Solutions to
Understanding Machine Learning
by Shai Shalev-Shwartz and Shai Ben-David

Chapter 18 (Decision Trees)

Zahra Taheri[1]

May 2020

Exercise 18.1

1. Let $h : \{0,1\}^d \to \{0,1\}$ be a binary classifier. Consider $(x_1 = 0?)$ as the root node of the tree, and $(x_{i+1} = 0?)$ as all nodes at depth i, for all $1 \leq i \leq d-1$. Therefore by definition, it has 2^d leaves. For all $x = (x_1, \ldots, x_d) \in \{0,1\}^d$, there exists a path from the root to a leaf of the tree corresponded to x, namely the path on nodes $(x_1 = 0?), (x_2 = 0?), \ldots, (x_d = 0?)$, leaf.

We define $h(x)$ as the leaf's value connected to such a path. This completes the proof.

2. By part (1), the class of decision trees over the domain $\{0,1\}^d$ can shatter the whole domain. So, VC dimension of such a class is 2^d.

Exercise 18.2

1. Let $S = \{((1,1,1),1),((1,0,0),1),((1,1,0),0),((0,0,1),0)\}$. Suppose that we want to use S in order to build a decision tree of depth 2 using ID3 algorithm. Let $H(Z)$ denote the entropy of a random variable Z. At the beginning, we must pick a feature with the highest information gain, $IG(X) = H(Y) - H(Y|X)$, or equivalently with the lowest value of $H(Y|X)$, as the root node.

(1) $H(Y|x_1) = -\frac{3}{4}\left(\frac{2}{3}log\frac{2}{3} + \frac{1}{3}log\frac{1}{3}\right) - \frac{1}{4}\left(0\ log\ 0 + 1\ log\ 1\right) = -\frac{3}{4}\left(\frac{2}{3}log\frac{2}{3} + \frac{1}{3}log\frac{1}{3}\right) < 1$

(2) $H(Y|x_2) = -\frac{1}{2}\left(\frac{1}{2}log\frac{1}{2} + \frac{1}{2}log\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}log\frac{1}{2} + \frac{1}{2}log\frac{1}{2}\right) = log\ 2 = 1$

(3) $H(Y|x_3) = -\frac{1}{2}\left(\frac{1}{2}log\frac{1}{2} + \frac{1}{2}log\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}log\frac{1}{2} + \frac{1}{2}log\frac{1}{2}\right) = log\ 2 = 1$

So, $(x_1 = 0?)$ is picked by the ID3 algorithm as the root. If we choose $(x_2 = 0?)$ as the node at depth 1, then as we can see in the following decision tree, either $(1,1,1)$ or $(1,1,0)$ is mislabeled. Therefore, the training error is at least $1/4$, because $|S| = 4$. If we choose $(x_3 = 0?)$ as the node at depth 1, with a similar discussion the statement is true (Figure 1).

[1] `https://github.com/zahta/Exercises-Understanding-Machine-Learning`

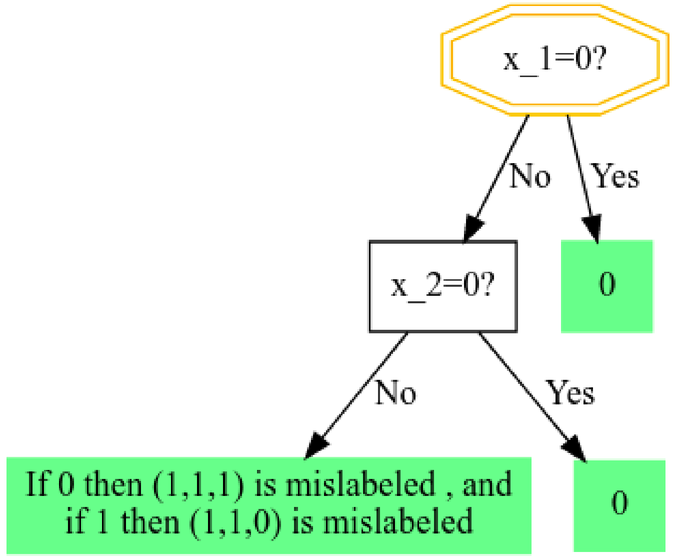

FIGURE 1.

2. The decision tree of depth 2 in Figure 2 have zero training error.

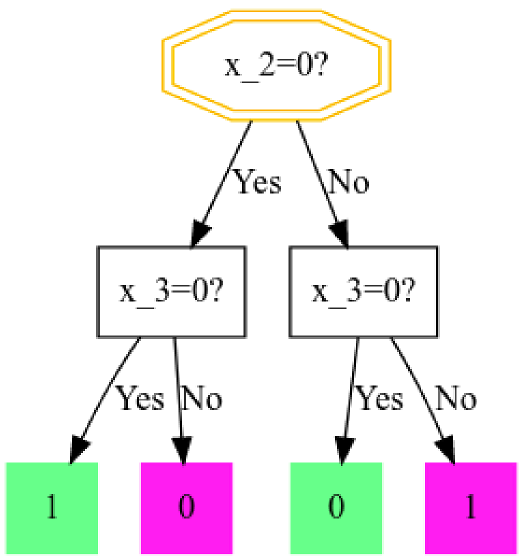

FIGURE 2.

Python codes for diagrams

Figure 1 codes:

```
from graphviz import Digraph
styles = {
'top': {'shape': 'doubleoctagon','color':'orange'},
'yes': {'shape': 'square', 'style': 'filled', 'color': 'lightgreen'},
'qst': {'shape': 'rect'},
'qst2': {'shape': 'rect','style': 'filled','color': 'lightgreen'}
}
tree1 = Digraph()
tree1.node('top', "x_1=0?", styles['top'])
tree1.node('q1', "x_2=0?", styles['qst'])
tree1.node('q2', "If 0 then (1,1,1) is mislabeled ,
and\n if 1 then (1,1,0) is mislabeled", styles['qst2'])
tree1.node('yes1', '0', styles['yes'])
tree1.node('yes2', '0', styles['yes'])
tree1.edge('top', "yes1", "Yes")
tree1.edge('top', "q1", "No")
tree1.edge('q1', "yes2", "Yes")
tree1.edge('q1', 'q2', "No")
tree1
```

Figure 2 codes:

```
from graphviz import Digraph
styles = {
'top': {'shape': 'doubleoctagon','color':'orange'},
'yes': {'shape': 'square', 'style': 'filled', 'color': 'lightgreen'},
'no': {'shape': 'square', 'style': 'filled', 'color': 'violet'},
'qst': {'shape': 'rect'}
}
tree1 = Digraph()
tree1.node('top', "x_2=0?", styles['top'])
tree1.node('q1', "x_3=0?", styles['qst'])
tree1.node('q2', "x_3=0?", styles['qst'])
tree1.node('yes1', '1', styles['yes'])
tree1.node('yes2', '0', styles['yes'])
tree1.node('no1', '0', styles['no'])
tree1.node('no2', '1', styles['no'])
tree1.edge('top', "q1", "Yes")
tree1.edge('top', "q2", "No")
tree1.edge('q1', "yes1", "Yes")
tree1.edge('q1', "no1", "No")
tree1.edge('q2', "yes2", "Yes")
tree1.edge('q2', "no2", "No")
tree1
```