MAT274 Lecture Notes 2/8/2006

3.1 2^{nd} order linear homogeneous equation w/ constant coef.

A second order ODE has form

$$y'' + p(t)y' + q(t)y = g(t)$$

if $g(t)=0$ for all t the equation homogenous (HODE)

In this section we will look at homogenous equation w/ constant coefficient.

$$ay'' + by' + cy = 0 \qquad a,b,c = \text{numbers}$$

<u>Initial conditions</u>: typically have the form

$$y(t_0)=y_0 \quad y'(t_0)=y_0'$$

Thus the solution passes thru (t_0, y_0) and the slope of tangent at t_0 is y_0'

<u>example</u>

$y''-y=0$ $\quad$ 2^{nd} order linear

Solutions $\quad y=e^t \quad y_2=e^{-t}$

In general $y(t)=c_1e^t+c_2e^{-t}$ is a solution w/ c_1 and c_2 any constant ← called general solution

$y'(t)= c_1e^t - c_2e^{-t}$

$y''(t)= c_1e^t + c_2e^{-t}$

$y''-y= c_1e^t + c_2e^{-t} - (c_1e^t+c_2e^{-t}) = 0$

Now lets add initial conditions $\quad y(0)=3 \quad y'(0)=1$

find c_1 and c_2

$y(0)=3 \rightarrow t=0$ and $y=3 \qquad 3=c_1+c_2$

$y' = c_1e^t - c_2e^{-t} \rightarrow y'(0)=1 \qquad 1=c_1-c_2$

solve $\begin{aligned} 3 &= c_1 + c_2 \\ 1 &= c_1 - c_2 \\ \hline 4 &= 2c_1 \rightarrow c_1 = 2 \quad c_2 = 1 \end{aligned}$

$$\boxed{y = 2e^{t} + e^{-t}}$$

In general

To solve a 2nd order HODE $ay'' + by' + cy = 0$ (*)
we begin by assuming a solution of the form
$$y = e^{rt}$$

plug in $y = e^{rt}$ $\quad y' = re^{rt}$ $\quad y'' = r^2 e^{rt}$

$$ay'' + by' + cy = ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$
$$e^{rt}(ar^2 + br + c) = 0$$
exponential is never 0

$\boxed{ar^2 + br + c = 0} \rightarrow$ characteristic equation of HODE

solve the equation for r

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we have 3 possibilities

- $b^2 - 4ac > 0 \Rightarrow$ 2 real solutions (r_1 and r_2)
  $\Rightarrow y = e^{r_1 t} \quad y = e^{r_2 t}$
- $b^2 - 4ac = 0 \Rightarrow$ one real solution r_1 (we will see @ 3.5)
- $b^2 - 4ac < 0 \Rightarrow$ 2 complex solutions (sec 3.4)

In this section we will assume we have 2 real solutions r_1 and r_2
$\Rightarrow$ general solution $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

example

find general solution of

$$y''+y'-12y=0$$

characteristic equation:

$$r^2+r-12=0$$

$$(r-3)(r+4)=0$$

$$r_1=3 \quad r_2=-4$$

general solution: $y=C_1e^{3t}+C_2e^{-4t}$

example

$4y''-y=0 \qquad y(-2)=1 \qquad y'(-2)=1$

be careful

characteristic equation $\quad 4r^2-1=0$

$$r=\pm\frac{1}{2}$$

$$y=C_1e^{\frac{t}{2}}+C_2e^{-\frac{t}{2}}$$

Now find C_1 and C_2 using IC

$y(-2)=1 \rightarrow 1=C_1e^{-1}+C_2e$

$y'=\frac{C_1}{2}e^{t/2}-\frac{C_2}{2}e^{-t/2}$

$y'(-2)=-1 \rightarrow -1=\frac{C_1}{2}e^{-1}-\frac{C_2}{2}e^{1}$

$$\begin{cases} 1=C_1e^{-1}+C_2e \\ -1=\frac{C_1}{2}e^{-1}-\frac{C_2}{2}e^{1} \end{cases} \longrightarrow \begin{aligned} 1&=Ce^{-1}+C_2e \\ -2&=C_1e^{-1}-C_2e \end{aligned}$$

$$-1=2C_1e^{-1}$$

$$C_1=-\frac{e}{2}\rightarrow C_2=\frac{3}{2}e^{-1}$$

plug in C_1 and C_2 into general solution

$$y=C_1e^{t/2}+C_2e^{-t/2}=-\frac{e}{2}e^{t/2}+\frac{3}{2}e^{-1}e^{-t/2}$$

$$\boxed{y=-\frac{e^{1+\frac{t}{2}}}{2}+\frac{3}{2}e^{-1-\frac{t}{2}}} \longrightarrow \text{Note it goes to } -\infty$$

example

$2y'' - 3y' = 0 \quad y(0)=1 \quad y'(0)=3$

$2r^2 - 3r = 0 \quad r=0, r=\frac{3}{2}$

$y = c_1 + c_2 e^{3/2t}$

$y(0)=1 \rightarrow 1 = c_1 + c_2$

$y' = \frac{3}{2} c_2 e^{\frac{3}{2}t} \quad y'(0)=3 \rightarrow 3 = \frac{3}{2} c_2$

$c_2 = 2 \rightarrow c_1 = -1$

$y = -1 + 2e^{3/2t}$

15.2

MA227 Lecture Notes 10/13/2005

Iterated Integrals over rectangles

suppose $f(x,y)$ is continuous over a rectangle ($a \le x \le b$; $c \le y \le d$)

want to evaluate $\iint_R f(x,y)\,dA$

1) $\iint_R f(x,y)\,dA = \int_{x=a}^{b} \int_{y=c}^{y=d} f(x,y)\,dy\,dx$

assume x is constant, integrate y

example

$\iint_R x+y\,dA$, $R = 0 \le x \le 1 \quad 2 \le y \le 3$

$\int_0^1 \boxed{\int_2^3 x+y\,dy}\,dx = \int_0^1 xy + \frac{y^2}{2}\Big]_2^3 dx = \int_0^1 3x + \frac{9}{2} - 2x - 2\,dx$

$= \int_0^1 x + \frac{5}{2}\,dx = \frac{x^2}{2} + \frac{5}{2}x\Big]_0^1 = \boxed{3}$

we can ~~ato~~ also reverse the order!

Fubini's theorem

$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$

example

$\iint_R \frac{xy^2}{x^2+1}\,dA \quad R: 0 \le x \le 1 \quad -3 \le y \le 3$

$\int_0^1 \int_{-3}^{3} \frac{xy^2}{x^2+1}\,dy\,dx = \int_0^1 \frac{x}{x^2+1} \cdot \frac{y^3}{3}\Big]_{-3}^{3} dx = \int_0^1 18\left(\frac{x}{x^2+1}\right)dx$ →

$= 18\int_0^1 \frac{x}{x^2+1}\,dx = \frac{18}{2}\int \frac{1}{u}\,du = \frac{18}{2}\ln(x^2+1)\Big]_0^1 = 9\ln(2) - \ln(1)$

$= \boxed{9\ln(2)}$

$u = x^2+1$
$du = 2x$

Now change order

$\int_{-3}^{3}\int_0^1 \frac{xy^2}{x^2+1}\,dx\,dy = \int_{-3}^{3} y^2 \boxed{\int_0^1 \frac{x}{x^2+1}\,dx}\,dy$

$= \int_{-3}^{3} y^2 \cdot \frac{1}{2}\ln(x^2+1)\Big]_0^1\,dy =$

$= \int_{-3}^{3} y^2 (\frac{1}{2}\ln 2\,dy = \frac{\ln 2}{2}\int_{-3}^{3} y^2\,dy$

$= \frac{\ln 2}{2} y^3\Big|_{-3}^{3} = \frac{18}{2}\ln(2) = \boxed{9\ln(2)}$

example

$\iint_R y\sin(xy)\,dA \quad 1 \le x \le 2 \quad 0 \le y \le \frac{\pi}{2}$

$\int_0^{\frac{\pi}{2}}\int_1^2 y\sin(xy)\,dx\,dy = \int_0^{\frac{\pi}{2}} \cancel{y}\left[-\frac{\cos(xy)}{\cancel{y}}\right]\Big|_{x=1}^{x=2}\,dy$

$\int_0^{\frac{\pi}{2}} -\cos(2y) + \cos(y)\,dy = \frac{-\sin(2y)}{2} + \sin(y)\Big|_0^{\frac{\pi}{2}}$

$-\frac{\sin(\frac{\pi}{2})}{2} + \sin(\frac{\pi}{2}) \ldots = \boxed{\sin\frac{\pi}{2} = 1}$