COMP/EECE 7/8740 Neural Networks

Topics:

- Linear Classifiers
- Loss Functions
- Logistic regression

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Linear Classification Approaches

Parametric Approach

Image

Array of **32x32x3** numbers
(3072 numbers total)

f(**x**,**W**)

10 numbers giving
class scores

W
parameters
or weights

Parametric Approach: Linear Classifier

$$f(x,W) = Wx$$

Image

Array of **32x32x3** numbers
(3072 numbers total)

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10 numbers giving
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parameters
or weights

Slide Credit: Fei-Fei Li, Justin Johnson, Serena Yeung, CS 231n

Parametric Approach: Linear Classifier

3072x1

$$\boxed{f(x,W)} = \boxed{W}\boxed{x}$$

10x1 10x3072

Image

Array of **32x32x3** numbers
(3072 numbers total)

$\longrightarrow$ f(**x**,**W**) $\longrightarrow$ **10** numbers giving class scores

$\uparrow$

W

parameters
or weights

Parametric Approach: Linear Classifier

3072x1

$$\boxed{f(x,W)} = \boxed{W}\boxed{x} + \boxed{b}$$

10x1

10x1 10x3072

Image

Array of **32x32x3** numbers
(3072 numbers total)

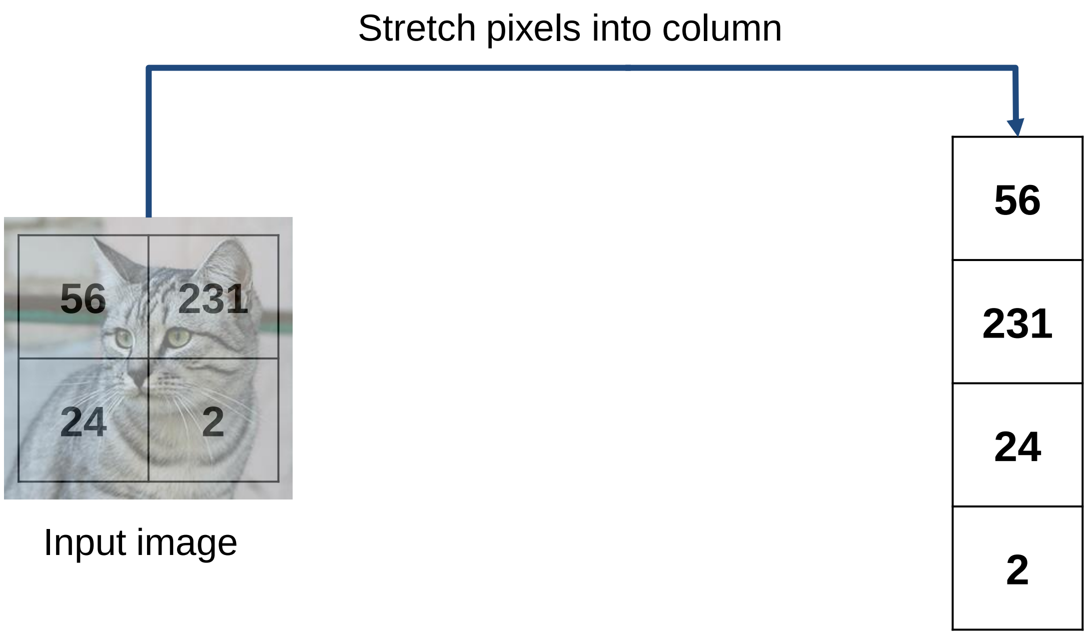
$\longrightarrow$ f(x,W) $\longrightarrow$ **10** numbers giving class scores

$\uparrow$

W

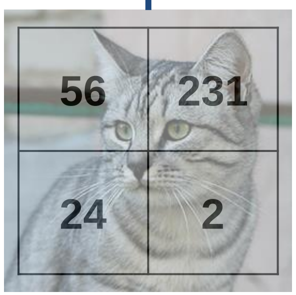
parameters
or weights

Example with an image with 4 pixels, and 3 classes (cat/dog/ship)



Example with an image with 4 pixels, and 3 classes (cat/dog/ship)

Stretch pixels into column



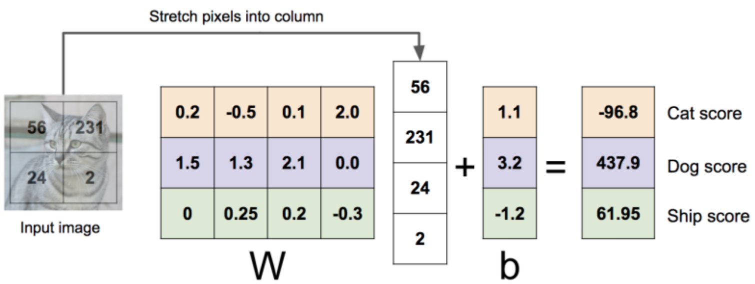
Input image

$$
\underbrace{\begin{bmatrix} 0.2 & -0.5 & 0.1 & 2.0 \\ 1.5 & 1.3 & 2.1 & 0.0 \\ 0 & 0.25 & 0.2 & -0.3 \end{bmatrix}}_{W}
\begin{bmatrix} 56 \\ 231 \\ 24 \\ 2 \end{bmatrix}
+
\underbrace{\begin{bmatrix} 1.1 \\ 3.2 \\ -1.2 \end{bmatrix}}_{b}
=
\begin{array}{rl} \begin{bmatrix} -96.8 \\ 437.9 \\ 61.95 \end{bmatrix} & \begin{array}{l} \text{Cat score} \\ \text{Dog score} \\ \text{Ship score} \end{array} \end{array}
$$

Example with an image with 4 pixels, and 3 classes (cat/dog/ship)

Algebraic Viewpoint

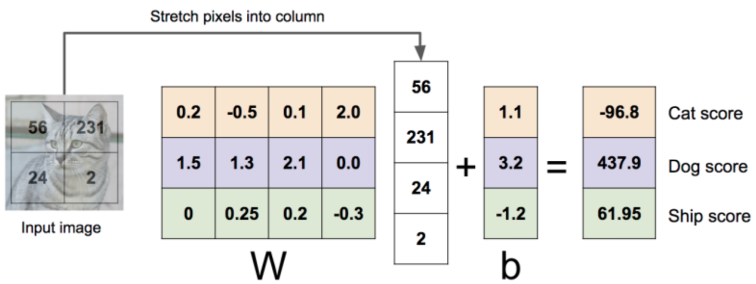
f(x,W) = Wx

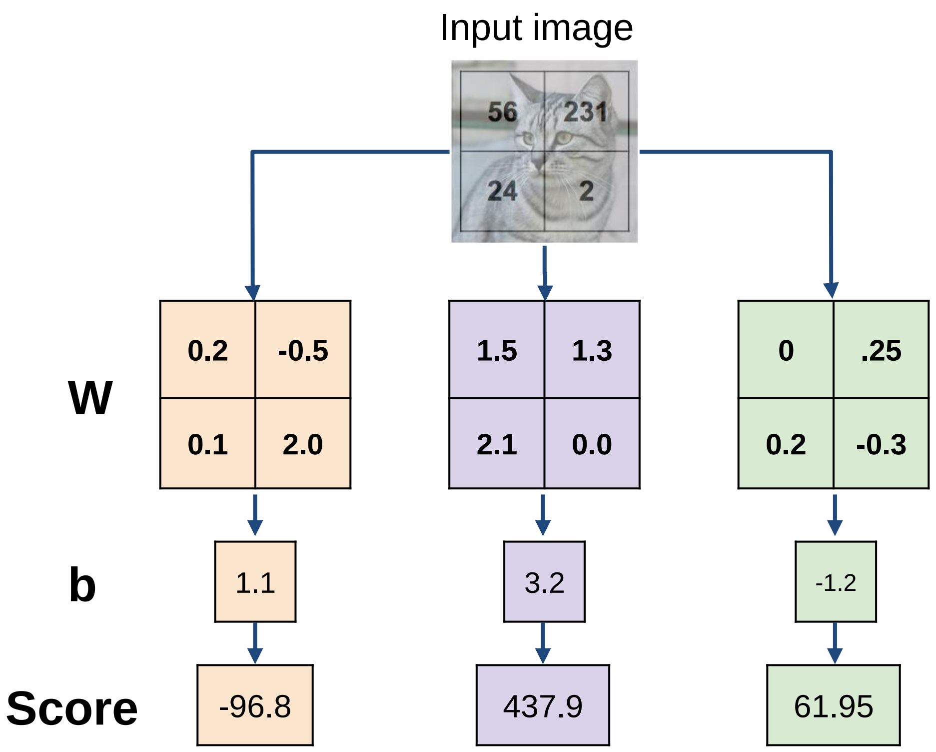


Example with an image with 4 pixels, and 3 classes (cat/dog/ship)

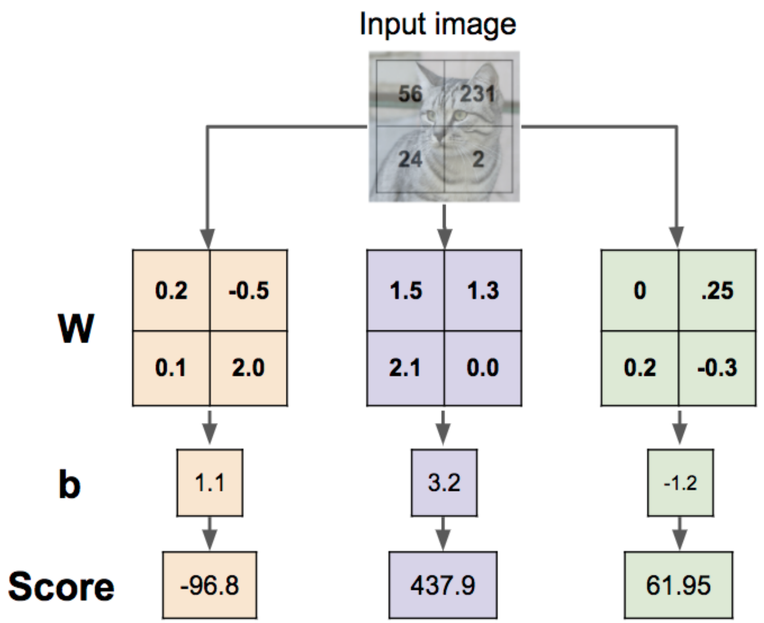
Algebraic Viewpoint

f(x,W) = Wx

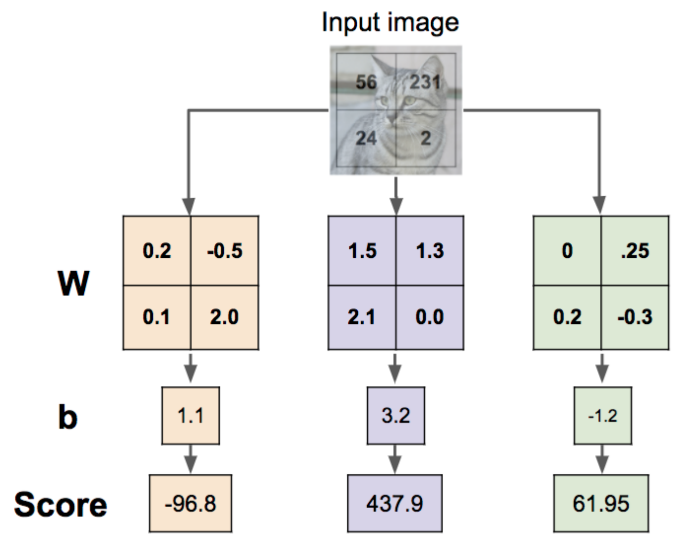




Interpreting a Linear Classifier

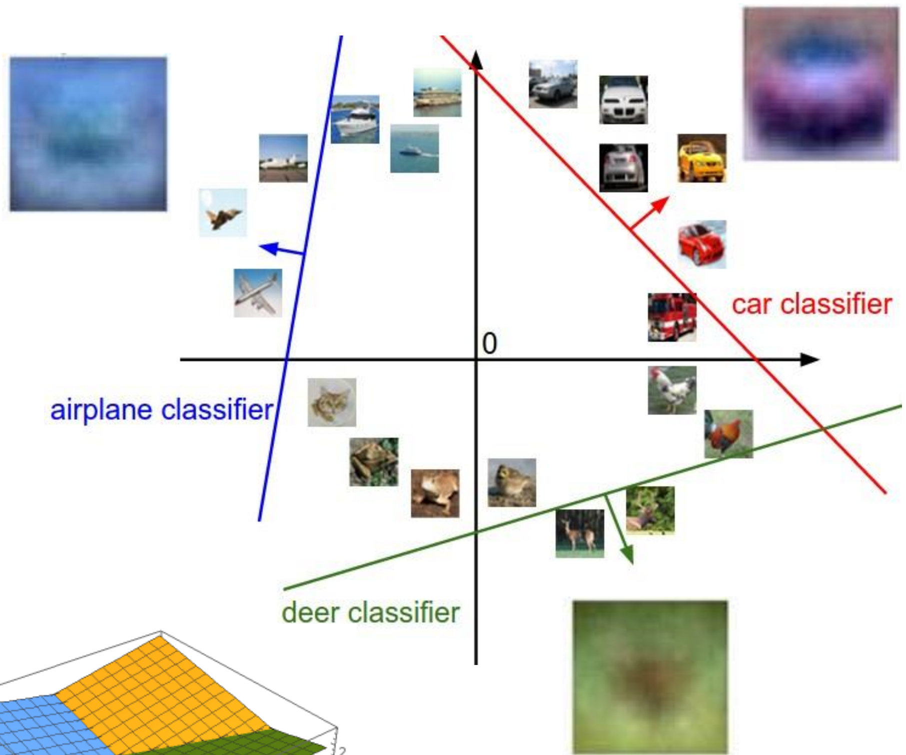


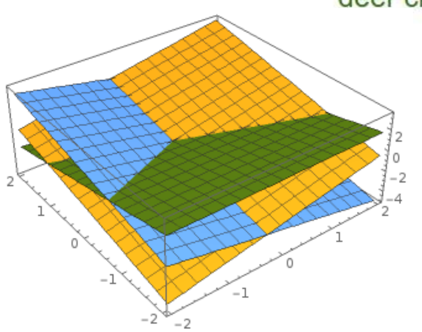
Interpreting a Linear Classifier: Visual Viewpoint



Slide Credit: Fei-Fei Li, Justin Johnson, Serena Yeung, CS 231n

Interpreting a Linear Classifier: Geometric Viewpoint





Plot created using Wolfram Cloud

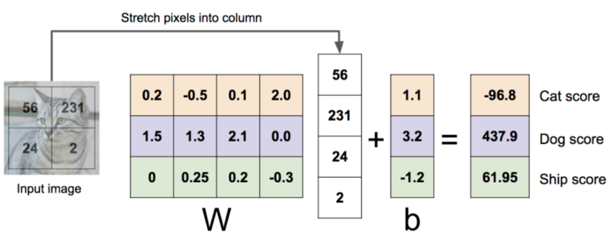
$$f(x,W) = Wx + b$$

Array of **32x32x3** numbers
(3072 numbers total)

Linear Classifier: Three Viewpoints

Algebraic Viewpoint

$f(x,W) = Wx$

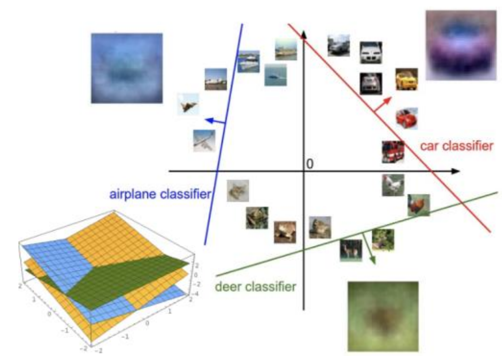


Visual Viewpoint

One template
per class

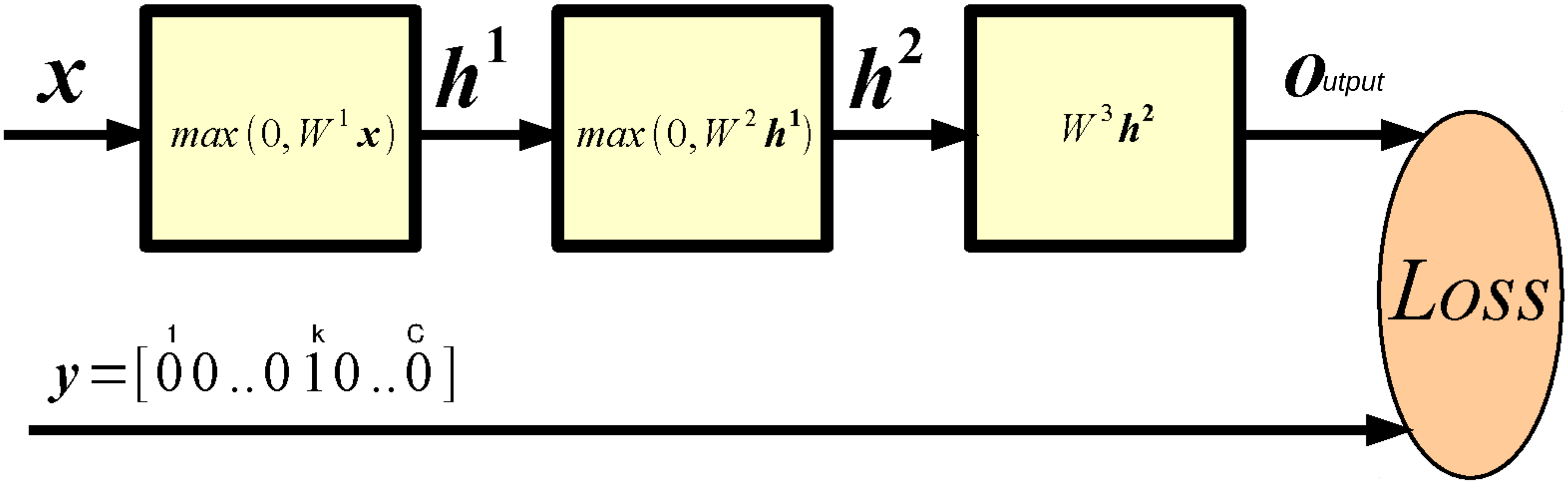
Geometric Viewpoint

Hyperplanes
cutting up space



Loss Functions

How Good is a Network?



What is an appropriate loss?

- Compare training class to output class
- Zero-one loss (per class)

$$L(\hat{y}, y) = I(\hat{y} \neq y),$$

- Is it good?

What is an appropriate loss?

- Compare training class to output class
- Zero-one loss (per class)

$$L(\hat{y}, y) = I(\hat{y} \neq y),$$

- Is it good?
  - Nope – it’s a step function.
  - I need to compute the gradient of the loss.
  - This loss is not differentiable, and ‘flips’ easily.

So far: Defined a (linear) <u>score function</u>

$$f(x,W) = Wx + b$$

airplane	-3.45	-0.51	3.42
automobile	-8.87	**6.04**	4.64
bird	0.09	5.31	2.65
cat	**2.9**	-4.22	5.1
deer	4.48	-4.19	2.64
dog	8.02	3.58	5.55
frog	3.78	4.49	**-4.34**
horse	1.06	-4.37	-1.5
ship	-0.36	-2.09	-4.79
truck	-0.72	-2.93	6.14

Example class scores for 3 images for some W:

How can we tell whether this **W is good or bad**?

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So far: Defined a (linear) score function



airplane	-3.45	-0.51	3.42
automobile	-8.87	**6.04**	4.64
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horse	1.06	-4.37	-1.5
ship	-0.36	-2.09	-4.79
truck	-0.72	-2.93	6.14

TODO:

1. Define **a loss function that quantifies our unhappiness with the scores** across the training data.

2. Come up with **a way of efficiently finding the parameters that minimize the loss function. (optimization)**

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x, W) = Wx$ are:



cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**

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A **loss function tells how good our current classifier is**

Given a dataset of examples

$$\{(x_i, y_i)\}_{i=1}^{N}$$

Where x_i is image and y_i is (integer) label

Loss over the dataset is a sum of loss over examples:

$$L = \frac{1}{N} \sum_i L_i(f(x_i, W), y_i)$$

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Multiclass SVM loss:

Given an example (x_i, y_i)
where x_i is the image and
where y_i is the (integer) label,

and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \begin{cases} 0 & \text{if } s_{y_i} \geq s_j + 1 \\ s_j - s_{y_i} + 1 & \text{otherwise} \end{cases}$$

$$= \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

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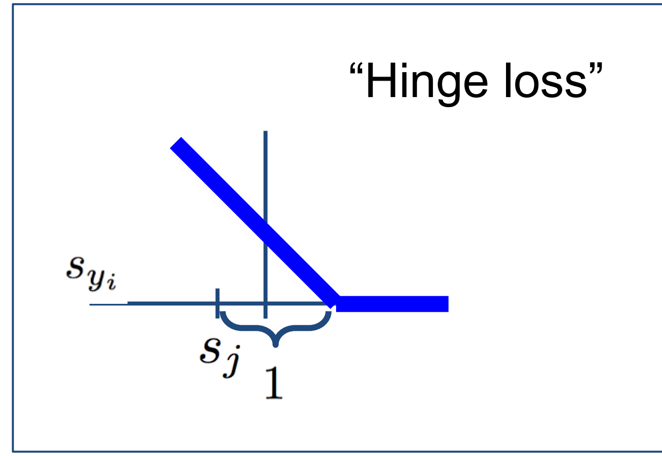

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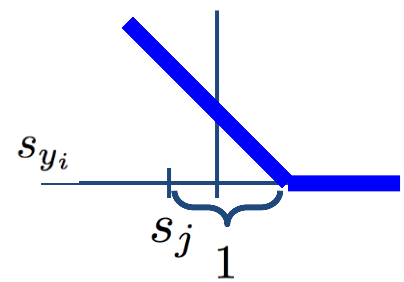
frog	-1.7	2.0	**-3.1**

Multiclass SVM loss:

"Hinge loss"



$$L_i = \sum_{j \neq y_i} \begin{cases} 0 & \text{if } s_{y_i} \geq s_j + 1 \\ s_j - s_{y_i} + 1 & \text{otherwise} \end{cases}$$

$$= \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

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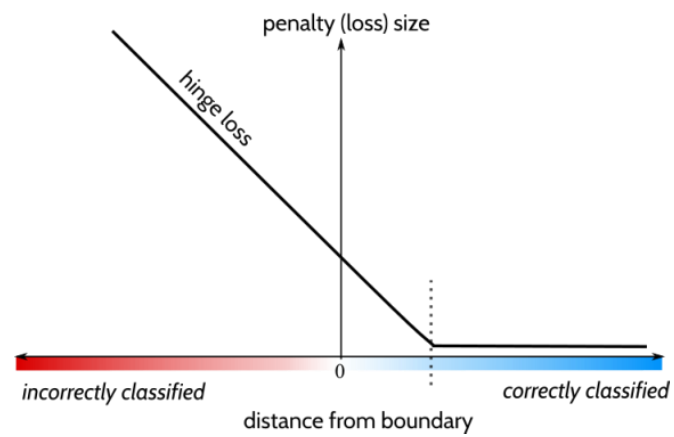

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x, W) = Wx$ are:

			
cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Losses:	2.9		

Multiclass SVM loss:

Given an example (x_i, y_i)
where x_i is the image and
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and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$\boxed{L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

= max(0, 5.1 - 3.2 + 1)
+max(0, -1.7 - 3.2 + 1)
= max(0, 2.9) + max(0, -3.9)
= 2.9 + 0
= 2.9

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x, W) = Wx$ are:

  

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Losses:	2.9	0	

Multiclass SVM loss:

Given an example (x_i, y_i)
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where y_i is the (integer) label,

and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$\boxed{L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

= max(0, 1.3 - 4.9 + 1)
+max(0, 2.0 - 4.9 + 1)
= max(0, -2.6) + max(0, -1.9)
= 0 + 0
= 0

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x, W) = Wx$ are:

  

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Losses:	2.9	0	12.9

Multiclass SVM loss:

Given an example (x_i, y_i)
where x_i is the image and
where y_i is the (integer) label,

and using the shorthand for the
scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$\boxed{L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

= max(0, 2.2 - (-3.1) + 1)
+max(0, 2.5 - (-3.1) + 1)
= max(0, 6.3) + max(0, 6.6)
= 6.3 + 6.6
= 12.9

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x, W) = Wx$ are:

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Losses:	2.9	0	12.9

Multiclass SVM loss:

Given an example (x_i, y_i)
where x_i is the image and
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scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$\boxed{L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

Loss over full dataset is average:

$$L = \frac{1}{N} \sum_{i=1}^{N} L_i$$

L = (2.9 + 0 + 12.9)/3
= **5.27**

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x, W) = Wx$ are:



cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Losses:	2.9	0	12.9

Multiclass SVM loss:

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where y_i is the (integer) label,

and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$\boxed{L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

Q: What happens to loss if car image scores change a bit?

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x, W) = Wx$ are:

  

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Losses:	2.9	0	12.9

Multiclass SVM loss:

Given an example (x_i, y_i)
where x_i is the image and
where y_i is the (integer) label,

and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$\boxed{L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

Q2: what is the min/max possible loss?

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x, W) = Wx$ are:



cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Losses:	2.9	0	12.9

Multiclass SVM loss:

Given an example (x_i, y_i)
where x_i is the image and
where y_i is the (integer) label,

and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$\boxed{L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

Q6: What if we used

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)^2$$

Multiclass SVM Loss

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

```python
def L_i_vectorized(x, y, W):
  scores = W.dot(x)
  margins = np.maximum(0, scores - scores[y] + 1)
  margins[y] = 0
  loss_i = np.sum(margins)
  return loss_i
```

$$f(x, W) = Wx$$

$$L = \frac{1}{N} \sum_{i=1}^{N} \sum_{j \neq y_i} \max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + 1)$$

E.g. Suppose that we found a W such that L = 0.
Is this W unique?

$$f(x, W) = Wx$$

$$L = \frac{1}{N} \sum_{i=1}^{N} \sum_{j \neq y_i} \max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + 1)$$

E.g. Suppose that we found a W such that L = 0.
Is this W unique?

No! 2W is also has L = 0!

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x,W) = Wx$ are:

  

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car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Losses:	2.9	0	

$$\boxed{L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

Before:

= max(0, 1.3 - 4.9 + 1)
+max(0, 2.0 - 4.9 + 1)
= max(0, -2.6) + max(0, -1.9)
= 0 + 0
= 0

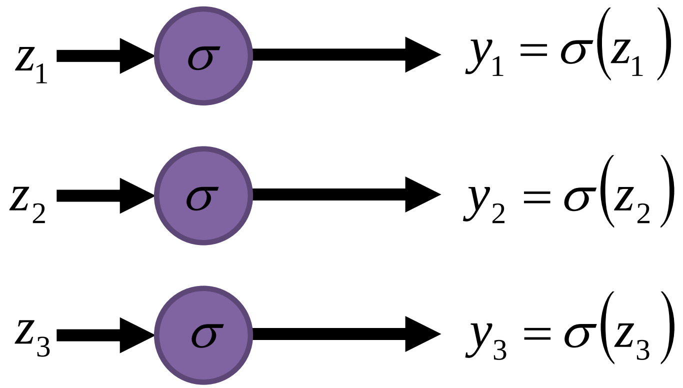
With W twice as large:

= max(0, 2.6 - 9.8 + 1)
+max(0, 4.0 - 9.8 + 1)
= max(0, -6.2) + max(0, -4.8)
= 0 + 0
= 0

Softmax Classifier (Multinomial Logistic Regression)

- Softmax layer as the output layer

<u>Ordinary Layer</u>



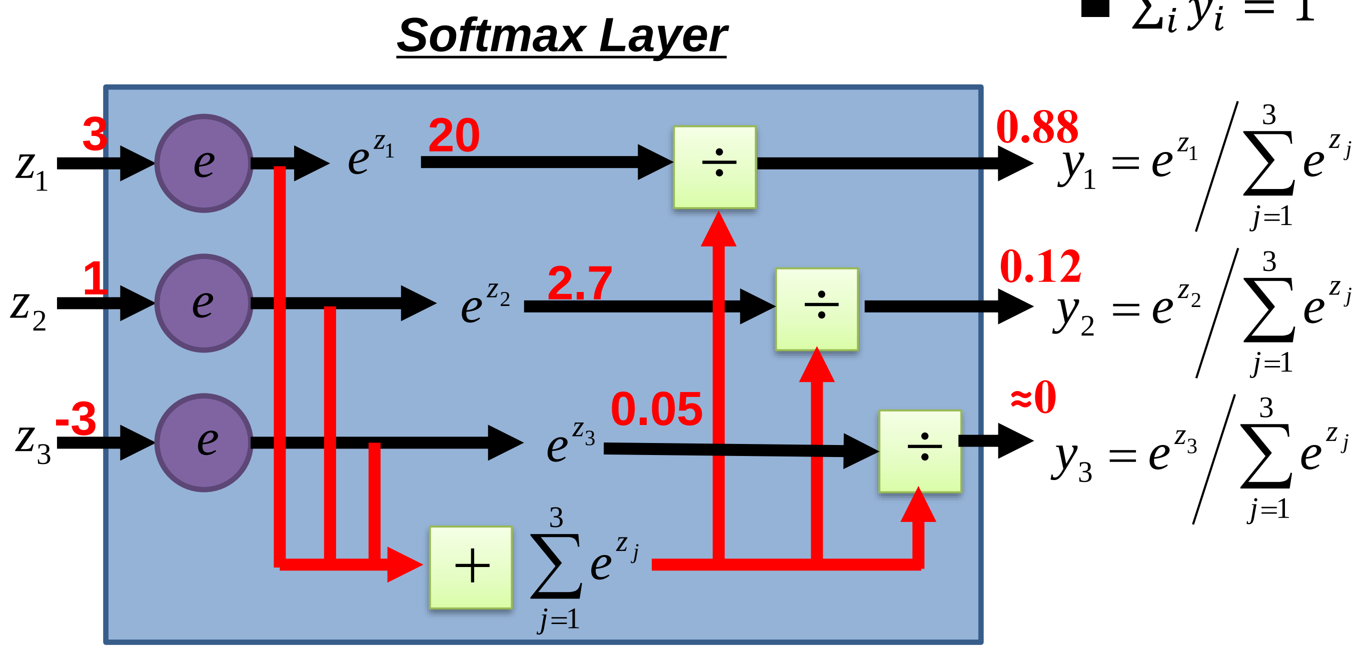
- In general, the output of network can be any value.
- May not be easy to interpret

Softmax Classifier (Multinomial Logistic Regression)

- Softmax layer as the output layer

Probability:

- $1 > y_i > 0$
- $\sum_i y_i = 1$



Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

cat	**3.2**
car	5.1
frog	-1.7

Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$\boxed{s = f(x_i; W)}$$

$$\boxed{P(Y = k|X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}}$$

Softmax Function

cat	**3.2**
car	5.1
frog	-1.7

Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$\boxed{s = f(x_i; W)}$$

$$\boxed{P(Y = k|X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}}$$

Softmax Function

Probabilities must be >= 0

cat	**3.2**		**24.5**
car	5.1	$\xrightarrow{\exp}$	164.0
frog	-1.7		0.18

unnormalized probabilities

Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$

Softmax Function

	scores	Probabilities must be >= 0	Probabilities must sum to 1
cat	**3.2**	**24.5**	**0.13**
car	5.1	164.0	0.87
frog	-1.7	0.18	0.00

$\xrightarrow{\text{exp}}$ unnormalized probabilities $\xrightarrow{\text{normalize}}$ probabilities

Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$\boxed{s = f(x_i; W)}$$

$$\boxed{P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}}$$ Softmax Function

Probabilities must be >= 0

Probabilities must sum to 1

	Unnormalized log-probabilities / logits		unnormalized probabilities		probabilities
cat	**3.2**	exp →	**24.5**	normalize →	**0.13**
car	5.1		164.0		0.87
frog	-1.7		0.18		0.00

Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$\boxed{s = f(x_i; W)}$$

$$\boxed{P(Y = k|X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}}$$ Softmax Function

$$L_i = -\log P(Y = y_i|X = x_i)$$

Probabilities must be >= 0

Probabilities must sum to 1

		exp		normalize		
cat	**3.2**	→	**24.5**	→	**0.13**	→ L_i = -log(0.13) = **0.89**
car	5.1		164.0		0.87	
frog	-1.7		0.18		0.00	
	Unnormalized log-probabilities / logits		unnormalized probabilities		probabilities	

Slide Credit: Fei-Fei Li, Justin Johnson, Serena Yeung, CS 231n

Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$\boxed{s = f(x_i; W)}$$

$$\boxed{P(Y = k|X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}}$$ Softmax Function

$$L_i = -\log P(Y = y_i|X = x_i)$$

	Unnormalized log-probabilities / logits	$\xrightarrow{\text{exp}}$	unnormalized probabilities	$\xrightarrow{\text{normalize}}$	probabilities
			Probabilities must be >= 0		Probabilities must sum to 1
cat	**3.2**		**24.5**		**0.13**
car	5.1		164.0		0.87
frog	-1.7		0.18		0.00

→ $L_i = -\log(0.13)$ **= 2.04**

Maximum Likelihood Estimation
Choose probabilities to maximize the likelihood of the observed data

Softmax Classifier (Multinomial Logistic Regression)

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Kullback–Leibler divergence

$$D_{KL}(P \| Q) = \sum_y P(y) \log \frac{P(y)}{Q(y)}$$

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Cross Entropy

$$H(P, Q) = H(p) + D_{KL}(P \| Q)$$

Slide Credit: Fei-Fei Li, Justin Johnson, Serena Yeung, CS 231n

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Softmax Function

Maximize probability of correct class

$$L_i = -\log P(Y = y_i | X = x_i)$$

Putting it all together:

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

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Q: What is the min/max possible loss L_i?

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A: min 0, max infinity

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Q2: At initialization all s will be approximately equal; what is the loss?

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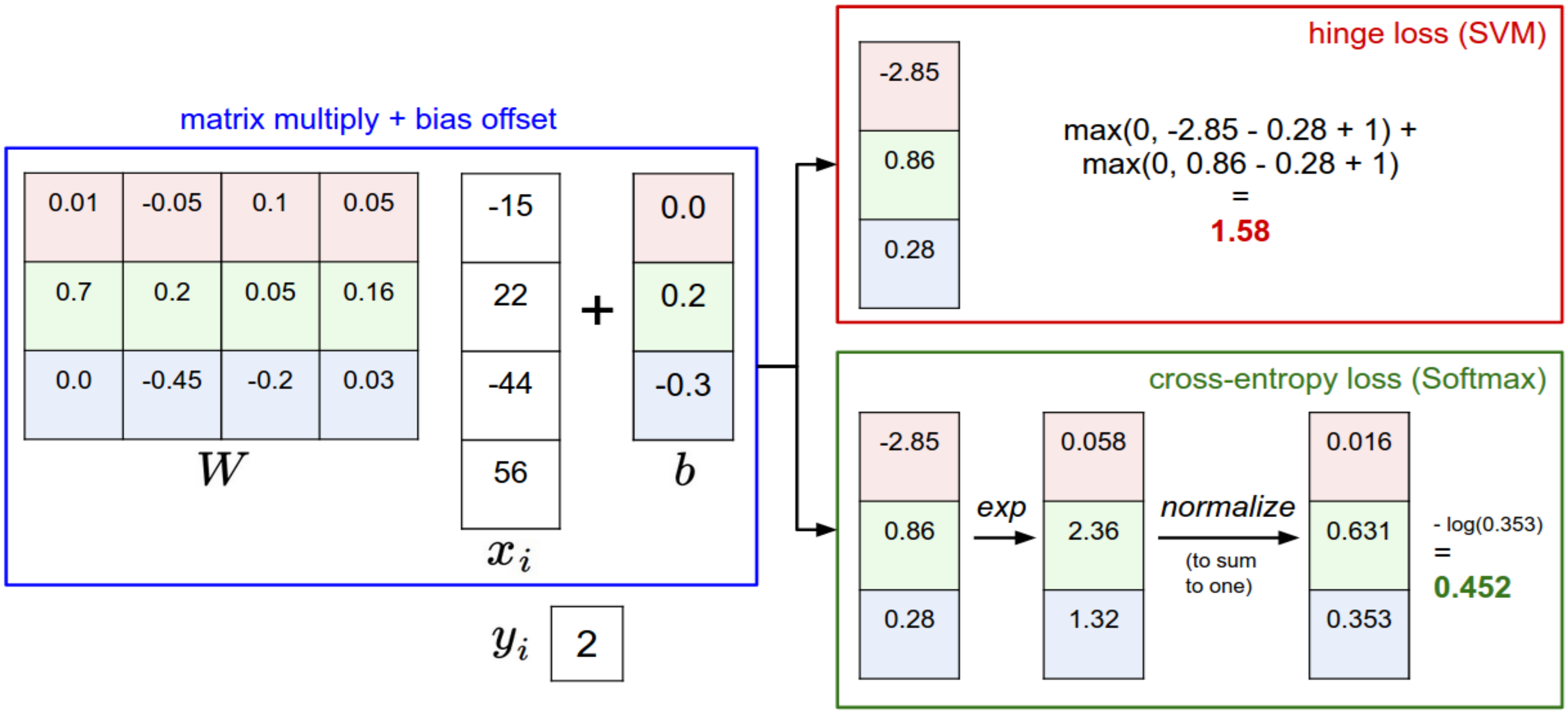
Q2: At initialization all s will be approximately equal; what is the loss?
A: log(C), eg log(10) ≈ 2.3

Softmax vs. SVM



$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

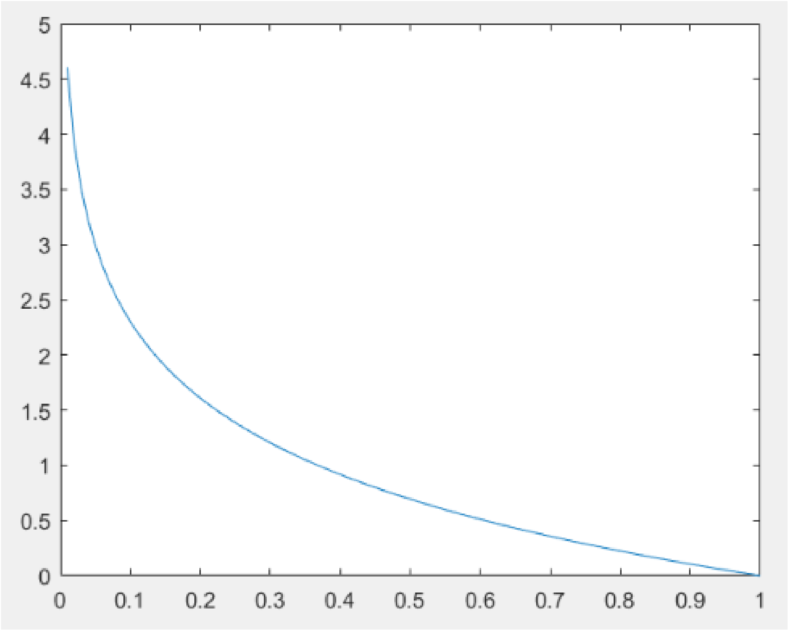
Slide Credit: Fei-Fei Li, Justin Johnson, Serena Yeung, CS 231n

Cross-entropy loss function

- Negative log-likelihood

$$L(\boldsymbol{x}, y; \boldsymbol{\theta}) = -\sum_j y_j \log p(c_j|\boldsymbol{x})$$

- Is it a good loss?
  - Differentiable
  - Cost decreases as probability increases



Examples

ground truth $= y$, prediction $= \hat{y}$

Mean Squared Error Loss

$$L = \frac{1}{m}\sum(y_i - \hat{y_i})^2$$

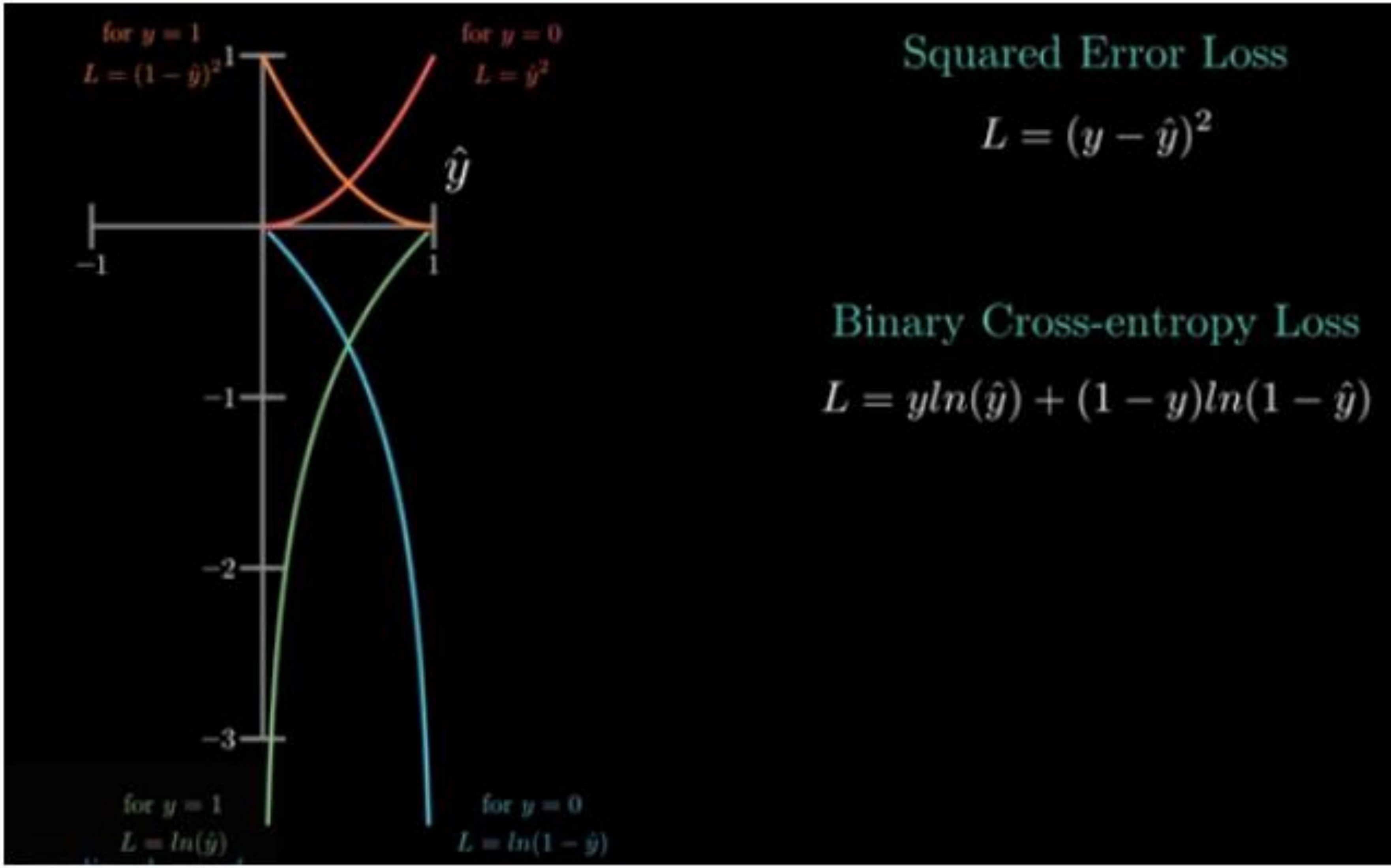
$$L = (y - \hat{y})^2$$

Binary Cross-entropy Loss

$$L = -\frac{1}{m}\sum[y_i ln(\hat{y_i}) + (1 - y_i)ln(1 - \hat{y_i})]$$

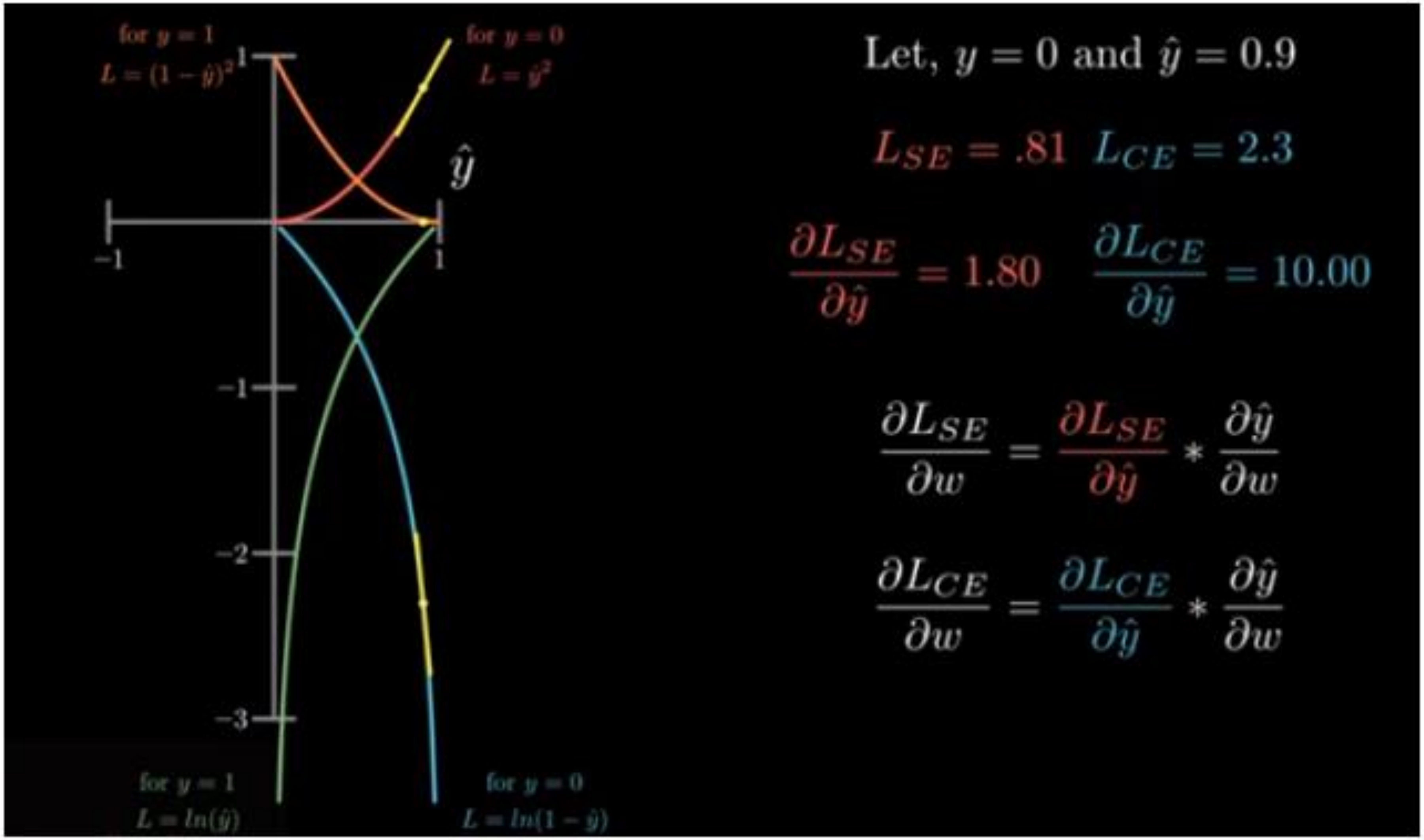
$$L = yln(\hat{y}) + (1 - y)ln(1 - \hat{y})$$

Examples



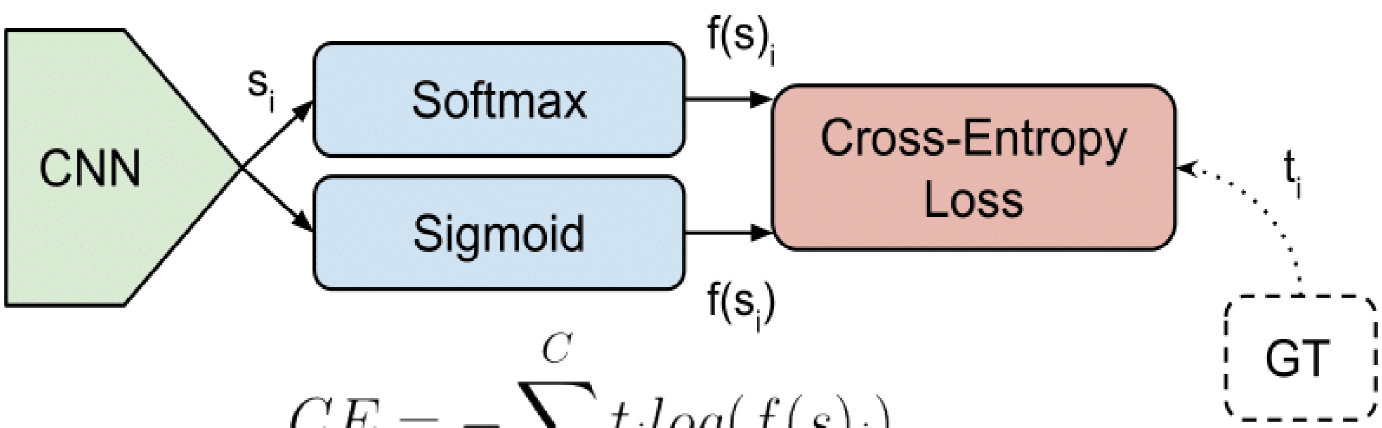
for $y = 1$
$L = (1 - \hat{y})^2$
for $y = 0$
$L = \hat{y}^2$
1
$\hat{y}$
-1
1
-1
-2
-3
for $y = 1$
$L = ln(\hat{y})$
for $y = 0$
$L = ln(1 - \hat{y})$
Squared Error Loss
$L = (y - \hat{y})^2$
Binary Cross-entropy Loss
$L = yln(\hat{y}) + (1 - y)ln(1 - \hat{y})$

Examples


for $y = 1$
$L = (1 - \hat{y})^2$
for $y = 0$
$L = \hat{y}^2$
$\hat{y}$
1
−1
−2
−3
for $y = 1$
$L = ln(\hat{y})$
for $y = 0$
$L = ln(1 - \hat{y})$
Let, $y = 0$ and $\hat{y} = 0.9$
$L_{SE} = .81$ $L_{CE} = 2.3$
$\frac{\partial L_{SE}}{\partial \hat{y}} = 1.80$ $\frac{\partial L_{CE}}{\partial \hat{y}} = 10.00$
$\frac{\partial L_{SE}}{\partial w} = \frac{\partial L_{SE}}{\partial \hat{y}} * \frac{\partial \hat{y}}{\partial w}$
$\frac{\partial L_{CE}}{\partial w} = \frac{\partial L_{CE}}{\partial \hat{y}} * \frac{\partial \hat{y}}{\partial w}$

Different Cross-Entropy Losses

- Categorical Cross-Entropy Loss
- Binary Cross-Entropy Loss
- Focal Loss and
- All those confusing names

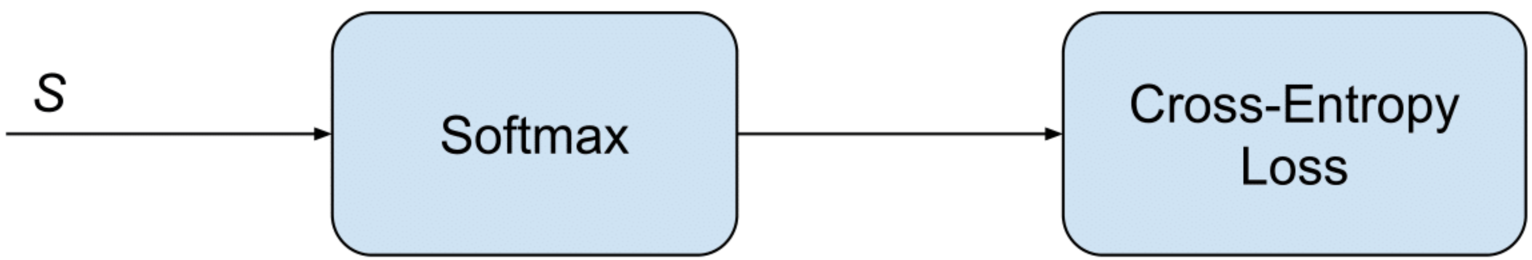


$$CE = -\sum_{i}^{C} t_i log(f(s)_i)$$

$$CE = -\sum_{i=1}^{C'=2} t_i log(f(s_i)) = -t_1 log(f(s_1)) - (1 - t_1) log(1 - f(s_1))$$

Categorical Cross-Entropy Loss

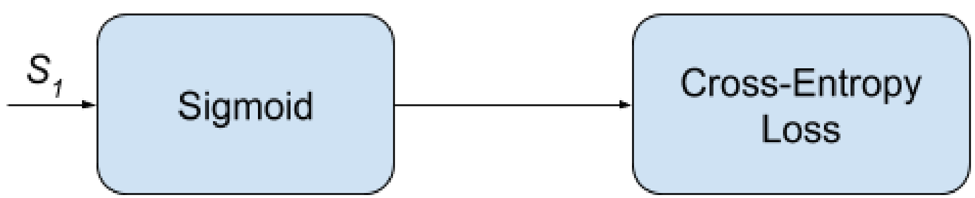
- Also called **Softmax Loss**. It is a **Softmax activation** plus a **Cross-Entropy loss**
- We use this loss when we train a DNN or CNN to output a probability over the C classes for each image (multi-class classification).



$$f(s)_i = \frac{e^{s_i}}{\sum_j^C e^{s_j}} \qquad CE = -\sum_i^C t_i log(f(s)_i)$$

Binary Cross-Entropy Loss

- Also called **Sigmoid Cross-Entropy loss**. It is a **Sigmoid activation** plus a **Cross-Entropy loss**.
- It's called **Binary Cross-Entropy Loss** because it sets up a binary classification problem between C′=2 classes for every class in C.



$$CE = -t_1 log(f(s_1)) - (1 - t_1) log(1 - f(s_1))$$

$$f(s_i) = \frac{1}{1 + e^{-s_i}}$$

Focal Loss

- **Focal Loss** was introduced by Lin et al., from Facebook. They claim to improve one-stage object detectors using **Focal Loss** to train a detector named "RetinaNet".
- **Focal loss** is a **Cross-Entropy Loss** that weights the contribution of each sample to the loss based in the classification error.
- **Focal loss** could also be considered a **Binary Cross-Entropy Loss** (Sigmoid activations + Cross-Entropy Loss)

$$FL = -\sum_{i=1}^{C=2}(1 - s_i)^{\gamma} t_i log(s_i) \qquad \gamma >= 0$$

γ=0, Focal Loss is equivalent to Binary Cross Entropy Loss.

Summary

- Linear classifiers
- Loss functions: Hinge and Softmax
- **Different loss functions**
- **Examples?**

- What's next?
  - Convolutional operations
  - Convolutional Neural Networks (CNN)