Exact Differential Equation:

Theorem 1:

Consider the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad -(1)$$

where M and N have continuous first partial derivatives at all points (x,y) in a rectangular domain D.

The differential equation is exact iff

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

Theorem 2:

Suppose the differential equation (1) satisfies the differentiability requirements of theorem 1 and is exact in a rectangular domain D. Then a one parameter family of solutions of this differential equation is given by $F(x,y)=c$ where F is a function such that,

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

for all $(x,y) \in D$.

where c is an arbitrary constant.

$\Rightarrow g(y) = y^2 + c_1$

Now from (iii) we can write

$$f(x,y) = x^3 + 2x^2y + y^2 + c_1$$

Therefore the solution of the given diff. equ^n is

$$f(x,y) = x^3 + 2x^2y + y^2 + c_1 = c_2$$

$$\Rightarrow x^3 + 2x^2y + y^2 = c \quad [\text{where } c = c_1 - c_2]$$

Ans.

2. Evaluate $(x^3 + xy^2)dx + (y^3 + x^2y)dy = 0$

Solution:

Given that,

$$(x^3 + xy^2)dx + (y^3 + x^2y)dy = 0 \quad \text{——— (1)}$$

Let, $M = x^3 + xy^2$ and $N = y^3 + x^2y$

$$\frac{\partial M}{\partial y} = 2xy \qquad \frac{\partial N}{\partial x} = 2xy$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So the given equation is exact and the solution of (1) is,

$$\underbrace{\int (x^3 + xy^2)\,dx}_{y \text{ const.}} + \underbrace{\int y^3\,dy}_{x \text{ free term}} = 0$$

$$\Rightarrow \frac{x^4}{4} + \frac{y^2x^2}{2} + \frac{y^4}{4} = c_1$$

$\therefore x^4 + 2x^2y^2 + y^4 = c$ [where $c = 4c_1$]

Ans.

3. Solve $(1+e^{x/y})dx + e^{x/y}(1-x/y)dy = 0$

Solution:

Given that,

$$\left(1+e^{x/y}\right)dx + e^{x/y}(1-x/y)\,dy = 0 \quad \text{--- (1)}$$

Let, $M = (1+e^{x/y})$ and $N = e^{x/y}(1-x/y)$

$$\frac{\partial M}{\partial y} = -e^{x/y}\, x/y^2$$

$$= -\frac{x}{y^2}e^{x/y}$$

$$\frac{\partial N}{\partial x} = e^{x/y}\left(-\frac{1}{y}\right) + (1-x/y)\,e^{x/y}\left(+\frac{1}{y}\right)$$

$$= -\frac{1}{y}e^{x/y} + \frac{1}{y}e^{x/y} - \frac{x}{y^2}e^{x/y}$$

$$= -\frac{x}{y^2}e^{x/y}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

so the equation is exact and its solution is.

$$\int(1+e^{x/y})\,dx + \int 0\,dy = 0$$

$$\Rightarrow x + \frac{e^{x/y}}{1/y} = c$$

$$\therefore x + ye^{x/y} = c. \text{ Ans.}$$

E · 4. $y\sin 2x\,dx - (y^2+\cos^2 x)\,dy = 0$

Solution :

Here $M = y\sin 2x$ $N = -(y^2+\cos^2 x)$

$\frac{\partial M}{\partial y} = \sin 2x$ $\frac{\partial N}{\partial x} = -(2\cos x(-\sin x))$

$= 2\cos x \sin x$

$= \sin 2x$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

So the equation is exact and its integration form be,

$\int y\sin 2x\,dx - \int y^2\,dy = c$

$\Rightarrow -y\cos 2x \cdot \frac{1}{2} - \frac{y^3}{3} = c$

$\Rightarrow \frac{3}{2} y\cos 2x \cancel{+} y^3 = c.$

$\Rightarrow 3y\cos 2x \cancel{+} 2y^3 + c_1 = 0$ Ans.

5. $(2x-y+1)\,dx + (2y-x-1)\,dy = 0$

Solution: Let $M = 2x-y+1$, $N = 2y-x-1$

$\frac{\partial M}{\partial y} = -1$ $\frac{\partial N}{\partial x} = -1$

Non-exact differential equation Make exact :

Integrating factor :

When a de is not exact but becomes exact by multiplying a factor. This factor is called integrating factor.

Rule - 1 :

If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, only function of x,

then I.F. $= e^{\int f(x)dx}$

Rule - 2 : If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = f(y)$, only function of y

then I.F. $= e^{\int f(y)dy}$

Rule - 3 :

If M and N are both homogeneous function in x, y of degree n, then

I.F. $= \dfrac{1}{Mx+Ny}$, $Mx+Ny \neq 0$

Rule - 4 : If the equation of the form

$y f(xy)\, dx + x g(xy) dy = 0$

$$I.F. = \frac{1}{Mx - Ny}, \quad Mx - Ny \neq 0$$

1. $(2x^2+y^2+x)\,dx + xy\,dy = 0$

Solution: $(2x^2+y^2+x)\,dx + xy\,dy = 0$ ——— (I)

Let $M = 2x^2+y^2+x$ $\qquad N = xy$

$$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = y$$

Here, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

So the given DE is not exact.

Here,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-y}{xy} = \frac{2-1}{x} = \frac{1}{x}$$

Therefore $I.F. = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

Now multiplying (I) by x

$(2x^3 + xy^2 + x^2)\,dx + x^2y\,dy = 0$ ——— (II)

Now, $M' = 2x^3 + xy^2 + x^2$

$N' = x^2y$

$$\frac{\partial M'}{\partial y} = 2xy \qquad \frac{\partial N'}{\partial x} = 2xy$$

$$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$$

Solution of DE (ii) is,

$$\int (2x^3 + xy^2 + x^2)\, dx + \int x^2 y\, dy = c$$

$$\Rightarrow \cancel{2} \cdot \frac{x^4}{\cancel{4}_2} + \frac{x^2y^2}{2} + \frac{x^3}{3} + 0 = c$$

$$\Rightarrow \frac{3x^4 + 3x^2y^2 + 2x^3}{6} = c$$

$$\therefore 3x^4 + 3x^2y^2 + 2x^3 = c_1$$. Ans.

2. $x\,dx + y\,dy + (x^2+y^2)\,dy = 0$

Solution: $x\,dx + y\,dy + (x^2+y^2)\,dy = 0$ —— (1)

$$\Rightarrow x\,dx + (y + x^2 + y^2)\,dy = 0$$

Let $M = x$ $\quad$ $N = y + x^2 + y^2$

$\frac{\partial M}{\partial y} = 0$ $\quad$ $\frac{\partial N}{\partial x} = 2x$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

So this is not an exact diff. equ^n.

Here, $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = \frac{0-2x}{-x} = 2$$

$\therefore$ I.F. $= e^{\int 2 dy} = e^{2y}$

So from (i) we get,

$$x e^{2y} dx + (y + x^2 + y^2) e^{2y} dy = 0 \quad \text{——(ii)}$$

So the solution of (ii) is,

$$\int x e^{2y} dx + \int (y + y^2) e^{2y} dy = c$$

$$\Rightarrow e^{2y} \frac{x^2}{2} + (y + y^2) \int e^{2y} dy - \int \left\{ \frac{d}{dy}(y + y^2) \int e^{2y} dy \right\} dy = c$$

$$\Rightarrow \frac{1}{2} x^2 e^{2y} + \frac{1}{2}(y + y^2) e^{2y} - \int \frac{1+2y}{2} e^{2y} dy = c$$

$$= \frac{1}{2} x^2 e^{2y} + \frac{1}{2}(y + y^2) e^{2y} - \left[\left(\frac{1}{2} + y\right) \frac{e^{2y}}{2} - \int \left\{ \frac{d}{dy}\left(\frac{1}{2} + y\right) \int e^{2y} dy \right\} dy \right] = c$$

$$= \frac{1}{2}(y + x^2 + y^2) e^{2y} - \frac{1+2y}{4} e^{2y} + \int \frac{e^{2y}}{2} dy = c$$

$$\Rightarrow \frac{1}{2}(y + x^2 + y^2) e^{2y} - \frac{1+2y}{4} e^{2y} + \frac{e^{2y}}{4} = c$$

$$\Rightarrow e^{2y}\left(\cancel{\frac{1}{2} y} + \frac{1}{2} x^2 + \frac{1}{2} y^2 - \cancel{\frac{1}{4}} - \cancel{\frac{1}{2} y} + \cancel{\frac{1}{4}} \right) = c$$

$$\Rightarrow \frac{1}{2} e^{2y} (x^2 + y^2) = c.$$

Ans.

3. $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

Solution: $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$ ——— (1)

Here, $M = x^2y - 2xy^2$ $\quad N = -x^3 + 3x^2y$

$$\frac{\partial M}{\partial y} = x^2 - 4xy \qquad \frac{\partial N}{\partial x} = 6xy - 3x^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Now, $Mx + Ny = x^3y - 2x^2y^2 + 3x^2y^2 - x^3y$

$= x^2y^2 \neq 0$

$$\therefore \text{I.F.} = \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$$

Multiplying (1) by $\frac{1}{x^2y^2}$, we have,

$$\frac{1}{x^2y^2}(x^2y - 2xy^2)\,dx - \frac{1}{x^2y^2}(x^3 - 3x^2y)\,dy = 0$$

$$\Rightarrow \left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$$

which is an exact equation.

So the required solution is,

$$\int\left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}\,dy = c$$

$$\Rightarrow \frac{x}{y} - 2\ln x + 3\ln y = c. \quad \text{Ans.}$$

4. $(1-xy)y\,dx - x(1+xy)\,dy = 0$

Solution:

$$(1-xy)y\,dx - x(1+xy)\,dy = 0 \quad \text{--- (1)}$$

$M = (1-xy)y$ $\qquad N = -x(1+xy)$

$$\frac{\partial M}{\partial y} = 1 - xy - xy \qquad \frac{\partial N}{\partial x} = -(1+xy) - x \cdot y$$

$$= 1 - 2xy \qquad = -1 - 2xy$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$Mx - Ny = (1-xy)xy + (1+xy)xy$$
$$= 2xy$$

Now, $\quad I.F. = \dfrac{1}{Mx-Ny} = \dfrac{1}{2xy}$

Multiplying (1) by $\frac{1}{2xy}$ we get,

$$\frac{1}{2xy}(1-xy)y\,dx - \frac{x}{2xy}(1+xy)\,dy = 0$$

$$\Rightarrow \frac{1-xy}{2x}\,dx - \frac{1}{2y}(1+xy)\,dy = 0$$

$$\Rightarrow \left(\frac{1}{2x} - \frac{1}{2}y\right)dx - \left(\frac{1}{2y} + \frac{1}{2}x\right)dy = 0$$

So the solution is,

$$\int\left(\frac{1}{2x} - \frac{1}{2}y\right)dx - \int \frac{1}{2y}\,dy = c$$

$$\Rightarrow \frac{1}{2}\ln x - \frac{1}{2}xy - \frac{1}{2}\ln y = c. \quad \text{Ans.}$$

Home work for Exact diff. equⁿ:

$(y^4 + 4x^3y + 3x)\,dx + (x^4 + 4xy^3 + y + 1)\,dy = 0$

$x(x^2 + y^2 - a^2)\,dx + y(x^2 - y^2 - b^2)\,dy = 0$

Home work for Non-exact diff. equⁿ:

$x^2y\,dx - (x^3 + y^3)\,dy = 0$

$(x^4 + y^4)\,dx - xy^3\,dy = 0$

$y^2\,dx + (x^2 - xy - y^2)\,dy = 0$