Mathematical Statistics Project

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Part I

MLE for Weibull Distribution

Given a sample $x_1, x_2, \ldots, x_n$ from a Weibull distribution with probability density function.
$X_1, \ldots, X_n \sim W(\alpha, \beta)$

$$f_{\alpha,\beta}(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}}, \quad \alpha, \beta, x > 0$$

a) Define $\mathbf{L}(\alpha, \beta) := \mathbf{f}\alpha, \beta(\mathbf{x})$

$$L(\alpha,\beta) := f_{\alpha,\beta}(x) = \prod_{i=1}^{n} f_{\alpha,\beta}(x_i) = \prod_{i=1}^{n} \frac{\alpha}{\beta}\left(\frac{x_i}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}} = \frac{\alpha^n}{\beta^{n\alpha}} \prod_{i=1}^{n} x_i^{\alpha-1} e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha}}$$

$$\ln L(\alpha,\beta) = \ln\left(\frac{\alpha^n}{\beta^{n\alpha}}\right) + \ln\left(\prod_{i=1}^{n} x_i^{\alpha-1}\right) + \ln\left(e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha}}\right)$$

$$\ln L(\alpha,\beta) = n\ln\alpha - n\alpha\ln\beta + \sum_{i=1}^{n} \ln x_i^{\alpha-1} - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha}$$

$$\ln L(\alpha,\beta) = n\ln\alpha - n\alpha\ln\beta + (\alpha-1)\sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha} \qquad (1)$$

According to equation (1), we were able to obtain the likelihood function.

b) MLE (α, β) based on $L(\alpha, \beta)$

$$\ln L(\alpha,\beta) = n\ln\alpha - n\alpha\ln\beta + (\alpha-1)\sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha}$$

$$\hat{\beta} := \frac{\partial \ln L(\alpha,\beta)}{\partial\beta} = 0 \Rightarrow -n\alpha\beta^{-1} + \alpha\beta^{-\alpha-1}\sum_{i=1}^{n}x_i^{\alpha} = 0$$

$$n\alpha\beta^{-1} = \alpha\beta^{-\alpha-1}\sum_{i=1}^{n}x_i^{\alpha} \Rightarrow n\alpha = \alpha\beta^{-\alpha}\sum_{i=1}^{n}x_i^{\alpha}$$

$$\beta^{\alpha} = \frac{\sum_{i=1}^{n}x_i^{\alpha}}{n}$$

$$\hat{\beta} = \left(\frac{\sum_{i=1}^{n}x_i^{\alpha}}{n}\right)^{\frac{1}{\alpha}} \qquad (2)$$

- $\ln L(\alpha,\beta) = n\ln(\alpha) - n\alpha\ln(\beta) + (\alpha-1)\sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha}$

$$\hat{\alpha} := \frac{\partial \ln L(\alpha,\beta)}{\partial\alpha} = 0 \Rightarrow \frac{n}{\alpha} - n\ln(\beta) + \sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha}\ln\left(\frac{x_i}{\beta}\right) = 0$$

$$\frac{n}{\alpha} - \sum_{i=1}^{n}\ln(\beta_0) + \sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha}\ln\left(\frac{x_i}{\beta}\right) = 0$$

$$\frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left(\frac{x_i}{\beta}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha}\ln\left(\frac{x_i}{\beta}\right) = 0$$

$$\Rightarrow \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left(\frac{x_i}{\beta}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha}\ln\left(\frac{x_i}{\beta}\right) = 0$$

$$\frac{n}{\alpha} = \sum_{i=1}^{n}\ln\left(\frac{x_i}{\beta}\right)\left(\left(\frac{x_i}{\beta}\right)^{\alpha} - 1\right) \Rightarrow \hat{\alpha} = \frac{n}{\sum_{i=1}^{n}\ln\left(\frac{x_i}{\beta}\right)\left(\left(\frac{x_i}{\beta}\right)^{\alpha} - 1\right)}$$

$$\hat{\alpha} = \left(\frac{\sum_{i=1}^{n}\ln\left(\frac{x_i}{\beta}\right)\left(\left(\frac{x_i}{\beta}\right)^{\alpha} - 1\right)}{n}\right)^{-1}$$

$$\hat{\alpha} = \left(\frac{\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha}\ln(x_i)}{n} - \frac{\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha}\ln(\beta)}{n} - \frac{\sum_{i=1}^{n}\ln(x_i)}{n} + \frac{\sum_{i=1}^{n}(\ln\beta)}{n}\right)^{-1}$$

$$\hat{\beta} = \left(\frac{\sum_{i=1}^{n}(x_i)^{\alpha}}{n}\right)^{\frac{1}{\alpha}} \Rightarrow \hat{\alpha} = \left(\frac{\sum_{i=1}^{n}(x_i)^{\alpha}\ln(x_i)}{\sum_{i=1}^{n}(x_i)^{\alpha}} - \frac{\sum_{i=1}^{n}\ln(x_i)}{n}\right)^{-1} \; (3)$$

c) Let β be known

$$L_\beta(\alpha) := L(\alpha, \beta_0) := f_{\alpha,\beta_0}(\boldsymbol{x}) = \prod_{i=1}^{n} f_{\alpha,\beta_0}(x_i) = \prod_{i=1}^{n} \frac{\alpha}{\beta_0}\left(\frac{x_i}{\beta_0}\right)^{\alpha-1} e^{-\left(\frac{x_i}{\beta_0}\right)^\alpha}$$

$$\frac{\alpha^n}{\beta_0^{n\alpha}} \cdot \prod_{i=1}^{n} x_i^{\alpha-1} \cdot e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\beta_0}\right)^\alpha}$$

$$\hat{\alpha}(\beta) := \frac{\partial \ln L(\alpha,\beta_0)}{\partial \alpha} = 0 \Rightarrow n\alpha^{-1} - n\ln\beta_0 + \sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\left(\frac{x_i}{\beta_0}\right)^\alpha \ln\left(\frac{x_i}{\beta_0}\right) = 0$$

$$\frac{n}{\alpha} - \sum_{i=1}^{n}\ln(\beta_0) + \sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n}\left(\frac{x_i}{\beta_0}\right)^\alpha \ln\left(\frac{x_i}{\beta_0}\right) = 0$$

$$\frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left(\frac{x_i}{\beta_0}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\beta_0}\right)^\alpha \ln\left(\frac{x_i}{\beta_0}\right) = 0$$

$$\Rightarrow \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left(\frac{x_i}{\beta_0}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\beta_0}\right)^\alpha \ln\left(\frac{x_i}{\beta_0}\right) = 0$$

$$\frac{n}{\alpha} = \sum_{i=1}^{n}\ln\left(\frac{x_i}{\beta_0}\right)\left(\left(\frac{x_i}{\beta_0}\right)^\alpha - 1\right) \Rightarrow \hat{\alpha} = \frac{n}{\sum_{i=1}^{n}\ln\left(\frac{x_i}{\beta_0}\right)\left(\left(\frac{x_i}{\beta_0}\right)^\alpha - 1\right)}$$

$$\hat{\alpha_{(\beta)}} = \left(\frac{\sum_{i=1}^{n}\left(\frac{x_i}{\beta_0}\right)^\alpha \ln(x_i)}{n} - \frac{\sum_{i=1}^{n}\left(\frac{x_i}{\beta_0}\right)^\alpha \ln(\beta_0)}{n} - \frac{\sum_{i=1}^{n}\ln(x_i)}{n} + \frac{\sum_{i=1}^{n}(\ln\beta_0)}{n}\right)^{-1}$$

$$\frac{\partial^2 \ln L(\alpha,\beta_0)}{\partial \alpha^2} = \frac{-n}{\alpha^2} - \sum_{i=1}^{n}\ln\left(\frac{x_i}{\beta_0}\right)^2\left(\frac{x_i}{\beta_0}\right)^\alpha < 0 \Rightarrow \alpha < \sqrt{\frac{\sum_{i=1}^{n}\ln\left(\frac{x_i}{\beta_0}\right)^2\left(\frac{x_i}{\beta_0}\right)^\alpha}{n}} \quad (4)$$

The above statement (4) must be true so that α can take its maximum value.

d) Define $L(\beta) := L(\hat{\alpha}(\beta), \beta)$ Find $\hat{\beta}$ based on $L(\beta)$

$$L(\beta) = L\left(\alpha_{\hat{(\beta)}}, \beta\right) = \frac{\alpha_{\hat{(\beta)}}{}^{n}}{\beta^{n\alpha(\beta)}} \cdot \prod_{i=1}^{n} x_i^{\alpha_{\hat{(\beta)}}-1} \cdot e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha_{\hat{(\beta)}}}}$$

$$\ln L\left(\alpha_{\hat{(\beta)}}, \beta\right) = n\ln\left(\alpha_{\hat{(\beta)}}\right) - n\alpha_{\hat{(\beta)}}\ln(\beta) + \left(\alpha_{\hat{(\beta)}} - 1\right)\sum_{i=1}^{n}\ln\left(x_i\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha_{\hat{(\beta)}}}$$

$$\hat{\beta} := \frac{\partial \ln L\left(\alpha_{\hat{(\beta)}}, \beta\right)}{\partial \beta} = 0 \Rightarrow -\frac{n\alpha_{\hat{(\beta)}}}{\beta} - \frac{-\alpha_{\hat{(\beta)}}\beta^{\alpha_{\hat{(\beta)}}-1}}{\beta^{2\alpha_{\hat{(\beta)}}}}\sum_{i=1}^{n}\left(x_i\right)^{\alpha} = 0$$

$$\frac{n\alpha_{\hat{(\beta)}}}{\beta} = \alpha_{\hat{(\beta)}}\beta^{-1-\alpha_{\hat{(\beta)}}}\sum_{i=1}^{n}\left(x_i\right)^{\alpha_{\hat{(\beta)}}} \Rightarrow \beta^{\alpha_{\hat{(\beta)}}} = \frac{\sum_{i=1}^{n}\left(x_i\right)^{\alpha_{\hat{(\beta)}}}}{n}$$

$$\hat{\beta} = \left(\frac{\sum_{i=1}^{n}\left(x_i\right)^{\alpha_{\hat{(\beta)}}}}{n}\right)^{\frac{1}{\alpha_{\hat{(\beta)}}}}$$

$$\alpha_{\hat{(\beta)}} = \frac{n}{\sum_{i=1}^{n}\ln\left(\frac{x_i}{\beta_0}\right)\left(\left(\frac{x_i}{\beta_0}\right)^{\alpha} - 1\right)} \not< 0$$

$$\hat{\beta} = \left(\frac{\sum_{i=1}^{n}(x_i)^{\frac{n}{\sum_{i=1}^{n}\ln(\frac{x_i}{\beta_0})((\frac{x_i}{\beta_0})^{\alpha}-1)}}}{n}\right)^{\frac{1}{\frac{n}{\sum_{i=1}^{n}\ln(\frac{x_i}{\beta_0})((\frac{x_i}{\beta_0})^{\alpha}-1)}}} = \left(\frac{\sum_{i=1}^{n}(x_i)^{\frac{n}{\sum_{i=1}^{n}\ln(\frac{x_i}{\beta_0})((\frac{x_i}{\beta_0})^{\alpha}-1)}}}{n}\right)^{\frac{\sum_{i=1}^{n}\ln(\frac{x_i}{\beta_0})((\frac{x_i}{\beta_0})^{\alpha}-1)}{n}}$$

$$\frac{\partial^2 \ln L\left(\alpha_{\hat{(\beta)}}, \beta\right)}{\partial \beta^2} = n\alpha_{\hat{(\beta)}}\beta^{\alpha_{\hat{(\beta)}}-1} \not< 0$$

e) Compare $\hat{\beta}$ in section (b) and (d).

In part (b), we see that the likelihood function depends on the parameter α But this is not the case in part (d) and it is independent of the parameter.

Part II

a) Is $L(\beta)$ at Part I in section (d) Likelihood?

No, it is basically Profile Likelihood. Because we fix a parameter and replace it with the relevant function and maximize the resulting probability.
In Likelihood Function we consider all the parameters.
But in Function Likelihood Profile, we remove disturbing parameters by expressing them as a function of desired parameters.

b) Let $(\alpha = 2, \beta = 1)$ generate 10000 and Compare $\mathbf{L}(\beta), \mathbf{L}(\alpha, \beta)$

likelihood: $L(\alpha, \beta) = \frac{\alpha^n}{\beta^{n\alpha}} \cdot \prod_{i=1}^{n} x_i^{\alpha-1} \cdot e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha}}$
$= 2^{10000} \cdot \prod_{i=1}^{10000} x_i \cdot e^{-\sum_{i=1}^{1000} x_i^2}$
Profile likelihood: $L(\beta) = L\left(\hat{\alpha}_{\hat{(\beta)}}, \beta\right) = \frac{\alpha_{(\hat{\beta})}{}^{n}}{\beta^{n\alpha(\hat{\beta})}} \cdot \prod_{i=1}^{n} x_i^{\alpha_{(\hat{\beta})}-1} \cdot e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha(\hat{\beta})}}$

$$L(\beta) = \alpha_{\hat{(\beta)}} \cdot \prod_{i=1}^{10000} x_i^{\alpha\hat{(\beta)}-1} \cdot e^{-\sum_{i=1}^{10000} x_i^{\alpha\hat{(\beta)}}}$$

$$\alpha_{\hat{(\beta)}} = \frac{n}{\sum_{i=1}^{n} \ln\left(\frac{x_i}{\beta_0}\right)\left(\left(\frac{x_i}{\beta_0}\right)^{\alpha} - 1\right)} = \frac{10000}{\sum_{i=1}^{10000} \ln(x_i)\left((x_i)^2 - 1\right)}$$

In [7]:

```
#B)
# Load necessary library
library(ggplot2)

# Set parameters
alpha <- 2
beta <- 1
n <- 10000

# Generate samples
set.seed(123)
samples <- rweibull(n, shape = alpha, scale = beta)
```

In [8]:

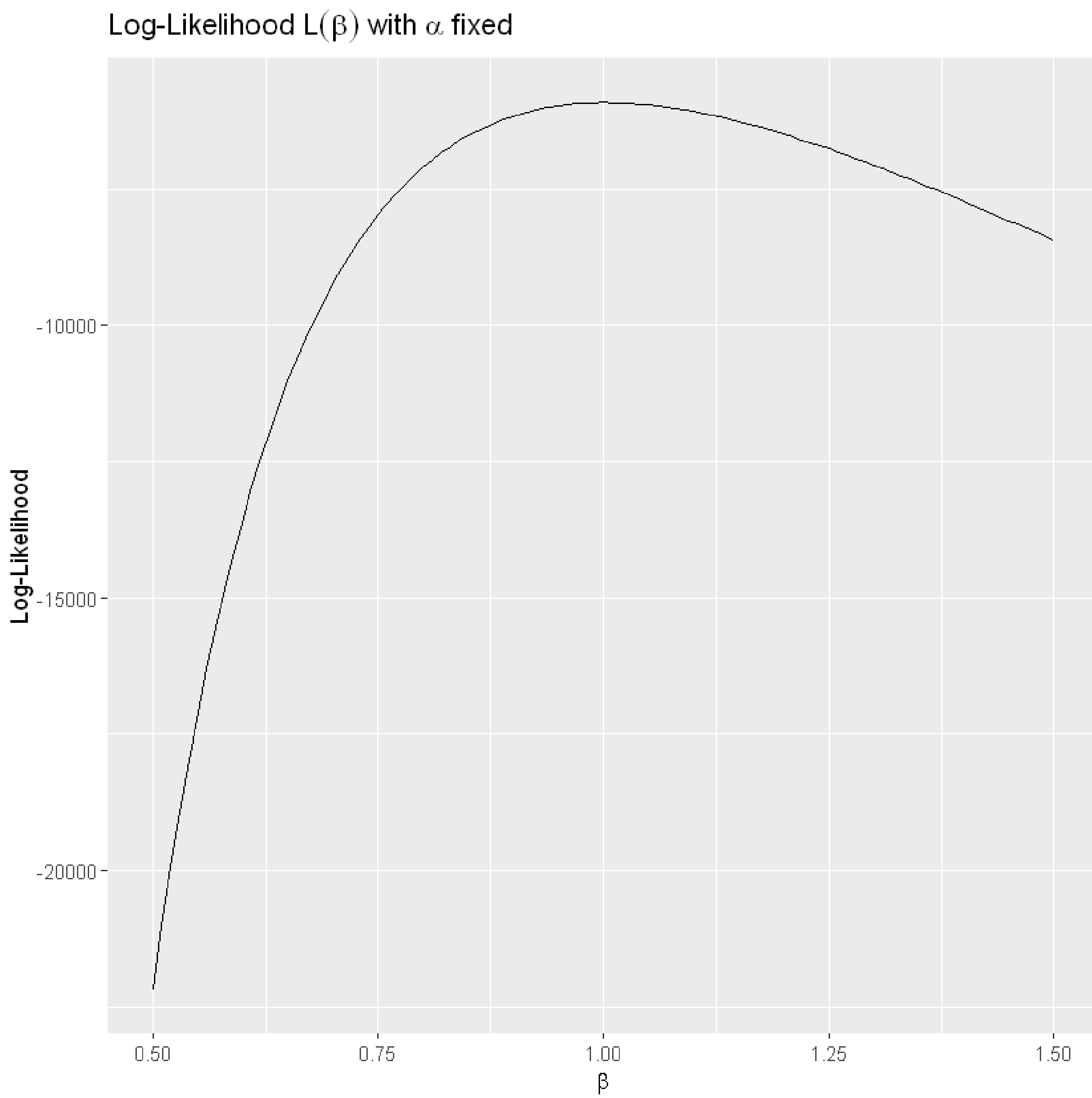
```
# Define the log-likelihood functions
logL_alpha_beta <- function(alpha, beta, x) {
  n <- length(x)
  logL <- n * log(alpha) - n * alpha * log(beta) + (alpha - 1) * sum(log(x)) - sum((x
  return(logL)
}

logL_beta <- function(beta, alpha, x) {
  n <- length(x)
  logL <- n * log(alpha) - n * alpha * log(beta) + (alpha - 1) * sum(log(x)) - sum((x
  return(logL)
}
```

In [9]:

```
# Define a range for beta
beta_range <- seq(0.5, 1.5, by = 0.01)

# Compute log-likelihood values for L(beta) with alpha fixed
logL_beta_values <- sapply(beta_range, function(b) logL_beta(b, alpha, samples))

# Compute log-likelihood values for L(alpha, beta)
logL_alpha_beta_values <- outer(beta_range, beta_range, Vectorize(function(a, b) logL_

# Convert to data frame for plotting
logL_beta_df <- data.frame(beta = beta_range, logL = logL_beta_values)
```

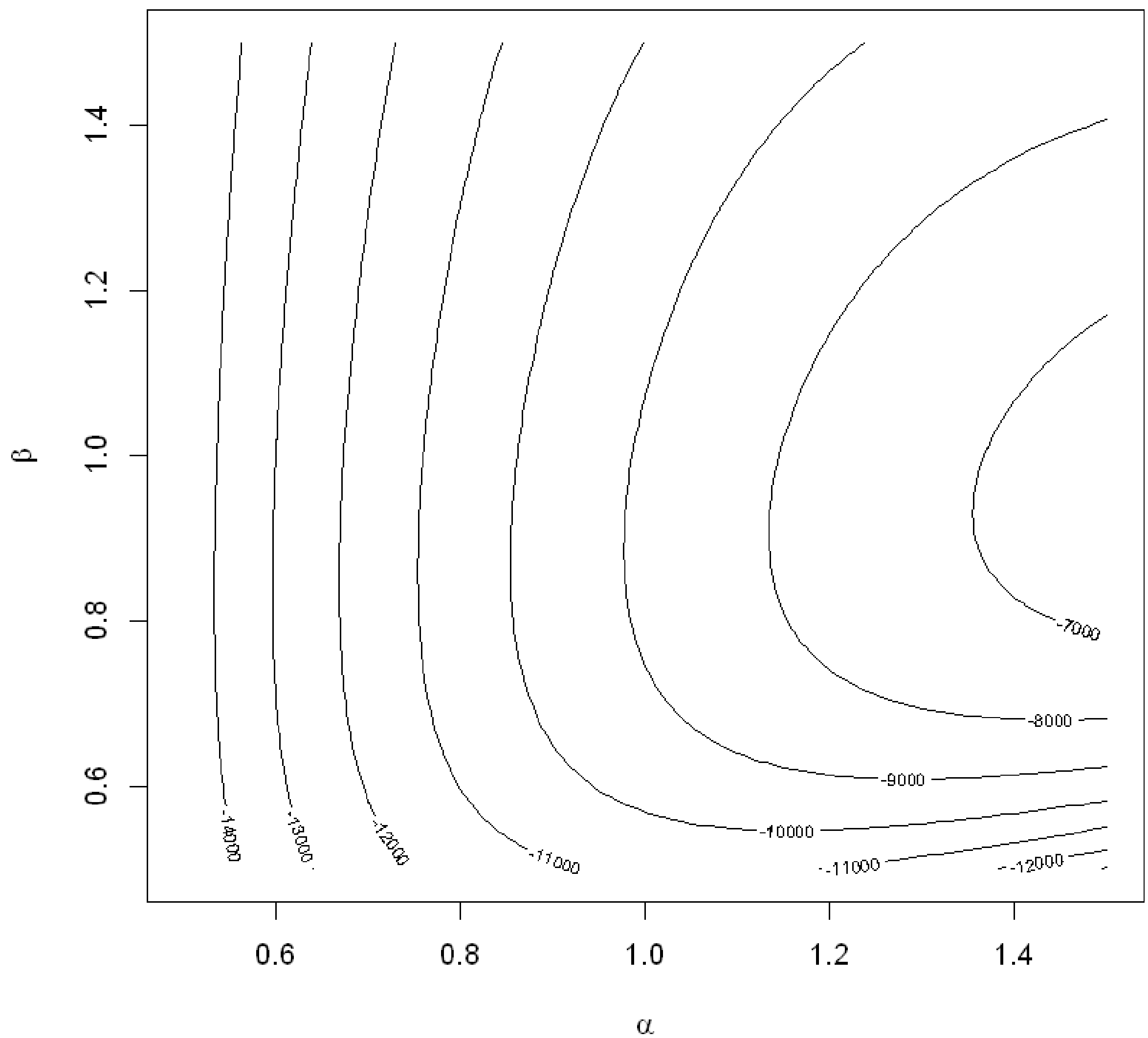
In [10]:

```
# Plot log-likelihood for L(beta)
ggplot(logL_beta_df, aes(x = beta, y = logL)) +
  geom_line() +
  ggtitle(expression(paste("Log-Likelihood ", L(beta), " with ", alpha, " fixed"))) +
  xlab(expression(beta)) +
  ylab("Log-Likelihood")
```



In [15]:
```
# Plot log-likelihood for L(alpha, beta)
contour(beta_range, beta_range, logL_alpha_beta_values, xlab = expression(alpha), ylab
        main = expression(paste("Log-Likelihood ", L(alpha, beta))),
        levels = pretty(range(logL_alpha_beta_values), 10))
```

Log-Likelihood $L(\alpha, \beta)$



c) based on (b), explain Why Profile likelihood proppsed in Part I works?

By increasing the number of samples we take, the Profile likelihood function tends to the Likelihood function.
That means, increasing the number of samples will fill the empty space of the parameter that we deleted.