ST2334 Cheatsheet

for AY24/25 Semester 1
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Basic Concepts of Probability

Multiplication Principle
If r different experiments are to be performed sequentially with n outcomes, there are $n_1 n_2 \dots n_r$ number of possible outcomes.

Addition Principles
If an experiment can be done in k different procedures that **do not overlap**, there are $k_1 + k_2 \dots$.

Permutation
The selection and arrangement of r objects out of n. Order **is taken into consideration**
$P^n_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)..(n-(r-1))$

Combination
The selection of r objects out of n where order is **not taken into consideration**
$\binom{n}{r} = \frac{P^n_r}{P^r_r} = \frac{n!}{r!(n-r)!}$

Inclusion Exclusion Principle

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Conditional probability of B given A: $P(B|A) = \frac{P(A \cap B)}{P(A)}$
Independence: $A \perp B \Leftrightarrow (P(A \cap B) = P(A)P(B))$
Law of Total Probability:
$P(B) = \sum_{i=1}^n P(B \cap A_i) = \sum_{i=1}^n P(A_i)P(B|A_i)$
Bayes Theorem: $P(A_k|B) = \frac{P(A_k)P(B|A_k)}{\sum_{i=1}^n P(A_i)P(B|A_k)}$

Event operations

- $A \cap A' = \emptyset$
- $A \cap \emptyset = \emptyset$
- $A \cup A' = S$
- $(A')' = A$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive)
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive)
- $A \cup B = A \cup (B \cap A')$
- $A = (A \cap B) \cup (A \cap B')$ (special case of Law of Total Probability)

DeMorgan's law: For any n events $A_1, \dots, A_n$:

- $\bigcup A_i = (\bigcap A_i')'$
  - $A \cup B = (A' \cap B')'$
- $\bigcap A_i = (\bigcup A_i')'$
  - $A / capB = (A' \cup B')'$

Probability functions

Discrete Random Variables
Properties:

1. $f(x_i) \geq 0$ for all $x_i \in R_X$
2. $f(x) = 0$ for all $x \notin R_X$
3. $\sum_{i=1}^\infty f(x_i) = 1$ or $\sum_{x_i \in R_X} f(x_i) = 1$

Continuous Random Variables
Properties:

1. **Non-negativity** $f(x) \geq 0$ for all $x \in R_X$, $f(x) = 0$ for $x \notin R_X$
2. **Sum of all probabilities add up to 1** $\int_{R_X} f(x)dx = 1$.
   This particular condition can be represented as $\int_{-\infty}^\infty f(x)dx = 1$
3. For any a, b where $a \leq b$, $P(a \leq X \leq b) = \int_a^b f(x)dx$

Cumulative distribution function (cdf)

Properties
Non-decreasing: $x_1 < x_2 \implies F(x_1) \leq F(x_2)$
Right-continuous: $F(a) = \lim_{x \to a^+} F(x)$
Convergence to 0 and 1 in limits: $\lim_{x \to -\infty} F(x) = 0, \lim_{x \to \infty} F(x) = 1$

Discrete random variables: $\sum_{t \in R_X; t \leq X} P(X = t)$
Continuous random variables: $F(x) = \int_\infty^x f(t)dt$

Expectation

$$\mu_X = E(X) = \int_{-\infty}^\infty x f(x)dx = \int_{x \in R^X} x f(x)dx$$

Properties

1. $E(aX + b) = aE(X) + b$
2. $E(X + Y) = E(X) + E(Y)$
3. $E[g(X)] = \sum_{x \in R_X} g(x)f(x)$ if X is discrete,
   $E[g(X)] = \int_{R_X} g(x)f(x)$ if X continuous

Variance

$$\sigma_x^2 = V(X) = E(X - \mu_X)^2$$
$$V(X) = E(X^2) - [E(X)]^2$$

- $E(X^2) = V(X) + E(X)^2$

With $f(x)$,

- $V(X) = \int_{-\infty}^\infty (x - \mu_X)^2 f(x)dx$ (continuous)
- $V(X) = \sum_{x \in R_X} (x - \mu_X)^2 f(x)$ (discrete)

Discrete Uniform Distribution

$$f_X(x) = \begin{cases} \frac{1}{k} & x = x_1, \dots, x_k \\ 0 & otherwise \end{cases}$$

$$\mu_X = E(X) = \sum_{i=1}^k x_i f_X(X_i) = \frac{1}{k}\sum_{i=1}^k x_i$$

$$\sigma_X^2 = V(X) = E(X^2) - (E(X))^2 = \frac{1}{k}\sum_{i=1}^k x_i^2 - \mu_X^2$$

Bernoulli Random Variable

$$f_X(x) = P(X = x) = \begin{cases} p & x = 1; \\ 1 - p & x = 0; \end{cases}$$

$$\mu_X = E(X) = p$$
$$\sigma_X^2 = V(X) = p(1-p)$$

Binomial Distribution

$X \sim Bin(n, p)$

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \text{for } x = 0, 1, .., n$$

$$E(X) = np$$
$$V(X) = np(1-p)$$

$$F_X(x) = P(X \leq x) = \sum_{i=0}^n p^i (1-p)^{n-i}$$

Negative Binomial Distribution

Number of trials needed until k^{th} success occurs.
$X \sim NB(n, p)$

$$f_X(x) = P(X = x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}, \text{for } x = k, k+1, \ldots$$

$$E(X) = \frac{k}{p}$$

$$V(X) = \frac{(1-p)k}{p^2}$$

Geometric Distribution

Number of trials needed until first success occurs.
$X \sim Geom(p)$

$$f_x(X) = P(X = x) = (1-p)^{x-1}p$$

$$E(X) = \frac{1}{p}$$

$$V(X) = \frac{1-p}{p^2}$$

Poisson random variable

Denotes number of events occuring in a fixed period of time
$X \sim Poisson(\lambda)$

$$f_X(k) = P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

$$F_X(k) = P(X \leq k) = e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!}$$

Poisson Approximation to Binomial

$X \sim Bin(n, p)$. Suppose that $n \to \infty, p \to 0$ such that $\lambda = np$ remains a constant.
Approximately, $X \sim Poisson(np)$.

Good approximation:

$n \geq 20, p \leq 0.05$ or $n \geq 100, np \leq 10$

$$\lim_{p \to 0; n \to \infty} P(X = x) = \frac{e^{-np}(np)^x}{x!}$$

Continuous Uniform Distribution

$X \sim U(a, b)$.

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & otherwise \end{cases}$$

$$E(X) = \frac{a+b}{2}$$

$$V(X) = \frac{(b-a)^2}{12}$$

$$F_X(x) = P(X \leq x) = \frac{1}{b-a}x$$

Exponential Distribution

$X \sim Exp(\lambda)$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x \leq 0 \end{cases}$$

Alternative form

$$f_X(x) = \begin{cases} \frac{1}{\mu} e^{\frac{-x}{\mu}} & x \geq 0 \\ 0 & x \leq 0 \end{cases}$$

where $E(X) = \mu$

$$E(X) = \frac{1}{\lambda}$$

$$V(X) = \frac{1}{\lambda^2}$$

$$F(X) = P(X \leq x) = 1 - e^{-\lambda x}$$

Theorem 15: Memoryless property

Suppose that X has an exponential distribution with $\lambda > 0$.

For any two positive numbers s, t,

$$P(X > s + t | X > s) = P(X > t)$$

Useful summations

$$\text{if } |r| < 1,$$

$$\sum_{k=0}^{n} r^n = \frac{1 - r^n}{1 - r}$$

Finding total number of permutations given combinations

There are 6 people, A, B, C, D, E, F. Find the total number of ways to arrange them, if C must come after A and B, and F must come after E.

Total number of ways: $6! = 720$

Total number of ways, if C comes after A and B, is equal to total number of ways where A > B > C, and B > A > C (and the rest can be anywhere) - 2 combinations out of the possible $3! = 6$.
$\therefore$ C comes after A and B = $\frac{720}{3!} \times 2 = 240$

Total number of ways, if F must come after E = F > E, 1 combination out of the possible $2! = 2$
$\therefore$ F comes after E = $\frac{240}{3} \times 1 = 120$

Final answer = 120.