Micro Theory IV - Comparative Statics

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Comparative statics are representations of how endogeneous variables respond to exogeneous ones. For instance, I might be interested in $\frac{dh_i}{dp_i}$, the substitution effect. I happen to know the sign of this, since the expenditure function is concave. But other comparative statics are harder to sign: for instance, I don't know dh_i/dp_j or dx_i/dp_j. I need some techniques for signing comparative statics.

In general I have three approaches:

1. Make an explicit functional form. Then we can solve for x_i, h_i etc and differentiate. This is nice when feasible. But sometimes we may have a functional form that can't be solved, or we may not want to pick one at all.
2. The implicit function theorem
3. Techniques of "Monotone Comparative Statics"

Implicit Function Theorem Comparative Statics

We're maximizing $u(x_1, x_2)$ subject to $w = p_1x_1 + p_2x_2$.

We have some FOCs:

$$\begin{aligned} u_1(x_1(p,w), x_2(p,w)) + \lambda(p,w)p_1 = 0 \\ u_2(x_1(p,w), x_2(p,w)) + \lambda(p,w)p_2 = 0 \\ p_1x_1(p,w) + p_2x_2(p,w) - w = 0 \end{aligned}$$

Note that I've already maximized this, i.e. x_1, x_2, λ are such that the FOCs hold. That makes these identities. So I can differentiate them all with respect to p_1 (say).

$$\begin{aligned} -\lambda &= u_{11}\frac{dx_1}{dp_1} + u_{12}\frac{dx_2}{dp_1} + p_1\frac{d\lambda}{dp_1} \\ 0 &= u_{21}\frac{dx_1}{dp_1} + u_{22}\frac{dx_2}{dp_1} + p_2\frac{d\lambda}{dp_1} \\ -x_1 &= p_1\frac{dx_1}{dp_1} + p_2\frac{dx_2}{dp_1} \end{aligned}$$

We can write this in matrix form:

$$\begin{bmatrix} u_{11} & u_{12} & p_1 \\ u_{21} & u_{22} & p_2 \\ p_1 & p_2 & 0 \end{bmatrix} \begin{bmatrix} dx_1/dp_1 \\ dx_2/dp_1 \\ d\lambda/dp_1 \end{bmatrix} = \begin{bmatrix} -\lambda \\ 0 \\ -x_1 \end{bmatrix}$$

This allows for a Cramer's Rule setup - i.e., for the i^{th} variable, take a ratio of determinants: the det. of the coefficient matrix with the i^{th} column replaced by the constants divided by the determinant of the coefficients matrix.

$$\frac{dx_1}{dp_1} = \frac{\begin{vmatrix} -\lambda & u_{12} & p_1 \\ 0 & u_{22} & p_2 \\ -x_1 & p_2 & 0 \end{vmatrix}}{\begin{vmatrix} u_{11} & u_{12} & p_1 \\ u_{21} & u_{22} & p_2 \\ p_1 & p_2 & 0 \end{vmatrix}}$$

The bottom matrix is sometimes called the Bordered Hessian. We can say it is ≥ 0 from the first-order conditions.
The top has determinant $\lambda p_2^2 - x_1(p_2u_{12} - p_1u_{22})$.

This first term λp_2^2 is negative: we defined $\lambda \leq 0$, and the square is positive.

We can then talk about our derivatives purely in terms of the signs of the second derivatives of u.

The cross derivatives:

$$\frac{dx_2}{dp_1} = \frac{\begin{vmatrix} u_{11} & -\lambda & p_1 \\ u_{21} & 0 & p_2 \\ p_1 & -x_1 & 0 \end{vmatrix}}{\begin{vmatrix} u_{11} & u_{12} & p_1 \\ u_{21} & u_{22} & p_2 \\ p_1 & p_2 & 0 \end{vmatrix}} = (1/|H|)\left[\lambda p_1p_2 - x_1(p_1u_{12} - p_2u_{11})\right]$$

$$\frac{dx_1}{dp_2} = \frac{\begin{vmatrix} 0 & u_{12} & p_1 \\ -\lambda & u_{22} & p_2 \\ -x_2 & p_2 & 0 \end{vmatrix}}{\begin{vmatrix} u_{11} & u_{12} & p_1 \\ u_{21} & u_{22} & p_2 \\ p_1 & p_2 & 0 \end{vmatrix}} = (1/|H|)\left[\lambda p_1 p_2 - x_2(p_1 u_{22} - p_2 u_{12})\right]$$

Then we have this symmetric term $\lambda p_1 p_2$ which is negative.

Implicit Function Theorem

For this purpose we can think of the Implicit Function Theorem as:

- a bunch of equations $f_1, \ldots f_n = 0$, with arguments (x, t)
  - here x is endogeenous variables and t is exogeneous.
  - One equation for each x (endogeneous variable).
  - Suppose you can define $x(t)$ st. $f_1(x(t), t) = 0$; then I can start differentiating.

But how do I know that's allowed?

The IFT tells me when I can do that: specifically, if $|H|$ is nonzero, then there is some $x(t)$ that makes this work in the neighborhood of t.

There are a number of conditions required for the IFT, but it's still very handy, and frequently used. Works well when all exogeneous variables are in the constraint.

Monotone Comparative Statics

MCS is a more generalized method with a different set of conditions. Notably, it requires no differentiability.

Let $f(X \times T) \to \mathbb{R}$. Once more, we can think of X as endogeneous variables and T as exogeneous; we are ultimately interested in how the X change when T changes.

Increasing Differences

Choose $x' > x$ and $t' > t$. Then f exhibits increasing differences if:

$$f(x', t') - f(x, t') \geq f(x', t) - f(x, t)$$
$$f(x', t') - f(x', t) \geq f(x, t') - f(x, t)$$

This is simply a discrete version of a criterion that $d^2 f / dx dt \geq 0$: when t is higher, an increase in x has a greater effect on f. Strictly increasing differences means that the above inequalities are strict.

Then we have a useful result:

Theorem

Let f exhibit strictly increasing differences; let $t' > t$, $x' \in \arg\max f(\cdot, t')$ and $x \in \arg\max f(\cdot, t)$. Then $x' \geq x$.

In other words: if x' maximizes under t' and x maximizes under t, then $t' > t \implies x' \geq x$.
We can't quite get to $x' > x$; for instance, suppose $x \in \{0, 1\}$ (e.g. loan approval decision).

This is a very easy method! It involves no calculations and no structure. Why is it so powerful? The increasing differences criterion has more information in it than the IFT, which only used $|H| \geq 0$.

It's simple to have a multidimensional t; just do this for every t. For multi dimensional X it's more complex. Let X be the Cartesian product of X_i, and T of T_j, with X_i and T_j in $\mathbb{R}$. Then, if f has SID in each pair x_i, t_j, and ID in each x_i, x_j $t' > t \implies x(t') \geq x(t)$.

Single Crossing

We may prefer an ordinal property, i.e. one that doesn't care about specific values of f. For instance, when we work with utility functions, all that matters is the order. Then we have the single crossing criterion: For $x' > x$, $t' > t$,

$$[f(x', t) \geq f(x, t)] \implies [f(x', t') \geq f(x, t')]$$

Intuitively, this means 'signs are preserved': the difference of x moves iwth the difference of t. Same generalization as above.

Supermodularity

A further generalization of our results: suppose X is a lattice (ie. each pair has a $\sup$ and $\inf$), T a partially ordered set. Let P be the ordering (eg xPy meaning x comes before y).
Define $x \wedge y$ as the least upper bound of (x, y); define $x \vee y$ as the greatest lower bound, meaning:

- $(x \wedge y)Px$, $(x \wedge y)Py$, and zPx, zPy means $zP(x \wedge y)$.

Then, f is supermodular in x if:

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$$

Quasi supermodularity is an ordinal condition:

$$f(x) \geq f(x \wedge y) \implies f(x \vee y) \geq f(y)$$

If X and T are $\mathbb{R}^j, \mathbb{R}^k$, and f is twice-differentiable, then we can say $x \wedge y$ is the elementwise maximum, and this maps to $f_{xy} \geq 0$ for all x, y. So we can see we have an even stronger condition.