Pair of straight line :

Every 1st degree equation in x and y represent a straight line and conversely every straight line can be represented by a 1st degree equation in x & y.

The general equation of the 1st degree is of the form $ax+by+c=0$, where a, b, c are constants.

1. The homogeneous quadratic equation $ax^2+2hxy+by^2=0$ always represents a pair of straight lines, real or imaginary through the origin.

Solution :

Proof :

Let us consider the equation,

$$ax^2+2hxy+by^2=0 \quad \text{——— (1)}$$

Multiplying both sides by a;

$$a^2x^2+2ahxy+aby^2=0 \quad , \text{ if } a \neq 0$$

$$\Rightarrow a^2x^2+2ahxy+h^2y^2-(h^2-ab)y^2=0$$

$$\Rightarrow (ax+hy)^2-\left(y\sqrt{h^2-ab}\right)^2=0$$

$$\Rightarrow \{(ax+hy)+y\sqrt{h^2-ab}\}\{(ax+hy)-y\sqrt{h^2-ab}\}=0$$

$\therefore$ The equation (1) therefore, represents the two straight lines whose equations are

$$ax + hy + y\sqrt{h^2 - ab} = 0 \quad \text{——} \ (2)$$

$$\text{and} \quad ax + hy - y\sqrt{h^2 - ab} = 0 \quad \text{——} \ (3)$$

$ax + by$

$y = mx$

Each of which passes through the origin.

Note:

These two straight lines are real and different if $h^2 > ab$, real and coincident (parallel) if $h^2 = ab$ and imaginary if $h^2 < ab$.

2. Prove that a homogeneous equation of the n-th degree represents n straight lines, real or imaginary, which all passes through the origin.

Proof: Consider the homogeneous equation,

$$y^n + a_1 x y^{n-1} + a_2 x^2 y^{n-2} + a_3 x^3 y^{n-3} + \dots + a_r x^r y^{n-r} + \dots + a_n x^n = 0 \quad \text{——} \ (1)$$

This can be written as (dividing by x^n)

$$\left(\frac{y}{x}\right)^n + a_1 \left(\frac{y}{x}\right)^{n-1} + a_2 \left(\frac{y}{x}\right)^{n-2} + a_3 \left(\frac{y}{x}\right)^{n-3} + \dots + a_r \left(\frac{y}{x}\right)^{n-r} + \dots + a_n = 0 \quad \text{——} \ (2)$$

Since this is an equation of the n-th degree in $\frac{y}{x}$ it must have n roots. Let the roots of this equation be,

$m_1, m_2, m_3; \dots, m_r, \dots, m_n$.

Then ② can be written as,

$$\left(\frac{y}{x}-m_1\right)\left(\frac{y}{x}-m_2\right)\cdots\left(\frac{y}{x}-m_r\right)\cdots\left(\frac{y}{x}-m_n\right)=0$$

or, $(y-m_1x)(y-m_2x)\cdots(y-m_rx)\cdots(y-m_nx)=0$

Thus, $y-m_1x=0$, $y-m_2x=0$, ..., $y-m_rx=0$,

...... $y-m_nx=0$

Which all passes through the origin.

Angle between the lines represented by the equation

$ax^2+2hxy+by^2=0$ —— ①

$$ax^2+2hxy+by^2=0$$

$y^2-(m_1+m_2)xy+m_1m_2x^2=0$

$by^2+2hxy+ax^2=0$

$y^2+\frac{2h}{b}xy+\frac{a}{b}x^2=0$

Dividing by x^2b we get,

$$\left(\frac{y}{x}\right)^2+2\frac{h}{b}\cdot\frac{y}{x}+\frac{a}{b}=0$$

Let the lines be, $y-m_1x=0$ and $y-m_2x=0$

So, $(y-m_1x)(y-m_2x)=ax^2+2hxy+by^2$

$\therefore m_1+m_2=-\frac{2h}{b}$ and $m_1m_2=\frac{a}{b}$

If θ is the angle between the straight lines then,

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$= \frac{\sqrt{(m_1 + m_2)^2 - 4 m_1 m_2}}{1 + m_1 m_2}$$

$$= \frac{\sqrt{\left(-\frac{2h}{b}\right)^2 - 4 \cdot \frac{a}{b}}}{1 + \frac{a}{b}}$$

$$= \frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{\frac{a+b}{b}}$$

$$= \frac{2\sqrt{\frac{h^2 - ab}{b^2}}}{\frac{a+b}{b}}$$

$$= \frac{\frac{2}{b}\sqrt{h^2 - ab}}{\frac{a+b}{b}}$$

$$\therefore \tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b} \quad \text{---} (2)$$

Where θ is the angle between the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$

Condition of perpendicularity:

If the straight lines are perpendicular to each other then $\theta = 90^\circ$ hence $\tan\theta = \infty$

then from (2), $\boxed{a + b = 0}$

Condition of Parallelism:

If the lines are coincident or parallel then $\theta = 0$

$\therefore h^2 - ab = 0$

$\Rightarrow \boxed{h^2 = ab}$

General equation of 2nd degree:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{--- (1)}$$

3) Find the condition that the general equation of the second degree may represent a pair of straight lines.

Solution: General equation of 2nd degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ (1)

If we transfer the origin to a point (α, β), the point of intersection of two straight lines and keep the direction of the axes unaltered, then (1) becomes,

$$a(x'+\alpha)^2 + 2h(x'+\alpha)(y'+\beta) + b(y'+\beta)^2 + 2g(x'+\alpha) + 2f(y'+\beta) + c = 0$$

$$\Rightarrow ax'^2 + 2hx'y' + by'^2 + 2(a\alpha + h\beta + g)x' + 2(h\alpha + b\beta + f)y' + a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0 \quad \text{--- (11)}$$

Then eq^n (11) may represent a pair of straight lines, if it is reduced to a homogeneous equation in x and y. This is possible if the co-efficients of x' and y' and the constant terms are separately zero.

i.e.
$$a\alpha + h\beta + g = 0 \quad \text{———} \quad (3)$$
$$h\alpha + b\beta + f = 0 \quad \text{———} \quad (4)$$
and
$$a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0 \quad \text{—} \quad (5)$$

(5) can be written as,

$$\alpha(a\alpha + h\beta + g) + \beta(h\alpha + b\beta + f) + g\alpha + f\beta + c = 0$$

$$\Rightarrow \quad g\alpha + f\beta + c = 0 \quad \text{———} \quad (6)$$

If we eliminate α, β from (3), (4) and (6), the required condition is,

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \Delta \text{ (let)}$$

$$\Rightarrow \Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

which is the required condition.

Prove that the following equations re[illegible] two straight lines. Also find their pair of intersection and the angle between them.

i. $3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0$

ii. $6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$

iii. $2y^2 + 3xy - 5y - 6x + 2 = 0$. $(-1, 2)$, $\tan^{-1} 3/4$.

Solution:

1. The general equation of 2nd degree is,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{--- (i)}$$

Given that,

$$-3x^2 - 8xy + 3y^2 - 29x + 3y - 18 = 0 \quad \text{--- (ii)}$$

Comparing (i) and (ii) we get,

$a = -3$, $b = 3$, $c = -18$, $f = \frac{3}{2}$, $g = -\frac{29}{2}$,

$h = -4$.

We know that, if the equation represent a pair of straight line then $\Delta = 0$.

$$\therefore \begin{vmatrix} -3 & -4 & -\frac{29}{2} \\ -4 & 3 & 3/2 \\ -\frac{29}{2} & \frac{3}{2} & -18 \end{vmatrix}$$

$$= -3\left(-54 - \frac{9}{4}\right) + 4\left(72 + \frac{87}{4}\right) - \frac{29}{2}\left(-6 + \frac{29 \times 3}{2}\right)$$

$$= \frac{675}{4} + 375 - \frac{2175}{4} = 0.$$

30) (ii) represent a pair of straight line.

Let, $F(x,y) = 3y^2 - 8xy - 3x^2 - 29x + 3y - 18$

$$\frac{\partial F}{\partial x} = -8y - 6x - 29 \quad , \quad \frac{\partial F}{\partial y} = 6y - 8x + 3$$

By cross product we get,

$$\frac{x}{-24 + 174} = \frac{y}{232 + 18} = \frac{1}{-36 - 64}$$

$$\Rightarrow \frac{x}{150} = \frac{y}{250} = \frac{1}{-100}$$

$\therefore x = -3/2 \ , \quad y = -5/2 .$

Let θ be the angle between two pair of straight line.

$$\tan\theta = \cancel{\tan} \ \frac{2\sqrt{h^2 - ab}}{a+b} = \frac{2\sqrt{16+9}}{-3+3} = \infty$$

$\Rightarrow \tan\theta = \tan \pi/2$

$\therefore \theta = \pi/2$

(ii) $a = 6, \ b = -6, \ c = 4, \ h = -5/2, \ g = 7,$

$f = 5/2$

$$\Delta = \begin{vmatrix} 6 & -5/2 & 7 \\ -5/2 & -6 & 5/2 \\ 7 & 5/2 & 4 \end{vmatrix}$$

$$= 6\left(-24 - \frac{25}{4}\right) + \frac{5}{2}\left(-10 - \frac{35}{2}\right) + 7\left(-\frac{25}{4} + \ldots\right.$$

$$= -\frac{726}{4} - \frac{275}{4} + \frac{1001}{4}$$

$$= \frac{-1001 + 1001}{4} = 0$$

$$\frac{\partial F}{\partial x} = 12x - 5y + 14, \quad \frac{\partial F}{\partial y} = -5x - 12y + 5$$

By cross product we get,

$$\frac{x}{-25+168} = \frac{y}{-70-60} = \frac{1}{-144-25}$$

$$\Rightarrow \frac{x}{143} = \frac{y}{-130} = \frac{1}{-169}$$

$$\therefore x = -\frac{11}{13}, \quad y = \frac{10}{13}$$

Let θ be the angle between two pair of straight line.

$$\therefore \tan\theta = \frac{2\sqrt{\frac{25}{4} + 36}}{6-6} = \frac{1}{0} = \infty = \tan \pi/2$$

$$\therefore \theta = \pi/2.$$

(ii)

3. Find the value of K so that the following equations represent pair of straight lines:

a. $2x^2+xy-y^2-2x-5y+K=0$

Solution:

Given that,

$$2x^2+xy-y^2-2x-5y+K=0 \quad \text{—— (I)}$$

The general equation of second degree is,

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{—— (II)}$$

Now comparing (I) and (II) we get,

$$a=2,\ b=-1,\ h=\frac{1}{2},\ g=-1,\ f=-\frac{5}{2},\ c=K$$

Now (I) represents a pair of straight lines if

$$\Delta = abc+2fgh-af^2-bg^2-h^2c=0$$

$$\Rightarrow -2K+2\left(-\frac{5}{2}\right)(-1)\frac{1}{2}-2\cdot\frac{25}{4}-(-1)(-1)^2-\frac{1}{4}K=0$$

$$\Rightarrow -2K+\frac{5}{2}-\frac{25}{2}+1-\frac{1}{4}K=0$$

$$\Rightarrow \frac{-8K-K}{4}+\frac{5-25+2}{2}=0$$

$$\Rightarrow \frac{-9K}{4}=+\frac{18}{2}$$

$$\Rightarrow \frac{-9K}{4}=9$$

$$\Rightarrow -9K=36$$

$$\therefore K=-4. \quad \text{Ans.}$$

Find the value of λ

$12x^2-10xy+2y^2+11x-5y+\lambda=0$

Ans: $\lambda=2$

6. Prove that the lines joining the origin to the intersection of $ax^2+2hxy+by^2+2gx=0$ and $a'x^2+2h'xy+b'y^2+2g'x=0$ are at right angles if $\frac{a+b}{g} = \frac{a'+b'}{g'}$

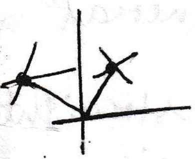
C.W.

Solution:

Given that,

$$ax^2+2hxy+by^2+2gx=0 \quad \text{---(i)}$$

$$a'x^2+2h'xy+b'y^2+2g'x=0 \quad \text{---(ii)}$$



(i) $\times g'$ − (ii) $\times g$ we get,

$$ag'x^2+2g'hxy+bg'y^2+2gg'x-a'gx^2-2gh'xy-b'gy^2-2gg'x=0$$

$$\Rightarrow (ag'-a'g)x^2+2(g'h-gh')xy+(bg'-b'g)y^2=0 \quad \text{---(iii)}$$

(iii) is a equation of pair of straight lines.

The two lines represented by (iii) will be at right angle if ~~the~~

* coefficient of x^2 + coefficient of $y^2=0$

$ax^2+2hxy+by^2=0$ Perpendicular if $a+b=0$

$$\Rightarrow (ag'-a'g)+(bg'-b'g)=0$$

$$\Rightarrow (a+b)g'-(a'+b')g=0$$

$\Rightarrow (a+b)g' = (a'+b')g$

$\Rightarrow \frac{a+b}{g} = \frac{a'+b'}{g'}$ [~~Divided by gg'~~]

$\therefore \frac{a+b}{g} = \frac{a'+b'}{g'}$ (Proved)

7. Show that the equation

$ax^2+2hxy+by^2+2gx+2fy+c=0$

represents two parallel line if $\frac{a}{h}=\frac{h}{b}=\frac{g}{f}$:

Solution:

We know the condition that the general equation of 2nd degree represent two straight lines if $\Delta=0$ and ~~if~~ represents two parallel lines if $h^2=ab$ or $h=\sqrt{ab}$.

Now, $\Delta=0$

$\Rightarrow abc+2fgh-af^2-bg^2-ch^2=0$ [$h^2=ab$]

$\Rightarrow \cancel{h^2c}+2fgh-af^2-bg^2-\cancel{ch^2}=0$

$\Rightarrow 2fg\sqrt{ab}-af^2-bg^2=0$

$\Rightarrow bg^2-2fg\sqrt{ab}+af^2=0$

$\Rightarrow (\sqrt{b}g)^2-2\cdot\sqrt{b}g\cdot\sqrt{a}f+(\sqrt{a}f)^2=0$

$\Rightarrow (\sqrt{b}g-\sqrt{a}f)^2=0$

$$\Rightarrow \sqrt{b}g - \sqrt{a}f = 0$$

$$\Rightarrow \sqrt{b}\,g = \sqrt{a}\,f$$

$$\Rightarrow \frac{g}{f} = \frac{\sqrt{a}}{\sqrt{b}}$$

$$\Rightarrow \frac{g}{f} = \sqrt{\frac{a}{b}}$$

$$\Rightarrow \frac{g}{f} = \sqrt{\frac{ab}{b^2}}$$

$$\Rightarrow \frac{g}{f} = \frac{\sqrt{h^2}}{\sqrt{b^2}}$$

$$\Rightarrow \frac{g}{f} = \frac{h}{b} \quad \text{——— (i)}$$

Again; $\quad ab = h^2$

$$\Rightarrow \frac{a}{h} = \frac{h}{b} \quad \text{——— (ii)}$$

Now from (i) and (ii) we get,

$$\frac{a}{h} = \frac{g}{f} = \frac{h}{b} \quad \text{(Proved)}.$$

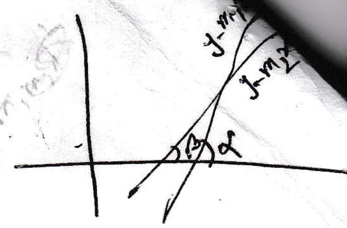
C.W. 8. If the straight lines represented by $x^2(\tan^2\theta + \cos^2\theta) - 2xy\tan\theta + y^2\sin^2\theta = 0$ makes angle α and β with x axis, show that $\tan\alpha - \tan\beta = 2$.

Solution:

$y^2-(m_1+m_2)xy+m_1m_2x^2=0$

$y^2-\frac{2\tan\theta}{\sin^2\theta}xy+\frac{\tan^2\theta+\cos^2\theta}{\sin^2\theta}x^2=0$

Suppose the lines are,



$y-m_1x=0$

and $y-m_2x=0$

Now, $m_1+m_2=\frac{-2h}{b}=\frac{2\tan\theta}{\sin^2\theta}$

$m_1m_2=\frac{a}{b}=\frac{\tan^2\theta+\cos^2\theta}{\sin^2\theta}$

Now, $\tan\alpha-\tan\beta=m_1-m_2$

$$=\sqrt{(m_1+m_2)^2-4m_1m_2}$$

$$=\sqrt{\frac{4\tan^2\theta}{\sin^4\theta}-\frac{4(\tan^2\theta+\cos^2\theta)}{\sin^2\theta}}$$

$$=2\sqrt{\frac{\tan^2\theta-\tan^2\theta\sin^2\theta-\cos^2\theta\sin^2\theta}{\sin^4\theta}}$$

$$=\frac{2\sqrt{\tan^2\theta(1-\sin^2\theta)-\cos^2\theta\sin^2\theta}}{\sin^2\theta}$$

$$=\frac{2\sqrt{\tan^2\theta\cos^2\theta-\cos^2\theta\sin^2\theta}}{\sin^2\theta}$$

$$=\frac{2\sqrt{\frac{\sin^2\theta}{\cos^2\theta}\cos^2\theta-\cos^2\theta\sin^2\theta}}{\sin^2\theta}$$

$$=\frac{2\sqrt{\sin^2\theta-\cos^2\theta\sin^2\theta}}{\sin^2\theta}$$

$$= \frac{2\sqrt{\sin^2\theta(1-\cos^2\theta)}}{\sin^2\theta}$$

$$= \frac{2\sqrt{\sin^4\theta}}{\sin^2\theta}$$

$$= \frac{2\sin^2\theta}{\sin^2\theta}$$

$$= 2$$

$\therefore \tan\alpha - \tan\beta = 2.$ (Proved)