...ution of 1st order differential equation:

Integrating factor For Linear diff. equⁿ:

A 1st order ordinary differential equation is linear in dependent variable y where x is independent variable if it can be written as,

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{—(1)}$$

A linear equation has an integrating factor,

$$I.F. = e^{\int P(x)dx}$$

Multiplying (1) by I.F., we may write

$$\frac{dy}{dx} e^{\int P(x)dx} + P(x)y e^{\int P(x)dx} = Q(x) e^{\int P(x)dx}$$

$$\Rightarrow \frac{d}{dx}\left[y e^{\int P(x)dx}\right] = Q(x) e^{\int P(x)dx} \quad \text{—(2)}$$

Integrating (2) we get,

$$y e^{\int P(x)dx} = \int Q(x) e^{\int P(x)dx} dx + c$$

$$\Rightarrow y = e^{-\int P(x)dx} \int Q(x) e^{\int P(x)dx} dx + c\, e^{-\int P(x)dx}$$

where c is constant.

1. Solve $\dfrac{dy}{dx} + \dfrac{2x+1}{x} y = e^{-2x}$

Solution: Here, $P(x) = \dfrac{2x+1}{x}$

$$\therefore \text{I.F.} = e^{\int \frac{2x+1}{x} dx} = e^{\int (2+\frac{1}{x}) dx} = e^{2x+\ln x}$$

$$= e^{2x} \cdot e^{\ln x}$$

$$= x e^{2x}$$

Multiplying the given ODE we get,

$$x e^{2x} \frac{dy}{dx} + x e^{2x} \cdot \frac{2x+1}{x} y = x e^{2x} \cdot e^{-2x}$$

$$\Rightarrow x e^{2x} \frac{dy}{dx} + e^{2x}(2x+1) y = x$$

$$\Rightarrow \frac{d}{dx} (x e^{2x} y) = x$$

$$\Rightarrow x e^{2x} y = \frac{x^2}{2} + c$$

$$\Rightarrow y = \frac{x}{2} e^{-2x} + \frac{c}{x} e^{-2x} \quad \text{Ans.}$$

(2). $(1-x^2) \frac{dy}{dx} - xy = 1$

Solution:

Given that,

$$(1-x^2) \frac{dy}{dx} - xy = 1$$

$$\Rightarrow \frac{dy}{dx} - \frac{xy}{1-x^2} = \frac{1}{1-x^2} \quad \text{--- (1)}$$

Integrating factor $= e^{\int -\frac{x}{1-x^2} dx}$

$$= e^{\frac{1}{2} \int \frac{-2x}{1-x^2} dx}$$

$$= e^{\frac{1}{2} \ln (1-x^2)}$$

$$= e^{\ln(1-x^2)^{1/2}}$$

$$= (1-x^2)^{1/2}$$

Multiplying bothsides by $\sqrt{1-x^2}$ we get ;

$$\sqrt{1-x^2}\,\frac{dy}{dx} - \frac{xy}{1-x^2}\sqrt{1-x^2} = \frac{\sqrt{1-x^2}}{1-x^2}$$

$$\Rightarrow \sqrt{1-x^2}\,\frac{dy}{dx} - \frac{xy}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{d}{dx}\left[y\sqrt{1-x^2}\right] = \frac{1}{\sqrt{1-x^2}}$$

Integrating w.r. to x we get ,

$$y\sqrt{1-x^2} = \int \frac{dx}{\sqrt{1-x^2}} + c$$

$$\Rightarrow y\sqrt{1-x^2} = \sin^{-1}x + c.$$

$$\therefore y = \sin^{-1}x\,(1-x^2)^{-1/2} + c\,(1-x^2)^{-1/2}$$

3) $(1+x^2)\frac{dy}{dx} + y = \tan^{-1}x$

Solution: Given that ,

$$(1+x^2)\frac{dy}{dx} + y = \tan^{-1}x$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{1+x^2} = \frac{\tan^{-1}x}{1+x^2} \quad \text{——— (1)}$$

$$\therefore \text{I.F.} = e^{\int \frac{dx}{1+x^2}} = e^{\tan^{-1}x}$$

Multiplying bothsides by $e^{\tan^{-1}x}$ we get,

$$\frac{dy}{dx}e^{\tan^{-1}x} + \frac{y}{1+x^2}e^{\tan^{-1}x} = \frac{\tan^{-1}x}{1+x^2}e^{\tan^{-1}x}$$

$$\Rightarrow \frac{d}{dx}\left[y e^{\tan^{-1}x}\right] = \frac{\tan^{-1}x \, e^{\tan^{-1}x}}{1+x^2}$$

Integreating w.r.to x

$$y e^{\tan^{-1}x} = \int \frac{\tan^{-1}x \, e^{\tan^{-1}x}}{1+x^2} dx \longrightarrow (2)$$

$$\Rightarrow y e^{\tan^{-1}x} = \int z e^z dz$$

Let, $\tan^{-1}x = z$

$\Rightarrow \frac{dx}{1+x^2} = dz$

$$\Rightarrow y e^{\tan^{-1}x} = z\int e^z dz - \int\left\{\frac{dz}{dz}\int e^z dz\right\} dz$$

$$\Rightarrow y e^{\tan^{-1}x} = z e^z - \int e^z dz$$

$$\Rightarrow y e^{\tan^{-1}x} = z e^z - e^z + c$$

$$\Rightarrow y e^{\tan^{-1}x} = (z-1)e^z + c$$

$$\Rightarrow y e^{\tan^{-1}x} = (\tan^{-1}x - 1)e^{\tan^{-1}x} + c$$

$$\therefore y = \tan^{-1}x - 1 + c e^{-\tan^{-1}x}$$

Ans

\# Solve $(2+y^2)dx = (xy + 2y + y^3)dy$

Ans: $x = 2 + y^2 + c\sqrt{2+y^2}$

Bernoulli Equation:

The equation, $\frac{dy}{dx} + P(x)y = Q(x)y^n$ is known as Bernoulli equation.

1. Solve $\frac{dy}{dx} + \frac{y}{x} = y^2$

Solution: Given that,

$$\frac{dy}{dx} + \frac{y}{x} = y^2$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{y}{xy^2} = 1$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = 1 \quad \text{—— (i)}$$

Let, $\frac{1}{y} = z$

$$\Rightarrow \frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$$

$$\therefore \frac{1}{y^2}\frac{dy}{dx} = -\frac{dz}{dx}$$

Now putting this value in (i)

$$-\frac{dz}{dx} + \frac{z}{x} = 1$$

$$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = -1 \quad \text{—— (ii)}$$

Here the equation is linear in z. So

$$I.F. = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$
$$= \frac{1}{x}$$

Now (11) $\times \frac{1}{x} \Rightarrow$

$$\frac{1}{x}\frac{dz}{dx} - \frac{z}{x^2} = -\frac{1}{x}$$

$$\Rightarrow \frac{d}{dx}\left(z \cdot \frac{1}{x}\right) = -\frac{1}{x}$$

$$\Rightarrow z \cdot \frac{1}{x} = \int -\frac{1}{x}\,dx + c$$

$$\Rightarrow \frac{z}{x} = -\ln x + c$$

$$\Rightarrow z = cx - x\ln x$$

$$\therefore \frac{1}{y} = cx - x\ln x. \quad \text{Ans.}$$

2. $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

Solution:

Given that,

$$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{y}{x} \times \frac{1}{y^2} = \frac{y^2}{x^2} \cdot \frac{1}{y^2}$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \frac{1}{x^2} \quad \text{--- (1)}$$

Let $\frac{1}{y} = z$

$$\Rightarrow \frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} = -\frac{dz}{dx}$$

Now Putting the value in (i),

$$-\frac{dz}{dx} + \frac{z}{x} = \frac{1}{x^2}$$

$$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x^2} \quad \text{—— (ii)}$$

This eqn is linear in z, we get

$$\text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = x^{-1} = \frac{1}{x}$$

(ii) $\times \frac{1}{x} \Rightarrow$

$$\frac{1}{x}\cdot\frac{dz}{dx} - \frac{z}{x^2} = -\frac{1}{x^3}$$

$$\Rightarrow \frac{d}{dx}\left(z\frac{1}{x}\right) = -\frac{1}{x^3}$$

$$\Rightarrow z\cdot\frac{1}{x} = -\int \frac{1}{x^3}dx + c$$

$$\Rightarrow \frac{z}{x} = \frac{1}{2x^2} + c$$

$$\Rightarrow \frac{1}{xy} = \frac{1}{2x^2} + c. \quad \text{Ans.}$$

3. $\frac{dy}{dx} + \frac{1}{x} \sin 2y = x^3 \cos^2 y$

Solution:

Given that,

$$\frac{dy}{dx} + \frac{1}{x} \sin 2y = x^3 \cos^2 y$$

$$\Rightarrow \frac{1}{\cos^2 y} \frac{dy}{dx} + \frac{1}{x} \frac{2 \sin y \cos y}{\cos^2 y} = x^3$$

$$\Rightarrow \sec^2 y \frac{dy}{dx} + \frac{2}{x} \tan y = x^3 \quad \text{--- (i)}$$

Let $\tan y = z$

$$\therefore \sec^2 y \frac{dy}{dx} = \frac{dz}{dx}$$

Putting this value in (i) we get,

$$\frac{dz}{dx} + \frac{2}{x} z = x^3 \quad \text{--- (ii)}$$

Now (ii) is linear in z.

$$\therefore I.F. = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = e^{\ln x^2} = x^2$$

Now (ii) $\times x^2 \Rightarrow$

$$x^2 \frac{dz}{dx} + \frac{2}{x} z \cdot x^2 = x^5$$

$$\Rightarrow x^2 \frac{dz}{dx} + 2xz = x^5$$

$$\Rightarrow \frac{d}{dx}(zx^2) = x^5$$

$$zx^2 = \int x^5 dx + c$$

$$\Rightarrow zx^2$$

$$\Rightarrow zx^2 = \frac{x^6}{6} + c$$

$$\Rightarrow z = \frac{x^4}{6} + \frac{c}{x^2}$$

$$\therefore \tan y = \frac{x^4}{6} + \frac{c}{x^2}$$

Solve $\frac{dy}{dx} + \frac{y}{x} \ln y = \frac{y}{x^2 (\ln y)^2}$

Ans: $(x \ln y)^3 = \frac{3}{2} x^2 + c$