Partial Differential equations

z is a function of two independent variables x and y we write

$$z = f(x, y)$$

$$p = \frac{\partial z}{\partial x} \quad , \quad p = \frac{\partial^2 z}{\partial x^2}$$

$$q = \frac{\partial z}{\partial y} \quad , \quad s = \frac{\partial^2 z}{\partial y \partial x}$$

$$t = \frac{\partial^2 z}{\partial y^2}$$

Questions:

Eliminate arbitrary constant form the followings:

1. $z = (x+a)(y+b)$
2. $z = (x^2+a)(y^2+b)$
3. $2z = (ax+y)^2 + b$
4. $ax^2 + by^2 + z^2 = 1$
5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Solution 1: $z = (x+a)(y+b)$

$$\frac{\partial z}{\partial x} = y + b$$

$$\frac{\partial z}{\partial y} = x + a$$

Now $\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = (x+b)(x+a)$

$\Rightarrow pq = z$ which is the result.

Solution 2: $z = (x^2+a)(y^2+b)$

$$\frac{\partial z}{\partial x} = 2x(y^2+b)$$

$$\frac{\partial z}{\partial y} = 2y(x^2+a)$$

Now $\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = 2x(y^2+b) \cdot 2y(x^2+a)$

$\Rightarrow pq = 4xy(x^2+a)(y^2+b)$

$\Rightarrow pq = 4xyz$ which is the result.

Solution 3: $2z = (ax+y)^2 + b$

$\Rightarrow 2z = a^2x^2 + 2axy + y^2 + b$ ——— (1)

Differentiating w.r.to x

$$2\frac{\partial z}{\partial x} = 2a^2x + 2ay$$

$$\Rightarrow \frac{\partial z}{\partial x} = a(ax+y)$$

$$\Rightarrow p = a^2x + ay$$

$$\Rightarrow px = a^2x^2 + axy \quad \text{——— (2)}$$

Differentiating (1) w.r.to y

$$2\frac{\partial z}{\partial y} = 2ax + 2y$$

$$\Rightarrow q = ax + y$$

$$\Rightarrow qy = axy + y^2 \quad \text{——— (3)}$$

Again $q^2 = (ax+y)^2$ ——— (4)

Adding (2) & (3)

$$px + qy = a^2x^2 + axy + axy + y^2$$

$$\Rightarrow px + qy = a^2x^2 + 2axy + y^2$$

$$\Rightarrow px + qy = (ax+y)^2$$

$$\Rightarrow px + qy = q^2 \quad [\text{using (4)}]$$

which is the result.

Solution 4: $ax^2 + by^2 + z^2 = 1$

Differentiating w.r.t to x

$$2ax + 2z \cdot \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow ax + zp = 0 \quad \text{——(I)}$$

Diff. w.r.t to x

$$a + z \cdot \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow a + z \cdot q + p \cdot p = 0$$

$$\Rightarrow ax + zqx + xp^2 = 0 \quad \text{——(II)}$$

(I) − (II) we get,

$$ax + zp - ax - zqx - p^2x = 0$$

$$\Rightarrow pz = p^2x + qxz \text{ which is the result.}$$

Solution 5: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Diff. w.r.t to x

$$\frac{2x}{a^2} + \frac{2z \cdot \frac{\partial z}{\partial x}}{c^2} = 0$$

$$\Rightarrow \frac{x}{a^2} + \frac{zp}{c^2} = 0 \quad \text{——(I)}$$

Diff w.r.t to x

$$\frac{1}{a^2} + \frac{1}{c^2}\left(z \cdot \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow \frac{1}{a^2} + \frac{1}{c^2}(z \cdot q + p \cdot p) = 0$$

$$\Rightarrow \frac{x}{a^2} + \frac{zqx}{c^2} + \frac{p^2x}{c^2} = 0 \quad \text{——(II)}$$

(I) − (II) we get, $\frac{zp}{c^2} = \frac{zqx}{c^2} + \frac{p^2x}{c^2}$

$$\Rightarrow zp = zqx + p^2x \text{ which is the result.}$$