Robustness of Model Reference Adaptive Controllers: An Input–Output Approach

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Abstract—**A general input–output approach is employed to study the direct model reference adaptive control problem in the presence of parametric and unstructured uncertainty. A sufficient condition for the boundedness of the closed-loop signals is developed, relating an appropriately defined size of the unstructured uncertainty with the size of parametric uncertainty. This condition involves the intuitively appealing gains of various sensitivity operators and allows for the study of the effect of the design parameters on the boundedness properties of the closed-loop system. The results of this work indicate a direction towards a unified and systematic theory in adaptive control.**

I. Introduction

The issue of robustness of adaptive controllers was the topic of numerous studies, publications, and discussions during the late 1980's. In this period, it became evident that adaptive control systems can successfully operate in the presence of plant unmodeled dynamics, provided that the modeling error is "small" relative to the rest of the signals in the closed loop [1]–[8]. On the other hand, it was also evident that, especially in the case of direct model reference adaptive control, the available analytical tools did not provide a clear characterization of the class and size of the unmodeled dynamics for which boundedness can be guaranteed without some additional assumptions (e.g., relative degree one nominal plants, persistently exciting inputs [9]–[11]). Further efforts in this direction, [12], [13], revealed that frequency domain bounds can also characterize the allowable plant uncertainty; however, the procedure to obtain these bounds was quite complicated, relying on uniform continuity arguments, and offering little insight on the effects of the design parameters on the closed-loop boundedness properties.

In this paper, we consider the robust stability problem in continuous-time direct model reference adaptive control using an adaptive law with normalization and projection modifications [2], [12]. This adaptive law (or variations of it) has often been used in the adaptive control problem to ensure boundedness in the presence of perturbations [14], [2], [12], [5] with the key observation, first made by Praly, that this normalizing signal bounds the contribution of the unmodeled dynamics for a wide class of uncertainty operators. Following a general I/O approach in an exponentially weighted L_2 space, we develop a sufficient condition for boundedness relating the size of the unstructured uncertainty with the size of the parametric one. The former is defined as an appropriate gain of the dynamically-weighted uncertainty operator, while the latter is the diameter of the set of parametric uncertainty translated in the controller parameters.

An attractive feature of our boundedness condition, attributed to some key technical results presented in the Appendix, is that it is expressed in terms of the sensitivity operators of a nominal (or "tuned") closed-loop system. In this form, various weighting techniques can be used to reduce conservatism and recover in the limit, results from nonadaptive (zero-parametric uncertainty) and "ideal" adaptive cases (zero-unstructured uncertainty). Furthermore, our approach not only simplifies the analysis and derivations (e.g., compare to [12], [5], [13]) but also allows for the numerical evaluation of the boundedness condition. Although at present such an evaluation may be cumbersome for design purposes, it is expected to provide valuable insight on the global properties in adaptive control with arbitrary inputs, complementing the more specialized results of [9]–[11].

A. Notation and Acronyms

$(x)_t$: truncation of a signal x at t;

$\|\cdot\|$: function norm; $|\cdot|$: vector norm (2 or ∞);

$L_p(\delta)$: the exponentially weighted L_p space [20], [27] $\{x: \|x\|_{p,\delta} \triangleq \|\mathscr{E}_\delta x\|_p < \infty\}$ with norm $\|\cdot\|_{p,\delta}\}$, where $p \in [1,\infty]\|\cdot\|_p$ is the usual L_p norm and $\mathscr{E}_\delta$ is the multiplication operator $(\mathscr{E}_\delta x)(t) = e^{\delta t}x(t)$;

$\gamma_{p,\delta}[H]$: the induced gain of an operator $H: L_p(\delta) \mapsto L_p(\delta)$; when H is described by a transfer function $H(s)$, $\gamma_{2,\delta}[H] = \|H(s-\delta)\|_\infty$;

$g_{p,\delta}[H]$: the induced gain of an operator $H: L_p(\delta) \mapsto L_\infty(\delta)$; when H is described by a transfer function $H(s)$, $g_{2,\delta}[H] = \|H(s-\delta)\|_2$;

$\boldsymbol{N}, \boldsymbol{R}, \boldsymbol{R}_+, \boldsymbol{C}$: the sets of natural, real, nonnegative real, and complex numbers, respectively;

$\Sigma(a) \triangleq \{s \in \boldsymbol{C}: \operatorname{Re}(s) \le -a;\ a \in \boldsymbol{R}\}$;

$\epsilon_a(t)$: an exponentially decaying term, $|\epsilon_a(t)| \le ce^{-at}$; $a > 0$;

MRAC: model reference adaptive control; BIBO: bounded-input bounded-output; SISO: single-input single-output; LTI: linear time-invariant; I/O: input–output; UB: uniformly bounded.

II. The Adaptive Control System

Let $n, n^*, N \in \boldsymbol{N}$, and $\alpha > 0$ and consider a family of casual LTI operators, say $\{\boldsymbol{P}_p\}$, parametrized by a parameter vector $p \in \mathscr{P} \subset \boldsymbol{R}^N$ as follows. For any member $P \in \{\boldsymbol{P}_p\}$ there exists $p \in \mathscr{P}$ such that P is described by a transfer

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function $P_0(s;p)$, having order n, relative degree n^*, positive high-frequency gain, and zeros in $\Sigma(\alpha)$. Further, consider a SISO plant with I/O description

$$y_p = P[u_p] \tag{2.1}$$

where (u_p, y_p) is the I/O pair of the plant and suppose that $P \in \{P_p\}$ for which n, n^* are known.

For such a plant, the MRAC objective is to design an adaptive controller such that:

MRC.OBJ ("control objective") The closed-loop plant is BIBO stable and the plant output y_p tracks, as closely as possible, the output of a reference model y_m given by

$$y_m = W_m[r] \tag{2.2}$$

where W_m is an LTI operator with an exponentially stable, minimum-phase transfer function $W_m(s)$, of order $\leq n$, relative degree n^*, and positive high-frequency gain and $r(t)$ is a piecewise continuous, UB reference input signal. ■

General solutions to this problem can be found in, e.g., [15], [16]. On the other hand, in more realistic situations, one can only expect that (2.1) with $P \in \{P_p\}$ is merely an approximate—in some sense—description of the plant I/O properties. This consideration gives rise to some very interesting robustness questions, related to the violation of some or all of the plant assumptions. In order to formulate such a robustness problem, let us assume the following.

Assumption 2.1: ("Plant") The I/O map of the SISO plant is given by (2.1) with P being a causal I/O operator of the form

$$P = P_0(1 + \Delta) \tag{2.3}$$

where $P_0 \in \{P_p\}$ is the modeled part of the plant, for which n, n^* are known and Δ is a causal multiplicative uncertainty. ■

The control objective for the plant (2.1), (2.3) is the same as before (MRC.OBJ) with the only difference being that the reference model is selected based on the modeled part of the plant.

In this framework we may pose the MRAC robustness as an analysis problem, that is, given an MRAC designed for the modeled part of the plant, find a characterization of Δ for which BIBO stability is preserved. From a different point of view, one may think of the same problem as to design a control law of the model reference type for a nominal plant $P_0(s;\, p_*)$, $p_* \in \mathscr{P}$, subject to parametric and nonparametric (unstructured) uncertainty. In any case, it should be emphasized that not only P_0, but also the unstructured uncertainty Δ, depends on the parameter vector p, since (2.3) implies that different p's give rise to different modeled parts of the plant and consequently different uncertainty operators Δ. In the same vein, notice that in contrast to the robust (nonadaptive) control case, Δ needs not account for imprecise knowledge of the poles of P_0, something that can be incorporated in the parametric uncertainty $\mathscr{P}$.

In the present study, we will consider multiplicative uncertainties Δ satisfying the following assumption.

Assumption 2.2: ("Unstructured Uncertainty") There exist an *a priori* known constant $\delta_* > 0$ and an LTI operator W_p such that for some $p \in \mathscr{P}$:

- the transfer function of W_p has poles and zeros in $\Sigma(\delta_*)$ and relative degree $< n^*$;
- the operator $W_p\Delta$ has a bounded and measurable in t state-space description satisfying a globally Lipschitz condition in the states and input;
- $W_p\Delta\colon L_2(\delta_*) \mapsto L_2(\delta_*)$ and $\gamma_{2,\delta_*}[W_p\Delta] < \infty$. ■

Further, let us consider the following control law that can be used to achieve the MRC objective for any fixed, known $p \in \mathscr{P}$ and $\Delta = 0$:

$$w = \begin{bmatrix} (sI-F)^{-1}q_1 & 0 \\ 0 & (sI-F)^{-1}q_2 \end{bmatrix}[U] \triangleq G[U] \tag{2.4}$$

$$u_p = \theta^{\mathrm{T}} w + c_0 r; \quad U = [u_p, y_p]^{\mathrm{T}}$$

where

- F is an $n \times n$ Hurwitz matrix whose eigenvalues include the zeros of $W_m(s)$;
- (F, q_i); $i = 1, 2$, are completely controllable pairs;
- θ is a $2n$-dimensional control parameter vector;
- c_0 is a scalar, taken as k_m/k_p where k_m, k_p are the high frequency gains of $W_m(s)$, $P_0(s;\, p)$, respectively.

Under Assumption 2.1 ($P_0 \in \{P_p\}$), it follows that for any $p \in \mathscr{P}$ there exists a vector θ_* and a scalar c_{0*} such that when $\theta = \theta_*$, $c_0 = c_{0*}$, and $\Delta = 0$, the controller (2.4) satisfies MRC.OBJ with the closed-loop transfer function $r \mapsto y_p$ being equal to $W_m(s)$. Thus, the (plant) parametric uncertainty set $\mathscr{P}$ induces a (controller) parametric uncertainty set, say $\mathscr{M}_* \times \mathscr{C}_*$, where $\mathscr{M}_*$ and $\mathscr{C}_*$ correspond to the uncertainty in θ_* and c_{0*}, respectively. We may now state our assumption regarding the parametric uncertainty $\mathscr{P}$, in terms of the controller parameters, our intention being to update the controller parameters directly.

Assumption 2.3: ("Parametric Uncertainty") There exist an *a priori* known, bounded, convex set $\mathscr{M} \supseteq \mathscr{M}_*$ with smooth boundary and an *a priori* known bounded interval $\mathscr{C} = [c_{0\min}, c_{0\max}] \supseteq \mathscr{C}_*$ such that $c_{0\min} > 0$. ■

To simplify the presentation, we will take $\mathscr{M}$ to be of the form $\mathscr{M} = \{\theta\colon |\theta - \theta_o| \leq M_0\}$ where $|\cdot|$ is an appropriate norm in $\boldsymbol{R}^{2n}$, θ_o is a known constant vector, and M_0 is a nonnegative constant indicating the size of the parametric uncertainty.

Next, we will use the following adaptive law to update the parameter vector θ and c_0.

$$\dot{\theta} = -\gamma \Pi_\theta \frac{\epsilon_1 \zeta}{m_2}; \qquad \theta(0) \in \mathscr{M} \tag{2.5}$$

$$\dot{c}_0 = -\gamma \Pi_c \frac{\epsilon_1 y_p}{m_2}; \qquad c_0(0) \in \mathscr{C}$$

$$\epsilon_1 = c_0 y_p + \theta^{\mathrm{T}}\zeta - W_m[u_p]; \qquad \zeta = W_m[w] \tag{2.6}$$

$$\dot{m}_2 = -2\delta_0 m_2 + |QU|_2^2 + q_r r^2 + q_e; \qquad m_2(0) > 0 \tag{2.7}$$

where

- γ, q_r, q_e are positive design parameters;
- Q is a positive definite weighting matrix (a typical selection is diag $[1, q]$);
- δ_0 is a positive constant such that $W_m(s-\delta_0)$, $G(s-\delta_0)$ are exponentially stable and $\delta_0 \leq \delta_*$;
- finally, Π_θ, Π_c are projection operators, designed to guarantee the boundedness of the parameter estimates θ and c_0. In particular, Π_c must also guarantee that c_0 is bounded away from zero. Essentially, Π_θ is the identity when θ is in the interior of $\mathscr{M}$ and projects $-\epsilon_1 \zeta$ on the tangent hyperplane at $\theta(t)$ when the latter is on the boundary of $\mathscr{M}$ and the vector field points towards the exterior of $\mathscr{M}$, e.g., see [17], [18]. Similarly, Π_c projects the vector field of c_0 in the interval $\mathscr{C}$. Some additional provisions should be taken so that the vector fields in (2.5) are locally Lipschitz, in order to avoid any questions on the existence and uniqueness of solutions. This can be accomplished by slightly increasing the size of $\mathscr{M}$ and $\mathscr{C}$ by ϵ_* as shown in the following example.

Suppose that the set $\mathscr{M}$ is defined in terms of the minimum and maximum values of the components of θ_*, say $\theta_{i*\min}$ and $\theta_{i*\max}$, respectively, a simple construction of Π_θ is given by decomposing the set $\mathscr{M}$ into a Cartesian product of intervals and taking $\Pi_\theta = \text{diag}\,[\Pi_{\theta i}]$ where

$$\Pi_{\theta i} = \begin{cases} \max\left\{0, \min\left[1, 1 + \dfrac{\theta_i - \theta_{i*\min}}{\epsilon_*}\right]\right\} & \\ \qquad \text{when } \epsilon_1 \zeta_i > 0 & \\ \max\left\{0, \min\left[1, 1 + \dfrac{\theta_{i*\max} - \theta_i}{\epsilon_*}\right]\right\} & \\ \qquad \text{when } \epsilon_1 \zeta_i < 0 & \\ 1 \quad \text{otherwise} \end{cases} \tag{2.8}$$

where ϵ_* is an arbitrary, small positive constant ensuring that $\Pi_{\theta i}\epsilon_1\zeta_i / m_2$ is locally Lipschitz. A similar projection can be used for c_0 and $\mathscr{C}$ where we will also require $c_{0\min} - \epsilon_* > 0$.

Under Assumptions 2.1, 2.2, and 2.3 such estimator guarantees that the estimates θ and c_0 will remain within distance ϵ_* of the sets $\mathscr{M}$ and $\mathscr{C}$, respectively, and if the unstructured (nonparametric) uncertainty operator is small, in the sense that $\gamma_{2,\delta_*}[W_p\Delta]$ is small, then the normalized prediction error $\epsilon_1/\sqrt{m_2}$, $\dot{\theta}$, and $\dot{c}_0$ will be small in the mean-square sense. This statement is made precise in the following section.

III. Robustness Analysis of the MRAC

A. Generalities

We begin the robustness analysis of the adaptive controller that was presented in Section II with the description of certain fundamental I/O properties of the closed-loop system. Among our objectives is to allow for the subsequent derivation of the various bounds in a systematic and—as much as possible—"tight" fashion, relating the final BIBO stability condition to the classical notions of sensitivity operators and loop gains.

At this point and in order to simplify the notation, let us define the following quantities.

1) The parameter errors $\phi = \theta - \theta_*$ and $\tilde{c}_0 = c_0 - c_{0*}$ where the subscript $*$ denotes the MRC parameters corresponding to the modeled part of the plant P_0 for a given $p \in \mathscr{P}$. The concatenated parameters, control, and estimation signals are denoted by $\bar{\cdot}$, i.e.,

$$\bar{\phi} = [\tilde{c}_0, \phi^T]^T; \quad \bar{\theta} = [c_0, \theta^T]^T;$$
$$\bar{w} = [r, w^T]^T; \quad \bar{\zeta} = [y_p, \zeta^T]^T.$$

2) The sensitivity operators of the "nominal" closed-loop system for a given $p \in \mathscr{P}$ [1]

$$S_1 : y_1 \mapsto y_p; \quad S_T : y_1 \mapsto u_p; \quad S_u : \phi^T w \mapsto u_p;$$
$$S_y (= c_{0*}^{-1} W_m) : \phi^T w \mapsto y_p.$$

3) The weighted unstructured uncertainty output $\bar{y} = W[y_1] = W\Delta[u_p]$ where the transfer function of W has poles and zeros in $\Sigma(\delta_*)$ and relative degree $\leq n^*$.

4) The subscript Q is reserved to denote weighted operators, mapping QU to the respective output ($U = [u_p, y_p]^T$); for example, $w = G[U] \triangleq G_Q[QU]$, $\bar{y} = W\Delta_Q[QU]$.

Thus, we arrive at the following description of the closed-loop system, shown in Fig. 1, in terms of the signals of interest u_p, y_p—expressed in the more convenient form QU — and the "perturbation" signals $\bar{\phi}^T\bar{w}$ and $\bar{y}$ [2]

$$QU = Q\underbrace{\begin{pmatrix} S_u & S_T W^{-1} \\ S_y & S_1 W^{-1} \end{pmatrix}}_{H} \begin{bmatrix} \bar{\phi}^T \bar{w} + c_{0*} r \\ \bar{y} \end{bmatrix} \tag{3.1}$$

$$\begin{pmatrix} \bar{\phi}^T \bar{w} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \phi^T G_Q \\ W\Delta_Q \end{pmatrix} [QU] + \begin{pmatrix} \tilde{c}_0 r \\ 0 \end{pmatrix}. \tag{3.2}$$

Invoking Assumption 2.1, we have that H is proper and exponentially stable for all $p \in \mathscr{P}$. Furthermore, for any $\delta < \min[\delta_0, \alpha]$, [3] $\gamma_{2,\delta}[H] < \infty$, uniformly in p.

For the I/O system (3.1), one may easily verify that the prediction error, defined by (2.6), is expressed in terms of the parameter error as

$$\epsilon_1 = \bar{\phi}^T \bar{\zeta} + c_{0*} S_1 \Delta[u_p] + \epsilon_\delta(t) \tag{3.3}$$

where the term $\epsilon_\delta(t)$ is included to describe the effect of nonzero initial conditions. Using (3.3) in the adpative law, the following result regarding the estimated parameters is established.

Corollary 3.1: ("Parameter Estimation") Suppose that for a plant satisfying Assumptions 2.1, 2.2, and 2.3, $U_t \in L_\infty$ $\forall t \in [t_0, t_0 + T]$. Then, the estimator (2.5) guarantees that $\bar{\theta}, \dot{\bar{\theta}}$ are UB on $[t_0, t_0 + T]$ (in this context, UB also means that the bounds are independent of t_0, T) and $\bar{\theta}$ remains

[1] i.e., the sensitivities of the closed loop of $P_0(s; p)$ together with the corresponding MRC $u_p = \bar{\theta}_*^T \bar{w}$.

[2] Similar derivations can also be found in the adaptive literature, e.g., [9], [19], [11], [10], [2].

[3] α, defined in Section II, is such that all members of $\{P_p\}$ have zeros in $\Sigma(\alpha)$.

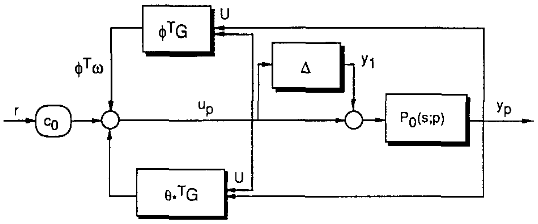

Fig. 1. Block diagram of the closed loop.

within distance ϵ_* from the set $\mathscr{C} \times \mathscr{M}$. Further, for any constant $\epsilon_c > 0$, there exists a constant C_0 depending on the initial conditions and $1/\epsilon_c$ but independent of T, t_0, such that for any interval $I = [t_I, t_I + T_I] \subseteq [t_0, t_0 + T]$

1)

$$\int_{t_I}^{t_I+T_I} \frac{(\bar{\phi}^T\bar{\zeta})^2(t)}{m_2(t)}\, dt \le C_0 + (1+\epsilon_c)\, g_{2,\delta_0}^2[c_{0*}S_1\Delta_Q]\, T_I;$$

(also valid for ϵ_1^2/m_2)

2)

$$|\dot{\theta}(t)| \le \gamma g_{2,\delta_0}[W_m G_Q] \frac{|\epsilon_1|}{\sqrt{m_2}} + \epsilon_\delta(t);$$

$$|\dot{c}_0(t)| \le \gamma K_y \frac{|\epsilon_1|}{\sqrt{m_2}} + \epsilon_\delta(t);$$

where, $K_y = c_{0*}^{-1} K_\phi g_{2,\delta_0}[W_m]\gamma_{2,\delta_0}[G_Q] + g_{2,\delta_0}[S_1\Delta_Q] + (1 + K_c/c_{0*}q_r) g_{2,\delta_0}[W_m]$ and K_ϕ, K_c are the bounds of $\phi, \tilde{c}_0$, satisfying $K_\phi \le 2M_0 + \epsilon_*$, $K_c \le c_{0\,\max} - c_{0\,\min} + \epsilon_*$. $\nabla\nabla$

Proof: (Outline) The proof follows as in [17], noticing that the perturbation term $c_{0*}S_1\Delta[u_p]$ is bounded by $\sqrt{m_2}$ (see Lemma A.1). Furthermore, the g_{2,δ_0} gain of $c_{0*}S_1\Delta_Q$ is well defined and finite since $g_{2,\delta_0}[S_1\Delta] \le g_{2,\delta_0}[S_1 W_p^{-1}]\gamma_{2,\delta_0}[W_p\Delta] \le g_{2,\delta_0}[S_1 W_p^{-1}]\gamma_{2,\delta_*}[W_p\Delta]$ the latter being finite by Assumption 2.2 and the former being finite since the poles of S_1 are those of W_m and G. The constant K_y represents a bound of $y_p/\sqrt{m_2}$ after the transient period and is obtained from (3.1) using Lemma A.1. Finally, the constant ϵ_c arises from using the Cauchy inequality $(x+y)^2 \le (1+\epsilon_c)x^2 + (1 + (1/\epsilon_c))y^2$ to square summations of terms involving U and exponentially decaying terms due to initial conditions. The latter, being integrable, can be incorporated in the constant C_0 whose size will not affect the final result, as long as it is finite. Also notice that C_0 depends on $1/\gamma$, indicating that small adaptation gains may have an adverse effect on the instantaneous value of $(\bar{\phi}^T\bar{\zeta})^2/m_2$. □□

At this point, it becomes apparent that it is possible to develop a stability result for the closed-loop system (3.1), (3.2) by a straightforward application of the small gain theorem [20]. This approach, however, would restrict the gain of the operator $\phi^T G_Q$ to be small which in turn, without any additional assumptions, would require the parametric uncertainty to be small. On the other hand, most of the interest in adaptive control arises exactly when the parametric uncertainty is large. This issue, being the focal point of this paper, is investigated in the following subsection.

B. Robustness of MRAC

The development of a sufficient condition for the BIBO stability of the closed-loop system (3.1), (3.2), with the adaptive law (2.5), can now be summarized as follows.

- Obtain a bound on the $L_2(\delta)$-norm of the truncated signal QU in terms of the $L_2(\delta)$-norms of the truncated signals $\bar{\phi}^T\bar{w}$ and $\bar{y}$.
- Use Lemmas A.2 and A.3 to extract a bound on $\bar{\phi}^T\bar{w}$ in terms of $\bar{\phi}^T\bar{\zeta}$ and $\dot{\theta}$, $\dot{c}_0$.
- Employ the Bellman–Gronwall lemma and the properties of the adpative law, given in Corollary 3.1, to obtain a boundedness condition in terms of the parametric and unstructured uncertainty.

Although this procedure is somewhat long and tedious, it is quite straightforward especially in view of Lemma A.1. The final result is stated next as a theorem, where in order to reduce the conservatism of the bounding process we define the following quantities.

1) W_1, W_2, the "swapping operators" corresponding to the reference model W_m. With $c_m(sI - A_m)^{-1}b_m = W_m(s)$ these operators can be defined in terms of their transfer matrix by

$$W_1(s) = c_m(sI - A_m)^{-1}; \quad W_2(s) = (sI - A_m)^{-1}b_m.$$

2) $\delta \in (0, \min[\delta_0\ \alpha])$, $\rho \in [0,1]$, and $Q_d = \mathrm{diag}(1, q_d)$ with $q_d > 0$, are "free" parameters to be selected.

3) Λ, Λ_1, fictitious operators such that:

- $\Lambda(s)$ is stable, with relative degree $\ge n^*$, unity dc gain and poles in $\Sigma(\delta_0)$,
- $\Lambda_1(s)$ is such that $\Lambda_1(s)s = 1 - \Lambda(s)$.

A simple choice Λ is $\Lambda(s) = [a/(s+a)]^{n^*}$, where $a \ge \delta_0$ is a "free" parameter. (Note that $\gamma_{2,\delta}(\Lambda_1) = O(1/a)$ and $\gamma_{2,\delta}(\Lambda W_m^{-1}) = O(a^{n^*})$.)

4) The constants $K_W = (1 + K_\phi c_{0*}^{-1}) g_{2,\delta}[W_2]\gamma_{2,\delta}[G_Q]$ and K_ϕ, K_c, K_y as defined in Corollary 3.1.

5) $\Gamma_H = \gamma_{2,\delta}[QHQ_d^{-1}]$.

6) $\Gamma_0 = K_\phi\gamma_{2,\delta}[\Lambda_1]\gamma_{2,\delta}[sG_Q] + \rho K_\phi\gamma_{2,\delta}[\Lambda]\gamma_{2,\delta}[G_Q] + [(1-\rho)c_{0*}/(c_{0\,\min} - \epsilon_*)]K_c\gamma_{2,\delta}[\Lambda W_m^{-1}]\gamma_{2,\delta}[S_1\Delta_Q]$.

7) $\Gamma_1 = [(1-\rho)c_{0*}/(c_{0\,\min} - \epsilon_*)]\gamma_{2,\delta}[\Lambda W_m^{-1}]$.

8) $\Gamma_2 = \gamma_{2,\delta}[\Lambda_1]g_{2,\delta}[G_Q] + [(1-\rho)c_{0*}/(c_{0\,\min} - \epsilon^*)]K_W\gamma_{2,\delta}[\Lambda W_m^{-1}W_1]$.

9) $\Gamma_3 = [(1-\rho)c_{0*}/(c_{0\,\min} - \epsilon_*)^2]K_W K_\phi\gamma_{2,\delta}[\Lambda W_m^{-1}W_1]$.

Theorem 3.2: ("MRAC Robustness") Consider a plant satisfying Assumptions 2.1, 2.2, 2.3 and suppose that there exist $p \in \mathscr{P}$, $\delta \in (0, \min[\delta_0, \alpha])$, $\rho \in [0,1]$, $q_d > 0$, Λ and W with the previously defined properties such that

$$\Gamma_H^2\left\{ q_d^2\gamma_{2,\delta}^2[W\Delta_Q] + \left(\Gamma_0 + \frac{\Gamma_1 + \gamma(\Gamma_2 g_{2,\delta_0}[W_m G_Q] + \Gamma_3 K_y)}{\sqrt{2\delta}} \cdot g_{2,\delta_0}[c_{0*}S_1\Delta_Q]\right)^2\right\} < 1.$$

Then, for any UB reference input, all signals in the closed-loop plant with the controller (2.4) and the update law (2.5) are UB, $\forall t \geq 0$.

Furthermore, the tracking error $e_1 \triangleq y_p - y_m$ satisfies

$$\int_{t_0}^{t_0+T} \frac{e_1^2(t)}{m_2(t)}\, dt \leq C_0 + (1 + \epsilon_c)\left(\frac{c_{0*} + K_c}{(c_{0\min} - \epsilon_*)} + \gamma K_s\right)^2 \cdot g_{2,\delta_0}^2[S_1 \Delta_Q] T; \qquad \forall t_0, T \geq 0$$

where K_s is a constant due to the swapping terms,[4] ϵ_c is a "Cauchy" constant, and C_0 is a constant depending on the initial conditions and $1/\epsilon_c, 1/\gamma$. $\nabla\nabla$

Proof: In the Appendix.

C. Discussion and Interpretations

Theorem 3.2 gives a sufficient condition for the BIBO stability of an MRAC system which, although conservative, can be used to provide some interesting interpretations and guidelines for the selection of the adaptation and controller parameters. It should be pointed out that this condition relies on rather weak assumptions regarding the estimation process and the relative degree of the modeled part of the plant. That is, the estimator is only required to keep the parameters bounded, the prediction error small in a mean-squared, normalized sense, and the speed of the parameter variations small in the mean square. These results can be established through a variety of estimation schemes without imposing any excitation conditions on the reference input, or strict positive realness on the reference model. Furthermore, the estimator can be kept "live" at all times, a desirable property in the case where the plant parameters may vary with time. On the other hand, such weak assumptions are naturally expected to produce weak results which can be improved in several special cases. In this sense, Theorem 3.2 complements the (discrete-time) results of [12] and the more specialized ones of [11] (persistent excitation, least-squares estimation) and [10] (local analysis using averaging techniques and persistent/dominant excitation; passivity analysis with relative degree one nominal plants).

Further, an important feature and contribution of Theorem 3.2 is that the involved quantities are given in the form of intuitively appealing gains of the closed-loop sensitivity operators. The implication of this feature is that the allowable unstructured uncertainty is characterized in a conceptually simple way by its exponentially weighted gain, while standard weighting techniques from nonadaptive robust control can be used to reduce the conservatism of the result (e.g., [21]). Such an approach leads to some interesting interpretations of the role of the parameters q_d, ρ and the weights Λ and W as well as the selection of the adaption parameters which will be discussed below.

- First and in order to appreciate the elegance of the results, let us employ Theorem 3.2 to re-drive some well-known results in adaptive and nonadaptive control. Consider for example, the case where no unstructured plant uncertainty is present [16], that is $\Delta(s) = 0$. In this case, the condition for signal boundedness in Theorem 3.2 becomes simply

$$\Gamma_H \Gamma_0 < 1$$

which is satisfied for any K_ϕ by letting a in the fictitious filter Λ be large enough and $\rho = 0$. Note that since $\Delta = 0$ the boundedness of the parameter estimates can be guaranteed for arbitrarily large projection sets $\mathscr{M}$ (i.e., $\mathscr{M}$ can be large enough so that the projection is never active[5]) although the bound will depend on the initial conditions. And since the result is independent of δ_0, we can allow any normalization, bounded by m_2, in the adaptive law provided that ϕ is bounded and $\|\dot{\phi}\| \in L_2$[16]. Finally, from the second part of Theorem 3.2, the easily concluded uniform continuity of e_1 implies that $e_1 \to 0$.

On the other hand, let us assume that the parametric uncertainty is zero, i.e., the sets $\mathscr{P}$ and consequently $\mathscr{C} \times \mathscr{M}$ collapse to a point. In this case, taking $\epsilon_* \to 0$, we have $K_\phi, K_c \to 0$. Further, letting $\rho \to 1$, $\delta \to 0$, and $a \to \infty$ with appropriate rates, we obtain that boundedness is assured provided that

$$\gamma_{2,\delta}[QHQ_d^{-1}]\, q_d \gamma_{2,\delta}[W\Delta_Q] < 1.$$

Setting $Q = \operatorname{diag}[1, q]$ and letting q, q_d approach 0 we recover the well-known condition for robustness with respect to multiplicative uncertainty $\gamma_2[S_T W^{-1}]\gamma_2[W\Delta] < 1$ (e.g., see [22], [23]). It must be noted, however, that the technical Assumption 2.2, i.e., $\gamma_{2,\delta_0}[W_p \Delta] < \infty$ should also be relaxed in order to fully recover the robustness results of the nonadaptive case. This indicates that turning off the adaptive controller requires not only $\gamma \to 0$ but also M_0, $(c_{0\max} - c_{0\min})$, ϵ_*, and $\delta_0 \to 0$ (see also [24] treating the $\delta_0 = 0$ case analog in discrete time).

Next, let us consider the case of mixed parametric and unstructured uncertainty Δ such that $\gamma_{2,\delta_0}[W_p\Delta] = O(\mu)$ for some known $\delta_0 > 0$ and suppose that we would like to know if an MRAC can tolerate plant uncertainty for which μ is nonzero [5]. In this case, for any fixed parametric uncertainty $(K_\phi, K_c < \infty)$

$$\Gamma_0 = O[a^{-1}] + O[(1-\rho)a^{n^*}\mu] + O[\rho];$$
$$\Gamma_1 = O[(1-\rho)a^{n^*}]$$
$$\Gamma_2 = O[a^{-1}] + O[(1-\rho)a^{n^*}];$$
$$\Gamma_3 = O[(1-\rho)a^{n^*}].$$

It follows immediately from Theorem 3.2 that, for some fixed large enough a and ρ sufficiently small, the closed-loop signals are uniformly bounded for nonzero values of μ such that μa^{n^*} is small enough.

Thus, the role of the (constant) weights Q, Q_d can be interpreted as a balancing among the gains of the components of H and the unstructured uncertainty, while W_p, W serve to "reshape" the unstructured uncertainty operator; for exam-

[4] An expression for K_s is included in the proof.

[5] Our approach, however, still requires the use of a lower bound on c_{0*}; the removal of this condition would require the presence of an additional parameter as in [16].

ple, if Δ is LTI, W_p can be thought as an operator such that $W_p\Delta$ has constant magnitude along the axis $-\delta + j\omega$, $\omega \in \boldsymbol{R}$, while $S_T W$ is "all-pass." On the other hand, Λ (or a, for the simple form used above) and ρ are weights related to the estimation part of the MRAC. Since in a direct MRAC the estimation of the unknown parameters is performed through $\bar{\phi}^T\bar{\zeta}$, all information supplied by the estimator on the "quality" of identification is given in terms of $\bar{\phi}^T\bar{\zeta}$ and $\bar{\theta}$. Hence, in order to extract the necessary information regarding the signal $\bar{\phi}^T\bar{w}$ which acts as a perturbation in the nominal closed-loop system, we must "invert" the reference model over the frequency range where $\bar{\phi}^T\bar{w}$ is large. The role of $\Lambda(s)$ can now be interpreted as an indicator of the worst-case frequency content of the perturbation $\bar{\phi}^T\bar{w}$ while ρ is an indicator of its magnitude. Finally, δ is an estimate of the minimum stability margin (rate of energy dissipation) of the nominal closed-loop system required to preserve boundedness in the presence of the perturbation $\bar{\phi}^T\bar{w}$ (observe that the operator $\phi^T G$ is not necessarily of small gain).

- A remarkable property of direct adaptive schemes, following from Theorem 3.2 is that their robustness properties are given in terms of the best possible parameter vector $p \in \mathscr{P}$ for which the boundedness condition is satisfied. In other words, the adpative controller performs a search in the set $\mathscr{P}$, indirectly through θ, to obtain a plant representation with uncertainty and nominal loop sensitivities satisfying the above condition. It should be noted, however, that the MRAC is able to tolerate only smaller amounts of nonparametric uncertainty than a "perfectly tuned" nonadaptive scheme, i.e., $\bar{\theta} = \bar{\theta}_*$, corresponding to $p \in \mathscr{P}$ with the smallest $\gamma_{2,\delta}$-gain of $W\Delta$.[6] This is due to the fact that Δ introduces a perturbation in both the closed-loop system and the estimator, causing the appearance of the additional perturbation term $\bar{\phi}^T\bar{w}$. Another interesting feature of the direct MRAC is that its tracking performance in a mean-square, normalized sense is $O(g^2_{2,\delta_0}[S_1\Delta_Q])$ while that of a nonadaptive controller would be order of both the size of $S_1\Delta$ and the diameter of the set $\mathscr{C}_* \times \mathscr{M}_*$. Of course, this performance measure does not give a complete picture as it allows the appearance of large instantaneous values of the output ("bursts") which, without any additional assumptions, will depend on $g_{2,\delta_0}[S_1\Delta_Q]$ as well as the diameter of $\mathscr{C} \times \mathscr{M}$ and $1/\gamma$.
- From Theorem 3.2 it can be easily seen that the bound on the maximum size of nonparametric uncertainty that can be tolerated by a given MRAC, with respect to the adaptation gain, occurs for $\gamma \to 0$. We may thus state the following simpler and quite more instructive version of the above theorem for the special case of "slow" adaptation.

Corollary 3.3: Under the conditions of Theorem 3.2, suppose that there exist $p \in \mathscr{P}$, $\delta \in (0, \min[\delta_0, \alpha])$, $\rho \in [0, 1]$, $q_d > 0$, Λ, and W such that

$$\Gamma_H^2\left\{q_d^2\gamma_{2,\delta}^2[W\Delta_Q] + \left(\Gamma_0 + \frac{\Gamma_1}{\sqrt{2\delta}}g_{2,\delta_0}[c_{0*}S_1\Delta_Q]\right)^2\right\} < 1.$$

Then, there exists $\gamma_* > 0$ such that for any $\gamma \in (0, \gamma_*)$ and any UB reference input, all signals in the closed-loop plant with the controller (2.4) and the update law (2.5) are UB, $\forall t \geq 0$. $\nabla\nabla$

Not to be misinterpreted, we should emphasize that very small adaptation gains may improve the robust stability properties of an adaptive controller for an LTI system but have serious drawbacks when the transient (adaptation) behavior or systems with time-varying parameters are considered.

- In the special case where additional considerations allow us to conclude that K_ϕ is small, e.g., when the plant parameters are known within a small error, one may let $\rho \to 1$, $\delta \to 0$, and $a \to \infty$ yielding the following result.

Corollary 3.4: Under the conditions of Theorem 3.2, suppose that there exist $p \in \mathscr{P}$, $q_d > 0$, and W such that

$$\Gamma_H^2\{q_d^2\gamma_2^2[W\Delta_Q] + K_\phi^2\gamma_2^2[G_Q]\} < 1.$$

Then, for any UB reference input, all signals in the closed-loop plant with the controller (2.4) and the update law (2.5) are UB, $\forall t \geq 0$. $\nabla\nabla$

(See also previous remarks on zero parametric uncertainty.)

- The arguments used in Theorem 3.2 can be extended in a straightforward manner to develop a boundedness condition for different characterizations of the nonparametric uncertainty (e.g., additive or stable-factor perturbations) and a variety of popular MRAC structures, employing different adaptive laws (e.g., σ or switching σ-modifications [8], dead-zones [25], [7]) or different control laws. In particular, when the control law of [16] is considered, the control input is evaluated by

$$u_p = \theta^T G'[U] + c_0 r; \quad G' = \begin{pmatrix} G \\ [0, 1] \end{pmatrix}$$

where θ is a $(2n-1)$-dimensional vector and G is as in (2.4) except that it is generated by a $(n-1) \times (n-1)$ matrix F. In this case, the control input contains a direct throughput of y_p which does not allow the evaluation of Γ_0 and Γ_2 in their current form, both requiring $G'(s)$ to be strictly proper. Alternative expressions for Γ_0 and Γ_2 can be obtained by using (3.1) to express y_p as the output of strictly proper operators with input depending on u_p and y_p. Letting,

$$\begin{aligned}\Gamma_0' &= \Gamma_0 + \rho K_\phi\gamma_{2,\delta}[\Lambda](q^{-1}) + K_\phi\gamma_{2,\delta}[\Lambda_1] \\ &\quad\cdot\{K_\phi\gamma_{2,\delta}[c_{0*}^{-1}sW_m]\gamma_{2,\delta}[G_Q'] + \gamma_{2,\delta}[sS_1\Delta_Q]\} \\ \Gamma_2' &= \Gamma_2 + \gamma_{2,\delta}[\Lambda_1]\{c_{0*}^{-1}K_\phi g_{2,\delta}[W_m]\gamma_{2,\delta}[G_Q'] \\ &\quad + g_{2,\delta}[S_1\Delta_Q]\} \qquad (3.4)\end{aligned}$$

and replacing Γ_0, Γ_2 by Γ_0', Γ_2' and G_Q by G_Q' in the rest of the expressions, the results of Theorem 3.2 are still valid. An apparent drawback of this control law is an increased susceptibility to high-frequency noise due to the direct throughput term, introducing a larger loop-gain at high frequencies.

- Our analysis so far, has excluded the important case of plant uncertainty expressed as external bounded disturbances. Such disturbances have a severe effect on the estimator as

[6] Needless to say, the presence of nonzero parametric uncertainty makes the construction of such a controller purely conceptual.

they introduce an additional term $c_{0*}S[d]$ in the prediction error, where d is the disturbance and S is the closed-loop sensitivity operator from the disturbance to the output. It is straightforward to see that this term will cause the appearance of an additional contribution of the form

$$\int_{t_I}^{t_I+T_I} \frac{\left(c_{0*}S[d]\right)^2(t)}{m_2(t)}\,dt \tag{3.5}$$

in the expressions given by Corollary 3.1. In general, $(c_{0*}S[d])^2/m_2$ is not uniformly small and consequently the previous arguments do not apply. Such cases are typically handled in the literature by either employing a contradiction argument as in [26] to show boundedness, or using a fixed dead-zone to avoid adaptation when the prediction error is too small [25]. On the other hand, it is interesting to observe that given an upper bound of $S[d]$, one can always design the constant bias q_e of the normalization signal to be sufficiently large so that the contribution of the term (3.5) is sufficiently small in the mean and a slightly modified version of Theorem 3.2 is applicable. In other words, q_e can be interpreted as the normalization signal level needed to bring the size of the effective external disturbance $c_{0*}S[d]/\sqrt{m_2}$ below a certain threshold and ensure that the parameter misadjustment during periods of poor signal-to-noise ratio is small.[7]

D. Example

Let us now consider a simple example which, nevertheless, demonstrates most of the points raised in our discussion.

Suppose that the plant to be controlled is described by

$$P = \frac{1}{s+a_p}[1+\Delta] \tag{3.6}$$

where the parameter a_p is in the interval $(-1, 1)$ with nominal value 0 and Δ is a multiplicative uncertainty. For the model reference control objective, we assume that

$$W_m(s) = \frac{a_m}{s+a_m}. \tag{3.7}$$

For simplicity, we consider a simple compensator composed of a scalar feedback gain θ. In a nonadaptive design, the compensator that satisfies the control objective for $\Delta = 0$ is $\theta = a_m - a_p$, i.e., $\theta = a_m$ when the design is performed based on the nominal value of a_p. Moreover, the closed-loop complementary sensitivity is $S_T(s) = \theta/(s+\theta+a_p)$, resulting in a condition for the BIBO stability of the nonadaptive loop

$$\frac{\theta}{a_m}\gamma_2\left[\frac{s+a_m}{s+\theta+a_p}\right]\gamma_2[W_m\Delta] < 1.$$

On the other hand, using the same *a priori* information, we would like to design an MRAC, based on the previous control law, and determine the class and size of uncertainty Δ for which the closed-loop signals are guaranteed to be bounded. We begin our investigation by taking $M_0 = 1$ and $\theta_o = a_m$ where θ_o is determined by the knowledge of the nominal value of a_p and the projection in the adaptive law takes effect whenever $|\theta - \theta_o| > M_0 + \epsilon_*$ for some small constant ϵ_*. Further, in order to simplify the computations, we let $\delta_0 = 0.5$, $W = W_m$, $\Lambda = a/(s+a)$, and $Q = \mathrm{diag}[1, q]$. We also let $D = \gamma_{2,\delta_0}[\Delta]$ which, by virtue of Assumption 2.2, is finite ($n^* = 1$). Employing Corollary 3.3, we find that, for sufficiently small γ and ϵ_* and treating q as a free parameter,[8] a sufficient condition for signal boundedness is

$$\inf_{q,\,q_d,\,\rho,\,a} \Gamma_H^2\left\{\frac{q_d^2 a_m^2}{(a_m-\delta)^2}D^2 + X^2\right\} < 1$$

where X is calculated using the expressions (3.4),

$$X = \frac{K_\phi}{(a-\delta)}\left(\frac{K_\phi}{q} + D + \frac{\rho a}{q}\right) + (1-\rho)\frac{aD}{2\sqrt{\delta(a_m-\delta_0)}}\max\left[1, \frac{a_m-\delta}{a-\delta}\right].$$

Selecting first $a_m = 1$, for which the parametric uncertainty is large relative to $1/\gamma_2[H]$, we obtain with $\rho = 0$

$$a_p = 0;\quad a_m = 1;\quad D_{\max} = 0.026$$
$$a_p \to -1;\quad a_m = 1;\quad D_{\max} = 0.0096.$$

We note that for this case, the optimum value $D_{\max}$ occurs for $\delta = 0.25 < \delta_0$ and thus, a reduction of δ_0 would expand the class of allowable nonparametric uncertainty (see Assumption 2.2). Moreover, even when a_p approaches -1, where a fixed controller designed for the nominal plant approaches its stability limits, the adaptive controller retains a small robustness margin.

To improve the robustness of the MRAC and guided by our stability condition, we increase the value of a_m to find that

$$a_p = 0;\quad a_m = 5;\quad D_{\max} = 0.478$$
$$a_p = 0;\quad a_m = 20;\quad D_{\max} = 0.733$$

where $D_{\max}$ is obtained for $\rho = 1$ due to the smallness of the parametric uncertainty relative to $1/\gamma_2[H]$. We should emphasize that, in general, a faster reference model may not improve robustness since it requires larger control gains (θ) something that affects adversely the gains of H and possibly K_ϕ.

IV. Conclusions

In this paper we considered the direct MRAC problem in the presence of multiplicative unstructured uncertainty. Using an adaptive law with projection and the normalization of [1], [12], we developed a general sufficient condition for the closed-loop BIBO stability, relating the size of the parametric uncertainty—diameter of the projection set—and the size of

[7] This is not an efficient approach to analyze the case of bounded disturbances; it is mentioned merely as an interpretation of the role of q_e.

[8] The value of $D_{\max}$ will change slightly if q is restricted to be constant.

the unstructured uncertainty, defined as an exponentially weighted gain of the frequency-weighted uncertainty operator. Among the contributions of this work are the intuitive appeal of the stability condition, involving gains of sensitivity operators and a substantial simplification of the (global) analysis of direct MRAC through the use of exponentially weighted L_2 spaces [12]. The general nature and elegance of the results indicate a direction towards a unified and systematic theory in adaptive control.

Complementing the results obtained through a local analysis and/or persistently exciting inputs [10], [11], our approach also allows for the ability to evaluate in a rather straightforward, albeit laborious, manner the quantities involved in the stability condition and study the effects of the design parameters on the robust stability of an MRAC. However, before such a condition becomes an effective design tool, the issue of robust performance should be addressed in more detail; at this point, it seems that additional assumptions (e.g., persistently exciting inputs) will be necessary in order to establish the robust performance of an MRAC in a practically meaningful sense. Such assumptions may also contribute to the simplification and conservatism reduction of the robust stability condition.

Appendix

A. Some Useful Results

Lemma A.1: ("Normalization" [1]) Consider a causal operator $H: L_p^e \mapsto L_\infty^e$ $(H: L_p^e \mapsto L_p^e)$, $p \in [1, \infty)$, and let $y = Hu$ and $m_p(t)$ be the signal defined by

$$\dot{m}_p = -p\delta m_p + |u|^p + q_e; \qquad t \geq 0;\ m(0) > 0$$

where δ, q_e are positive constants and assume that $u \in L_p^e$ on an interval $J \subseteq \boldsymbol{R}_+ (0 \in J)$.

a) If $g_{p,\delta}(H)$ is finite, then there exists a constant $\beta_0 > 0$ such that

$$\frac{|y(t)|}{[m_p(t)]^{1/p}} \leq g_{p,\delta}(H) + \beta_0 e^{-\delta t}, \qquad \forall t \in J.$$

b) If $\gamma_{p,\delta}(H)$ is finite, then there exists a constant $\beta_0 > 0$ such that

$$\left\{ \frac{\int_0^t [e^{-\delta(t-\tau)} |y(\tau)|]^p \, d\tau}{m_p(t)} \right\}^{1/p} \leq \gamma_{p,\delta}(H) + \beta_0 e^{-\delta t}, \qquad \forall t \in J. \quad \nabla$$

Proof: Straightforward, using the notion of induced gains [20].

Lemma A.2: ("Swapping" [17]) Consider the system $\dot{x} = Ax + Bu$, $y = Cx$, $x(t_0) = 0$ and suppose that $u(t) = w(t)\theta(t)$ where θ is absolutely continuous, w is (absolutely) integrable on an interval $J = [t_0, t_0 + T]$, $T > 0$ and A, B, C, u, w, θ are matrices of compatible dimensions. Further, define the (causal) operators

$$H: v \mapsto z: \quad z(t) = \int_{t_0}^t C\Phi(t,\tau)Bv(\tau)\, d\tau$$

$$W_1: v_z \mapsto z_1: \quad z_1(t) = \int_{t_0}^t C\Phi(t,\tau)v_1(\tau)\, d\tau$$

$$W_2: v_2 \mapsto z_2: \quad z_2(t) = \int_{t_0}^t \Phi(t,\tau)Bv_2(\tau)\, d\tau$$

where $\Phi(\cdot,\cdot)$ is the state transition matrix corresponding to A. Then, for all $t \in J$

$$\begin{aligned} y(t) &= (H[w\theta])(t) \\ &= (H[w])(t)\theta(t) - (W_1\{W_2[w]\dot{\theta}\})(t). \quad \nabla\nabla \end{aligned}$$

Lemma A.3: ("Operator Inversion") Suppose $W(s)$ is a stable, minimum-phase transfer function of relative degree n^* and let $\rho \in [0, 1]$ be an arbitrary constant and

- $\Lambda(s)$ denote an arbitrary transfer function such that $\Lambda(s)$ is stable, with relative degree $\geq n^*$ and unity dc gain;
- $\Lambda_1(s)$ denote the transfer functions such that $\Lambda_1(s)s = 1 - \Lambda(s)$.

Further, let (u, y) be the I/O pair of $W(s)$ and suppose that in an interval $[0, T]$, u is absolutely continuous. Then, in the same interval

$$u = \Lambda_1(s)[\dot{u}] + (1-\rho)\Lambda(s)W^{-1}(s)\{W(s)[u]\} + \rho\Lambda(s)[u]. \quad \nabla\nabla$$

Proof: Straightforward from the operator identity

$$[1 - \Lambda(s)] + (1-\rho)\Lambda(s)W^{-1}(s)W(s) + \rho\Lambda(s) = 1. \quad \square\square$$

B. Proof of Theorem 3.2

Observe first that under the assumptions of the theorem it follows that there exists $T > 0$ such that the closed-loop ODE has a unique, continuous, and continuously differentiable almost everywhere solution in $[t_0, t_0 + T]$ (for simplicity we will take $t_0 = 0$). Let $t \in [0, T]$, $\delta \in (0, \min[\delta_0, \alpha])$ and define the fictitious signal m_f by

$$\dot{m}_f = -2\delta m_f + |QU|_2^2 + q_r r^2 + q_e;$$
$$m_f(0) = m_2(0).$$

Note that somewhat less conservative but more complicated results can be similarly obtained by using a fictitious signal $Q_f U$ to describe the closed-loop system (3.1) and replacing QU in the definition of m_f by $Q_f U$. Q_f, being an arbitrary positive definite matrix, can then be selected to maximize the region of unstructured uncertainty for which BIBO stability is ensured.

Thus, a bound on the $L_p(\delta)$ norm of the truncated signal $(|QU(\cdot)|_2)$, denoted simply by $\|(QU)_t\|_{2,\delta}$ can be evaluated from (3.1) as follows:

$$\|(QU)_t\|_{2,\delta}^2 \leq (1+\epsilon_c)\Gamma_H^2\{q_d^2\|\bar{y}_t\|_{2,\delta}^2 + \|(\overline{\phi}^{\mathrm{T}}\overline{w})_t\|_{2,\delta}^2\} + Ce^{2\delta t} \quad \text{(B.1)}$$

where ϵ_c is a Cauchy constant and C is a constant incorporating the effects of initial conditions and bounded inputs. Without loss of generality, and in order to avoid an unnecessary proliferation of symbols, we will use ϵ_c to denote the—arbitrarily small—Cauchy constants which appear in squaring sums of possibly unbounded signals with bounded ones and C to denote finite constants, possibly depending on $1/\epsilon_c$.

Further, from Assumption 2.2, $\gamma_{2,\delta}[W\Delta_Q]$ is finite and therefore,

$$\|\bar{y}_t\|_{2,\delta}^2 \le (1+\epsilon_c)\gamma_{2,\delta}^2[W\Delta_Q]\,\|(QU)_t\|_{2,\delta} + C. \qquad \text{(B.2)}$$

Next, in order to obtain a bound for $\|(\bar{\phi}^{\mathrm{T}}\bar{w})_t\|_{2,\delta}$ we will invoke Lemma A.3 to write

$$\begin{aligned}\|(\bar{\phi}^{\mathrm{T}}\bar{w})_t\|_{2,\delta} \le{}& Ce^{\delta t} + \|(s\Lambda_1[\phi^{\mathrm{T}}w])_t\|_{2,\delta}\\ &+(1-\rho)\|(\Lambda W_m^{-1}W_m[\bar{\phi}^{\mathrm{T}}\bar{w}])_t\|_{2,\delta}\\ &+\rho\|(\Lambda[\phi^{\mathrm{T}}w])_t\|_{2,\delta} \qquad \text{(B.3)}\end{aligned}$$

where the (bounded) contribution of $\tilde{c}_0 r$ in the second and fourth terms has been incorporated in C. Observing that

$$\bar{\phi}^{\mathrm{T}}\bar{w} = \frac{c_{0*}}{c_0}\bar{\phi}^{\mathrm{T}}\hat{w}; \quad \hat{w} \triangleq \left[\left(\frac{1}{c_{0*}}\bar{\phi}^{\mathrm{T}}\bar{w}+r\right), w^{\mathrm{T}}\right]^{\mathrm{T}}$$

and employing Lemma A.2, it follows that

$$\begin{aligned}\|(\bar{\phi}^{\mathrm{T}}\bar{w})_t\|_{2,\delta} \le{}& Ce^{\delta t} + \|(s\Lambda_1[\phi^{\mathrm{T}}w])_t\|_{2,\delta}\\ &+(1-\rho)\left\|\left(\Lambda W_m^{-1}\left\{\frac{c_{0*}}{c_0}\bar{\phi}^{\mathrm{T}}W_m[\hat{w}]\right\}\right)_t\right\|_{2,\delta}\\ &+(1-\rho)c_{0*}\\ &\cdot\left\|\left(\Lambda W_m^{-1}W_1\left\{W_2[\hat{w}^{\mathrm{T}}]\frac{d}{dt}\left[\frac{\bar{\phi}}{c_0}\right]\right\}\right)_t\right\|_{2,\delta}\\ &+\rho\|(\Lambda[\phi^{\mathrm{T}}w])_t\|_{2,\delta}. \qquad \text{(B.4)}\end{aligned}$$

Further, from (3.1), we have that $W_m[\hat{w}] = \bar{\zeta} - [\tilde{c}_0 S_1\Delta u_p, 0]^{\mathrm{T}} + \epsilon_\delta$ which substituted in (B.4) yields after some straightforward calculations (writing $s\phi^{\mathrm{T}}w$ as $\dot{\phi}^{\mathrm{T}}w + \phi^{\mathrm{T}}\dot{w}$, and using Lemma A.1 to derive bounds in terms of operator gains)

$$\begin{aligned}\|(\bar{\phi}^{\mathrm{T}}\bar{w})_t\|_{2,\delta} \le{}& Ce^{\delta t} + \Gamma_0\|(QU)_t\|_{2,\delta} + \Gamma_1\|(\bar{\phi}^{\mathrm{T}}\bar{\zeta})_t\|_{2,\delta}\\ &+\Gamma_2\|([\,|\dot{\theta}|+\epsilon_\delta]m_f^{1/2})_t\|_{2,\delta}\\ &+\Gamma_3\|(|\dot{c}_0|\,m_f^{1/2})_t\|_{2,\delta} \qquad \text{(B.5)}\end{aligned}$$

where the constants Γ_i are as defined in Section III-B. Taking squares of both sides of (B.5) we get

$$\begin{aligned}\|(\bar{\phi}^{\mathrm{T}}\bar{w})_t\|_{2,\delta}^2 \le{}& Ce^{2\delta t} + (1+\epsilon_c)\{\lambda_0^2\|(QU)_t\|_{2,\delta}^2\\ &+\lambda_1^2\|(\bar{\phi}^{\mathrm{T}}\bar{\zeta})_t\|_{2,\delta}^2\\ &+\lambda_2^2\|([\,|\dot{\theta}|+\epsilon_\delta]m_f^{1/2})_t\|_{2,\delta}^2\\ &+\lambda_3^2\|(|\dot{c}_0|\,m_f^{1/2})_t\|_{2,\delta}^2\} \qquad \text{(B.6)}\end{aligned}$$

where $\lambda_i^2 = (1+q_{ci})\Gamma_i^2$ and q_{ci} are Cauchy constants. Notice that for these terms we need to keep track of the corresponding Cauchy constants in order to reduce the conservatism in the final boundedness condition, as much as possible.

Rewriting (B.1) in terms of $e^{2\delta t}m_f$ and using (B.2) and (B.6), we obtain

$$\begin{aligned}e^{2\delta t}m_f(t) \le{}& Ce^{2\delta t} + (1+\epsilon_c)\Gamma_H^2\Bigg\{(\lambda_\Delta^2+\lambda_0^2)e^{2\delta t}m_f(t)\\ &+\int_0^t \lambda_1^2\frac{(\bar{\phi}^{\mathrm{T}}\bar{\zeta}(\tau))^2}{m_f(\tau)}e^{2\delta\tau}m_f(\tau)\,d\tau\\ &+\int_0^t\left[\lambda_2^2[\,|\dot{\theta}(\tau)|+\epsilon_\delta(\tau)]^2+\lambda_3^2\dot{c}_0^2(\tau)\right]\\ &\cdot e^{2\delta\tau}m_f(\tau)\,d\tau\Bigg\} \qquad \text{(B.7)}\end{aligned}$$

where, again, the bounds of $q_r r^2$ and q_e are incorporated in C and $\lambda_\Delta^2 = q_d^2\gamma_{2,\delta}^2[W\Delta_Q]$. In view of Corollary 3.1, a straightforward application of the Bellman–Gronwall lemma on (B.7) and a subsequent optimization with respect to the Cauchy constants q_{ci}, yield that m_f (and therefore m_2 since $m_0 < m_f$) will be UB on $[0, T]$ provided that the condition stated in the theorem holds, with a bound independent of T.[9] Thus, the overall state vector of the closed-loop system will be UB on $[0, T]$ and therefore [28] the solution can be extended on an interval $[T, T+T_\delta]$ for some $T_\delta > 0$. Since the bounds of m_f and θ were independent of T, the same bounds will be valid on $[T, T+T_\delta]$ and therefore, the solution can be extended to an interval $[T+T_\delta, T+2T_\delta]$, etc., from which the first part of the theorem follows.

Finally, for the second part of the theorem, similar techniques can be used to decompose e_1 as $\bar{\phi}^{\mathrm{T}}\bar{\zeta}$ and a "swapping" term [see (3.1)]. Thus, the inequality of the theorem follows immediately from Corollary 3.1 with

$$K_s = \frac{c_{0*}}{(c_{0\,\min}-\epsilon_*)}\frac{K_{W0}g_{2,\delta'}[W_1]}{2(\delta'-\delta_0)}\cdot\left[g_{2,\delta_0}[W_mG_Q] + \frac{K_\phi K_y}{(c_{0\,\min}-\epsilon_*)}\right]$$

where $\delta' > \delta_0$ is such that $G(s-\delta')$, $W_m(s-\delta')$ are exponentially stable and K_{W0} is the same as K_W except that it is evaluated for $\delta = \delta_0$. □□

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[9] Notice that the constants ϵ_c, being absorbed under the strict inequality sign, do not appear in the final condition for signal boundedness.

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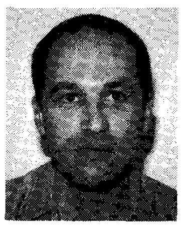

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