(2,4) Trees

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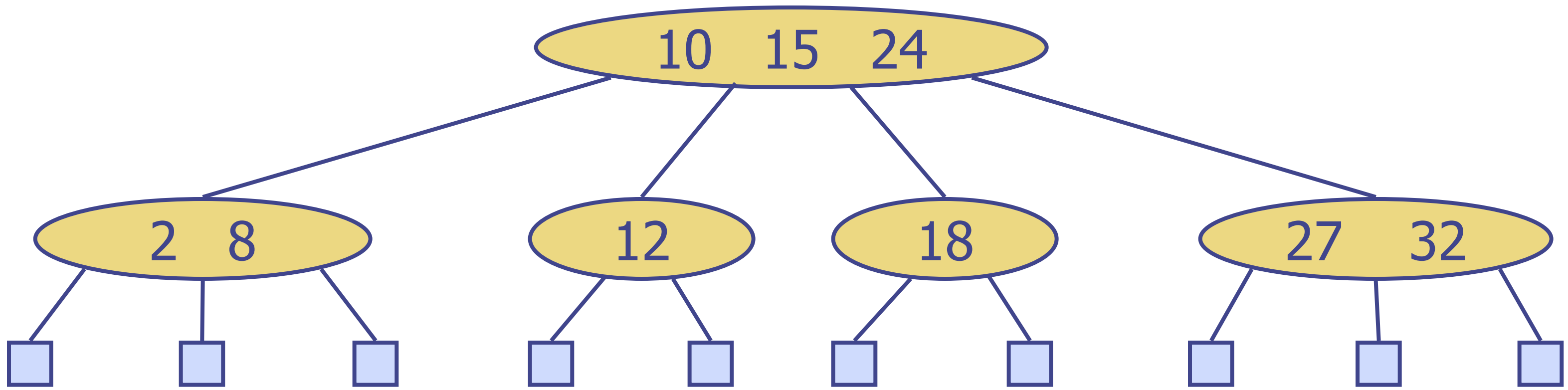

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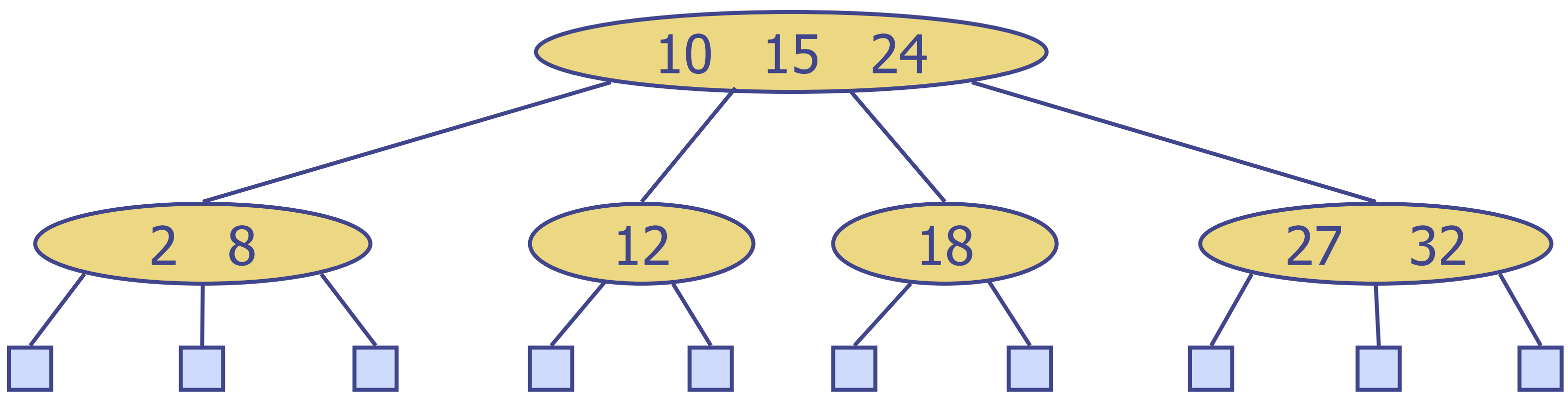
  - Node-Size Property: every internal node has 2, 3, or 4 children



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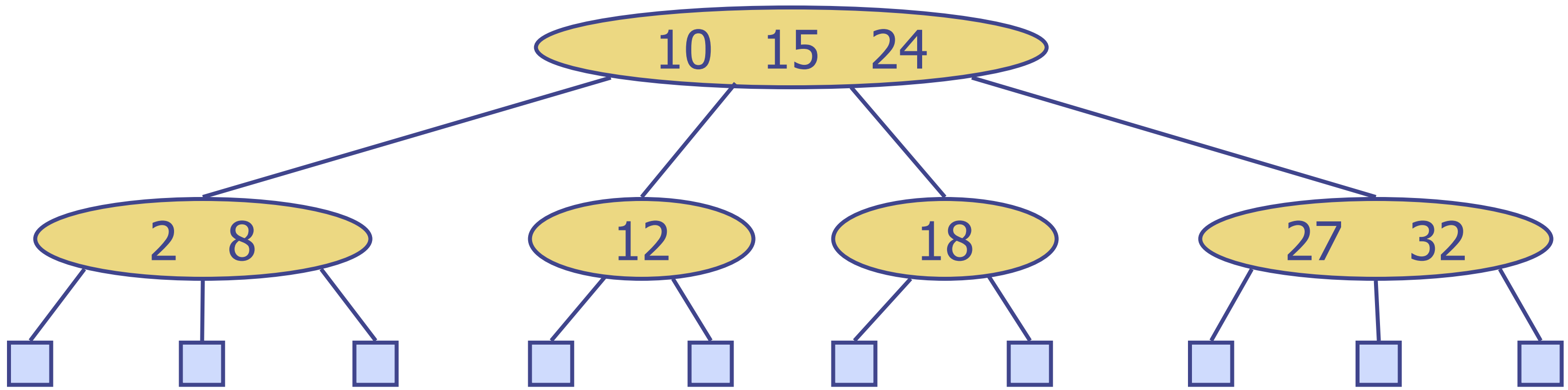

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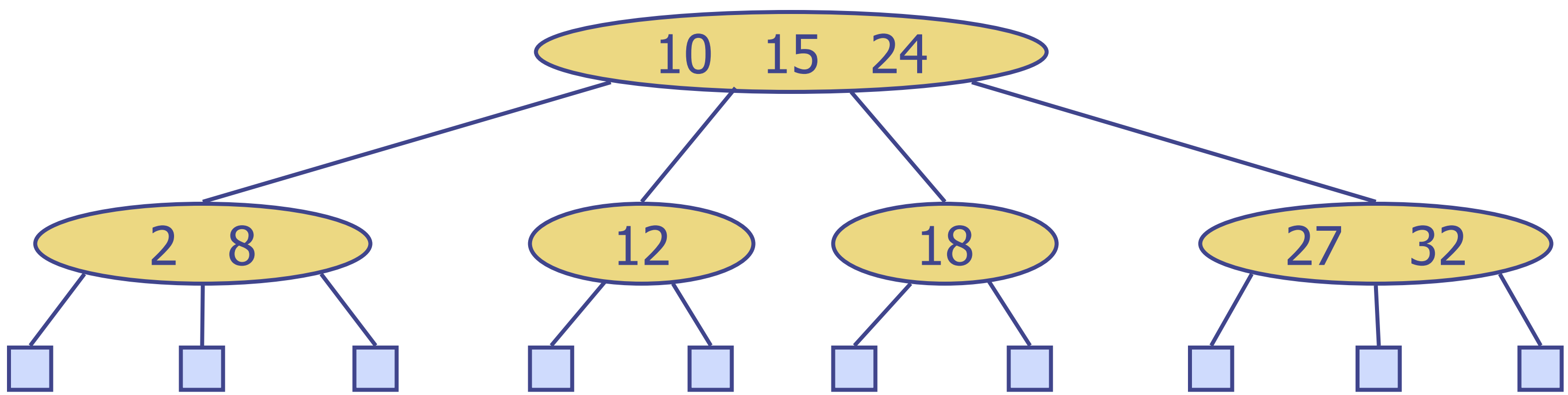
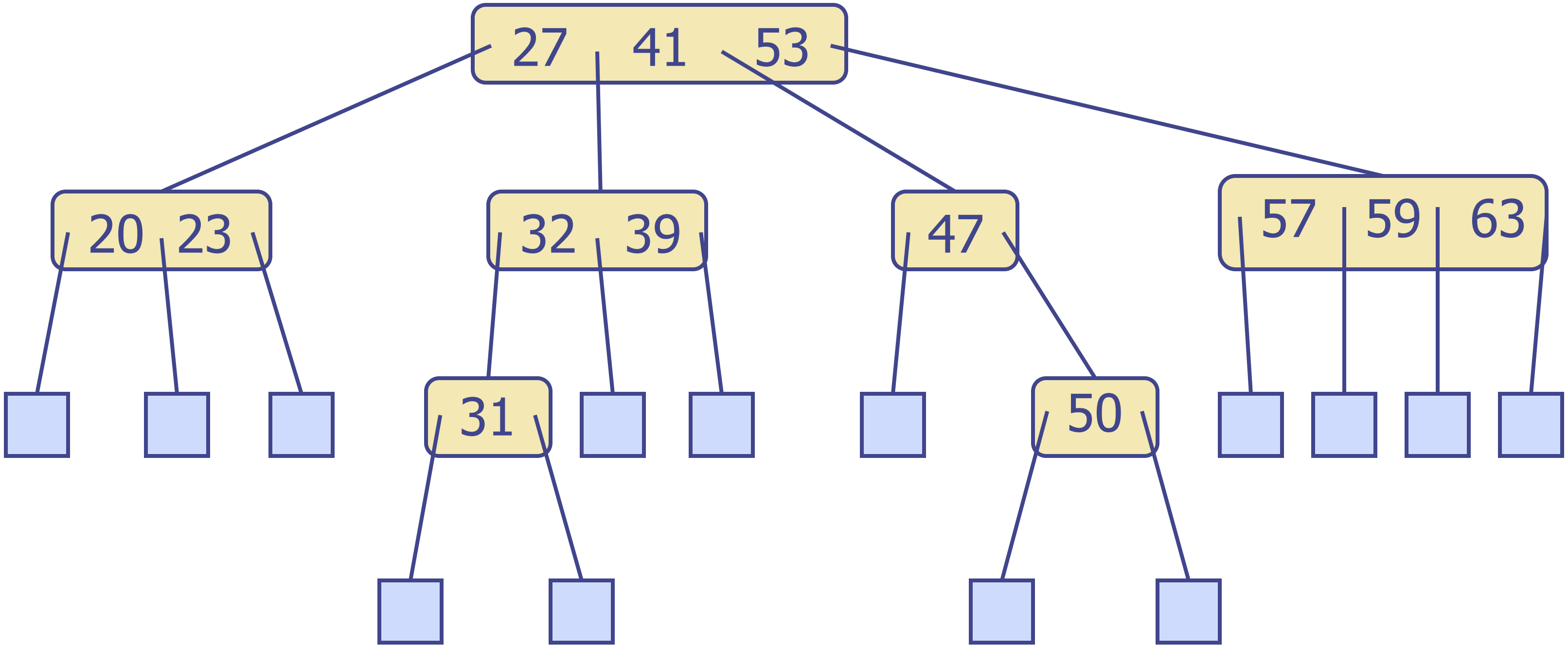
  - Node-Size Property: every internal node 2, 3, or 4children
  - Depth Property: all the leaves are in the same level
- Depending on the number of children, an internal node of a (2,4) tree is called a 2-node, 3-node or 4-node



(2,4) Tree?



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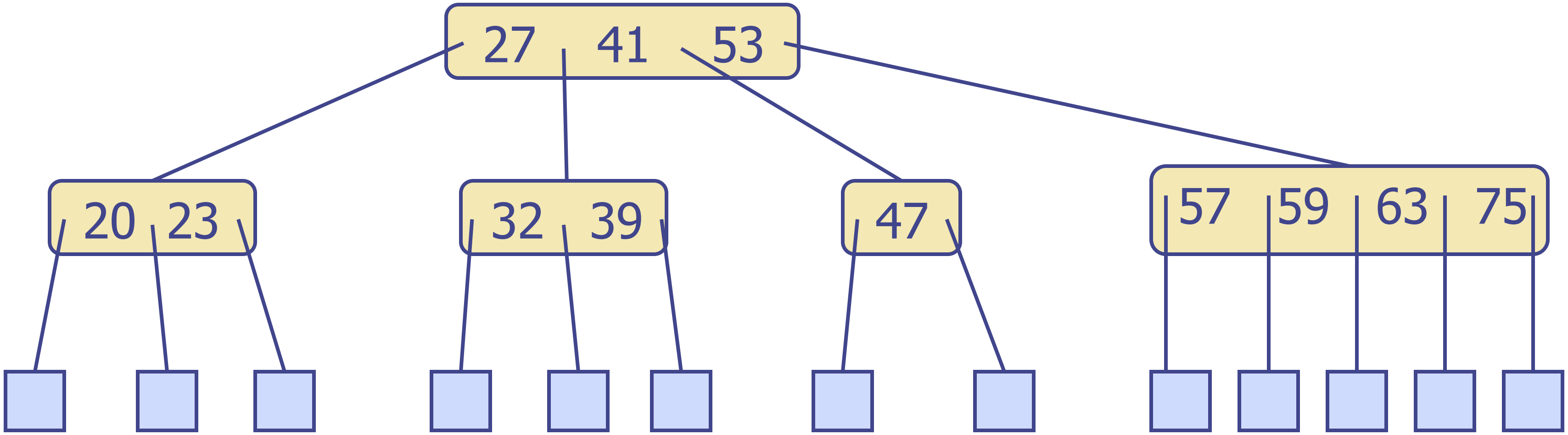


What is the Maximum Height of a (2,4) Tree?

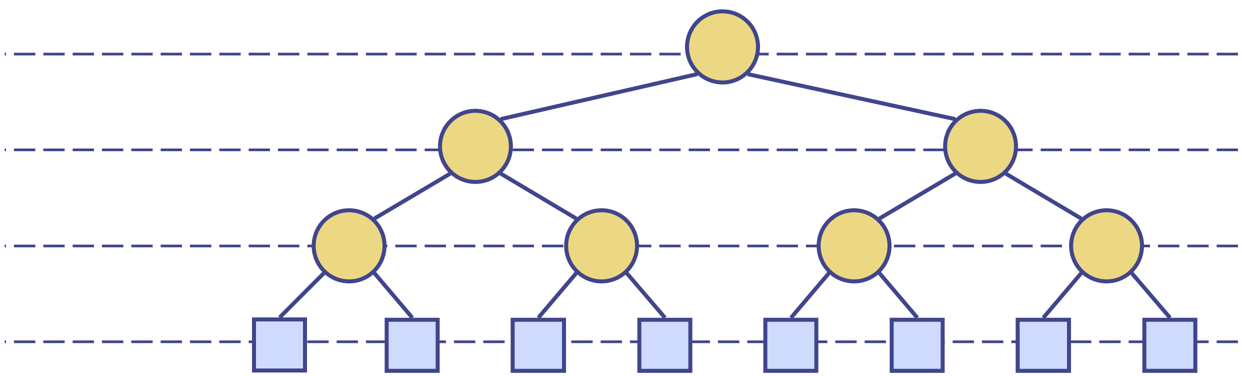
Height of a (2,4) Tree

- Theorem: A (2,4) tree storing n items has height $O(\log n)$

  Proof:
  - Let h be the height of a (2,4) tree with n items
  - Since there are at least 2^i items at depth $i = 0, \ldots, h-1$ and no items at depth h, we have

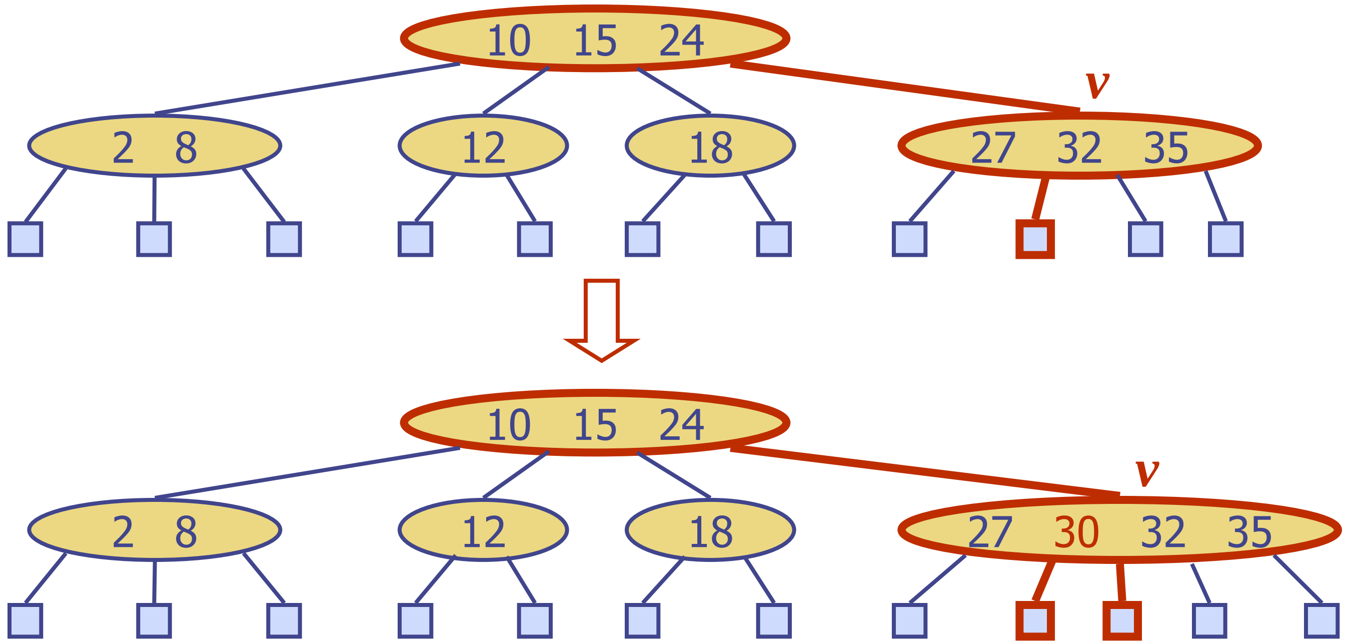
    $$n \geq 1 + 2 + 4 + \ldots + 2^{h-1} = 2^h - 1$$
  - Thus, $h \leq \log(n+1)$
- Searching in a (2,4) tree with n items takes $O(\log n)$ time

depth	items
0	1
1	2
$h-1$	2^{h-1}
h	0



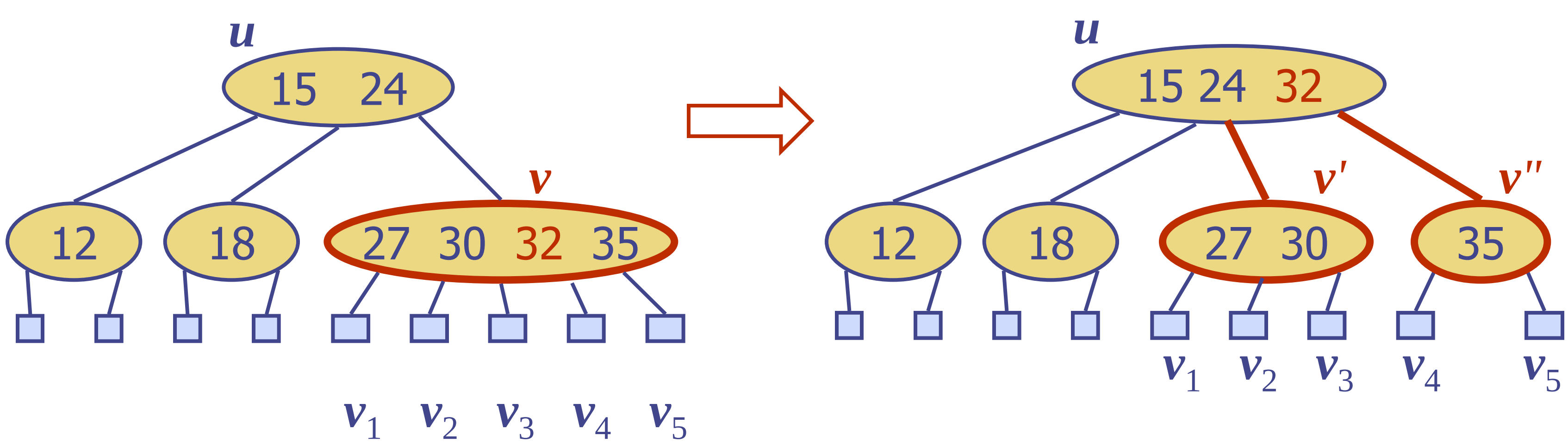
Insertion

- We insert a new item (k, o) at the parent v of the leaf reached by searching for k
  - We preserve the depth property but
  - We may cause an overflow (i.e., node v may become a 5-node)
- Example: inserting key 30 causes an overflow



Overflow and Split

- We handle an overflow at a 5-node v with a split operation:
  - let $v_1 \ldots v_5$ be the children of v and $k_1 \ldots k_4$ be the keys of v
  - node v is replaced nodes v' and v''
    - v' is a 3-node with keys k_1 k_2 and children v_1 v_2 v_3
    - v'' is a 2-node with key k_4 and children v_4 v_5
  - key k_3 is inserted into the parent u of v (a new root may be created)
- The overflow may propagate to the parent node u



Algorithm ***put (r,k,o)***

In: Root r of a (2,4) tree, data item (k,o)

Out: {Insert data item (k,o) in (2,4) tree}

Algorithm ***put* (*r*,*k*,*o*)**

In: Root *r* of a (2,4) tree, data item (*k*,*o*)

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Search for ***k*** to find the lowest insertion internal node ***v***

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if *v* is the root **then**

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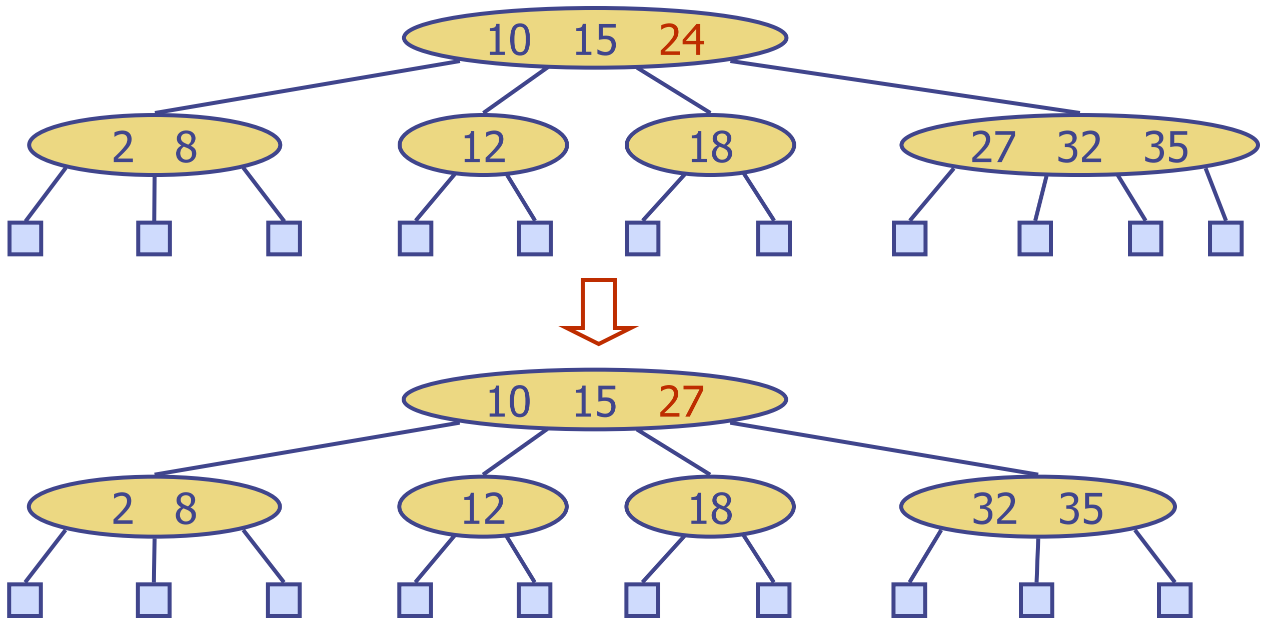
Time complexity of put is O(log n)

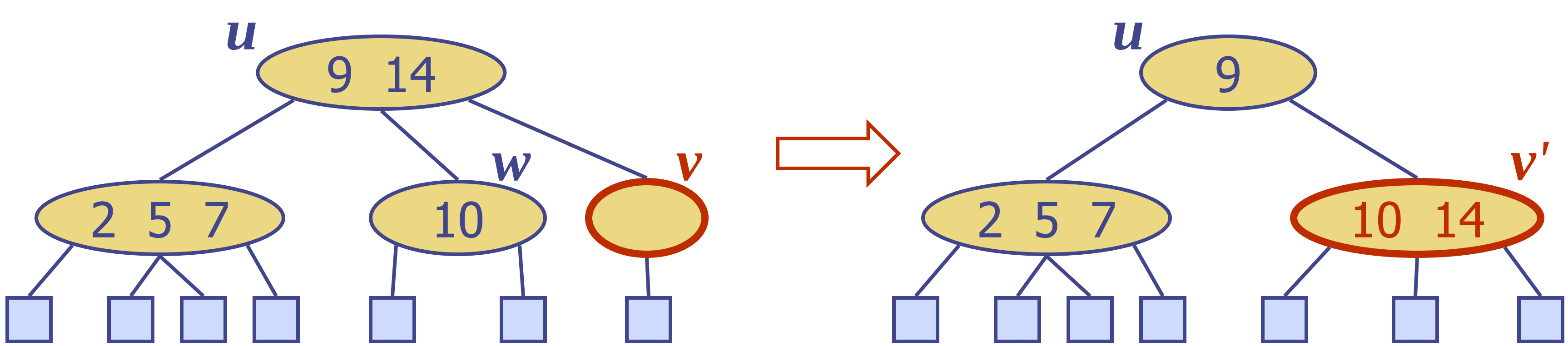
Deletion

- We reduce deletion of an entry to the case where the item is at the node with leaf children
- Otherwise, we replace the entry with its inorder successor (or, equivalently, with its inorder predecessor) and delete the latter entry
- Example: to delete key 24, we replace it with 27 (inorder successor)



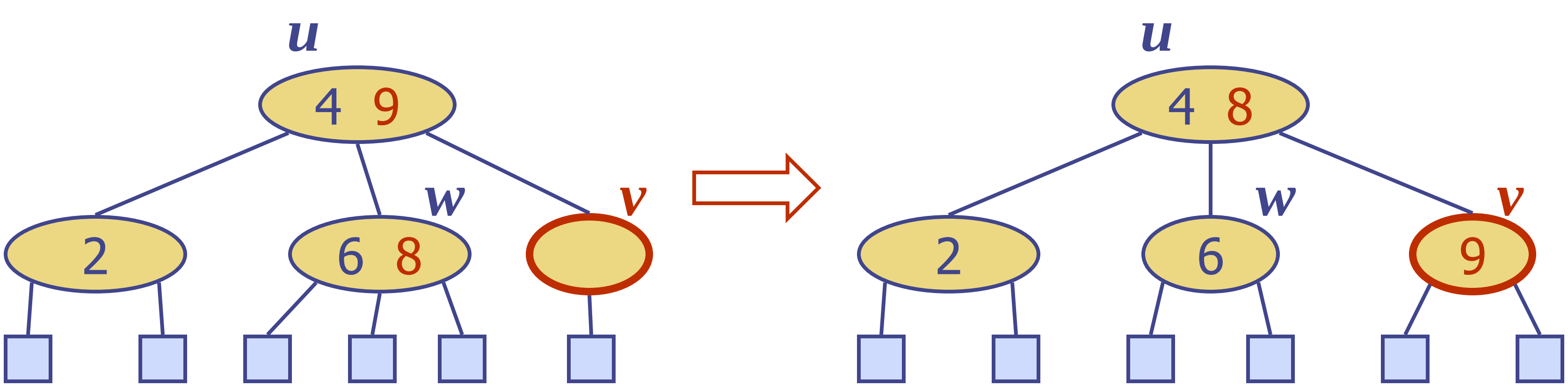
Underflow and Fusion

- Deleting an entry from a node v may cause an underflow, where node v becomes a 1-node with one child and no keys
- To handle an underflow at node v with parent u, we consider two cases
- Case 1: the adjacent siblings of v are 2-nodes
  - Fusion operation: we merge v with an adjacent sibling w and move an entry from u to the merged node v'
  - After a fusion, the underflow may propagate to the parent u



Underflow and Transfer

- To handle an underflow at node v with parent u, we consider two cases
- Case 2: an adjacent sibling w of v is a 3-node or a 4-node
  - Transfer operation:
    1. we move a child of w to v
    2. we move an item from u to v
    3. we move an item from w to u
  - After a transfer, no underflow occurs



Algorithm ***remove*(*r*,*k*)**

In: Root *r* of a (2,4) tree, key *k*

Out: {remove data item with key *k* from the tree}

Find the node ***v*** storing key ***k***

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perform a transfer operation

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$v \leftarrow$ parent of v

}

O(1)

}

Time complexity of remove: O(log n)