# Micro Theory II - Utility Maximization

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Econ 500 Micro Theory Index

Why not just start here? Well, it's good to have the foundations. And in various analytic contexts one might prefer to use choice functions or preferences.

## Basic Properties

We have a constrained optimization problem:

- Maximize $U : X \to \mathbb{R}$
- subject to $px \leq w$

Can I solve this? Yes - thanks to the Weierstrass Extreme Value Theorem, which tells us that a continuous function from a compact set to a subset of $\mathbb{R}$ has a minimum/maximum. It's nice for us that the budget set is compact. Generally we assume continuity of $U$, also strict increasingness. We don't need to have strict increasingness of utility/strict monotonicity of preferences. We like to have $px = w$ so we can do optimization.

Write $x(p, w)$ is the maximizer of utility given $p, w$. We can write two conditions on $x$:

- Homogeneous of degree zero. $x(\lambda p, \lambda w) = \lambda^0 x(p, w)$. (later we will have homogeneity of degree >0).
- Walras' law: Every $px_i > 0$ means that $px = w$ for all $x \in x(p, w)$. In other words, you spend all your money. We get this easily from monotonicity.

### Implications

#### *Implication 1*

Homogeneity of degree zero can be differentiated wrt $\lambda$:

$$\begin{aligned}
0 &= x(\lambda p, \lambda w) - x(p, w) \\
&= x_\ell(\lambda p, \lambda w) - x_\ell(p, w) \\
\frac{\partial}{\partial \lambda} 0 &= \frac{\partial}{\partial \lambda} x_\ell(\lambda p, \lambda w) - \frac{\partial}{\partial \lambda} x_\ell(p, w) \\
0 &= p \frac{\partial x_\ell(p, w)}{\partial p} + w \frac{\partial x_\ell(p, w)}{\partial w} \\
&= \sum_k p_k \frac{\partial x_\ell(p, w)}{\partial p_k} + w \frac{\partial x_\ell(p, w)}{\partial w}
\end{aligned}$$

Lec slides say 'evaluate at λ = 1'. why do I have to do that? I don't want to

Suppose I want to express this in terms of the elasticities of good $\ell$. I can do that by dividing thru by $x_\ell(p, w)$ to get:

$$\sum_k \epsilon_{\ell k} + \epsilon_{\ell w} = 0$$

Intuitively, this means that if I were to change all the prices a tiny percentage amount and also change the income the same tiny percentage amount, the demand for good $\ell$ wouldn't change.

#### *Implication 2*

Walras' law can be differentiated by some $p_k$:

$$\begin{aligned}
w &= px(p, w) \\
\frac{\partial w}{\partial p_k} &= \frac{\partial px(p, w)}{\partial p_k} \\
0 &= p \frac{\partial x(p, w)}{\partial p_k} + x(p, w) \frac{\partial p}{\partial p_k} \\
0 &= \sum_\ell p_\ell \frac{\partial x_\ell(p, w)}{\partial p_k} + x_k(p, w)
\end{aligned}$$

recalling that $p, x$ are vectors.

This can be turned into an elasticity with annoying stuff

$$
\begin{aligned}
0 &= \sum_\ell p_\ell \frac{\partial x_\ell(p,w)}{\partial p_k} + x_k(p,w) \\
&= \frac{p_k}{w} \sum_\ell p_\ell \frac{x_\ell(p,w)}{x_\ell(p,w)} \frac{\partial x_\ell(p,w)}{\partial p_k} + \frac{p_k}{w} x_k(p,w) \\
&= \sum_\ell \frac{p_\ell x_\ell(p,w)}{w} \left[ \frac{\partial x_\ell(p,w)}{\partial p_k} \frac{p_k}{x_\ell(p,w)} \right] + \frac{p_k x_k(p,w)}{w} \\
&= \sum_\ell \frac{p_\ell x_\ell(p,w)}{w} \epsilon_{\ell k} + \frac{p_k x_k(p,w)}{w} \\
&= \sum_\ell b_\ell \epsilon_{\ell k} + b_k
\end{aligned}
$$

where $b_i$ is the budget share of good $i$.

### *Implication 3*

While we're at it we can differentiate Walras' law by $w$ to get

$$
\begin{aligned}
w &= px(p,w) \\
1 &= p \frac{\partial x(p,w)}{\partial w} \\
&= \sum_\ell p_\ell \frac{\partial x_\ell(p,w)}{\partial w}
\end{aligned}
$$

Predictably we can turn this into an elasticity:

$$
\begin{aligned}
1 &= \sum_\ell p_\ell \frac{\partial x_\ell(p,w)}{\partial w} \\
&= \sum_\ell p_\ell \frac{x_\ell(p,w) w}{x\ell(p,w) w} \frac{\partial x_\ell(p,w)}{\partial w} \\
&= \sum_\ell \frac{p_l x_l(p,w)}{w} \left[ \frac{\partial x_\ell(p,w)}{\partial w} \cdot \frac{w}{x_\ell(p,w)} \right] \\
&= \sum_\ell b_\ell \epsilon_{\ell w}
\end{aligned}
$$

in other words; the sum of budget shares times elasticities is $1$.

## Proposition 5

If $u$ is quasiconcave then $x(p,w)$ is convex. If $u$ is SQCV then $x(p,w)$ is a singleton satisfying WARP.

**TODO: fill in prop 5 proof,**

#status/section/🚧

## First-Order Conditions

The problem is $\max_x u(x)$ subject to $px \leq w$.
For an interior solution - on the budget set - we can have FOC that are

$$
\begin{aligned}
\lambda p_i &= \frac{\partial u}{\partial x_i} \\
w &= \sum p_i x_i
\end{aligned}
$$

where $\lambda$ is a Lagrange multiplier.

In this case we conventionally get rid of $\lambda$ by dividing it away. But we could also not do that, and interpret $\lambda$ as the marginal utility of income:

$$
\lambda = \frac{\partial u}{\partial x_\ell} \frac{1}{p_\ell} = \frac{\partial u}{\partial x_\ell} \frac{\partial x_\ell(p,w)}{\partial w}
$$

Here we use an Envelope Theorem-type result. If I am maximizing utility, that means I must be indifferent between the marginal expenditure on any good. Given a tiny amount of money I can spend it anywhere. So if $e_\ell$ is expenditure on good $\ell$ that means $\frac{\partial e_\ell}{\partial w} = \frac{\partial e_k}{\partial w}$ So then given an additional dollar, spending on $x_\ell$ will be $1/p_\ell$, so $1/p_\ell$ is simply $\partial x_\ell(p,w)/\partial w$.