On the Consistency of Velocity in Riemann Solvers for Godunov-Type Lagrangian Methods

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September 25, 2009

1 Introduction

Let interface fluxes be $\mathbf{f}(\mathbf{u}_L, \mathbf{u}_R) = \hat{\mathbf{f}}$ and internal fluxes $\mathbf{f}(\mathbf{u}) = \mathbf{f}$. Define the interface velocity as u^*. I define consistency of velocity between Arbitrary Lagrangian-Eulerian (ALE) and Lagrangian fluxes in the sense $\hat{\mathbf{f}}_{\text{ALE}}(\mathbf{u}_L, \mathbf{u}_R, u) = \hat{\mathbf{f}}_{\text{Lag}}(\mathbf{u}_L, \mathbf{u}_R)$ for the consistent choice of u. In general, I can only enforce this in one flux. Since Lagrangian methods require constant mass in each cell, consistency will refer to

$$\hat{\mathbf{f}}_{\rho,\text{ALE}}(\mathbf{u}_L, \mathbf{u}_R, u) = 0 = \hat{\mathbf{f}}_{\rho,\text{Lag}}(\mathbf{u}_L, \mathbf{u}_R)$$

2 Exact Riemann Solver

Let the Solution to the Exact Riemann Solver be $\mathbf{u}^*$. The flux on a Lagrangian grid is then

$$\hat{\mathbf{f}}_{\text{ex,ALE}} = \mathbf{f}(\mathbf{u}^*) - u^*\mathbf{u}^* = \begin{bmatrix} 0 \\ p^* \\ u^*p^* \end{bmatrix} = \hat{\mathbf{f}}_{\text{ex,Lag}}$$

This is consistent, provided that u^* and the flux are evaluated at the same time location.

3 Roe's Riemann Solver

3.1 Lagrangian

First, Roe's Riemann Solver in the Lagrangian Framework is

$$\hat{\mathbf{f}}_{\text{Roe,Lag}} = \begin{bmatrix} 0 \\ p^*_{\text{Roe}} \\ u^*_{\text{Roe}}p^*_{\text{Roe}} \end{bmatrix}$$

for some choice of vertex velocity u^* and pressure p^*. Roe suggests

$$\begin{aligned} u^*_{\text{Roe}} &= \frac{1}{2}(u_L + u_R) - \frac{\Delta p}{2\tilde{\rho}\tilde{a}} \\ p^*_{\text{Roe}} &= \frac{1}{2}(p_L + p_R) - \frac{1}{2}\tilde{\rho}\tilde{a}\Delta u \end{aligned}$$

where I have Roe-averaged variables as

$$\begin{aligned}
\tilde{\rho} &= \sqrt{\rho_L \rho_R} \\
\tilde{u} &= \frac{\sqrt{\rho_L} u_L + \sqrt{\rho_R} u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \\
\tilde{H} &= \frac{\sqrt{\rho_L} H_L + \sqrt{\rho_R} H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \\
\tilde{a} &= \sqrt{(\gamma - 1)\left(\tilde{H} - \frac{1}{2}\tilde{u}^2\right)}
\end{aligned}$$

I'll refer to this u^*_{Roe} as the 'normal' choice of velocity.

3.2 Arbitrary Lagrangian-Eulerian

In the ALE framework, define the wavestrengths as

$$\begin{aligned}
\alpha_1 &= \frac{1}{2\tilde{a}^2}(\Delta p - \tilde{\rho}\tilde{a}\Delta u) \\
\alpha_2 &= -\frac{1}{\tilde{a}^2}(\Delta p - \tilde{a}^2 \Delta \rho) \\
\alpha_3 &= \frac{1}{2\tilde{a}^2}(\Delta p + \tilde{\rho}\tilde{a}\Delta u)
\end{aligned}$$

Observe that for the choice of $u^* = \tilde{u}$

$$\hat{\mathbf{f}}_\rho \neq 0 \tag{1}$$

but that

$$\hat{\mathbf{f}}_\rho = \frac{1}{2}\left(\rho_L(u_L - \tilde{u}) + \rho_R(u_R - \tilde{u}) - \frac{\Delta p}{\tilde{a}}\right)$$

To find a consistent choice of velocity, u^*_c, I have

$$\hat{\mathbf{f}}_\rho = \frac{1}{2}\left(\rho_L u_L + \rho_R u_R - u^*_c(\rho_L + \rho_R) - |\tilde{u} - \tilde{a} - u^*_c|\alpha_1 - |\tilde{u} - u^*_c|\alpha_2 - |\tilde{u} + \tilde{a} - u^*_c|\alpha_3\right)$$

Write this in terms of the Roe-averaged velocity, such that

$$u^*_c = \tilde{u} + \dot{u}$$

This simplifies the math a bit and leads to

$$\begin{aligned}
\hat{\mathbf{f}}_\rho &= \frac{1}{2}\left(\rho_L(u_L - \tilde{u}) + \rho_R(u_R - \tilde{u}) - \dot{u}(\rho_L + \rho_R) - |-\tilde{a} - \dot{u}|\alpha_1 - |\dot{u}|\alpha_2 - |\tilde{a} - \dot{u}|\alpha_3\right) \\
&= \frac{1}{2}\left(\rho_L(u_L - \tilde{u}) + \rho_R(u_R - \tilde{u}) - \dot{u}(\rho_L + \rho_R)\right. \\
&\quad \left. -\text{sign}(\tilde{a} + \dot{u})(\tilde{a} + \dot{u})\alpha_1 - \text{sign}(\dot{u})\dot{u}\alpha_2 - \text{sign}(\tilde{a} - \dot{u})(\tilde{a} - \dot{u})\alpha_3\right) \\
&= \frac{1}{2}\left(\rho_L(u_L - \tilde{u}) + \rho_R(u_R - \tilde{u}) - \dot{u}(\rho_L + \rho_R)\right. \\
&\quad \left. -\tilde{a}(\text{sign}(\tilde{a} + \dot{u})\alpha_1 + \text{sign}(\tilde{a} - \dot{u})\alpha_3) - \dot{u}(\text{sign}(\tilde{a} + \dot{u})\alpha_1 - \text{sign}(\tilde{a} - \dot{u})\alpha_3 + \text{sign}(\dot{u})\alpha_2)\right)
\end{aligned}$$

where the sign function has been introduced to handle various cases. Setting $\hat{\mathbf{f}}_\rho = 0$ gives the result that

$$\dot{u} = \frac{\rho_L(u_L - \tilde{u}) + \rho_R(u_R - \tilde{u}) - \tilde{a}(\mathrm{sign}(\tilde{a} + \dot{u})\alpha_1 + \mathrm{sign}(\tilde{a} - \dot{u})\alpha_3)}{(\rho_L + \rho_R) + (\mathrm{sign}(\tilde{a} + \dot{u})\alpha_1 - \mathrm{sign}(\tilde{a} - \dot{u})\alpha_3 + \mathrm{sign}(\dot{u})\alpha_2)}$$

3.2.1 Cases 1 and 2: $|\dot{u}| < \tilde{a}$

If I assume that $|\dot{u}| < a$, there are two cases: $\dot{u} > 0$ and $\dot{u} < 0$. Both cases are represented by a single sign function.

$$\begin{aligned}
\dot{u} &= \frac{\rho_L(u_L - \tilde{u}) + \rho_R(u_R - \tilde{u}) - \tilde{a}(\alpha_1 + \alpha_3)}{(\rho_L + \rho_R) + (\alpha_1 - \alpha_3 + \mathrm{sign}(\dot{u})\alpha_2)} \\
&= \frac{\rho_L(u_L - \tilde{u}) + \rho_R(u_R - \tilde{u}) - \frac{\Delta p}{\tilde{a}}}{(\rho_L + \rho_R) - \frac{\tilde{\rho}\Delta u}{\tilde{a}} - \mathrm{sign}(\dot{u})\left(\frac{\Delta p}{\tilde{a}^2} - \Delta\rho\right)}
\end{aligned}$$

3.2.2 Case 3: $\dot{u} > \tilde{a}$

$$\dot{u} = \frac{\rho_L(u_L - \tilde{u}) + \rho_R(u_R - \tilde{u}) + \tilde{\rho}\Delta u}{(\rho_L + \rho_R) - \Delta\rho}$$

3.2.3 Case 4: $\dot{u} < -\tilde{a}$

$$\dot{u} = \frac{\rho_L(u_L - \tilde{u}) + \rho_R(u_R - \tilde{u}) - \tilde{\rho}\Delta u}{(\rho_L + \rho_R) + \Delta\rho}$$

I'm sure there's a very good reason why cases 3 and 4 don't occur, but I don't know it off the top of my head.

4 Sod Problem

Start with $[\rho, u, p]_L = [1.0, 0.0, 1.0]$ and $[\rho, u, p]_R = [0.125, 0.0, 0.1]$. For comparison, from the initial conditions at the interface, I have

$$\begin{aligned}
\tilde{\rho} &= 0.35355 \\
\tilde{u} &= 0 \\
\tilde{a} &= 1.1519
\end{aligned}$$

The value from the exact Riemann Solver is $u^*_{\mathrm{ex}} = 0.92745$. I also have that $u^*_{\mathrm{Roe,Lag}} = 1.105$ and that $u^*_c = 0.8417$.

All results have

1. CFL = 0.8, where

$$\Delta t = \mathrm{CFL} \times \min_{i \in \mathrm{cells}} \left(\frac{\Delta x_i}{|u_i| + a_i + \max(|u^*_{i-\frac{1}{2}}|, |u^*_{i+\frac{1}{2}}|)} \right)$$

2. 400 initially uniform cells.

3. Final time t = 0.15.

4. A Roe-type Riemann Solver, (Lagrangian or Leaky-Lagrangian).

5. Data with the velocity chosen as 'normal', u^*_{Roe}, or 'consistent', $u^*_c = \tilde{u} + \dot{u}$.

6. Density and Internal Energy plots, and a zoomed in view around the contact.

7. All first-order results with Forward Euler, and second-order results with RK2, since in the ALE framework, RK2 is consistent while Hancock is not.

Note: In the Roe Lagrangian, when the consistent velocity is used, our flux is then

$$\hat{\mathbf{f}}_{\text{Roe,Lag}} = \begin{bmatrix} 0 \\ p^*_{\text{Roe}} \\ u^*_c p^*_{\text{Roe}} \end{bmatrix}$$

4.1 Observations

In Figures 1-6, for both Lagrangian and Leaky Lagrangian, the consistent choice improves the error at the contact, however only slightly. In Figure 7, despite Hancock not being consistent, I get a much better result in the Lagrangian Scheme. Consistency is less of an issue with the pure Lagrangian, since I have enforced zero mass flux. In Figures 8-10, I can say that Roe performs as well as the Exact solver in the Lagrangian case, which is interesting since we generally treat the Exact Solver as the best we can achieve

5 Extension to 2D

Seems like I can solve a least squares consistency type of problem, to get cell velocity based on this condition.

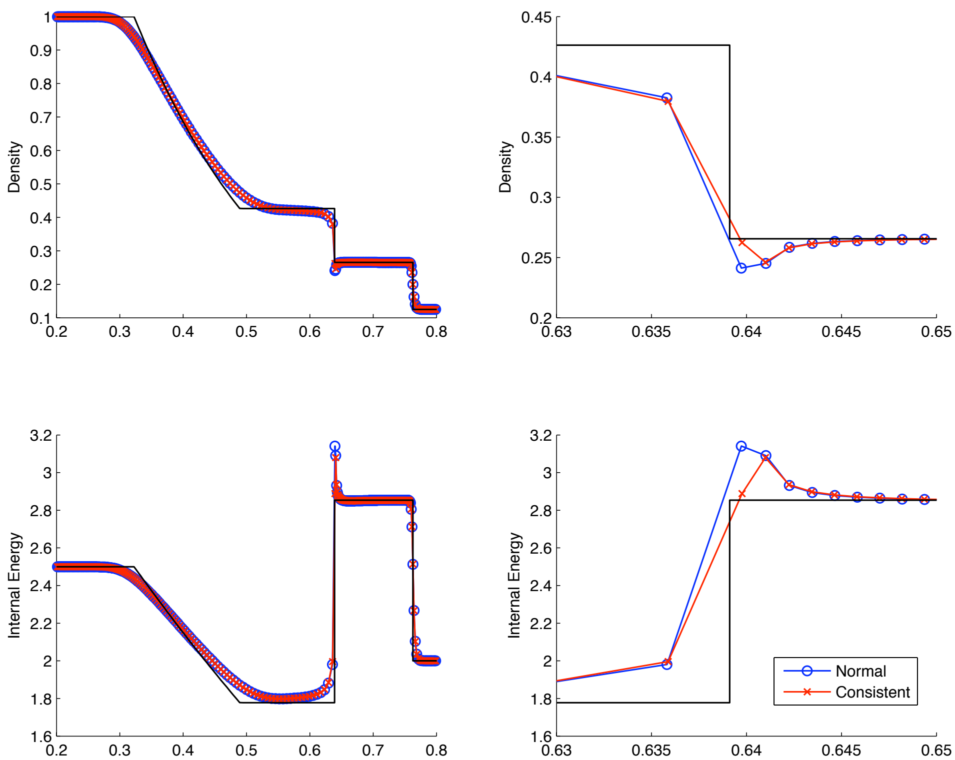

Figure 1: First-Order Lagrangian Method

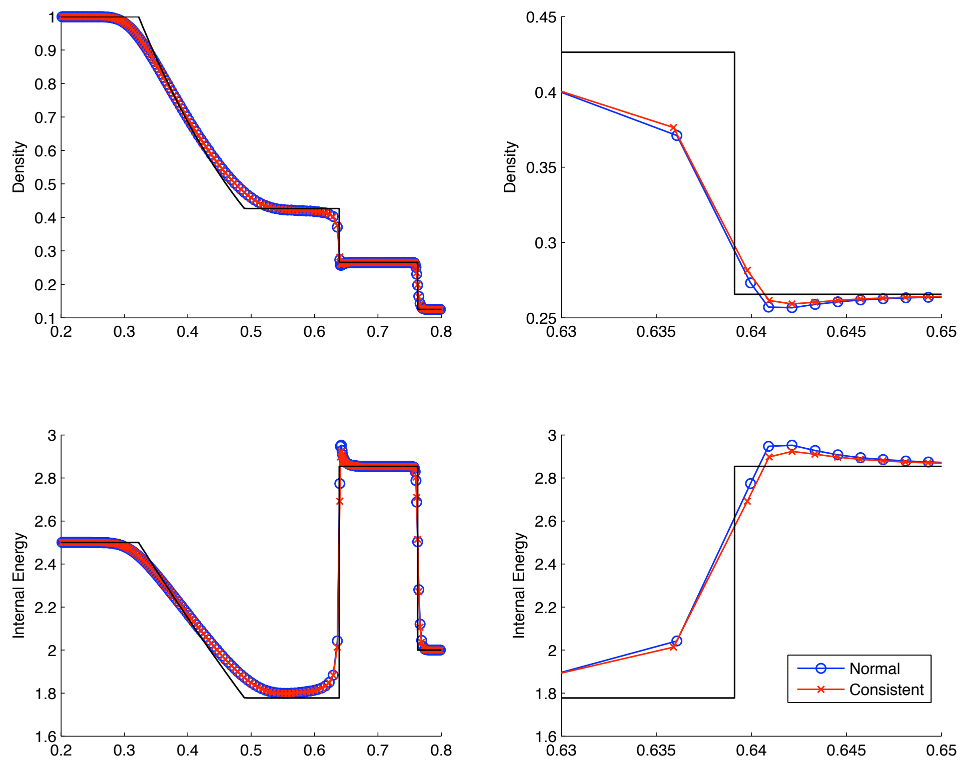

Figure 2: First-Order Leaky Lagrangian Method

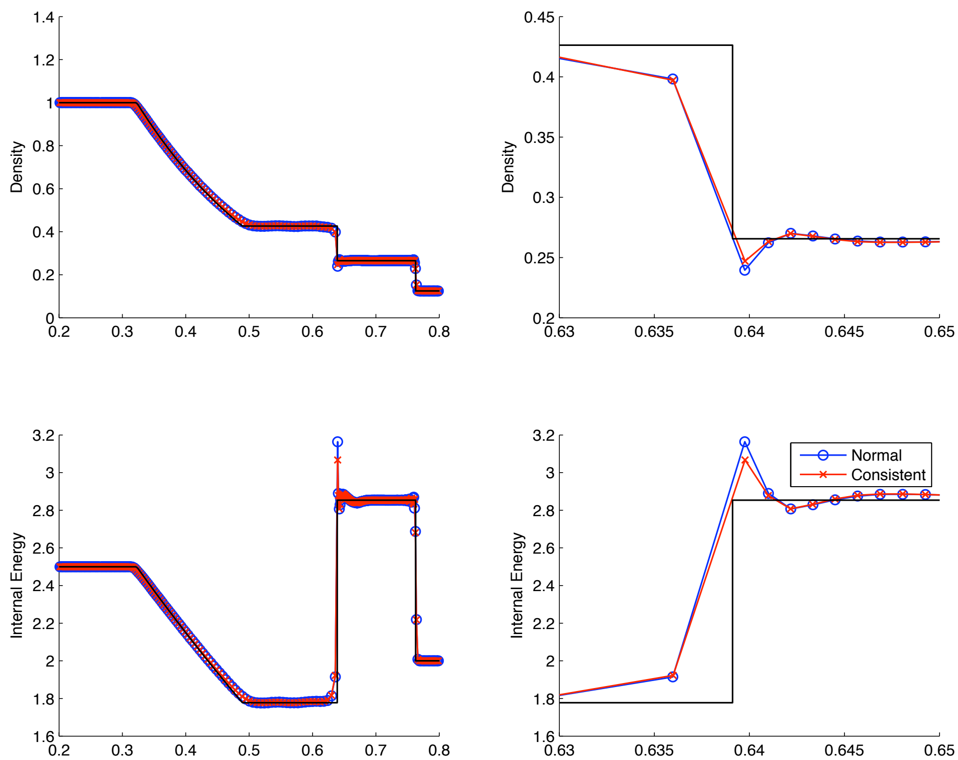

Figure 3: Second-order Lagrangian Method, harmonic limiter.

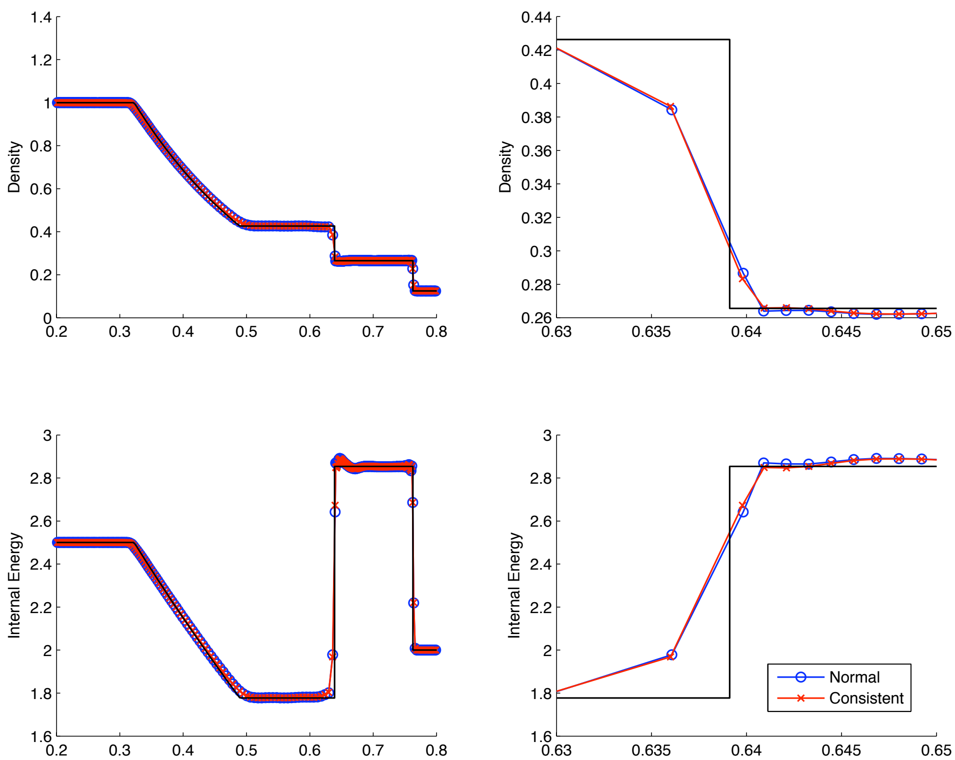

Figure 4: Second-order Leaky Lagrangian Method, harmonic limiter.

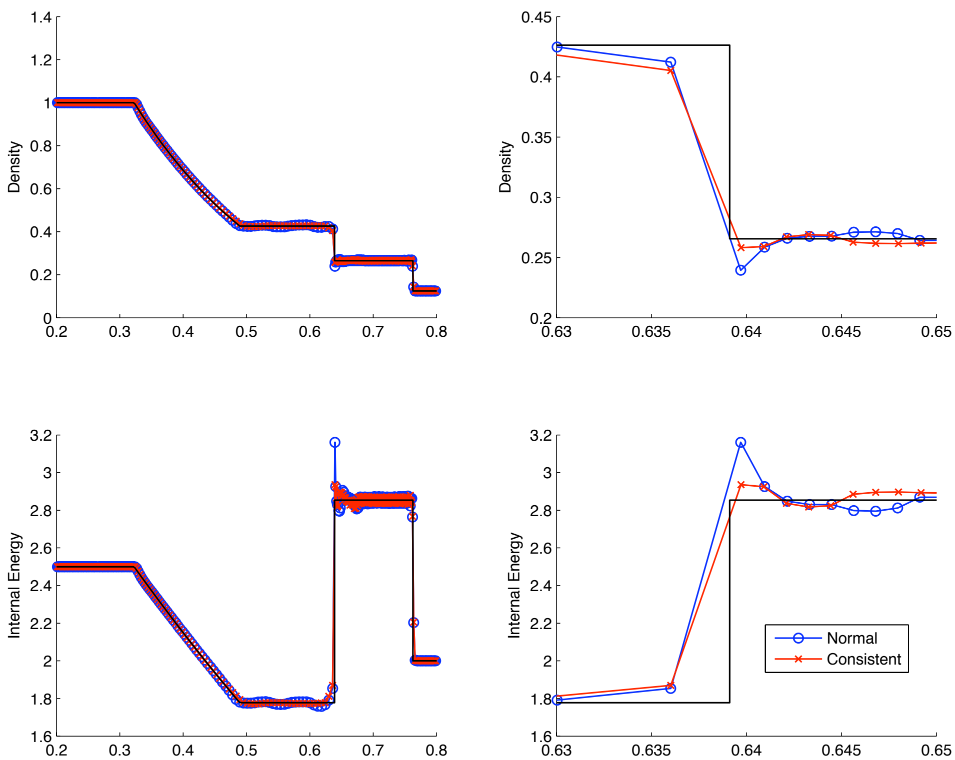

Figure 5: Second-order Lagrangian Method, superbee limiter.

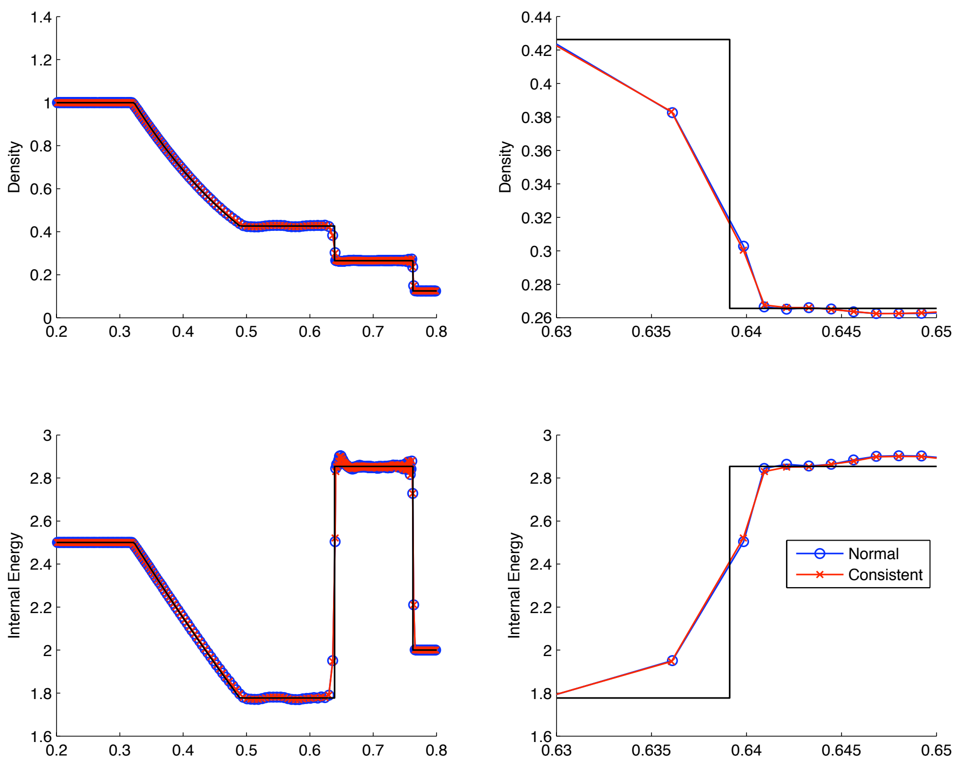

Figure 6: Second-order Leaky Lagrangian Method, superbee limiter.

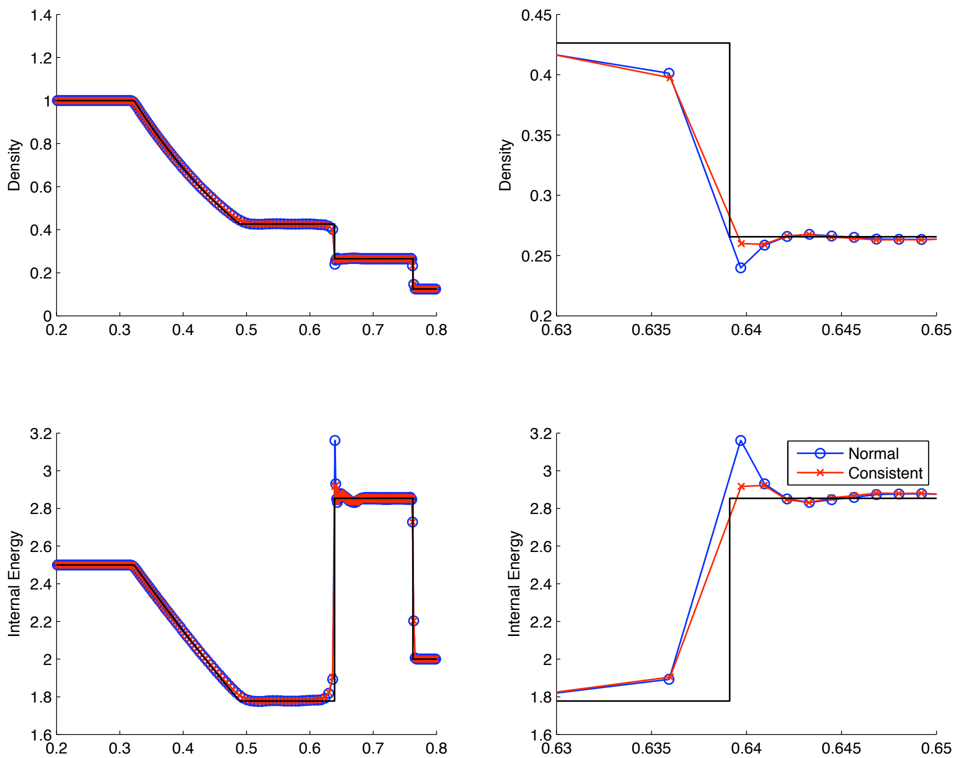

Figure 7: Second-order Lagrangian Method, harmonic limiter with Hancock timestepping instead of RK2.

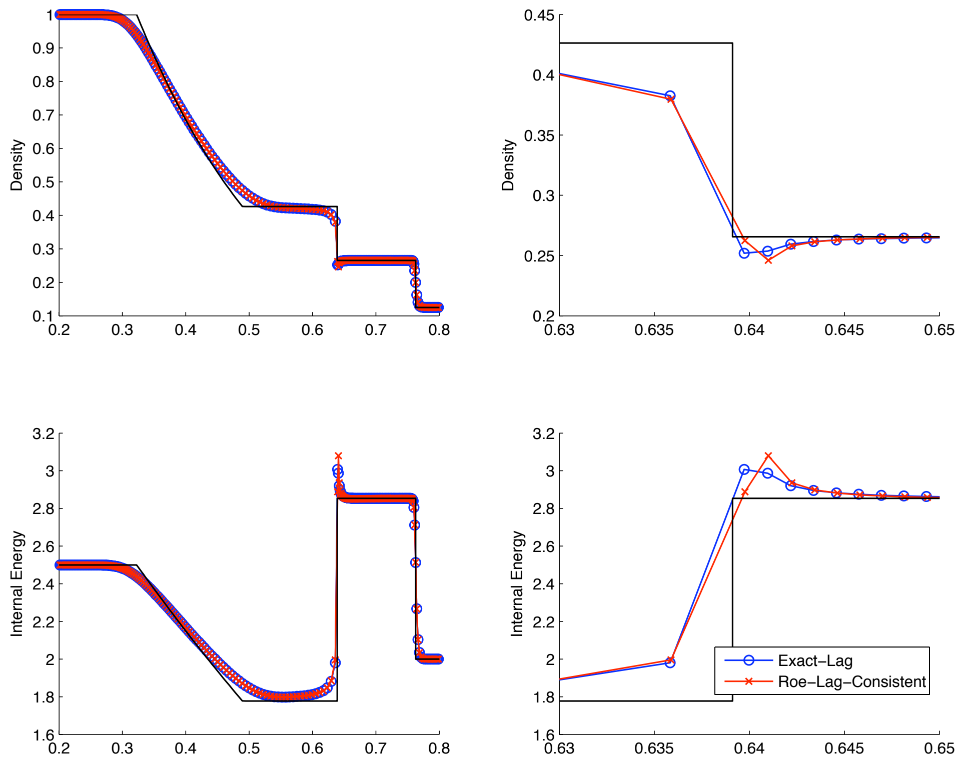

Figure 8: First-order Comparison of Exact Lagrangian and Roe Lagrangian with Consistent Choice

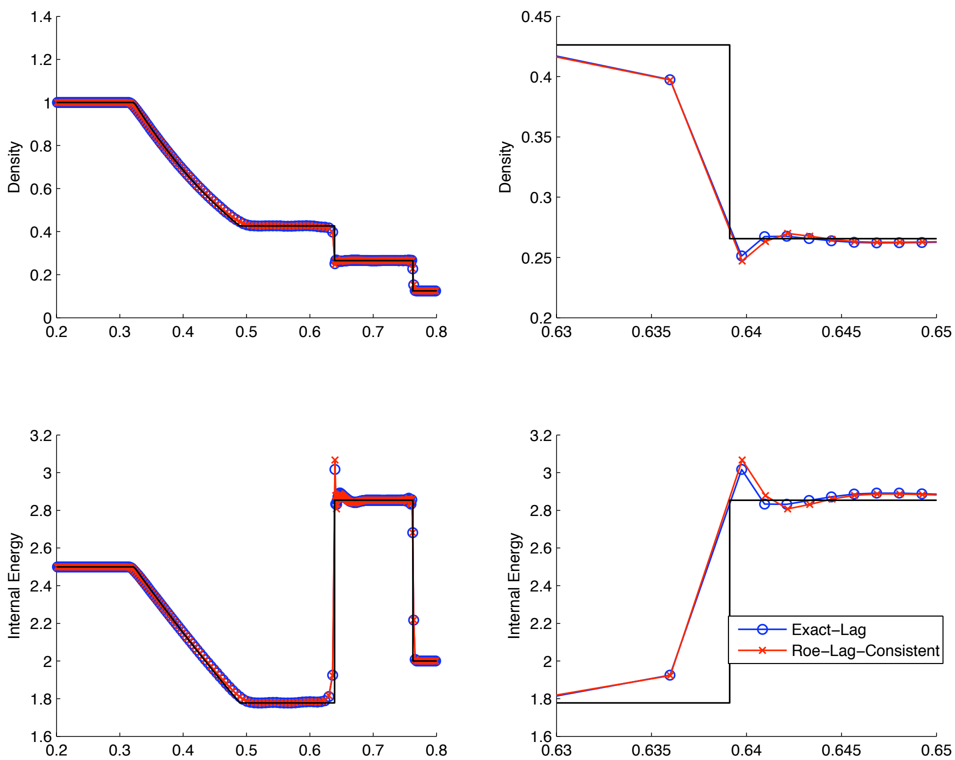

Figure 9: Second-order Comparison of Exact Lagrangian and Roe Lagrangian with Consistent Choice with RK2 timestepping and harmonic limiting.

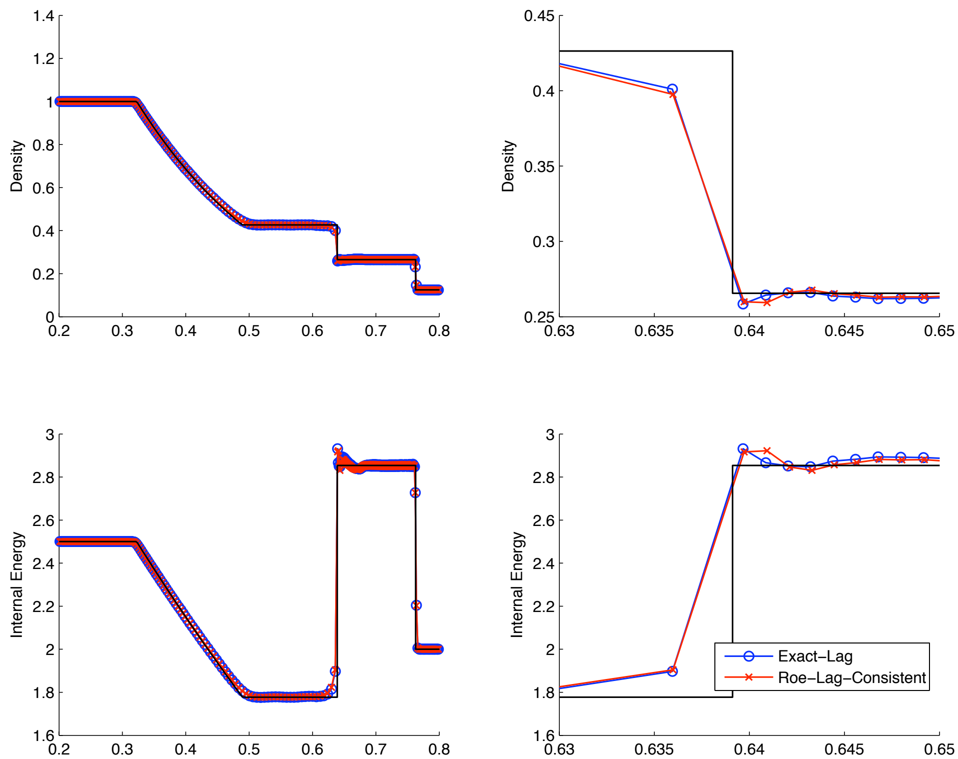

Figure 10: Second-order Comparison of Exact Lagrangian and Roe Lagrangian with Consistent Choice with Hancock timestepping and harmonic limiting.