Application of linear 1st order differential equⁿ:

1. A culture initially has P_0 number of bacteria. At $t = 1h$ the number of bacteria is measured to be $\frac{3}{2}P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t, determine the time necessary for the number of bacteria to triple.

Solution:

The rate of growth is proportional to the number of bacteria $P(t)$ at time (t),

$$\therefore \frac{dP}{dt} \propto P$$

$$\Rightarrow \frac{dP}{dt} = KP$$

$$\Rightarrow \frac{dP}{P} = Kdt$$

$$\Rightarrow \int \frac{dP}{P} = K\int dt + c$$

$$\Rightarrow \ln P = Kt + c$$

$$\Rightarrow P = e^{Kt+c}$$

$$\Rightarrow P(t) = e^{c} \cdot e^{Kt}$$

$$\therefore P(t) = Ae^{Kt} \quad [A = e^{c}] \quad \text{——— (1)}$$

Given that,

$$P(0) = P_0$$

$$P(1) = \frac{3}{2} P_0$$

Now put $t=0$ in (i)

$$P(0) = Ae^{K.0}$$

$$\Rightarrow P_0 = A \cdot 1$$

$$\therefore P_0 = A$$

Now (i) become,

$$P(t) = P_0 e^{Kt} \quad \text{—— (ii)}$$

Put $t=1$ in equ^n (ii),

$$P(1) = P_0 e^K$$

$$\Rightarrow \frac{3}{2} P_0 = P_0 e^K$$

$$\Rightarrow e^K = \frac{3}{2}$$

$$\Rightarrow K = \ln\left(\frac{3}{2}\right) \approx 0.41$$

Let at time t_1 the number of bacteria become triple.

$$P(t_1) = 3P_0$$

Then (ii) becomes,

$$P(t_1) = P_0 e^{Kt_1}$$

$$\Rightarrow 3P_0 = P_0 e^{kt_1}$$

$$\Rightarrow e^{kt_1} = 3$$

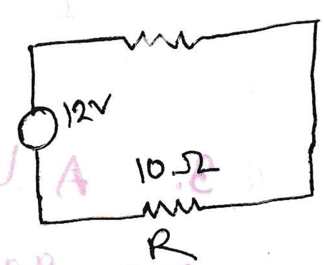
$$\Rightarrow kt_1 = \ln 3$$

$$\Rightarrow t_1 = \frac{\ln 3}{k}$$

$$= \frac{\ln 3}{0.41}$$

$$\therefore t_1 = 2.71 \text{ hours (approximately)}.$$

E 2. A 12 Volt battery is connected to the series circuit in which the inductance is $\frac{1}{2}$ henry and the resistance is 10 ohms. Determine the current I if the initial current is zero.



Solution:

We know,

$$\therefore \frac{dI}{dt} + RI = E$$

$$\Rightarrow \frac{1}{2}\frac{dI}{dt} + 10I = 12$$

$$\Rightarrow \frac{dI}{dt} + 20I = 24 \quad \text{——} (1)$$

Integrating factor $I.F. = e^{\int 20\,dt} = e^{20t}$

$(1) \times e^{20t} \Rightarrow$

$$e^{20t}\frac{dI}{dt} + e^{20t}.20I = 24 \times e^{20t}$$

$$\Rightarrow \frac{d}{dt}\left(Ie^{20t}\right) = 24e^{20t}$$

$$\Rightarrow Ie^{20t} = 24\int e^{20t}\,dt + c$$

$$\Rightarrow Ie^{20t} = \frac{24}{20}e^{20t} + c$$

$$\therefore I(t) = \frac{6}{5} + ce^{-20t} \quad \text{--- (ii)}$$

Given that, $I(0) = 0$

put $t = 0$ in (ii)

$$0 = \frac{6}{5} + c$$

$$\therefore c = -\frac{6}{5}$$

(ii) $\Rightarrow$

$$I(t) = \frac{6}{5} - \frac{6}{5}e^{-20t} \quad \text{Am.}$$

4 3. A large tank is filled to capacity with 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped into the tank at a rate of 5 gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt at time t and at time 5 min.

Solution:

$$\frac{dA}{dt} = R_{in} - R_{out} \quad \text{--- (i)}$$

Pure water $A(0) = 0$.

$R_{in} = 5 \text{ gal/min} \cdot 2 \text{ lb/gal}$

$= 10 \text{ lb/min}$

$R_{out} = \cancel{A} \; 5 \text{ gal/min} \cdot \dfrac{A}{500} \text{ lb/gal}$

$= \dfrac{A}{100} \text{ lb/min}$.

total salt A
water 500.
mixed soletions
মধ্যে $\dfrac{A}{500}$

$\therefore$ (1) becomes,

$$\frac{dA}{dt} = 10 - \frac{A}{100}$$

$$\Rightarrow \frac{dA}{dt} + \frac{A}{100} = 10 \quad \text{——— (II)}$$

$$\therefore I.F. = e^{\int \frac{1}{100} dt}$$

$$= e^{t/100}$$

Now (II) $\times e^{t/100} \Rightarrow$

$$e^{t/100} \times \frac{dA}{dt} + e^{t/100} \times \frac{A}{100} = 10 \times e^{t/100}$$

$$\Rightarrow \frac{d}{dt}\left(A \cdot e^{t/100}\right) = 10 e^{t/100}.$$

$$\Rightarrow A e^{t/100} = \int 10 e^{t/100} dt + c$$

$$\Rightarrow A e^{t/100} = 10 \cdot 100 \, e^{t/100} + c$$

$$\therefore A = 1000 + c e^{-t/100} \quad \text{——— (III)}$$

If $t = 0$ then $A(0) = 1000 + c$

$\therefore c = -1000$

$\therefore$ (iii) $\Rightarrow$

$$A = 1000 - 1000e^{-t/100}$$

Now when $t = 5$ then,

$$A = 1000 - 1000e^{-5/100}$$

$$= 48.77 \text{ lb}$$

7. A 100 volt electromagnetic force is applied to an RC series circuit in which the resistance is 200 ohms and the capacitance is 10^{-4} farad. Find the charge $q(t)$ on the capacitor if $q(0)=0$. Find the current $i(t)$.

Solution:

We know,

$$RI + \frac{q}{c} = E$$

$$\Rightarrow R\frac{dq}{dt} + \frac{q}{c} = E$$

$$\Rightarrow 200\frac{dq}{dt} + \frac{q}{10^{-4}} = 100$$

$$\Rightarrow \frac{dq}{dt} + 50q = \frac{1}{2} \quad \text{—— (I)}$$

Now, I.F. $= e^{\int 50\,dt} = e^{50t}$

(I) $\times e^{50t} \Rightarrow$

$$e^{50t}\frac{dq}{dt} + 50q \times e^{50t} = \frac{1}{2} \times e^{50t}$$

$$\Rightarrow \frac{d}{dt}\left(q e^{50t}\right) = \frac{1}{2} e^{50t}$$

$$\Rightarrow q e^{50t} = \frac{1}{2}\int e^{50t}\,dt + c$$

$$\Rightarrow q e^{50t} = \frac{1}{100} e^{50t} + c$$

$$\therefore q(t) = \frac{1}{100} + c e^{-50t} \quad \text{—— (II)}$$

Given $q(0) = 0$

$$\Rightarrow q(0) = \frac{1}{100} + c e^{0}$$

$$\Rightarrow 0 = \frac{1}{100} + c$$

$$\therefore c = -\frac{1}{100}$$

$$\text{(II)} \Rightarrow \quad q(t) = \frac{1}{100} - \frac{1}{100} e^{-50t}$$

$$\therefore I(t) = \frac{dq}{dt} = -\frac{1}{100}(-50) e^{-50t}$$

$$= \frac{1}{2} e^{-50t}$$

Ans.