Rotman School of Management
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Group Number: N/A___________________________

Assignment Title: Project #1 ___ _________________

Course Code: RSM2310HS_____________________

Section #: 1 2 3 4

Course Name: Analysis and Management of Fixed Income Securities

Professor Name: Prof. Redouane Elkamhi___ ____

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Fixed Income Project 1 Report

Nik Ristic, Zaid Abid, Matthew Belmore, and Jack Evans

Enclosed: 2 Excel Documents (BDT_Model_Final.xlsm, VASICEK_Model_Final.xlsm). Each operates independently because together the runtime is not feasible.

Project Goal: Build a pricing model for options on interest rates using both Vasicek and Black-Derman-Toy models for a client.

Part 1: Data
Excel Pricing Model Sheet: Instalment 1&2.xlsm - Raw Data

The data used in this analysis are weekly U.S. government bond yields collected from the St. Louis Federal Reserve Bank's FRED database on February 8, 2019. Weekly historical data was collected for the range February 10, 2006 to February 8, 2019. The maturities collected were 1, 3, and 6 months and 1, 2, 3, 5, 7, 10, 20, and 30 years. The US Treasury Department produces their yield curves using a quasi-cubic hermite spline function and various other methods to smooth the term structure released to the public. These are reported as constant maturity yields which are assumed to pay a semi-annual coupon and are quoted in bond-equivalent yields and assume a 365-day year.

Part 2: Cubic Spline Interpolation
Excel Pricing Model Sheet: Instalment 1&2.xlsm - Cubic Spline Model & Interpolated Yields (output)

Cubic spline interpolation is a method for interpolating bond yields or prices for which data is unavailable or not reported. We chose to use the cubic spline method for interpolation because we wanted to more accurately fit the shape of the yield curve. Linear interpolation would have caused substantial errors as this would be assuming a linear relation for a polynomial function. As in the class notes, the optimal number of buckets was chosen to be 4. The buckets spanned maturities of less than one year, 2 to 5 years, 6 to 10 years, and 11 to 30 years respectively. Utilizing cubic spline interpolation on the original weekly data and given maturities produced weekly observations for all maturities from 1 month to 30 years in semi-annual increments which will be then bootstrapped to find zero coupon bond rates.

Part 3: Bootstrapping to Obtain Zero-Coupon Bond Yields
Excel Pricing Model Sheet: Instalment 1&2.xlsm - Bootstrapped Rates (output)

Assumption: Bond yields are equal to the coupon rate.

The cubic-spline-interpolated data generated in part 2 still contain semi-annual coupons which need to be stripped to obtain zero-coupon bond yields for option pricing. Bootstrapping was completed in Matlab instead of Excel for efficiency and the code may be found in appendix one. We assume that the yields are equal to the coupons as the collected data are generated as par yield curves by the U.S. Treasury Department. With the completion of bootstrapping, we are left with a matrix of semi-annual zero-coupon bond yields with maturities from 1 month to 30 years.

Part 4: The Nelson-Siegel-Svensson Model (NSS)

Excel Pricing Model Sheet: Instalment 1&2.xlsm - NSS Model & Post-NSS Rates (output)

For the purposes of option pricing, underlying bond maturities must be as granular as possible. In an ideal setting, underlying term structure data should be at least daily, if not hourly for the most accurate pricing possible. From the guidance of this assignment, however, the required granularity is monthly and therefore, monthly data must be extracted. Our group elected to use the Nelson-Siegel-Svensson model rather than the Nelson-Siegel model for this application. The original Nelson-Siegel model (1987) interpolates a smooth term structure by optimizing three beta parameters to minimize the sum of squared differences between the actual yield curve and the interpolated curve. Svensson (1994) added two additional parameters, lambda 1 and 2, to the original model. He found that the addition of the two lambda parameters significantly improved the fit of the term structure and this is now the preferred method of many world central banks. Due to the high yield volatility of our zero-rates, the NSS model was an obvious choice, as many researchers in the literature have found the NSS to fit more a volatile series better. NSS is used to produce monthly yields on all bonds and this output is the final data matrix for our pricing model.

Discussion: Interpolation - NSS VS. Cubic Spline

The Nelson-Siegel-Svensson model was a clear winner in this application for three main reasons: simplicity, shape, and volatility. First, the NSS model is the simpler of the two to run, as it is a fairly straightforward sum of squared errors minimization problem with limited parameters. Cubic spline interpolation requires the determination of an optimal number of buckets which has a meaningful effect on the results. Spline interpolation requires a higher degree of trial and error. Second, NSS produces very smooth term structures which fit the real yield data impeccably. The shape is stable and reasonable with no surprising troughs, peaks, or kinks which would lead to difficulties in the latter installments. Cubic spline interpolation produced irregular kinks and multiple humps which is something that the Black-Derman-Toy model struggles to deal with. Finally, we believe that the cubic spline model introduced volatility to the yield data. Through additional analysis, we believe that the cubic spline interpolation procedure approximately doubled the volatility present in the yield data using the natural logarithm of the returns. The NSS model did not appear to be augmenting volatility to the same degree. For these three reasons, the NSS model is preferred to cubic spline interpolation.

Part 5: Calculating Yield Volatility - A Significant Issue

Excel Pricing Model Sheet: BDT_Model_Final.xlsm - Yield Volatility

For the purposes of interest rate option pricing, a confident estimation of realized volatility in zero-coupon bond yields is crucial. Volatility is estimated by finding the continuously compounded returns and then taking the annual standard deviation of these returns. Henceforth, the main issue should be the optimal time period for which the volatility should be estimated from. However, the problem of volatility was more difficult than it appeared. Irrespective of the time period chosen, the calculated volatilities were unreasonably high with the short end of the curve displaying volatilities upwards of 400% per year. One expects a volatility between 50% and 90% in the short end of the curve. Hand selecting historical returns which give reasonable volatility is not acceptable, and even more importantly, volatilities from 10 years ago likely do not have the same inter-maturity relationship as they do today. The calculated volatilities can be found in the yield volatility sheet in the model but are henceforth replaced by proxies.

We use the volatilities provided in the sample BDT model from Professor Redouane. These are for bonds up to 5.5 years of maturity in semi-annual increments. NSS is then utilized to interpolate monthly volatilities which are much more reasonable. Henceforth, these interpolated proxy yields are used for the BDT model to give reasonable results. Therefore, sensitivity analysis based on the time period for historical volatility is not possible, so we supplement this with parallel shifts in the volatility curve instead. From this exercise, it became apparent that a smooth and downward sloping volatility curve with no humps was imperative for the BDT model to find a solution.

Part 6: Pricing - Vasicek Analytical Model

Excel Pricing Model Sheet: VASICEK_Model_Final.xlsm - Vasicek Analytical

The Vasicek model is a continuous time model which allows for closed-form derivative pricing formulas. It is a one-factor short rate model which models interest rate movements driven only by one source of market risk. In this model, the key parameters are “a” (speed of mean reversion), “b” (long-term mean), and “sigma” (instantaneous volatility). These parameters and r_0 (initial interest rate) completely characterize the model. Vasicek's model relies on the assumption that interest rates are mean reverting. A main disadvantage is that negative interest rates are possible which was later tackled in the Cox-Ingersoll-Ross model. Additionally, the model only uses a short-term rate to model the entire term structure. Therefore, the Vasicek model is better-suited for short-term option pricing and may produce substantial errors for long maturities. Excel solver is used to calibrate the parameters using observed bond prices and r_0. Calibrated parameters are then entered into the closed-form call option pricing formula to price the interest rate option. Put-call parity is then used to determine the put option price.

Part 7: Pricing - Vasicek Simulation Model

Excel Pricing Model Sheet: VASICEK_Model_Final.xlsm - Vasicek-Simulation-Fitting

The Vasicek simulation model makes use of the original Vasicek equation to simulate interest rates multiple times. The Wiener process introduces uncertainty in the model due to a random normal variable generator. We use the values of a, b and sigma calibrated in the analytical model as the model parameters and start with an initial r_0. We simulate 115 times which produces 115 bond prices at each maturity. These prices are then averaged and taken to be the observed bond prices in the analytical model in order to price the options as done in part 6. We find that the prices generated by the analytical and simulation models are virtually identical.

Part 8: Pricing - Black-Derman-Toy Model (BDT)

Excel Pricing Model Sheet: BDT_Model_Final.xlsm - BDT

Developed in 1990, the BDT model is a short rate model used for interest rate option pricing. The short-term interest rate (short rate) determines the evolution of the interest rate tree. This model assumes mean reversion as with the Vasicek. The model is an extension of the binomial option pricing model and works in a similar fashion. Current term structure data and observed volatility is used to calibrate the interest rate tree which is then used to discount the bond pricing tree and option prices. The BDT model does not allow for negative interest rates and can be easily improved by using additional calibration data. A main strength of this approach is that the tree only needs to be calculated once and can then be used to price many derivatives. However, if any of the calibration data changes, the model must be recalibrated which is time-intensive.

The BDT model is where our group had the most trouble. Two main issues hindered our progress. First, the model is very finicky. Excel solver had many issues if the starting point of an iteration was too far from the solution and scaling the model is difficult with the possibility of making small errors very high. Because this model is solved using backwardation, a small error in a previous step can undermine the entire model. Second, volatility most be smooth, downward sloping, and have only one hump. Additionally, the input volatilities must be reasonably small. If these criteria are not met, the BDT in Excel cannot find a solution. In our case, the market volatilities were too high and steep in the short run to find solutions. Hence, we substituted NSS interpolated volatilities based on the sample BDT model provided in class.

Discussion: Pricing Differences - Vasicek VS. BDT

The prices given by the Vasicek and BDT models are very similar with all price differences being less than 75 cents on $100 principal. We believe that there are two main factors accounting for the differences. First, interpolation errors have a potentially large effect on BDT option valuation. The problem of high volatilities is discussed earlier as well as the use of proxies. The volatilities used are not from the dataset, therefore, we expect some error in BDT pricing as incorrect volatilities are being used. Second, the relatively long maturities (1 to 4 years) is likely too long for the Vasicek model and therefore tends to inflate option prices at longer maturities. This stems again from the reliance on a single short rate to populate the entire model. In this exercise, we have more confidence in the results of the Vasicek model, because of the significant problems with volatilities in the BDT setting. However, for pricing, we price 1- and 2-year maturity options Vasicek and 3- and 4-year maturities with BDT. The pricing differences are found below.

Pricing Differences (BDT Subtract Vasicek)

Call Pricing Differences

		Strike Price				
		$ 0.90	$ 0.93	$ 0.95	$ 0.98	$ 1.00
Option	1.00	-0.599	-0.212	-0.010	0.000	0.000
	2.00	0.021	-0.468	-0.185	-0.010	0.000
	3.00	0.106	0.006	-0.370	-0.070	0.000
	4.00	-0.051	-0.062	-0.068	-0.233	0.000

Put Pricing Differences

		Strike Price				
		0.9	0.925	0.95	0.975	1
Option	1.000	-0.843	-0.469	-0.262	-0.256	-0.260
	2.0000	-0.227	-0.721	-0.433	-0.264	-0.259
	3.0000	0.001	-0.103	-0.470	-0.173	-0.105
	4.0000	0.000	0.000	0.004	-0.160	0.074

Part 9: Sensitivity Analysis - Interest Rate Volatility

Sensitivity analysis is conducted on both the Vasicek and BDT option prices due to interest rate volatility. In the Vasicek model, sigma captures the instantaneous volatility and we see that very extreme volatilities, either high or low, significantly decrease the option prices. This is unexpected, as theory suggests that higher volatility is compensated for by high option values. Sensitivity of the BDT model due to volatility was conducted by applying a parallel shift in the volatility curve as inputs in the model. This exercise requires the tree to be calibrated each time and then use the pricing equation to find the option value. Here, we find that option prices increase with volatility as expected. All sensitivity analysis assumes a 5-year bond as the underlying asset, a two-year option maturity, and a strike price of $0.95. The results are reported below.

Vasicek Sensitivity

Call Option Sensitivity Analysis

			Sigma				
			Extreme (+)	High	Base	Low	Extreme (-)
			0.0311	0.0259	0.0207	0.0156	0.0104
r(o)	Extreme (+)	0.0273	$ 1.18	$ 1.79	$ 1.79	$ 1.04	$ 0.12
	High	0.0261	$ 0.40	$ 0.74	$ 0.74	$ 0.32	$ 0.01
	Base	0.0237	$ 0.10	$ 0.25	$ 0.25	$ 0.07	$ 0.00
	Low	0.0213	$ 0.02	$ 0.09	$ 0.09	$ 0.02	$ 0.00
	Extreme (-)	0.0201	$ 0.01	$ 0.04	$ 0.04	$ 0.00	$ 0.00

Put Option Sensitivity Analysis

			Sigma				
			Extreme (+)	High	Base	Low	Extreme (-)
			0.0311	0.0259	0.0207	0.0156	0.0104
r(o)	Extreme (+)	0.0273	$ 4.47	$ 5.08	$ 5.08	$ 4.32	$ 3.41
	High	0.0261	$ 4.13	$ 4.47	$ 4.47	$ 4.05	$ 3.74
	Base	0.0237	$ 3.83	$ 3.98	$ 3.98	$ 3.80	$ 3.73
	Low	0.0213	$ 3.27	$ 3.33	$ 3.33	$ 3.26	$ 3.24
	Extreme (-)	0.0201	$ 2.58	$ 2.61	$ 2.61	$ 2.57	$ 2.57

BDT Sensitivity to Volatility

Sensitivity Analysis

	Put Price	Call Price
High Volatility	$ 2.84	$ 0.45
Moderate Increase	$ 2.64	$ 0.31
Original Volatility	$ 2.49	$ 0.16
Moderate Decrease	$ 2.39	$ 0.06
Low Volatility	$ 2.34	$ 0.01

Part 10: Sensitivity Analysis - BDT Time Step

Again, using a 5-year bond as the underlying asset, a two-year option maturity, and a strike price of $0.95, we run sensitivity analysis of the BDT model with respect to time step choices. We test 1 month, quarterly, semi-annual, and annual time steps. We find a significant difference in the price observed by changing the time steps which was unexpected. The larger the time steps, the higher the call price and the lower the put price. In practice, option traders use the smallest possible time step for the most accurate pricing. Therefore, we use monthly time steps for our pricing platform. The results of this sensitivity analysis are shown below.

BDT Sensitivity to Time Step Changes

Sensitivity Analysis		
Time Step	Price	Option Type
0.08	$ 2.34	Put
	$ 0.01	Call
0.25	$ 2.72	Put
	$ 0.00	Call
0.50	$ 3.31	Put
	$ -	Call
1.00	$ 4.50	Put
	$ -	Call

Part 11: Official Quotes

We price interest rate options on maturities of 1 to 4 years and quote prices with strike prices $0.90, $0.93, $0.95, $0.98, and $1.00. We quote 1- and 2-year maturities with Vasicek and 3- and 4-year maturities with BDT. We have chosen to charge a constant 8% spread. This is reasonable given that it is an OTC derivative and liquidity does not play a role in competitive pricing here.

The official quotes can be found below.

Quotes

1-Year Maturity		
Call	Strike	Put
$ 1.33	$ 0.90	$ 2.38
$ 0.24	$ 0.93	$ 2.91
N/A	$ 0.95	$ 5.30
N/A	$ 0.98	$ 7.93
N/A	$ 1.00	$ 10.56

2-Year Maturity		
Call	Strike	Put
$ 2.62	$ 0.90	$ 0.28
$ 1.10	$ 0.93	$ 1.33
N/A	$ 0.95	$ 2.99
N/A	$ 0.98	$ 5.37
N/A	$ 1.00	$ 7.93

3-Year Maturity		
Call	Strike	Put
$ 4.84	$ 0.90	N/A
$ 2.37	$ 0.93	N/A
$ 0.39	$ 0.95	$ 0.55
N/A	$ 0.98	$ 2.66
N/A	$ 1.00	$ 5.17

4-Year Maturity		
Call	Strike	Put
$ 7.05	$ 0.90	N/A
$ 4.61	$ 0.93	N/A
$ 0.39	$ 0.95	N/A
N/A	$ 0.98	$ 0.43
N/A	$ 1.00	$ 2.71

Appendix 1 (Bootstrapping: Matlab Code)

```
clear all

load('interpolated_data.mat');
%loads interpolated rates as a matrix

final = zeros(679,60);

for z = 1:679
    %creating an upper triangular cash flow matrix
    for i = 1:60
        for j = 1:60
            if j < i
                A(i,j) = sampledata(z,i)/2;
            elseif j > i
                A(i,j) = 0;
            else
                A(i,j) = sampledata(z,i)/2 + 1;
            end
        end
    end
    p = ones(60,1);
    %assuming par value bonds so prices = 1
    r = A^(-1) * p;
    %calculates the zero coupon bond prices
    t = (0.5:0.5:30)';
    rates = log(r)./-t;
    %calculates rates from bond prices
    final(z,:) = rates;
end
```

Final Page

Grade:________