Compatible system of 1st order equation.

Consider 1st order partial differential equation

$$f(x, y, z, p, q) = 0 \quad \text{(i)}$$

and $$g(x, y, z, p, q) = 0 \quad \text{(ii)}$$

Equation (i) and (ii) are known as compatible when every solⁿ of one is also a solⁿ of the other.

Condition for (i) and (ii) to be compatible

$$[f, g] = 0$$

$$\Rightarrow \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0$$

where $dz = p\,dx + q\,dy$.

1. Show that the equations $xp = yq$ and $z(xp+yq) = 2xy$ are compatible and solve them.

Solution: Let, $f(x, y, z, p, q) = xp - yq = 0 \quad \text{(1)}$

$g(x, y, z, p, q) = z(xp+yq) - 2xy = 0 \quad \text{(2)}$

$$\therefore \frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} p & x \\ zp-2y & zx \end{vmatrix} = pxz - pxz + 2xy$$

$$= 2xy$$

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & x \\ xp+yq & xz \end{vmatrix} = -x^2p - xyq$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} -q & -y \\ zq-2x & zy \end{vmatrix} = -zqy + zqy - 2xy$$

$$= -2xy$$

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -y \\ xp+yq & zy \end{vmatrix} = xyp + qy^2$$

$$\therefore [f,g] = \frac{\partial(f,g)}{\partial(x,p)} + p\,\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\,\frac{\partial(f,g)}{\partial(z,q)}$$

$$= 2xy + p(-x^2p - xyq) - 2xy + q(xyp + qy^2)$$

$$= \cancel{2xy} - x^2p^2 - \cancel{xypq} - \cancel{2xy} + \cancel{xypq} + q^2y^2$$

$$= -\{(xp)^2 - (yq)^2\}$$

$$= -(xp+yq)(xp-yq)$$

$$= 0 \quad [\text{using (i)}]$$

Hence (i) & (ii) are compatible.

Solving (i) and (2) for p and q,

$$\text{(i)} \Rightarrow xp = yq \quad \text{—— (iii)}$$

$$\text{(ii)} \Rightarrow z(yq + yq) = 2xy$$

$$2yq = \frac{2xy}{z}$$

$$\Rightarrow q = \frac{x}{z} \quad \text{—— (iv)}$$

(iii) $\Rightarrow$

$$xp = y \cdot \frac{x}{z}$$

$$\Rightarrow p = \frac{y}{z} \quad \text{--- (v)}$$

Now using (iv) and (v) in

$dz = pdx + qdy$ we have

$$\Rightarrow dz = \left(\frac{y}{z}\right)dx + \left(\frac{x}{z}\right)dy$$

or, $zdz = ydx + xdy$

$$\Rightarrow zdz = d(xy)$$

Integrating,

$$\frac{z^2}{2} = xy + \frac{c}{2}$$

$$\Rightarrow z^2 = 2xy + c$$

where c is an arbitrary constant.

H.W. 2. Show that the equations $xp - yq = x$ and $x^2p + q = xz$ are compatible and find their solution.

Solution: Let $f(x, y, z, p, q) = xp - yq - x = 0$ --- (i)

$g(x, y, z, p, q) = x^2p + q - xz = 0$ --- (ii)

Charpit's method:

This is general method for solving equations with two independent variables. This method is applied to solve equations which can not be reduced to any of the standard forms through previous method.

Let the given equation be,

$$f(x, y, z, p, q) = 0 \quad \text{---} \ (1)$$

We know that, $dz = p\,dx + q\,dy \quad \text{---} \ (2)$

The auxiliary equation is,

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}}$$

$$= \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0}$$

Finding the value of p and q we will put this in (2).

~~1. Find a complete integral of $q = 3p^2$.~~

1. Find a complete integral of $q = 3p^2$

Solution:

Here, the given equation is,

$$f(x,y,z,p,q) \equiv 3p^2 - q = 0 \quad \text{——} \ (1)$$

$\therefore$ Charpit's Auxiliary equations are,

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}}$$

$$= \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{0 + p.0} = \frac{dq}{0 + q.0} = \frac{dz}{-p.6p - q(-1)}$$

$$= \frac{dx}{-6p} = \frac{dy}{-1} \quad \text{——} \ (2)$$

Taking 1st ~~two~~ fraction,

$dp = 0$ → like as Lagrange type 3.

$\cancel{\frac{xdx + ydy + zdz}{0}}$

$\therefore \cancel{xdx + ydy + zdz = 0}$

$\cancel{p = q + a}$

$\cancel{\Rightarrow q = p - a}$

$\cancel{3p = p + a}$

$\Rightarrow p = a$ ——— (3)

Substituting this value in (1)

$3a^2 - q = 0$

$\therefore q = 3a^2$

Putting these value of p & q in,

$dz = p\,dx + q\,dy$ we get,

$$dz = a\,dx + 3a^2\,dy$$

Integrating we get,

$$z = ax + 3a^2 y + b$$

which is a complete integral, a, b being arbitrary constant.

2. Find a complete integral of $zpq = p + q$.

Solution: Here, given equation,

$$f(x,y,z,p,q) = zpq - p - q = 0 \quad \text{——①}$$

∴ Charpit's Auxiliary equations are,

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}}$$

$$= \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{0 + p(pq)} = \frac{dq}{0 + q(pq)} = \frac{dz}{-p(zq-1) - q(zp-1)}$$

$$= \frac{dx}{-(zq-1)} = \frac{dy}{-(zp-1)} \quad \text{——②}$$

Taking 1st two fractions,

$$\frac{dp}{p^2q} = \frac{dq}{pq^2}$$

$$\Rightarrow \frac{dp}{p} = \frac{dq}{q}$$

Integreating both sides,

$$\ln p = \ln q + \ln a$$

$$\Rightarrow p = aq \quad \text{———} (3)$$

Putting the value of p in (1),

$$zaq \cdot q - aq - q = 0$$

$$\Rightarrow zaq^2 = (a+1)q$$

$$\Rightarrow zaq = 1+a$$

$$\therefore q = \frac{1+a}{az}$$

$\therefore$ (3) $\Rightarrow$

$$p = a \cdot \frac{1+a}{az}$$

$$\therefore p = \frac{1+a}{z}$$

Now, $dz = pdx + qdy$

$$\Rightarrow dz = \frac{1+a}{z}dx + \frac{1+a}{az}dy$$

$$\Rightarrow zdz = (1+a)\left[dx + \frac{1}{a}dy\right]$$

Integrating,

$$\frac{z^2}{2} = (1+a)\left(x + \frac{1}{a}y\right) + b$$

$$z^2 = 2(1+a)\left(x + \frac{1}{a}y\right) + 2b.$$

Ans.

3. Find a complete integral of $px + qy = pq$

Ans: $az = \frac{1}{2}(ax+y)^2 + b$

Particular Integral :

A particular integral is obtained by giving particular values of a & b in the complete integral.

Singular Integral :

we have to use,

$$\phi(x, y, z, a, b) = 0$$

$$\frac{\partial \phi}{\partial a} = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial b} = 0$$

In general it is distinct from the complete integral. However, in exceptional cases it may be contained in the comple integral that is, singular integral may be obtained by giving particular values to the constants in the complete integral.

1. Find a complete and ~~singular~~ particular integrals of $2xz - px^2 - 2qxy + pq = 0$

Solution:

Here given equation is,

$$f(x,y,z,p,q) \equiv 2xz - px^2 - 2qxy + pq = 0 \quad \text{--- (1)}$$

Charpit's auxiliary equations are,

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}}$$

$$= \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{2z - 2px - 2qy + p2x} = \frac{dq}{-2qx + q \cdot 2x} = \frac{dz}{-p(-x^2+q) - q(-2xy+p)}$$

$$= \frac{dx}{-(-x^2+q)} = \frac{dy}{-(-2xy+p)}$$

$$\Rightarrow \frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dz}{px^2 + 2xyq - 2pq}$$

$$= \frac{dx}{x^2 - q} = \frac{dy}{2xy - p}$$

The 2nd fraction gives,

$$dq = 0$$

$$\Rightarrow q = a$$

Putting $q = a$ in ①

$$2xz - px^2 - 2axy + ap = 0$$

$$\Rightarrow p(a - x^2) = 2x(ay - z)$$

$$\Rightarrow p(x^2 - a) = 2x(z - ay)$$

$$\therefore p = \frac{2x(z - ay)}{x^2 - a}$$

Putting the value of p & q in

$dz = pdx + qdy$ we get,

$$dz = \frac{2x(z - ay)}{x^2 - a} dx + a dy$$

$$\Rightarrow \frac{dz - a dy}{z - ay} = \frac{2x}{x^2 - a} dx$$

Integrating both sides,

$$\ln(z - ay) = \ln(x^2 - a) + \ln b$$

$$\Rightarrow z - ay = b(x^2 - a)$$

$$\Rightarrow z = ay + b(x^2 - a) \quad \text{—— ②}$$

which is the complete integral, a & b being arbitrary constant.

Differentiating (2) partially w.r. to a & b we get,

$0 = y - b$ and $0 = x^2 - a$ —— (3)

Solving (3)

$b = y$, $a = x^2$

Substituting a & b in (2) we get,

$$z = x^2y + y(x^2 - x^2)$$

$$\Rightarrow z = x^2y.$$

Which is the required singular integral.