COMPUTATIONAL METHODS IN FREEFORM OPTICAL TECHNOLOGIES

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EE 575 PROJECT - COMPUTATIONAL DIFFERENTIAL GEOMETRY

OUTLINE

I. INTRODUCTION & DESCRIPTION

II. THEORETICAL SETUP & FRAMEWORK

III. IMPLEMENTATION

IV. CONCLUSIONS & FUTURE DIRECTIONS

Introduction

- Optical Applications
  - Corrective Lenses
  - Image Formation
  - AR/VR Systems
  - Telescopes
  - Precision Cameras
  - LED Display

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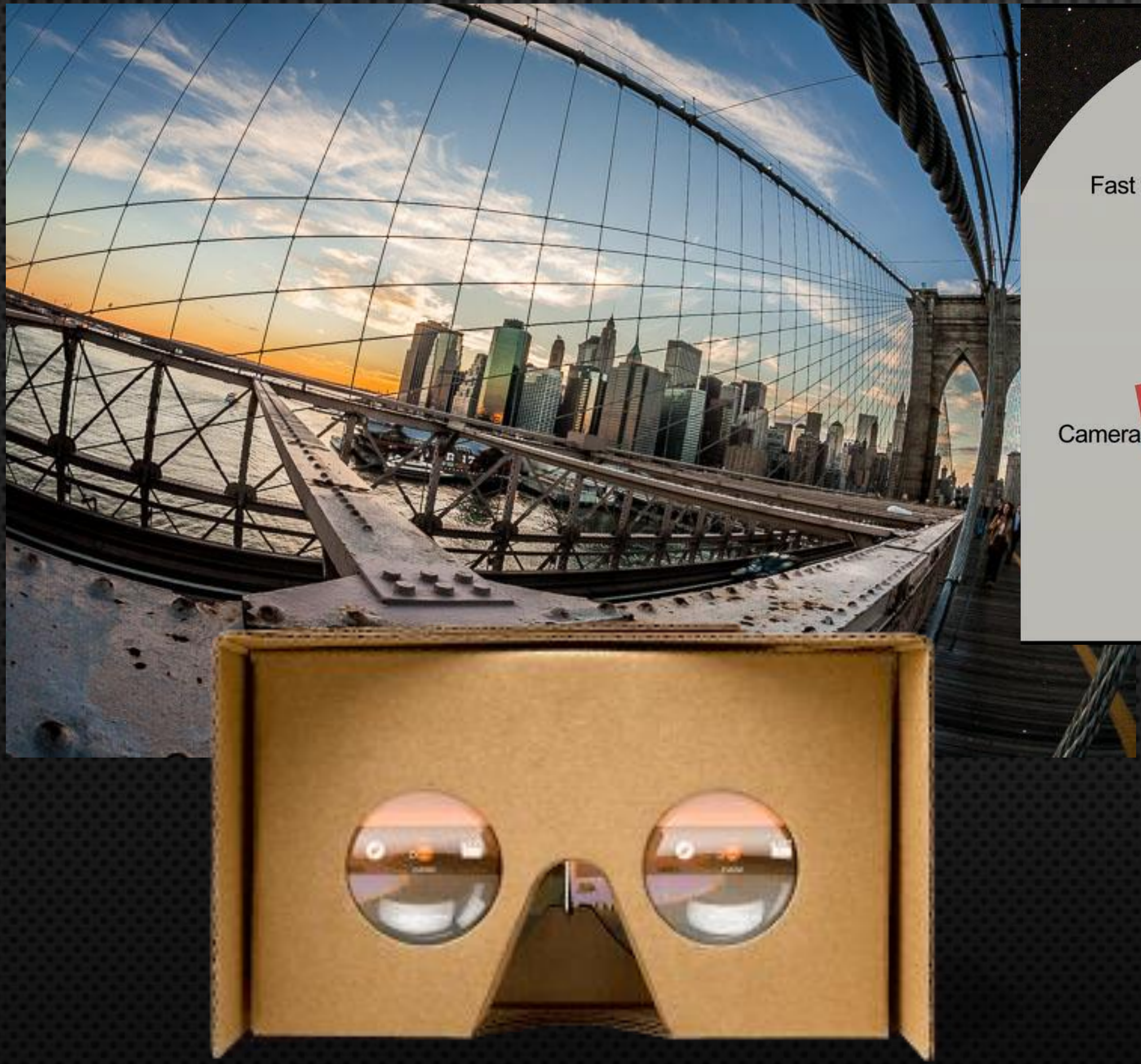

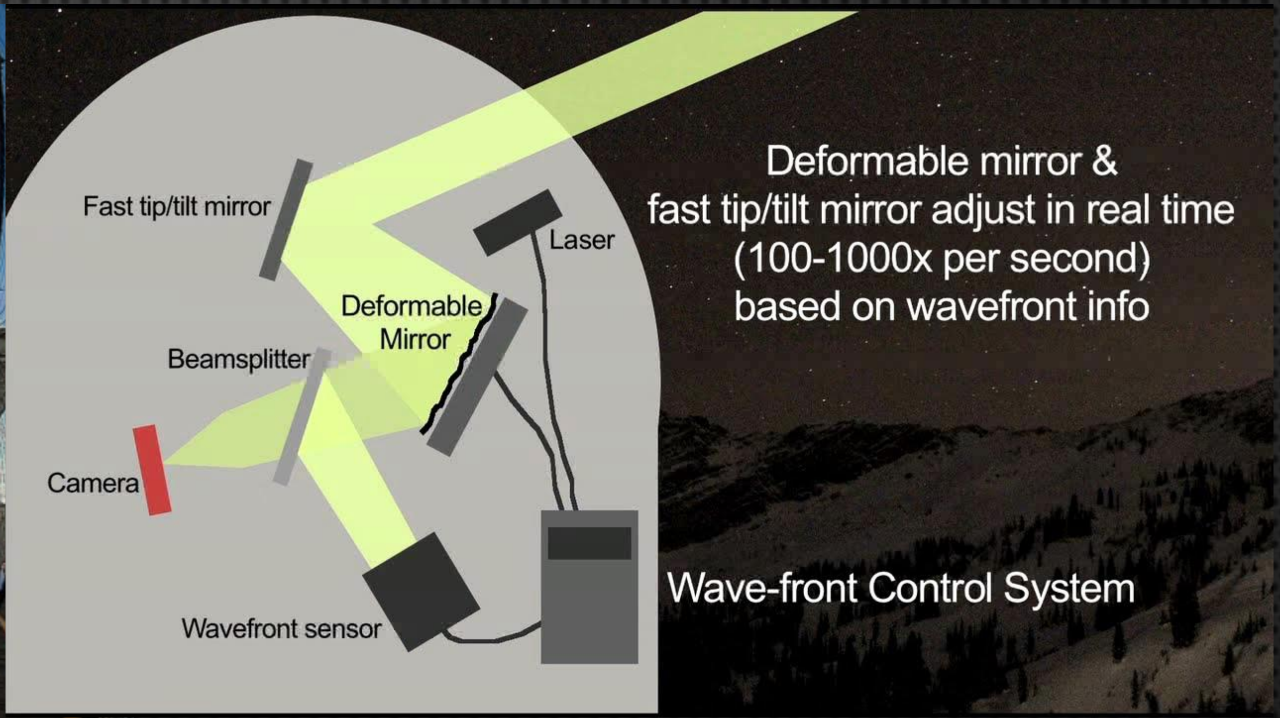

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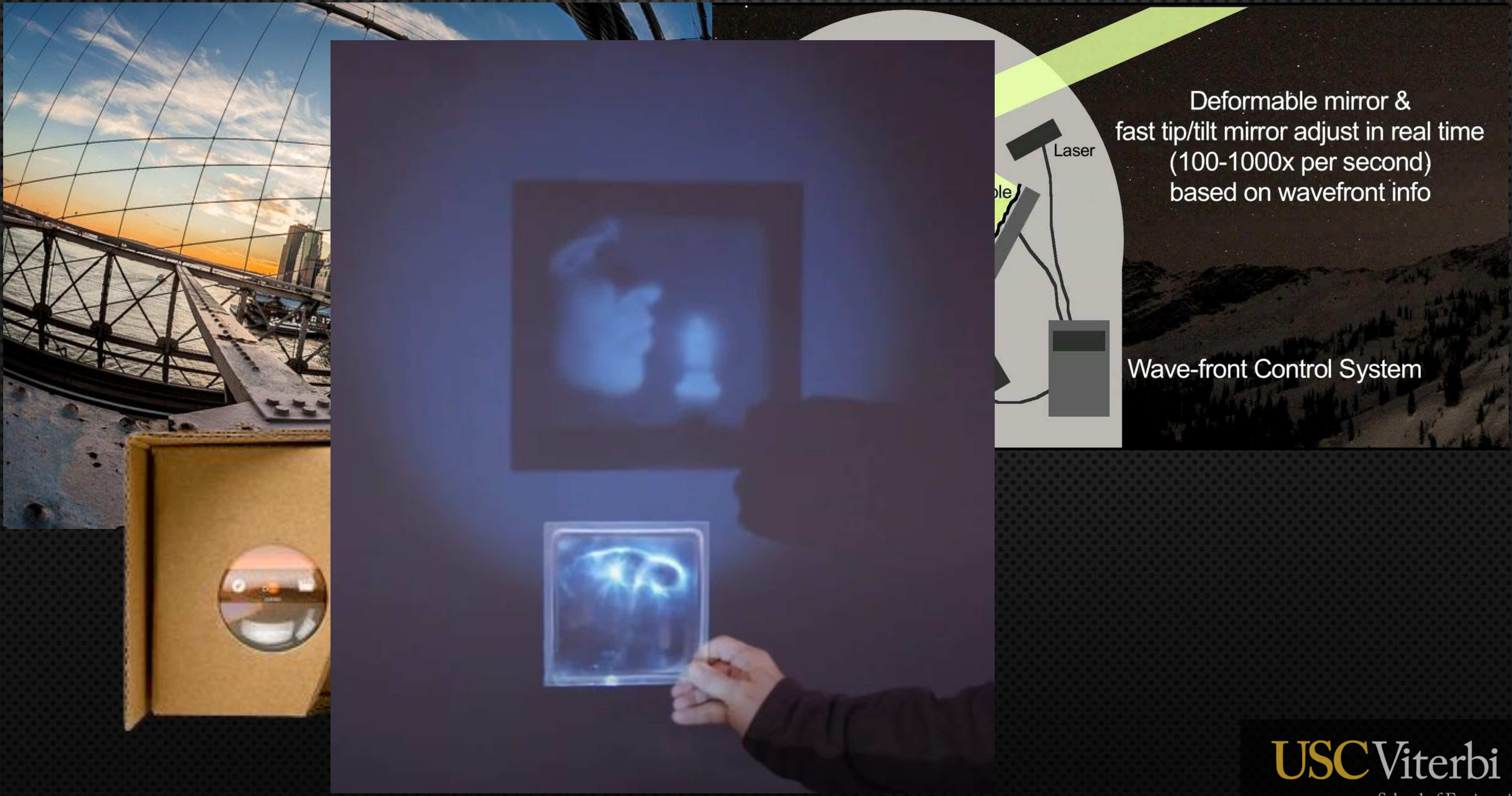

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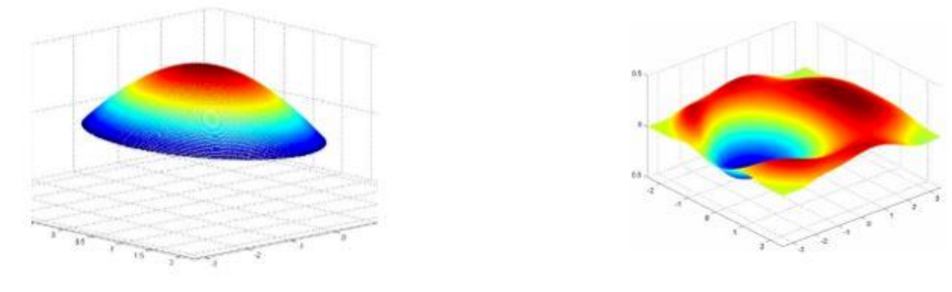
Introduction

- Freeform Lenses
  - In optics, freeform lenses are optical surfaces that are designed without rigid radial dimensions such as translational or rotational symmetry about axes normal to any mean plane.



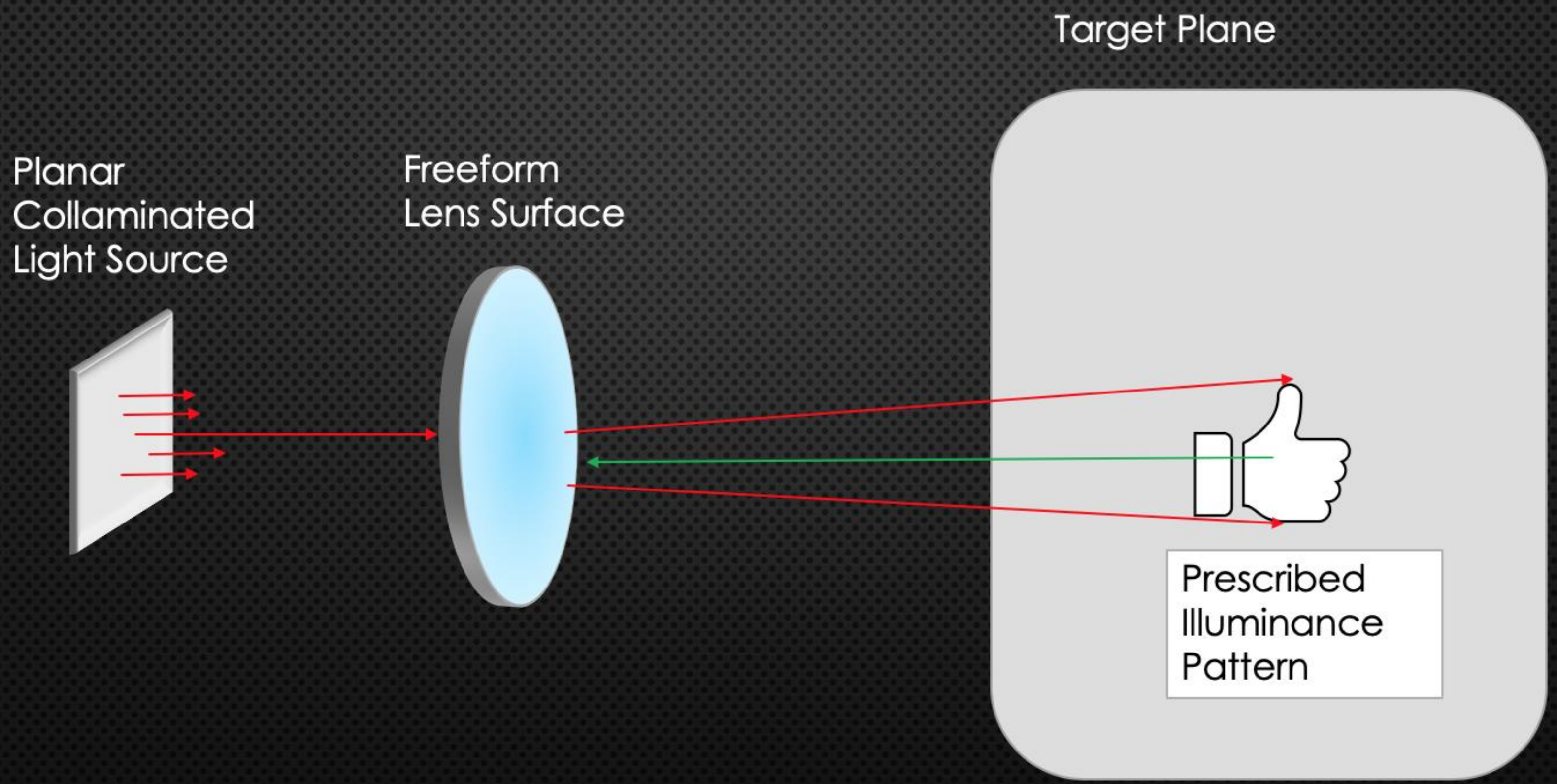
Traditional Rotationally Symmetrical Lens

Not a free-form lens

Non-rotationally Symmetrical Lens

Free-form Lens

GENERAL SETUP



GIVEN AN ILLUMINATION SOURCE AND AN IRRADIANCE DISTRIBUTION ON A PLANE. HOW CAN WE COMPUTE THE SURFACE OF THE LENS NEEDED TO GENERATE THE TARGET DISTRIBUTION?

FRAMEWORK

EIKONAL EQUATION:

$$\nabla^2 E = c^2(x, y, z)$$

SOLUTIONS TO THE 3D WAVE EQUATION THAT REPRESENT LIGHT WAVEFRONT AT DIFFERENT TIMES.

INTEGRATING THE EIKONAL AT DIFFERENT SPATIAL POINTS YIELDS AN OPTICAL RAY PATH LENGTH

$$\int_A^B \nabla^2 E \; de \approx \sum_{i \in [A,B]} n_i d_i$$

FIND A REFRACTIVE OPTICAL SURFACE WHICH GIVES US RAY PATHS THAT SATISFY THE CONDITIONS FROM THE SOURCE (A) AND TARGET ENDPOINT LOCATIONS (B).

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FRAMEWORK

THE COORDINATES OF A LIGHT RAY ON THE TARGET PLANE ARE RELATED TO THE SOURCE POINTS (X,Y) BY A VECTOR VALUED FUNCTION Z(X, Y) SO THAT:

$$x_t = z_1(x, y) \; ; y_t = \; z_2(x, y)$$

SNELL'S LAW AND THE JACOBIAN DEFINES A CONFORMAL MAPPING FROM THE FREEFORM LENS SURFACE TO THE TARGET PLANE:

$$dx_t \, dy_t = |\, J(\mathrm{Z})|\ \mathrm{DXDY}$$

WHERE:

$$J(z) = \begin{bmatrix} \frac{\partial z_1}{\partial x} & \frac{\partial z_1}{\partial y} \\ \frac{\partial z_2}{\partial x} & \frac{\partial z_2}{\partial x} \end{bmatrix}$$

MORE PRECISELY, THE PHYSICS INTERPRETATION DESCRIBES THE TRANSFORM BETWEEN IRRADIANCE DISTRIBUTIONS OF THE TARGET AND FREEFORM SURFACES:

$$dI_t(x_t, y_t) = |\, J(\mathrm{Z})| dI(\mathrm{X, Y})$$

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FRAMEWORK

GIVEN:

$$dI_t(x_t, y_t) = |J(\mathrm{Z})|\, \mathrm{DI}(\mathrm{X}, \mathrm{Y}) \qquad \text{(Conformal Map of Irradiance Distributions)}$$

$$\iint I_t(x_t, y_t)\ dx_t dy_t = \iint I(x, y)\, dxdy \qquad \text{(Conservation of Energy)}$$

THESE CONDITIONS IMPLY A NONLINEAR MONGE-AMPERE ELLIPTICAL PDE :

$$\det D^2 z = K(\bar{x})(1 + |Dz|^2)^{\frac{n+2}{2}} \qquad \text{(Prescribed Curvature Problem)}$$

ALSO EXPRESSED IN A LOCALLY LINEARIZED FORM AS:

$$A\left(z_{xx}z_{yy} - {z_{xy}}^2\right) + Bz_{xx} + Cz_{yy} + Dz_{xy} + E = 0$$

$$BC: \begin{cases} x_t = f_1(x, y, z, z_x, z_y) \\ y_t = f_2(x, y, z, z_x, z_y) \end{cases} : \partial S \rightarrow \partial T$$

WHERE THE TARGET COORDINATES ARE RE-EXPRESSED AS FUNCTIONS OF THE RAY SOURCE AND THEIR INCIDENCE AT THE OPTICAL SURFACE.

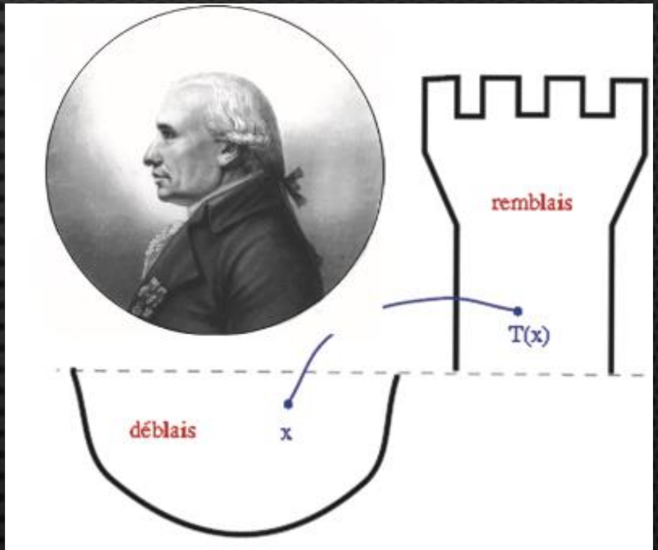
FRAMEWORK

Taking the view of light as a photon mass, Energy conservation also implies an Optimal Mass Transport problem (Monge-Kantorovich Formulation):

$$\int c(\bar{x}, T(\bar{x}))I(\bar{x})\,d\bar{x}$$

S/T:

$$T: I(\bar{x}) \rightarrow I_t(\bar{x}_t)$$



Where the mass-preserving transform $T(\bar{x})$ expresses a set of constraints for the Irradiance correspondence between each source to target spatial pairing.

$T(\bar{x})$ has also been shown to be the gradient of the MA PDE solutions.

Existing Methods in Literature

- Simultaneous Multiple Surface Method
- MA PDE and Newton Methods
- MK Transport as Linear Program
- NURBS Approaches

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FRAMEWORK

THE GOAL IS TO COMPUTE A CHART CONSISTING OF LOCAL MONGE PATCHES DEFINED AS

$m: U \to \mathbb{R}^3, \mathrm{U} \subseteq \mathbb{R}^2$ WHERE $m(x,y) = \big(x, y, z(x,y)\big)$ UNIQUELY SATISFIES THE BOUNDARY CONDITIONS.

FROM THE ELLIPTICAL MA PDE AND THE FUNDAMENTAL FORMS, THE MONGE PATCHES SHOULD HAVE:

$$\det D^2 z = K(\bar{x})(1 + |Dz|^2)^{\frac{n+2}{2}} \Leftrightarrow K_{Gaussian} = \frac{z_{xx}z_{yy} - {z_{xy}}^2}{(1 + {z_x}^2 + {z_y}^2)^2}$$

IN OUR CASE, WE FOUND AN EXPLICIT EXPRESSION OF MONGE-AMPERE USED IN LENS DESIGN:

$$\left|z_{Lxx}z_{Lyy} - {z_{Lxy}}^2\right| = |J(z)| \frac{\sqrt{1 + ({z_{Lx}}^2 + {z_{Ly}}^2)(1 - n^2)}}{K(z_{Lx}, z_{Ly})^2}$$

FRAMEWORK

Using the Lens Expression of Monge-Ampere we compute the surface as a function of Finite Difference Approximations:

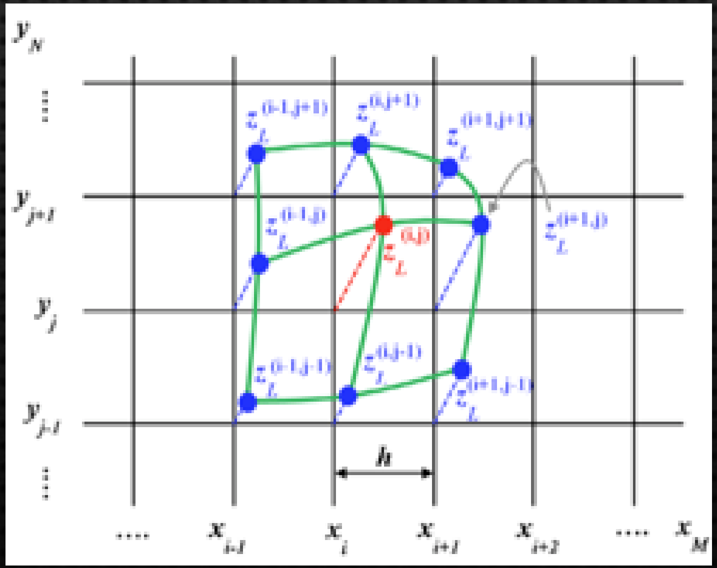
$$z_L^{(i,j)} = \frac{av_1^{(i,j)} + av_2^{(i,j)} - \sqrt{(av_1^{(i,j)} - av_2^{(i,j)})^2 + \left(\frac{av_3^{(i,j)} - av_4^{(i,j)}}{2}\right)^2 + h^4 g^{(i,j)}}}{2}$$

Solving each point using An iterative Method Based on Gauss-Seidel.

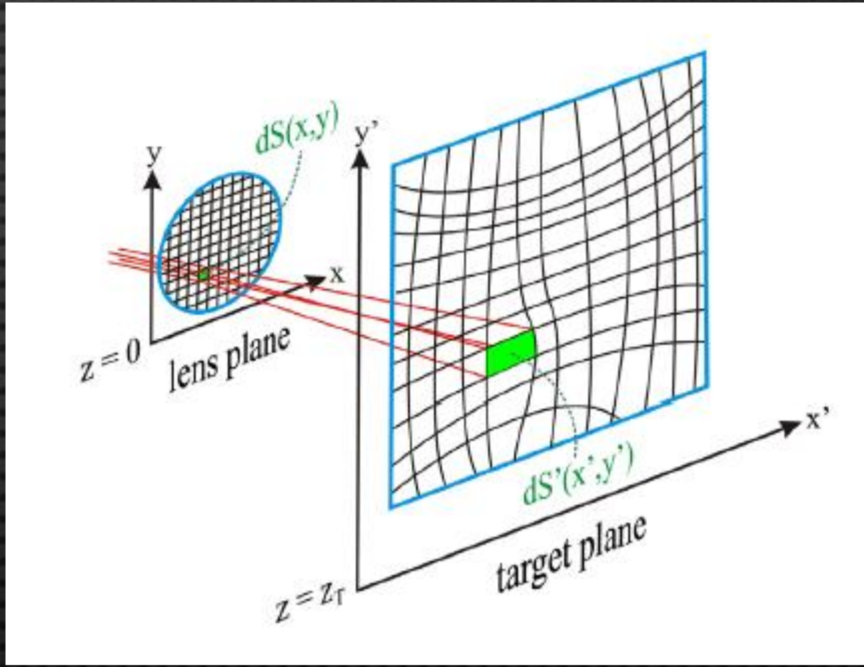
For t = (1:Iters)

$$z_L^{t+1} = f(z_L^t, z_L^{t-1});$$

End



IMPLEMENTATION



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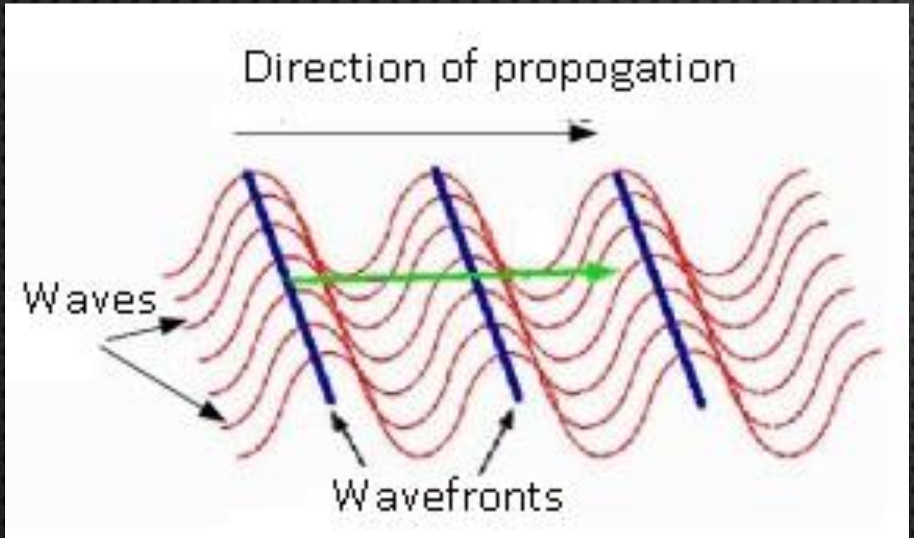
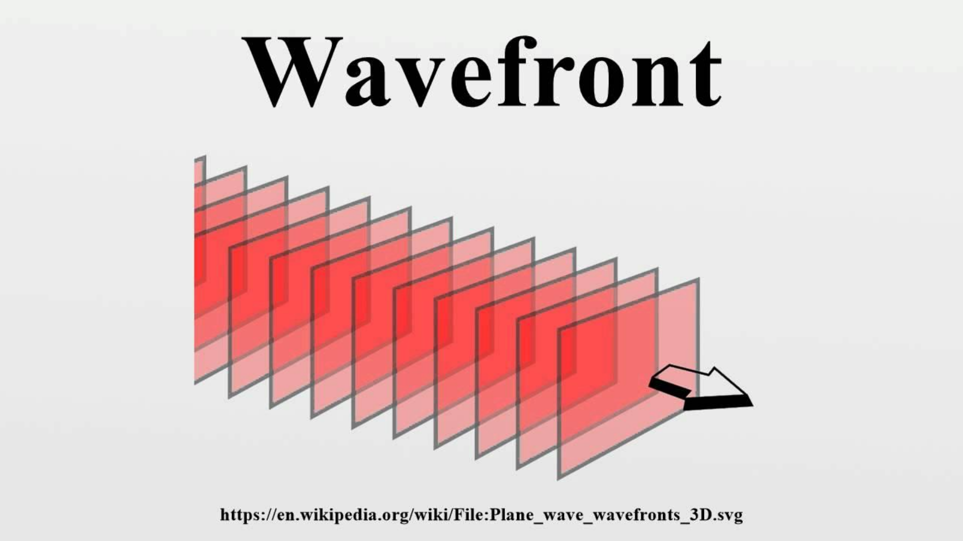
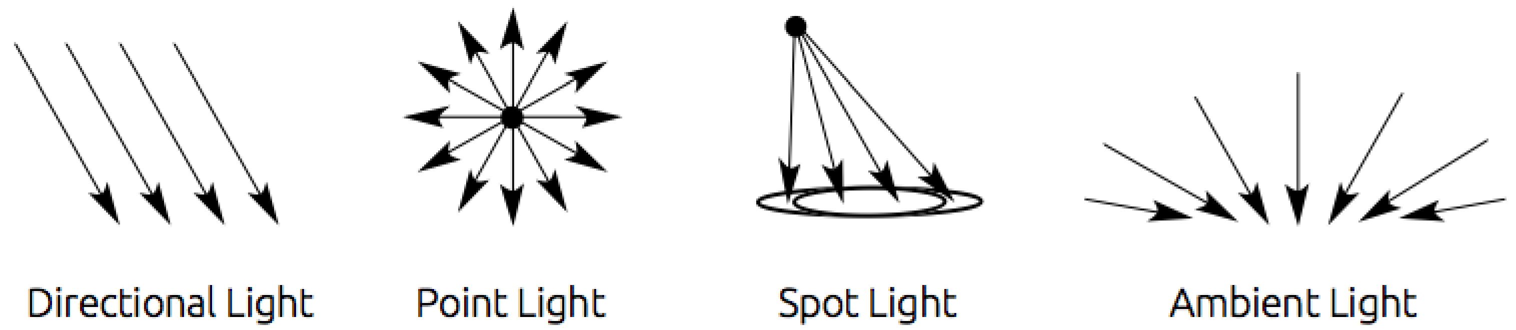
IMPLEMENTATION







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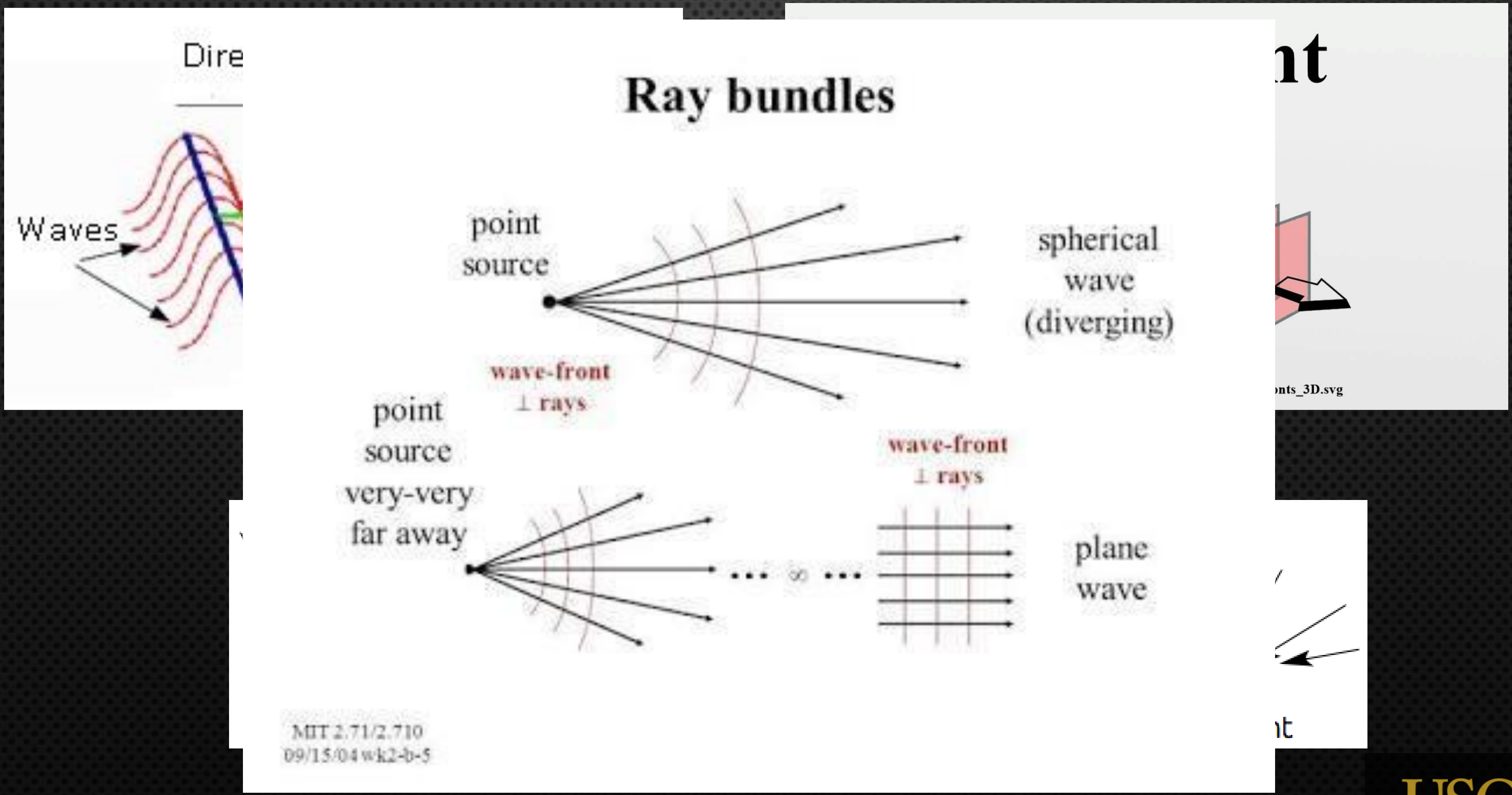

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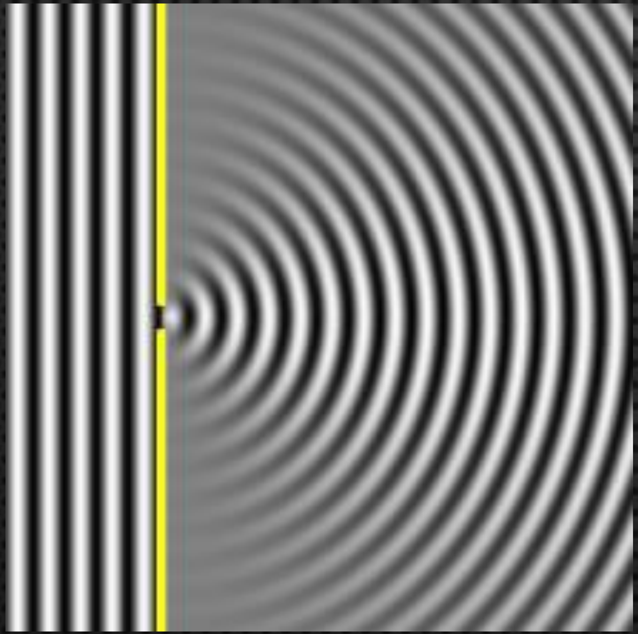
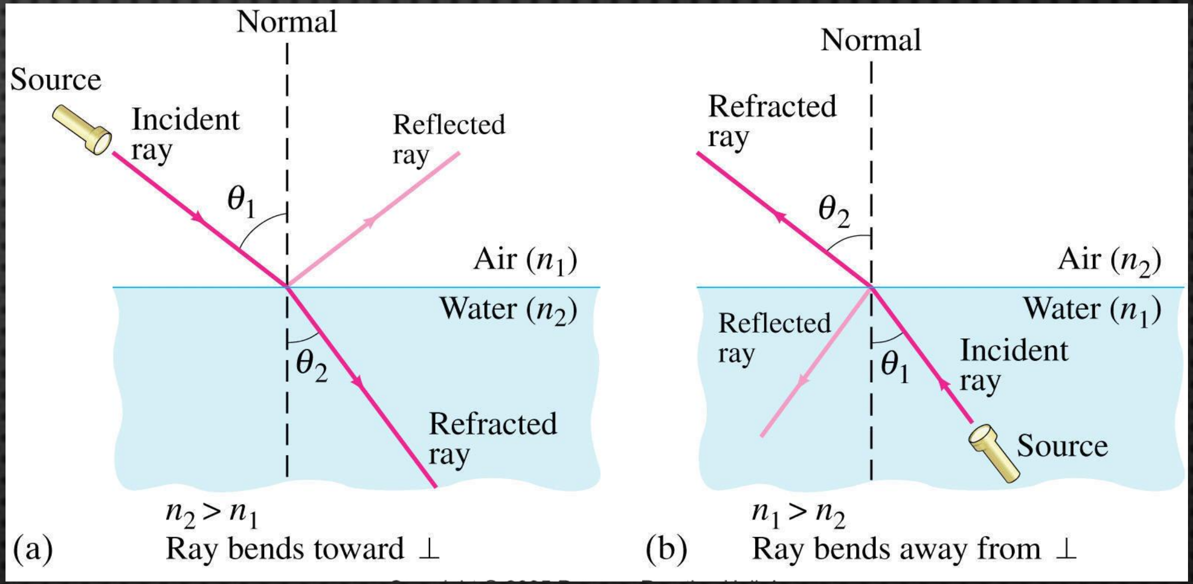
IMPLEMENTATION





Snell's Law:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

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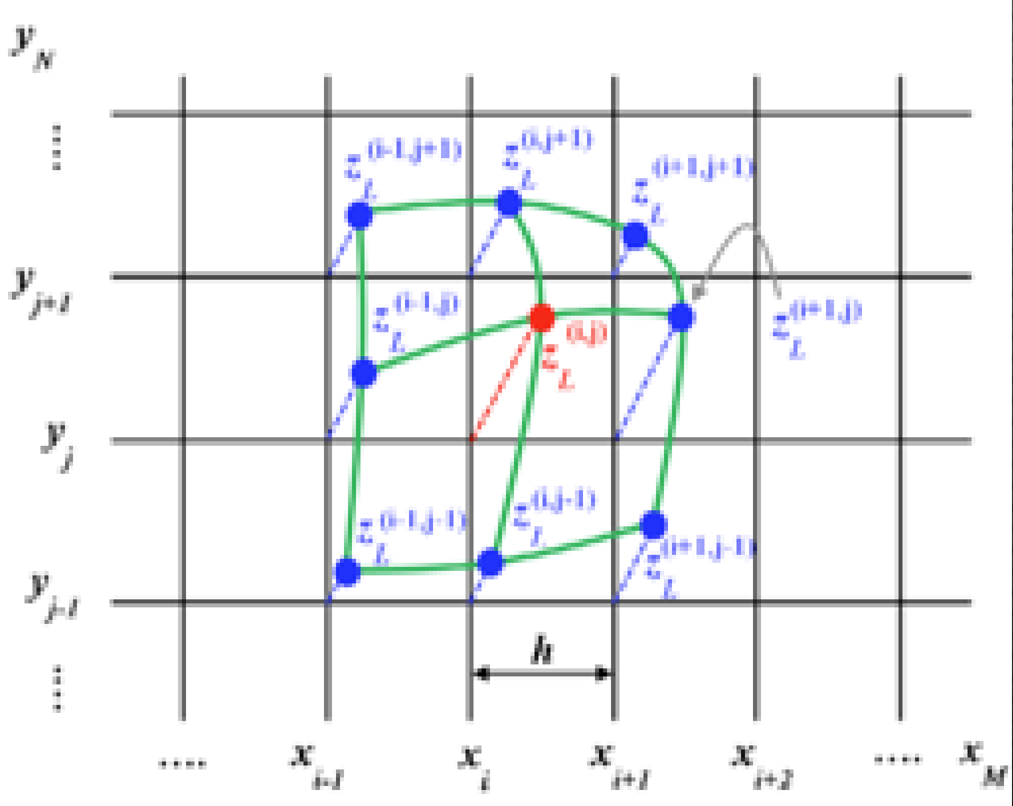
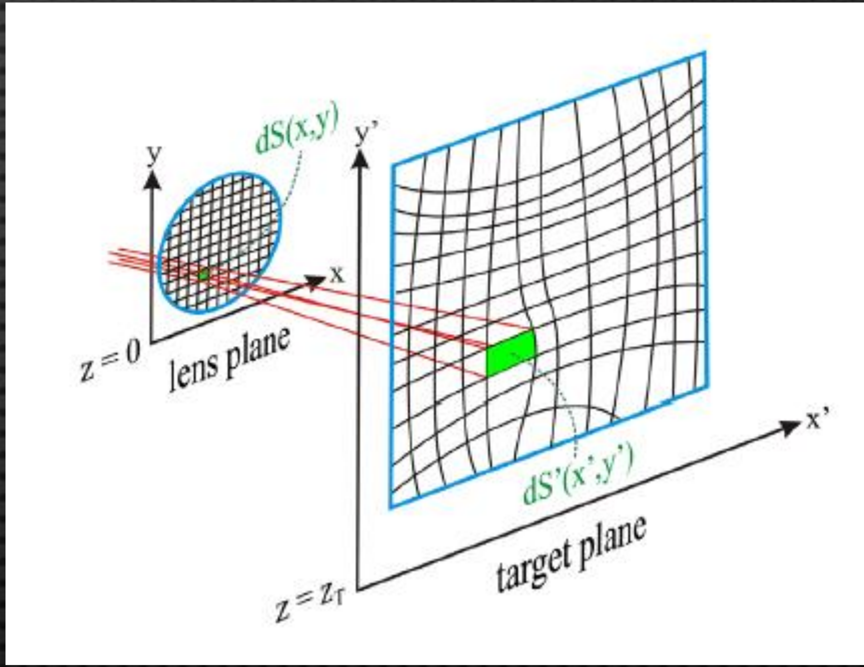
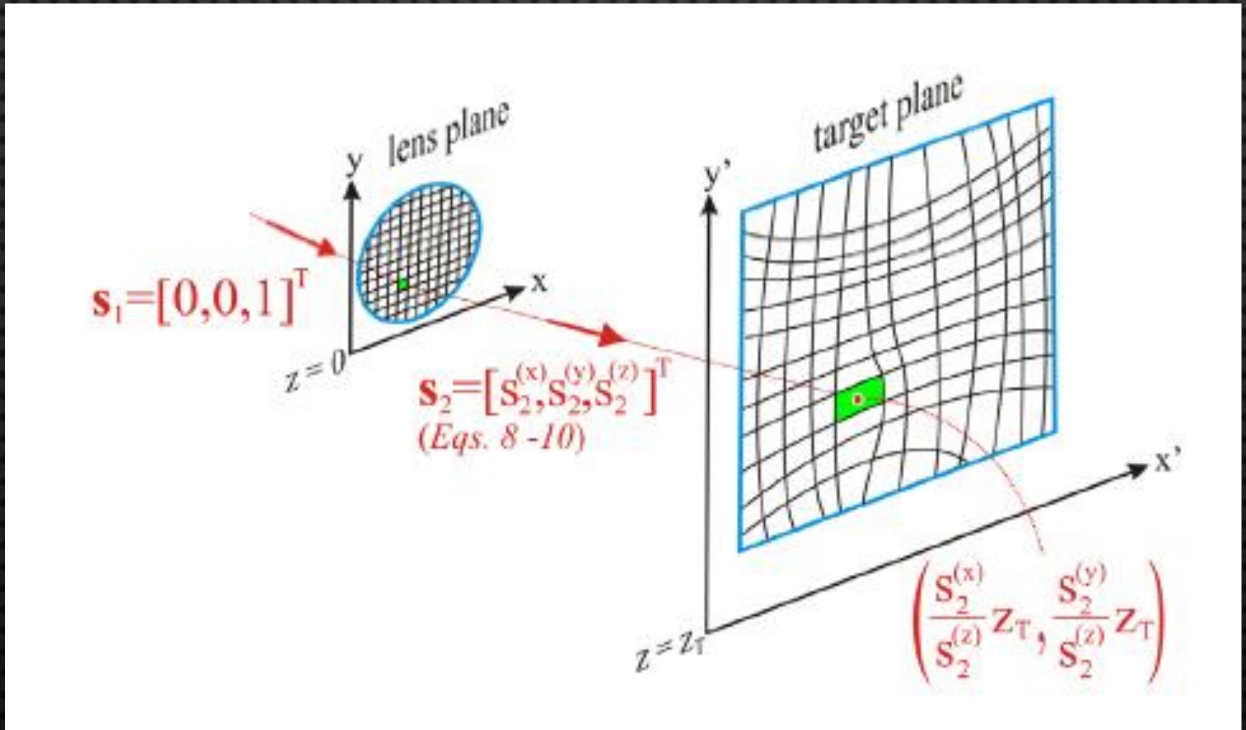
IMPLEMENTATION





IMPLEMENTATION



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IMPLEMENTATION


```
clf;
clear all;

img = imread("USC_logo2.jpg");
imshow(img);
```

IMPLEMENTATION

NxN x 3

IMPLEMENTATION

IMPLEMENTATION


```
img_bw = im2bw(img);

global target_img;
target_img = imcomplement(img_bw);


%Turn the following section on to lower
%resolution of target image
downSampleScalar = 100;
target_img = imresize(target_img, [floor(size(target_img, 1)/downSampleScalar), floor(size(target_img, 2)/downSampleScalar)]);

target_img = imcomplement(target_img);
imshow(target_img);
```

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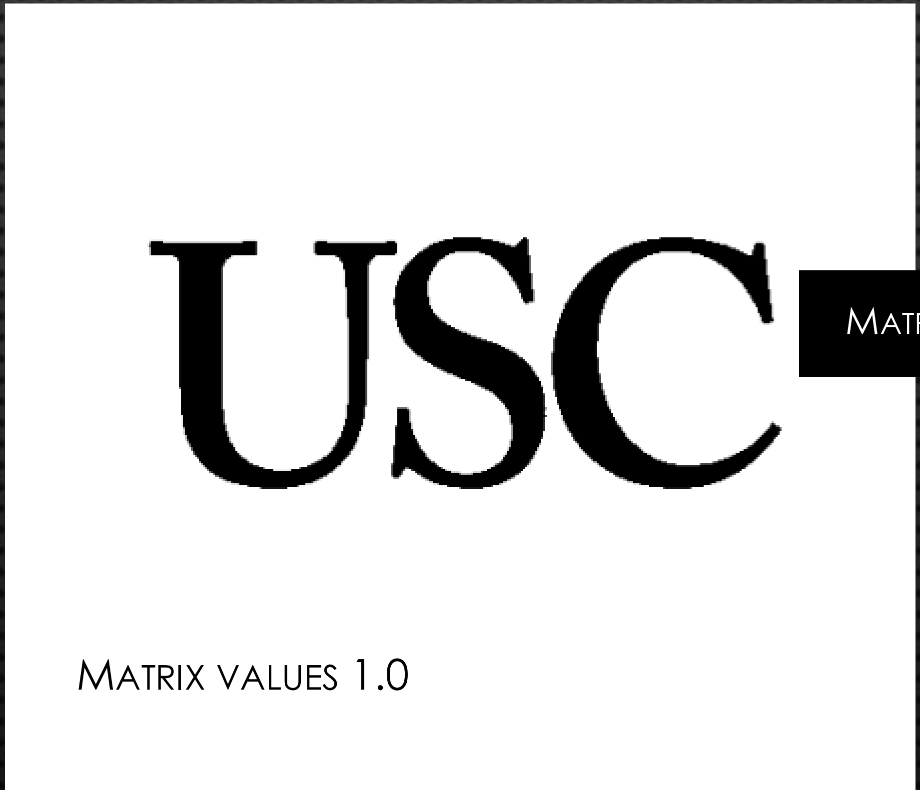
IMPLEMENTATION

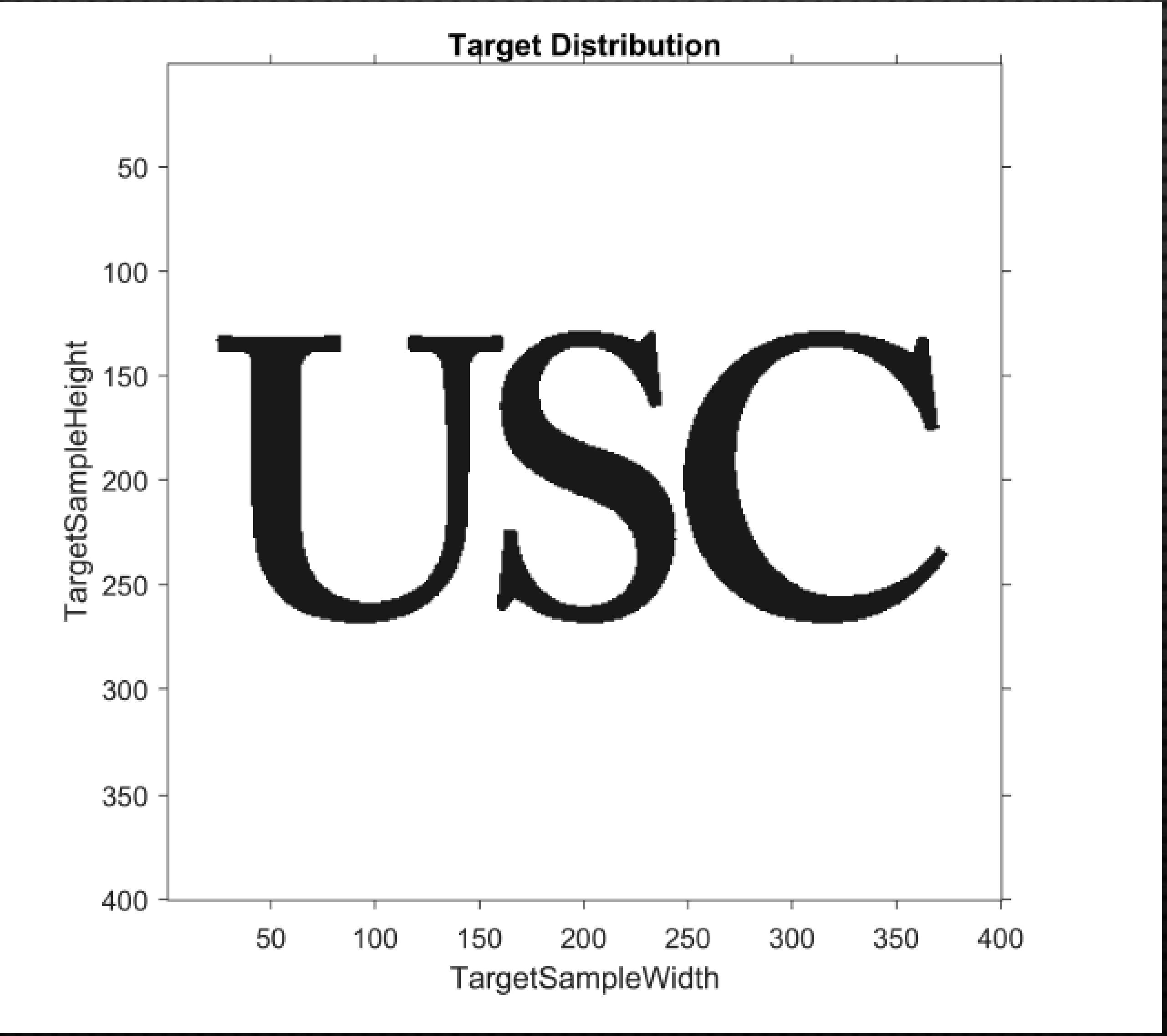
IMPLEMENTATION



MATRIX VALUES 0.10

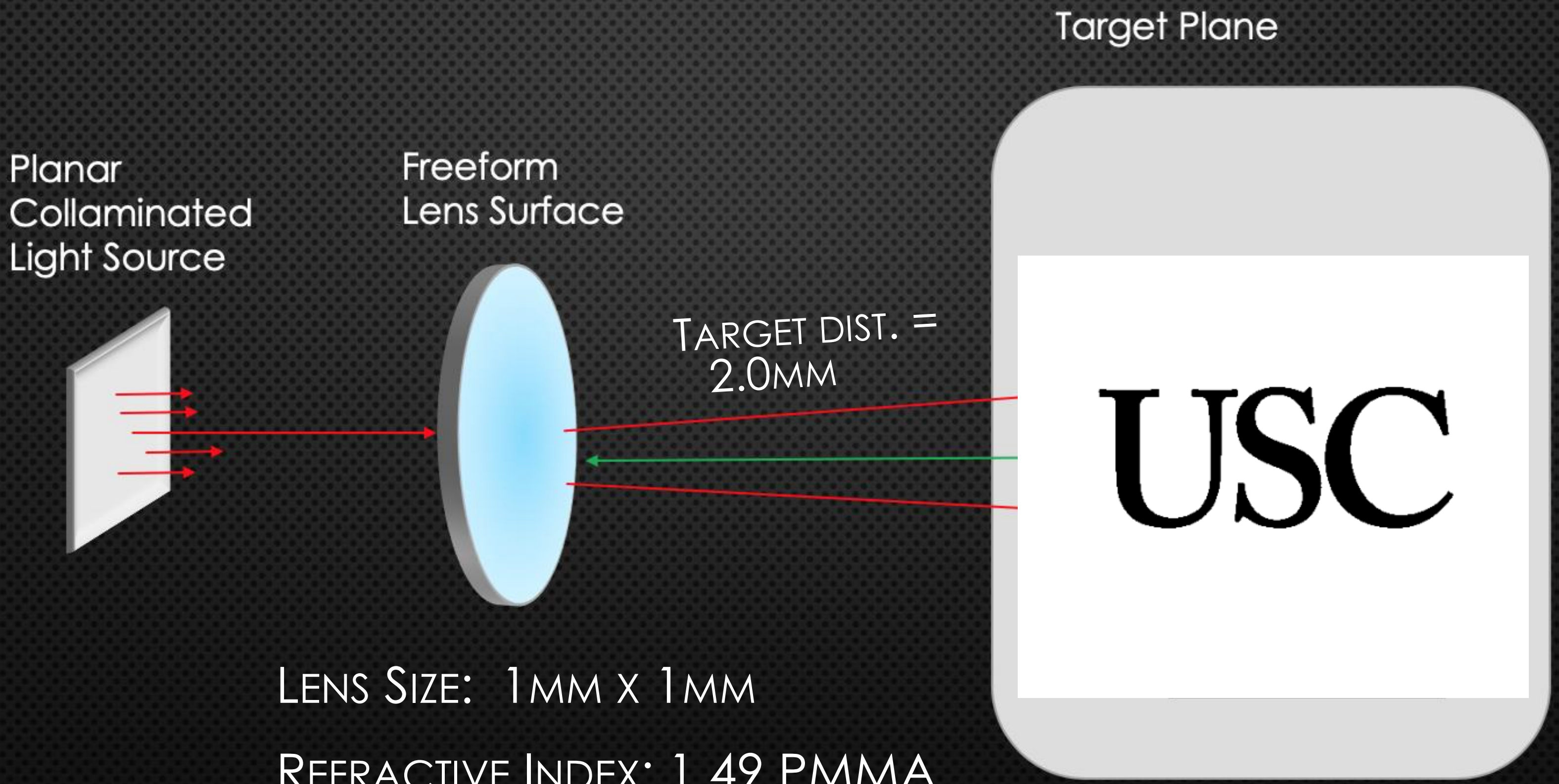
```
%Turn this on if you wish to add noise to the target image:
target_img = target_img + 0.1;
target_img(target_img>1) = 1.0;
imshow(target_img)
```

IMPLEMENTATION



IMPLEMENTATION

RESULTS:



LENS SIZE: 1MM X 1MM

REFRACTIVE INDEX: 1.49 PMMA

LENS SIZE: 20MM X 20MM

Next we define our hyperparameters:

stepsize:

```
global h;
h = 1;
```

lens surface sample dimensions:

```
global sampleSize;
sampleSize = 99;
global stepSizeH;
%stepSizeH = 1;
stepSizeH = double(0.01);
```

targetDistance

```
global zT;
%zT = sampleSize*2;
zT = 2.0;
%zT = 20;
```

targetSize

```
global Tmax;
Tmax = 10;
%Tmax = 500;
```

Refractive Indicies:

```
refractiveIndexIn = 1.49; %PMMA
refractiveIndexOut = 1.0; %Vacuum
global n;
n = refractiveIndexIn/refractiveIndexOut;
```

IMPLEMENTATION

```
global i1;
global j1;

for itr = 1: total_itr
    zLensPrev = zLens;
    for i1 = 2:sampleSize-1
        for j1 = 2:sampleSize-1

            av1 = avfunc(zLens(i1+1,j1), zLens(i1-1,j1));
            av2 = avfunc(zLens(i1,j1+1), zLens(i1,j1-1));
            av3 = avfunc(zLens(i1+1,j1+1), zLens(i1-1,j1-1));
            av4 = avfunc(zLens(i1-1,j1+1), zLens(i1+1,j1-1));
            g = gfunc(i1,j1);
            zLens(i1,j1) = 0.5*(av1 + av2 - sqrt( ( (av1-av2)^2 ) + (((av3-av4)*0.5)^2) + (stepSizeH^4) * (g) )  );
        end
    end
    stabilityContainer(itr) = stabilityfunc(zLensPrev, zLens);
end
```

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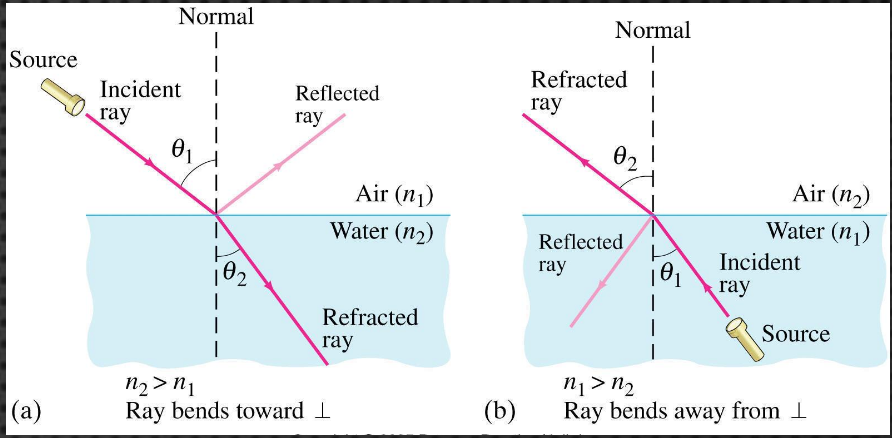
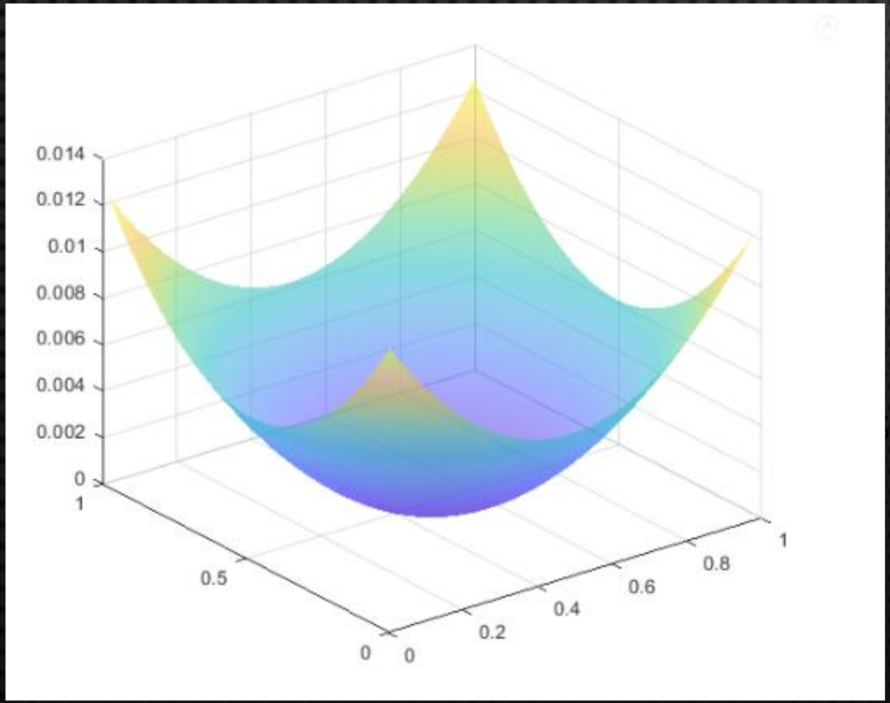
IMPLEMENTATION

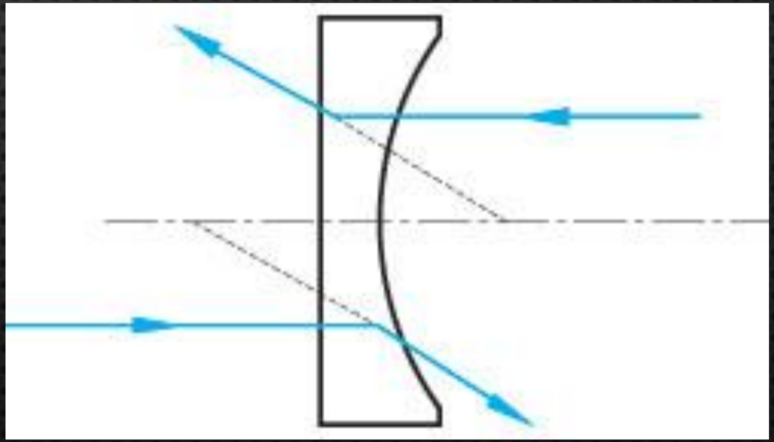
Lens Size = 1mm x 1mm | | | D0 1mm

Target distance Zt= 2.0mm

Target max dimensions = 20mm x 20mm | Tmax = 10mm







Snell's Law:

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IMPLEMENTATION

LENS SIZE = 1MM X 1MM | | | D0 1MM

TARGET DISTANCE ZT= 2.0MM

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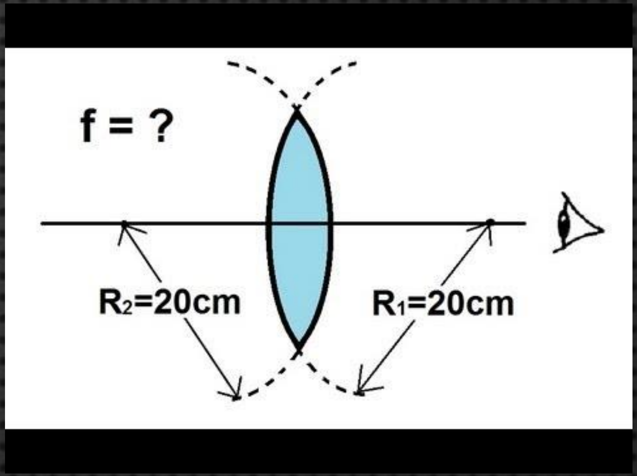
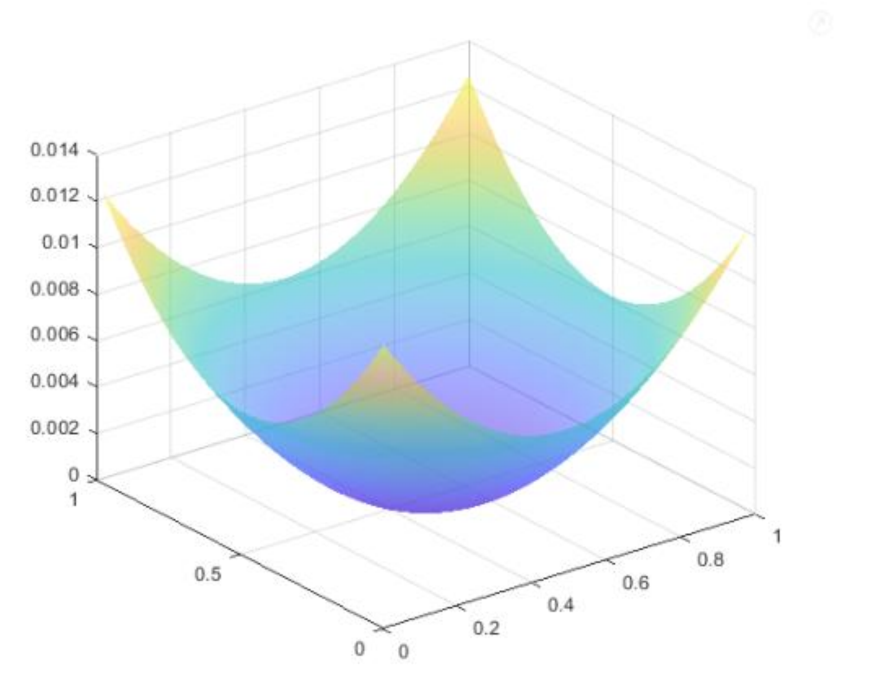
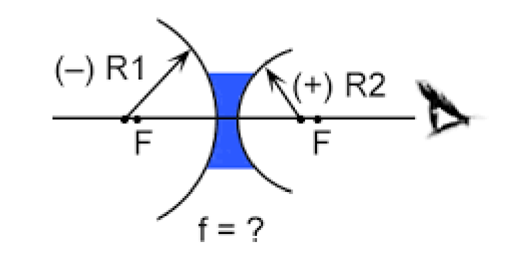






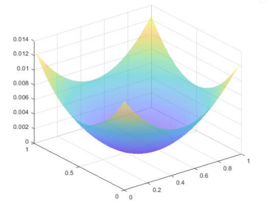
$$\frac{1}{f} \approx (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

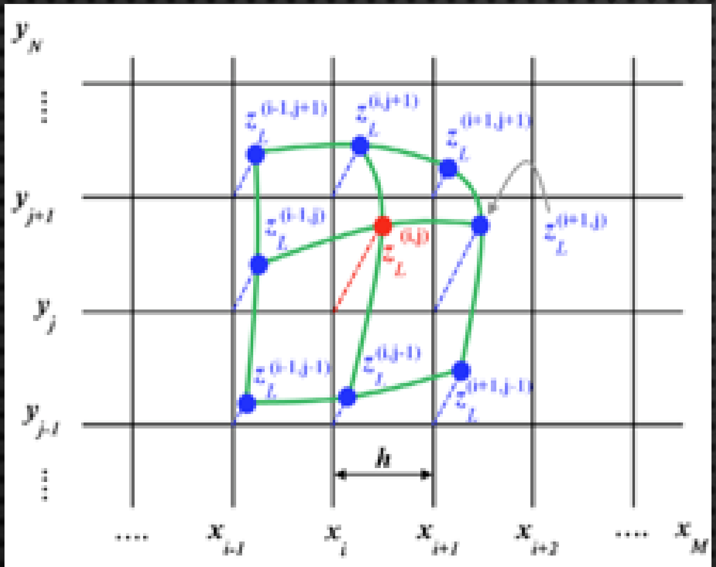
$$R_c = \frac{D_0(n-1)z_T}{2T_{\max}}$$

$$z_{L0}^{(i,j)} = \frac{x_{i,j}^2 + y_{i,j}^2}{\frac{D_0(n-1)z_T}{2T_{\max}} + \sqrt{\left(\frac{D_0(n-1)z_T}{2T_{\max}}\right)^2 - (x_{i,j}^2 + y_{i,j}^2)}}$$

IMPLEMENTATION

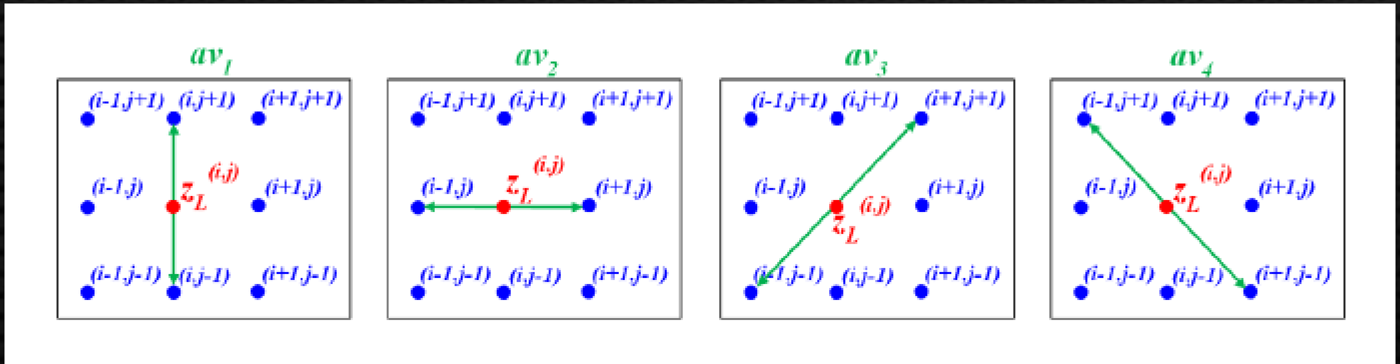
$$z_L^{(i,j)} = \frac{av_1^{(i,j)} + av_2^{(i,j)} - \sqrt{\left(av_1^{(i,j)} - av_2^{(i,j)}\right)^2 + \left(\frac{av_3^{(i,j)} - av_4^{(i,j)}}{2}\right)^2 + h^4 g^{(i,j)}}}{2}$$





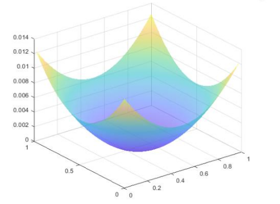
Average:

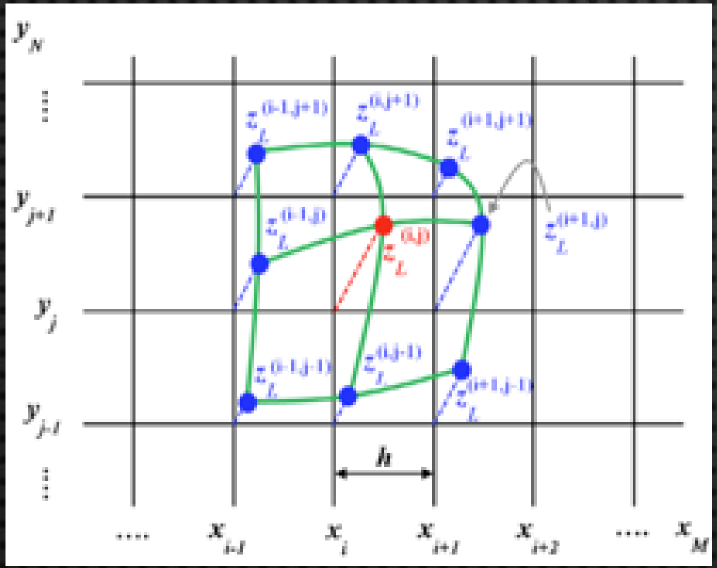
```
function avCalc= avfunc(input1, input2)
avCalc = double(input1 + input2)/2;
end
```



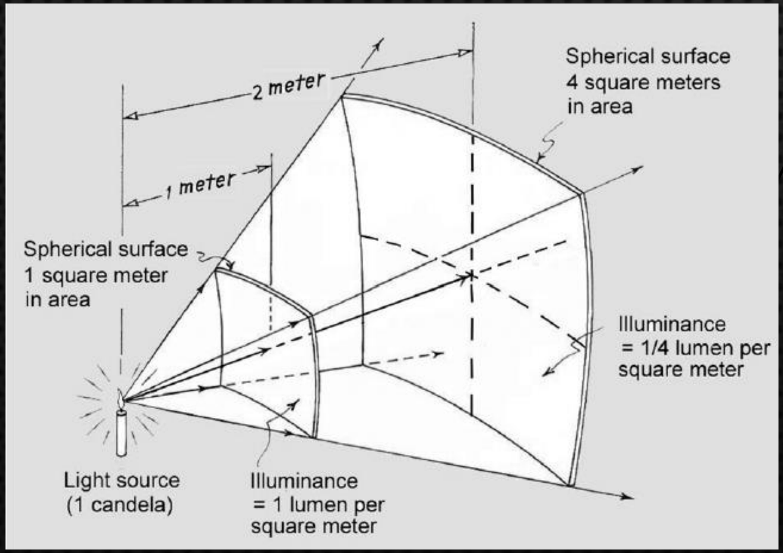
IMPLEMENTATION

$$z_L^{(i,j)} = \frac{av_1^{(i,j)} + av_2^{(i,j)} - \sqrt{\left(av_1^{(i,j)} - av_2^{(i,j)}\right)^2 + \left(\frac{av_3^{(i,j)} - av_4^{(i,j)}}{2}\right)^2 + h^4 g^{(i,j)}}}{2}$$

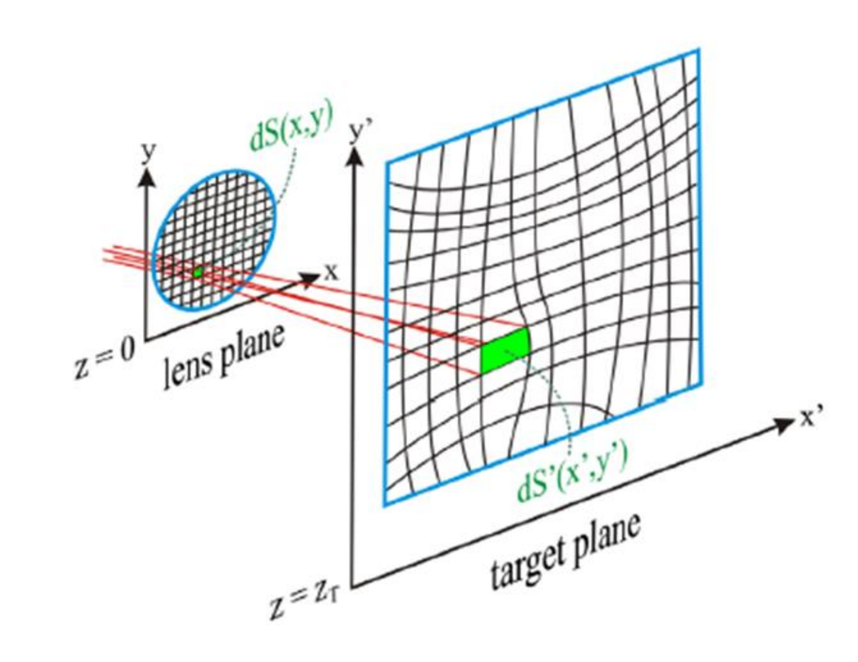




$$g^{(i,j)} = \frac{I_0(x_i, y_i)}{I_T(f_1(x_i, y_i), f_2(x_i, y_i))} \frac{\sqrt{1 + \left(z_{Lx}^{(i,j)2} + z_{Ly}^{(i,j)2}\right)\left(1 - n^2\right)}}{K(z_{Lx}^{(i,j)}, z_{Ly}^{(i,j)})^2}$$



MATRIX



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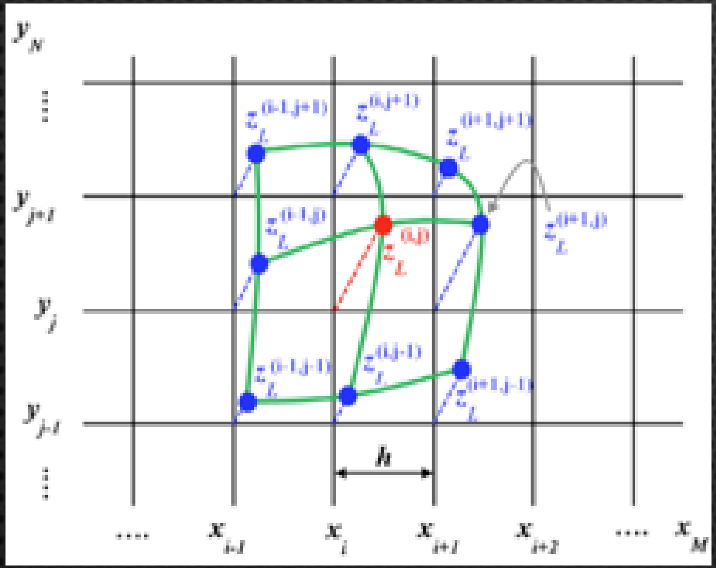
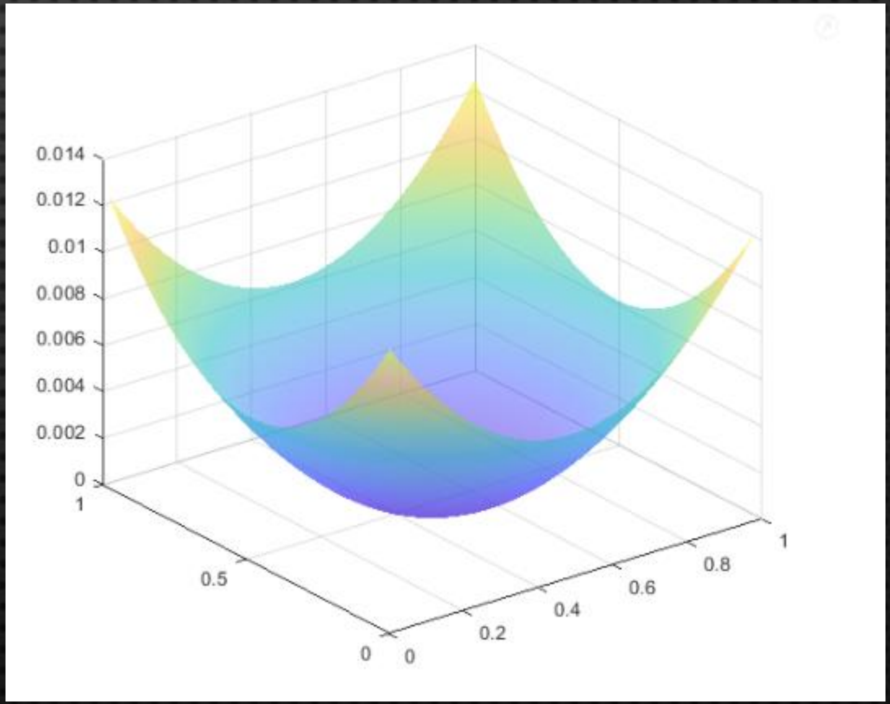
IMPLEMENTATION



$$z_L^{(i,j)} = \frac{av_1^{(i,j)} + av_2^{(i,j)} - \sqrt{\left(av_1^{(i,j)} - av_2^{(i,j)}\right)^2 + \left(\frac{av_3^{(i,j)} - av_4^{(i,j)}}{2}\right)^2 + h^4 g^{(i,j)}}}{2}$$



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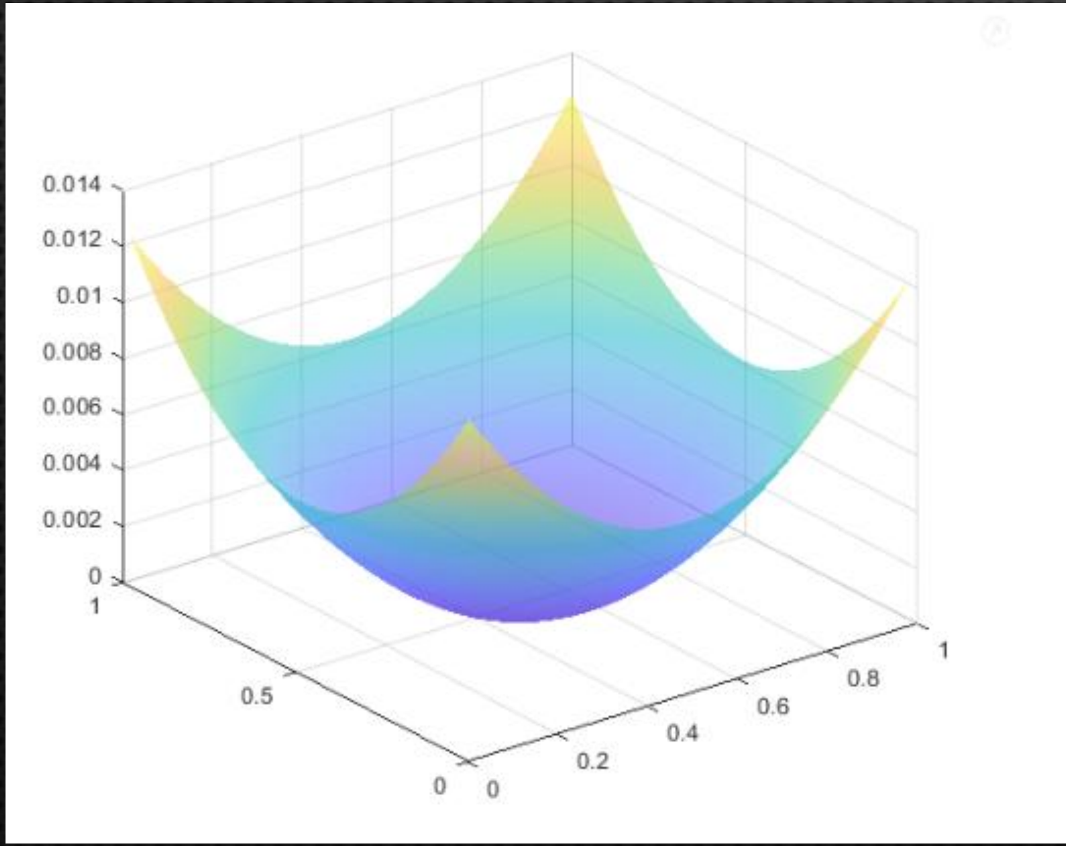
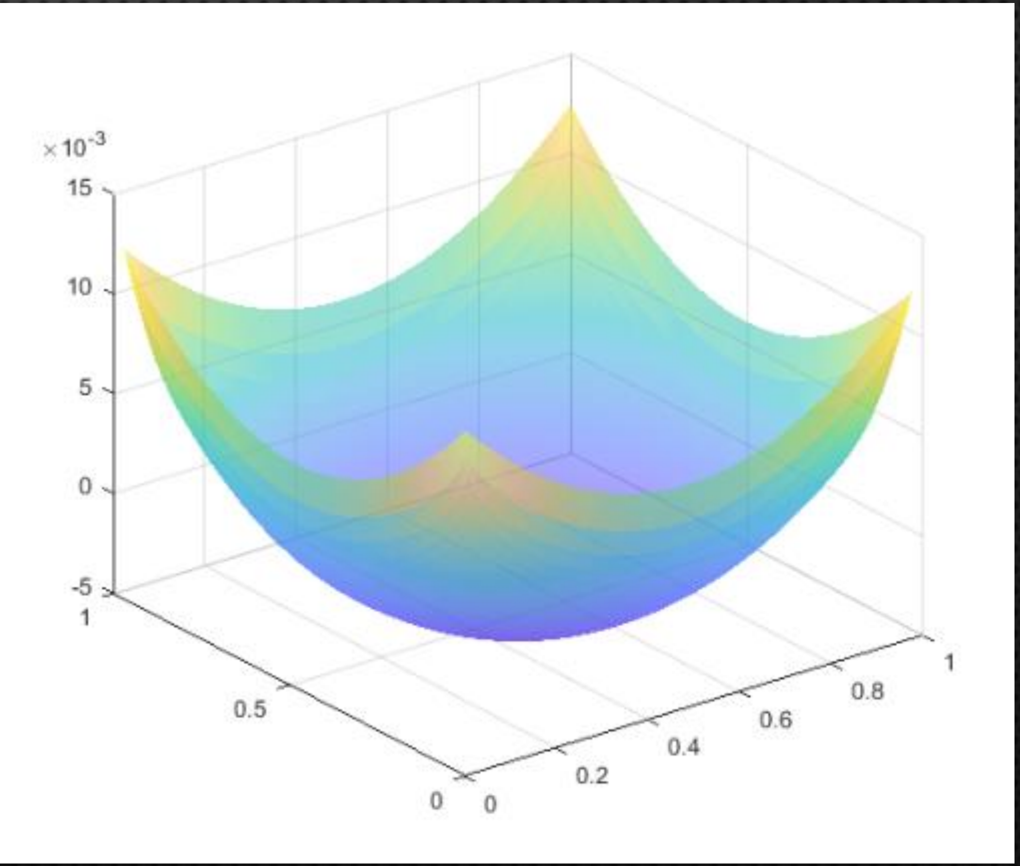
IMPLEMENTATION

Initial Lens:



Lens after 25 Iterations:



IMPLEMENTATION

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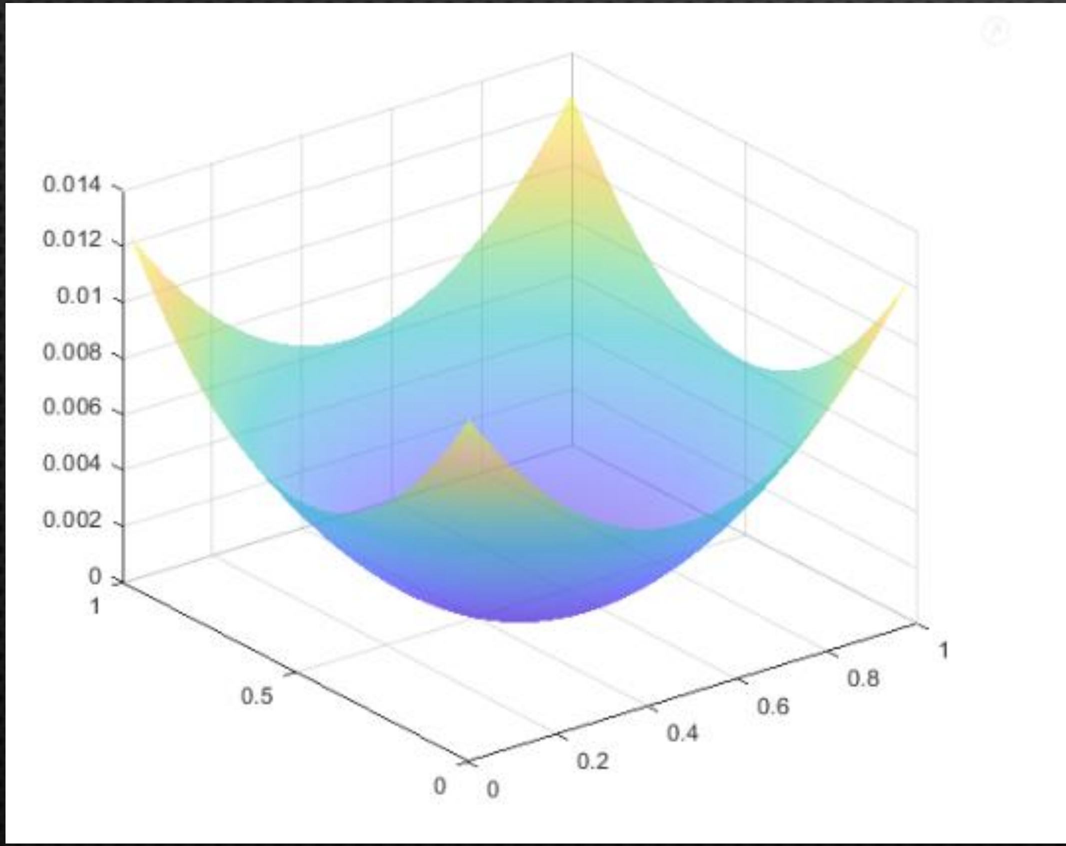
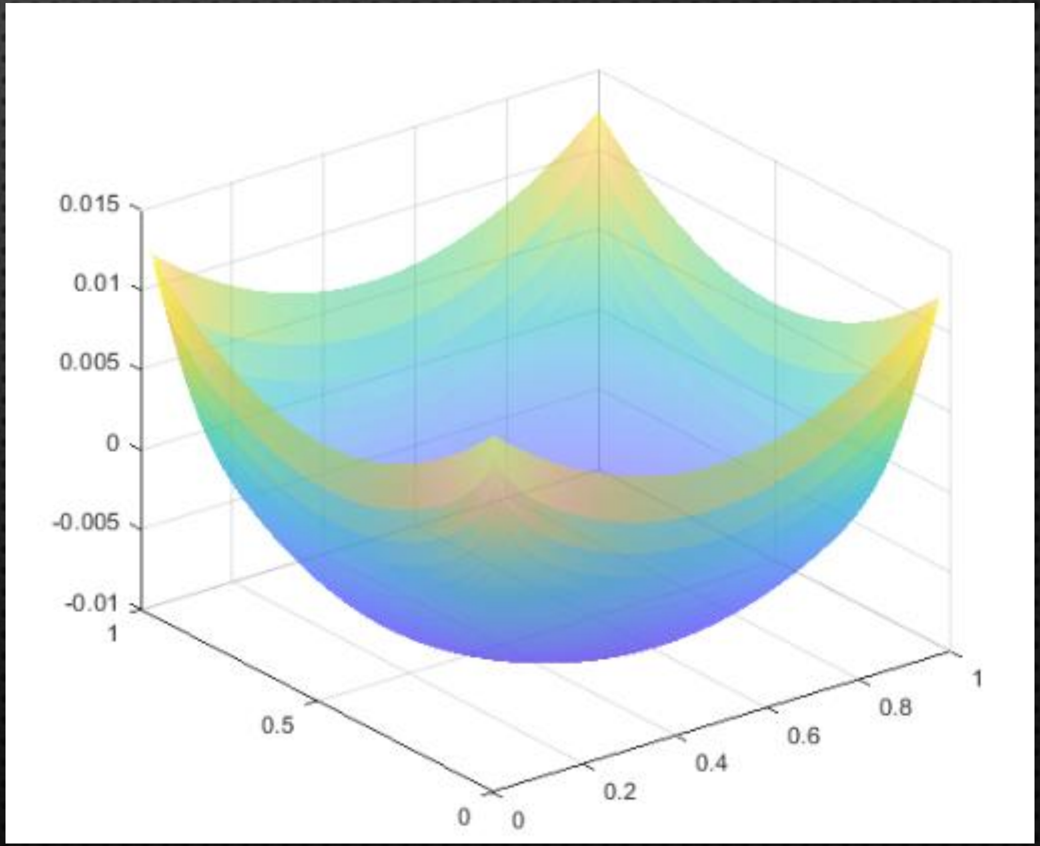


Lens after 50 Iterations:



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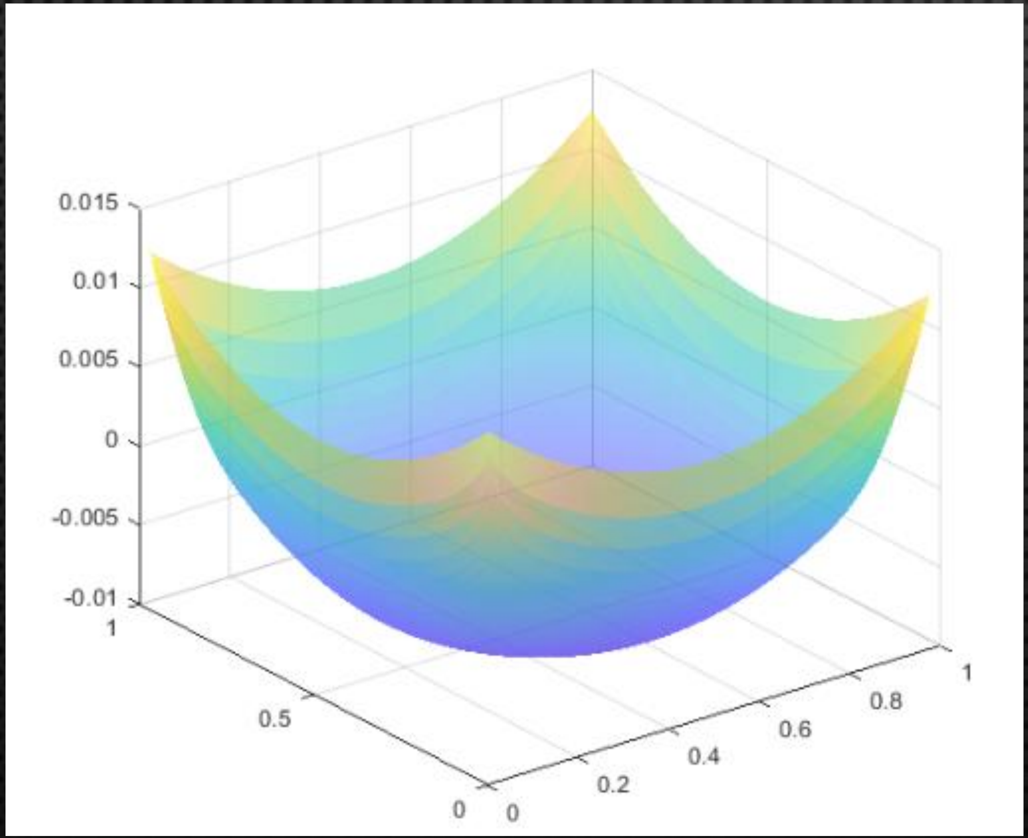
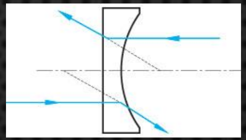
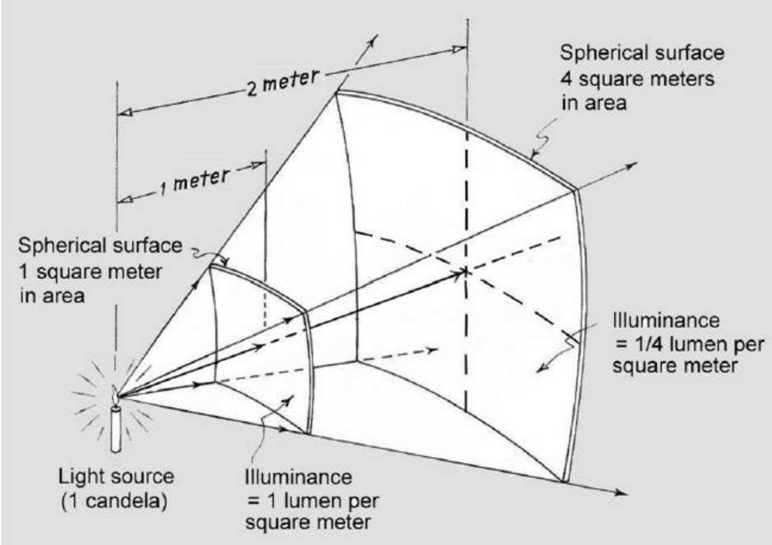
IMPLEMENTATION





IMPLEMENTATION - Challenges



$$z_L^{(i,j)} = \frac{av_1^{(i,j)} + av_2^{(i,j)} - \sqrt{\left(av_1^{(i,j)} - av_2^{(i,j)}\right)^2 + \left(\frac{av_3^{(i,j)} - av_4^{(i,j)}}{2}\right)^2 + h^4 g^{(i,j)}}}{2}$$

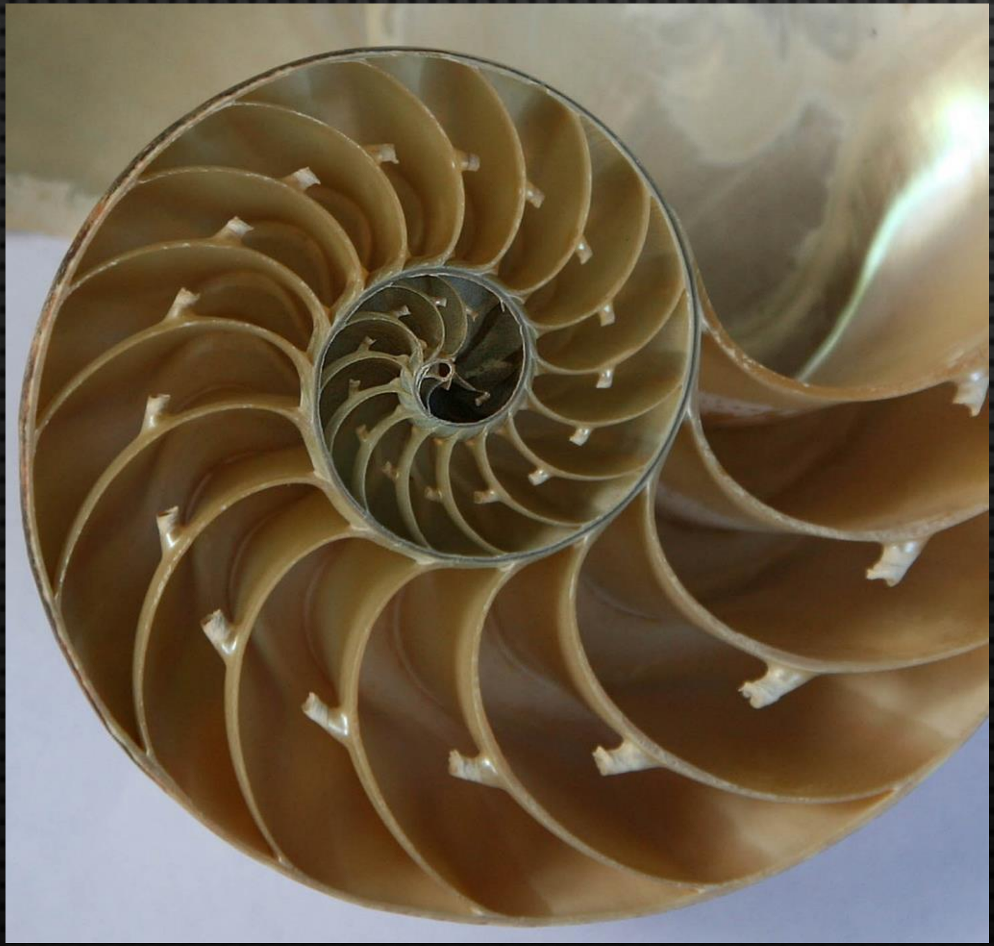
$$g^{(i,j)} = \frac{I_0(x_i, y_i)}{I_T(f_1(x_i, y_i), f_2(x_i, y_i))} \frac{\sqrt{1 + \left(z_{Lx}^{(i,j)2} + z_{Ly}^{(i,j)2}\right)(1 - n^2)}}{K(z_{Lx}^{(i,j)}, z_{Ly}^{(i,j)2})}$$

$$I_T(x', y') \cdot |J(x, y)| = I(x, y)$$

Matrix values 1.0

IMPLEMENTATION

IMPLEMENTATION



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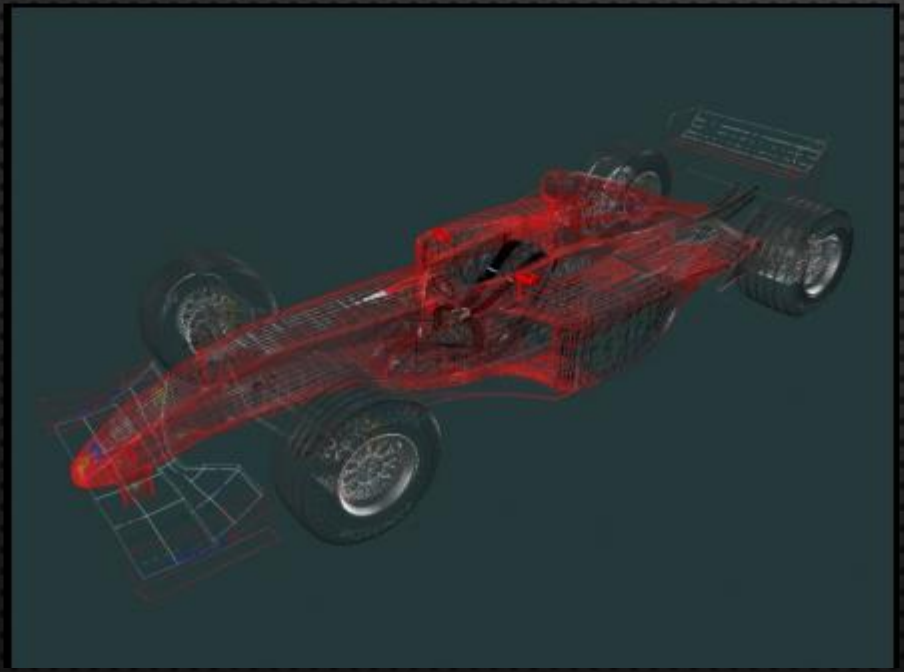
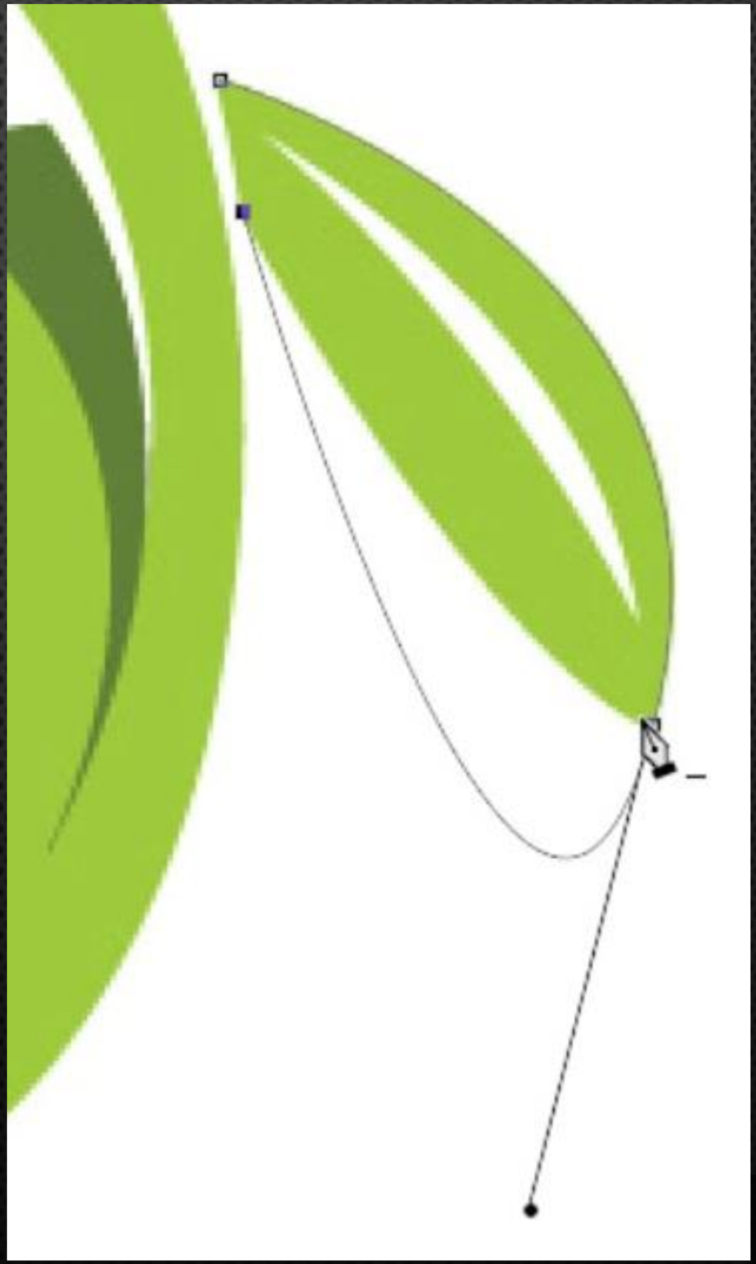
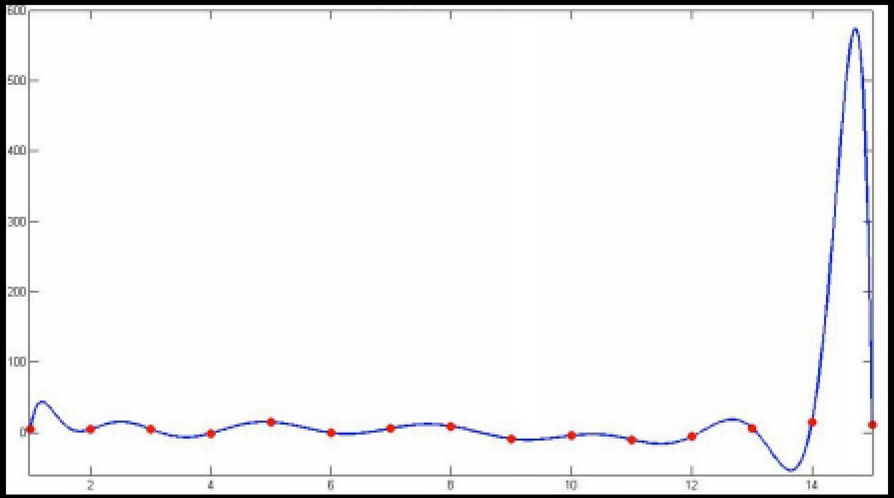
IMPLEMENTATION SPLINES





IMPLEMENTATION SPLINES

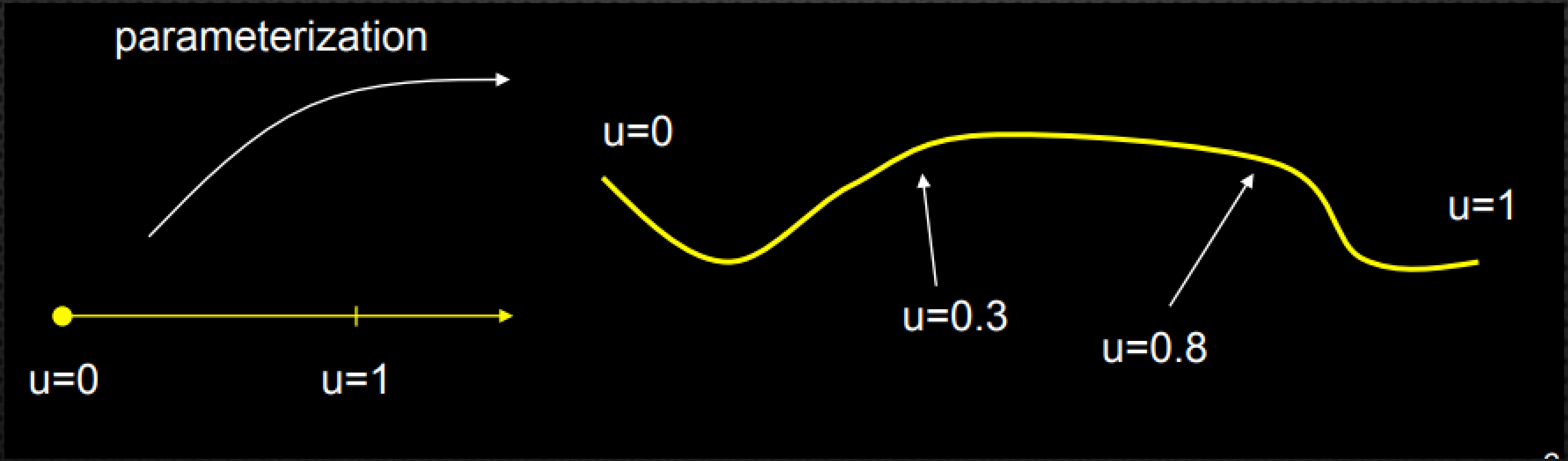
- An *n*-th degree polynomial fits a curve to n+1 points
  - called Lagrange Interpolation
  - result is a curve that is too wiggly, change to any control point affects entire curve (non-local)
  - *this method is poor*
- We usually want the curve to be as smooth as possible
  - minimize the wiggles
  - high-degree polynomials are bad



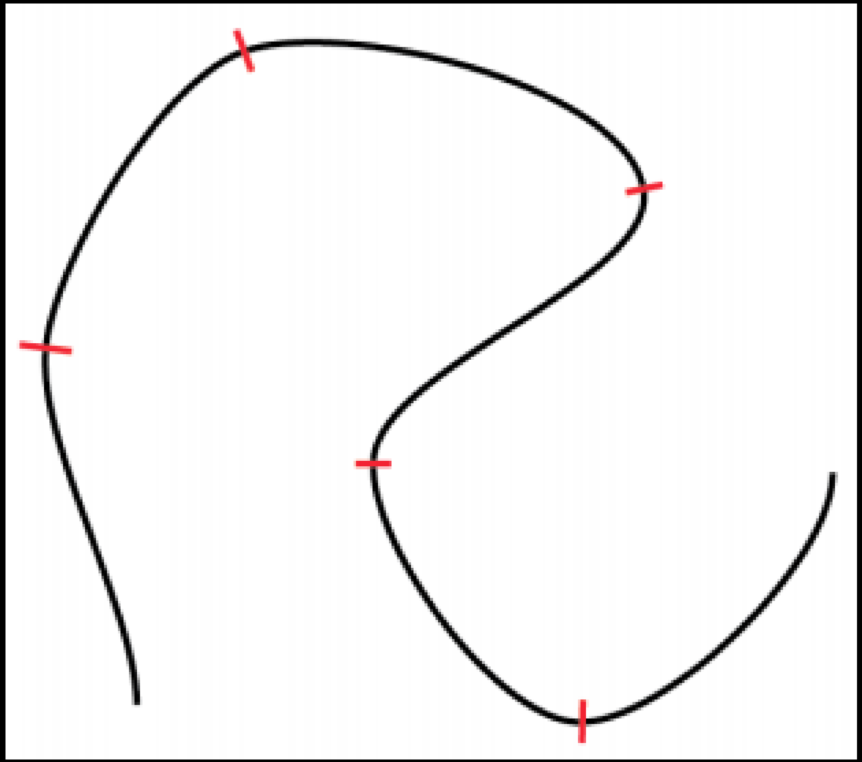
source: Wikipedia

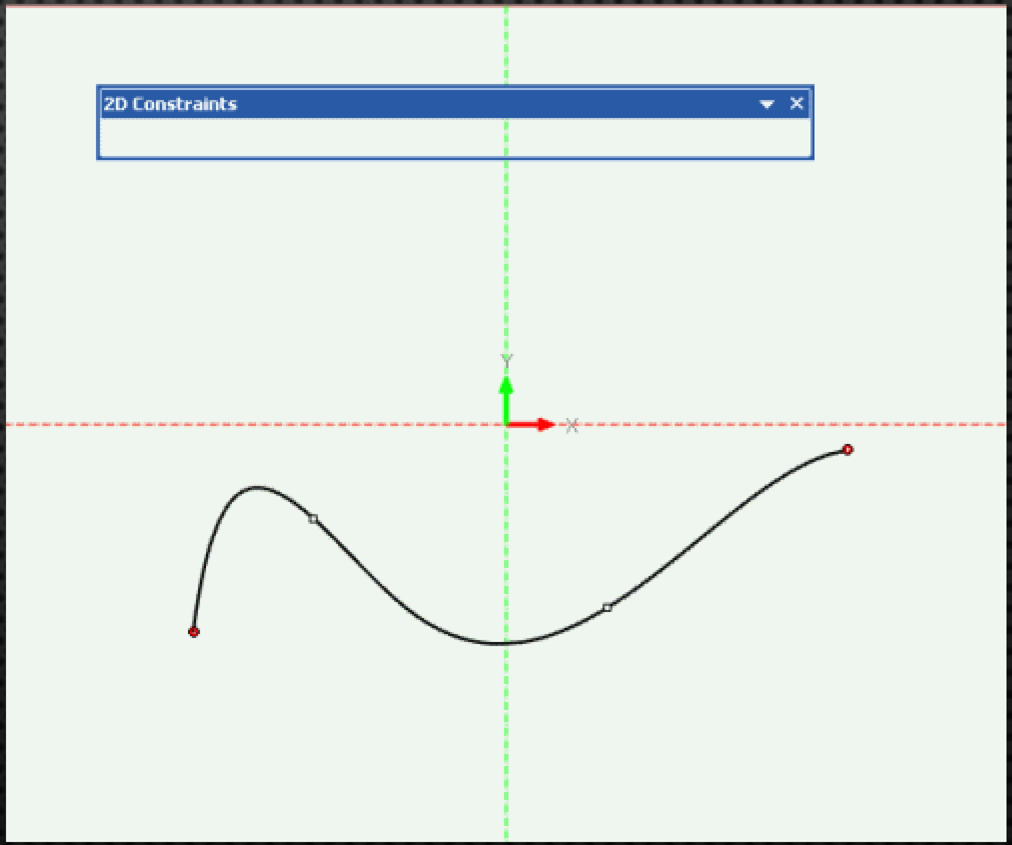
Lagrange interpolation, degree=15

IMPLEMENTATION SPLINES



IMPLEMENTATION SPLINES


a spline



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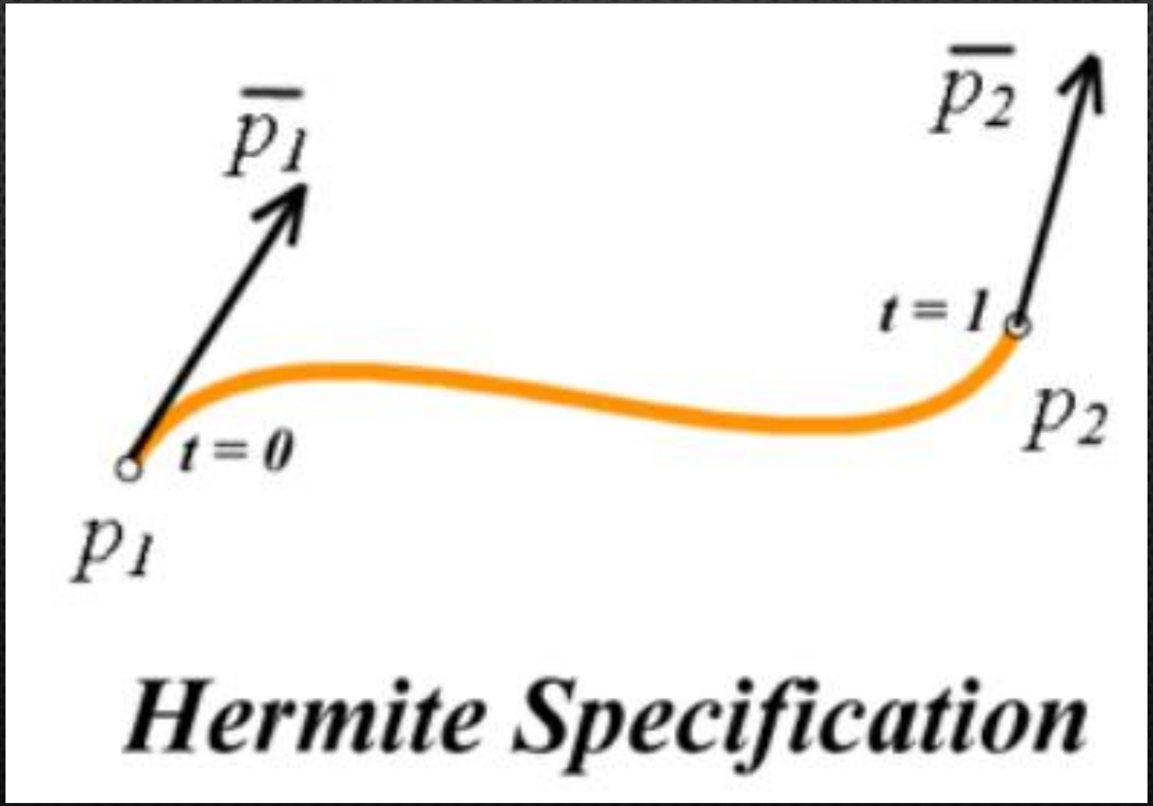
IMPLEMENTATION SPLINES

- Cubic polynomial:
  - $p(u) = au^3+bu^2+cu+d = [u^3 \;\; u^2 \;\; u \;\; 1]\,[a \;\; b \;\; c \;\; d]^T$
  - a,b,c,d are 3-vectors, u is a scalar
- Three cubic polynomials, one for each coordinate:
  - $x(u) = a_x u^3 + b_x u^2 + c_x u + d_x$
  - $y(u) = a_y u^3 + b_y u^2 + c_y u + d_y$
  - $z(u) = a_z u^3 + b_z u^2 + c_z u + d_z$
- In matrix notation:

$$\begin{bmatrix} x(u) & y(u) & z(u) \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$$

- Or simply: $p = [u^3 \; u^2 \; u \; 1]\, A$

IMPLEMENTATION – Hermite SPLINES



IMPLEMENTATION – Hermite SPLINES

- Four constraints: value and slope (in 3-D, position and tangent vector) at beginning and end of interval [0,1] :

$$p(0) = p_1 = (x_1, y_1, z_1)$$
$$p(1) = p_2 = (x_2, y_2, z_2)$$
$$p'(0) = \bar{p}_1 = (\bar{x}_1, \bar{y}_1, \bar{z}_1)$$
$$p'(1) = \bar{p}_2 = (\bar{x}_2, \bar{y}_2, \bar{z}_2)$$

the user constraints

- Assume cubic form: $p(u) = au^3 + bu^2 + cu + d$
- Four unknowns: a, b, c, d

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IMPLEMENTATION – Hermite SPLINES

- Assume cubic form: $p(u) = au^3 + bu^2 + cu + d$

$$p_1 = p(0) = d$$

$$p_2 = p(1) = a + b + c + d$$

$$\overline{p_1} = p'(0) = c$$

$$\overline{p_2} = p'(1) = 3a + 2b + c$$

- Linear system: 12 equations for 12 unknowns
(however, can be simplified to 4 equations for 4 unknowns)
- Unknowns: *a, b, c, d* (each of *a*, *b*, *c*, *d* is a 3-vector)

IMPLEMENTATION – Hermite SPLINES

$$d = p_1$$
$$a + b + c + d = p_2$$
$$c = \overline{p}_1$$
$$3a + 2b + c = \overline{p}_2$$

Rewrite this 12x12 system as a 4x4 system:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \overline{x}_1 & \overline{y}_1 & \overline{z}_1 \\ \overline{x}_2 & \overline{y}_2 & \overline{z}_2 \end{bmatrix}$$

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IMPLEMENTATION – Hermite SPLINES

- After inverting the 4x4 matrix, we obtain:

$$\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \bar{x}_1 & \bar{y}_1 & \bar{z}_1 \\ \bar{x}_2 & \bar{y}_2 & \bar{z}_2 \end{bmatrix}$$

point on the spline — parameter vector — basis — control matrix (what the user gets to pick)

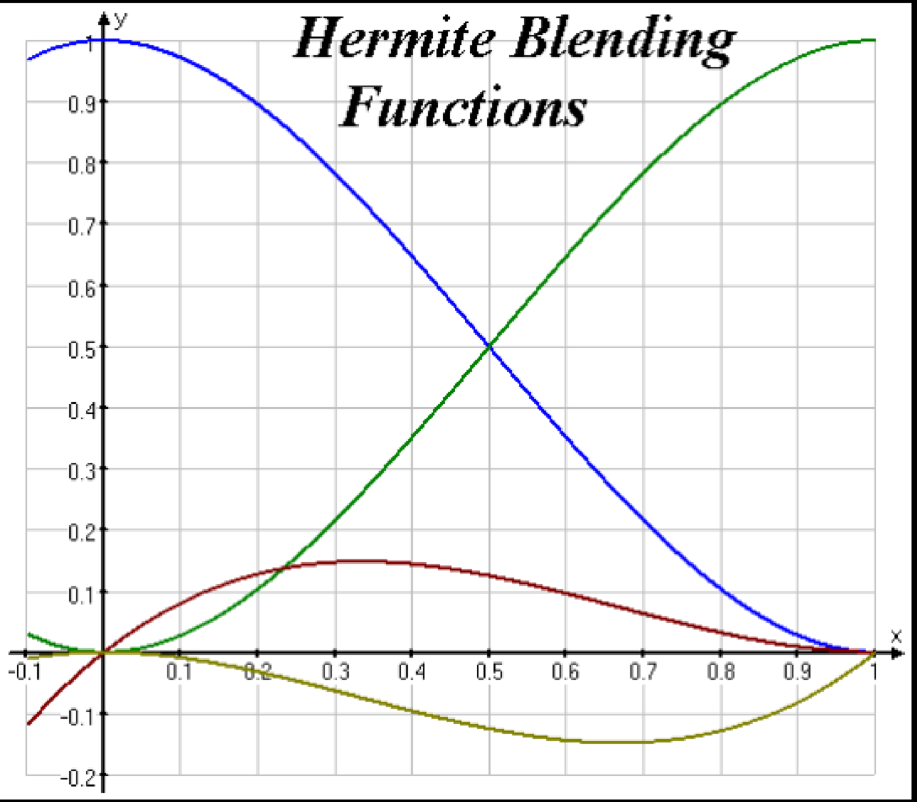
- This form is typical for splines
  - basis matrix and meaning of control matrix change with the spline type

IMPLEMENTATION – Hermite SPLINES

transpose

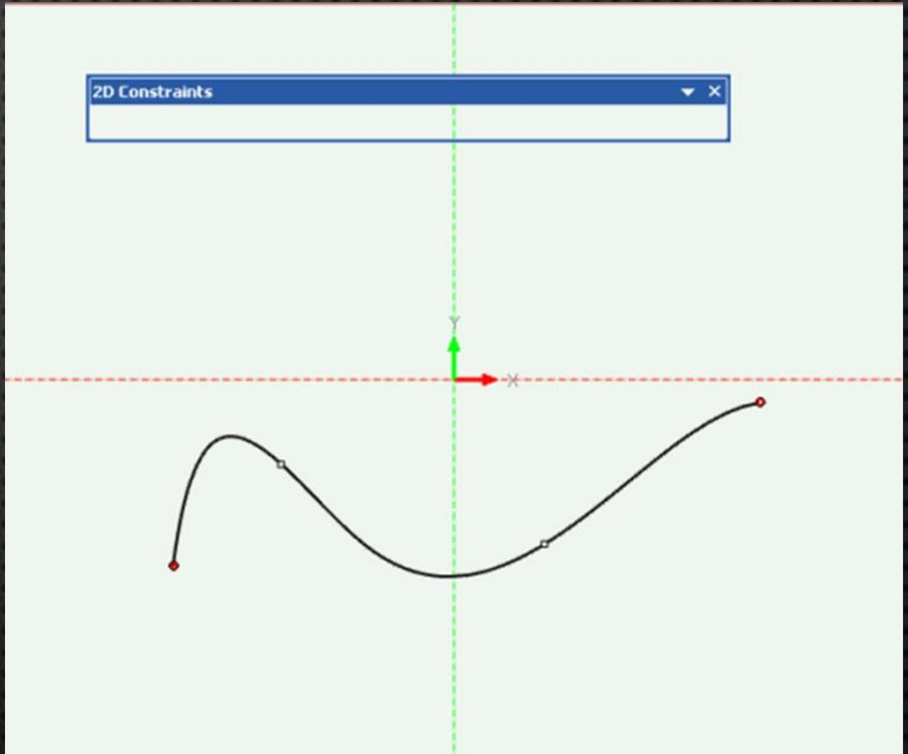
$$p(u) = \begin{bmatrix} 2u^3 - 3u^2 + 1 \\ -2u^3 + 3u^2 \\ u^3 - 2u^2 + u \\ u^3 - u^2 \end{bmatrix}^T \begin{bmatrix} p_1 \\ p_2 \\ \bar{p}_1 \\ \bar{p}_2 \end{bmatrix}$$

4 Basis Functions



Every cubic Hermite spline is a linear combination (blend) of these 4 functions.

IMPLEMENTATION – B-Splines



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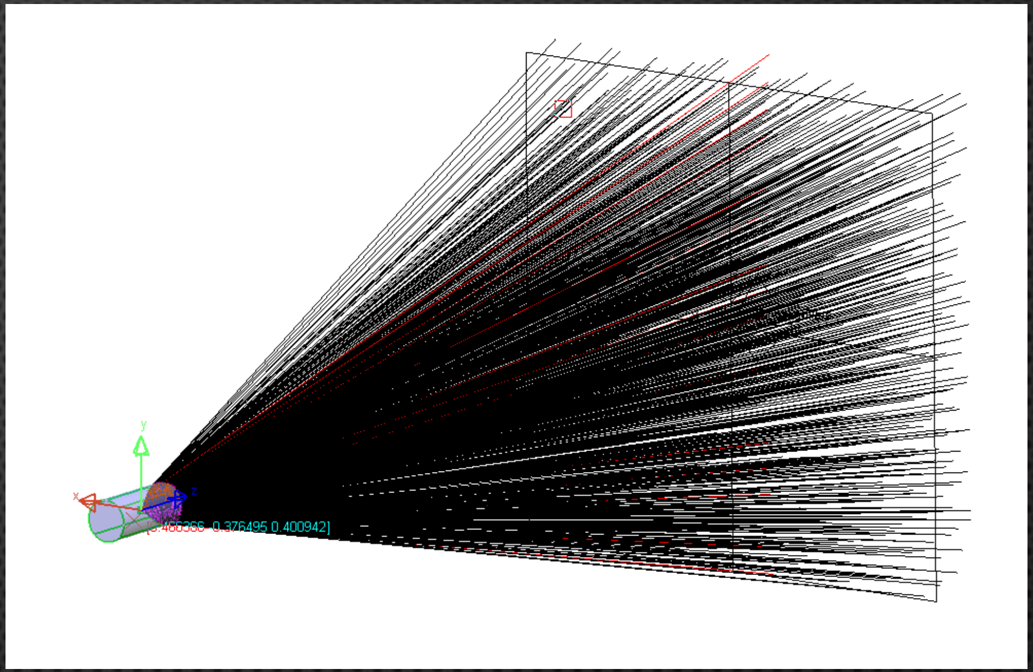
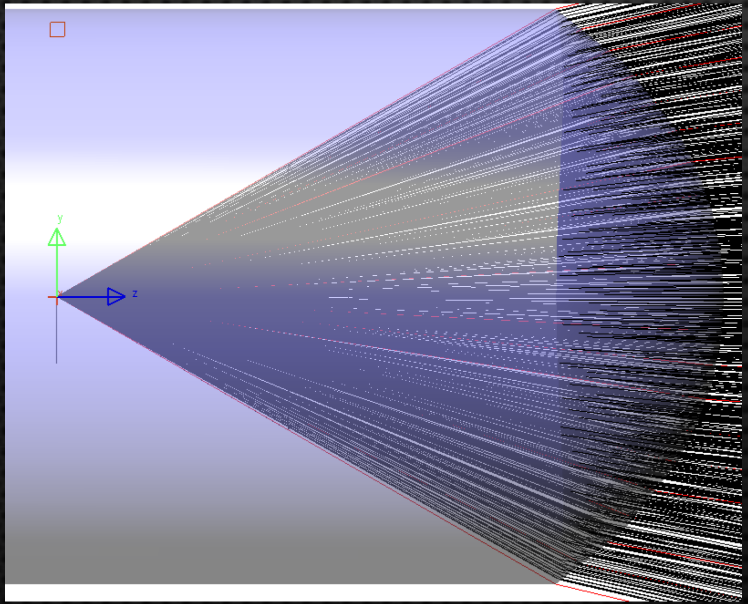
IMPLEMENTATION

LIGHT TOOLS





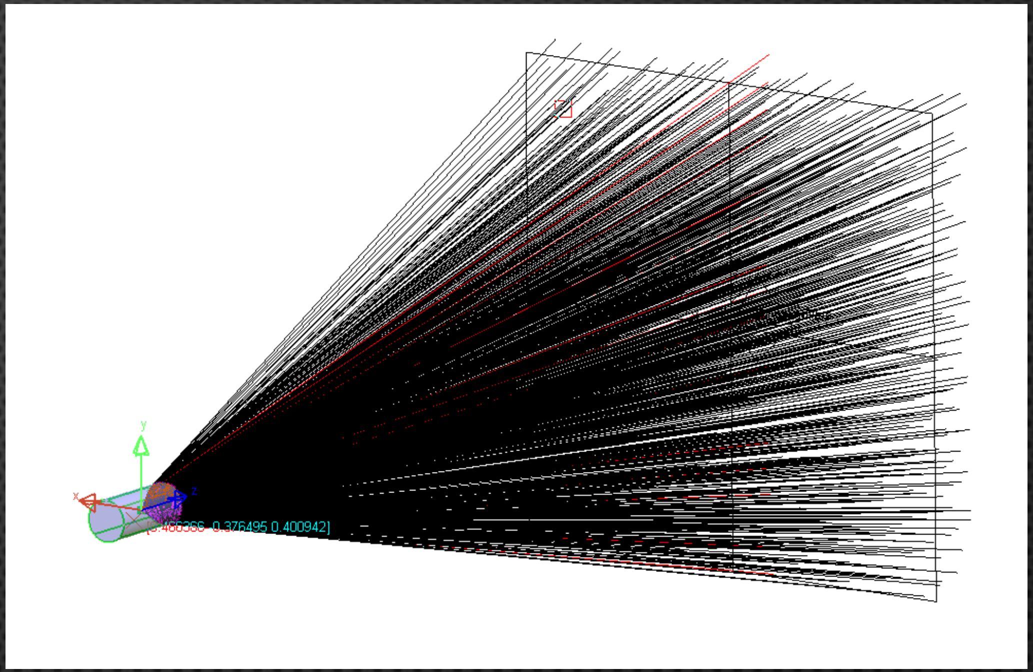
LENS SIZE = 1MM X 1MM | | | D0 1MM

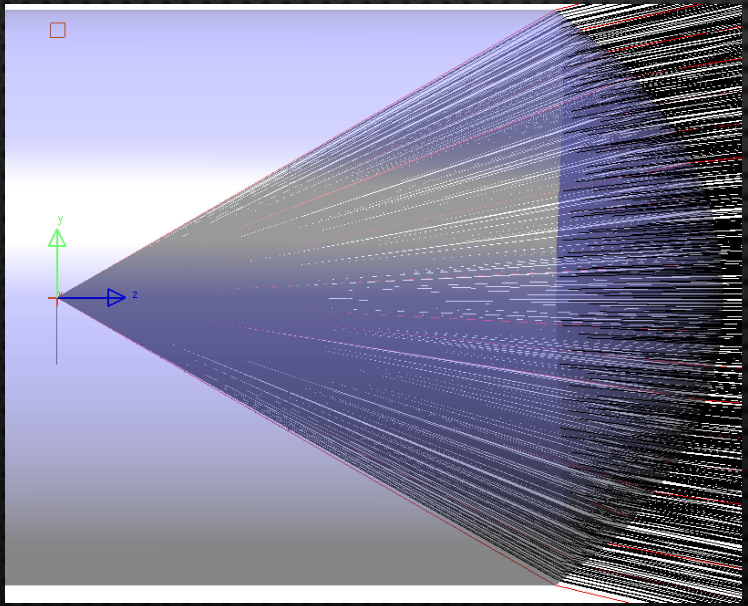
TARGET DISTANCE ZT= 2.0MM

TARGET MAX DIMENSIONS = 20MM X 20MM | TMAX = 10MM

IMPLEMENTATION

LIGHT TOOLS



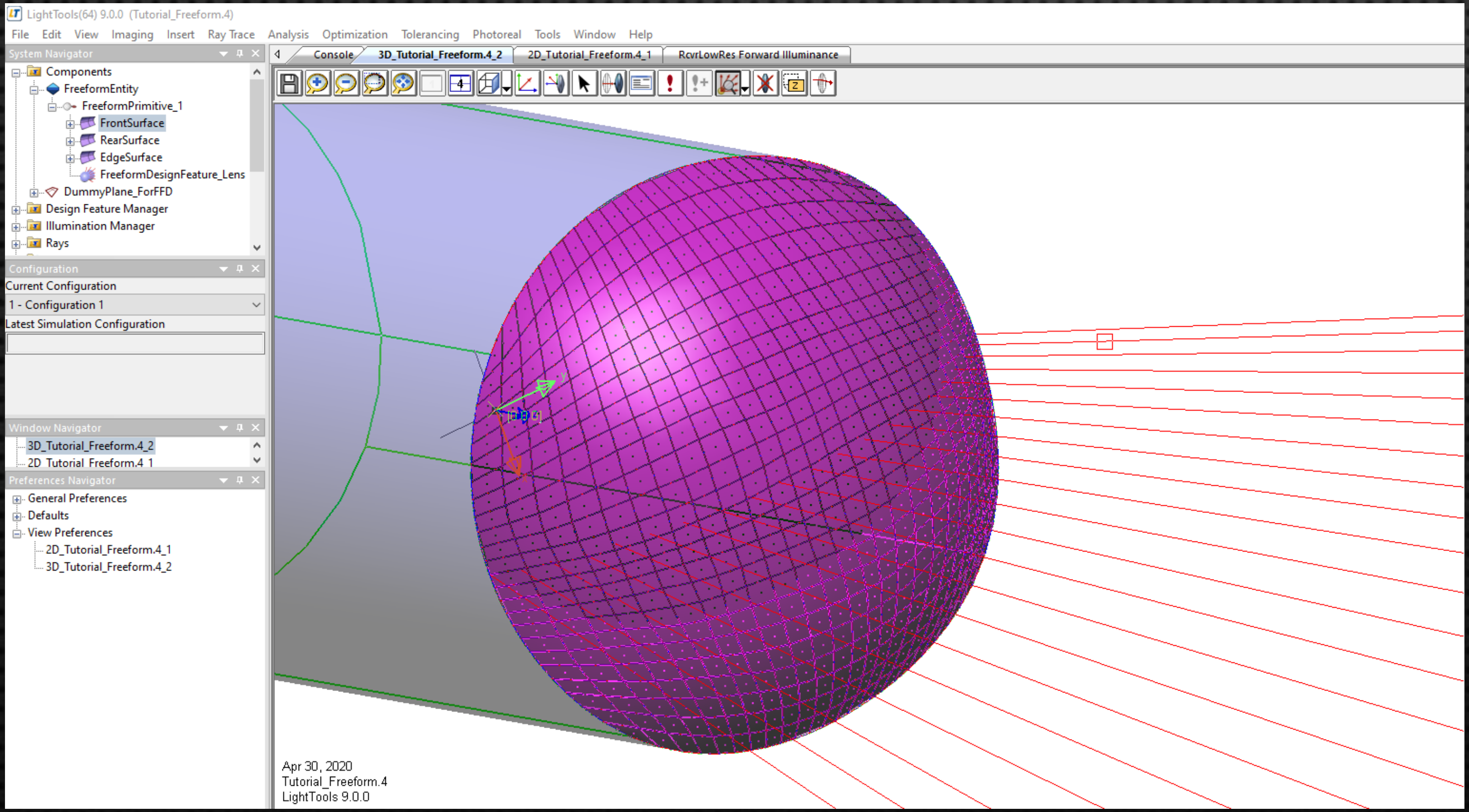


Lens Size = 1mm x 1mm | | | D0 1mm

Target distance Zt= 2.0mm

Target max dimensions = 20mm x 20mm | Tmax = 10mm

IMPLEMENTATION



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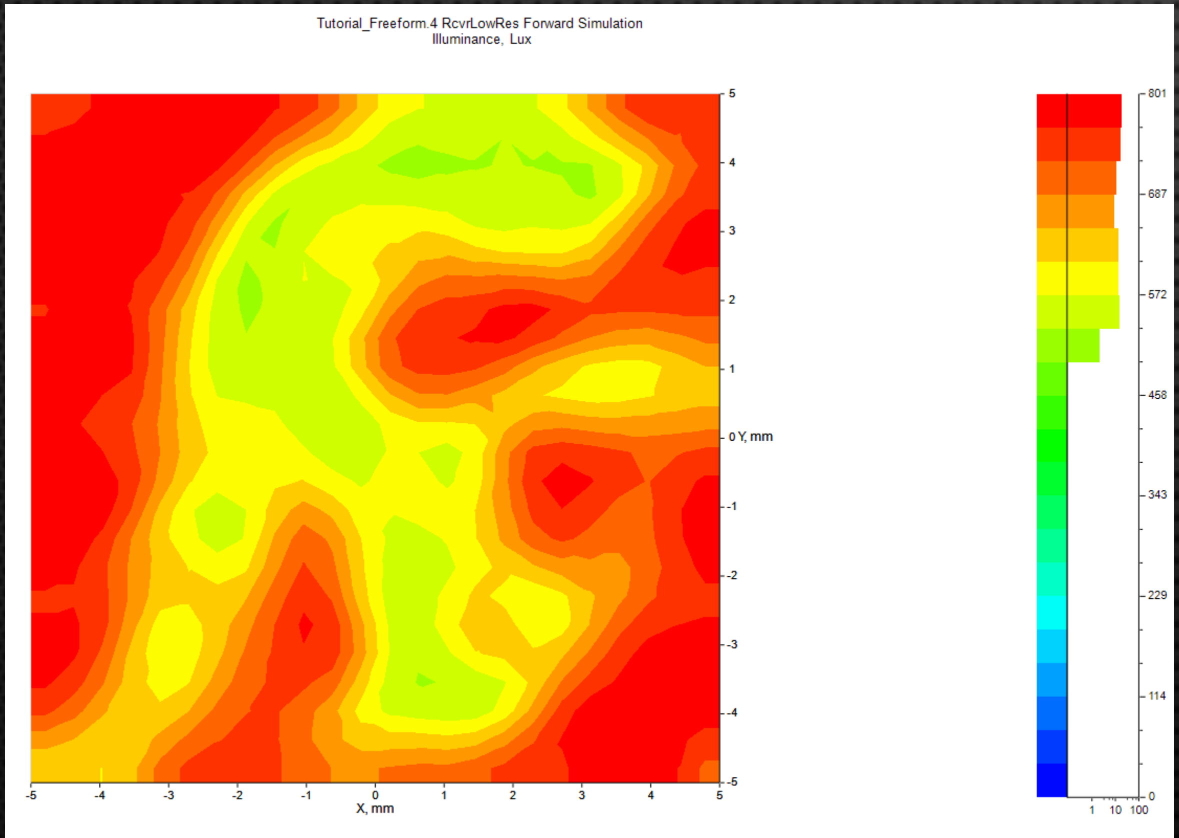
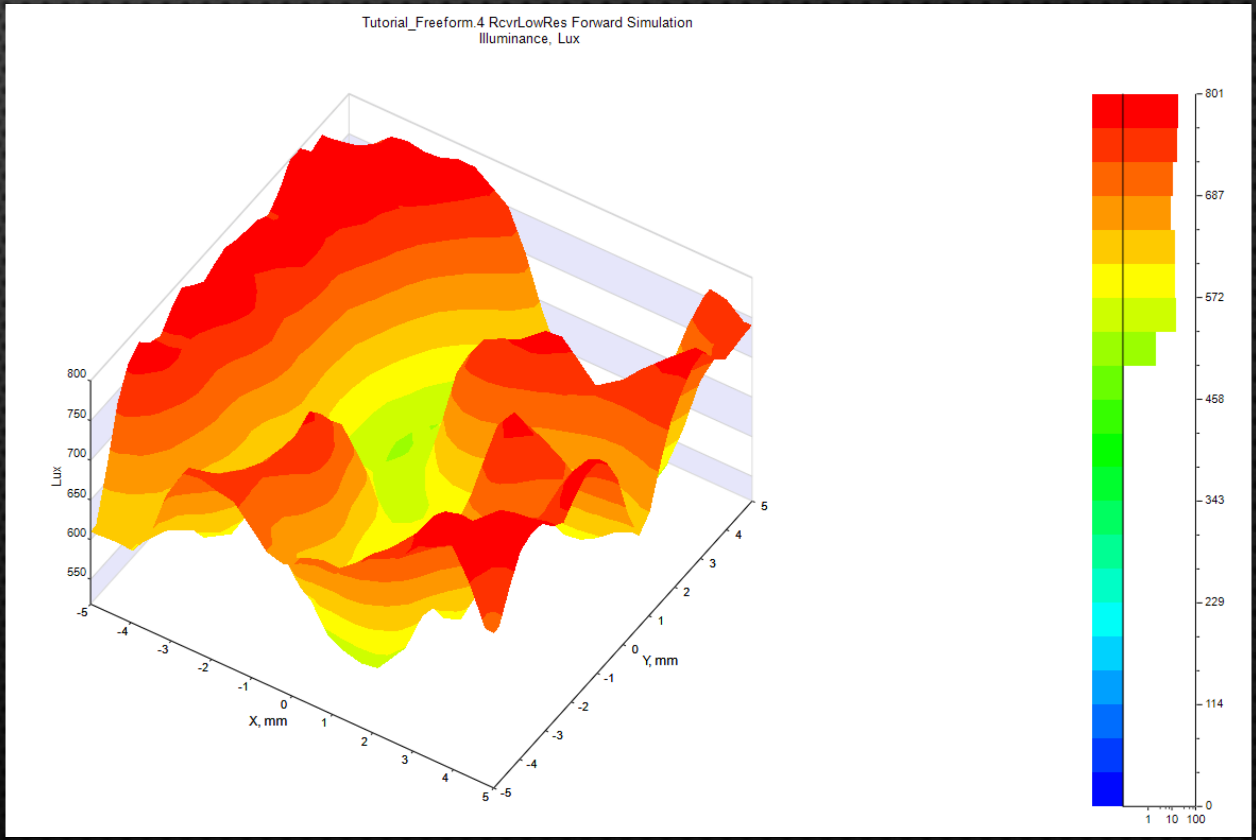
IMPLEMENTATION





Discretizing the lens surface into 25x25 samples

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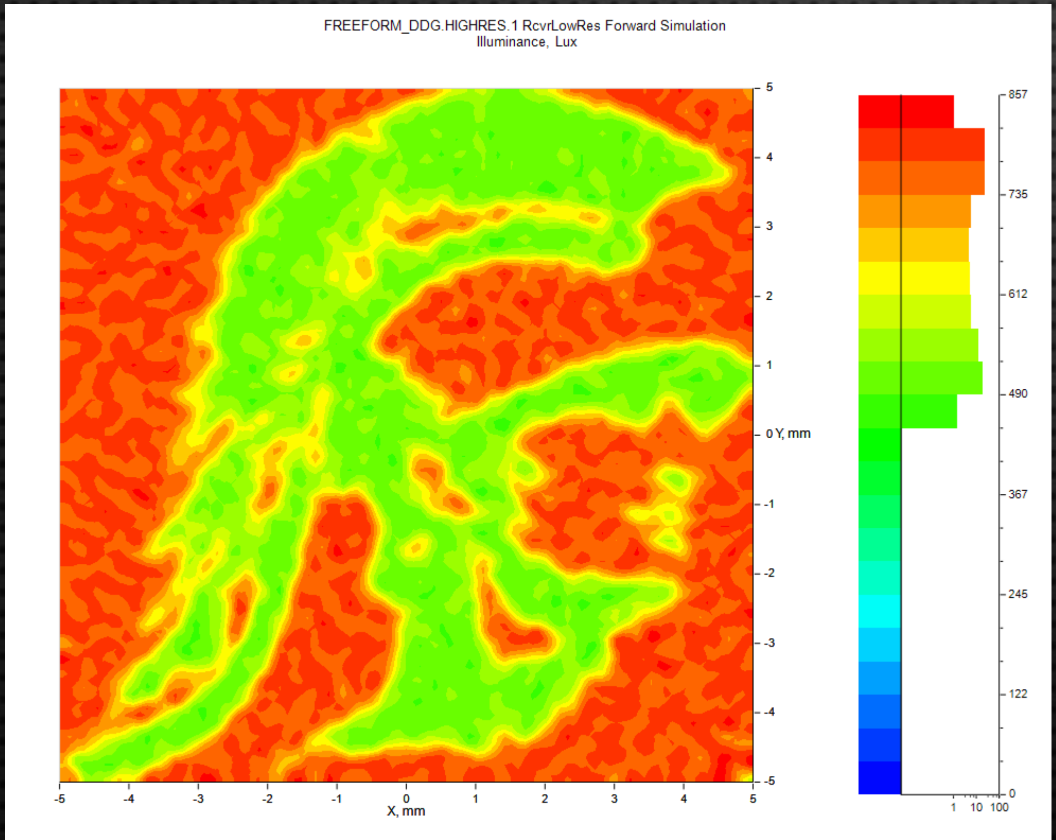
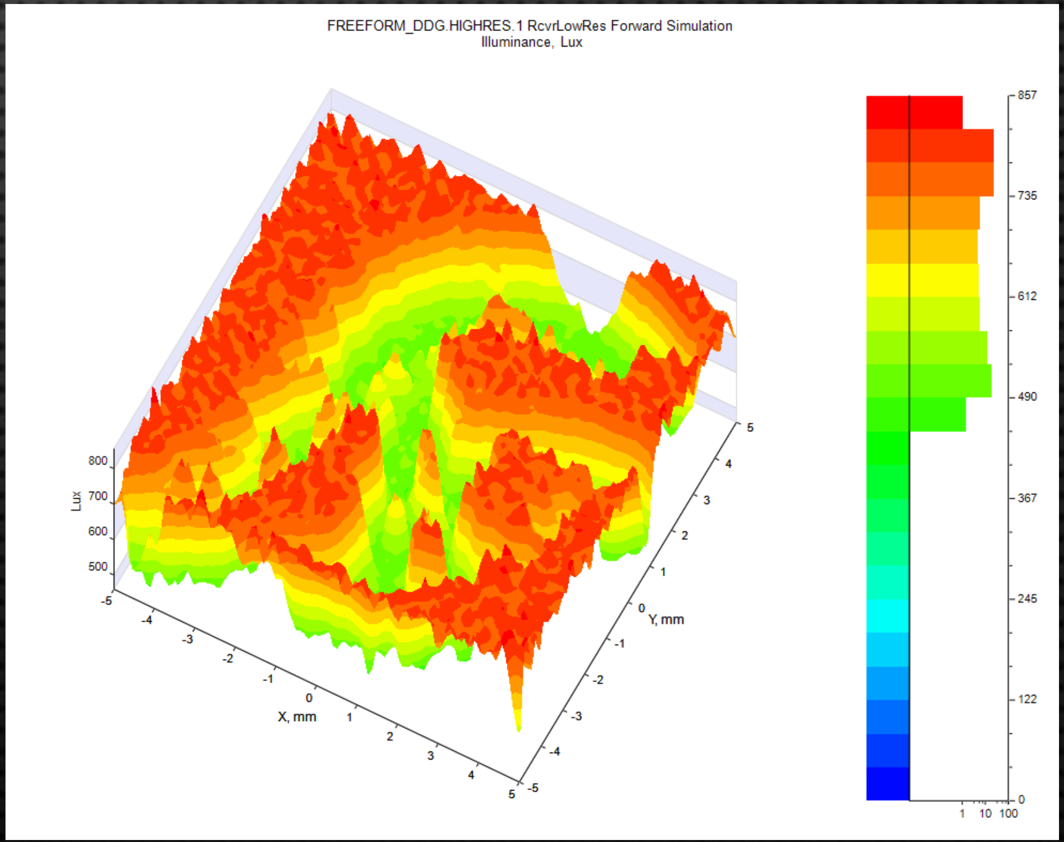
IMPLEMENTATION





Discretizing the lens surface into 100x100 samples

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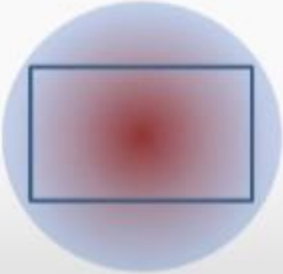
Concluding Remarks

Fact: Rotationally symmetrical lenses optimize image quality in a circle

The Opportunity: Optimize image quality almost <u>any way you want</u>

(within the laws of physics of course...)



Today: Image Quality from Conventional Rotationally Symmetric Lens

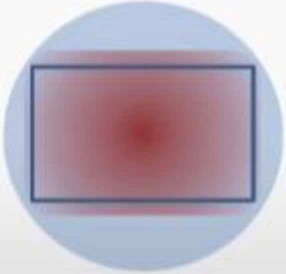


Image Quality Optimized with using free-form lenses

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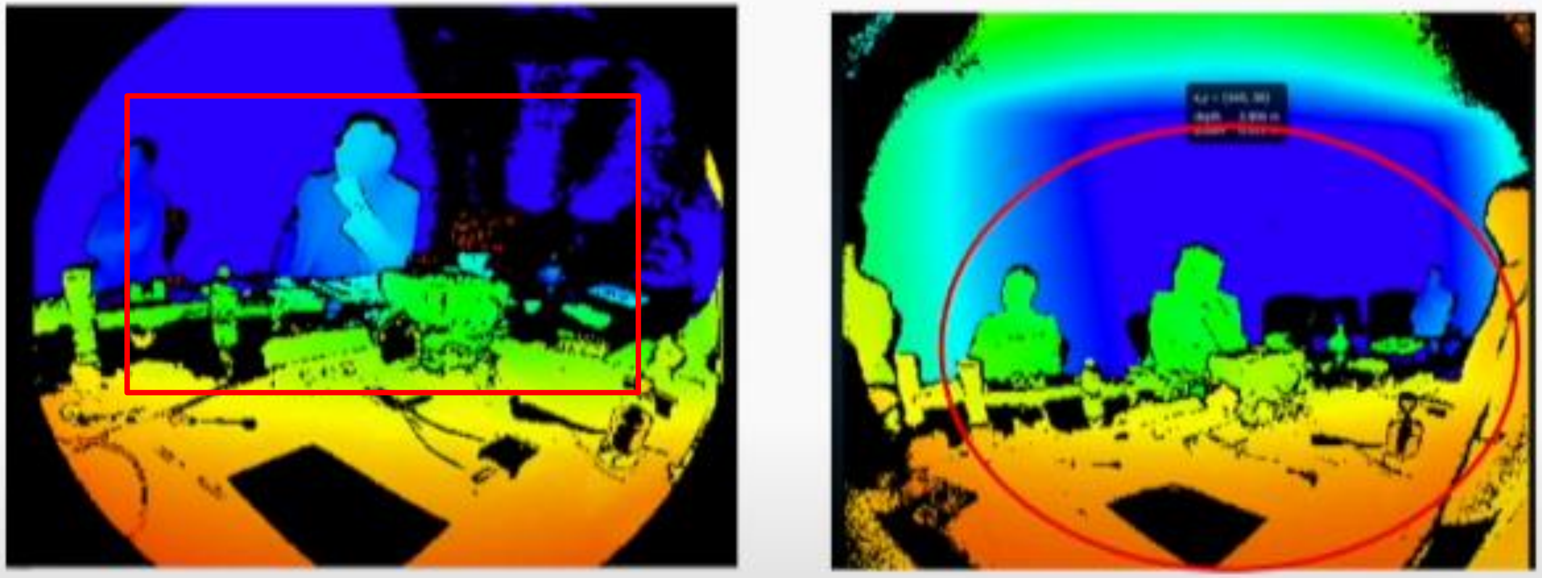
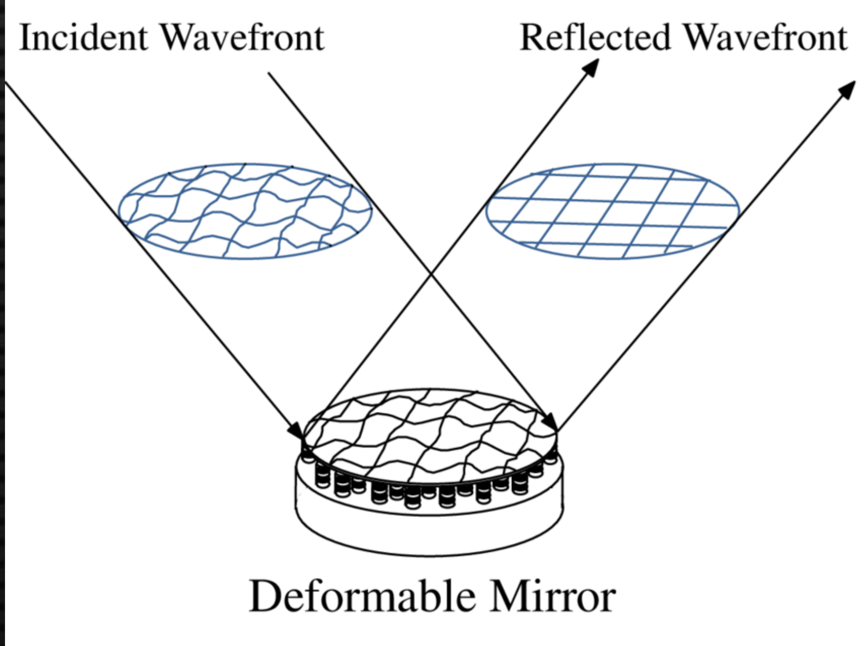
Concluding Remarks

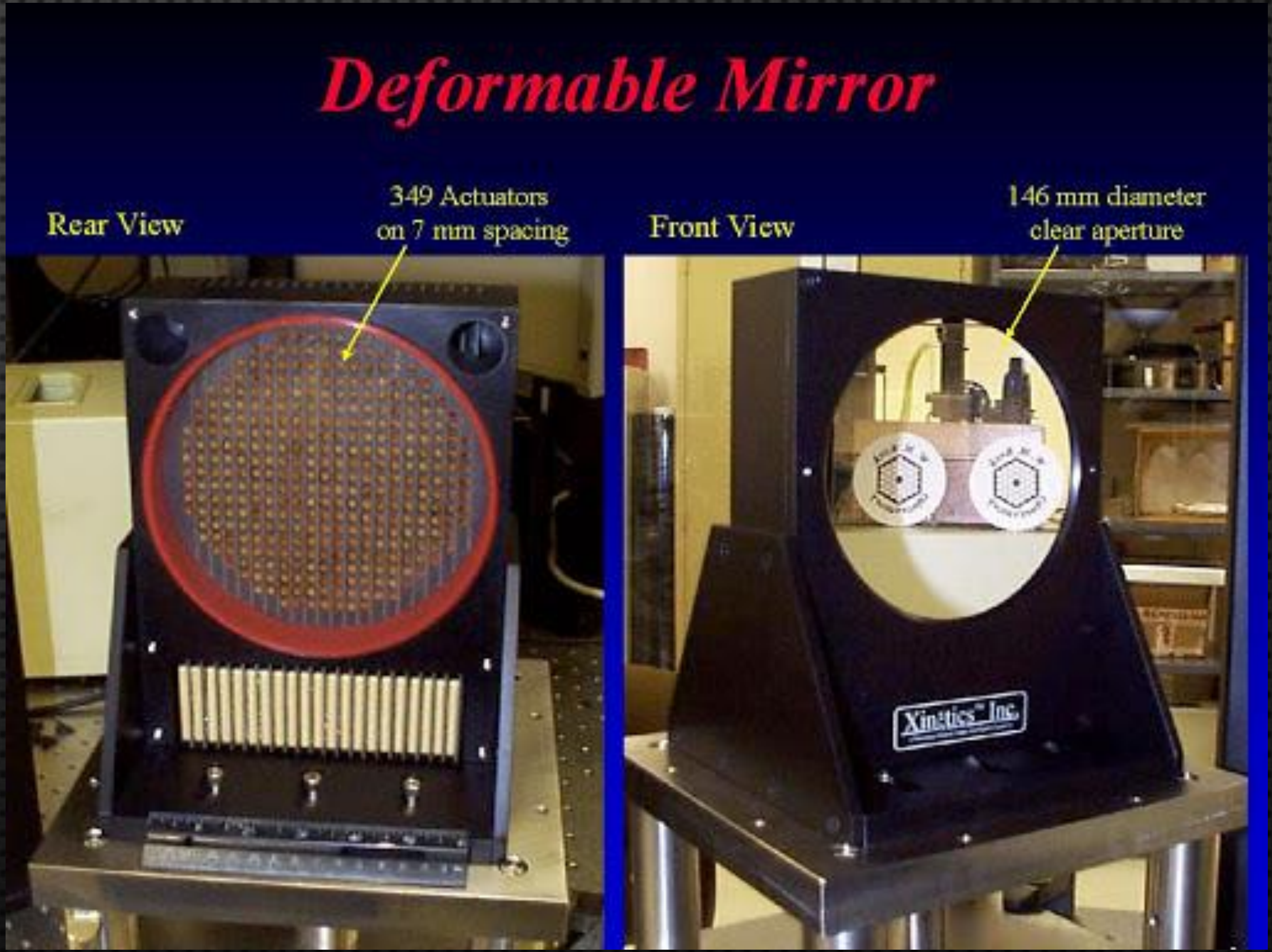


100 deg horizontal | 120 deg horizontal

Concluding Remarks

Concluding Remarks





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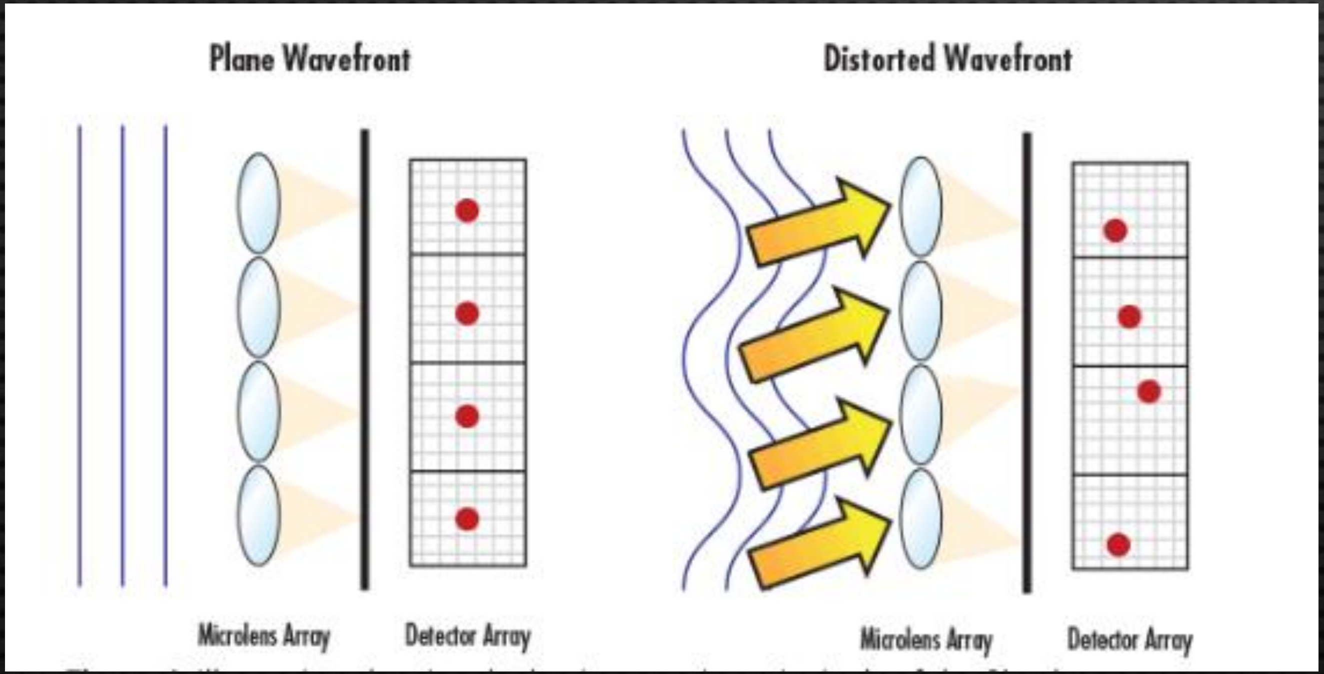
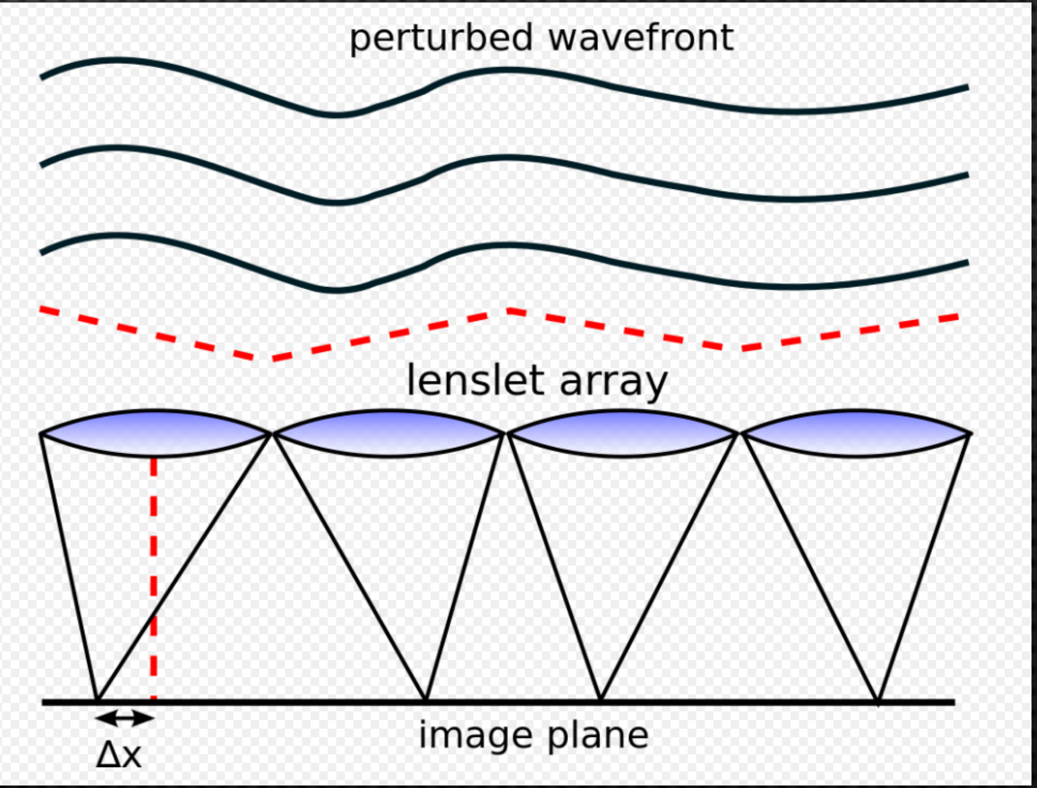
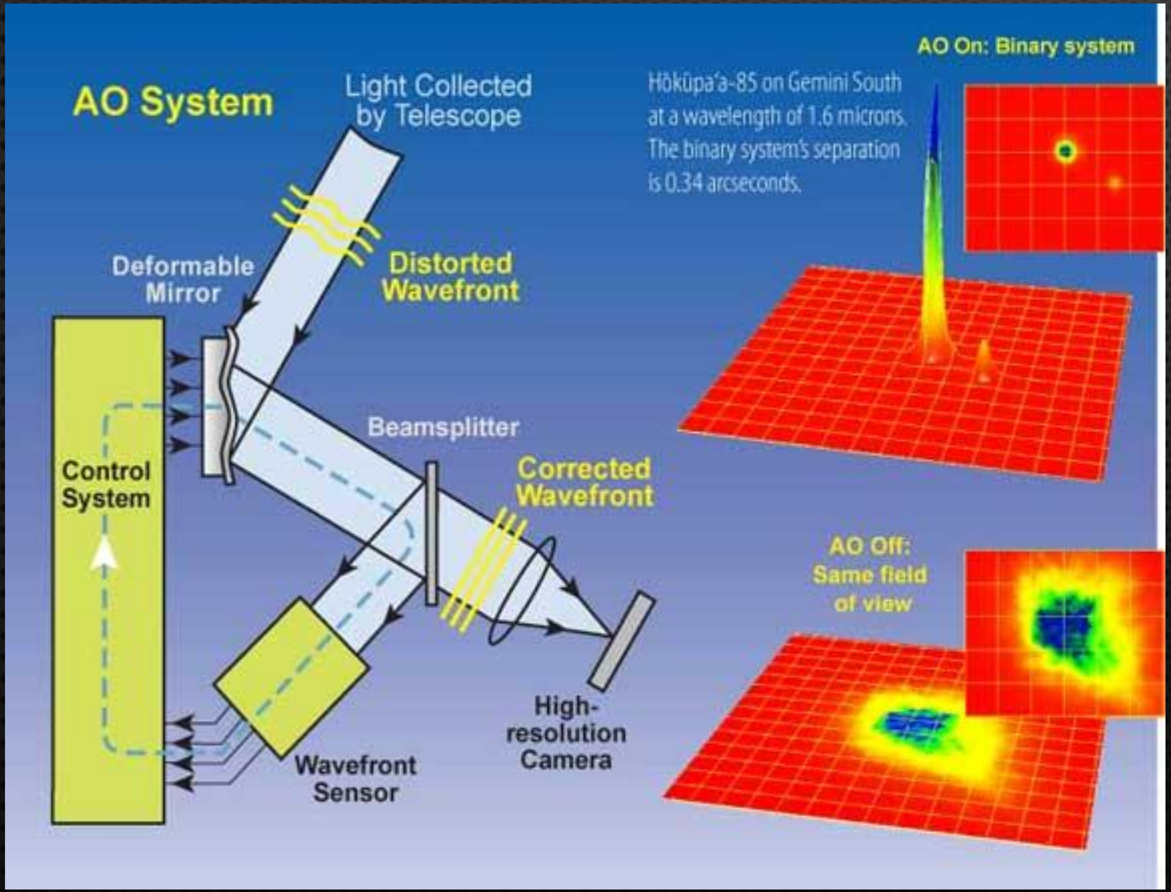
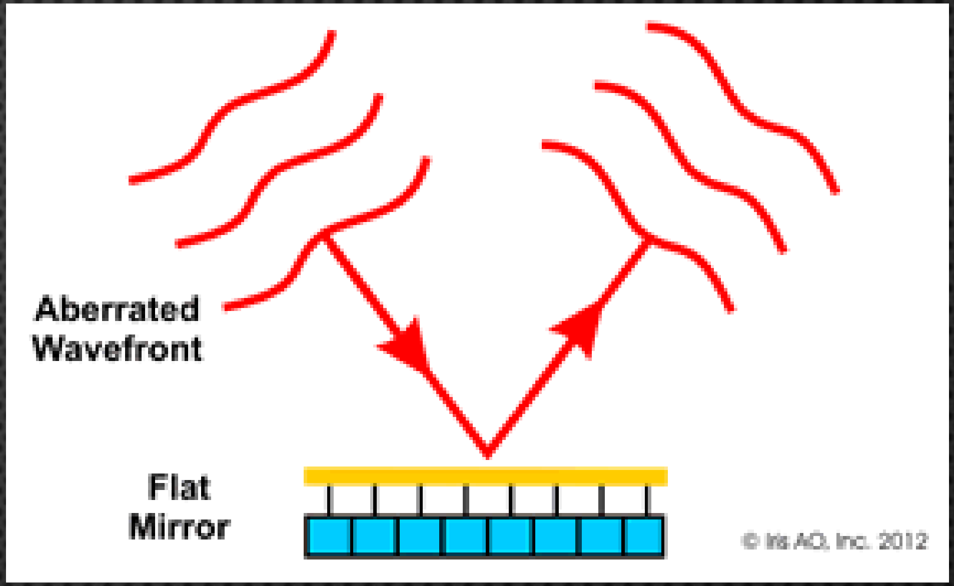
Concluding Remarks

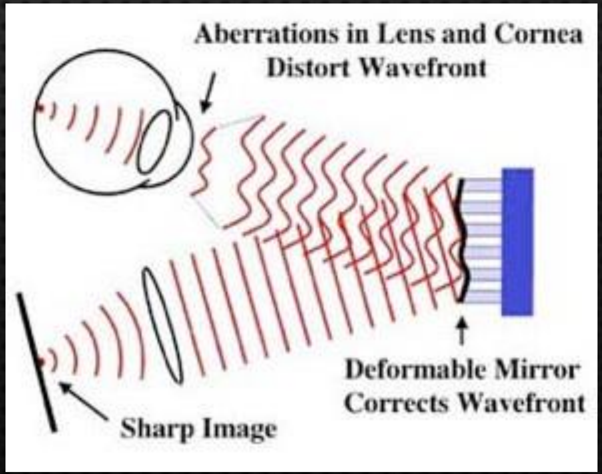




Concluding Remarks







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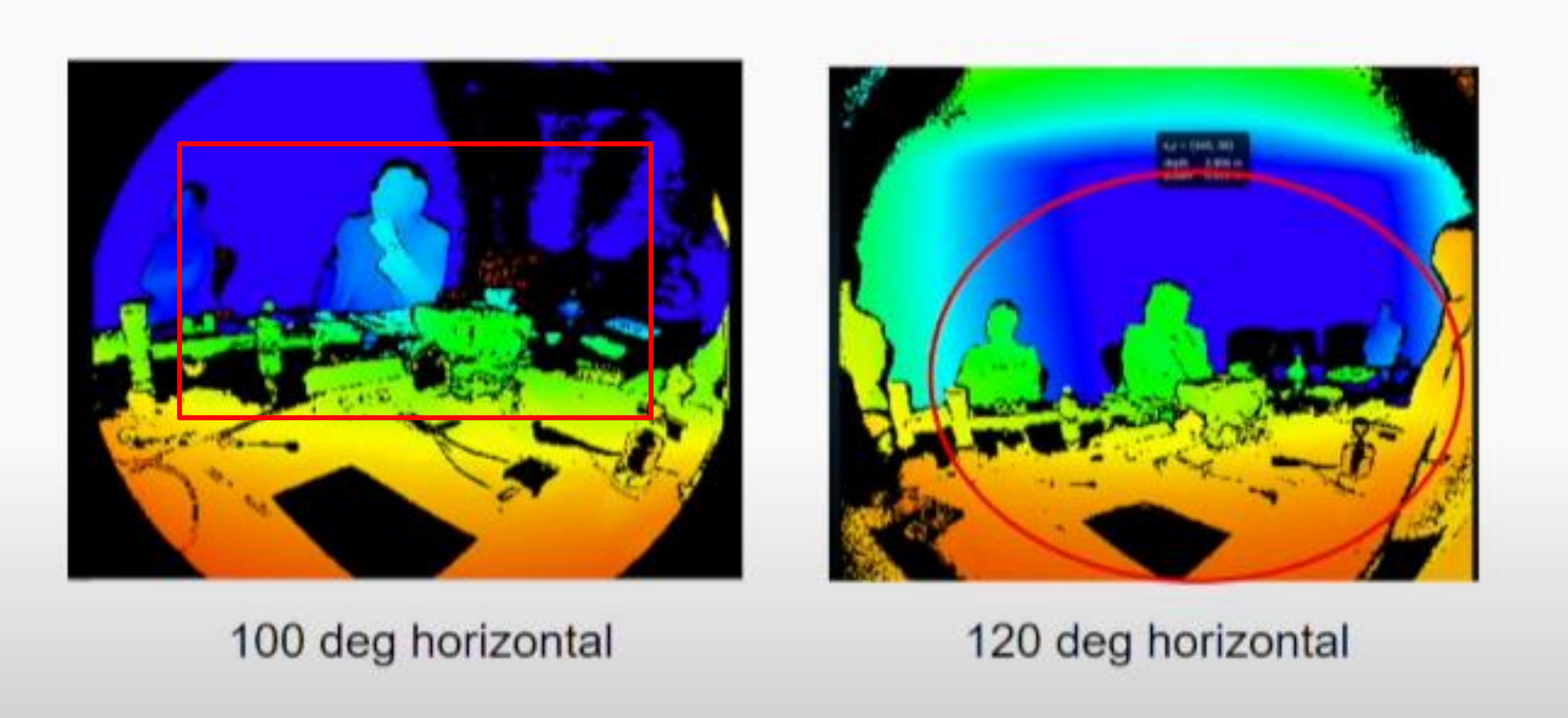
Concluding Remarks



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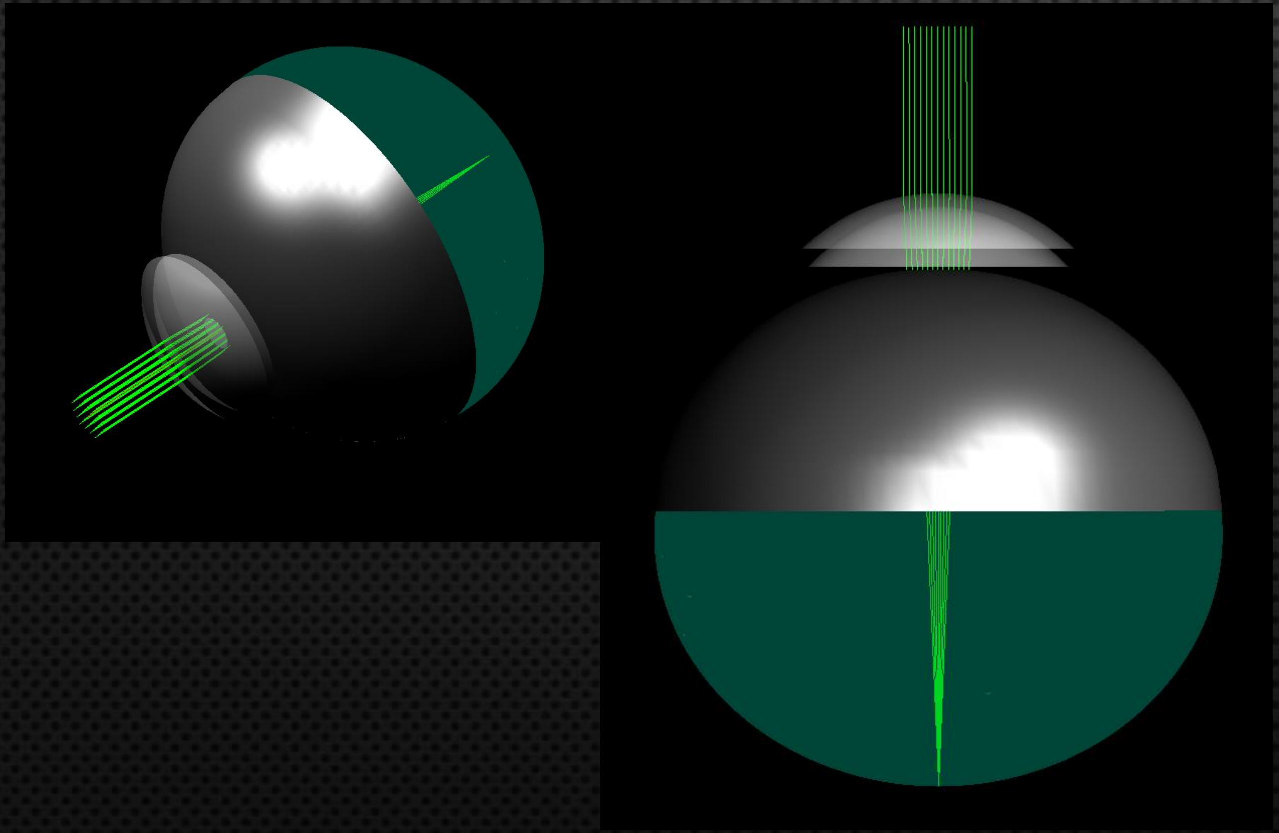
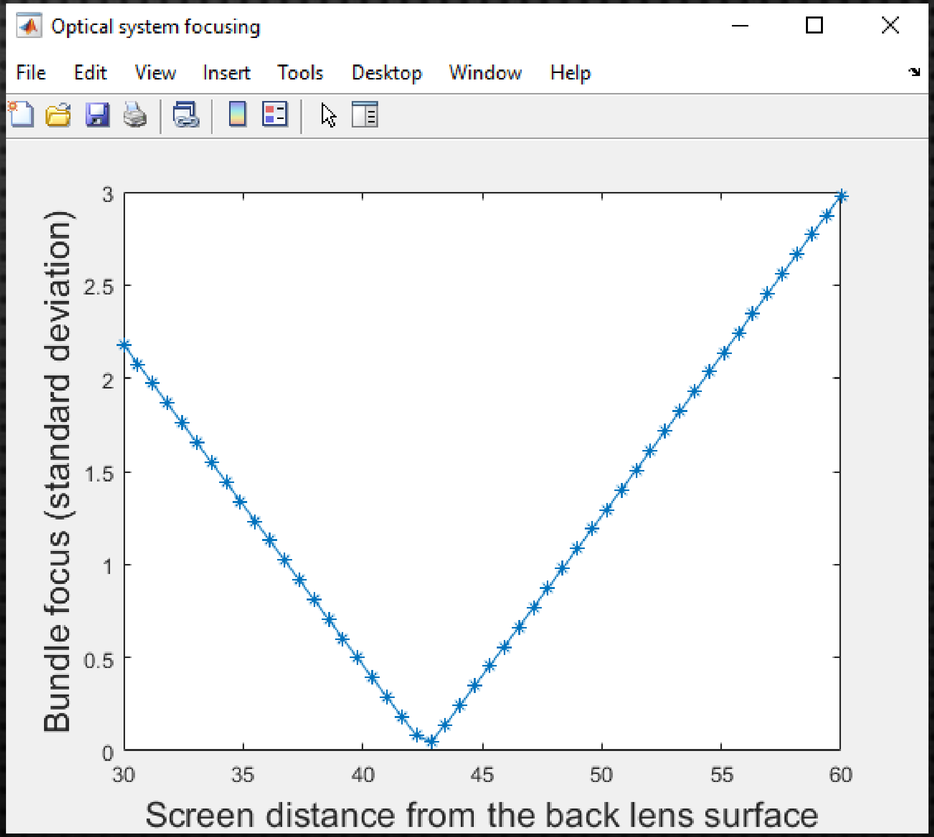
CONCLUSIONS





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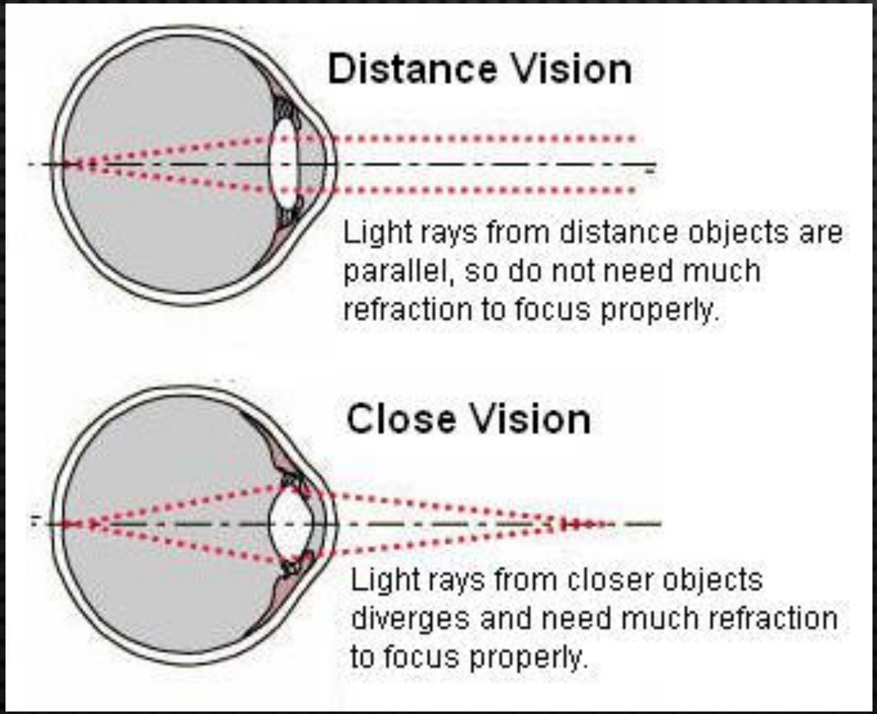
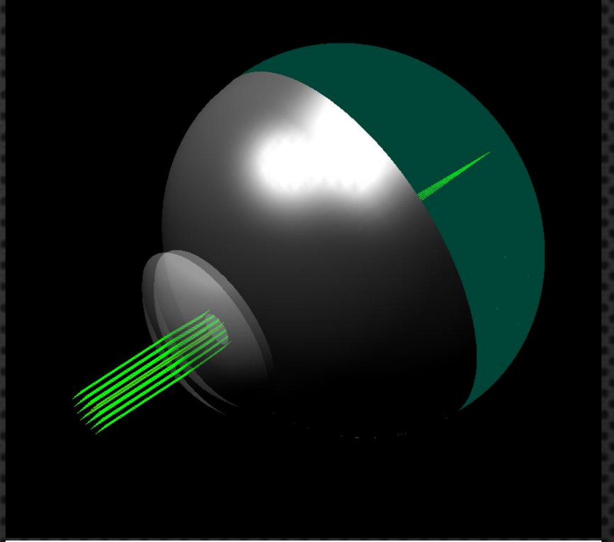
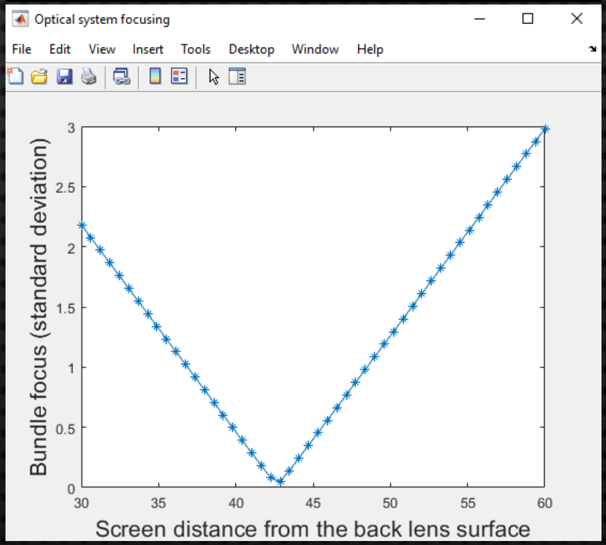
CONCLUSIONS







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Q & A

END

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