First Order Differential Equation Solution Method:

Homogeneous Differential Equation:

A first order differential equation,

$$M(x,y)dx + N(x,y)dy = 0$$

is said to be homogeneous if we write it in the form,

$$\frac{dy}{dx} = f(x,y)$$

There exists a function g such that $f(x,y)$ can be expressed in the form $\left(\frac{y}{x}\right)$.

Test a differential equation homogeneous or not:

We will choose $M(x,y)$ and $N(x,y)$ as,

$$M(tx,ty) = t^n M(x,y)$$

$$N(tx,ty) = t^n N(x,y)$$

Where t is just a constant number. If in both case we get same n, then the given ODE is homogeneous.

Example: Test the following differential equation is homogeneous or not.

$$(x^2 + xy)dx + (x^2y - x^3)dy = 0$$

Solution: Given,

$(x^2 + xy)dx + (x^2y - x^3)dy = 0$----------------(1)

Comparing equation (1) with the equation,

$$M(x,y)dx + N(x,y)dy = 0$$

We can write,

$$M(x,y) = x^2 + xy$$

$$N(x,y) = x^2y - x^3$$

For any t,

$$M(tx,ty) = (tx)^2 + (tx)(ty)$$

$$= t^2x^2 + t^2xy$$

Here n=2

$$= t^2(x^2 + xy)$$

Again

$$N(tx, ty) = (t \quad x)^3$$

Here n=3

$$= t^3x^2y$$

$$= t^3(x^2y - x^3)$$

In both case the value of n is not equal. So the given first order equation is not homogeneous.

Try now: Test the following differential equation is homogeneous or not.

$$(3x + 5y)dx + (4x + 6y)dy = 0$$

Solution Rules of Homogeneous Differential Equation:

Step 1: At first, test the given differential equation is homogeneous or not.

If homogeneous then go for step 2.

Step 2: Transform the equation into new variable v and $x$by $= vx$.

Then the equation will in separable of variable form

Example: Solve the following differential equation,

$$\frac{dy}{dx} = \frac{x - y}{x + y}$$

Or $$-(x - y)dx + (x + y)dy = 0$$

Solution:Given,

$-(x - y)dx + (x + y)dy = 0$-----------------------(1)

Comparing equation(1) with the following,

$$M(x,y)dx + N(x,y)dy = 0$$

We get,

$$M(x,y) = -(x-y)$$

$$N(x,y) = (x+y)$$

For any t,

$$M(tx,ty) = -(tx-ty) = -t(x-y)$$

and

$$N(tx,ty) = (tx+ty) = t(x+y)$$

In both equation the power of t is 1. So the given differential equation is homogeneous. We can go for the next step.

Writing equation (1) in the following form,

$$-(x-y)dx + (x+y)dy = 0$$

$$=> (x+y)dy = (x-y)dx$$

$$=> \frac{dy}{dx} = \frac{x-y}{x+y}$$

Now writing $y = vx$,

$$\frac{x-vx}{x+vx} = \frac{d}{dx}(vx)$$

$$=> \frac{x(1-v)}{x(1+v)} = v + x\frac{dv}{dx}$$

$$=> \frac{1-v}{1+v} - v = x\frac{dv}{dx}$$

$$=> \frac{1-v-v-v^2}{1+v} = x\frac{dv}{dx}$$

$$=> \frac{1-2v-v^2}{1+v} = x\frac{dv}{dx}$$

$$=> \frac{-(v^2+2v-1)}{1+v} = x\ \frac{dv}{dx}$$

$$=> \ -\frac{1+v}{(v^2+2v-1)} = \frac{1}{x}\frac{dx}{dv}$$

$$=> \ -\frac{1+v}{(v^2+2v-1)} = \ \frac{1}{x}\frac{dx}{dv}$$

$$=> \ -\frac{1}{2}\frac{2+2v}{(v^2+2v-1)}\ dv = \ \frac{1}{x}\ dx$$

$$=> \frac{1}{2}\frac{2+2v}{(v^2+2v-1)}\ dv = \ -\frac{1}{x}\ dx$$

$=> \frac{1}{2}\frac{2+2v}{(v^2+2v-1)}\ dv + \frac{1}{x}\ dx = 0$

$=> \frac{2+2v}{(v^2+2v-1)}\ dv + \frac{2}{x}\ dx = 0$

Integrating both side we get,

$$\int \frac{2+2v}{(v^2+2v-1)}\ dv\ + \int \frac{2}{x}\ dx = \int 0$$

$$=> \ln(v^2+2v-1) + 2\ln x = \ln c$$

$$=> \ln(v^2+2v-1) + \ln x^2 = \ln c$$

$$=> x^2(v^2+2v-1) = c$$

$$=> x^2\left(\frac{y}{x^2}^2 + 2\frac{y}{x} - 1\right) = c$$

$$=> x^2\left(\frac{y^2+2xy-x^2}{x^2}\right) = C$$

$y^2 + 2xy - x^2 = c$ (ans)

Example 2: $(2xy+3y^2)dx - (2xy+x^2)dy = 0$

Solution:

step1: $M(x,y) = 2xy + 3y^2$

$$=> M(tx,ty) = 2tx \times ty + 3(ty)^2$$

$$= t^2(2xy + 3y^2)$$

$$N(x,y) = -(2xy + x^2)$$

$$=> N(tx,ty) = -\left(2tx \times ty + (tx^2)\right)$$

$$= -t^2(2xy + x^2)$$

The given DE is homogeneous.

Step2: $(2xy + 3y^2)dx = (2xy + x^2)dy$

$=> \frac{dy}{dx} = \frac{2xy+3y^2}{2xy+x^2}$------------(1)

Letting $y = vx,$

Differentiating w.r.t x

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$=> \frac{2x.vx+3v^2.x^2}{2x.vx+x^2} = v + x\frac{dv}{dx}$(using eqn (1))

$$=> \frac{2x^2.v + 3x^2.v^2}{2x^2v + x^2} = v + x\frac{dy}{dx}$$

$$=> \frac{2v + 3v^2}{2v + 1} = v + x\frac{dy}{dx}$$

$$=> x\frac{dv}{dx} = \frac{2v + 3v^2}{2v + 1}$$

$$=> x\frac{dv}{dx} = \frac{v^2 + v}{2v + 1}$$

$$=> \frac{2v + 1}{v^2 + v}dv = \frac{1}{x}dx$$

$$=> \int \frac{2v + 1}{v^2 + v}dv = \int \frac{1}{x}dx$$

$$=> \ln(v^2 + v) = lnx + lnc$$

$$=> v^2 + v = cx$$

$$=> y^2 + xy = cx^3$$

Which is the required equation (ans.)

Exercises :

1) $\frac{dy}{dx} = \frac{2x-3y}{3x-2y}$ H.W

2) $(x - 2y)dx + (2x + y)dy = 0$

3) $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$

4) $xydx + (x^2 + y^2)dy = 0$