Chapter One

General Solution Method of Higher Order Linear Differential Equation with Constant Coefficient

The general form of higher order linear differential equation with constant co-efficient is,

$$\boldsymbol{a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + a_{n-1}\frac{dy}{dx} + a_n = Q(x)} \text{---------}\textbf{(1)}$$

Here, $a_0,, a_1, a_2, \ldots, a_n$ are constants.

Note That: If $Q(x) = 0$ in equation (1), then the differential is called higher order homogeneous linear differential equation with constant coefficient.

Solution Method:

To solve higher order differential equations we divide the equations into two parts and then find solutions from that parts. Then finally we obtain the general solutions.

$$\boldsymbol{a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + a_{n-1}\frac{dy}{dx} + a_n = 0}$$ **Part one**

$\boldsymbol{Q(x)}$ **Part two**

Finding solution from part one:

$$\boldsymbol{a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + a_{n-1}\frac{dy}{dx} + a_n = 0} \quad \text{------}\textbf{(2)}$$

Let $y = e^{mx}$ is the trial solution of equation(2), then auxiliary equation will be

$$\boldsymbol{a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \ldots\ldots + a_{n-1}m + a_n = 0} \quad \text{------}\textbf{(3)}$$

If we solve equation (3),depending on the values of m the following three cases arises,

Case 1: (Roots of eqn (3) are real and unequal) :

If $m = \alpha_1, \alpha_2, \ldots., \alpha_n$ are real and unequal, then the particular solution of eqn (2) is

$$y_p = C_1 e^{\alpha_1 x} + C_2 e^{\alpha_2 x} + \cdots + C_n e^{\alpha_n x}$$

Case 2: (Roots of eqn (3) are real and equal) :

If $m = \alpha_1, \alpha_2, \ldots., \alpha_n$ are real and equal, then the particular solution of eqn (2) is

$$y_p = (C_1 + C_2 x + C_3 x^2 + \cdots + C_n x^n)\, e^{\alpha x}$$

Case 3: (Roots of eqn (3) are imaginary) :

If $m = \alpha_1 + i\beta_1, \alpha_2 + i\beta_2, \ldots., \alpha_n + i\beta_n$ are imaginary, then the particular solution of eqn (2) is

$$y_p = (C_1 \cos\beta_1 x + C_2 \sin\beta_1 x)\, e^{\alpha_1 x} + (C_3 \cos\beta_2 x + C_4 \sin\beta_2 x)\, e^{\alpha_2 x} + \ldots..$$

$$\ldots + (C_{n-1} \cos\beta_n x + C_n \sin\beta_n x)\, e^{\alpha_n x}$$

Note That: The solution we get from part one is known as particular solution.

Examples:

1) $2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$

2) $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0$

3) $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 25y = 0$

4) $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 18y = 0$

Excercise:

Solve the following higher order differential equations:

1. $2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$
2. $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0$
3. $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 25y = 0$
4. $\frac{d^4y}{dx^4} + 4y = 0$
5. $\frac{d^3y}{dx^3} + 8y = 0$
6. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$
7. $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$
8. $\frac{d^2s}{dt^2} + 8\frac{ds}{dt} + 25s = 0, \quad t = 0, \quad s = -4, \quad \frac{ds}{dt} = 4$
9. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0, \quad x = 0, \quad y = 1, \quad \frac{dy}{dx} = 0$
10. $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0, \quad t = 0, \quad x = 2, \quad \frac{dx}{dt} = 0$