Solution of 1st order differential equations by various Method:

First order & 1st degree differential equ^n:

A differential equation of the form

$M + N\frac{dy}{dx} = 0$ or $Mdx + Ndy = 0$ is called first order and first degree differential equation, where both M & N are functions of x & y. They are divided mainly into 6 catagories:

(i) Seperation of variables

(ii) Homogeneous equation

(iii) Equation reducible to homogeneous

(iv) Exact equation

(v) Linear equation

(vi) Reducible to linear equation.

Solution by integration:

$$\frac{dy}{dx} = g(x)$$

$$\Rightarrow \int dy = \int g(x)dx$$

$$\therefore y = G(x) + c$$

Example:

$$\frac{dy}{dx} = 1 + e^{2x}$$

$$\Rightarrow dy = (1+e^{2x})\,dx$$

$$\Rightarrow y = \int (1+e^{2x})\,dx + c$$

$$\therefore y = x + \frac{1}{2}e^{2x} + c.$$

Seperation of variable:

If the equation of $M(x,y)dx + N(x,y)dy = 0$ can be written in this form $f(x)dx + g(y)dy = 0$ and then it can be solved easily term by term and the solution is,

$$\int f(x)dx + \int g(y)dy = c$$

1. Solve $\frac{dy}{dx} = \frac{x^2+x+1}{y^2+y+1}$

Solution:

$$\frac{dy}{dx} = \frac{x^2+x+1}{y^2+y+1}$$

$$\Rightarrow (y^2+y+1)\,dy = (x^2+x+1)\,dx$$

$$\Rightarrow \frac{y^3}{3} + \frac{y^2}{2} + y = \frac{x^3}{3} + \frac{x^2}{2} + x + c$$

$$\Rightarrow \frac{1}{3}(y^3 - x^3) + \frac{1}{2}(y^2 - x^2) + y - x = c$$

2. Solve the 1st order differential eq^n $\frac{dy}{dx} = \frac{2y}{x}$

Solution:

Here, $\frac{dy}{dx} = \frac{2y}{x}$

$$\Rightarrow \frac{dy}{y} = \frac{2dx}{x}$$

Now integrating both sides,

$$\int \frac{dy}{y} = 2\int \frac{dx}{x} + \ln c$$

$$\Rightarrow \ln y = 2\ln x + \ln c$$

$$\Rightarrow \ln y = \ln x^2 + \ln c$$

$$\Rightarrow \ln y = \ln(x^2 c)$$

$$\Rightarrow y = cx^2$$ (taking exponential on both sides)

3. Solve $(x-y)^2 \frac{dy}{dx} = a^2$

Solution:

Given that,

$$(x-y)^2 \frac{dy}{dx} = a^2 \quad \text{—(1)}$$

Let $x-y = z$

$\therefore y = x - z$

$$\frac{dy}{dx} = 1 - \frac{dz}{dx}$$

From (1) we get,

$$z^2\left(1 - \frac{dz}{dx}\right) = a^2$$

$$\Rightarrow \left(1 - \frac{dz}{dx}\right) = \frac{a^2}{z^2}$$

$$\Rightarrow \frac{dz}{dx} = 1 - \frac{a^2}{z^2}$$

$$\Rightarrow \frac{dz}{dx} = \frac{z^2-a^2}{z^2}$$

$$\Rightarrow \frac{z^2}{z^2-a^2}\,dz = dx$$

Now integrating both sides,

$$\int \frac{z^2 dz}{z^2-a^2} = \int dx + c$$

$$\Rightarrow \int \frac{(z^2-a^2+a^2)dz}{z^2-a^2} = \int dx + c$$

$$\Rightarrow \int \frac{z^2-a^2}{z^2-a^2}\,dz + \int \frac{a^2 dz}{z^2-a^2} = \int dx + c$$

$$\Rightarrow \int dz + a^2 \int \frac{dz}{z^2-a^2}\,dz = \int dx + c$$

$$\Rightarrow z + a^2 \frac{1}{2a} \ln\left|\frac{z-a}{z+a}\right| = x + c$$

$$\Rightarrow z + \frac{a}{2} \ln\left|\frac{z-a}{z+a}\right| = x + c$$

$$\Rightarrow x-y + \frac{a}{2} \ln\left|\frac{x-y-a}{x-y+a}\right| = x + c$$

$$\Rightarrow \frac{a}{2} \ln\left|\frac{x-y-a}{x-y+a}\right| - y = c.$$

Ans.

*4) solve $3e^x \tan y\, dx + (1-e^x)\sec^2 y\, dy = 0$

Solution:

Given, $3e^x \tan y\, dx + (1-e^x)\sec^2 y\, dy = 0$

$$\Rightarrow 3e^x \tan y\, dx = (e^x-1)\sec^2 y\, dy$$

$$\Rightarrow \frac{e^x dx}{e^x-1} = \frac{\sec^2 y}{3\tan y}\,dy$$

Now integrating both sides,

$$\int \frac{e^x dx}{e^x - 1} = \int \frac{\sec^2 y \, dy}{3 \tan y} + \ln c$$

$$\Rightarrow \ln(e^x - 1) = \frac{1}{3} \ln(\tan y) + \ln c$$

$$\Rightarrow \ln(e^x - 1) = \ln(\tan y)^{1/3} + \ln c$$

$$\Rightarrow \ln(e^x - 1) = \ln\{c(\tan y)^{1/3}\}$$

$$\Rightarrow e^x - 1 = c(\tan y)^{1/3}$$

$$\therefore e^x = c(\tan y)^{1/3} + 1. \quad \text{Ans.}$$

$e^x - 1 = z$

$e^x dx = dz$

$\tan y = \theta$

$\sec^2 y = \frac{d\theta}{dy}$

$\sec^2 y \, dy = d\theta$

$\int \frac{dz}{z} = \frac{1}{3} \int \frac{d\theta}{\theta}$

5. Solve $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

Solution:

Given that,

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

$$\Rightarrow \frac{dy}{dx} = e^{-y}(e^x + x^2)$$

$$\Rightarrow e^y dy = (e^x + x^2) dx$$

Integrating both sides,

$$\int e^y dy = \int (e^x + x^2) dx + c$$

$$\Rightarrow \int e^y dy = \int e^x dx + \int x^2 dx + c$$

$$\Rightarrow e^y = e^x + \frac{x^3}{3} + c$$

$$\therefore e^y - e^x - \frac{x^3}{3} = c. \quad \text{Ans.}$$

11. Solve $\sin^{-1}\left(\frac{dy}{dx}\right) = x+y$

Solution:

Given,

$$\sin^{-1}\left(\frac{dy}{dx}\right) = x+y$$

$$\Rightarrow \frac{dy}{dx} = \sin(x+y) \quad \text{—— (i)}$$

let $x+y = z$

$$\Rightarrow 1+\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1 \quad \text{—— (ii)}$$

from equ^n (i)

$$\frac{dz}{dx} - 1 = \sin z$$

$$\Rightarrow \frac{dz}{dx} = 1 + \sin z$$

$$\Rightarrow \frac{dz}{1+\sin z} = dx$$

Integrating both sides,

$$\int \frac{dz}{1+\sin z} = \int dx$$

$$\Rightarrow \int \frac{(1-\sin z)\,dz}{(1+\sin z)(1-\sin z)} = \int dx$$

$$\Rightarrow \int \frac{(1-\sin z)\,dz}{1-\sin^2 z} = \int dx$$

$$\Rightarrow \int \frac{(1-\sin z)\,dz}{\cos^2 z} = \int dx$$

$$\Rightarrow \int \{\sec^2 z\,dz - \tan z \sec z\,dz\} = \int dx$$

$$\Rightarrow \tan z - \sec z = x + c$$

$$\Rightarrow \tan(x+y) - \sec(x+y) = x + c.$$ Ans.

Solve $5e^{-5x}\sin y\,dx + (e^{-5x}-3)\cos y\,dy = 0$

Ans: $3 - e^{-5x} = c\sin y$

Solve $\frac{dy}{dx} = e^{x+y} + x^2 e^{x^3+y}$

Ans: $e^{x} + e^{-y} + \frac{1}{3}e^{x^3} + c = 0$

$\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0$

Ans: $\tan x \tan y = c$