Ordinary and Partial Differential Equations and Coordinate Geometries (MAT 103)

Topic Which Will Cover Today: Formation of Differential Equations

Questions:

1. Find the differential equation of the followings
   - $Ae^{x} + Be^{-x} = y$
   - $Ax^2 + By^2 = 1$
   - $y = Ax^2 + Bx + C$ (Home Work)
   - $y = C_1e^{-x} + C_2e^{-3x}$

2. Find the differential equation of all circles passing through the origin and having their circles on the x axis.

3. Find the differential equation of all circles passing through the origin and having their circles on the y axis (Home work)

4. In each of the following eliminate arbitrary constant,

   a) $y = (A\cos 2x + B\sin 2x)e^{x}$
   b) $y^2 = 4a(x+a)$
   c) $y = a + \log bx + c(\log x)^2 + 3x^2$
   d) $y = Ax + \dfrac{B}{x}$ H.W
   e) $y = Ae^{3x} + Be^{-2x} + \sin 5x$

Working Rule:

The main concern of your problem here is to eliminate the arbitrary constants by continuously differentiating the given equation.

Question 1:

Find the differential equation of the followings

a) $Ae^x + Be^{-x} = y$
b) $Ax^2 + By^2 = 1$
c) $y = Ax^2 + Bx + C$
d) $y = C_1e^{-x} + C_2e^{-3x}$

Solutions:

a) Given,

$$Ae^x + Be^{-x} = y$$

Differentiating the above equation with respect to,

$$A\frac{d}{dx}(e^x) + B\frac{d}{dx}(e^{-x}) = \frac{dy}{dx}$$

$$=> \quad A\,e^x - Be^{-x} = \frac{dy}{dx}$$

$$=> \quad A\frac{d}{dx}(e^x) - B\frac{d}{dx}(e^{-x}) = \frac{d^2y}{dx^2}$$

$$=> \quad Ae^x + Be^{-x} = \frac{d^2y}{dx^2}$$

$\therefore\ y = \frac{d^2y}{dx^2}$ (Answer)

b) Given,

$$Ax^2 + By^2 = 1$$

Differentiating the above equation with respect to,

$$=> A\frac{d}{dx}(x^2) + B\frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$

$$=> 2xA + 2yB\frac{dy}{dx} = 0$$

$$=> \frac{y}{x}\frac{dy}{dx} = -\frac{A}{B}$$

$$=> \frac{y}{x}\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2}\right) = 0$$

$$=> \frac{y}{x}\frac{d^2y}{dx^2} + \frac{1}{x}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x^2}\frac{dy}{dx} = 0$$

$=> xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$ (answer)

c) Home work

d) Given,

$y = C_1e^{-x} + C_2e^{-3x}$ --------------------(1)

Differentiating equation (1) with respect to x,

$\frac{dy}{dx} = C_1\frac{d}{dx}(e^{-x}) + C_2\frac{d}{dx}(e^{-3x})$

$=> \frac{dy}{dx} = -C_1e^{-x} - 3C_2e^{-3x}$ ------------------------(2)

$=> \frac{d^2y}{dx^2} = C_1e^{-x} + 9C_2e^{-3x}$ ------------------------(3)

Adding equation (2) and equation (3)

$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 6C_2e^{-3x}$ --------------------------(4)

Adding equation (1) and equation (2)

$y + \frac{dy}{dx} = -2C_2e^{-3x}$ ------------------------(5)

Now to remove C_2 , 3×eqn(5) +eqn(4),

$$3y + 3\frac{dy}{dx} + \frac{dy}{dx} + \frac{d^2y}{dx^2} = 0$$

∴ $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0$ (answer)

Question 2: Find the differential equation of all circles passing through the origin and having their circles on the x axis.

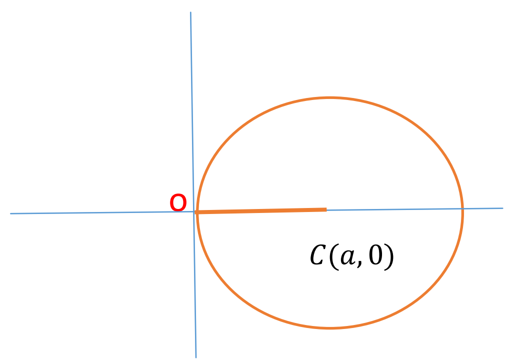
Solution: We know,

The equation of a circle with centers at (h, k) with radius a is,

$(x - h)^2 + (y - k)^2 = a^2$ ---------------(1)

When the circle passes through origin and centers lies on x axis,

$h = a$ and $k = 0$



so, equation (1) becomes,

$$(x - a)^2 + y^2 = a^2$$

$$=> x^2 - 2ax + a^2 + y^2 = a^2$$

$=> x^2 - 2ax + y^2 = 0$ -----------------(2)

$$=> 2x - 2a - 2y\frac{dy}{dx} = 0$$

$=> x - a + y\frac{dy}{dx} = 0$ ------------------(3)

From equation (2)

$$x^2 - 2ax + y^2 = 0$$

$$=> \ -2ax = \ -(x^2 - y^2)$$

$$=> a = \frac{x^2 - y^2}{2x}$$

Substituting the value of 'a' in equation (3),

$$x - \frac{x^2 + y^2}{2x} + y\frac{dy}{dx} = 0$$

$$=> \frac{2x^2 - x^2 - y^2}{2x} + y\frac{dy}{dx} = 0$$

$$=> \frac{x^2 - y^2}{2x} + y\frac{dy}{dx} = 0$$

$$=> y\frac{dy}{dx} = \ - \frac{(x^2 - y^2)}{2x}$$

$\therefore \ \frac{dy}{dx} = \ -\frac{(x^2-y^2)}{2xy}$ (answer)

Question 3: In each of the following eliminate arbitrary constant,

$$a)\ x^2y = 1 + Cx$$

b)$y = C_1 \cos x + C_2 \sin x$(H.W)

c) $y = x + C_1e^{-x} + C_2e^{-3x}$

Solution:

a) Given,

$$x^2y = 1 + Cx$$

Differentiating with respect to x,

$$2xy + x^2\frac{dy}{dx} = 0 + C$$

Again differentiating with respect to x

$$=> 2\left(y + x\frac{dy}{dx}\right) + 2x\ \frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = 0$$

$$=> \ 2y + 2x\ \frac{dy}{dx} + 2x\ \frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = 0$$

$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0$ (ans.)

c) Given,

$y = x + C_1e^{-x} + C_2e^{-3x}$ ------------------------(1)

$\frac{dy}{dx} = 1 - C_1e^{-x} - 3C_2e^{-3x}$ -------------------------(2)

$=> \frac{d^2y}{dx^2} = 0 + \ C_1e^{-x} + 9C_2e^{-3x}$ --------------------(3)

Adding equation (2) and equation (3),

$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 6C_2e^{-3x} + 1$--------------------------(4)

Adding equation (1) and equation (2),

$y + \frac{dy}{dx} = x + 1 - 2C_2e^{-3x}$ ------------------------(5)

Now to remove C_2 , 3×eqn(5) +eqn(4),

$$3y + 3\frac{dy}{dx} + \frac{dy}{dx} + \frac{d^2y}{dx^2} = 3x + 3 + 1$$

$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3x + 4$ (ans.)

Short Questions:

1. What is the main concept we should keep in mind to form DE(differential equations)?