0.5$
$k_2 = 0.75$

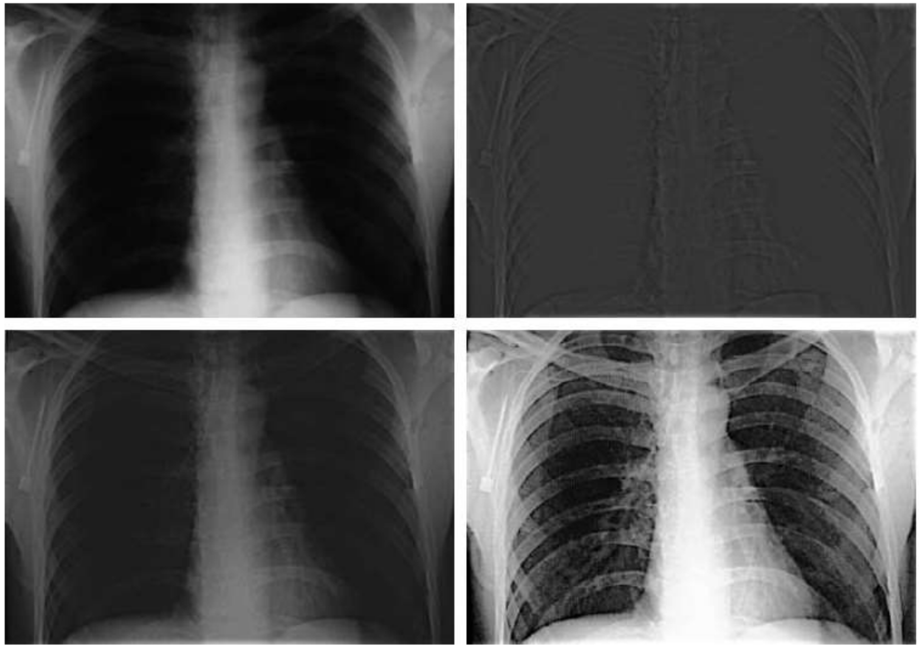
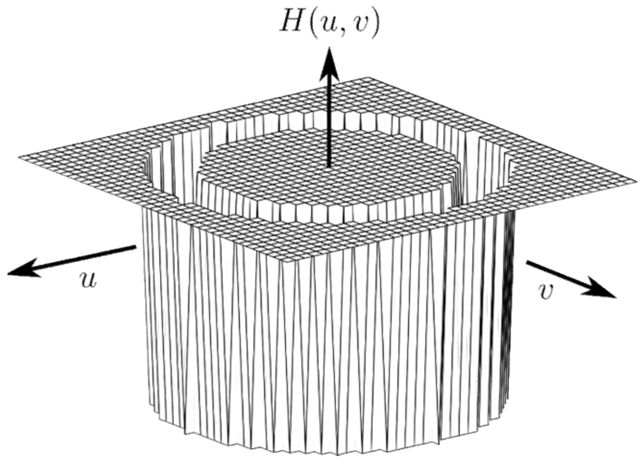


FIGURE 4.59 (a) A chest X-ray image. (b) Result of highpass filtering with a Gaussian filter. (c) Result of high-frequency-emphasis filtering using the same filter. (d) Result of performing histogram equalization on (c). (Original image courtesy of Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

10. Selective Filtering

- Bandreject and Bandpass Filters
  - Ideal Bandreject Filters

$$H(u,v) = \begin{cases} 0 & \text{if } D_0 - \dfrac{W}{2} \leq D \leq D_0 + \dfrac{W}{2} \\ 1 & \text{otherwise} \end{cases}$$

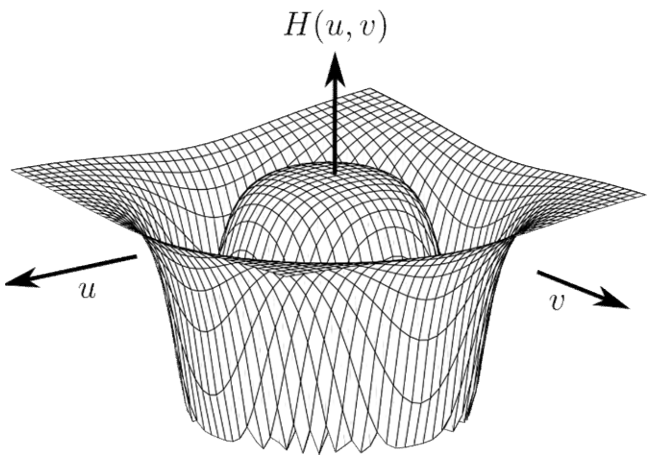


10. Selective Filtering

- Bandreject and Bandpass Filters

  ➢ Butterworth Bandreject Filters

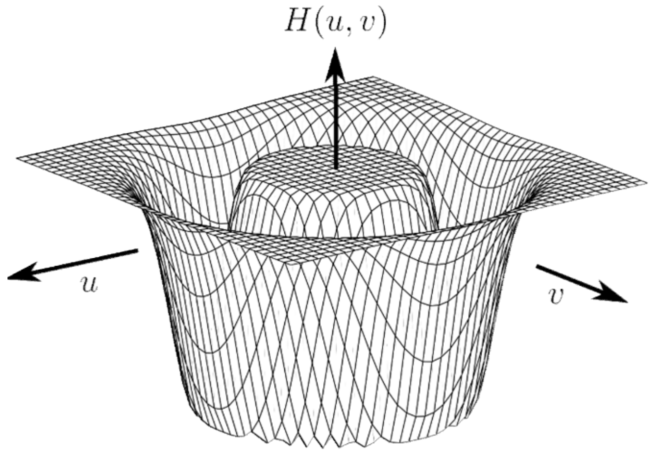
$$H(u,v) = \frac{1}{1 + \left[\dfrac{DW}{D^2 - D_0^2}\right]^{2n}}$$



10. Selective Filtering

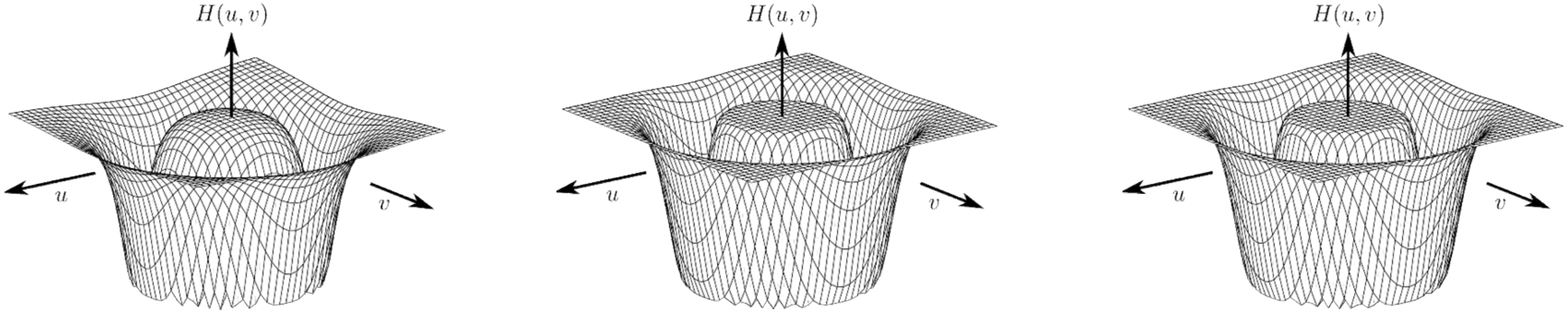
- Bandreject and Bandpass Filters
  - Gaussian Bandreject Filters

$$H(u,v) = 1 - e^{-\left[\frac{D^2 - D_0^2}{DW}\right]^2}$$



10. Selective Filtering

- Bandreject and Bandpass Filters



Ideal	Butterworth	Gaussian
$H(u,v)=\begin{cases}0 & \text{if } D_0-\frac{W}{2}\leq D\leq D_0+\frac{W}{2}\\ 1 & \text{otherwise}\end{cases}$	$H(u,v)=\dfrac{1}{1+\left[\dfrac{DW}{D^2-D_0^2}\right]^{2n}}$	$H(u,v)=1-e^{-\left[\frac{D^2-D_0^2}{DW}\right]^2}$

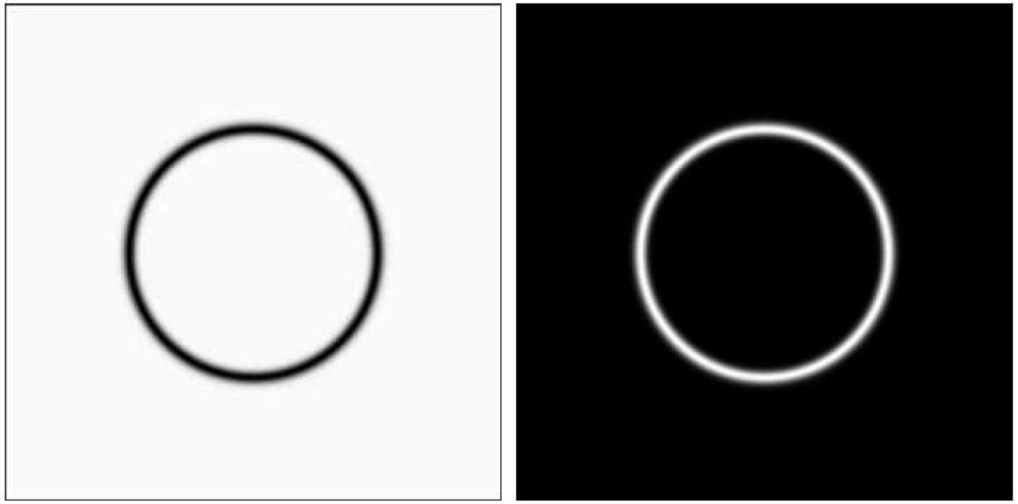
- D is the distance *D(u,v)* from the center of the filter, D_0 is the radial center of the band, W is the width of the band (upper-lower cutoff frequency) and *n* is the order of the Butterworth filter.

10. Selective Filtering

- Bandreject and Bandpass Filters

  - A bandpass filter is obtained from a bandreject filter in the same manner that we obtained a highpass filter from a lowpass filter:

$$H_{\mathrm{BP}}(u, v) = 1 - H_{\mathrm{BR}}(u, v)$$



a b

FIGURE 4.63 (a) Bandreject Gaussian filter. (b) Corresponding bandpass filter. The thin black border in (a) was added for clarity; it is not part of the data.

10. Selective Filtering

- Notch Filters

  - Notch filters rejects (or passes) frequencies in a predefined neighborhood about the center of the frequency rectangle.

  - They are symmetric about the origin, so a notch with center at (u_0, v_0) must have a corresponding notch at location ($-u_0$, $-v_0$).

  - *Notch filters rejects* are constructed as products of highpass filters whose centers have been translated to the centers of the notches. The general form is:

$$H_{\mathrm{NR}}(u, v) = \prod_{k=1}^{Q} H_k(u, v) H_{-k}(u, v)$$

  - where H_k and H_{-k} are highpass filters whose centers are at (u_k, v_k) and ($-u_k$, $-v_k$).

10. Selective Filtering

- Notch Filters

  - These centers are specified with respect to the center of the frequency rectangle, (M/2, N/2).

  - The distance computations for each filter are thus carried out using the expressions

$$D_k(u,v) = \left[(u - M/2 - u_k)^2 + (v - N/2 - v_k)^2\right]^{1/2}$$

and

$$D_{-k}(u,v) = \left[(u - M/2 + u_k)^2 + (v - N/2 + v_k)^2\right]^{1/2}$$

- For example, the following is a Butterworth notch reject filter of order n, containing three notch pairs

$$H_{\mathrm{NR}}(u,v) = \prod_{k=1}^{3}\left[\frac{1}{1 + [D_{0k}/D_k(u,v)]^{2n}}\right]\left[\frac{1}{1 + [D_{0k}/D_{-k}(u,v)]^{2n}}\right]$$

10. Selective Filtering

- Notch Filters

  - A notch pass filter is obtained from a notch reject filter using the expression

$$H_{\mathrm{NP}}(u, v) = 1 - H_{\mathrm{NR}}(u, v)$$

  - Example

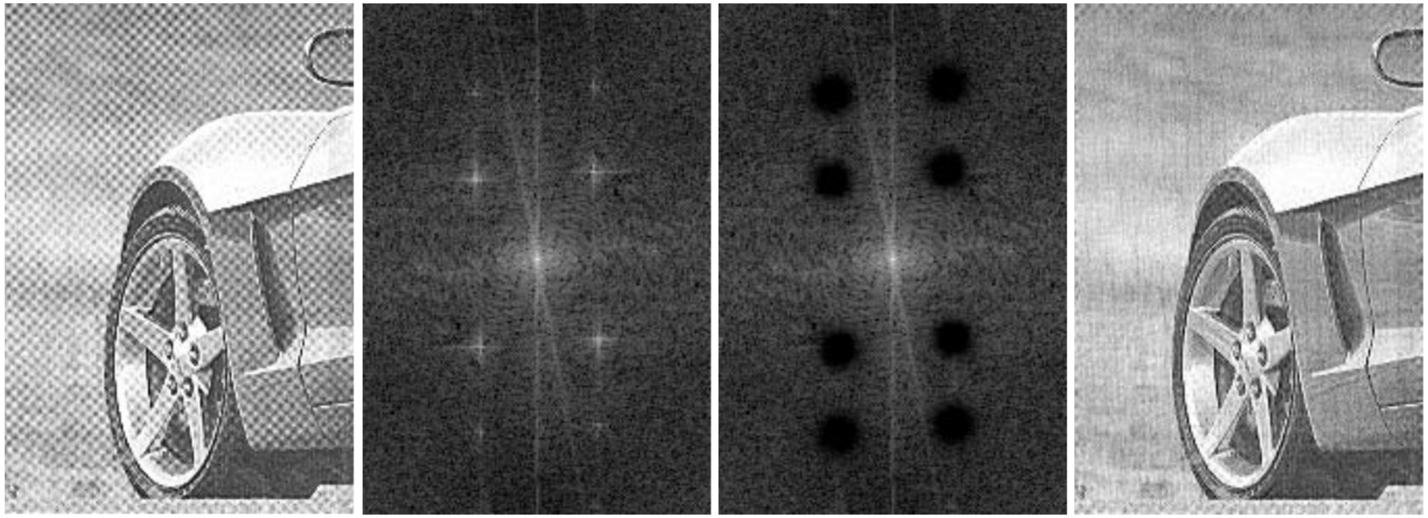


FIGURE 4.64 (a) Sampled newspaper image showing a moiré pattern. (b) Spectrum. (c) Butterworth notch reject filter multiplied by the Fourier transform. (d) Filtered image.

10. Selective Filtering

- MATLAB: s186Notch.m