Lagrange's method of solving the quasi-linear Partial differential equation of order one namely $Pp + Qq = R$, Known as Lagrange equation.

The general solution of the linear Partial differential equation,

$$Pp + Qq = R$$

is $\phi(u, v) = 0$

where ϕ is an arbitrary function and

$u(x, y, z) = c_1$ and $v(x, y, z) = c_2$

form a solution of the equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

Type-1: In this type taking two fraction we get the integral

1. Solve $(y^2z/x)p + xzq = y^2$

Solution:

Given that,

$$\left(\frac{y^2z}{x}\right)p + xzq = y^2 \quad \text{——} (1)$$

∴ The Lagrange's auxiliary equations for (1) are,

$$\frac{dx}{y^2z/x} = \frac{dy}{xz} = \frac{dz}{y^2} \quad \text{——} (2)$$

Taking 1st two fraction of (2) we get,

$$\frac{dx}{y^2z/x} = \frac{dy}{xz}$$

$$\Rightarrow \frac{dx}{y^2/x} = \frac{dy}{x}$$

$$\Rightarrow \frac{x\,dx}{y^2} = \frac{dy}{x} \quad \text{—— (3)}$$

$$\Rightarrow x^2dx - y^2dy = 0$$

Integreating (3) we get,

$$\frac{1}{3}x^3 - \frac{1}{3}y^3 = \frac{1}{3}c_1$$

$$\Rightarrow x^3 - y^3 = c_1 \quad \text{—— (4)}$$

Now taking 1st and the last fraction of (2) we get,

$$\frac{dx}{y^2z/x} = \frac{dz}{y^2}$$

$$\Rightarrow \frac{x\,dx}{z} = \frac{dz}{1}$$

$$\Rightarrow x\,dx - z\,dz = 0 \quad \text{—— (5)}$$

Integreating (3) we get,

$$\frac{1}{2}x^2 - \frac{1}{2}z^2 = \frac{1}{2}c_2$$

$$\therefore x^2 - z^2 = c_2 \quad \text{—— (6)}$$

Now from (4) and (6) the required general integral is,

$$\phi\left(x^3-y^3,\ x^2-y^2\right)=0$$

where ϕ is an arbitrary function.

2. Solve $y^2p-xyq=x(z-2y)$

Solution:

Given that,

$$y^2p-xyq=x(z-2y) \quad \text{--- (1)}$$

The Lagrange's auxiliary equations are,

$$\frac{dx}{y^2}=\frac{dy}{-xy}=\frac{dz}{x(z-2y)} \quad \text{--- (2)}$$

Taking the 1st two fractions of (2) we get,

$$\frac{dx}{y^2}=\frac{dy}{-xy}$$

$$\Rightarrow \frac{dx}{y}=\frac{dy}{-x}$$

$$\Rightarrow x\,dx+y\,dy=0 \quad \text{--- (3)}$$

Integrating both sides we get,

$$\frac{x^2}{2}+\frac{y^2}{2}=\frac{c_1}{2}$$

$$\Rightarrow x^2+y^2=c_1 \quad \text{--- (4)}$$

Now taking the last two fraction of (2) we get,

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\Rightarrow \frac{dy}{y} = \frac{dz}{2y-z}$$

$\Rightarrow 2ydy - zdy = ydz$

$\Rightarrow zdy + ydz = 2ydy$

$\Rightarrow d(zy) = 2ydy$

In.

$zy = 2 \cdot \frac{y^2}{2} + c_2$

$\Rightarrow zy - y^2 = c_2$

$$\Rightarrow \frac{dz}{dy} = \frac{2y-z}{y}$$

$$\Rightarrow \frac{dz}{dy} = -\frac{1}{y}z + 2$$

$$\Rightarrow \frac{dz}{dy} + \frac{1}{y}z = 2 \quad \text{------ (5)}$$

Which is linear.

$$\therefore I.F. = e^{\int \frac{1}{y} dy} = e^{\ln y} = y.$$

Now from (5) we get,

$$\frac{dz}{dy} \cdot y + \frac{1}{y} z \cdot y = 2y$$

$$\Rightarrow \frac{d}{dy}(zy) = 2y$$

Integreating both sides,

$$zy = 2 \cdot \frac{y^2}{2} + c_2$$

$$\Rightarrow zy - y^2 = c_2 \quad \text{------ (6)}$$

Now from (4) and (6) the general integreal is of the form,

$\phi(x^2+y^2, 2y-y^2) = 0$, where ϕ is an arbitrary function.

Type-2: Here 1st integral is obtained by type 1. But for 2nd integral the result of 1st integral is used.

3. Solve $p+3q = 5z + \tan(y-3x)$

Solution:

Given that,

$$p+3q = 5z + \tan(y-3x) \quad \text{—— (1)}$$

$\therefore$ The Lagrange's auxiliary equations of (1) are,

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z+\tan(y-3x)} \quad \text{—— (2)}$$

Taking 1st two fraction of (2) we get,

$$\frac{dx}{1} = \frac{dy}{3}$$

$$\Rightarrow dy - 3dx = 0$$

Integrating both sides we get,

$$y - 3x = c_1 \quad \text{—— (3)}$$

Now, taking 1st and last fraction of (2) we get,

$$\frac{dx}{1} = \frac{dz}{5z+\tan(y-3x)}$$

$$\Rightarrow \frac{dx}{1} = \frac{dz}{5z+\tan(c_1)} \quad [\text{using (3)}]$$

$$\Rightarrow \frac{dx}{1} = \frac{1}{5} \frac{5\,dz}{5z + \tan c_1}$$

Integrating both sides we get,

$$x = \frac{1}{5} \ln[5z + \tan c_1] + \frac{1}{5} c_2$$

$$\Rightarrow 5x - \ln[5z + \tan(y-3x)] = c_2 \quad \text{--- (4)}$$

Now from (3) and (4) the required general integral is,

$\phi(y-3x,\ 5x - \ln(5z + \tan(y-3x))) = 0$, where ϕ is an arbitrary function.

$\therefore\ 5x - \ln[5z + \tan(y-3x)] = \phi(y-3x)$, ϕ is an ...

4. Solve $z(z^2+xy)(px - qy) = x^4$

Solutions:

Given that,

$$z(z^2+xy)(px-qy) = x^4$$

$$\Rightarrow zx(z^2+xy)p - yz(z^2+xy)q = x^4 \quad \text{--- (1)}$$

The Lagranges auxiliary equation for (1) are,

$$\frac{dx}{xz(z^2+xy)} = \frac{dy}{-yz(z^2+xy)} = \frac{dz}{x^4} \quad \text{--- (2)}$$

Taking the 1st two fraction of (2) we get,

$$\frac{dx}{xz(z^2+xy)} = \frac{dy}{-yz(z^2+xy)}$$

$$\Rightarrow \frac{dx}{xz} = \frac{dy}{-yz}$$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{-y}$$

$$\Rightarrow \frac{dx}{x} + \frac{dy}{y} = 0$$

Integrating both sides we get,

$$\ln x + \ln y = \ln c_1$$

$$\Rightarrow \ln(xy) = \ln c_1$$

$$\therefore xy = c_1 \quad \text{——— (3)}$$

Now taking the 1st and last fraction we get,

$$\frac{dx}{xz(z^2+xy)} = \frac{dz}{x^4}$$

$$\Rightarrow \frac{dx}{xz(z^2+c_1)} = \frac{dz}{x^4} \quad [\text{using } (3)]$$

$$\Rightarrow \frac{dx}{z(z^2+c_1)} = \frac{dz}{x^3}$$

$$\Rightarrow x^3 dx - (z^3 + c_1 z)\, dz = 0$$

Now integrating both sides we get,

$$\frac{1}{4}x^4 - \frac{1}{4}z^4 - \frac{1}{2}c_1 z^2 = \frac{1}{4}c_2$$

$$\Rightarrow x^4 - z^4 - 2c_1 z^2 = c_2$$

$$\therefore x^4 - z^4 - 2xyz^2 = c_2 \quad \text{——— (4)}$$

Now from (3) and (4) the required general integral is

$\phi(xy,\ x^4 - z^4 - 2xyz^2) = 0$, where ϕ being an arbitrary function.

5. Solve $xyp + y^2 q = 2xy - 2x^2$.

Ans: $\phi\left(\frac{x}{y},\ x - \ln\left(z - 2\frac{x}{y}\right)\right) = 0$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{---} ①$$

Type-3: Let P_1, Q_1, R_1 be functions of x, y, and z. Then by well-known principle of algebra, each fraction in ① will be equal to

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R} \quad \text{---} ②$$

If $P_1 P + Q_1 Q + R_1 R = 0$ then we know that the numerator of ② is also zero.

$$\therefore P_1 dx + Q_1 dy + R_1 dz = 0$$

which can be integrated to give $u_1(x, y, z) = c_1$.

This method can be repeated to get another integral $u_2(x, y, z) = c_2$. P_1, Q_1, R_1 are called multipliers. These can be constant also. Sometimes only one integral is possible by this method, another is obtained by using type 1 or 2.

6. Solve $z(x+y)p + z(x-y)q = x^2 + y^2$

Solution:

Given that,

$$z(x+y)p + z(x-y)q = x^2 + y^2 \quad \text{---} ①$$

The Lagrange's auxiliary equations of ① are

$$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2} \quad \text{---} ②$$

choosing $x, -y, -z$ as multipliers, each fraction

$$= \frac{x\,dx - y\,dy - z\,dz}{xz(x+y) - yz(x-y) - z(x^2+y^2)} = \frac{x\,dx - y\,dy - z\,dz}{0}$$

$$\therefore x\,dx - y\,dy - z\,dz = 0$$

Integrating both sides we get,

$$\frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2} = \frac{c_1}{2}$$

$$\therefore x^2 - y^2 - z^2 = c_1 \quad \text{——— (3)}$$

Again choosing $y, x, -z$ as multipliers, each fraction

$$= \frac{y\,dx + x\,dy - z\,dz}{yz(x+y) + xz(x-y) - z(x^2+y^2)} = \frac{y\,dx + x\,dy - z\,dz}{0}$$

$$\therefore y\,dx + x\,dy - z\,dz = 0$$

$$\Rightarrow d(xy) - z\,dz = 0$$

Integrating both sides we get,

$$xy - \frac{1}{2}z^2 = \frac{1}{2}c_2$$

$$\Rightarrow 2xy - z^2 = c_2. \quad \text{——— (4)}$$

From (3) & (4) the required general solution is given by,

$\phi(x^2 - y^2 - z^2,\ 2xy - z^2) = 0$, where ϕ being an arbitrary function.

7. Solve $x(y^2-z^2)p - y(z^2+x^2)q = z(x^2+y^2)$

Solution:

Given that,

$$x(y^2-z^2)p - y(z^2+x^2)q = z(x^2+y^2) \quad \text{--- (1)}$$

The Lagrange's auxiliary equations of the given equation are,

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)} \quad \text{--- (2)}$$

choosing x, y, z as multipliers, each fraction of (2)

$$= \frac{xdx+ydy+zdz}{x^2(y^2-z^2)-y^2(z^2+x^2)+z^2(x^2+y^2)} = \frac{xdx+ydy+zdz}{0}$$

$$\Rightarrow xdx+ydy+zdz=0$$

Integreating both sides we get,

$$\frac{1}{2}x^2+\frac{1}{2}y^2+\frac{1}{2}z^2=\frac{1}{2}c_1$$

$$\Rightarrow x^2+y^2+z^2=c_1 \quad \text{--- (3)}$$

Again choosing $\frac{1}{x}, -\frac{1}{y}, -\frac{1}{z}$ as multipliers, each fraction of (2)

$$= \frac{\frac{1}{x}dx-\frac{1}{y}dy-\frac{1}{z}dz}{y^2-z^2+z^2+x^2-(x^2+y^2)} = \frac{\frac{1}{x}dx-\frac{1}{y}dy-\frac{1}{z}dz}{0}$$

$$\Rightarrow \frac{1}{x}dx-\frac{1}{y}dy-\frac{1}{z}dz=0$$

Now integrating both sides we get,

$$\ln x - \ln y - \ln z = \ln c_2$$

$$\Rightarrow \ln\left(\frac{x}{yz}\right) = \ln c_2$$

$$\therefore \frac{x}{yz} = c_2$$

Hence the required general solution is given by,

$$\phi\left(x^2+y^2+z^2, \frac{x}{yz}\right) = 0,$$ ϕ being an arbitrary function.

8. Solve $(y-zx)p + (x+yz)q = x^2+y^2$

Ans: $\phi(x^2-y^2+z^2, xy-z) = 0$

Solution: