Higher Order linear differential equations with constant coefficient :

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_{n-1} \frac{dy}{dx} + P_n y = Q(x) \quad \text{—①}$$

Above equation is higher order linear differential equation. if all P's are constant then we say this is higher order linear differential equation with constant coefficient.

Usually equ^n ① with $Q(x)=0$ has the solution $y=e^{mx}$ type. If $y=e^{mx}$ is a solution of

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_{n-1} \frac{dy}{dx} + P_n y = 0 \quad \text{—②}$$

Then it ~~must~~ satisfies the equ^n ① i.e. we get,

$$(m^n + P_1 m^{n-1} + P_2 m^{n-2} + \dots + P_{n-1} m + P_n)\, y = 0$$

$$\Rightarrow m^n + P_1 m^{n-1} + P_2 m^{n-2} + \dots + P_{n-1} m + P_n = 0 \quad \text{—③}$$

Equation ③ is known as auxiliary equation, then we have to solve equation ③ for m, Once we get the value of m, then the solution of ② will be obtained, suppose those solutions are

$y_1, y_2, \ldots, y_n$ (which depend on the solution type of m, will be discussed later). Then general solution of ② will be.

$$y_c = c_1y_1 + c_2y_2 + \cdots + c_ny_n .$$

This is known as complementary function of equation ①.

Let $D \equiv \frac{d}{dx}$, $D^2 \equiv \frac{d^2}{dx^2}$, $\ldots$ $D^n \equiv \frac{d^n}{dx^n}$. Then equation ① can be written as,

$$F(D)y = Q(x)$$

where,

$$F(D) = D^n + P_1D^{n-1} + P_2D^{n-2} + \cdots + P_{n-1}D + P_n$$

Then,

$y_p = \frac{Q(x)}{F(D)}$ will be a solution which is known as particular solution of (1). The rules of finding $\frac{Q(x)}{F(D)}$ will also be discussed later.

The general solution of (1) will be

$$y = y_c + y_p .$$

The rules of finding the complementary function:

To explain the rule let us consider 2^{nd} order DE.

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = Q(x) \quad \text{--- (4)}$$

If $y = e^{mx}$ be a solution of (4) then the auxiliary equation will be,

$$m^2 + P_1 m + P_2 = 0 \quad \text{--- (5)}$$

If the roots of (5)

(1) are distinct i.e. $m = m_1, m_2$ then the complementary function of (4) will be

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

(2) are equal say, $m = m_1 = m_2$, then the complementary function of (4) will be

$$y_c = c_1 e^{mx} + c_2 x e^{mx}$$

(3) are complex say, $m = \alpha \pm i\beta$ then the complementary function of (4) will be

$$y_c = e^{\alpha x}(c_1 \sin\beta x + c_2 \cos\beta x).$$

The rules of finding the Particular solution:

1. If the RHS is a Polynomial of x:

Let a differential equation be $F(D)y = X$, X is polynomial of x of order m.

Say $X(x) = x^m$.

Then $y_p = \frac{x^m}{F(D)}$, $F(D)$ can be expressed as a series,

Example:

$(D^2+1)y = x^3$ then

$$y_p = \frac{x^3}{D^2+1} = (1+D^2)^{-1}x^3 = (1-D^2+D^4\cdots)x^3$$
$$= x^3 - 6x$$

3. If the RHS is the type $V(x)e^{ax}$, a is a constant and $V(x)$ is a function of x.

$F(D)y = Ve^{ax}$ then $y_p = \frac{V(x)e^{ax}}{F(D)} = e^{ax}\frac{1}{F(D+a)}V(x)$

Example:

$(D+1)^2 y = x^3 e^{-x}$ then

$$y_p = \frac{x^3 e^{-x}}{(D+1)^2} = e^{-x}\frac{1}{(D-1+1)^2}x^3$$
$$= e^{-x}\frac{1}{D^2}x^3 = \frac{1}{20}x^5 e^{-x}$$

(i) ① If the RHS is the type Ae^{ax}, A and a are both constants.

$$F(D)y = Ae^{ax} \quad \text{then} \quad y_p = \frac{Ae^{ax}}{F(D)} = A\frac{e^{ax}}{F(a)},$$

if $F(a) \neq 0$

Example:

$$(D^2+D+1)y = 2e^{-x} \quad \text{then}$$

$$y_p = \frac{2e^{-x}}{D^2+D+1} = \frac{2e^{-x}}{(-1)^2+(-1)+1} = 2e^{-x}$$

(ii) ⑪ If the RHS is the type Ae^{ax}, A and a are both constants.

$$F(D)y = Ae^{ax} \quad \text{then} \quad y_p = \frac{Ae^{ax}}{F(D)} = Ae^{ax}\frac{1}{F(D+a)}\cdot 1$$

if $F(a) = 0$.

Example:

$$(D^2+3D+2)y = 3e^{-2x} \quad \text{then}$$

$$y_p = \frac{3e^{-2x}}{(D^2+3D+2)} = 3e^{-2x}\frac{1}{(D-2)^2+3(D-2)+2}$$

$$= 3e^{-2x}\frac{1}{D^2-4D+4+3D-6+2}\cdot 1$$

$$= 3e^{-2x}\frac{1}{D^2-D}\cdot 1$$

$$= -3e^{-2x} \frac{1}{D(1-D)} \cdot 1$$

$$= -3e^{-2x} \frac{1}{D} (1-D)^{-1} \cdot 1$$

$$= -3e^{-2x} \frac{1}{D} (1+D+D^2 - \cdots) \cdot 1$$

$$= -3e^{-2x} \frac{1}{D} \cdot 1$$

$$= -3xe^{-2x}$$

(4) When the r.h.s. is the trigonometric function like $\sin ax / \cos ax$ then put $D^2 = -a^2$.

1. solve $\frac{d^2y}{dx^2}+4y = x^2+2x+3$

Solution:

Given that,

$$\frac{d^2y}{dx^2}+4y = x^2+2x+3 \quad \text{—(I)}$$

The homogeneous equation of (I)

$$\frac{d^2y}{dx^2}+4y=0 \quad \text{—(II)}$$

If $y=e^{mx}$ is the solution of (II), then the auxiliary equation will be,

$$m^2+4=0$$

$$\Rightarrow m^2=-4$$

$$\Rightarrow m = 4i^2$$

$$\therefore m = 2i$$

Here m is a complex number. so,

$$y_c = e^{0\cdot x}(c_1 \sin 2x + c_2 \cos 2x)$$

$$= c_1 \sin 2x + c_2 \cos 2x$$

Here, the Particular solution is,

$$y_p = \frac{x^2+2x+3}{D^2+4}$$

$$= \frac{x^2+2x+3}{4(1+\frac{D^2}{4})}$$

$$= \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}(x^2+2x+3)$$

$$= \frac{1}{4}\left(1-\frac{D^2}{4}+\frac{D^4}{16}-\cdots\right)(x^2+2x+3)$$

$$= \frac{1}{4}\left(x^2+2x+3-\frac{1}{4}\cdot 2\right)$$

$$= \frac{1}{4}\left(x^2+2x+3-\frac{1}{2}\right)$$

$$= \frac{1}{4}\left(\frac{2x^2+4x+6-1}{2}\right)$$

$$= \frac{2x^2+4x+5}{8}$$

$(1+x)^{-1} = 1-x+x^2-x^3+\cdots$

$(1-x)^{-1} = 1+x+x^2+x^3+\cdots$

$\therefore$ The general solution of (i) is

$$y = y_c + y_p$$

$$= c_1 \sin 2x + c_2 \cos 2x + \frac{2x^2+4x+5}{8}$$

C.W.

2. $(D+3)^2 y = 4e^{2x} + 5e^{-3x}$

Solution: Given that,

$$(D+3)^2 y = 4e^{2x} + 5e^{-3x} \quad \text{—— (i)}$$

$$\Rightarrow (D^2+6D+9)y = 4e^{2x} + 5e^{-3x}$$

$$\Rightarrow \frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 4e^{2x} + 5e^{-3x} \quad \text{—— (ii)}$$

Here, the homogeneous equ^n of (ii),

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0 \quad \text{—— (iii)}$$

If $y = e^{mx}$ is the solution of (iii) then the auxiliary equ^n is,

$$m^2 + 6m + 9 = 0$$

$$\Rightarrow m^2 + 3m + 3m + 9 = 0$$

$$\Rightarrow m(m+3) + 3(m+3) = 0$$

$$\therefore (m+3)(m+3) = 0$$

$$\therefore m = -3, -3$$

The values of m are same, so

$$y_c = c_1 e^{-3x} + c_2 x e^{-3x}$$

Here, the particular solution is,

$$y_p = \frac{4e^{2x} + 5e^{-3x}}{(D+3)^2}$$

$$= \frac{4e^{2x}}{(D+3)^2} + \frac{5e^{-3x}}{(D+3)^2}$$

$$= 4e^{2x} \times \frac{1}{(2+3)^2} + 5e^{-3x} \frac{1}{(D-3+3)^2} \cdot 1$$

$$= \frac{4e^{2x}}{25} + 5e^{-3x} \times \frac{1}{D^2} \cdot 1$$

$$= \frac{4e^{2x}}{25} + 5 \cdot e^{-3x} \times \frac{x^2}{2}$$

$$= \frac{4e^{2x}}{25} + \frac{5x^2 e^{-3x}}{2}$$

$$\therefore y = y_c + y_p$$

$$= c_1 e^{-3x} + c_2 x e^{-3x} + \frac{4e^{2x}}{25} + \frac{5x^2 e^{-3x}}{2}$$

C.W. 1. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = e^{3x} + xe^{2x}$ —— (1)

Solution: Given that,

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = e^{x} + xe^{2x} \quad \text{—— (1)}$$

The homogeneous equⁿ of (1),

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0 \quad \text{—— (11)}$$

If $y = e^{mx}$ is the solution of (11), the the auxiliary equ^n of (11),

$$m^2 - 4m + 3 = 0$$

$$\Rightarrow m^2 - 3m - m + 3 = 0$$

$$\Rightarrow m(m-3) - 1(m-3) = 0$$

$$\Rightarrow (m-3)(m-1) = 0$$

$$\therefore m = 1, 3$$

Here, the values of m are distinct, so the equ^n is,

$$y_c = c_1 e^{x} + c_2 e^{3x}$$

The Particular solution,

$$y_p = \frac{e^{3x} + x e^{2x}}{D^2 - 4D + 3}$$

$$= \frac{e^{3x}}{D^2 - 4D + 3} + \frac{x e^{2x}}{D^2 - 4D + 3}$$

$$= e^{3x} \frac{1}{(D+3)^2 - 4(D+3) + 3} + e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 3} \cdot x$$

$$= e^{3x} \frac{1}{D^2 + 6D + 9 - 4D - 12 + 3} + e^{2x} \frac{1}{D^2 + 4D + 4 - 4D - 8 + 3} x$$

$$= e^{3x} \frac{1}{D^2+2D} \cdot 1 + e^{2x} \frac{1}{D^2-1} x$$

$$= e^{3x} \frac{1}{2D(1+\frac{D}{2})} \cdot 1 - e^{2x} \frac{1}{(1-D^2)} x$$

$$= \frac{1}{2} e^{3x} \left(1+\frac{D}{2}\right)^{-1} \frac{1}{D} \cdot 1 - e^{2x} (1-D^2)^{-1} x$$

$$= \frac{1}{2} e^{3x} \left(1 - \frac{D}{2} + \frac{D^2}{4} - \cdots\right) x - e^{2x}(1 + D^2 + D^4 + \cdots) x$$

$$= \frac{1}{2} e^{3x} \left(x - \frac{1}{2}\right) - e^{2x} \cdot x$$

$$= \frac{1}{2} x e^{3x} - \frac{1}{4} e^{3x} - x e^{2x}$$

The general solution of (1) is,

$$y = y_c + y_p$$

$$= c_1 e^{x} + c_2 e^{3x} + \frac{1}{2} x e^{3x} - \frac{1}{4} e^{3x} - x e^{2x}.$$

Ans.

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = \cos 2x$

Solution:

The auxiliary equation is,

$m^2 + 2m + 2 = 0$

$\Rightarrow m = \frac{-2 \pm \sqrt{2^2 - 8}}{2.1} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm \sqrt{4i^2}}{}$

$= \frac{2(-1 \pm i)}{2} = -1 \pm i$

$\therefore$ The complementary function is,

$y_c = e^{-x}(c_1 \sin x + c_2 \cos x)$

The particular sol^n is:

$y_p = \frac{1}{D^2 + 2D + 2} \cos 2x$

$= \frac{1}{-2^2 + 2D + 2} \cos 2x$

$= \frac{1}{2D - 2} \cos 2x$

$= \frac{1}{2(D-1)} \cos 2x$

$= \frac{1}{2} \frac{D+1}{(D-1)(D+1)} \cos 2x$

$= \frac{1}{2} \frac{D+1}{D^2 - 1} \cos 2x$

$$= \frac{1}{2} \frac{D+1}{-2^2-1} \cos 2x$$

$$= -\frac{1}{10}(D+1)\cos 2x$$

$$= -\frac{1}{10}\left[D(\cos 2x) + \cos 2x\right]$$

$$= -\frac{1}{10}(-\sin 2x \cdot 2 + \cos 2x)$$

$$= \frac{1}{10}(2\sin 2x - \cos 2x)$$

$\therefore$ The general sol^n is,

$$y = y_c + y_p$$

$$= e^{-x}(c_1 \sin x + c_2 \cos x) + \frac{1}{10}(2\sin 2x - \cos 2x)$$

Solve the following higher order diff. equ^n:

(i) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2$

(ii) $\frac{d^3y}{dx^3} + y = 3 + e^{-x} + 5e^{2x}$

Ans: $c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} + 3 + \frac{x}{3} e^{-x} + \frac{5}{9} e^{2x}$

(iii) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = \sin 2x$

$y = c_1 e^{-2x} + c_2 x e^{-2x} - \frac{1}{8}\cos 2x.$