Homogeneous Differential Equation:

Reducible to homogeneous equation:

If a 1st order and 1st degree differential equation can be written in the following form

$$\frac{dy}{dx} = f(y/x) \quad \text{——} (1)$$

then the differential equation is known to be homogeneous.

Solution method:

Put $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then the equation (1) becomes,

$$v + x\frac{dv}{dx} = f(v)$$

$$\Rightarrow x\frac{dv}{dx} = f(v) - v$$

$$\Rightarrow \frac{dv}{f(v)-v} = \frac{dx}{x}$$

Now the above equation can be solved by the method separation of variables.

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(1) $(x^2+y^2)\,dx - 2xy\,dy = 0$. Solve it.

Solution:

Given that,

$$(x^2+y^2)\,dx - 2xy\,dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2+y^2}{2xy} \quad \text{--- (1)}$$

$$\frac{x^2(1+y^2/x^2)}{2xy} = \frac{x(1+y^2/x^2)}{2y} = \frac{1+y^2/x^2}{2\,y/x}$$

Now put $y = vx$.

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then equ^n (1) becomes,

$$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2vx^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{2v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\Rightarrow \frac{dx}{x} - \frac{2v}{1-v^2}\,dv = 0$$

$$\Rightarrow \ln x + \ln(1-v^2) = \ln c$$

$$\Rightarrow x(1-v^2) = c$$

$$\Rightarrow x\left(1-\frac{y^2}{x^2}\right) = c$$

$$\Rightarrow x^2 - y^2 = cx$$

Ans.

2. Solve $x + y\frac{dy}{dx} = 2y$

$$\frac{dy}{dx} = \frac{2y - x}{y}$$

Solution:

$$x + y\frac{dy}{dx} = 2y \quad \text{——} (1)$$

Put $y = vx$ then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now, equ^n (1) becomes,

$$x + vx\left(v + x\frac{dv}{dx}\right) = 2vx$$

$$\Rightarrow vx^2\frac{dv}{dx} = 2vx - v^2x - x$$

$$\Rightarrow \frac{dv}{dx} = \frac{-x(v^2 - 2v + 1)}{vx^2}$$

$$\Rightarrow \frac{dv}{dx} = \frac{-(v-1)^2}{vx}$$

$$\Rightarrow \frac{v}{(v-1)^2}dv = -\frac{dx}{x}$$

$$\Rightarrow \frac{v-1+1}{(v-1)^2}dv = -\frac{dx}{x}$$

$$\Rightarrow \frac{dv}{v-1} + \frac{dv}{(v-1)^2} = -\frac{dx}{x}$$

$$\Rightarrow \ln(v-1) - \frac{1}{v-1} = -\ln x + c$$

$$\Rightarrow \ln\{x(v-1)\} = \frac{1}{v-1} + c$$

$$\Rightarrow \ln\left\{x\left(\frac{y}{x} - 1\right)\right\} = \frac{1}{\frac{y}{x} - 1} + c$$

$$\Rightarrow \ln(y - x) = \frac{x}{y - x} + c$$

$$\Rightarrow y - x = e^{\frac{x}{y-x}} \cdot e^{c}$$

$$\Rightarrow y - x = c_1 e^{\frac{x}{y-x}}$$

$$\therefore y = x + c_1 e^{\frac{x}{y-x}}$$

3. Solve $\frac{dy}{dx} = \frac{y(y+x)}{x(y-x)}$

Solution:

Let $y = vx$ then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then the given de becomes,

$$v + x\frac{dv}{dx} = \frac{vx(vx+x)}{x(vx-x)}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v(v+1)}{(v-1)} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2+v-v^2+v}{v-1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v}{v-1}$$

$$\Rightarrow \frac{(v-1)\,dv}{2v} = \frac{dx}{x}$$

$$\Rightarrow \left[\frac{1}{2} - \frac{1}{2v}\right] dv = \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}v - \frac{1}{2}\ln(v) = \ln(x) + c$$

$$\Rightarrow v - \ln(v) = 2\ln(x) + c_1 \quad [\text{let } 2c = c_1]$$

$$\Rightarrow v = \ln(x^2) + \ln(v) + c_1$$

$$\Rightarrow v = \ln(vx^2) + c_1$$

$$\Rightarrow vx^2 = e^{v-c_1}$$

$$\Rightarrow xy = c_2 e^{y/x}$$ Ans.

4. Solve $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

Solution:

Given that,

$$x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx \quad \text{---} (1)$$

Let $y = vx$

$$\Rightarrow dy = v\,dx + x\,dv$$

$\therefore (1) \Rightarrow$

$$x(v\,dx + x\,dv) - vx\,dx = \sqrt{x^2+v^2x^2}\,dx$$

$$\Rightarrow x\cancel{v\,d}x + x^2\,dv - x\cancel{v}\,dx = x\sqrt{1+v^2}\,dx$$

$$\Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{x\,dx}{x^2}$$

$$\Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

Now integrating both sides we get

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$

$\int \frac{dx}{\sqrt{1+x^2}} = \ln\left(x+\sqrt{1+x^2}\right)$

$$\Rightarrow \ln\left(v+\sqrt{1+v^2}\right) = \ln x + \ln c$$

$$\Rightarrow v + \sqrt{1+v^2} = cx$$

$$\Rightarrow y/x + \frac{\sqrt{x^2+y^2}}{x} = cx$$ Ans.

$$\Rightarrow y + \sqrt{x^2+y^2} = cx^2$$

Solve $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$

Ans: $\sin\frac{y}{x} = cx$

Solve $(x^2+y^2)\,dy - xy\,dx = 0$

Ans: $\frac{1}{2}\frac{x^2}{y^2} = \ln y + c$