MAT274 Lecture Notes 2/8/2006

3.1 2nd order linear homogeneous Equation w/ constant coef.

A second order ODE has form

$$y'' + p(t)y' + q(t)y = g(t)$$

if $g(t)=0$ for all t the equation homogenous (HODE)

In this section we will look at homogenous equation w/ constant coefficient.

$$ay'' + by' + cy = 0 \qquad a, b, c = \text{numbers}$$

Initial conditions: typically have the form

$$y(t_0) = y_0 \quad y'(t_0) = y_0'$$

Thus the solution passes thru (t_0, y_0) and the slope of tangent at t_0 is y_0'

example

$y'' - y = 0$ 2nd order linear

solutions $y = e^t$ $y_2 = e^{-t}$

In general $y(t) = c_1 e^t + c_2 e^{-t}$ is a solution w/ c_1 and c_2 any constant

called general solution

$y'(t) = c_1 e^t - c_2 e^{-t}$

$y''(t) = c_1 e^t + c_2 e^{-t}$

$y'' - y = c_1 e^t + c_2 e^{-t} - (c_1 e^t + c_2 e^{-t}) = 0$

Now lets add initial conditions $y(0) = 3$ $y'(0) = 1$

find c_1 and c_2

$y(0) = 3 \rightarrow t=0$ and $y=3$ $3 = c_1 + c_2$

$y' = c_1 e^t - c_2 e^{-t} \rightarrow y'(0) = 1$ $1 = c_1 - c_2$

Solve $3 = c_1 + c_2$
$1 = c_1 - c_2$
$4 = 2c_1 \Rightarrow c_1 = 2 \quad c_2 = 1$

$$\boxed{y = 2e^{t} + e^{-t}}$$

In general

To solve a 2nd order HODE $ay'' + by' + cy = 0$ (*)
we begin by assuming a solution of the form
$y = e^{rt}$

plug in $y = e^{rt}$ $y' = re^{rt}$ $y'' = r^2 e^{rt}$

$ay'' + by' + cy = ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$

$e^{rt}(ar^2 + br + c) = 0$

exponential is never 0

$\boxed{ar^2 + br + c = 0} \rightarrow$ characteristic equation of HODE

Solve the equation for r

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we have 3 possibilities

- $b^2 - 4ac > 0 \Rightarrow$ 2 real solutions (r_1 and r_2)
  $\Rightarrow y = e^{r_1 t} \quad y = e^{r_2 t}$
- $b^2 - 4ac = 0 \Rightarrow$ one real solution r_1 (we will see @ 3.5)
- $b^2 - 4ac < 0 \Rightarrow$ 2 complex solutions (sec 3.4)

In this section we will assume we have 2 real solutions r_1 and r_2
$\Rightarrow$ general solution $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

example

find general solution of

$$y'' + y' - 12y = 0$$

characteristic equation:

$$r^2 + r - 12 = 0$$

$$(r-3)(r+4) = 0$$

$$r_1 = 3 \quad r_2 = -4$$

general solution: $y = C_1 e^{3t} + C_2 e^{-4t}$

example

$$4y'' - y = 0 \qquad y(-2) = 1 \qquad y'(-2) = 1$$

be careful

characteristic equation $4r^2 - 1 = 0$

$$r = \pm \frac{1}{2}$$

$$y = C_1 e^{\frac{t}{2}} + C_2 e^{-\frac{t}{2}}$$

Now find C_1 and C_2 using IC

$y(-2) = 1 \rightarrow 1 = C_1 e^{-1} + C_2 e$

$y' = \frac{C_1}{2} e^{t/2} - \frac{C_2}{2} e^{-t/2}$

$y'(-2) = -1 \rightarrow -1 = \frac{C_1}{2} e^{-1} - \frac{C_2}{2} e^{1}$

$$\begin{cases} 1 = C_1 e^{-1} + C_2 e \\ -1 = \frac{C_1}{2} e^{-1} - \frac{C_2}{2} e^{1} \end{cases} \rightarrow$$

$1 = C_1 e^{-1} + C_2 e$

$-2 = C_1 e^{-1} - C_2 e$

$-1 = 2C_1 e^{-1}$

$C_1 = -\frac{e}{2} \rightarrow C_2 = \frac{3}{2} e^{-1}$

plug in C_1 and C_2 into general solution

$$y = C_1 e^{t/2} + C_2 e^{-t/2} = -\frac{e}{2} e^{t/2} + \frac{3}{2} e^{-1} e^{-t/2}$$

$$\boxed{y = -\frac{e^{1+\frac{t}{2}}}{2} + \frac{3}{2} e^{-1-\frac{t}{2}}}$$ → Note it goes to $-\infty$

example

$2y'' - 3y' = 0 \quad y(0) = 1 \quad y'(0) = 3$

$2r^2 - 3r = 0 \quad r = 0, r = \frac{3}{2}$

$y = C_1 + C_2 e^{3/2 t}$

$y(0) = 1 \rightarrow 1 = C_1 + C_2$

$y' = \frac{3}{2} C_2 e^{\frac{3}{2}t} \quad y'(0) = 3 \rightarrow 3 = \frac{3}{2} C_2$

$C_2 = 2 \rightarrow C_1 = -1$

$y = -1 + 2e^{3/2 t}$

15.2

MA272 Lecture Notes 10/13/2005

Iterated Integrals over rectangles

suppose $f(x,y)$ is continuous over a rectangle $(a \le x \le b;\ c \le y \le d)$

want to evaluate $\iint_R f(x,y)\,dA$

1) $$\iint_R f(x,y)\,dA = \int_{x=a}^{b}\int_{y=c}^{y=d} f(x,y)\,dy\,dx$$

assume x is constant, integrate y

example

$$\iint_R x+y\,dA,\quad R= 0\le x\le 1\quad 2\le y\le 3$$

$$\int_0^1 \boxed{\int_2^3 x+y\,dy}\,dx = \int_0^1 xy+\frac{y^2}{2}\Big]_2^3\,dx = \int_0^1 3x+\frac{9}{2}-2x-2\,dx$$

$$=\int_0^1 x+\frac{5}{2}\,dx = \frac{x^2}{2}+\frac{5}{2}x\Big]_0^1 = 3$$

we can ~~do~~ also reverse the order!

Fubini's theorem

$$\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

example

$$\iint_R \frac{xy^2}{x^2+1}\,dA\quad R: 0\le x\le 1\quad -3\le y\le 3$$

$$\int_0^1\int_{-3}^{3}\frac{xy^2}{x^2+1}\,dy\,dx = \int_0^1 \frac{x}{x^2+1}\cdot\frac{y^3}{3}\Big]_{-3}^{3}\,dx = \int_0^1 18\left(\frac{x}{x^2+1}\right)dx \rightarrow$$

$$= 18\int_0^1 \frac{x}{x^2+1}\,dx = \frac{18}{2}\int \frac{1}{u}\,du = \frac{18}{2}\ln(x^2+1)\Big]_0^1 = 9\ln(2) - \ln(1)$$

$$= \boxed{9\ln(2)}$$

$u = x^2+1$
$du = 2x$

Now change order

$$\int_{-3}^{3}\int_0^1 \frac{xy^2}{x^2+1}\,dx\,dy = \int_{-3}^{3} y^2 \boxed{\int_0^1 \frac{x}{x^2+1}\,dx}\,dy$$

$$= \int_{-3}^{3} y^2 \cdot \frac{1}{2}\ln(x^2+1)\Big]_0^1\,dy =$$

$$= \int_{-3}^{3} y^2 \left(\tfrac{1}{2}\ln 2\right) dy = \frac{\ln 2}{2}\int_{-3}^{3} y^2\,dy$$

$$= \frac{\ln 2}{2}\, y^3\Big|_{-3}^{3} = \frac{9}{2}\ln(2) = \boxed{9\ln(2)}$$

example

$$\iint_R y\sin(xy)\,dA \qquad 1 \le x \le 2 \quad 0 \le y \le \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}}\int_1^2 y\sin(xy)\,dx\,dy = \int_0^{\frac{\pi}{2}} \cancel{y}\left[-\frac{\cos(xy)}{\cancel{y}}\right]\Bigg|_{x=1}^{x=2} dy$$

$$\int_0^{\frac{\pi}{2}} -\cos(2y) + \cos(y)\,dy = \frac{-\sin(2y)}{2} + \sin(y)\Big|_0^{\frac{\pi}{2}}$$

$$-\frac{\sin\left(\frac{\pi}{2}\right)}{2} + \sin\left(\frac{\pi}{2}\right) \ldots = \boxed{\sin\frac{\pi}{2} = 1}$$