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HIGH GAIN OBSERVERS APPLIED TO PROBLEMS IN THE STABILIZATION OF UNCERTAIN LINEAR SYSTEMS, DISTURBANCE ATTENUATION AND H^∞ OPTIMIZATION

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SUMMARY

In recent years, a number of approaches have been developed for solving the following problem in linear systems theory: design a state feedback control so that the norm of a specified transfer function is less than a given bound over all frequencies. This paper deals with the case in which not all systems states can be measured. It shows that if a certain minimum phase condition is satisfied, then the transfer function bound achievable using state feedback can be asymptotically recovered using a high gain observer. These results have application to the problems of disturbance attenuation, H^∞ optimization and the stabilization of uncertain linear systems.

KEY WORDS Linear systems theory State feedback Transfer functions Minimum phase condition High gain observer Disturbance attenuation H^∞ optimization Stabilization

1. INTRODUCTION

In designing a feedback control system, the ability to measure all of the system states often greatly aids the design process. One example in which this situation holds is the disturbance attenuation problem. The disturbance attenuation problem is concerned with designing a feedback control law which ensures the effect of disturbances acting on a linear system is reduced to an acceptable level.

References 1–3 provide different approaches to the disturbance attenuation problem for the case in which all of the system states can be measured. However, much less is known about the disturbance attenuation problem for the case in which some of the system states cannot be measured.[4] In this paper we present a new approach to this problem.

A problem related to the disturbance attenuation problem is the problem of stabilizing an uncertain linear system. The uncertain linear systems under consideration are described by state equations containing time-varying unknown-but-bounded uncertain parameters; see e.g. Reference 5. The connection between the disturbance attenuation problem and the problem of

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stabilizing an uncertain system has been pointed out in References 3 and 6–8. As with the disturbance attenuation problem, there exist a number of approaches to the problem of stabilizing an uncertain system for the full state feedback case; see e.g. References 5 and 7–10. However, when not all of the state variables are available for measurement, there are only a few results available on the uncertain system stabilization problem; see e.g. References 11–13.

The main result of this paper is an approach to the disturbance attenuation problem for the case in which some of the state variables cannot be measured. Our design procedure begins with a full state feedback design and then the system states are reconstructed using an observer. Given the connection between the disturbance attenuation problem and the problem of stabilizing an uncertain system, this procedure will also lead to a method for stabilizing an uncertain system for the measurement feedback case.

The approach taken to designing the required observer is related to the 'loop transfer recovery' method described in Reference 14. In Reference 14, a high gain observer is used to obtain gain and phase margins arbitrarily close to those which can be achieved using full state LQ optimal control.[15] In contrast, we use a high gain observer in order to achieve a level of disturbance attenuation which is arbitrarily close to that which can be achieved using full state feedback. Thus previous results on disturbance attenuation which applied only in the full state feedback case can now be applied to a much larger class of systems. However, as in Reference 14, a key assumption which will be required is a certain minimum phase condition.

Another area in which the results of this paper can be applied is the H^∞ optimization problem.[16] In the H^∞ optimization problem, a feedback compensator is designed to minimize the bound on the norm of the specified transfer function. By applying the results of this paper successively, we can construct a feedback compensator which achieves a bound arbitrarily close to the H^∞ optimum (provided the minimum phase condition is satisfied). Thus the results of this paper provide an alternative approach to a class of H^∞ optimization problems. Furthermore, previous approaches to the H^∞ optimization problem have led to quite complicated design procedures. In contrast, the design procedure proposed in this paper is a simple scheme based on the algebraic Riccati equation.

2. PROBLEM FORMULATION: DISTURBANCE ATTENUATION

Consider a linear system described by the state equations

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{D}\mathbf{d}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)\\ \mathbf{w}(t) &= \mathbf{E}\mathbf{x}(t)\end{aligned} \tag{Σ}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the *state*, $\mathbf{d}(t) \in \mathbb{R}^p$ is the *disturbance input*, $\mathbf{u}(t) \in \mathbb{R}^m$ is the *control input*, $\mathbf{y}(t) \in \mathbb{R}^l$ is the *measured output* and $\mathbf{w}(t) \in \mathbb{R}^q$ is the *controlled output*.

Assumptions. Throughout the sequel, it will be assumed that the system (Σ) satisfies the following assumptions:

A1. $(\mathbf{C}, \mathbf{A})$ is detectable.
A2. $(\mathbf{A}, \mathbf{D})$ is stabilizable.
A3. $\mathbf{C}$ is full rank.
A4. $\mathbf{D}$ is full rank.
A5. $p \leqslant l \leqslant n$.

A6. The transfer function $\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{D}$ is left invertible and strictly minimum phase; i.e.

$$\operatorname{rank}\begin{bmatrix}\mathbf{A}-s\mathbf{I} & \mathbf{D}\\ \mathbf{C} & \mathbf{0}\end{bmatrix}=n+p$$

for all $\operatorname{Re}(s)\geqslant 0$.

Remarks. Assumptions A3 and A4 can be made without loss of generality. Assumption A2 can be removed if the minimum phase condition A6 is replaced by a more general geometric condition.[17,18]

Definition 2.1 (see also Reference 2). Let the constant $\sigma>0$ be given. Then the system (Σ) is said to be *stabilizable via state feedback with disturbance attenuation* σ if there exists a state feedback matrix $\mathbf{K}\in\mathbb{R}^{n\times m}$ such that the following conditions are satisfied:

(i) The matrix $\bar{\mathbf{A}}\triangleq\mathbf{A}+\mathbf{BK}$ is a stability matrix; i.e. all eigenvalues of $\bar{\mathbf{A}}$ lie in the open left half-plane.
(ii) The transfer function matrix

$$\mathbf{H}_1(s)\triangleq\mathbf{E}(s\mathbf{I}-\bar{\mathbf{A}})^{-1}\mathbf{D} \tag{1}$$

satisfies the bound

$$\|\mathbf{H}_1(\mathrm{j}\omega)\|\leqslant\sigma$$

for all $\omega\in\mathbb{R}$.

In the above definition and throughout the sequel, $\|\cdot\|$ will denote the induced matrix norm. That is, $\|\mathbf{H}_1(\mathrm{j}\omega)\|^2$ is the maximum eigenvalue of the matrix $\mathbf{H}_1(-\mathrm{j}\omega)^{\mathrm{T}}\mathbf{H}_1(\mathrm{j}\omega)$.
Full state disturbance attenuation is illustrated in the block diagram shown in Figure 1.

Remark. If (Σ) is stabilizable with disturbance attenuation σ, then under state feedback the system state is asymptotically stable. Moreover, given any disturbance input with Laplace transform $\mathbf{d}(s)$, the transform of the controlled output $\mathbf{w}(s)$ satisfies

$$\|\mathbf{w}(\mathrm{j}\omega)\|\leqslant\sigma\|\mathbf{d}(\mathrm{j}\omega)\|$$

for all $\omega\in\mathbb{R}$. That is, (Σ) has guaranteed disturbance attenuation σ. In Section 5 we show that the disturbance attenuation problem can be recast as one of guaranteed stabilization (of an uncertain linear system). Presently, we shall focus on achieving disturbance attenuation via measurement feedback.

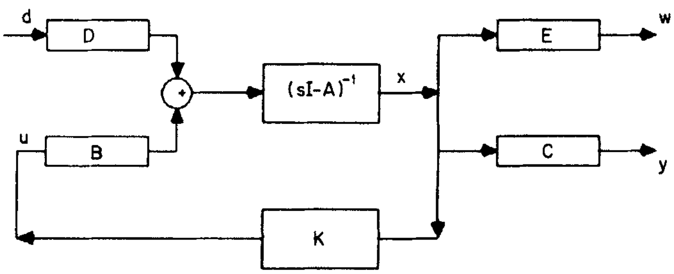

Figure 1. Full state feedback

Definition 2.2. Let the constant $\sigma > 0$ be given. Then the system (Σ) is said to be *stabilizable via measurement feedback with disturbance attenuation* σ if there exists a dynamic compensator

$$\begin{aligned} \dot{\boldsymbol{\eta}}(t) &= \mathbf{F}\boldsymbol{\eta}(t) + \mathbf{G}\mathbf{y}(t) \\ \mathbf{u}(t) &= \mathbf{H}\boldsymbol{\eta}(t) + \mathbf{J}\mathbf{y}(t) \end{aligned} \qquad (\Sigma_c)$$

such that:

(i) The resulting closed-loop system matrix

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A} + \mathbf{BJC} & \mathbf{BH} \\ \mathbf{GC} & \mathbf{F} \end{bmatrix}$$

is a stability matrix.

(ii) The transfer function matrix

$$\mathbf{H}_2(s) \triangleq [\mathbf{E} \quad \mathbf{0}](s\mathbf{I} - \tilde{\mathbf{A}})^{-1}\begin{bmatrix} \mathbf{D} \\ \mathbf{0} \end{bmatrix} \tag{2}$$

satisfies the bound

$$\| \mathbf{H}_2(j\omega) \| \leqslant \sigma \tag{3}$$

for all $\omega \in \mathbb{R}$.

In this paper we shall consider the case in which this dynamic compensator is based on a full state observer. That is, we consider a compensator described by the state equations.

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{BK}\hat{\mathbf{x}}(t) + \mathbf{L}(\mathbf{y}(t) - \mathbf{C}(\hat{\mathbf{x}}(t)) \\ \mathbf{u}(t) &= \mathbf{K}\hat{\mathbf{x}}(t) \end{aligned} \qquad (\Sigma_0)$$

where $\hat{\mathbf{x}}(t) \in \mathbb{R}^n$ is the *observer state*, $\mathbf{K} \in \mathbb{R}^{m \times n}$ is a suitable *controller gain matrix* and $\mathbf{L} \in \mathbb{R}^{n \times l}$ is a suitable *observer gain matrix*. This situation is illustrated in the block diagram shown in Figure 2.

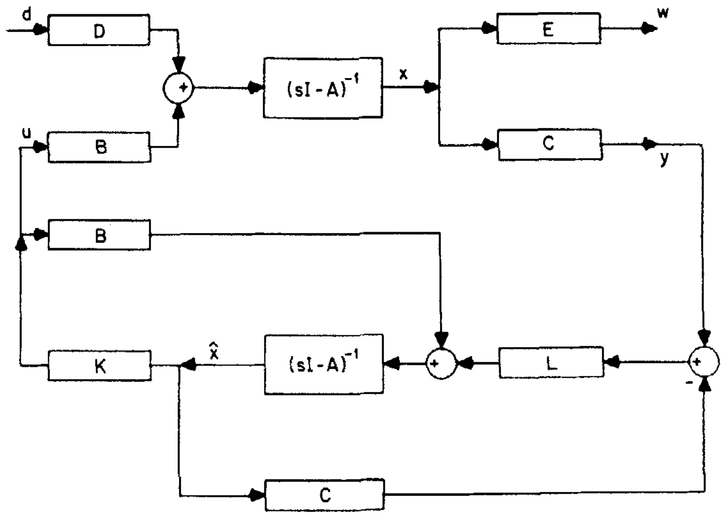

Figure 2. Observer-based feedback

We now consider the state equations of the closed-loop system which results when compensator (Σ_0) is applied to the system (Σ). Indeed, we obtain the state equations

$$\begin{aligned}\dot{\mathbf{z}}(t) &= \tilde{\mathbf{A}}\mathbf{z}(t) + \tilde{\mathbf{D}}\mathbf{d}(t)\\ \mathbf{w}(t) &= \tilde{\mathbf{E}}\mathbf{z}(t)\end{aligned} \qquad (\Sigma_{\mathrm{cl}})$$

where

$$\mathbf{z} = \begin{bmatrix}\mathbf{x}\\ \hat{\mathbf{x}}\end{bmatrix} \qquad \tilde{\mathbf{A}} = \begin{bmatrix}\mathbf{A} & \mathbf{BK}\\ \mathbf{LC} & \mathbf{A}+\mathbf{BK}-\mathbf{LC}\end{bmatrix} \qquad \tilde{\mathbf{D}} = \begin{bmatrix}\mathbf{D}\\ \mathbf{0}\end{bmatrix} \qquad \tilde{\mathbf{E}} = [\mathbf{E} \quad 0]$$

The system (Σ_{cl}) is said to be *stable with disturbance attenuation* σ if $\tilde{\mathbf{A}}$ is a stability matrix and

$$\| \tilde{\mathbf{E}}(\mathrm{j}\omega\mathbf{I} - \tilde{\mathbf{A}})^{-1}\tilde{\mathbf{D}} \| \leqslant \sigma$$

for all $\omega \in \mathbb{R}$.

Using the results of References 2 and 6, we can test if the system (Σ_{cl}) is stable with disturbance attenuation σ by repeatedly solving an algebraic Riccati equation.

Theorem 2.1 (see Reference 2 for proof). Suppose that the system (Σ_{cl}) is stable with disturbance attenuation σ. Then, given any $\tilde{\sigma} > \sigma$, there exists an $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$ the Riccati equation

$$\tilde{\mathbf{A}}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\tilde{\mathbf{A}} + \frac{1}{\sigma}\mathbf{P}\tilde{\mathbf{D}}\tilde{\mathbf{D}}^{\mathrm{T}}\mathbf{P} + \frac{1}{\sigma}\tilde{\mathbf{E}}^{\mathrm{T}}\tilde{\mathbf{E}} + \varepsilon\mathbf{I} = \mathbf{0} \qquad (4)$$

has a positive-definite symmetric solution $\mathbf{P}$.

Conversely, suppose that there exists an $\varepsilon > 0$ such that the Riccati equaiton

$$\tilde{\mathbf{A}}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\tilde{\mathbf{A}} + \frac{1}{\sigma}\mathbf{P}\tilde{\mathbf{D}}\tilde{\mathbf{D}}^{\mathrm{T}}\mathbf{P} + \frac{1}{\sigma}\tilde{\mathbf{E}}^{\mathrm{T}}\tilde{\mathbf{E}} + \varepsilon\mathbf{I} = \mathbf{0}$$

has a positive-definite symmetric solution $\mathbf{P}$. Then the system (Σ_{cl}) is stable with disturbance attenuation σ.

The dual system

The key assumption required in this paper is the minimum phase condition A6. We now introduce the linear system dual to (Σ). This will enable us to replace Assumption A.6 by an alternative minimum phase condition which is dual to A.6. Towards this end, consider the dual system (Σ_{D}) defined by the state equations

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}^{\mathrm{T}}\mathbf{x}(t) + \mathbf{E}^{\mathrm{T}}\mathbf{d}(t) + \mathbf{C}^{\mathrm{T}}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{B}^{\mathrm{T}}\mathbf{x}(t)\\ \mathbf{w}(t) &= \mathbf{D}^{\mathrm{T}}\mathbf{x}(t)\end{aligned} \qquad (\Sigma_{\mathrm{D}})$$

Theorem 2.2. The system (Σ) is stabilizable via measurement feedback with disturbance attenuation σ if and only if the dual system (Σ_{D}) is stabilizable via measurement feedback with disturbance attenuation σ.

Proof. Suppose that the system (Σ_{D}) is stabilizable via measurement feedback with disturbance attenuation σ. Then there exists a compensator

$$\dot{\eta}(t) = \bar{\mathbf{F}}\eta(t) + \bar{\mathbf{G}}\mathbf{y}(t)$$
$$\mathbf{u}(t) = \bar{\mathbf{H}}\eta(t) + \bar{\mathbf{J}}\mathbf{y}(t)$$

such that

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A}^{\mathrm{T}} + \mathbf{C}^{\mathrm{T}}\bar{\mathbf{J}}\mathbf{B}^{\mathrm{T}} & \mathbf{C}^{\mathrm{T}}\bar{\mathbf{H}} \\ \bar{\mathbf{G}}\mathbf{B}^{\mathrm{T}} & \bar{\mathbf{F}} \end{bmatrix}$$

is a stability matrix and the transfer function

$$\bar{\mathbf{H}}_2(s) \triangleq [\mathbf{D}^{\mathrm{T}} \quad \mathbf{0}]\,(s\mathbf{I} - \bar{\mathbf{A}})^{-1}\begin{bmatrix} \mathbf{E}^{\mathrm{T}} \\ \mathbf{0} \end{bmatrix}$$

satisfies the bound

$$\|\bar{\mathbf{H}}_2(\mathrm{j}\omega)\| \leqslant \sigma$$

for all $\omega \in \mathbb{R}$. We now let $\mathbf{F} = \bar{\mathbf{F}}^{\mathrm{T}}, \mathbf{G} = \bar{\mathbf{H}}^{\mathrm{T}}, \mathbf{H} = \bar{\mathbf{G}}^{\mathrm{T}}$ and $\mathbf{J} = \bar{\mathbf{J}}^{\mathrm{T}}$. It follows that

$$\tilde{\mathbf{A}} \triangleq \bar{\mathbf{A}}^{\mathrm{T}} = \begin{bmatrix} \mathbf{A} + \mathbf{BJC} & \mathbf{BH} \\ \mathbf{GC} & \mathbf{F} \end{bmatrix}$$

is a stability matrix. Furthermore, the transfer function matrix

$$\mathbf{H}_2(s) \triangleq \bar{\mathbf{H}}_2^{\mathrm{T}}(s) = [\mathbf{E} \quad \mathbf{0}]\,(s\mathbf{I} - \tilde{\mathbf{A}})^{-1}\begin{bmatrix} \mathbf{D} \\ \mathbf{0} \end{bmatrix}$$

satisfies the bound

$$\|\mathbf{H}_2(\mathrm{j}\omega)\| = \|\bar{\mathbf{H}}_2(\mathrm{j}\omega)\| \leqslant \sigma$$

for all $\omega \in \mathbb{R}$. Thus the system (Σ) is stabilizable via measurement feedback with disturbance attenuation σ.

The proof of the converse part of the theorem is identical to the above. This completes the proof of the theorem. □

Observation. The system (Σ_{D}) satisfies Assumptions A1–A6 if and only if the system (Σ) satisfies the following set of assumptions:

A1′. $(\mathbf{A}, \mathbf{B})$ is stabilizable.
A2′. $(\mathbf{E}, \mathbf{A})$ is detectable.
A3′. $\mathbf{B}$ is full rank.
A4′. $\mathbf{E}$ is full rank.
A5′. $q \leqslant m \leqslant n$.
A6′. The transfer function $\mathbf{E}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$ is right invertible and strictly minimum phase.

3. THE MAIN RESULT ON DISTURBANCE ATTENUATION

We are now in a position to state our main result on disturbance attenuation. In Section 4 we will show how these results apply to the stabilization of uncertain linear systems.

Theorem 3.1. Let the constant $\sigma > 0$ be given and suppose that the system (Σ) satisfies Assumptions A1–A6 and is stabilizable via state feedback with disturbance attenuation σ.

Then, given any $\varepsilon > 0$, the system (Σ) is stabilizable via measurement feedback with disturbance attenuation $\sigma + \varepsilon$.

Construction of the observer

In order to establish that the system (Σ) is stabilizable via measurement feedback with disturbance attenuation $\sigma + \varepsilon$, we shall use a dynamic compensator based on the full state observer (Σ_0). The controller gain $\mathbf{K}$ to be used will be the gain matrix used in the full state feedback case. The observer gain $\mathbf{L}$ will be constructed as follows. Let $\mathbf{Q}$ be a given positive-definite symmetric weighting matrix and let δ and $q > 0$ be given constants (the procedure for the construction of δ and q will be given in the sequel). Furthermore, suppose $\mathbf{\Sigma}$ is a positive-definite symmetric solution to the Riccati equation

$$\mathbf{A\Sigma} + \mathbf{\Sigma A}^{\mathrm{T}} + \mathbf{Q} + q^2 \frac{DD^{\mathrm{T}}}{\delta} - \mathbf{\Sigma C}^{\mathrm{T}}\mathbf{C\Sigma} + \frac{\mathbf{\Sigma E}^{\mathrm{T}}\mathbf{E\Sigma}}{q^2\delta} = \mathbf{0} \tag{5}$$

Then the observer gain $\mathbf{L}$ is given by

$$\mathbf{L} \triangleq \mathbf{\Sigma C}^{\mathrm{T}} \tag{6}$$

The following lemma shows that given any $\delta > 0$, Riccati equation (5) will have a positive-definite solution provided $q > 0$ is chosen to be sufficiently large.

Lemma 3.1 (see Appendix for proof). Suppose that the system (Σ) satisfies Assumptions A1–A6. Then, given any $\delta > 0$, there exists a $q^* > 0$ such that Riccati equation (5) has a positive-definite symmetric solution for all $q \geqslant q^*$. Furthermore, with $\mathbf{L}$ defined as in (6), the transfer function $\mathbf{G}(s) \triangleq \mathbf{E}(s\mathbf{I} - \mathbf{A} + \mathbf{LC})^{-1}\mathbf{D}$ satisfies the bound

$$\| \mathbf{G}(\mathrm{j}\omega) \| \leqslant \delta$$

for all $\omega \in \mathbb{R}$.

Notation and preliminary lemmas

In order to show that the observer gain matrix $\mathbf{L}$ defined above will indeed lead to the desired disturbance attenuation, we first introduce some additional notation and preliminary lemmas.

Notation. The transfer function matrix $\boldsymbol{\phi}(s)$ is defined by

$$\boldsymbol{\phi}(s) \triangleq (s\mathbf{I} - \mathbf{A})^{-1} \tag{7}$$

Standing assumptions. In the following lemmas it will be assumed that the system (Σ) satisfies Assumptions A1–A6. Furthermore, it will be assumed that the constant $\delta > 0$ has been prespecified and that $q^* > 0$ has been defined as in Lemma 3.1. Finally, it will be assumed that for given $q \geqslant q^*$, the observer gain $\mathbf{L}$ has been constructed according to equations (5) and (6).

Lemma 3.2 (see Appendix for proof). Suppose the observer-based compensator (Σ_0) is applied to system (Σ). Then the transfer function $\mathbf{H}_2(s)$ defined in (2) is given by

$$\mathbf{H}_2(s) = \mathbf{E}[\boldsymbol{\phi}(s)^{-1} - \mathbf{BK}]^{-1}\mathbf{L}(\mathbf{I} + \mathbf{C}\boldsymbol{\phi}(s)\mathbf{L})^{-1}\mathbf{C}\boldsymbol{\phi}(s)\mathbf{D} + \mathbf{E}[\boldsymbol{\phi}(s)^{-1} + \mathbf{LC}]^{-1}\mathbf{D}$$

Lemma 3.3 (see Appendix for proof). There exists a matrix $\mathbf{N} \in \mathbb{R}^{p \times l}$ such that $\mathbf{NN}^{\mathrm{T}} = \mathbf{I}$ and $\mathbf{L}/q \to \mathbf{DN}$ as $q \to \infty$.

In the light of this lemma we now rewrite $\mathbf{H}_2(s)$ as

$$\mathbf{H}_2(s) = \mathbf{E}[\boldsymbol{\phi}(s)^{-1} - \mathbf{BK}]^{-1}(\mathbf{L}/q)(\mathbf{I}/q + \mathbf{C}\boldsymbol{\phi}(s)\mathbf{L}/q)^{-1}\mathbf{C}\boldsymbol{\phi}(s)\mathbf{D} + \mathbf{E}[\boldsymbol{\phi}(s)^{-1} + \mathbf{LC}]^{-1}\mathbf{D} \tag{8}$$

In the following lemmas we show that the terms on the right-hand side of (8) can be suitably bounded.

Lemma 3.4 (see Appendix for proof). Given any $q \geqslant q^*$,

$$\|[\mathbf{I}/q + \mathbf{C}\boldsymbol{\phi}(j\omega)\mathbf{L}/q]^{-1}\mathbf{C}\boldsymbol{\phi}(j\omega)\mathbf{D}\| \leqslant 1$$

for all $\omega \in \mathbb{R}$.

Lemma 3.5 (see Appendix for proof). Suppose the matrix $\mathbf{K} \in \mathbb{R}^{n \times m}$ is such that $\mathbf{A} + \mathbf{BK}$ is a stability matrix and the transfer function $\mathbf{H}_1(s) = \mathbf{E}(s\mathbf{I} - \mathbf{A} - \mathbf{BK})^{-1}\mathbf{D}$ satisfies the bound

$$\|\mathbf{H}_1(j\omega)\| \leqslant \sigma \tag{9}$$

for all $\omega \in \mathbb{R}$. Then, given any $\varepsilon > 0$, there exists a constant $\tilde{q} \geqslant q^*$ such that if $q \geqslant \tilde{q}$ then

$$\|\mathbf{E}(j\omega\mathbf{I} - \mathbf{A} - \mathbf{BK})^{-1}\mathbf{L}/q\| \leqslant \sigma + \varepsilon/2$$

for all $\omega \in \mathbb{R}$.

Using the above lemmas, we are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Given that the system (Σ) is stabilizable via state feedback with disturbance attenuation σ, it follows that there exists a matrix $\mathbf{K}$ such that $\mathbf{A} + \mathbf{BK}$ is a stability matrix and

$$\|\mathbf{H}_1(j\omega)\| = \|\mathbf{E}(j\omega\mathbf{I} - \mathbf{A} - \mathbf{BK})^{-1}\mathbf{D}\| \leqslant \sigma \tag{10}$$

for all $\omega \in \mathbb{R}$.

We now let $\varepsilon > 0$ be given and let $\delta = \varepsilon/2$. Furthermore, the constant $q^* > 0$ is defined as in Lemma 3.1 and $\tilde{q} \geqslant q^*$ is defined as in Lemma 3.5. Now consider the compensator (Σ_0) applied to the system (Σ). The controller gain $\mathbf{K}$ is taken to be the same as for the full state feedback case. Moreover, the observer gain $\mathbf{L}$ is defined according to equations (5) and (6) with $q \geqslant \tilde{q}$. Using Lemma 3.2, it follows that the resulting transfer function $\mathbf{H}_2(s)$ is given by

$$\mathbf{H}_2(s) = \mathbf{E}[\boldsymbol{\phi}(s)^{-1} - \mathbf{BK}]^{-1}\mathbf{L}(\mathbf{I} + \mathbf{C}\boldsymbol{\phi}(s)\mathbf{L})^{-1}\mathbf{C}\boldsymbol{\phi}(s)\mathbf{D} + \mathbf{E}[\boldsymbol{\phi}(s)^{-1} + \mathbf{LC}]^{-1}\mathbf{D}$$

Hence, using the triangle inequality, the Schwarz inequality and Lemma 3.4, it follows that

$$\|\mathbf{H}_2(j\omega)\| \leqslant \|\mathbf{E}(j\omega\mathbf{I} - \mathbf{A} - \mathbf{BK})^{-1}\mathbf{L}/q\| + \|\mathbf{E}(j\omega\mathbf{I} - \mathbf{A} + \mathbf{LC})^{-1}\mathbf{D}\|$$

for all $\omega \in \mathbb{R}$. Furthermore, using Lemma 3.1, it follows that

$$\|\mathbf{H}_2(j\omega)\| \leqslant \|\mathbf{E}(j\omega\mathbf{I} - \mathbf{A} - \mathbf{BK})^{-1}\mathbf{L}/q\| + \varepsilon/2$$

for all $\omega \in \mathbb{R}$. Therefore, using inequality (10) and Lemma 3.5, we conclude

$$\|\mathbf{H}_2(j\omega)\| \leqslant \sigma + \varepsilon$$

for all $\omega \in \mathbb{R}$.

In order to complete the proof, it remains only to show that the closed-loop system matrix is a stability matrix. It is well known that using an observer-based compensator (Σ_0), the eigenvalues of the closed-loop system matrix will be those of the matrices $\mathbf{A} + \mathbf{BK}$ and $\mathbf{A} - \mathbf{LC}$;

see e.g. Reference 19. Furthermore, the matrix $\mathbf{K}$ has been constructed so that $\mathbf{A}+\mathbf{BK}$ is a stability matrix. To show that $\mathbf{A}-\mathbf{LC}$ is a stability matrix, we rewrite Riccati equation (5) as

$$(\mathbf{A}-\mathbf{LC})\Sigma+\Sigma(\mathbf{A}-\mathbf{LC})^{\mathrm{T}}+\mathbf{Q}+\frac{q^2\mathbf{DD}^{\mathrm{T}}}{\delta}+\Sigma\mathbf{C}^{\mathrm{T}}\mathbf{C}\Sigma+\frac{\Sigma\mathbf{E}^{\mathrm{T}}\mathbf{E}\Sigma}{q^2\delta}=\mathbf{0}$$

Given that $\boldsymbol{\Sigma}$ and $\mathbf{Q}$ are positive-definite symmetric matrices, it now follows from Lyapunov stability theory that $\mathbf{A}-\mathbf{LC}$ is a stability matrix; see e.g. Reference 19. This completes the proof of the theorem. □

Using Theorem 2.2, we now obtain a corollary to the above theorem in which Assumptions A1–A6 are replaced by the dual assumptions A1′ – A6′.

Corollary 3.1. Let the constant $\sigma>0$ be given and suppose that the system (Σ) satisfies Assumptions A1′–A6′. Furthermore, suppose that the dual system (Σ_{D}) is stabilizable via state feedback with disturbance attenuation σ. Then, given any $\varepsilon>0$, the system (Σ) is stabilizable via measurement feedback with disturbance attenuation $\sigma+\varepsilon$.

Proof. Suppose the system (Σ) satisfies Assumptions A1′–A6′ and furthermore suppose the dual system (Σ_{D}) is stabilizable via state feedback with disturbance attenuation σ. It follows that the system (Σ_{D}) satisfies Assumptions A1–A6. Hence, using Theorem 3.1, we conclude that given any $\varepsilon>0$, the system (Σ_{D}) is stabilizable via measurement feedback with disturbance attenuation $\sigma+\varepsilon$. The required result now follows directly from Theorem 2.2. □

Theorem 3.1 gives a sufficient condition for achieving a desired degree of disturbance attenuation via measurement feedback. We now give a necessary condition. There is a gap between this necessary condition and the sufficient condition given in Theorem 3.1. However, this gap involves only an infinitesimal change in the achievable disturbance attenuation.

Theorem 3.2. Suppose that the system (Σ) is stabilizable via measurement feedback with disturbance attenuation σ. Then, given any $\varepsilon>0$, the system (Σ) is stabilizable via state feedback with disturbance attenuation $\sigma+\varepsilon$.

Proof. Suppose that (Σ) is stabilizable via measurement feedback with disturbance attenuation σ and let $\varepsilon>0$ be given. It follows that there exists a dynamic compensator of the form (Σ_{c}) such that the resulting closed-loop system has the following properties:

(i)
$$\tilde{\mathbf{A}}=\begin{bmatrix}\mathbf{A}+\mathbf{BJC} & \mathbf{BH}\\ \mathbf{GC} & \mathbf{F}\end{bmatrix} \tag{11}$$

is a stability matrix.

(ii) The transfer function

$$\mathbf{H}_2(s)\triangleq[\mathbf{E}\quad \mathbf{0}](s\mathbf{I}-\tilde{\mathbf{A}})^{-1}\begin{bmatrix}\mathbf{D}\\ \mathbf{0}\end{bmatrix}\triangleq\tilde{\mathbf{E}}(s\mathbf{I}-\tilde{\mathbf{A}})^{-1}\tilde{\mathbf{D}}$$

satisfies the bound

$$\|\mathbf{H}_2(j\omega)\|\leqslant\sigma$$

for all $\omega\in\mathbb{R}$.

Letting $\tilde{\sigma}=\sigma+\varepsilon$, it follows from Theorem 2.1 that there exists a $\sigma>0$ such that the Riccati equation

$$\tilde{\mathbf{A}}^{\mathrm{T}}\mathbf{P}+\mathbf{P}\tilde{\mathbf{A}}+\frac{\mathbf{P}\tilde{\mathbf{D}}\tilde{\mathbf{D}}^{\mathrm{T}}\mathbf{P}}{\tilde{\sigma}}+\frac{\tilde{\mathbf{E}}^{\mathrm{T}}\tilde{\mathbf{E}}}{\tilde{\sigma}}+\sigma\mathbf{I}=\mathbf{0}$$

has a positive-definite symmetric solution $\mathbf{P}$. We now let $\mathbf{S}=\mathbf{P}^{-1}$. It follows that

$$\tilde{\mathbf{A}}\mathbf{S}+\mathbf{S}\tilde{\mathbf{A}}^{\mathrm{T}}+\frac{\tilde{\mathbf{D}}\tilde{\mathbf{D}}^{\mathrm{T}}}{\tilde{\sigma}}+\frac{\mathbf{S}\tilde{\mathbf{E}}^{\mathrm{T}}\tilde{\mathbf{E}}\mathbf{S}}{\tilde{\sigma}}+\sigma\mathbf{S}\mathbf{S}=\mathbf{0} \tag{12}$$

The matrix S is now partitioned to conform with (11); i.e.

$$\mathbf{S}=\begin{bmatrix}\mathbf{S}_{11} & \mathbf{S}_{12}\\ \mathbf{S}_{12}^{\mathrm{T}} & \mathbf{S}_{22}\end{bmatrix}$$

where $\mathbf{S}_{11}$ and $\mathbf{S}_{22}$ are positive-definite symmetric matrices. Substituting for $\mathbf{S}$ in (12) and taking the (1,1) block leads to the Riccati equation

$$\begin{aligned}&[\mathbf{A}+\mathbf{B}(\mathbf{J}\mathbf{C}+\mathbf{H}\mathbf{S}_{12}^{\mathrm{T}}\mathbf{S}_{11}^{-1})]\mathbf{S}_{11}+\mathbf{S}_{11}[\mathbf{A}+\mathbf{B}(\mathbf{J}\mathbf{C}+\mathbf{H}\mathbf{S}_{12}^{\mathrm{T}}\mathbf{S}_{11}^{-1})]^{\mathrm{T}}\\&\qquad+\frac{\mathbf{D}\mathbf{D}^{\mathrm{T}}}{\tilde{\sigma}}+\frac{\mathbf{S}_{11}\mathbf{E}^{\mathrm{T}}\mathbf{E}\mathbf{S}_{11}}{\tilde{\sigma}}+\delta(\mathbf{S}_{11}\mathbf{S}_{11}+\mathbf{S}_{12}\mathbf{S}_{12}^{\mathrm{T}})=\mathbf{0}\end{aligned}$$

Given that $\mathbf{S}_{11}$ is a positive-definite matrix, we now use Lyapunov stability theory to conclude that the matrix $\bar{\mathbf{A}}\triangleq\mathbf{A}+\mathbf{B}(\mathbf{J}\mathbf{C}+\mathbf{H}\mathbf{S}_{12}^{\mathrm{T}}\mathbf{S}_{11}^{-1})$ is a stability matrix. Furthermore, if we let $\bar{\mathbf{P}}=\mathbf{S}_{11}^{-1}$, then

$$\bar{\mathbf{A}}^{\mathrm{T}}\bar{\mathbf{P}}+\bar{\mathbf{P}}\bar{\mathbf{A}}+\frac{\bar{\mathbf{P}}\mathbf{D}\mathbf{D}^{\mathrm{T}}\bar{\mathbf{P}}}{\tilde{\sigma}}+\frac{\mathbf{E}^{\mathrm{T}}\mathbf{E}}{\tilde{\sigma}}+\bar{\mathbf{Q}}=\mathbf{0}$$

where $\bar{\mathbf{Q}}\triangleq\delta(\mathbf{I}+\bar{\mathbf{P}}\mathbf{S}_{12}\mathbf{S}_{12}^{\mathrm{T}}\bar{\mathbf{P}})$. Hence, using Theorem 3.1 of Reference 2, it follows that

$$\|\mathbf{E}(\mathrm{j}\omega\mathbf{I}-\bar{\mathbf{A}})^{-1}\mathbf{D}\|<\tilde{\sigma}=\sigma+\varepsilon$$

for all $\omega\in\mathbb{R}$. Therefore the system (Σ) is stabilizable via state feedback with disturbance attenuation $\sigma+\varepsilon$. □

The controller design algorithm

Using the above results, we are now in a position to present our algorithm for designing a compensator to achieve stability with disturbance attenuation $\sigma+\delta$. It is assumed that the system (Σ) satisfies Assumptions A1–A6. Furthermore, it is assumed that (Σ) is stabilizable via *state feedback* with disturbance attenuation σ.

- Step (i) Use the results of Reference 2 to design a full state feedback control law which gives closed-loop stability with disturbance attenuation σ.
- Step (ii) Initialize q to some starting value (e.g. $q=1$).
- Step (iii) Determine whether Riccati equation (5) has a positive-definite solution. In practice, one would simply apply a standard algorithm to (5) (see e.g. Reference 20) in order to determine if it yields a positive-definite symmetric solution. If a positive-definite symmetric solution exists, proceed to step (iv). If not, replace q by $2q$ and repeat this step.
- Step (iv) Using the controller gain obtained in step (i) and the observer gain given by (6), construct the compensator (Σ_0) and then form the resulting closed-loop system (Σ_{cl}).

Step (v) Initialize ε to some starting value (e.g. $\varepsilon = 1$) and let $\tilde{\sigma} = \sigma + \delta$.

Step (vi) Determine whether Riccati equation (4) has a positive-definite solution. If a positive-definite symmetric solution exists, we conclude that the compensator (Σ_0) is a suitable compensator. If not, proceed to step (vii).

Step (vii) Replace ε by $\varepsilon/2$. If ε is less than some computational accuracy ε_0, replace q by $2q$ and return to step (iii). Otherwise repeat step (vi).

Remark. In the above algorithm, we have used the Riccati equation test of Theorem 2.1 in order to determine if the closed-loop system is stable with disturbance attenuation σ. An alternative approach would be to simply plot $\|\mathbf{H}_2(j\omega)\|$ versus ω for the closed-loop system and determine graphically whether the desired degree of disturbance attenuation has been achieved.

H^∞ optimization

It should be noted that the design procedure described above can be used to construct a stabilizing controller which achieves a disturbance attenuation level which is arbitrarily close to the H^∞ optimum (for systems satisfying Assumptions A1–A6). Indeed, by using the results of Reference 2 one can obtain a stabilizing state feedback control which achieves a disturbance attenuation level arbitrarily close to the H^∞ optimum. Then, using the results of this paper, one can construct a stabilizing measurement feedback compensator (Σ_0) which achieves a disturbance attenuation level arbitrarily close to the H^∞ optimum.

4. STABILIZATION OF UNCERTAIN LINEAR SYSTEMS

In this section we apply the results of the previous sections to the problem of stabilizing an uncertain linear system. The uncertain linear systems under consideration will be described by the state equations

$$\begin{aligned}\dot{\mathbf{x}}(t) &= (\mathbf{A} + \mathbf{DF}(t)\mathbf{E})\mathbf{x}(t) + \mathbf{Bu}(t)\\ \mathbf{y}(t) &= \mathbf{Cx}(t)\\ \|\mathbf{F}(t)\| &\leqslant \bar{r}\end{aligned} \qquad (\Sigma_u)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the *state*, $\mathbf{u}(t) \in \mathbb{R}^m$ is the *control input*, $\mathbf{y}(t) \in \mathbb{R}^l$ is the *measured output*, $\mathbf{F}(t) \in \mathbb{R}^{p\times q}$ is a *matrix of uncertain parameters* and $\bar{r} > 0$ is the *uncertainty bound*. It is assumed that the matrix function $\mathbf{F}(\cdot)$ is Lebesgue measurable and $\|\mathbf{F}(t)\| \leqslant \bar{r}$ for all $t \geqslant 0$.

In order to stabilize the system (Σ), we will apply an observer-based controller of the form (Σ_0) described in Section 2. This will result in a closed-loop uncertain system of the form

$$\begin{aligned}\dot{\mathbf{z}}(t) &= (\tilde{\mathbf{A}} + \tilde{\mathbf{D}}\mathbf{F}(t)\tilde{\mathbf{E}})\mathbf{z}(t)\\ \|\mathbf{F}(t)\| &\leqslant \bar{r}\end{aligned} \qquad (\Sigma_{ucl})$$

Definition 4.1 (see also Reference 9). The uncertain system (Σ_{cl}) is said to be *quadratically stable* if there exists a positive-definite symmetric matrix $\mathbf{P}$ and a constant $\alpha > 0$ such that the following condition holds: Given any admissible uncertainty $\mathbf{F}(\cdot)$, the Lyapunov derivative corresponding to the Lyapunov function $V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{x}$ satisfies the bound

$$L(\mathbf{z}, t) \triangleq 2\mathbf{z}^{\mathrm{T}}\mathbf{P}(\tilde{\mathbf{A}} + \tilde{\mathbf{D}}\mathbf{F}(t)\tilde{\mathbf{E}})\mathbf{z} \leqslant -\alpha\|\mathbf{z}\|^2$$

for all $\mathbf{z} \in \mathbb{R}^{2n}$ and all $t \geqslant 0$.

In the following lemmas, which are proved in Reference 7, we relate the quadratic stability of (Σ_{ucl}) to the properties of the transfer function $\tilde{\mathbf{E}}(s\mathbf{I}-\tilde{\mathbf{A}})\tilde{\mathbf{D}}$. Similar results are also given in References 3, 6 and 8.

Lemma 4.1 (see Reference 7 for proof). Suppose that the uncertain system (Σ_{ucl}) is quadratically stable. Then:

(i) $\tilde{\mathbf{A}}$ is a stability matrix.
(ii) $\|\tilde{\mathbf{E}}(j\omega\mathbf{I}-\tilde{\mathbf{A}})^{-1}\tilde{\mathbf{D}}\| \leqslant 1/\bar{r}$ for all $\omega \in \mathbb{R}$.

Lemma 4.2 (see Reference 7 for proof). Suppose that the uncertain system (Σ_{ucl}) satisfies the following conditions:

(i) $\tilde{\mathbf{A}}$ is a stability matrix.
(ii) There exists a constant $\varepsilon > 0$ such that

$$\|\tilde{\mathbf{E}}(j\omega\mathbf{I}-\tilde{\mathbf{A}})^{-1}\tilde{\mathbf{D}}\| \leqslant (1-\varepsilon)/\bar{r}$$

for all $\omega \in \mathbb{R}$.

The the system (Σ_{ucl}) is quadratically stable.

Theorem 4.1 Suppose that the uncertain system (Σ_u) satisfies Assumptions A1–A6 given in Section 2. Furthermore, suppose that there exists a state feedback control $\mathbf{u}(t)=\mathbf{K}\mathbf{x}(t)$ such that the resulting closed-loop uncertain system is quadratically stable. Then, given any $\varepsilon > 0$, there exists a dynamic compensator of the form (Σ_0) such that the resulting closed-loop uncertain system is quadratically stable when we replace the uncertainty bound $\bar{r}$ by $\bar{r}-\varepsilon$.

Proof. Suppose (Σ_u) satisfies A1–A6 and there exists a control $\mathbf{u}(t)=\mathbf{K}\mathbf{x}(t)$ such that the resulting closed-loop system is quadratically stable. It then follows from Lemma 4.1 that $\mathbf{A}+\mathbf{BK}$ is a stability matrix and

$$\|\mathbf{E}(j\omega\mathbf{I}-\mathbf{A}-\mathbf{BK})^{-1}\mathbf{D}\| \leqslant 1/\bar{r}$$

for all $\omega \in \mathbb{R}$.

We now apply a compensator of the form (Σ_0) to the system(Σ_u). Indeed, let $\varepsilon > 0$ be given and let $\delta = \varepsilon/2$. The constant q^* is defined as in Lemma 3.1 and the constant $\tilde{q} > q^*$ is defined as in Lemma 3.5. We now construct the observer gain as in equations (5) and (6) with $q \geqslant \tilde{q}$. Furthermore, the controller gain $\mathbf{K}$ is taken to be the same as for the state feedback case. It follows from Theorem 3.1 that the transfer function $\mathbf{H}_2(s)$ for the resulting closed-loop system satisfies the bound

$$\begin{aligned}\|\mathbf{H}_2(j\omega)\| &= \left\| [\mathbf{E} \quad \mathbf{0}](j\omega\mathbf{I}-\tilde{\mathbf{A}})^{-1}\begin{bmatrix}\mathbf{D}\\ \mathbf{0}\end{bmatrix} \right\| \\ &\leqslant 1/\bar{r} + \varepsilon/2\bar{r}(\bar{r}-\varepsilon) \\ &= (1-\varepsilon/2\bar{r})/(\bar{r}-\varepsilon)\end{aligned}$$

for all $\omega \in \mathbb{R}$. Furthermore, the system matrix $\tilde{\mathbf{A}}$ for the resulting closed-loop system is a stability matrix. Hence, using Lemma 4.2, it follows that the closed-loop uncertain system is quadratically stable with uncertainty bound $\bar{r}-\varepsilon$. □

Observations. Suppose that the uncertain system (Σ_u) satisfies the conditions of the above theorem. Using a continuity argument, it is straightforward to establish that one can find a $\varepsilon > 0$ such that when state feedback is applied, the closed-loop system will be quadratically stable with uncertainty bound $\bar{r} + \varepsilon$. Hence, using the above theorem, we can construct a compensator (Σ_0) such that the resulting closed-loop system is quadratically stable with uncertainty bound $\bar{r}$. Thus the above theorem can be strengthened to remove the requirement that the uncertainty bound be reduced when using measurement feedback.

At this point we also observe that the results of Section 3 can be used in order to obtain a 'dual' version of Theorem 4.1. In this case Assumptions A1–A6 would be replaced by Assumptions A1′–A6′.

Remark. The original motivation for Theorem 4.1 came from the main result of Reference 13. However, Reference 13 does not establish quadratic stabilizability via measurement feedback for the original uncertain system (Σ_u). Indeed, owing to a technical difficulty, Reference 13 was only able to establish quadratic stabilizability for an uncertain system which is arbitrarily close to the original system. Thus Theorem 4.1 overcomes this shortcoming in the main result of Reference 13.

5. ILLUSTRATIVE EXAMPLE

In this section we show how the results of Section 4 can be applied to design a controller which ensures the longitudinal stability of the A4D aircraft.

The system

The nominal linearized longitudinal dynamics of the A4D at flight condition 0·9 Mach and 15 000 ft altitude (see p. 702 of Reference 21) is given by the state equation

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned} \tag{13}$$

where

$$\mathbf{A} = \begin{bmatrix} -0\cdot0605 & -32\cdot37 & 0 & 32\cdot2 \\ -0\cdot00014 & -1\cdot475 & 1 & 0 \\ -0\cdot0111 & -34\cdot72 & -2\cdot793 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ -0\cdot1064 \\ -33\cdot8 \\ 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

x_1 is the forward velocity (ft s^{-1}), x_2 is the angle of attack (rad), x_3 is the pitching velocity (rad s^{-1}), x_4 is the pitch angle (rad) and $\mathbf{u}(t)$ is the elevator deflection (deg).

Notes. This system was also considered in Reference 5. There, however, the sign of the (1, 2) element of matrix **A** was incorrect. Also it was assumed that all of the state variables could be measured. In this example we will assume that only the forward velocity x_1 and the pitch rate x_3 can be measured.

As noted in Reference 5, the (3, 2) entry of $\mathbf{A}$ represents the change in pitching moment with varying angle of attack. This parameter is commonly referred to as the 'longitudinal static stability derivative, M_α'. Furthermore, M_α is very sensitive to disturbances and thus there is a large degree of uncertainty in M_α. Thus we will assume M_α is an uncertain parameter whose values lie in the range $-34\cdot72 \pm 50$ (this range for M_α was also considered in Reference 5).

We now consider the corresponding uncertain system of the form (Σ_u), where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are as given in (13), $p = q = 1$,

$$\mathbf{D} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{E} = [0 \quad 1 \quad 0 \quad 0]$$

and $\bar{r} = 50$. In this case $\mathbf{F}(t)$ represents the time-varying uncertainty associated with M_α.

It is straightforward to verify that this uncertain system satisfies Assumptions A1–A6. Hence we can now apply Theorem 4.1 in order to generate the required stabilizing feedback compensator. Indeed, we first generate a state feedback matrix $\mathbf{K} \in \mathbb{R}^{m \times n}$ using the quadratic bound method described in Reference 5. As detailed in Reference 5, the matrix $\mathbf{K}$ is generated by solving the algebraic Riccati equation

$$\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} - \frac{1}{\varepsilon}\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P} + \bar{r}\mathbf{P}\mathbf{D}\mathbf{D}^{\mathrm{T}}\mathbf{P} + \bar{r}\mathbf{E}^{\mathrm{T}}\mathbf{E} + \varepsilon\mathbf{Q} = \mathbf{0} \tag{14}$$

where $\mathbf{R}$ and $\mathbf{Q}$ are suitable positive-definite weighting matrices and ε is a positive parameter. Choosing $\mathbf{Q} = \mathbf{I}$, $\mathbf{R} = \mathbf{1}$, $\bar{r} = 100$ and $\varepsilon = 0\cdot12$ we obtain the following positive definite solution to (14):

$$\mathbf{P} = \begin{bmatrix} -0\cdot44291 & -5\cdot90790 & 0\cdot03432 & 6\cdot66368 \\ -5\cdot90790 & 150\cdot37116 & -0\cdot72242 & -161\cdot56230 \\ 0\cdot03432 & -0\cdot72242 & 0\cdot02275 & 0\cdot98905 \\ 6\cdot66368 & -161\cdot56230 & 0\cdot98905 & 183\cdot44784 \end{bmatrix}$$

This leads to the desired stabilizing state feedback gain matrix

$$\mathbf{K} = -\frac{1}{\varepsilon}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P} = [1\cdot06312 \quad -16\cdot83668 \quad 1\cdot38415 \quad 32\cdot47965]$$

In order to construct the observer portion of the compensator, we now solve Riccati equation (5). Indeed, choosing $\mathbf{Q} = \mathbf{I}$, $\delta = 0\cdot01$ and $q = 16$ we obtain the following positive-definite solution to (5):

$$\mathbf{\Sigma} = \begin{bmatrix} 9\cdot60245 & 0\cdot04583 & -0\cdot01011 & 1\cdot48036 \\ 0\cdot04583 & 0\cdot73790 & 0\cdot81620 & 0\cdot75733 \\ -0\cdot01011 & 0\cdot81620 & 157\cdot05813 & 0\cdot81953 \\ 1\cdot48036 & 0\cdot75733 & 0\cdot81953 & 1\cdot20521 \end{bmatrix}$$

Using (6), this leads to the following observer gains:

$$\mathbf{L} = \mathbf{\Sigma}\mathbf{C}^{\mathrm{T}} = \begin{bmatrix} 9\cdot60245 & -0\cdot01011 \\ 0\cdot04583 & 0\cdot81620 \\ -0\cdot01011 & 157\cdot05813 \\ 1\cdot48036 & 0\cdot81953 \end{bmatrix}$$

Note. The parameters $\tilde{r}$ and δ were chosen so that $\delta + 1/\tilde{r} \leqslant 1/\bar{r} = 0\cdot02$. This ensures that for

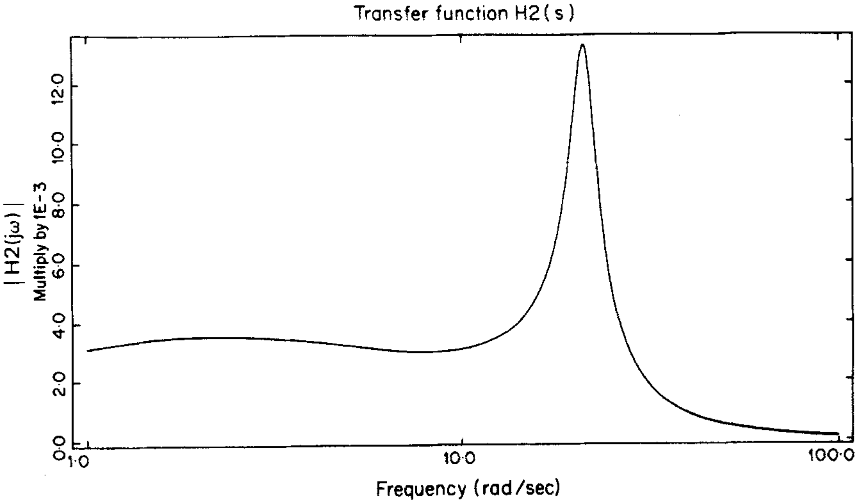

Figure 3. A4D with stabilizing control; magnitude of closed-loop transfer function $\mathbf{H}_2(j\omega)$

sufficiently large q, the closed-loop transfer function $\mathbf{H}_2(s)$ (defined in (2)) satisfies

$$|\mathbf{H}_2(j\omega)| \leqslant 0.02$$

for all ω; see Theorem 3.1. Indeed, for our example we can see from the plot shown in Figure 3 that this indeed holds for $q = 16$. Thus, using Theorem 4.2, it follows that the closed-loop uncertain system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= (\mathbf{A} + f(t)\mathbf{D}\mathbf{E})\mathbf{x}(t) + \mathbf{B}\mathbf{K}\hat{\mathbf{x}}(t) \\ \dot{\hat{\mathbf{x}}}(t) &= (\mathbf{A} + \mathbf{B}\mathbf{K})\hat{\mathbf{x}}(t) + \mathbf{L}\mathbf{C}(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) \\ |f(t)| &\leqslant 50 \end{aligned} \tag{15}$$

is quadratically stable.

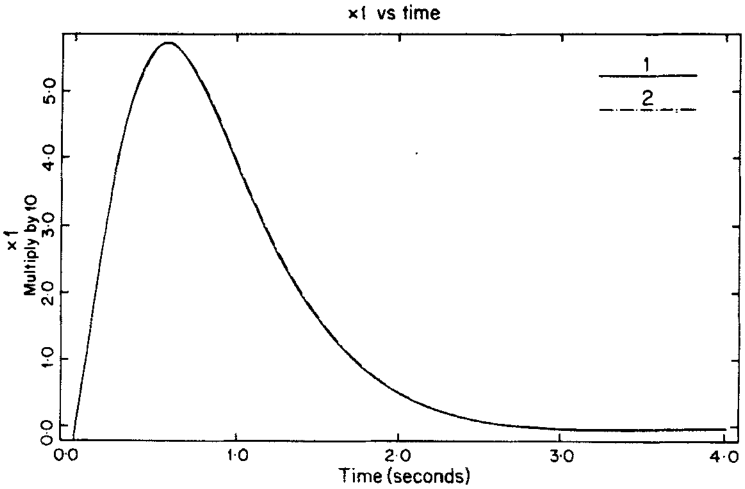

Figure 4. Simulation of A4D with stabilizing control; $\mathbf{x}(0) = [-1 \quad -1 \quad 2 \quad 4]^{\mathrm{T}}$, $\hat{\mathbf{x}}(0) = \mathbf{0}$, state x_1 versus time

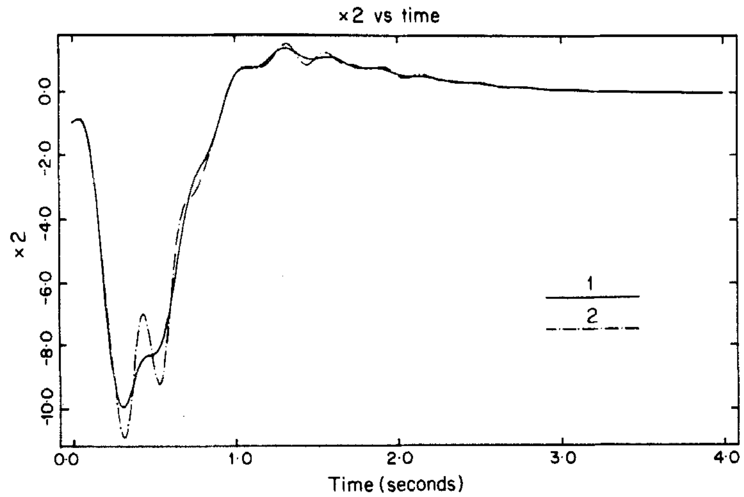

Figure 5. Simulation of A4D with stabilizing control; $\mathbf{x}(0) = [-1 \quad -1 \quad 2 \quad 4]^T$, $\hat{\mathbf{x}}(0) = \mathbf{0}$, state x_2 versus time

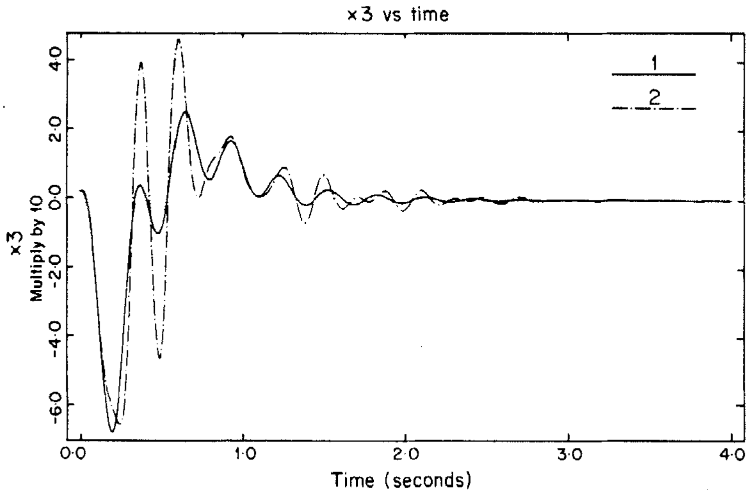

Figure 6. Simulation of A4D with stabilizing control; $\mathbf{x}(0) = [-1 \quad -1 \quad 2 \quad 4]^T$, $\hat{\mathbf{x}}(0) = \mathbf{0}$, state x_3 versus time

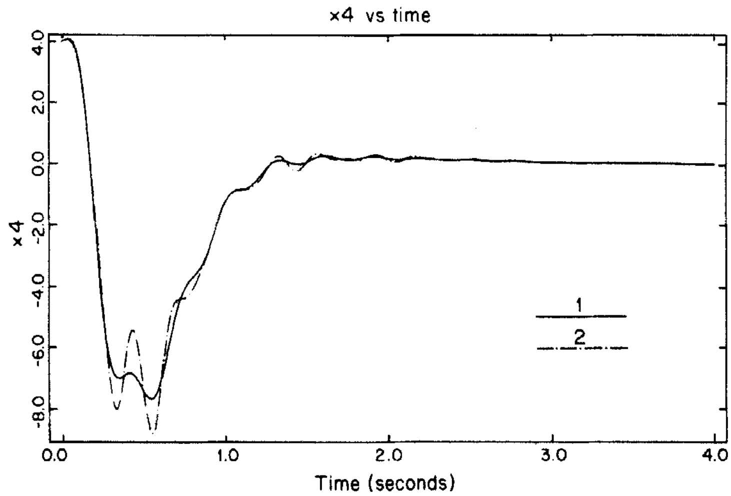

Figure 7. Simulation of A4D with stabilizing control; $\mathbf{x}(0) = [-1 \quad -1 \quad 2 \quad 4]^T$, $\hat{\mathbf{x}}(0) = \mathbf{0}$, state x_4 versus time

Simulations

Simulations are now presented for the closed-loop uncertain system (15). In these simulations the uncertain parameter $f(t)$ is taken to be $f(t) = 0$ (labelled '1') and $f(t) = 50\sin(2\pi 5t)$ (labelled '2') and the initial conditions $\mathbf{x}(0)$, $\hat{\mathbf{x}}(0)$ are taken to be $\mathbf{x}(0) = [-1 \quad -1 \quad 2 \quad 4]^{\mathrm{T}}$ and $\hat{\mathbf{x}}(0) = [0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}$. The results of these simulations are shown in Figures 4–7. From these simulations it can be seen that the closed-loop system is indeed stable.

ACKNOWLEDGEMENTS

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APPENDIX: PROOF OF LEMMAS 3.1–3.5

In order to prove Lemma 3.1, we will use some results from References 3 and 22. However, before using these results, we must introduce some notation.

Notation (see References 1 and 3). The maximal **A**, **B**-invariant subspace contained in ker **E** (denoted V^*) is defined to be the supremal subspace in the class

$$\{\mathbf{V} \subset \ker \mathbf{E}: \text{there exists an } \mathbf{F} \text{ such that } (\mathbf{A} + \mathbf{BF})V \subset V\}$$

The maximal stabilizing **A**, **B**-invariant subspace contained in ker **E** (denoted V^+) is defined to be the supremal subspace in the class

$$\{V \subset \ker \mathbf{E}: \text{there exists an } \mathbf{F} \text{ such that} (\mathbf{A} + \mathbf{BF})V \subset V \text{ and the operator} (\mathbf{A} + \mathbf{BF})|_V \text{ is stable}\}$$

In this definition the operator $\mathbf{A} + \mathbf{BF}|_V: V \rightarrow V$ is defined to be the restriction of the operator $\mathbf{A} + \mathbf{BF}$ to the subspace V.

The supremal $\mathbf{L}_p$-almost controllability subspace in ker **E** (denoted R_b^*) is defined to be the infimal subspace in the class

$$\{V \supset \operatorname{im}\mathbf{B}: \text{there exists an } \mathbf{F} \text{ such that} (\mathbf{A} + \mathbf{FE})V \subset V\}$$

The supremal stabilizing $\mathbf{L}_p$-almost invariant subspace in ker **E** (denoted V_b^+) is defined to be

$$V_b^+ = V + R_b^*$$

Lemma A.1. Suppose that the system (Σ) satisfies Assumptions A1′–A6′ and let V^* be defined as above. Then given any matrix **F** such that $(\mathbf{A} + \mathbf{BF})V^* \subset V^*$, the operator $(\mathbf{A} + \mathbf{BF})|_{V^*}$ is stable.

Proof (follows Theorem 6.8 of Reference 22). Let the matrix **F** be given such that $(\mathbf{A} + \mathbf{BF})V^* \subset V^*$. We observe that given any $s \in \mathbb{C}$

$$\begin{bmatrix} s\mathbf{I} - \mathbf{A} - \mathbf{BF} & \mathbf{B} \\ \mathbf{E} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} s\mathbf{I} - \mathbf{A} & \mathbf{B} \\ \mathbf{E} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{F} & \mathbf{I} \end{bmatrix}$$

Hence, using Assumption A6′, it follows that the matrix

$$\begin{bmatrix} s\mathbf{I} - \mathbf{A} - \mathbf{BF} & \mathbf{B} \\ \mathbf{E} & \mathbf{0} \end{bmatrix}$$

is of full rank for all s: $\mathrm{Re}(s) \geqslant 0$. Thus the matrix

$$\begin{bmatrix} s\mathbf{I} - \mathbf{A} - \mathbf{BF} \\ \mathbf{E} \end{bmatrix}$$

must be of full rank for all s: $\mathrm{Re}(s) \geqslant 0$. Therefore the pair $(\mathbf{E}, \mathbf{A} + \mathbf{BF})$ is detectable; see e.g. Reference 23.

We now recall that $(\mathbf{A} + \mathbf{BF})V^* \subset V^*$ and $V^* \subset \ker \mathbf{E}$. Thus V^* is contained in the $(\mathbf{E}, \mathbf{A} + \mathbf{BF})$-unobservable subspace. Hence, using the fact that the pair $(\mathbf{E}, \mathbf{A} + \mathbf{BF})$ is detectable, we conclude that the operator $\mathbf{A} + \mathbf{BF}\,|_{V^*}$ is stable. □

Lemma A.2. Suppose that the system (Σ) satisfies Assumptions A1′–A6′. Then the two subspaces V^+ and V^* defined above are equal.

Proof. The fact tht $V^+ \subset V^*$ follows from the definitions. To show that $V^+ \supset V^*$, we will show that

$V^* \in \{ V \subset \ker \mathbf{E}$: there exists an $\mathbf{F}$ such that $(\mathbf{A} + \mathbf{BF})V \subset V$ and $(\mathbf{A} + \mathbf{BF})\,|_V$ is stable$\}$

Indeed, let $\mathbf{F}$ be given such that $(\mathbf{A} + \mathbf{BF})V^* \subset V^*$ and $V^* \subset \ker \mathbf{E}$. Using Lemma A.1, it follows that $\mathbf{A} + \mathbf{BF}\,|_{V^*}$ is stable. Hence

$V^* \in \{ V \subset \ker \mathbf{E}$: there exists an $\mathbf{F}$ such that $(\mathbf{A} + \mathbf{BF})V \subset V$ and $(\mathbf{A} + \mathbf{BF})\,|_V$ is stable$\}$

as required. The result now follows. □

Lemma A.3. Suppose that the system (Σ) satisfies Assumptions A1′–A6′ and let V_b^* be defined as above. Then $V_b^+ = \mathbb{R}^n$.

Proof. In order to prove this lemma, we use the right invertibility of the transfer function $\mathbf{E}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$. Indeed, using this fact and Theorem 3.24 of Reference 22, it follows that $V^* + R_b^* = \mathbb{R}^n$. Hence, using Lemma A.2, we conclude that

$$V_b^+ = V^+ + R_b^* = V^* + R_b^* = \mathbb{R}^n \quad \square$$

Lemma A.4 (see Reference 3 for proof). Suppose that the system (Σ) is such that the pair $(\mathbf{A}, \mathbf{B})$ is stabilizable and $\mathrm{im}\mathbf{D} \subset V_b^+$. Then the following condition holds: given any $\varepsilon > 0$, there exists a matrix $\mathbf{F}$ such that

(i) $\bar{\mathbf{A}} = \mathbf{A} + \mathbf{BF}$ is a stability matrix
(ii) $\mathbf{G}(s) = \mathbf{E}(s\mathbf{I} - \bar{\mathbf{A}})^{-1}\mathbf{D}$ satisfies

$$\| \mathbf{G}(j\omega) \| \leqslant \varepsilon$$

for all $\omega \in \mathbb{R}$.

Proof of Lemma 3.1. Suppose that the system (Σ) satisfies Assumptions A1–A6. It follows that the dual system (Σ_D) satisfies Assumptions A1′–A6′. We now apply Lemmas A.3 and A.4 to the dual system (Σ_D). Indeed, using Lemma A.3, it follows that the set V_b^+ (defined for (Σ_D)) is equal to $\mathbb{R}^n$. Furthermore, using Lemma A.4, it now follows that the following condition holds: given any $\varepsilon > 0$, there exists a matrix $\mathbf{F}$ such that

(i) $\bar{\mathbf{A}} = \mathbf{A}^{\mathrm{T}} + \mathbf{C}^{\mathrm{T}}\mathbf{F}$ is a stability matrix

(ii) $\mathbf{G}(s) = \mathbf{D}^{\mathrm{T}}(s\mathbf{I} - \bar{\mathbf{A}}^{\mathrm{T}})^{-1}\mathbf{E}^{\mathrm{T}}$ satisfies

$$\|\mathbf{G}(\mathrm{j}\omega)\| \leqslant \varepsilon$$

for all $\omega \in \mathbb{R}$.

That is, given any $\varepsilon > 0$, the dual system (Σ_{D}) is stabilizable via state feedback with disturbance attenuation ε.

We now apply Theorem 3.1 of Reference 2 to the system (Σ_{D}). Indeed, let $\delta > 0$ be given. It follows from the above analysis that the system (Σ_{D}) is stabilizable via state feedback with disturbance attenuation $\delta/2$. Hence, using Theorem 3.1 of Reference 2, we conclude that the following condition holds.

There exists a constant $\varepsilon^* > 0$ such that the Riccati equation

$$\mathbf{AP} + \mathbf{PA}^{\mathrm{T}} - \frac{1}{\varepsilon}\mathbf{PC}^{\mathrm{T}}\mathbf{CP} + \frac{1}{\delta}\mathbf{PE}^{\mathrm{T}}\mathbf{EP} + \frac{1}{\delta}\mathbf{DD}^{\mathrm{T}} + \varepsilon\mathbf{Q} = 0 \tag{16}$$

has a positive-definite solution $\mathbf{P}$ for all $\varepsilon \in (0, \varepsilon^*]$. Letting $q^* = (\varepsilon^*)^{-1/2}$ and $\boldsymbol{\Sigma} = \mathbf{P}/\varepsilon$, it follows that Riccati equation (5) has a positive-definite solution $\boldsymbol{\Sigma}$ for all $q \geqslant q^*$. Furthermore, if we let $\mathbf{L} = \mathbf{PC}^{\mathrm{T}}/\varepsilon = \boldsymbol{\Sigma}\mathbf{C}^{\mathrm{T}}$ and $\hat{\mathbf{A}} = \mathbf{A} - \mathbf{LC}$, it follows from (16) that

$$\hat{\mathbf{A}}\mathbf{P} + \mathbf{P}\hat{\mathbf{A}}^{\mathrm{T}} + \frac{1}{\delta}\mathbf{PE}^{\mathrm{T}}\mathbf{EP} + \frac{1}{\delta}\mathbf{DD}^{\mathrm{T}} + \tilde{\mathbf{Q}} = \mathbf{0}$$

where $\tilde{\mathbf{Q}} = \mathbf{PC}^{\mathrm{T}}\mathbf{CP}/\varepsilon + \varepsilon\mathbf{Q}$. Hence, using Lemma 1 of Reference 24, it follows that the transfer function $\mathbf{H}(s) = \mathbf{D}^{\mathrm{T}}(sI - \hat{\mathbf{A}}^{\mathrm{T}})^{-1}\mathbf{E}^{\mathrm{T}}$ satisfies

$$\|\mathbf{H}(\mathrm{j}\omega)\| \leqslant \delta$$

for all $\omega \in \mathbb{R}$. This completes the proof of the lemma. □

Proof of Lemma 3.2. Let $\mathbf{d}(s)$ be the Laplace transform of the disturbance input signal applied to the closed-loop system and let $\mathbf{u}(s)$ be the Laplace transform of the control input signal. Furthermore, let $\mathbf{x}(s)$ be the Laplace transform of the system state. It then follows from Figure 2 that the Laplace transform $\hat{\mathbf{x}}(s)$ of the observer state satisfies the equation

$$\hat{\mathbf{x}}(s) = \boldsymbol{\phi}(s)\mathbf{Bu}(s) - \boldsymbol{\phi}(s)\mathbf{LC}\hat{\mathbf{x}}(s) + \boldsymbol{\phi}(s)\mathbf{LCx}(s)$$

Therefore

$$\boldsymbol{\phi}(s)^{-1}\hat{\mathbf{x}}(s) = \mathbf{Bu}(s) - \mathbf{LC}\hat{\mathbf{x}}(s) + \mathbf{LCx}(s)$$

and hence

$$\hat{\mathbf{x}}(s) = [\boldsymbol{\phi}(s)^{-1} + \mathbf{LC}]^{-1}[\mathbf{Bu}(s) + \mathbf{LCx}(s)]$$

Furthermore, using the matrix inversion lemma (see p. 656 of Reference 23), it follows that

$$\hat{\mathbf{x}}(s) = [\boldsymbol{\phi}(s) - \boldsymbol{\phi}(s)\mathbf{L}(\mathbf{I} + \mathbf{C}\boldsymbol{\phi}(s)\mathbf{L})^{-1}\mathbf{C}\boldsymbol{\phi}(s)]\,[\mathbf{Bu}(s) + \mathbf{LCx}(s)]$$

However,

$$\mathbf{x}(s) = \boldsymbol{\phi}(s)\mathbf{Bu}(s) + \boldsymbol{\phi}(s)\mathbf{Dd}(s)$$

so that

$$\begin{aligned}\hat{\mathbf{x}}(s) &= \boldsymbol{\phi}(s)[\mathbf{I} - \mathbf{L}(\mathbf{I} + \mathbf{C}\boldsymbol{\phi}(s)\mathbf{L})^{-1}\mathbf{C}\boldsymbol{\phi}(s)]\,[\mathbf{Bu}(s) + \mathbf{LC}\boldsymbol{\phi}(s)(\mathbf{Bu}(s) + \mathbf{Dd}(s))]\\ &= \boldsymbol{\phi}(s)[\mathbf{I} - \mathbf{L}(\mathbf{I} + \mathbf{C}\boldsymbol{\phi}(s)\mathbf{L})^{-1}\mathbf{C}\boldsymbol{\phi}(s) + (\mathbf{L} - \mathbf{L}(\mathbf{I} + \mathbf{C}\boldsymbol{\phi}(s)\mathbf{L})^{-1}\mathbf{C}\boldsymbol{\phi}(s)\mathbf{L})\mathbf{C}\boldsymbol{\phi}(s)]\,\mathbf{Bu}(s)\\ &\quad + \boldsymbol{\phi}(s)[\mathbf{L} - \mathbf{L}(\mathbf{I} + \mathbf{C}\boldsymbol{\phi}(s)\mathbf{L})^{-1}\mathbf{C}\boldsymbol{\phi}(s)\mathbf{L}]\,\mathbf{C}\boldsymbol{\phi}(s)\mathbf{Dd}(s)\end{aligned} \tag{17}$$

using $\mathbf{u}(s)=\mathbf{K}\hat{\mathbf{x}}(s)$ and substituting the easily verified identity

$$\mathbf{L}-\mathbf{L}(\mathbf{I}+\mathbf{C}\phi(s)\mathbf{L})^{-1}\mathbf{C}\phi(s)\mathbf{L}=\mathbf{L}(\mathbf{I}+\mathbf{C}\phi(s)\mathbf{L})^{-1}$$

into equation (17) gives

$$\begin{aligned}\hat{\mathbf{x}}(s)&=\phi(s)\mathbf{B}\mathbf{u}(s)+\phi(s)\mathbf{L}(\mathbf{I}+\mathbf{C}\phi(s)\mathbf{L})^{-1}\mathbf{C}\phi(s)\mathbf{D}\mathbf{d}(s)\\&=\phi(s)\mathbf{B}\mathbf{K}\hat{\mathbf{x}}(s)+\phi(s)\mathbf{L}(\mathbf{I}+\mathbf{C}\phi(s)\mathbf{L})^{-1}\mathbf{C}\phi(s)\mathbf{D}\mathbf{d}(s)\end{aligned}$$

from whence

$$\hat{\mathbf{x}}(s)=[\phi(s)^{-1}-\mathbf{B}\mathbf{K}]^{-1}\mathbf{L}(\mathbf{I}+\mathbf{C}\phi(s)\mathbf{D}\mathbf{d}(s)$$

However, we can see from Figure 2 that

$$\begin{aligned}\mathbf{x}(s)&=\phi(s)\mathbf{D}\mathbf{d}(s)+\phi(s)\mathbf{B}\mathbf{K}\hat{\mathbf{x}}(s)\\&=\phi(s)\mathbf{D}\mathbf{d}(s)+\phi(s)\mathbf{B}\mathbf{K}[\phi(s)^{-1}-\mathbf{B}\mathbf{K}]^{-1}\mathbf{L}(\mathbf{I}+\mathbf{C}\phi(s)\mathbf{L})^{-1}\mathbf{C}\phi(s)\mathbf{D}\mathbf{d}(s)\end{aligned}\tag{18}$$

Furthermore, it is straightforward to verify that

$$\mathbf{B}\mathbf{K}[\phi(s)^{-1}-\mathbf{B}\mathbf{K}]^{-1}=\phi(s)^{-1}[\phi(s)^{-1}-\mathbf{B}\mathbf{K}]^{-1}-\mathbf{I}$$

Hence it follows from (18) that

$$\begin{aligned}\mathbf{x}(s)=[\phi(s)^{-1}-\mathbf{B}\mathbf{K}]^{-1}\mathbf{L}(\mathbf{I}+\mathbf{C}\phi(s)\mathbf{L})^{-1}\mathbf{C}\phi(s)\mathbf{D}\mathbf{d}(s)&\\+[\mathbf{I}-\phi(s)\mathbf{L}(\mathbf{I}+\mathbf{C}\phi(s)\mathbf{L})^{-1}\mathbf{C}]\phi(s)\mathbf{D}\mathbf{d}(s)&\end{aligned}\tag{19}$$

We again use the matrix inversion lemma to obtain the result

$$\mathbf{I}-\phi(s)\mathbf{L}(\mathbf{I}+\mathbf{C}\phi(s)\mathbf{L})^{-1}\mathbf{C}=[\mathbf{I}+\phi(s)\mathbf{L}\mathbf{C}]^{-1}$$

Substituting this into (19) we obtain

$$\begin{aligned}\mathbf{x}(s)&=[\phi(s)^{-1}-\mathbf{B}\mathbf{K}]^{-1}\mathbf{L}(\mathbf{I}+\mathbf{C}\phi(s)\mathbf{L})^{-1}\mathbf{C}\phi(s)\mathbf{D}\mathbf{d}(s)+[\mathbf{I}+\phi(s)LC]^{-1}\phi(s)\mathbf{D}\mathbf{d}(s)\\&=[\phi(s)^{-1}-\mathbf{B}\mathbf{K}]^{-1}\mathbf{L}(\mathbf{I}+\mathbf{C}\phi(s)\mathbf{L})^{-1}\mathbf{C}\phi(s)\mathbf{D}\mathbf{d}(s)+[\phi(s)^{-1}+\mathbf{L}\mathbf{C}]^{-1}\mathbf{D}\mathbf{d}(s)\end{aligned}$$

The Laplace transform $\mathbf{w}(s)$ of the controlled output is thus given by

$$\begin{aligned}\mathbf{w}(s)&=\mathbf{E}\mathbf{x}(s)\\&=\mathbf{E}[\phi(s)^{-1}-\mathbf{B}\mathbf{K}]^{-1}\mathbf{L}(\mathbf{I}+\mathbf{C}\phi(s)\mathbf{L})^{-1}\mathbf{C}\phi(s)\mathbf{D}\mathbf{d}(s)+\mathbf{E}[\phi(s)^{-1}+\mathbf{L}\mathbf{C}]^{-1}\mathbf{D}\mathbf{d}(s)\end{aligned}$$

However, $\mathbf{w}(s)$ is also given by $\mathbf{w}(s)=\mathbf{H}_2(s)\mathbf{d}(s)$, and since $\mathbf{d}(s)$ was arbitrary, we conclude that

$$\mathbf{H}_2(s)=\mathbf{E}[\phi(s)^{-1}-\mathbf{B}\mathbf{K}]^{-1}\mathbf{L}(\mathbf{I}+\mathbf{C}\phi(s)\mathbf{L})^{-1}\mathbf{C}\phi(s)\mathbf{D}+\mathbf{E}[\phi(s)^{-1}+\mathbf{L}\mathbf{C}]^{-1}\mathbf{D}\quad\square$$

Proof of Lemma 3.3. Using Assumptions A1–A6, it follows from the main result of Reference 18 that the solution $\mathbf{\Sigma}$ to (5) satisfies $\mathbf{\Sigma}/q^2\to\mathbf{0}$ as $q\to\infty$. However, using (5) and (6) we have

$$\mathbf{L}\mathbf{L}^{\mathrm{T}}/q^2=\mathbf{A}(\mathbf{\Sigma}/q^2)+(\mathbf{\Sigma}/q^2)\mathbf{A}^{\mathrm{T}}+\mathbf{Q}/q^2+\mathbf{D}\mathbf{D}^{\mathrm{T}}$$

and hence

$$\mathbf{L}\mathbf{L}^{\mathrm{T}}/q^2\to\mathbf{D}\mathbf{D}^{\mathrm{T}}$$

as $q\to\infty$. Consequently, $\bar{\mathbf{L}}=\lim_{q\to\infty}\mathbf{L}/q$ exists and we can write $\bar{\mathbf{L}}\bar{\mathbf{L}}^{\mathrm{T}}=\mathbf{D}\mathbf{D}^{\mathrm{T}}$. Using Assumptions A4 and A5 and the lemma given in Reference 25, it now follows that there exists a matrix $\mathbf{N}$ such that $\mathbf{N}\mathbf{N}^{\mathrm{T}}=\mathbf{I}$ and $\bar{\mathbf{L}}=\mathbf{D}\mathbf{N}$. Thus $\mathbf{L}/q\to\mathbf{D}\mathbf{N}$ as $q\to\infty$. $\square$

Proof of Lemma 3.4. It follows from equations (5) and (6) that for any $s \in \mathbb{C}$

$$(s\mathbf{I}-\mathbf{A})\boldsymbol{\Sigma}+\boldsymbol{\Sigma}(-s\mathbf{I}-\mathbf{A}^{\mathrm{T}})+\mathbf{L}\mathbf{L}^{\mathrm{T}}=\mathbf{Q}+q^2\mathbf{D}\mathbf{D}^{\mathrm{T}}+\frac{1}{q^2}\boldsymbol{\Sigma}\mathbf{E}^{\mathrm{T}}\mathbf{E}\boldsymbol{\Sigma}$$

That is

$$\boldsymbol{\phi}(s)^{-1}\boldsymbol{\Sigma}+\boldsymbol{\Sigma}[\boldsymbol{\phi}(-s)^{-1}]^{\mathrm{T}}+\mathbf{L}\mathbf{L}^{\mathrm{T}}=\mathbf{Q}+q^2\mathbf{D}\mathbf{D}^{\mathrm{T}}+\frac{1}{q^2}\boldsymbol{\Sigma}\mathbf{E}^{\mathrm{T}}\mathbf{E}\boldsymbol{\Sigma}$$

premultiply this equation by $\mathbf{C}\boldsymbol{\phi}(s)$ and postmultiply by $\boldsymbol{\phi}(-s)^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}$. It follows that

$$\mathbf{C}\boldsymbol{\Sigma}\boldsymbol{\phi}(-s)^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}+\mathbf{C}\boldsymbol{\phi}(s)\boldsymbol{\Sigma}\mathbf{C}^{\mathrm{T}}+\mathbf{C}\boldsymbol{\phi}(s)\mathbf{L}\mathbf{L}^{\mathrm{T}}\boldsymbol{\phi}(-s)^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}+\mathbf{I}$$
$$=\mathbf{C}\boldsymbol{\phi}(s)\left[\mathbf{Q}+q^2\mathbf{D}\mathbf{D}^{\mathrm{T}}+\frac{\boldsymbol{\Sigma}\mathbf{E}^{\mathrm{T}}\mathbf{E}\boldsymbol{\Sigma}}{q^2}\right]\boldsymbol{\phi}(-s)^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}+\mathbf{I} \quad (20)$$

However, $\mathbf{L}=\boldsymbol{\Sigma}\mathbf{C}^{\mathrm{T}}$ and hence

$$[\mathbf{I}+\mathbf{C}\boldsymbol{\phi}(s)\mathbf{L}]\,[\mathbf{I}+\mathbf{C}\boldsymbol{\phi}(-s)\mathbf{L}]^{\mathrm{T}}$$
$$=\mathbf{I}+\mathbf{C}\boldsymbol{\Sigma}\boldsymbol{\phi}(-s)^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}+\mathbf{C}\boldsymbol{\phi}(s)\boldsymbol{\Sigma}\mathbf{C}^{\mathrm{T}}+\mathbf{C}\boldsymbol{\phi}(s)\mathbf{L}\mathbf{L}^{\mathrm{T}}\boldsymbol{\phi}(-s)^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}$$

Substituting this result into equation (20) we obtain the result

$$[\mathbf{I}+\mathbf{C}\boldsymbol{\phi}(s)\mathbf{L}]\,[\mathbf{I}+\mathbf{C}\boldsymbol{\phi}(-s)\mathbf{L}]^{\mathrm{T}}=\mathbf{I}+\mathbf{C}\boldsymbol{\phi}(s)\left[\mathbf{Q}+q^2\mathbf{D}\mathbf{D}^{\mathrm{T}}+\frac{1}{q^2}\boldsymbol{\Sigma}\mathbf{E}^{\mathrm{T}}\mathbf{E}\boldsymbol{\Sigma}\right]\boldsymbol{\phi}(-s)^{\mathrm{T}}\mathbf{C}^{\mathrm{T}} \quad (21)$$

for all $s \in \mathbb{C}$.

We now use Assumption A6 to conclude that there exists a transfer function matrix $\mathbf{Z}(s)$ such that

$$\mathbf{Z}(\mathrm{j}\omega)\mathbf{C}\boldsymbol{\phi}(\mathrm{j}\omega)\mathbf{D}=\mathbf{I} \quad (22)$$

for all $\omega \in \mathbb{R}$. Applying this result to equation (20), it follows that

$$\mathbf{Z}(\mathrm{j}\omega)\,\frac{[\mathbf{I}+\mathbf{C}\boldsymbol{\phi}(\mathrm{j}\omega)\mathbf{L}]}{q}\,\frac{[\mathbf{I}+\mathbf{C}\boldsymbol{\phi}(-\mathrm{j}\omega)\mathbf{L}]^{\mathrm{T}}}{q}\,\mathbf{Z}(-\mathrm{j}\omega)^{\mathrm{T}}$$
$$=\frac{\mathbf{Z}(\mathrm{j}\omega)\mathbf{Z}(-\mathrm{j}\omega)^{\mathrm{T}}}{q^2}+\mathbf{I}+\mathbf{Z}(\mathrm{j}\omega)\mathbf{C}\boldsymbol{\phi}(\mathrm{j}\omega)\left[\frac{Q}{q^2}+\frac{\boldsymbol{\Sigma}\mathbf{E}^{\mathrm{T}}\mathbf{E}\boldsymbol{\Sigma}}{q^4}\right]\boldsymbol{\phi}(-\mathrm{j}\omega)^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{Z}(-\mathrm{j}\omega)^{\mathrm{T}}$$
$$\geqslant \mathbf{I}>0$$

for all $\omega \in \mathbb{R}$. Therefore

$$\left\{\mathbf{Z}(\mathrm{j}\omega)\left[\frac{\mathbf{I}+\mathbf{C}\boldsymbol{\phi}(\mathrm{j}\omega)\mathbf{L}}{q}\right]\left\{\frac{\mathbf{I}+\mathbf{C}\boldsymbol{\phi}(-\mathrm{j}\omega)\mathbf{L}}{q}\right]^{\mathrm{T}}\mathbf{Z}(-\mathrm{j}\omega)^{\mathrm{T}}\right\}^{-1}<\mathbf{I}$$

for all $\omega \in \mathbb{R}$. However, it follows from (22) that

$$\left\{\mathbf{Z}(\mathrm{j}\omega)\left[\frac{\mathbf{I}+\mathbf{C}\boldsymbol{\phi}(\mathrm{j}\omega)\mathbf{L}}{q}\right]\left[\frac{\mathbf{I}+\mathbf{C}\boldsymbol{\phi}(-\mathrm{j}\omega)\mathbf{L}}{q}\right]^{\mathrm{T}}\mathbf{Z}(-\mathrm{j}\omega)\right\}^{-1}$$
$$=[\mathbf{C}\boldsymbol{\phi}(-\mathrm{j}\omega)\mathbf{D}]^{\mathrm{T}}\left[\frac{\mathbf{I}+\mathbf{C}\boldsymbol{\phi}(-\mathrm{j}\omega)\mathbf{L}}{q}\right]^{\mathrm{T}^{-1}}\left[\frac{\mathbf{I}+\mathbf{C}\boldsymbol{\phi}(\mathrm{j}\omega)\mathbf{L}}{q}\right]^{-1}\mathbf{C}\boldsymbol{\phi}(\mathrm{j}\omega)\mathbf{D}$$

and thus

$$[\mathbf{C}\boldsymbol{\phi}(-\mathrm{j}\omega)\mathbf{D}]^{\mathrm{T}}\left[\frac{\mathbf{I}+\mathbf{C}\boldsymbol{\phi}(-\mathrm{j}\omega)\mathbf{L}}{q}\right]^{\mathrm{T}^{-1}}\left[\frac{\mathbf{I}+\mathbf{C}\boldsymbol{\phi})\mathrm{j}\omega)\mathbf{L}}{q}\right]^{-1}\mathbf{C}\boldsymbol{\phi}(\mathrm{j}\omega)\mathbf{D}\leqslant\mathbf{I}$$

for all $\omega \in \mathbb{R}$. That is

$$\|[\mathbf{I}/q + \mathbf{C}\boldsymbol{\phi}(j\omega\mathbf{L}/q)^{-1}\mathbf{C}\boldsymbol{\phi}(j\omega)\mathbf{D}\| \leqslant 1$$

for all $\omega \in \mathbb{R}$. □

Proof of Lemma 3.5. It follows from Lemma 3.3 that we can write $\mathbf{L} = q\mathbf{DN} + q\,\boldsymbol{\Delta}(q)$ where $\boldsymbol{\Delta}(q) \to \mathbf{0}$ as $q \to \infty$. Now, using (9), the triangle inequality, the Schwarz inequality and the fact that $\mathbf{NN}^{\mathrm{T}} = \mathbf{I}$, it follows that

$$\begin{aligned}&\|\mathbf{E}(j\omega\mathbf{I} - \mathbf{A} - \mathbf{BK})^{-1}(\mathbf{DN} + \boldsymbol{\Delta}(q))\|\\ &\leqslant \|\mathbf{E}(j\omega\mathbf{I} - \mathbf{A} - \mathbf{BK})^{-1}\mathbf{D}\| \cdot \|\mathbf{N}\| + \|\mathbf{E}(j\omega\mathbf{I} - \mathbf{A} - \mathbf{BK})^{-1}\| \cdot \|\boldsymbol{\Delta}(q)\|\\ &\leqslant \sigma + \|\mathbf{E}(j\omega\mathbf{I} - \mathbf{A} - \mathbf{BK})^{-1}\| \cdot \|\boldsymbol{\Delta}(q)\|\end{aligned}$$

for all $\omega \in \mathbb{R}$. Furthermore, since $\mathbf{A} + \mathbf{BK}$ is a stability matrix, the transfer function $\mathbf{E}(s\mathbf{I} - \mathbf{A} - \mathbf{BK})^{-1}$ is a strictly proper transfer function with no poles on the $j\omega$-axis. Hence there exists a constant $\mu > 0$ such that

$$\|\mathbf{E}(j\omega\mathbf{I} - \mathbf{A} - \mathbf{BK})^{-1}\| \leqslant \mu$$

for all $\omega \in \mathbb{R}$. It now follows that

$$\|\mathbf{E}(j\omega\mathbf{I} - \mathbf{A} - \mathbf{BK})^{-1}(\mathbf{DN} + \boldsymbol{\Delta}(q))\| \leqslant \sigma + \mu\|\boldsymbol{\Delta}(q)\| \tag{23}$$

for all $\omega \in \mathbb{R}$.

We now let $\varepsilon > 0$ be given and choose $\tilde{q} \geqslant q^*$ such that $q \geqslant \tilde{q}$ implies

$$\|\boldsymbol{\Delta}(q)\| \leqslant \varepsilon/2\mu$$

Therefore, if $q \geqslant \tilde{q}$, it follows from (23) that

$$\|\mathbf{E}(j\omega\mathbf{I} - \mathbf{A} - \mathbf{BK})^{-1}\mathbf{L}/q\| \leqslant \sigma + \varepsilon/2$$

for all $\omega \in \mathbb{R}$. □

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