Linear Integral Curves Through an Equation of State

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I. Experimental Math

Given the condition for a linear integral curve, that

$$\frac{\partial r_i^k}{\partial u_j} r_j^k = 0 \qquad |r^k| = 1$$

for the k^{th} integral curve , the non-normalized form of this becomes

$$\frac{1}{|r^k|^2}\left[\frac{\partial r_i^k}{\partial u_j} - \frac{r_i^k}{|r^k|^2}\left(r_\ell^k \frac{\partial r_\ell^k}{\partial u_j}\right)\right] r_j^k = 0$$

If $r_I^k = 1$ for one I and all k, we get that $\frac{\partial r_I^k}{\partial u_j} = 0$ and that

$$r_\ell^k \frac{\partial r_\ell^k}{\partial u_j} r_j^k = 0$$

It can be shown that this results in

$$\frac{\partial r_i^k}{\partial u_j} r_j^k = 0 \qquad i \neq I$$

which in turn satisfies the previous equation, thus

$$\frac{\partial r_i^k}{\partial u_j} r_j^k = 0$$

holds, provided one $r_i^k = 1$.

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I.A. Isothermal Euler

Given the Isothermal Euler equations, start with $p = p(\rho, m)$ and get that

$$A = \begin{bmatrix} 0 & 1 \\ -u^2 + p_\rho & 2u + p_m \end{bmatrix}$$

and that one eigendecomposition is

$$R = \begin{bmatrix} u + \frac{p_m}{2} + a & u + \frac{p_m}{2} - a \\ u^2 - p_\rho & u^2 - p_\rho \end{bmatrix} \quad \Lambda = \begin{bmatrix} u + \frac{p_m}{2} - a & 0 \\ 0 & u + \frac{p_m}{2} + a \end{bmatrix}$$

for $a = \sqrt{\frac{p_m^2}{4} + up_m + p_\rho}$. Now, let $R = [r^1, r^2]$ and we have that

$$\begin{aligned}
\frac{\partial a}{\partial \rho} &= \frac{1}{2a}\left(p_{\rho m}\left(\frac{p_m}{2} + u\right) - \frac{u}{2\rho}p_m + p_{\rho\rho}\right) \\
\frac{\partial a}{\partial m} &= \frac{1}{2a}\left(p_{mm}\left(\frac{p_m}{2} + u\right) + \frac{1}{\rho}p_m + p_{m\rho}\right) \\
\frac{\partial r_1}{\partial \rho} &= -\frac{u}{\rho} + \frac{1}{2}p_{m\rho} \pm \frac{\partial a}{\partial \rho} \\
\frac{\partial r_1}{\partial m} &= \frac{1}{\rho} + \frac{1}{2}p_{mm} \pm \frac{\partial a}{\partial m} \\
\frac{\partial r_2}{\partial \rho} &= -\frac{2u^2}{\rho} - p_{\rho\rho} \\
\frac{\partial r_2}{\partial m} &= \frac{2u}{\rho} - p_{m\rho}
\end{aligned}$$

Putting this all together leads to a very, very complex system for general pressure equations. So lets assume $p = p(\rho)$. This leads to

$$A = \begin{bmatrix} 0 & 1 \\ -u^2 + p_\rho & 2u \end{bmatrix}$$

and that one eigendecomposition is

$$R = \begin{bmatrix} 1 & 1 \\ u - a & u + a \end{bmatrix} \quad \Lambda = \begin{bmatrix} u - a & 0 \\ 0 & u + a \end{bmatrix}$$

for $a = \sqrt{p_\rho}$. Now, let $R = [r^1, r^2]$ and we have that

$$\begin{aligned}
\frac{\partial a}{\partial \rho} &= \frac{p_{\rho\rho}}{2\sqrt{p_\rho}} \\
\frac{\partial r_1}{\partial \rho} &= 0 \\
\frac{\partial r_1}{\partial m} &= 0 \\
\frac{\partial r_2}{\partial \rho} &= -\frac{u}{\rho} \pm \frac{\partial a}{\partial \rho} \\
\frac{\partial r_2}{\partial m} &= \frac{1}{\rho}
\end{aligned}$$

This leads to $a = \frac{C}{\rho}$ and $p = -\frac{C^2}{\rho} = -Ca$. The hugoniot curve is then $u_R - u_L = \pm(a_R - a_L)$. The shock speeds are $S = u_L \pm a_R$. For expansion waves, we have that

$$d\left(u \pm \frac{C}{\rho}\right) = 0, \quad \text{along } \frac{dx}{dt} = u \pm \frac{C}{\rho}$$

But what should C be? In order for the Hugoniot's to be the same, choose $C = a\sqrt{\rho_L \rho_R}$. In order for the shock speed to be the same, $C = a\sqrt{\rho_L \rho_R}$, as well. The Roe-averaged variables for this system are

$$u = \frac{\sqrt{\rho_L} u_L + \sqrt{\rho_R} u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \qquad \rho = \sqrt{\rho_L \rho_R}$$

Comparing this to $p_{iso} = a^2 \rho$, we have that

$$\Delta p_{iso} = a^2(\rho_R - \rho_L) \qquad \Delta p = C^2 \left(\frac{\rho_R - \rho_L}{\rho_L \rho_R}\right)$$

For the choice of $C = a\sqrt{\rho_L \rho_R}$, $\Delta p = \Delta p_{iso}$. However, $p_L + p_R = -a^2(\rho_L + \rho_R)$, which is the opposite sign.

So here we have C as a constant. What if C is not constant? Use the system

$$\begin{bmatrix} \rho \\ \rho u \\ \rho c \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \rho u^2 - \frac{c^2}{\rho} \\ \rho u c \end{bmatrix}_x = 0$$

or, in conserved variable form

$$\begin{bmatrix} \rho \\ m \\ q \end{bmatrix}_t + \begin{bmatrix} m \\ \frac{m^2}{\rho} - \frac{q^2}{\rho^3} \\ \frac{mq}{\rho} \end{bmatrix}_x = 0$$

The flux jacobian is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{3c^2}{\rho^2} - u^2 & 2u & -\frac{2c}{\rho^2} \\ -uc & c & u \end{bmatrix}$$

with eigenvalues $\lambda = u - a, u, u + a$ for $a = \frac{c}{\rho}$

$$R = \begin{bmatrix} 1 & 1 & 1 \\ u - a & u & u + a \\ c & \frac{3}{2}c & c \end{bmatrix}$$

This system is not linearly degenerate, which is a bad thing.

I.B. Euler Equations

We get that

$$A = \begin{bmatrix} 0 & 1 & 0 \\ p_\rho - u^2 & 2u + p_m & p_E \\ u(p_\rho - H) & H + up_m & u(1 + p_E) \end{bmatrix}$$

The eigenvalues of this system are

$$\lambda = u, u + \tilde{u} \pm a$$

where

$$\tilde{u} = \frac{1}{2}(p_m + up_E) \qquad a^2 = \tilde{u}^2 + (p_\rho + up_m + Hp_E)$$

and

$$R = \begin{bmatrix} 1 & 1 & 1 \\ u & u + \tilde{u} - a & u + \tilde{u} + a \\ -\frac{p_\rho + up_m}{p_E} & H + u(\tilde{u} - a) & H + u(\tilde{u} + a) \end{bmatrix}$$

The requirements for linear integral curves for $\lambda = u + \tilde{u} + a$ are

$$\begin{aligned} \frac{\partial r_2}{\partial u_j} r_j &= \frac{\partial(u + \tilde{u} + a)}{\partial \rho} + \frac{\partial(u + \tilde{u} + a)}{\partial m}(u + \tilde{u} + a) + \frac{\partial(u + \tilde{u} + a)}{\partial E}(H + u\tilde{u} + ua) \\ \frac{\partial r_3}{\partial u_j} r_j &= \frac{\partial(H + u\tilde{u} + ua)}{\partial \rho} + \frac{\partial(H + u\tilde{u} + ua)}{\partial m}(u + \tilde{u} + a) + \frac{\partial(H + u\tilde{u} + ua)}{\partial E}(H + u\tilde{u} + ua) \end{aligned}$$

Further, we have that $\frac{\partial u}{\partial \rho} = -\frac{u}{\rho}$, $\frac{\partial u}{\partial m} = \frac{1}{\rho}$, $\frac{\partial H}{\partial \rho} = -\frac{H}{\rho} + \frac{p_\rho}{\rho}$, $\frac{\partial H}{\partial m} = \frac{p_m}{\rho}$, $\frac{\partial H}{\partial E} = \frac{1}{\rho} + \frac{p_E}{\rho}$ and $\frac{\partial \tilde{u}}{\partial \rho} = \frac{1}{2}\left(p_{\rho m} + u\left(p_{\rho E} + \frac{p_E}{\rho}\right)\right)$, $\frac{\partial \tilde{u}}{\partial m} = \frac{1}{2}\left(p_{mm} + up_{mE} + \frac{p_E}{\rho}\right)$, $\frac{\partial \tilde{u}}{\partial E} = \frac{1}{2}(p_{mE} + up_{EE})$ which results in

$$\begin{aligned} \frac{\partial r_2}{\partial u_j} r_j &= \frac{\partial(u + \tilde{u} + a)}{\partial \rho} + \frac{\partial(u + \tilde{u} + a)}{\partial m}(u + \tilde{u} + a) + \frac{\partial(u + \tilde{u} + a)}{\partial E}(H + u\tilde{u} + ua) \\ &= -\frac{u}{\rho} + \frac{\partial \tilde{u}}{\partial \rho} + \frac{\partial a}{\partial \rho} + \frac{u + \tilde{u} + a}{\rho} + (u + \tilde{u} + a)\frac{\partial \tilde{u}}{\partial m} + (u + \tilde{u} + a)\frac{\partial a}{\partial m} + (H + u\tilde{u} + ua)\frac{\partial \tilde{u}}{\partial E} + (H + u\tilde{u} + ua)\frac{\partial a}{\partial E} \\ &= \frac{\tilde{u} + a}{\rho} + \frac{\partial(\tilde{u} + a)}{\partial \rho} + (u + \tilde{u} + a)\frac{\partial(\tilde{u} + a)}{\partial m} + (H + u\tilde{u} + ua)\frac{\partial(\tilde{u} + a)}{\partial E} \\ \frac{\partial r_3}{\partial u_j} r_j &= \frac{\partial(H + u\tilde{u} + ua)}{\partial \rho} + \frac{\partial(H + u\tilde{u} + ua)}{\partial m}(u + \tilde{u} + a) + \frac{\partial(H + u\tilde{u} + ua)}{\partial E}(H + u\tilde{u} + ua) \\ &= \frac{\partial H}{\partial \rho} + u\frac{\partial(\tilde{u} + a)}{\partial \rho} + (\tilde{u} + a)\frac{\partial u}{\partial \rho} + (u + \tilde{u} + a)\frac{\partial H}{\partial m} + (u + \tilde{u} + a)(\tilde{u} + a)\frac{\partial u}{\partial m} + (u + \tilde{u} + a)u\frac{\partial(\tilde{u} + a)}{\partial m} \\ &\quad + (H + u\tilde{u} + ua)\frac{\partial H}{\partial E} + (H + u\tilde{u} + ua)u\frac{\partial(\tilde{u} + a)}{\partial E} \\ &= -\frac{H}{\rho} + \frac{p_\rho}{\rho} + u\frac{\partial(\tilde{u} + a)}{\partial \rho} + (\tilde{u} + a)\frac{\partial u}{\partial \rho} + (u + \tilde{u} + a)\frac{p_m}{\rho} + (u + \tilde{u} + a)(\tilde{u} + a)\frac{\partial u}{\partial m} + (u + \tilde{u} + a)u\frac{\partial(\tilde{u} + a)}{\partial m} \\ &\quad + (H + u\tilde{u} + ua)\left(\frac{1}{\rho} + \frac{p_E}{\rho}\right) + (H + u\tilde{u} + ua)u\frac{\partial(\tilde{u} + a)}{\partial E} \\ &= u\left(\frac{\tilde{u} + a}{\rho} + \frac{\partial(\tilde{u} + a)}{\partial \rho} + (u + \tilde{u} + a)\frac{\partial(\tilde{u} + a)}{\partial m} + (H + u\tilde{u} + ua)\frac{\partial(\tilde{u} + a)}{\partial E}\right) \\ &\quad + \frac{p_\rho}{\rho} + (u + \tilde{u} + a)\frac{p_m}{\rho} + (\tilde{u} + a)(\tilde{u} + a)\frac{1}{\rho} + (H + u\tilde{u} + ua)\frac{p_E}{\rho} \\ &= u\left(\frac{\tilde{u} + a}{\rho} + \frac{\partial(\tilde{u} + a)}{\partial \rho} + (u + \tilde{u} + a)\frac{\partial(\tilde{u} + a)}{\partial m} + (H + u\tilde{u} + ua)\frac{\partial(\tilde{u} + a)}{\partial E}\right) + \frac{2(\tilde{u} + a)^2}{\rho} \end{aligned}$$

Now what? From here, we get that $\tilde{u} + a = 0$, or that $\tilde{u}^2 = a^2 \;\; \rightarrow \;\; (p_\rho + u p_m + H p_E) = 0$. This can be rewritten as

$$\rho \frac{\partial p}{\partial \rho} + m \frac{\partial p}{\partial m} + E \frac{\partial p}{\partial E} + p \frac{\partial p}{\partial E} = 0$$

Enforcing that $-\frac{p_\rho + u p_m}{p_E} = \frac{1}{2} u^2$ along with this gives $p_E = 0$ and that

$$\rho \frac{\partial p}{\partial \rho} + m \frac{\partial p}{\partial m} = 0$$

This allows solutions of the form $p = A u^B + C$ and

II. Already Linear Hugoniot System

III. Governing Equations

Start with

$$\mathbf{u}_t + \mathbf{f}_x = 0$$

or expanded as

$$\begin{bmatrix} u \\ a \\ b \end{bmatrix}_t + \begin{bmatrix} \frac{1}{2}\left(u^2 + a^2 + b^2\right) \\ ua \\ ub \end{bmatrix}_x = 0$$

This can be decomposed as

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}} = \begin{bmatrix} u & a & b \\ a & u & 0 \\ b & 0 & u \end{bmatrix}$$

where further decomposition leads to $\mathbf{A} = \mathbf{R}\Lambda\mathbf{L}$. Define a propagation speed

$$c^2 = a^2 + b^2$$

and write the eigenvalue matrix as

$$\Lambda = \begin{bmatrix} u - c & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u + c \end{bmatrix}$$

and right and left eigenvector matrices as

$$\mathbf{R} \;\; = \;\; \begin{bmatrix} -c & 0 & c \\ a & -b & a \\ b & a & b \end{bmatrix} \tag{1}$$

and we have for this system that

$$\frac{\partial c}{\partial a} = \frac{a}{c} \qquad \frac{\partial c}{\partial b} = \frac{b}{c}$$

For $r^1 = [1, -a/c, -b/c]^T$,

$$\frac{\partial r_i^1}{\partial u_j} r_j^1 = 0 \quad \rightarrow \quad \begin{array}{c} \frac{\partial(a/c)}{\partial a}\frac{a}{c} + \frac{\partial(a/c)}{\partial b}\frac{b}{c} = 0 \\ \frac{\partial(b/c)}{\partial a}\frac{a}{c} + \frac{\partial(b/c)}{\partial b}\frac{b}{c} = 0 \end{array}$$

For $r^2 = [0, -1, -a/b]^T$,

$$\frac{\partial r_i^2}{\partial u_j} r_j^2 = 0 \quad \rightarrow \quad -\frac{\partial(a/b)}{\partial a} + \frac{\partial(a/b)}{\partial b}\frac{a}{b} = 1 + \frac{a^2}{b^2} = 0$$

so this does not have a linear integral curve.

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