Lagrange Method:

Equation $Pp + Qq = R$ is the standard form of Linear p.d.e

The general solution of L.p.de $Pp + Qq = R$ is $\varphi(u, v) = 0$ where φ is an arbitrary function and

$$u(x, y, z) = c_1$$
$$v(x, y, z) = c_2$$

are solution of equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

where ~~R~~ P, Q, R are function of x, y, z

Problem and solution:

1. $xzp + yzq = xy$

Solution:

$$xzp + yzq = xy$$

where $P = xz$, $Q = yz$, $R = xy$

Auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$$

From first two

$$\frac{dx}{dz} = \frac{dy}{yz}$$

$$\Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\Rightarrow \log x = \log y + \log c_1$$

$$\Rightarrow \log x - \log y = \log c_1$$

$$\Rightarrow \log\left(\frac{x}{y}\right) = \log c_1$$

$$\Rightarrow \frac{x}{y} = c_1$$

From last two

$$\frac{dy}{yz} = \frac{dz}{xy}$$

$$\Rightarrow \int x\,dy = \int z\,dz$$

$$\Rightarrow xy = \frac{z^2}{2} + c_2$$

$$\Rightarrow xy - \frac{z^2}{2} = c_2$$

Here the general solution is

$$\phi\left(\frac{x}{y},\ xy - \frac{z^2}{2}\right) = 0 \quad \underline{\text{Ans}}$$

2. $\frac{y^2 z}{x} p + xzq = y^2 \Rightarrow y^2 z p + x z q = y^2 x$

Where $P = \frac{y^2 z}{x}$, $Q = xz$, $R = y^2$

Auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}$$

From first two

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz}$$

$$\Rightarrow \frac{dx}{x \cdot \frac{y^2 z}{x}} = \frac{dy}{x \cdot xz}$$

$$\Rightarrow \frac{dx}{y^2} = \frac{dy}{x^2}$$

$$\Rightarrow \int x^2\, dx = \int y^2\, dy$$

$$\Rightarrow \frac{x^3}{3} = \frac{y^3}{3} + c_1$$

$$\Rightarrow x^3 - y^3 = c_1$$

From first and 3rd

$$\frac{dx}{\frac{\cancel{y^2} z}{x}} = \frac{dz}{\cancel{y^2}}$$

$$\Rightarrow \frac{dx}{\cancel{x} \cdot \frac{z}{\cancel{x}}} = \frac{dz}{x}$$

$$\Rightarrow \frac{dx}{z} = \frac{dz}{x}$$

$$\Rightarrow \int x\, dx = \int z\, dz$$

$$\Rightarrow \frac{x^2}{2} = \frac{z^2}{2} + c_2$$

$$\Rightarrow x^2 - z^2 = c_2$$

Here the general solution is

$$\phi\left(x^3 - y^3,\ x^2 - z^2\right) = 0 \quad \underline{\text{Ans}}$$

3. $(x^2 - yz)p + (y^2 - zx)q = (z^2 - xy)$

Where, $P = x^2 - yz$, $Q = y^2 - zx$, $R = z^2 - xy$

The auxiliary equation is

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

$$\Rightarrow \frac{dx - dy}{x^2 - yz - y^2 + zx} = \frac{dy - dz}{(y^2 - zx - z^2 + xy} = \frac{dz - dx}{z^2 - xy - x^2 + yz}$$

$$\Rightarrow \frac{dx - dy}{(x^2 - y^2) + z(x - y)} = \frac{dy - dz}{(y^2 - z^2) + x(y - z)} = \frac{dz - dx}{(z^2 - x^2) + y(z - x)}$$

$$\Rightarrow \frac{dx - dy}{(x - y)(x + y + z)} = \frac{dy - dz}{(y - z)(x + y + z)} = \frac{dz - dx}{(z - x)(x + y + z)}$$

$$\Rightarrow \frac{dx - dy}{x - y} = \frac{dy - dz}{y - z} = \frac{dz - dx}{z - x}$$

From first two

$$\int \frac{dx - dy}{x - y} = \int \frac{dy - dz}{y - z}$$

$$\Rightarrow \log(x - y) = \log(y - z) + \log c_1$$

$$\Rightarrow \log\left(\frac{x - y}{y - z}\right) = \log c_1$$

$$\Rightarrow \frac{x - y}{y - z} = c_1$$

$y\,dx - y\,dy - x\,dx + z\,dy$

$= x\,dy - x\,dz + y\,dz - y\,dy$

From last two

$$\int \frac{dy - dz}{y - z} = \int \frac{dz - dx}{z - x}$$

$$\Rightarrow \log(y - z) = \log(z - x) + \log c_2$$

$$\Rightarrow \log\left(\frac{y - z}{z - x}\right) = \log c_2$$

$$\Rightarrow \frac{y - z}{z - x} = c_2$$

Here the general solution is

$$\phi\left(\frac{x - y}{y - z}, \frac{y - z}{z - x}\right) = 0 \text{ Ans}$$

4. $(y + z)p + (z + x)q = x + y$

Where. $P = y + z$, $Q = z + x$, $R = x + y$

Auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{y + z} = \frac{dy}{z + x} = \frac{dz}{x + y}$$

$$\Rightarrow \frac{dy - dx}{z + x - y - z} = \frac{dz - dy}{x + y - z - x} = \frac{dx - dz}{y + z - x - y}$$

$$\Rightarrow \frac{dy - dx}{-(y - x)} = \frac{dz - dy}{-(z - y)} = \frac{dx - dz}{-(x - z)}$$

From first two

$$\frac{dy-dx}{y-x} = \frac{dz-dy}{z-y}$$

$$\Rightarrow \int \frac{dy-dx}{y-x} = \int \frac{dz-dy}{z-y}$$

$$\Rightarrow \log(y-x) = \log(z-y) + \log c_1$$

$$\Rightarrow \log\left(\frac{y-x}{z-y}\right) = \log c_1$$

$$\Rightarrow \frac{y-x}{z-y} = c_1$$

From last two

$$\Rightarrow \int \frac{dz-dy}{z-y} = \int \frac{dx-dz}{x-z}$$

$$\Rightarrow \log(z-y) = \log(x-z) + \log c_2$$

$$\Rightarrow \log\left(\frac{z-y}{x-z}\right) = \log c_2$$

$$\Rightarrow \frac{z-y}{x-z} = c_2$$

Here the general solution is

$$\phi\left(\frac{y-x}{z-y}, \frac{z-y}{x-z}\right) = 0$$ Ans

5. $(mz-ny)p+(nx-lz)q=ly-mx$

Where. $P=mz-ny$, $Q=nx-lz$, $R=ly-mx$

The auxiliary equation is

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\Rightarrow \frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$$

Using multipliers x, y, z

$$\frac{xdx}{mzx-nxy}=\frac{ydy}{nxy-lzy}=\frac{zdz}{lzy-mzx}$$

$$\Rightarrow xdx+ydy+zdz=0$$

Now integrating

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c_1$$

$$\Rightarrow x^2+y^2+z^2=c_1$$

Now using multipliers l, m, n

$$\frac{ldx}{lmz-lny}=\frac{mdy}{mnx-mlz}=\frac{ndz}{lny-mnx}$$

$$\Rightarrow ldx+mdy+ndz=0$$

Now integrating

$$lx+my+nz=c_2$$

Here the general solution is

$$\phi(x^2+y^2+z^2,\ lx+my+nz)=0 \text{ Ans}$$

6. $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$

Where, $P = x(y^2+z)$, $Q = -y(x^2+z)$, $R = z(x^2-y^2)$

The auxiliary equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

Using multipliers $x, y, -1$

$$x\,dx + y\,dy - dz = 0$$

$$\Rightarrow x^2 + y^2 - z = c_1$$

Using multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\Rightarrow \log x + \log y + \log z = \log c_2$$

$$\Rightarrow \log(xyz) = \log c_2$$

$$\Rightarrow xyz = c_2$$

Here the general solution is

$$\phi(x^2+y^2-z^2,\ xyz) = 0$$ Ans