A system of equations with a linear shock curve and a contact discontinuity.

1 Governing Equations

Start with

$$\mathbf{u}_t + \mathbf{f}_x = 0 \tag{1}$$

or expanded as

$$\begin{bmatrix} u \\ a \\ b \end{bmatrix}_t + \begin{bmatrix} \frac{1}{2}\left(u^2 + a^2 + b^2\right) \\ ua \\ ub \end{bmatrix}_x = 0 \tag{2}$$

This can be decomposed as

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}} = \begin{bmatrix} u & a & b \\ a & u & 0 \\ b & 0 & u \end{bmatrix} \tag{3}$$

where further decomposition leads to $\mathbf{A} = \mathbf{R}\Lambda\mathbf{L}$. Define a propagation speed

$$c^2 = a^2 + b^2 \tag{4}$$

and write the eigenvalue matrix as

$$\Lambda = \begin{bmatrix} u-c & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u+c \end{bmatrix} \tag{5}$$

and right and left eigenvector matrices as

$$\mathbf{R} = \begin{bmatrix} -c & 0 & c \\ a & -b & a \\ b & a & b \end{bmatrix} \tag{6}$$

$$\mathbf{L} = \frac{1}{2c^2}\begin{bmatrix} -c & a & b \\ 0 & -2b & 2a \\ c & a & b \end{bmatrix} \tag{7}$$

Looking at $\mathbf{L}\mathrm{d}\mathbf{u}$, across a contact, $\lambda = u$,

$$\mathrm{d}u = 0 \qquad \frac{\mathrm{d}b}{\mathrm{d}a} = -\frac{a}{b} \tag{8}$$

The second condition corresponds to c constant. Thus both conditions combine to give that u, c and $u \pm c$ is constant.

1.1 Hugoniot Curve

Start with

$$
\begin{aligned}
S[\mathbf{u}] &= [\mathbf{f}] && (9)\\
S[u] &= \frac{1}{2}[u^2+c^2] && (10)\\
S[a] &= [ua] && (11)\\
S[b] &= [ub] && (12)
\end{aligned}
$$

Cross multiply to get rid of S and get

$$
\begin{aligned}
[u][ua] &= \frac{1}{2}[u^2+c^2][a] && (13)\\
[u][ub] &= \frac{1}{2}[u^2+c^2][b] && (14)\\
[a][ub] &= [b][ua] && (15)
\end{aligned}
$$

Looking at the third equation,

$$(a_R - a_L)(u_R b_R - u_L b_L) - (b_R - b_L)(u_R a_R - u_L a_L) = 0 \tag{16}$$

$$[u](b_L a_R - b_R a_L) = 0 \tag{17}$$

This gives that

$$\frac{b_L}{a_L} = \frac{b_R}{a_R} \qquad \frac{c_L}{a_L} = \frac{c_R}{a_R} \tag{18}$$

now the first two equations give

$$(a_R + a_L)[u]^2 - [c^2][a] = 0 \tag{19}$$

$$(b_R + b_L)[u]^2 - [c^2][b] = 0 \tag{20}$$

Combining these, gives $[c^2] = c_R^2\left(1 - \frac{a_L^2}{a_R^2}\right)$ and with lots of rearrangement,

$$u_L \pm c_L = u_R \pm c_R \tag{21}$$

1.2 Expansion Curve

Define the characteristic variables as $\mathrm{d}\mathbf{w} = \mathbf{L}\mathrm{d}\mathbf{u}$

$$
\begin{aligned}
\mathrm{d}\mathbf{w} &= \frac{1}{2c^2}\begin{bmatrix} -c & a & b \\ 0 & -2b & 2a \\ c & a & b \end{bmatrix}\begin{bmatrix} \mathrm{d}u \\ \mathrm{d}a \\ \mathrm{d}b \end{bmatrix} && (22)\\
\mathrm{d}\mathbf{w} &= \frac{1}{2c^2}\begin{bmatrix} -c\mathrm{d}u + a\mathrm{d}a + b\mathrm{d}b \\ -2b\mathrm{d}a + 2a\mathrm{d}b \\ c\mathrm{d}u + a\mathrm{d}a + b\mathrm{d}b \end{bmatrix} && (23)
\end{aligned}
$$

Across wave λ_i, $\mathrm{d}\mathbf{w}_j = 0$ for $i \neq j$. So for a right moving expansion, $\lambda = u + c$,

$$-b\mathrm{d}a + a\mathrm{d}b = 0 \qquad -c\mathrm{d}u + a\mathrm{d}a + b\mathrm{d}b = -c\mathrm{d}u + c\mathrm{d}c = 0 \tag{24}$$

and

$$a\mathrm{d}b = b\mathrm{d}a \qquad \mathrm{d}u = \mathrm{d}c \tag{25}$$

which combine to give

$$\frac{a}{b} = \text{constant} \qquad u - c = \text{constant} \tag{26}$$

For the left moving expansion, the result is similar as

$$\frac{a}{b} = \text{constant} \qquad u + c = \text{constant} \tag{27}$$

As we can see, for this system, the jumps are the same across expansions and shocks

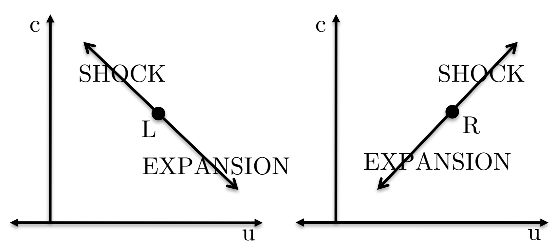

Figure 1: Expansion and Shock Curves

2 An Entropy

From Tadmor's work (1987), define a convex function $U = U(\mathbf{u})$ augmented with an entropy flux function $F = F(\mathbf{u})$ such that

$$U_{\mathbf{u}}^T \mathbf{A} = F_{\mathbf{u}}^T \tag{28}$$

and

$$U_{\mathbf{uu}} \mathbf{A} = [U_{\mathbf{uu}} \mathbf{A}]^T \tag{29}$$

Since $\mathbf{A}$ is already symmetric, the choice of $U_{\mathbf{uu}} = \mathbf{I}$ is acceptable. Using the entropy variables, $\mathbf{v}(\mathbf{u}) = U_{\mathbf{u}}$ leads to $\mathbf{v}_{\mathbf{u}} = U_{\mathbf{uu}} = \mathbf{I}$, and that $\mathbf{v}(\mathbf{u}) = \mathbf{u}$. With $U_{\mathbf{u}} = \mathbf{v} = \mathbf{u}$, $U = \frac{1}{2}\mathbf{u}^T\mathbf{u}$ and the entropy function is

$$U(\mathbf{u}) = \frac{1}{2}(u^2 + c^2) \tag{30}$$

Further,

$$\mathbf{u}^T\mathbf{A} = F_{\mathbf{u}}^T = [u^2 + c^2, 2ua, 2ub] \tag{31}$$

The simplest solution to this is

$$F = \frac{1}{3}u^3 + uc^2 \tag{32}$$

3 An Exact Riemann Solver

Dividing the spacetime into 4 regions, from left to right as 1,2,3,4, there are four cases. In all four cases, across a contact, $u_2 = u_3 = u^*$ and $c_2 = c_3 = c^*$, which leads to

$$u^* = \frac{u_1 + c_1 + (u_4 - c_4)}{2} \tag{33}$$

$$c^* = \frac{u_1 + c_1 - (u_4 - c_4)}{2} \tag{34}$$

3.1 Inside a Left Expansion

$$c(\xi) = \frac{1}{2}(u_1 + c_1 - \xi) \tag{35}$$

$$u(\xi) = \frac{1}{2}(u_1 + c_1 + \xi) \tag{36}$$

$$a(\xi) = \frac{a_1}{c_1}c \tag{37}$$

$$b(\xi) = \frac{b_1}{a_1}a \tag{38}$$

3.2 Inside a Right Expansion

$$c(\xi) = \frac{1}{2}(-u_4 + c_4 + \xi) \tag{39}$$

$$u(\xi) = \frac{1}{2}(u_4 - c_4 + \xi) \tag{40}$$

$$a(\xi) = \frac{a_4}{c_4}c \tag{41}$$

$$b(\xi) = \frac{b_4}{a_4}a \tag{42}$$

4 A Single Shock Problem

Start with states $\mathbf{u}_L$ and $\mathbf{u}_R$ connected by a single shock. Prescribe $\mathbf{u}_R$ and u_L to create a similar problem to that of Noh. Looking at the jump conditions,

$$S[\mathbf{u}] = [\mathbf{f}] \tag{43}$$

$$\begin{aligned} S(u_R - u_L) &= \frac{1}{2}(u_R^2 + c_R^2 - u_L^2 - c_L^2) && (44)\\ S(a_R - a_L) &= u_R a_R - u_L a_L && (45)\\ S(b_R - b_L) &= u_R b_R - u_L b_L && (46) \end{aligned}$$

The second two equations lead to

$$\begin{aligned} a_L &= a_R \frac{S - u_R}{S - u_L} && (47)\\ b_L &= b_R \frac{S - u_R}{S - u_L} && (48)\\ c_L^2 &= c_R^2 \left(\frac{S - u_R}{S - u_L}\right)^2 && (49)\\ c_L &= c_R \left|\frac{S - u_R}{S - u_L}\right| && (50) \end{aligned}$$

which gives

$$\begin{aligned} u_R^2 - u_L^2 + c_R^2 - c_R^2 \left(\frac{S - u_R}{S - u_L}\right)^2 - 2(u_R - u_L)S &= 0 && (51)\\ ([u^2] + c_R^2 - 2[u]S)(S - u_L)^2 - c_R^2(S - u_R)^2 &= 0 && (52)\\ ([u^2] + c_R^2 - 2[u]S)(S - u_L)^2 - c_R^2(S^2 - 2u_R S + u_R^2) &= 0 && (53)\\ ([u^2] - 2[u]S)(S - u_L)^2 + c_R^2(S^2 - 2u_L S + u_L^2 - S^2 + 2u_R S - u_R^2) &= 0 && (54)\\ ([u^2] - 2[u]S)((S - u_L)^2 - c_R^2) &= 0 && (55) \end{aligned}$$

which has solutions of

$$S = u_L \pm c_R, \frac{1}{2}(u_L + u_R) \tag{56}$$

Note that $u_L \pm c_R = u_R \pm c_L$. Now looking at if we know S, and solve for u_L,

$$
\begin{aligned}
u_R^2 - u_L^2 + c_R^2 - c_R^2\left(\frac{S-u_R}{S-u_L}\right)^2 - 2(u_R-u_L)S &= 0 \qquad (57)\\
(u_R^2 + c_R^2 - 2u_RS)(S-u_L)^2 \qquad\qquad &\\
+(2u_LS - u_L^2)(S-u_L)^2 - c_R^2(S-u_R)^2 &= 0 \qquad (58)\\
(u_R^2 - 2u_RS)(S-u_L)^2 + (2u_LS-u_L^2)(S-u_L)^2 &\\
+c_R^2(S^2 - 2u_LS + u_L^2 - S^2 + 2u_RS - u_R^2) &= 0 \qquad (59)\\
(u_R^2 - 2u_RS)(S-u_L)^2 + (2u_LS-u_L^2)(S-u_L)^2 &\\
+c_R^2(-2u_LS + u_L^2 + 2u_RS - u_R^2) &= 0 \qquad (60)\\
((S-u_L)^2 - c_R^2)(-u_L^2 + 2Su_L + u_R(u_R - 2S) &= 0 \qquad (61)
\end{aligned}
$$

which has roots of

$$u_L = S \pm c_R, \pm u_R \qquad (62)$$

5 An Entropy Condition

5.1 Part 1

Using the entropy conditions of

$$u_L - c_L > S > u_R - c_R \qquad (63)$$

$$u_L + c_L > S > u_R + c_R \qquad (64)$$

for left and right shocks, lets look at the solutions from above. First, for $S = \frac{1}{2}(u_L + u_R)$, this gives $c_L = c_R$ and the entropy condition leads to

$$\mp 2c_R > [u] \qquad \pm 2c_R > [u] \qquad (65)$$

which has no solution, so this violates the entropy condition and doesn't lead to a single shock. For $S = u_L + c_R$, simple math shows that for $u_L < u_R$, a single right shock is possible and similarly for $S = u_L - c_R$, and $u_L < u_R$, a single left shock is possible.

5.2 Part 2

Lets look at the steady case now, $S = 0$ for the quartic u_L case. For a left wave,

$$u_L - c_L > 0 > u_R - c_R \qquad (66)$$

From the right side, $c_R > u_R$ and from the left side $u_L|u_L| > c_R|u_R|$. Of the four choices, the only satisfying case is $u_L = c_R > u_R$. Conversely for a right wave, the only satisfying case is $u_L = -c_R > u_R$. Examining these two cases, we have that from the entropy equation that

$$[F] = \frac{1}{3}(u_R^3 - u_L^3) + u_R c_R^2 - u_L c_L^2 \tag{67}$$

$$[F] = \frac{1}{3}(u_R^3 - (\pm c_R)^3) + u_R c_R^2 - \pm c_R u_R^2 \tag{68}$$

$$[F] = \frac{1}{3}(u_R^3 \mp c_R^3) + u_R^R c_R^3 \mp c_R u_R^2 \tag{69}$$

$$[F] = \frac{1}{3}(u_R \mp c_R)^3 < 0 \tag{70}$$

5.3 An Expansion

Start with a right moving expansion with $\xi_L = u_L + c_L < 0 < u_R + c_R = \xi_R$ and integrate the entropy equation around the box $[\xi_L T, \xi_R T] \times [0, T]$ to get

$$\iint (U_t + F_x)\mathrm{d}t\mathrm{d}x \tag{71}$$

$$= \int_{\xi_L T}^{\xi_R T} (U(x,T) - U(x,0))\mathrm{d}x + \int_0^T (F(\xi_R T, t) - F(\xi_L T, t)\mathrm{d}t \tag{72}$$

$$= \int_{\xi_L T}^{\xi_R T} U(x,T)\mathrm{d}x - [U_R \xi_R - U_L \xi_L]T + (F_R - F_L)T \tag{73}$$

Examining the first term, $\int_{\xi_L T}^{\xi_R T} U(x,T)\mathrm{d}x$

$$\int_{\xi_L T}^{\xi_R T} U(x,T)\mathrm{d}x = T \int_{\xi_L}^{\xi_R} U(\xi)\mathrm{d}\xi \tag{74}$$

and

$$U(\xi) = \frac{1}{2}(u(\xi)^2 + c(\xi)^2) \tag{75}$$

From the derivations for a right expansion,

$$u(\xi) = \frac{1}{2}(u_R - c_R + \xi) \qquad c(\xi) = \frac{1}{2}(c_R - u_R + \xi) \tag{76}$$

$$U(\xi) = \frac{1}{4}(\xi^2 + (u_R - c_R)^2) \tag{77}$$

$$T \int_{\xi_L}^{\xi_R} U(\xi)\mathrm{d}\xi = \frac{T}{4}\left(\frac{1}{3}(\xi_R^3 - \xi_L^3) + (u_R - c_R)^2(\xi_R - \xi_L)\right) \tag{78}$$

Putting this together,

$$\begin{aligned}
\frac{1}{T}\iint(U_t+F_x)\mathrm{d}t\mathrm{d}x &= \frac{1}{4}\left(\frac{1}{3}(\xi_R^3-\xi_L^3)+(u_R-c_R)^2(\xi_R-\xi_L)\right) \\
&\quad +[U_R\xi_R-U_L\xi_L]+(F_R-F_L) \qquad (79)\\
&= \frac{1}{4}\left(\frac{1}{3}(\xi_R^3-\xi_L^3)+(u_R-c_R)^2(\xi_R-\xi_L)\right) \\
&\quad +\frac{1}{2}(u_R^2+c_R^2)\xi_R-\frac{1}{2}(u_L^2+c_L^2)\xi_L \\
&\quad +\frac{1}{3}(u_R^3+3u_Rc_R^2-u_L^3-3u_Lc_L^2) \qquad (80)
\end{aligned}$$

Which should equal zero, but the point is, this is a very difficult and time consuming thing to do.

6 Entropy Across a Single Shock

we have

$$U_t+F_x\le 0 \qquad (81)$$

which integrates into

$$[F]-S[U]\le 0 \qquad (82)$$

with from the jump condition on the first equation and the identities $[ab]=\overline{a}[b]+\overline{b}[a]$ with $\overline{a}=\frac{1}{2}(a_L+a_R)$ and $\overline{a^2}-\overline{a}^2=\frac{1}{4}[a]^2$. We also have that $[u]=\pm[c]$ or $[u]^2=[c]^2$.

$$S=\frac{[U]}{[u]} \qquad (83)$$

$$\begin{aligned}
S[U] &= \frac{[U]^2}{[u]} && (84)\\
&= \frac{([u^2]+[c^2])^2}{4[u]} && (85)\\
&= \frac{[u^2]^2+2[u^2][c^2]+[c^2]^2}{4[u]} && (86)\\
&= \overline{u}^2[u]+\overline{u}[c^2]+\frac{\overline{c}^2[c]^2}{[u]} && (87)\\
&= [u](\overline{u}^2+\overline{c}^2)+\overline{u}[c^2] && (88)
\end{aligned}$$

$$\begin{aligned}
[F] &= \frac{1}{3}[u^3]+[uc^2] && (89)\\
&= \frac{1}{3}[u^3]+\overline{u}[c^2]+\overline{c^2}[u] && (90)
\end{aligned}$$

Putting this together gives the result of

$$
\begin{aligned}
[F]-S[U] &= \frac{1}{3}[u^3]+\overline{u}[c^2]+\overline{c^2}[u]-[u](\overline{u}^2+\overline{c}^2)-\overline{u}[c^2] &(91)\\
&= \frac{1}{3}[u^3]+[u](\overline{c^2}-\overline{c}^2)-[u]\overline{u}^2 &(92)\\
&= \frac{1}{3}[u^3]+\frac{1}{4}[u]^3-[u]\bar{u}^2 &(93)\\
&= \frac{1}{3}[u]^3 &(94)
\end{aligned}
$$

7 Compound Waves

Compound Waves occur when λ_i is not a monotonic function of $\mathbf{u}$, or when $\frac{\partial\lambda_i}{\partial\mathbf{u}}$ does not change sign. When $\lambda = u$, the contact wave, the derivatives are all one, so this is fine. For the other waves,

$$
\frac{\partial(u\pm c)}{\partial u}=1 \qquad \frac{\partial(u\pm c)}{\partial a}=\pm\frac{a}{c} \qquad \frac{\partial(u\pm c)}{\partial b}=\pm\frac{b}{c} \tag{95}
$$

For a or b changing sign, these derivatives also change sign, thus compound waves are possible.

8 Alternate Equations

Start with

$$
a_t+(ua)_x=0 \qquad b_t+(ub)_x=0 \tag{96}
$$

and multiply by a and b respectively to get

$$
aa_t+a(ua)_x=0 \qquad bb_t+b(ub)_x=0 \tag{97}
$$

and, with $c^2=a^2+b^2$, $cdc=ada+bdb$, adding these together gives

$$
\begin{aligned}
0 &= aa_t+bb_t+u(aa_x+bb_x)+(a^2+b^2)u_x &(98)\\
&= cc_t+ucc_x+c^2u_x &(99)\\
&= c_t+uc_x+cu_x &(100)\\
0 &= c_t+(uc)_x &(101)
\end{aligned}
$$

and we get that c is transported similar to a and b. The equation for c^2 is $(c^2)_t+(uc^2)_x+c^2u_x=0$, which is not in conservation form. Taking the initial equations and multiplying by b and a respectively, and adding, gives

$$
\begin{aligned}
(ab)_t+b(ua)_x+a(ub)_x &= (ab)_t+u(ab)_x+2abu_x &(102)\\
&= (ab)_t+(uab)_x+abu_x &(103)
\end{aligned}
$$

which is a similar equation to that of c^2, and we can combine these to get that

$$((a+b)^2)_t = (u(a+b)^2)_x + (a+b)^2 u_x = 0$$

Combining the equations for u and c, gives the entropy equation $U_t + F_x = 0$ and the nonconservative equation

$$(u^2 - c^2)_t + (u^2 - c^2)u_x = 0$$

9 A Stationary Shock Test

Start with

$$\mathbf{u}_t + \mathbf{f}_x = 0 \tag{104}$$

or expanded as

$$\begin{bmatrix} u \\ a \\ b \end{bmatrix}_t + \begin{bmatrix} \frac{1}{2}\left(u^2 + a^2 + b^2\right) \\ ua \\ ub \end{bmatrix}_x = 0 \tag{105}$$

Create a one-point stationary shock, prescribing u_R, c_R, choosing an $S = u_R - c_L = 0$ shock such that $u_L = c_R, c_L = u_R$. Here, I have chosen

$$u_R = 1, c_R = \frac{1}{M}, a_R = b_R = \frac{1}{\sqrt{2}M}$$

The theory for this system says that every stationary shock should have

$$\mathbf{u}_M = \varepsilon \mathbf{u}_L + (1-\varepsilon)\mathbf{u}_R$$

Indeed, testing 1000 equally spaced points for $\varepsilon \in [0,1]$ shows that there is no movement at all. Since this is a $u - c$ shock, $u + c$ is constant across it. Now suppose we choose the middle state

$$\mathbf{u}_M = \begin{bmatrix} \varepsilon_u \\ \varepsilon_a \\ \varepsilon_b \end{bmatrix} \mathbf{u}_L + \left(1 - \begin{bmatrix} \varepsilon_u \\ \varepsilon_a \\ \varepsilon_b \end{bmatrix}\right) \mathbf{u}_R$$

for $\varepsilon_u, \varepsilon_a, \varepsilon_b$ chosen at random. From this, we get that

$$c_M = \sqrt{a_M^2 + b_M^2}$$

where c_M is a nonlinear function of ε_a and ε_b and the mach number, M. After letting the simulation run and reach a stable stationary shock position, we can calculate

$$\varepsilon_{u,f} = \frac{u_{M,f} - u_{R,f}}{u_{L,f} - u_{R,f}} \qquad \varepsilon_{a,f} = \frac{a_{M,f} - a_{R,f}}{a_{L,f} - a_{R,f}} \qquad \varepsilon_{b,f} = \frac{b_{M,f} - b_{R,f}}{b_{L,f} - b_{R,f}} \qquad \varepsilon_{c,f} = \frac{c_{M,f} - c_{R,f}}{c_{L,f} - c_{R,f}}$$

and make plots of $\varepsilon_{*,f}$ against ε_*, the final subcell position compared with the initial subcell position of each variable, to determine if there is any correlation, which would indicate the type(s) of waves being shed to achieve a stationary position. Here, we have chosen to use a 21 cell grid, with $\partial_x \mathbf{u} = 0$ at the boundaries. In these plots, the diagonal corresponds to the variable remaining the same.

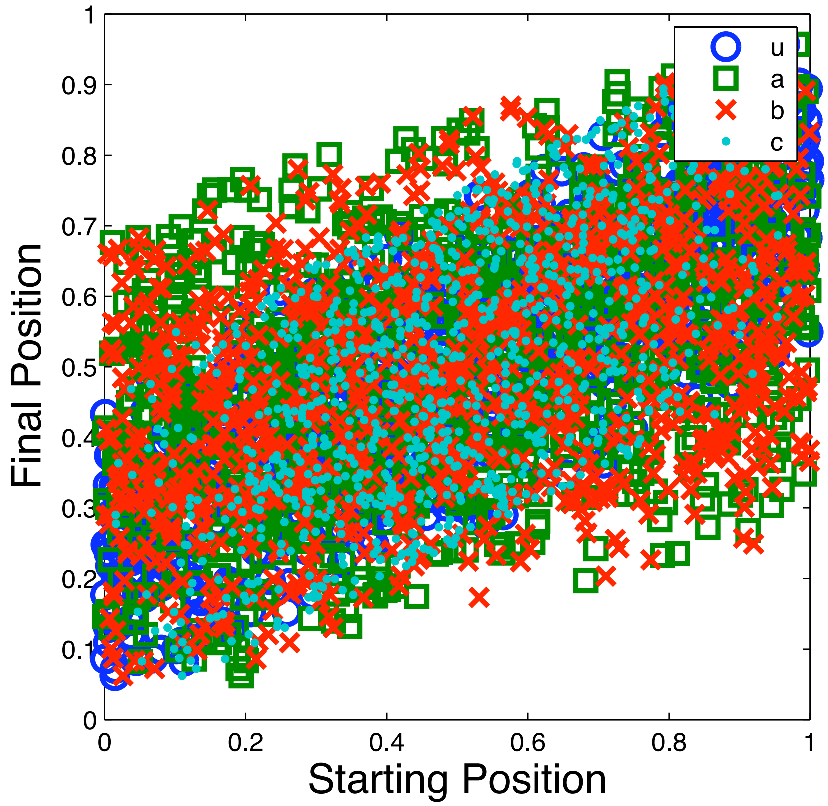

Figure 2: Starting subcell position vs Final subcell position, ‘primitive variables’, Mach 1.01

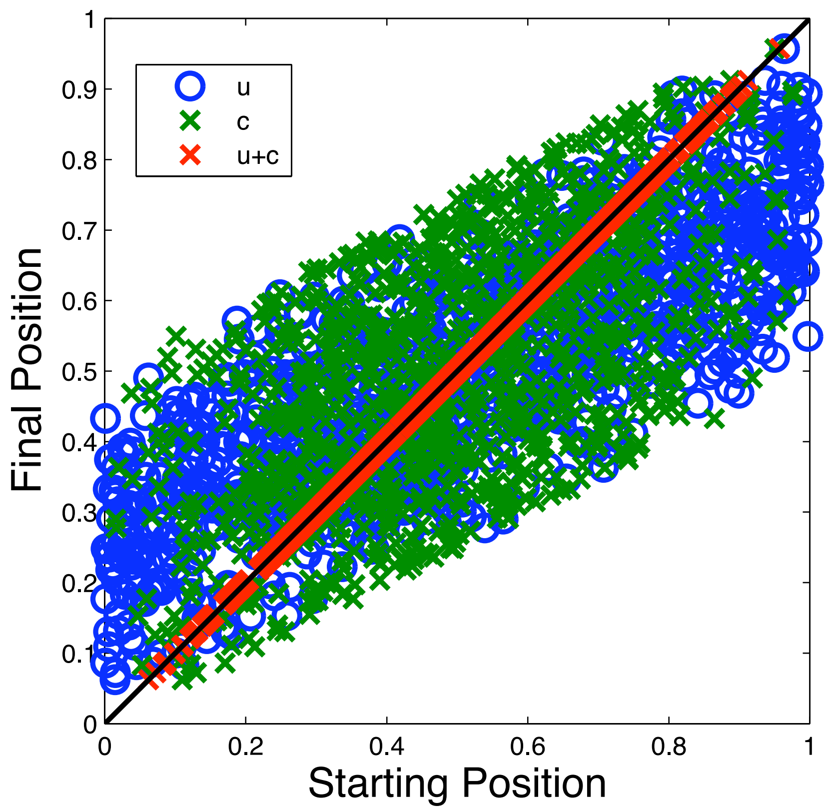

Figure 3: Starting subcell position vs Final subcell position, u, c and $u+c$, Mach 1.01

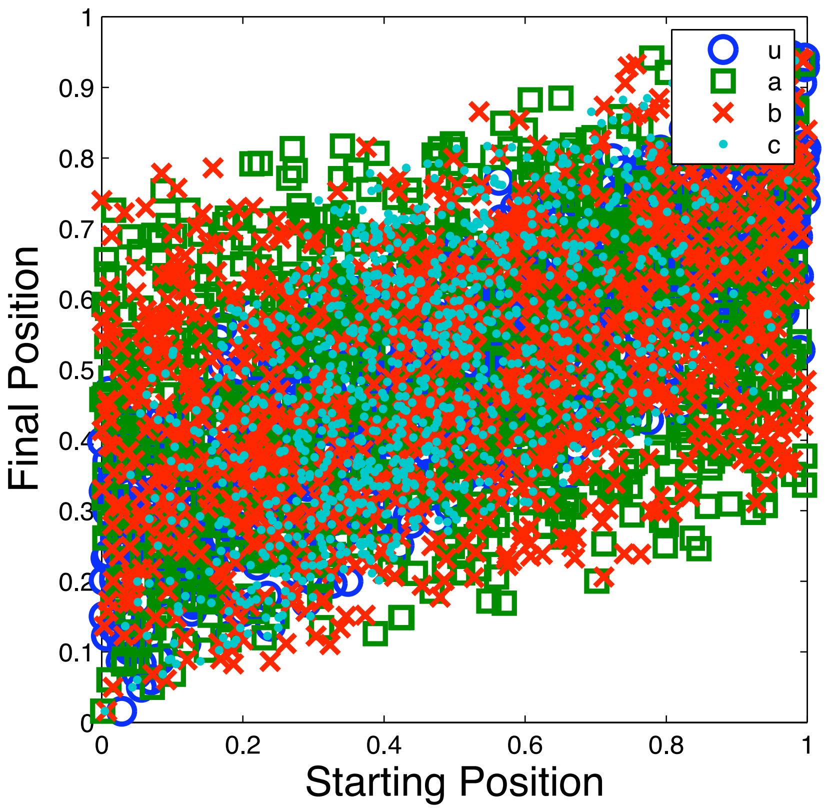

Figure 4: Starting subcell position vs Final subcell position, 'primitive variables', Mach 10

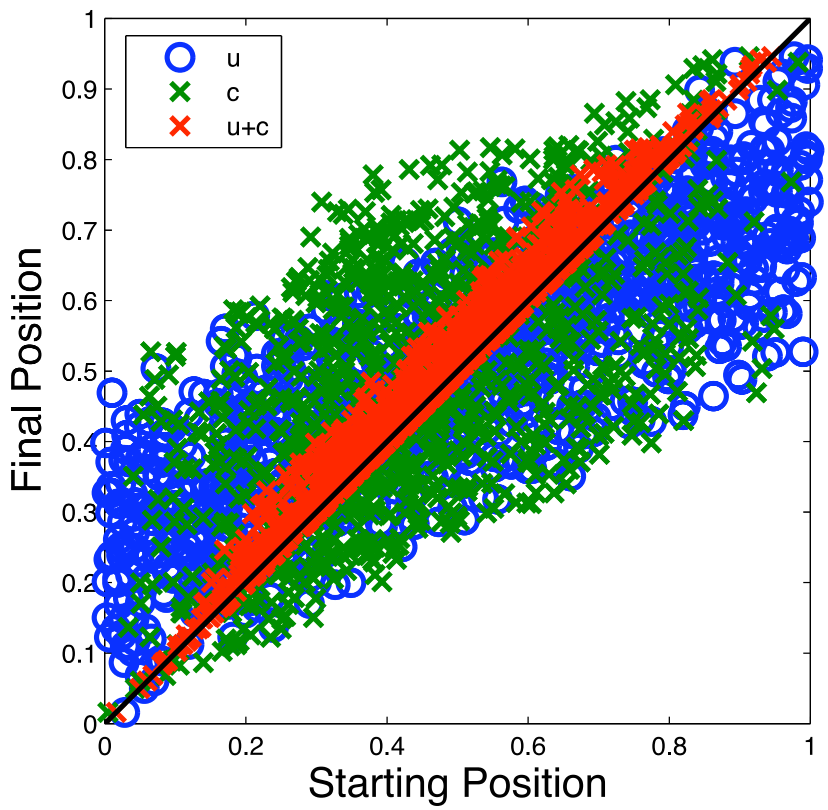

Figure 5: Starting subcell position vs Final subcell position, u, c and $u + c$, Mach 10