Dennis G. Zill. A First Course in diff. eqn's with modeling Applications (10th edi).

Differential Equation:

An equation containing the derivatives of one or more dependent variables with respect to one or more independent variables is said to be a differential eqn.

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0 \quad \text{or} \quad y'' + y' + y = 0$$

$$\frac{dy}{dx} + (1-y^2)\tan x = 0$$

Here y is dependent variable

x is independent "

Derivation of y w.r.t x is denoted by $\frac{dy}{dx}$.

Types: Mainly there are two types of diff. eqn

(i) Ordinary diff. eqn (ODE)

(ii) Partial " " (PDE)

Ordinary differential equation:

An equation involving only ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ODE.

For example,
$$\frac{dy}{dx} + 5y = e^x$$
$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + y = 0$$
$$\frac{dx}{dt} + \frac{dy}{dt} = 2x+y$$

Partial Differential equation:

An equation involving the partial derivatives of one or more dependent variables w.r.t two or more independent variables is called a PDE.

For example,
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 3z$$

Order of Differential equation:

The order is the highest derivative occured in the diff. eqn.

$$\underset{\text{2nd order}}{\frac{d^2y}{dx^2}} + 5\underset{\text{1st order}}{\left(\frac{dy}{dx}\right)^3} - 4y = e^x$$

Degree of Differential equation:

The degree or power of the highest order derivatives of a differential eq^n is called the degree of differential equ^n.

$$\cos x \frac{d^2y}{dx^2} + \sin x \left(\frac{dy}{dx}\right)^2 + 8y = \tan x \rightarrow \text{degree } 1$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2 \rightarrow \text{degree } 2.$$

Linear differential equation:

$$a_n(x) \frac{d^ny}{dx^n} + a_{n-1}(x) \frac{d^{(n-1)}y}{dx^{(n-1)}} + \dots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$$

is called linear if functions of x and $y, y', \dots, y^n$ are linear and each coefficient depends atmost on x.

$$(y-x)\,dx + 4x\,dy = 0$$

$$\frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$$

Non Linear Ordinary differential equation:

A nonlinear ODE is simply one that is not linear.

$$(1-y)y' + 2y = e^x \rightarrow \text{coefficient depends on } y$$

Linear & non-linear diff. equ^n:

A differential equation is called linear if (i) every dependent variable and every derivatives involved occurs in the 1st degree

(ii) No products of dependent variables or derivatives.

Otherwise the differential equation is called non-linear.

Formation of differential equation:

1. Find the diff. equ^n from a straight line $y = mx$.

Sol^n: Given $y = mx$ — (i).

Differentiating (i) w.r.t x

$$\frac{dy}{dx} = m \quad \text{— (ii)}$$

Putting (ii) into (i) we get,

$$y = \frac{dy}{dx} x$$

$\therefore y = x\frac{dy}{dx}$ is the required differential equ^n.

C.W 2. Form a differential equation of the relation

$y = A\cos x + B\sin x$

Solution:

Given that,

$$y = A\cos x + B\sin x \quad \text{——} (i)$$

Differentiating (i) w.r.t. to x

$$y' = -A\sin x + B\cos x \quad \text{——} (ii)$$

Differentiating (ii) we get,

$$y'' = -A\cos x - B\sin x$$

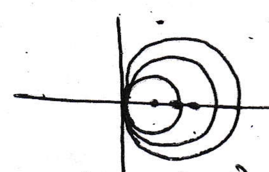
$$\Rightarrow y'' = -(A\cos x + B\sin x)$$

$$\Rightarrow y'' = -y$$

$$\therefore y'' + y = 0$$

which is the required differential eqn.

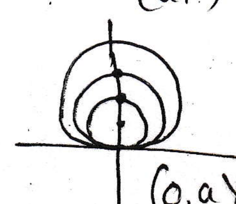
C.W 3. Form the differential equation of all circles passing through the origin and having their centres on the X-axis.



(a,0)

Solution:

Let the eqn of circle, whose centre is (a,0) and radius a be



(0,a)

$$(x-a)^2 + (y-0)^2 = a^2$$

$$\Rightarrow x^2 - 2ax + a^2 + y^2 = a^2$$

$$\Rightarrow x^2 + y^2 = 2ax \quad \text{--- (1)}$$

Differentiating (1) w.r.t. to x we get,

$$2x + 2y\frac{dy}{dx} = 2a$$

Multiplying both sides by x we get,

$$2x^2 + 2xy\frac{dy}{dx} = 2ax$$

$$\Rightarrow 2x^2 + 2xy\frac{dy}{dx} = x^2 + y^2 \quad [\text{using (1)}]$$

$$\Rightarrow 2x^2 + 2xy\frac{dy}{dx} - x^2 - y^2 = 0$$

$$\Rightarrow x^2 - y^2 + 2xy\frac{dy}{dx} = 0$$

Which is the required differential eqn.

5. Form the diff. equ^n of the relation $y = A\cos 2x + B\sin 2x$

Solution:

Given $y = A\cos 2x + B\sin 2x$

differentiating w.r. to x

$$\frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x$$

Again differentiating,

$$\frac{d^2y}{dx^2} = -4A\cos 2x - 4B\sin 2x$$

$$\Rightarrow \frac{d^2y}{dx^2} = -4\left(A\cos 2x + B\sin 2x\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -4y$$

$$\Rightarrow \frac{d^2y}{dx^2} + 4y = 0$$

Which is the required differential equ^n.

6. Find the differential equ^n for the family of curves $y = Ae^{2x} + Be^{-2x}$ for different values of A & B.

Solution:

Given, $y = Ae^{2x} + Be^{-2x}$ —— (1)

Differentiating equ^n (1) w.r. to x

$$y' = 2Ae^{2x} - 2Be^{-2x}$$
$$= 2\left(Ae^{2x} - Be^{-2x}\right)$$

Differentiating again,

$$y'' = 2\,[2A\,e^{2x} + 2B\,e^{-2x}]$$

$$= 4\,[A\,e^{2x} + B e^{-2x}]$$

$$\Rightarrow y'' = 4y$$

$$\Rightarrow y'' - 4y = 0$$

which is the required diff. equ^n.

7. If $y = c_1 e^{4x} + c_2 e^{-2x}$, where c_1 & c_2 are arbitrary constant, then show that, $y'' - 2y' - 8y = 0$

Solution:

Given that, $y = c_1 e^{4x} + c_2 e^{-2x}$ ——— (1)

Differentiating (1) w.r.to x we get,

$$y' = 4c_1 e^{4x} - 2c_2 e^{-2x}$$

Again differentiating

$$y'' = 16c_1 e^{4x} + 4c_2 e^{-2x}$$

Now $\quad$ L.H.S. $= y'' - 2y' - 8y$

$$= 16c_1 e^{4x} + 4c_2 e^{-2x} - 8c_1 e^{4x} + 4c_2 e^{-2x}$$
$$- 8c_1 e^{4x} - 8c_2 e^{-2x}$$

$$= 0$$

$=$ R.H.S.

$\therefore$ L.H.S. $=$ R.H.S.

(Showed)

8. Show that the diff. equ^n of $Ax^2+By^2=1$ is

$$x\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]=y\frac{dy}{dx}.$$

Solution:

Given, $Ax^2+By^2=1$ ⟶ (I)

Differentiating equ^n (I) w.r. to x

$$2Ax+2By\frac{dy}{dx}=0$$

$$\Rightarrow 2By\frac{dy}{dx}=-2Ax$$

$$\Rightarrow By\frac{dy}{dx}=-Ax$$

$$\Rightarrow \frac{y}{x}\frac{dy}{dx}=\frac{-A}{B} \longrightarrow \text{(II)}$$

Differentiating (II) w. r. to x

$$\frac{y}{x}\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(\frac{x.\frac{dy}{dx}-y}{x^2}\right)=0$$

$$\Rightarrow \frac{y}{x}\frac{d^2y}{dx^2}+\frac{1}{x^2}\left(x\frac{dy}{dx}-y\right)\frac{dy}{dx}=0$$

$$\Rightarrow xy\frac{d^2y}{dx^2}+\left(x\frac{dy}{dx}-y\right)\frac{dy}{dx}=0$$ [Multiplying by x^2]

$$\Rightarrow xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$$

$$\Rightarrow x\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]=y\frac{dy}{dx}$$

(Showed)

By eliminating the constants a and b, obtain a differential equ^n for which $xy = ae^x + be^{-x} + x^2$ is a solution.

Ans: $xy'' + 2y' - xy + x^2 - 2 = 0$

Find the differential equation corresponding to $y = c(x-c)^2$, where c is an arbitrary constant.

Solution:

Given that,

$$y = c(x-c)^2 \quad \text{—} (1)$$

Differentiating ① w.r. to x

$$y' = 2c(x-c) \quad \text{—— (ii)}$$

Dividing equ^n ② by ①,

Constant একটি, তাই একবার এর বেশি diff. করা যাবে না।

$$\frac{y'}{y} = \frac{2c(x-c)}{c(x-c)^2}$$

$$\Rightarrow \frac{y'}{y} = \frac{2}{x-c}$$

$$\Rightarrow (x-c) = \frac{2y}{y'}$$

$$\Rightarrow c = x - \frac{2y}{y'} \quad \text{—— (iii)}$$

Putting the value of (iii) in (ii) we get,

$$y' = 2\left(x - \frac{2y}{y'}\right)\left[x - \left(x - \frac{2y}{y'}\right)\right]$$

$$\Rightarrow y' = 2\left(x - \frac{2y}{y'}\right)\left(x - x + \frac{2y}{y'}\right)$$

$$\Rightarrow y' = 2\left(x - \frac{2y}{y'}\right)\frac{2y}{y'}$$

$$\Rightarrow y' = \frac{4y}{y'}\left(x - \frac{2y}{y'}\right)$$

$$\Rightarrow y' = \frac{4y}{y'}\left(\frac{xy' - 2y}{y'}\right)$$

$$\Rightarrow y' = \frac{4y(xy'-2y)}{(y')^2}$$

$$\Rightarrow (y')^3 = 4y(xy'-2y)$$

Which is the required diff. equ^n.