PDE:

Formation of Partial differential equation:

Type-1:

1. Find a Partial differential equation by eliminating a and b from the equation

(a) $z = ax + by + a^2 + b^2$.

Solution:

Given that,

$$z = ax + by + a^2 + b^2 \quad \text{--- (1)}$$

Differentiating (1) Partially with respect to x and y, we get,

$$\frac{\partial z}{\partial x} = a$$

and $$\frac{\partial z}{\partial y} = b$$

Now substituting the value of a and b in (1) we get,

$$z = \left(\frac{\partial z}{\partial x}\right) x + \left(\frac{\partial z}{\partial y}\right) y + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

which is the required partial differential equation.

2. Find a partial differential equation by eliminating a and b from the equation

(b) $z = axe^y + (1/2)a^2e^{2y} + b$.

Solution:

Given that,

$$z = axe^{y} + (1/2)a^2e^{2y} + b \quad \text{(i)}$$

Differentiating (i) partially w.r.to x and y

we get,

$$\frac{\partial z}{\partial x} = ae^{y} \quad \text{(ii)}$$

$$\frac{\partial z}{\partial y} = axe^{y} + \frac{1}{2}a^2e^{2y}.2$$

$$= axe^{y} + a^2e^{2y}$$

$$\therefore \frac{\partial z}{\partial y} = x(ae^{y}) + (ae^{y})^2 \quad \text{(iii)}$$

substituting the values ~~of~~ ~~ae^y~~ of ae^{y}

from (ii) in (iii) we get,

$$\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2$$

which is the required differential eqn.

(c) eliminating h and K from the equation

$$(x-h)^2 + (y-K)^2 + z^2 = \lambda^2$$

Solution:

Given that,

$$(x-h)^2 + (y-K)^2 + z^2 = \lambda^2 \quad \text{(i)}$$

Differentiating ① partially w.r.to. x & y

we get,

$$2(x-h)+2z\frac{\partial z}{\partial x}=0$$

$$\Rightarrow (x-h)=-z\frac{\partial z}{\partial x} \quad \text{—— (ii)}$$

and $2(y-k)+2z\frac{\partial z}{\partial y}$

$$\Rightarrow (y-k)=-z\left(\frac{\partial z}{\partial y}\right) \quad \text{—— (iii)}$$

Substituting the values of $(x-h)$ & $(y-k)$ from (ii) and (iii) in ① we get,

$$z^2\left(\frac{\partial z}{\partial x}\right)^2+z^2\left(\frac{\partial z}{\partial y}\right)^2+z^2=\lambda^2$$

$$\Rightarrow z^2\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1\right]=\lambda^2$$

which is the required partial diff. equn.

(d) $z=(x^2+a)(y^2+b)$.

Solution:

Given that,

$$z=(x^2+a)(y^2+b). \quad \text{—— ①}$$

Differentiating ① partially w.r. to x and y we get

$$\frac{\partial z}{\partial x}=2x(y^2+b) \Rightarrow y^2+b=\frac{1}{2x}\frac{\partial z}{\partial x} \quad \text{—— ②}$$

$$\frac{\partial z}{\partial y}=2y(x^2+a) \Rightarrow x^2+a=\frac{1}{2y}\frac{\partial z}{\partial y} \quad \text{—— ③}$$

Now using ② and ③ in ① we get,

$$z = \frac{1}{2y}\frac{\partial z}{\partial y}\cdot\frac{1}{2x}\frac{\partial z}{\partial x}$$

$$\Rightarrow 4xyz = \frac{\partial z}{\partial x}\frac{\partial z}{\partial y}.$$

e. $z = Ae^{pt}\sin px$ Ans: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$

Ans 1. $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Solution:

Given that,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{--- (1)}$$

Differentiating (1) partially w.r. to x and y

we get,

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{c^2 x + a^2 z \frac{\partial z}{\partial x}}{a^2 c^2} = 0$$

$$\text{or, } c^2 x + a^2 z \frac{\partial z}{\partial x} = 0 \quad \text{--- (2)}$$

$$\text{and } \frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0$$

$$\text{or, } c^2 y + b^2 z \frac{\partial z}{\partial y} = 0 \quad \text{--- (3)}$$

Differentiating (2) w.r.to x and (3) w.r.to y

we find,

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2 z \frac{\partial^2 z}{\partial x^2} = 0 \quad \text{--- (4)}$$

$$\text{and } c^2 + b^2\left(\frac{\partial z}{\partial y}\right)^2 + b^2 z \frac{\partial^2 z}{\partial y^2} = 0 \quad \text{--- (5)}$$

Now from (2) we get,

$$c^2 = -\frac{a^2 z}{x}\frac{\partial z}{\partial x} \quad \text{--- (6)}$$

Now putting the value of c^2 in (4) we get,

$$-\frac{a^2 z}{x}\frac{\partial z}{\partial x} + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2 z \frac{\partial^2 z}{\partial x^2} = 0$$

Now dividing by a^2 we get,

$$-\frac{z}{x}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = 0$$

$$\Rightarrow zx\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2 - z\frac{\partial z}{\partial x} = 0 \quad \text{---} (7)$$

Similarly from (3) and (5) we get,

$$zy\frac{\partial^2 z}{\partial y^2} + y\left(\frac{\partial z}{\partial y}\right)^2 - z\frac{\partial z}{\partial y} = 0 \quad \text{---} (8)$$

$\therefore$ (7) and (8) are two possible forms of the required equations.

Derivation of partial differential equation by the elimination of arbitrary function ϕ from equation $\phi(u,v)=0$ where u,v are functions of x,y and z.

$\partial z/\partial x = p \quad \partial z/\partial y = q$.

$Pp+Qq=R$ —— (1) where

$$P=\begin{vmatrix} \partial u/\partial z & \partial u/\partial y \\ \partial v/\partial z & \partial v/\partial y \end{vmatrix}, \quad Q=\begin{vmatrix} \partial u/\partial x & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial z \end{vmatrix}, \quad R=\begin{vmatrix} \partial u/\partial y & \partial u/\partial x \\ \partial v/\partial y & \partial v/\partial x \end{vmatrix}$$

Equation (1) is called differential equation of first order and first degree known as Lagranges Linear partial differential equation.

Form partial differential equations by eliminating arbitrary functions from the following relations.

C.W. (1) $\phi(x+y+z,\ x^2+y^2-z^2)=0$ H.W. (2) $\phi(x^2+y^2,\ z-xy)=0$

(1) Soln: We know that partial differential equation of $\phi(u,v)=0$ is $Pp+Qq=R$ —— (1)

where $P=\begin{vmatrix} \partial u/\partial z & \partial u/\partial y \\ \partial v/\partial z & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -2z & 2y \end{vmatrix} = 2(y+z)$

$Q=\begin{vmatrix} \partial u/\partial x & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial z \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2x & -2z \end{vmatrix} = -2(x+z)$

$$R = \begin{vmatrix} \partial u/\partial y & \partial u/\partial x \\ \partial v/\partial y & \partial v/\partial x \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2y & 2x \end{vmatrix} = 2(x-y)$$

$\therefore$ Now from ① we get

$$2(y+z)p - 2(x+z)q = 2(x-y)$$

$$\Rightarrow (y+z)p - (x+z)q = (x-y)$$ Ans.

② Ans: $xq - yp = x^2 - y^2$