A Second-Order IMEX Method for Radiation Hydrodynamics

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Introduction

- Some of the most challenging computational problems are those with coupled physics.
- Coupled physics has two major challenges:
  - Software implementation.
  - Disparities between temporal and spatial scales.
- This work focuses on disparities in temporal scales (stiffness).
- The effects of nonlinearity are also examined.
- For this work, the main intended application is Radiation-Hydrodynamics.

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Main Idea

- Use an explicit scheme on the slower timescale and an implicit scheme on the faster timescale.
- These are referred to as IMEX schemes (IMplicit-EXplicit).
- These schemes are often computationally less expensive than a purely implicit scheme.
- In this work, a scheme for the general equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u_t = E(u) + I(u)$$

is developed, where $E(u)$ and $I(u)$ are the explicit and implicit components, respectively.

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Outline

1. Theory
2. A Representative Test Problem
3. Radiation Hydrodynamics
4. Conclusions and Future Work

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Theory

- Conventional IMEX schemes tend to split the implicit operator, for example the second-order IMEX-BDF2 scheme (Hundsdorfer and Ruuth, 2007) is

$$\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} = 2E(u^n) - E(u^{n-1}) + I(u^{n+1})$$

- This scheme performs well in the implicit limit, but as $I(u) \to 0$, displays poor treatment of the explicit operator.
- To overcome poor performance in one limit, design IMEX schemes based on good explicit and implicit schemes.

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BDF2-RK2

1. Whenever $I(u) \equiv 0$, the method should reduce to the second order, explicit Runge-Kutta (RK-2) method. Specifically, in this case

$$v^{n+1} = u^n + \Delta t E(u^n)\,,$$
$$u^{n+1} = u^n + \Delta t[E(u^n) + E(v^{n+1})]/2\,.$$

2. Whenever $E(u) \equiv 0$, the method should reduce to the second order, backward-difference formula (BDF-2) method. For a constant time step Δt, we then have

$$\frac{3(u^{n+1} - u^n) - (u^n - u^{n-1})}{2\Delta t} = I(u^{n+1})\,.$$

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BDF2-RK2

- The following IMEX method satisfies these properties:

$$v^{n+1} = u^n + \Delta t[E(u^n) + I(v^{n+1})]\,,$$

$$\frac{3}{2}\left(\frac{u^{n+1} - u^n}{\Delta t} - \frac{1}{2}\left[E(u^n) + E(v^{n+1})\right]\right) - \frac{1}{2}\left(\frac{u^n - u^{n-1}}{\Delta t} - \frac{1}{2}\left[E(u^{n-1}) + E(v^n)\right]\right) = I(u^{n+1})\,.$$

- Since the first step requires a separate nonlinear solve, an alternative, without loss of accuracy, is

$$v^{n+1} = u^n + \Delta t[E(u^n) + I(u^{n+1})]$$

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A Representative ODE

- To examine this method, use a nonlinear generalization of Prothero and Robinson equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(T - T_e) = -\sigma(T - T_e)^p.$$

with equilibrium solution $T_e = 1 + \frac{1}{2}\sin(\pi t)$.

- This problem has two timescales: relaxation, $O(1/\sigma)$, and equilibrium, $O(1)$.
- The exact solution to this problem is

$$T(t) = \begin{cases} T_e(t) + (T(0) - T_e(0))e^{-\sigma t} & p = 1 \\ T_e(t) + \left((p-1)\sigma t + (T(0) - T_e(0))^{1-p}\right)^{\frac{1}{1-p}} & p > 1 \end{cases}$$

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Homogeneous System

- To avoid effects from the nonhomogeneity of the single ODE, it can be written as a three-equation system.
- The system governing T is

$$\begin{aligned} T' &= T_i - \sigma(T - T_e)^p \\ T_e' &= T_i \\ T_i' &= -\pi^2(T_e - 1) \end{aligned}$$

  with extra initial conditions $T_e(0) = 1, T_e'(0) = \pi/2$.
- This system can be made arbitrarily stiff, in that eigenvalues are

$$\lambda = \pm i\pi, -\sigma p(T - T_e)^{p-1}$$

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Implicit-Explicit Splitting

- The homogeneous system can be split into explicit and implicit components as

$$\mathbf{E}(\mathbf{T}) = \begin{bmatrix} T_i \\ T_i \\ -\pi^2(T_e - 1) \end{bmatrix}$$

$$\mathbf{I}(\mathbf{T}) = \begin{bmatrix} -\sigma(T - T_e)^p \\ 0 \\ 0 \end{bmatrix}$$

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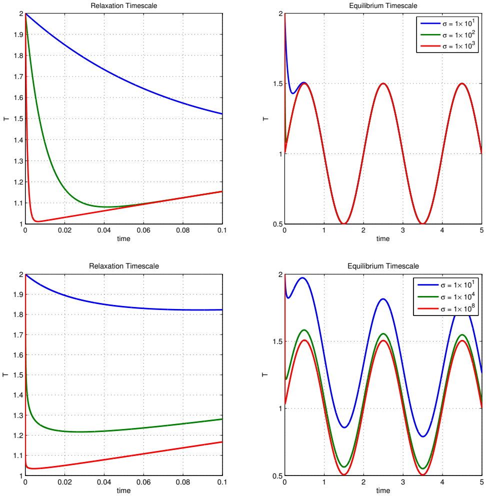
Sample Solutions

Linear

$$T(0) = 2,$$
$$\sigma = 10^1, 10^2, 10^3,$$
$$p = 1.$$

Nonlinear

$$T(0) = 2,$$
$$\sigma = 10^1, 10^4, 10^8,$$
$$p = 5.$$



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Numerical Results

- All results are done with full nonlinear convergence at each step.
- For two-step methods, the first step is done with backward euler as

$$T^1 = T^0 + \Delta t[E(T^1) + I(T^1)]$$

- BDF2-RK2 is compared with four implicit schemes: Crank-Nicolson (C-N), BDF2, Trapezoidal BDF2 (TR-BDF2), and Backward Euler (B-E).
- Two error norms are shown:
  1. Total: Error from $t = 0$ to $t = T$
  2. Equilibrium: Error from $t = 1$ to $t = T$

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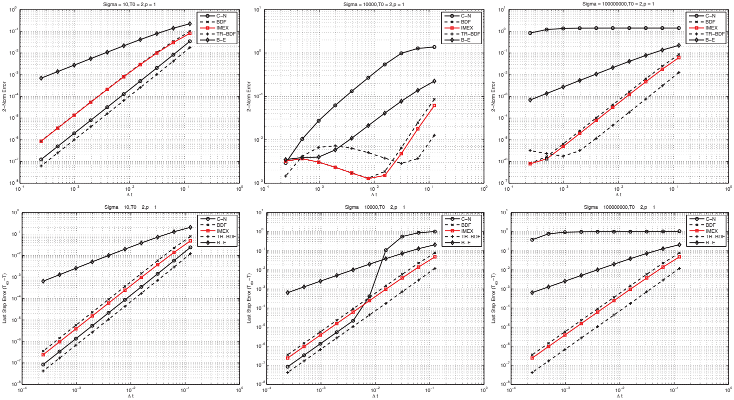
Error - Linear



Error Convergence for $\sigma = 10^1, 10^4, 10^8$ and $p = 1$.
(Top) Total Error. (Bottom) Equilibrium Error.

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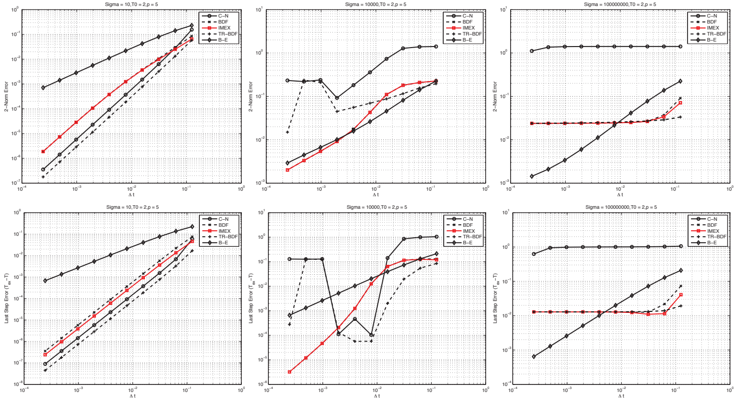
Total Error - Nonlinear



Error Convergence for $\sigma = 10^1, 10^4, 10^8$ and $p = 1$.
(Top) Total Error. (Bottom) Equilibrium Error.

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Variable Timestepping

- To resolve fast timescales, variable timestepping is often used.
- Within BDF2-RK2, it can be easily implemented by only accepting timesteps that satisfy the following inequality.

$$\frac{||u^{n+1} - v^{n+1}||}{\Delta t} < 1$$

- In the following results, the initial timestep is decreased by a factor of two until the inequality is satisfied.

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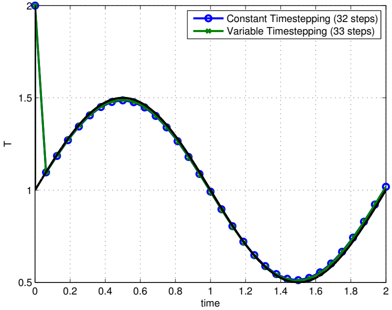
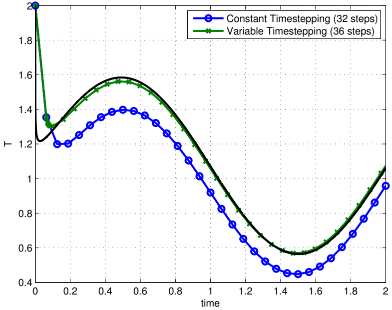
Variable Timestepping





(Left) $\sigma = 10^4$ and $p = 1$. (Right) $\sigma = 10^4$ and $p = 5$.

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P_1-Euler Radiation Hydrodynamics

- The rad-hydro model examined is the non-relativistic Euler equations with a grey P_1 radiation treatment without scattering, in nondimensional form as

$$\begin{aligned}
\partial_t\rho + \partial_x(\rho v) &= 0,\\
\partial_t(\rho v) + \partial_x(\rho v^2 + p) &= -\mathbb{P}S_F,\\
\partial_t(\rho E) + \partial_x(\rho E + p)v &= -\mathbb{P}S_E,\\
\partial_t E_r + \mathbb{C}\partial_x F_r &= S_E,\\
\partial_t F_r + \frac{1}{3}\mathbb{C}\partial_x E_r &= \mathbb{C}S_F,
\end{aligned}$$

$$\begin{aligned}
S_E &= \mathbb{C}\sigma_t\Phi + \sigma_t v F_{r0},\\
S_F &= -\sigma_t F_{r0} + \sigma_t\frac{v}{\mathbb{C}}\Phi,\\
\Phi &= T^4 - E_r,\\
F_{r0} &= F_r - \frac{4}{3}\frac{v}{\mathbb{C}}E_r.
\end{aligned}$$

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P_1-Euler Radiation Hydrodynamics

- The nondimensional parameters are

$$\mathbb{P} = \frac{a_r T_\infty^4}{\rho_\infty a_\infty^2}, \qquad \mathbb{C} = \frac{c}{a_\infty},$$

where a_r is the radiation constant, c the speed of light, and a_∞ the reference sound speed.

- The system may be manipulated to yield the following conservation statements:

$$\begin{aligned} \partial_t\left(\rho v + \frac{\mathbb{P}}{\mathbb{C}} F_r\right) + \partial_x\left(\rho v^2 + p + \frac{1}{3}\mathbb{P} E_r\right) &= 0 \\ \partial_t(\rho E + \mathbb{P} E_r) + \partial_x\left[(\rho E + p)v + \mathbb{P}\mathbb{C} F_r\right] &= 0 \end{aligned}$$

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Implicit-Explicit Splitting

- For this case, we choose to treat the hydrodynamics explicitly and all radiative effects implicitly.
- Specifically, we define the explicit and implicit operators as

$$\mathbf{E}(\mathbf{u}) = -\begin{pmatrix} \partial_x(\rho v) \\ \partial_x(\rho v^2 + p) \\ \partial_x(\rho E + p)v \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{I}(\mathbf{u}) = -\begin{pmatrix} 0 \\ \mathbb{P}S_F, \\ \mathbb{P}S_E, \\ \mathbb{C}\partial_x F_r - S_E \\ \frac{1}{3}\mathbb{C}\partial_x E_r - \mathbb{C}S_F \end{pmatrix}$$

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Test Problems

- To numerically test this timestepping scheme, manufactured solutions based on those of McClarren and Lowrie are used to verify accuracy in both the equilibrium diffusion limit (large opacity) and streaming limit (small opacity).
- For discretization in space, a Discontinuous Galerkin framework is used with Roe's approximate Riemann solver for the Riemann problem.

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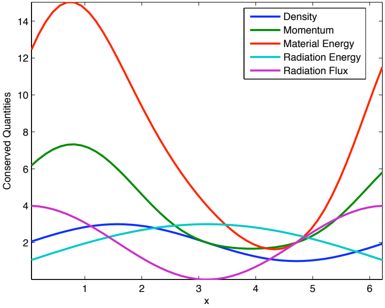
Equilibrium Diffusion Limit

Initial Solution

- The opacity is large and of $O(\mathbb{C})$.
- $\mathbb{C} = 1000$, $\sigma = 1000$, and $\mathbb{P} = 0.001$.



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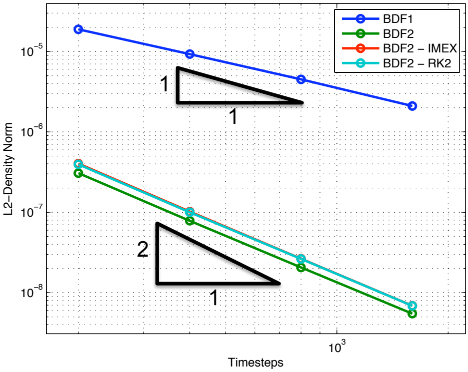
Equilibrium Diffusion Limit

Error Convergence



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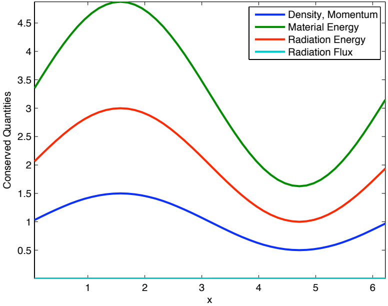
Streaming Limit

Initial Solution

- Radiation and hydrodynamics are weakly coupled, so that the radiation energy travels much faster than fluid energy.
- The material is optically thin and the radiation streams through.
- $\mathbb{C} = 10$, $\sigma = 1$, and $\mathbb{P} = 0.001$.



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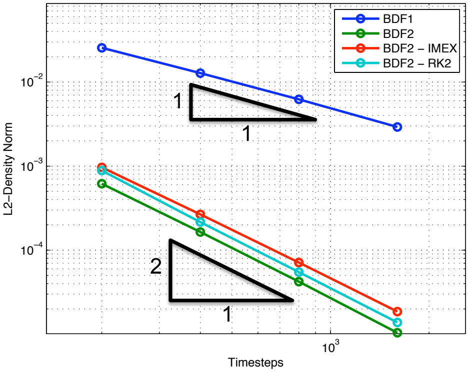
Streaming Limit

Error Convergence



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Conclusions and Future Work

- Conclusions
  - Linear analysis does not carry over to the nonlinear problem.
  - ‘Stepping over’, or under-resolving the initial layer of the fast timescale in the nonlinear case results in an initial error that persists much longer than a linear analysis predicts.
  - This scheme allows for the natural implementation of timestep control, which does help, however resolving the fast timescale is still necessary.
  - BDF2-RK2 performs as well as common implicit schemes on radiation hydrodynamics test problems.
- Future Work
  - Implement variable timestepping for Radiation Hydrodynamics.
  - Examine BDF2-RK2 in the presence of shocks and other discontinuities.

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- Professor Philip Roe
- Professor Ken Powell

MS437, Room 221. 10:30-10:55
Level Set Methods for Multimaterial Flows in Radiative Hydrodynamics - David Starinshak

Questions?

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