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Disturbance Attenuation and H^∞ Optimization: A Design Method Based on the Algebraic Riccati Equation

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Abstract—**This note presents a method for designing a state feedback control law to reduce the effect of disturbances on the output of a given linear system. The problem under consideration involves attenuating the disturbances to a prespecified level. The construction of the state feedback control law requires the solution of a certain algebraic Riccati equation. By applying the procedure with successively smaller values of the prespecified disturbance level, the result also provides an approach to the H^∞ optimization problem for the state feedback case.**

I. INTRODUCTION

When designing a control system, one often begins with a plant which is subject to external disturbances. A common design objective is to reduce the effect of these disturbances to an acceptable level. For example, in the "disturbance decoupling problem" described in [1], state feedback control is used to ensure that the disturbances are completely decoupled from the output. However, for some systems, it may be impossible to reduce the effect of the disturbances below a certain threshold value. Hence, the disturbance decoupling problem would have no solution in this case. In this note, we present a procedure for designing a stabilizing state feedback control which will reduce the effect of the disturbances to a prespecified level.

The results of this note can also be applied to a class of H^∞ optimization problems in which the effect of the disturbances is minimized in a certain sense. Indeed, by applying our procedure with successively smaller values of the prespecified disturbance level, we can construct a state feedback control which is arbitrarily close to the H^∞ optimum. Thus, the results of this note are closely related to recent results on the H^∞ optimization problem; see [2]. However, previous results on the H^∞ optimization problem have led to quite complicated design procedures. In contrast, the design procedure proposed in this note is a simple scheme based on the algebraic Riccati equation. It follows that the algebraic Riccati equation which was previously associated with LQG optimization also plays a role in H^∞ optimization.

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II. SYSTEMS AND DEFINITIONS

We consider a linear system described by the state equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Dw(t); \\ z(t) &= Ex(t); \\ y(t) &= Cx(t) \end{aligned} \qquad (\Sigma)$$

where $x(t) \in \mathbb{R}^n$ is the *state*, $u(t) \in \mathbb{R}^m$ is the *control*, $w(t) \in \mathbb{R}^p$ is the *disturbance*, $y(t) \in \mathbb{R}^r$ is the *measured output*, and $z(t) \in \mathbb{R}^q$ is the *controlled output*.

The standard H^∞ optimization problem is concerned with constructing a dynamic feedback compensator $u = F(s)y$ to minimize the H^∞ norm of the transfer function from w to z; e.g., see [2]. In this note, we consider the special case in which the full state can be measured; i.e., $C = I$. This motivates us to restrict attention to constant state feedback controllers of the form $u = Fx$. The problem of extending our results to the case in which the full state cannot be measured is the topic of current research. One idea being pursued is the use of an observer to reconstruct the state variables; see [3].

The first problem considered in this note is that of reducing the effect of the disturbances to a prespecified level. Hence, we introduce the following definition.

Definition 2.1: Let the constant $\gamma > 0$ be given. The system (Σ) is said to be *stabilizable with disturbance attenuation* γ if there exists a state feedback matrix $F \in \mathbb{R}^{n \times m}$ such that the following conditions are satisfied.

1) The matrix $\bar{A} = A + BF$ is a stability matrix. That is, all of the eigenvalues of $\bar{A}$ lie in the open left-half plane.

2) The transfer function matrix

$$H(s) \triangleq E(sI - \bar{A})^{-1}D$$

satisfies the bound[1]

$$H(-j\omega)'H(j\omega) \leqslant \gamma^2 I$$

for all $\omega \in \mathbb{R}$. That is, the H^∞ norm of $H(s)$ is less than or equal to γ.

We now present a condition which can be used to test whether a given system is stabilizable with disturbance attenuation γ. In order to test this condition, the designer must specify two positive-definite matrices Q and R. It will be shown in the sequel that the question of disturbance attenuation is independent of the choice of Q and R.

Condition 1: Let $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$ be given positive-definite matrices and let the constant $\gamma > 0$ be prespecified. Then the system (Σ) is said to satisfy *Condition* 1 (*with attenuation constant* γ) if there exists an $\epsilon > 0$ such that the Riccati equation

$$A'P + PA - \frac{1}{\epsilon}PBR^{-1}B'P + \frac{1}{\gamma}PDD'P + \frac{1}{\gamma}E'E + \epsilon Q = 0 \qquad (2.1)$$

has a positive-definite solution P.

Remark: Riccati equation (2.1) is of a type which also arises in linear quadratic differential games. Furthermore, the process of decreasing ϵ corresponds to a linear quadratic differential game with cheap control for the minimizing player; see [4]. Thus, we will see that there is a connection between the H^∞ optimization problem and linear quadratic differential games with cheap control.

Theorem 2.1: Suppose that there exist positive-definite matrices $Q \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{m \times m}$ and a constant $\epsilon > 0$ such that Riccati equation (2.1) *has a positive-definite solution. Then, given any positive-definite matrices $\tilde{Q} \in \mathbb{R}^{n \times n}$ and $\tilde{R} \in \mathbb{R}^{m \times m}$, there exists a constant ϵ^**

[1] Given two matrices $N, M \in \mathbb{R}^{n \times n}$, the notation $N \geqslant M$ refers to the fact that the matrix $N - M$ is positive semi-definite. Similarly, the notation $N > M$ refers to the fact that the matrix $N - M$ is positive-definite.

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> 0 *such that the Riccati equation*

$$A'P+PA-\frac{1}{\tilde{\epsilon}}PB\tilde{R}^{-1}B'P+\frac{1}{\gamma}PDD'P+\frac{1}{\gamma}E'E+\tilde{\epsilon}\tilde{Q}=0 \tag{2.2}$$

has a positive-definite solution for all $\tilde{\epsilon} \in (0, \epsilon^*]$.

Proof: If (2.1) has a positive-definite solution P, then the positive-definite matrix $S = P^{-1}$ satisfies the Riccati equation

$$-AS-SA'-S\left(\frac{1}{\gamma}E'E+\epsilon Q\right)S+\frac{BR^{-1}B'}{\epsilon}-\frac{DD'}{\gamma}=0.$$

We now choose the constant $\epsilon^* > 0$ such that $\tilde{R}^{-1}/\epsilon^* > R^{-1}/\epsilon$ and $\epsilon^*\tilde{Q} < \epsilon Q$. Hence, using [5, Lemma 1] and the results of [6], it follows that the Riccati equation

$$-AS-SA'-S\left(\frac{1}{\gamma}E'E+\tilde{\epsilon}\tilde{Q}\right)S+\frac{B\tilde{R}^{-1}B'}{\tilde{\epsilon}}-\frac{DD'}{\gamma}=0$$

has a maximal symmetric solution S^+ for all $\tilde{\epsilon} \in (0, \epsilon^*]$. Furthermore, using the main result of [7], it follows that $S^+ \geq S > 0$. However, the positive-definite matrix $P^+ = S^{+-1}$ must satisfy Riccati equation (2.2). This is the required result. □

III. The Main Result

Theorem 3.1: Suppose that the system (Σ) *satisfies Condition* 1 *with attenuation constant* γ. *Then* (Σ) *is stabilizable with disturbance attenuation* γ. *Furthermore, the required feedback gain matrix is given by*

$$F=-\frac{1}{2\epsilon}R^{-1}B'P \tag{3.1}$$

where P is the solution Riccati equation (2.1).

Conversely, let $\gamma > 0$ *be given and suppose that there exists a constant* $\delta \in (0, \gamma)$ *such that the system* (Σ) *is stabilizable with disturbance attenuation* $\gamma - \delta$. *Then the system* (Σ) *must satisfy Condition* 1 *with attenuation constant* γ.

Remark: It should be noted that there is a gap between the necessity part and the sufficiency part of the above theorem. In practice, this gap will not be important since it involves only an infinitesimal change in the achievable disturbance attenuation.

Proof of Theorem 3:1: Suppose that the system (Σ) satisfies Condition 1 with attenuation constant γ. Using the positive-definite solution P of Riccati equation (2.1), we construct the required state feedback matrix according to equation (3.1). That is

$$F=-\frac{1}{2\epsilon}R^{-1}B'P.$$

This results in a closed-loop system matrix $\bar{A} = A + BF$ such that

$$\bar{A}'P+P\bar{A}=A'P+PA-\frac{1}{2\epsilon}PBR^{-1}B'P-\frac{1}{2\epsilon}PBR^{-1}B'P$$
$$=-\frac{1}{\gamma}PDD'P-\frac{1}{\gamma}E'E-\epsilon Q.$$

Consequently, the positive-definite matrix P satisfies the Riccati equation

$$\bar{A}'P+P\bar{A}+\frac{1}{\gamma}PDD'P+\frac{1}{\gamma}E'E+\epsilon Q=0. \tag{3.2}$$

Therefore, the matrix $\bar{A}'P + P\bar{A}$ is negative-definite. It follows immediately that $\bar{A}$ is a stability matrix; e.g., see [1]. Furthermore, if we substitute $K = -P$ into (3.2), we arrive at the Riccati equation

$$\bar{A}'K+K\bar{A}-\frac{1}{\gamma}KDD'K-\frac{1}{\gamma}E'E-\epsilon Q=0.$$

Using [6, Lemma 1], we now conclude that

$$\gamma I-D'(-j\omega I-\bar{A}')^{-1}\left(\frac{E'E}{\gamma}+\epsilon Q\right)(j\omega I-\bar{A})^{-1}D \geqslant 0$$

for all $\omega \in \mathbb{R}$. Hence, the transfer function matrix $H(s) \triangleq E(sI - \bar{A})^{-1}D$ satisfies the inequality

$$H(-j\omega)'H(j\omega) \leqslant \gamma^2 I-\gamma\epsilon D'(-j\omega I-\bar{A}')^{-1}Q(j\omega I-\bar{A})^{-1}D \leqslant \gamma^2 I$$

for all $\omega \in \mathbb{R}$. Thus, we have established that the system (Σ) is stabilizable with disturbance attenuation γ. This completes the proof of the first part of the theorem.

Conversely, suppose that there exists a constant $\delta \in (0, \gamma)$ such that the system (Σ) is stabilizable with disturbance attenuation $\gamma - \delta$. Therefore, there exists a state feedback matrix $F \in \mathbb{R}^{n\times m}$ such that the resulting closed-loop system matrix $\bar{A} = A + BF$ has the following properties.

i) $\bar{A}$ is a stability matrix.

ii) The transfer function matrix $H(s) = E(sI - \bar{A})^{-1}D$ is such that $H(-j\omega)'H(j\omega) \leqslant (\gamma - \delta)^2 I$ for all $\omega \in \mathbb{R}$. Hence,

$$D'(-j\omega I-\bar{A}')^{-1}E'E(j\omega I-\bar{A})^{-1}D$$
$$\leqslant (\gamma^2-2\gamma\delta+\delta^2)I \leqslant (\gamma^2-\gamma\delta)I \tag{3.3}$$

for all $\omega \in \mathbb{R}$. We now define a transfer function matrix by $W(s) \triangleq (sI - \bar{A})^{-1}D$. This rational transfer function matrix is strictly proper and has no poles on the $j\omega$-axis. Hence, there exists a constant $\mu > 0$ such that $W(-j\omega)'W(j\omega) \leqslant \mu I$ for all $\omega \in \mathbb{R}$. Using this fact, it follows from (3.3) that

$$D'(-j\omega I-\bar{A}')^{-1}E'E(j\omega I-\bar{A})^{-1}D$$
$$\leqslant \left(\gamma^2-\frac{1}{2}\gamma\delta\right)I-\frac{\gamma\delta W(-j\omega)'W(j\omega)}{2\mu}$$

for all $\omega \in \mathbb{R}$. A nonsingular matrix $\tilde{E} \in \mathbb{R}^{n\times n}$ is now defined according to the expression $\tilde{E}'\tilde{E} = E'E + \gamma\delta I/(2\mu)$. Consequently,

$$D'(-j\omega I-\bar{A}')^{-1}\tilde{E}'\tilde{E}(j\omega I-\bar{A})^{-1}D \leqslant \gamma\left(\gamma-\frac{\delta}{2}\right)I$$

for all $\omega \in \mathbb{R}$. This implies that

$$\tilde{E}(j\omega I-\bar{A})^{-1}DD'(-j\omega I-\bar{A}')^{-1}\tilde{E}' \leqslant \gamma\left(\gamma-\frac{\delta}{2}\right)I \tag{3.4}$$

for all $\omega \in \mathbb{R}$. In a similar fashion to above, we can define a transfer function matrix $Y(s) \triangleq (sI + \bar{A})^{-1}\tilde{E}'$ and a constant $\eta > 0$ such that $Y(-j\omega)'Y(j\omega) \leq \eta I$ for all $\omega \in \mathbb{R}$. Using this fact, it follows from (3.4) that

$$\tilde{E}(-j\omega I+\bar{A})^{-1}DD'(j\omega I+\bar{A}')^{-1}\tilde{E}' \leqslant \gamma^2 I-\frac{\gamma\delta Y(-j\omega)'Y(j\omega)}{2\eta}$$

for all $\omega \in \mathbb{R}$. A nonsingular matrix $\tilde{D} \in \mathbb{R}^{n\times n}$ is now defined according to the expression $\tilde{D}\tilde{D}' = DD' + \gamma\delta I/(2\eta)$. It follows that

$$\tilde{E}(-j\omega I+\bar{A})^{-1}\tilde{D}\tilde{D}'(j\omega I+\bar{A}')^{-1}\tilde{E}' \leqslant \gamma^2 I$$

for all $\omega \in \mathbb{R}$, and hence

$$\tilde{D}'(-j\omega I-\bar{A}')^{-1}\frac{\tilde{E}'\tilde{E}}{\gamma}(j\omega I-\bar{A})^{-1}\tilde{D} \leqslant \gamma I$$

for all $\omega \in \mathbb{R}$. Using this inequality, it follows from [8, Theorem 2, p. 167] that the Riccati equation

$$\bar{A}'K+K\bar{A}-\frac{K\tilde{D}\tilde{D}'K}{\gamma}-\frac{\tilde{E}'\tilde{E}}{\gamma}=0$$

has a negative-definite symmetric solution K. Letting $\tilde{P} = -K$, it follows that

$$\bar{A}'\tilde{P}+\tilde{P}\bar{A}+\frac{\tilde{P}\tilde{D}\tilde{D}'\tilde{P}}{\gamma}+\frac{\tilde{E}'\tilde{E}}{\gamma}=0. \tag{3.5}$$

Therefore,

$$A'\tilde{P}+\tilde{P}A+F'B'\tilde{P}+\tilde{P}BF+\frac{1}{\gamma}\tilde{P}DD'\tilde{P}+\frac{\delta}{2\eta}\tilde{P}\tilde{P}+\frac{1}{\gamma}E'E+\frac{\delta}{2\mu}I=0.$$

Hence, given any nonzero $x \in \mathbb{R}^n$ such that $B'\tilde{P}x = 0$, it follows that

$$x'\left[A'\tilde{P}+\tilde{P}A+\frac{1}{\gamma}\tilde{P}DD'\tilde{P}+\frac{1}{\gamma}E'E\right]x=-\frac{\delta}{2\mu}x'x-\frac{\delta}{2\eta}x'\tilde{P}\tilde{P}x<0.$$

We now apply Finsler's theorem to this inequality; see [9]. It follows that there exists a constant $\sigma \geq 0$ such that the matrix

$$-\tilde{Q} \triangleq A'\tilde{P}+\tilde{P}A+\frac{1}{\gamma}\tilde{P}DD'\tilde{P}+\frac{1}{\gamma}E'E-\sigma\tilde{P}BB'\tilde{P}$$

is negative-definite. Therefore, the positive-definite matrix $\tilde{P}$ satisfies the Riccati equation

$$A'\tilde{P}+\tilde{P}A-\sigma\tilde{P}BB'\tilde{P}+\frac{1}{\gamma}\tilde{P}DD'\tilde{P}+\frac{1}{\gamma}E'E+\tilde{Q}=0. \tag{3.6}$$

Let $R \in \mathbb{R}^{m\times m}$ and $Q \in \mathbb{R}^{n\times n}$ be any positive-definite weighting matrices and choose the constant $\epsilon^* > 0$ such that $R^{-1}/\epsilon^* > \sigma I$ and $\epsilon^* Q < \tilde{Q}$. As in the proof of Theorem 2.1, we now use the results of [5]–[7]. That is, the fact that Riccati equation (3.6) has a positive-definite solution implies that the Riccati equation

$$A'P+PA-\frac{1}{\epsilon^*}PBR^{-1}B'P+\frac{1}{\gamma}PDD'P+\frac{1}{\gamma}E'E+\epsilon^*Q=0$$

will also have a positive-definite solution. Consequently, the system (Σ) satisfies Condition 1 with attenuation constant γ. □

Remarks: It can be seen from Theorems 2.1 and 3.1 that our design procedure involves solving a Riccati equation for successively smaller values of the constant ϵ. Thus, to apply our results in practice, one would simply apply a standard numerical algorithm to solve Riccati equation (2.1); e.g., see [10] and [11]. It should be noted that solving Riccati equation (2.1) with extremely small values of ϵ may lead to numerical problems in some cases. However, in most of the examples to which the method has been applied so far, no numerical problems were encountered; see [12].

H^∞ Optimization: It should be noted that the design procedure described above can be used to construct a stabilizing controller which achieves a disturbance attenuation level which is arbitrarily close to the H^∞ optimum. This is achieved by applying the procedure with successively smaller values of disturbance attenuation parameter γ.

Frequency Weighted Disturbance Attenuation: Another application for the design procedure described above is the frequency weighted disturbance attenuation problem. In this problem, frequency weighting transfer functions $V(s)$ and $W(s)$ are prespecified and the disturbance attenuation requirement $H(-j\omega)'H(j\omega) \leq \gamma^2 I$ is replaced by the requirement $V(-j\omega)'H(-j\omega)'W(-j\omega)'W(j\omega)H(j\omega)V(j\omega) \leq \gamma^2 I$. It is straightforward to verify that this problem can be solved by applying our results to an augmented system. This system is formed by augmenting the original system (Σ) with state-space realizations for $V(s)$ and $W(s)$. This approach will result in a dynamic state feedback compensator.

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Controllability of Generalized Linear Time-Invariant Systems

AMIT AILON

Abstract—**This note deals with the problem of *c*-controllability of generalized systems. A new criterion for determining *c*-controllability of generalized systems is established. With controllability established, it is shown that it is always possible to transfer the system from the initial position to any target by a control vector-valued function whose elements are polynomials of bounded degree. The polynomial coefficients are obtained as a solution to a set of linear equations.**

I. Introduction

In this study we consider the finite-dimensional generalized linear time-invariant system

$$E\dot{x}(t)=Ax(t)+Bu(t), \qquad t\geq 0 \tag{1.1}$$

where $x \in R^n$, $u \in R^m$, E and A are $n \times n$ matrices, possibly singular, and B is an $n \times m$ matrix.

The system (1.1) is a mixture of static and dynamic equations, and can serve as a useful model in analysis of many real and practical situations. Quite naturally, systems of the form (1.1) have been the subject of much attention. Recently, analysis of generalized systems resembling (1.1) for the case of the matrix E being singular has been given in several papers. To cite a few examples: Luenberger [1], [2], Dervisoğlu and Desoer [3], Verghese *et al.* [4], [5], Aplevich [6], Bernhard [7], Sincovec *et al.* [8] Yip and Sincovec [9], Cobb [10], and Lewis [11].

The problem considered in this note is that of complete controllability (c.c.) of the generalized system (1.1). Our major result shows that if the generalized system is completely controllable, then a polynomial control function that steers the system from its current position to the target can be computed by solving a set of linear equations. Their solution yields the polynomial coefficients. Hence, application of the theory of this note transforms the problem from a system consisting of a set of static and dynamic equations to a set of linear equations. This study also provides a new controllability criterion for singular systems.

This result can be implemented for use in design problems in singular systems. Firstly, by solving a set of linear equations one can determine a control function (which is in fact a polynomial) that ensures the transfer of the system from its current position to a desired target. Next, it is possible to implement the approach presented in the paper to parameterization procedures in order to obtain suboptimal control solutions. If one restricts the function $u(t)$ to a subset C_m (the set of all polynomials in the space of admissible controls) by specifying $u^*(t) = F(P, t)$ where the parameter

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