On Non-Conservative Pressure Diffusion in the Noh Problem

Daniel W. Zaide*, Philip L. Roe†

*University of Michigan, Ann Arbor
zaidedan@umich.edu

†University of Michigan, Ann Arbor

1 Governing Equations

The Euler Equations can be written in vector form as

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0 \tag{1}$$

or, expanded as

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho u\\ E\end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix}\rho u\\ \rho u^2+p\\ u(E+p)\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}, \tag{2}$$

with the equation of state $p = p(\rho, i)$. For an ideal gas as

$$p = (\gamma-1)\rho i, \qquad \rho i = \left(E - \frac{1}{2}\rho u^2\right) \qquad H = \frac{E+p}{\rho} \tag{3}$$

Alternative to the energy equation, we can also define a convective equation for pressure of the form

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\left(p_{\rm nc} - (\gamma-1)\left[E - \frac{1}{2}\rho u^2\right]\right) = 0 \tag{4}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)p_{\rm nc} + (p + (\gamma-1)p_{\rm nc})\frac{\partial u}{\partial x} = 0 \tag{5}$$

We can note that from pressure convexity that the actual pressure at the interpolated state is always greater than the linearly interpolated pressure. Thus the pressure from this convective equation will always be lower.

2 Basic Method

The first order method in space and time is

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \frac{\Delta t}{\Delta x}(\mathbf{f}(\mathbf{u})_{j+\frac{1}{2}} - \mathbf{f}(\mathbf{u})_{j-\frac{1}{2}}) \tag{6}$$

Within the finite volume context, the choice of $\mathbf{f}(\mathbf{u})_{j+\frac{1}{2}} = \mathbf{f}(\mathbf{u}_j, \mathbf{u}_{j+1})$ will affect the behaviour at the wall. Until we sort out the first-order in space and time method, higher-order methods will not be examined.

3 The Linearised Riemann Solver

The linearised Riemann Solver is used to determine the flux as

$$\mathbf{f}(\mathbf{u}_L, \mathbf{u}_R) = \frac{1}{2}(\mathbf{f}(\mathbf{u}_L) + \mathbf{f}(\mathbf{u}_R)) - \frac{1}{2}\mathbf{R}|\Lambda|\Delta\mathbf{v} \tag{7}$$

where

$$\mathbf{R} = \left[\begin{array}{c|c|c} 1 & 1 & 1 \\ u-a & u & u+a \\ H-au & \frac{1}{2}u^2 & H+au \end{array}\right] \tag{8}$$

$$\Delta\mathbf{v} = \frac{1}{a^2}\left[\begin{array}{c} \frac{1}{2}(\Delta p - \rho a \Delta u) \\ -(\Delta p - a^2\Delta\rho) \\ \frac{1}{2}(\Delta p + \rho a \Delta u) \end{array}\right]. \tag{9}$$

and $\Lambda = \mathrm{diag}(u-a, u, u+a)$ with ρ, u, H, a as density-averaged variables from

$$\rho = \sqrt{\rho_L \rho_R} \tag{10}$$

$$u = \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \tag{11}$$

$$H = \frac{\sqrt{\rho_L}H_L + \sqrt{\rho_R}H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \tag{12}$$

$$a = \sqrt{(\gamma - 1)\left(H - \frac{1}{2}u^2\right)} \tag{13}$$

4 Determining the Non-Conservative Pressure

Define the non-conservative pressure by p_{nc}.The non-conservative pressure can be determined from the waves at each step. Non-conserved pressure should always be lower, as indicated by Roe [Cite some unpublished note]. The following diagram describes the method used. This can be expressed as a sum.

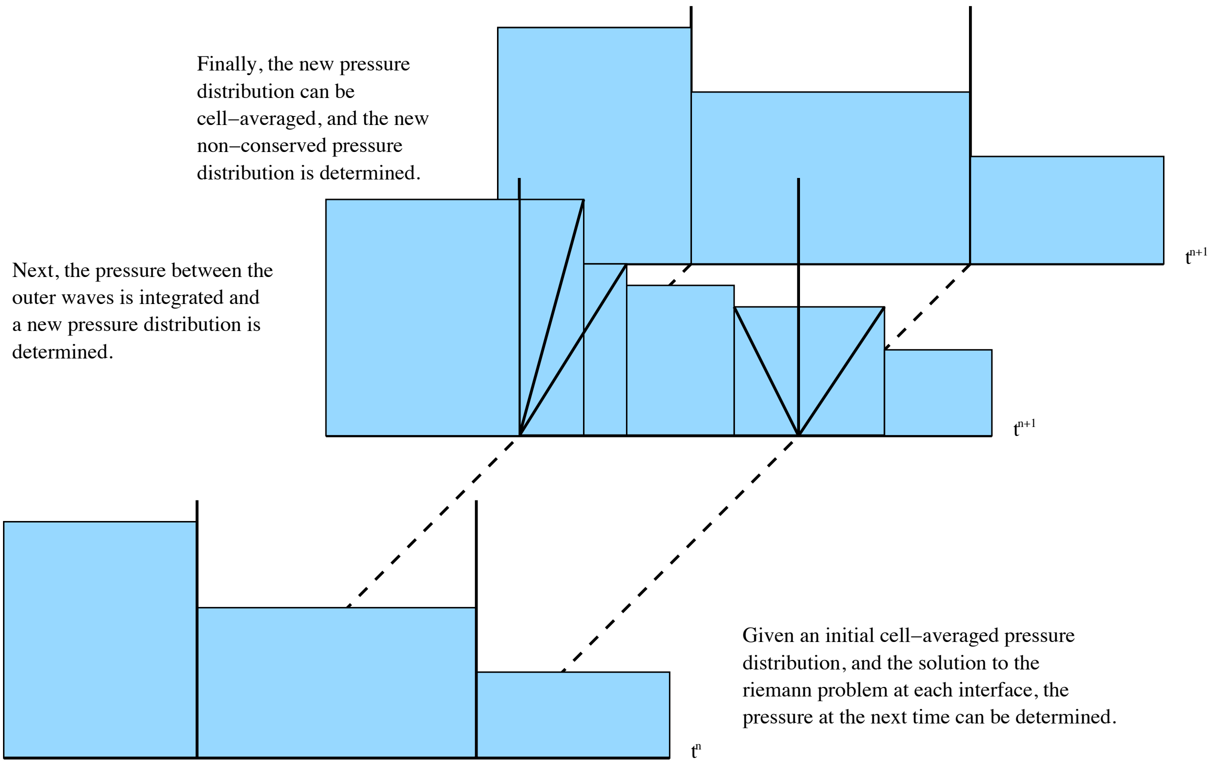

Figure 1: Method for determining new cell-averaged non-conservative pressure

In the case of the waves in the diagram above, this would be

$$\begin{aligned} p_{\mathrm{nc},M}^{n+1} &= \frac{1}{\Delta x}\left(p_L \Delta t(u-a)_{LM} + p_{LM}^{*}\Delta t((u+a)_{LM} - (u-a)_{LM}) + \right. && (14)\\ & p_M[\Delta x - \Delta t(u+a)_{LM} + \Delta t(u-a)_{MR}] - p_{MR}^{*}\Delta t(u-a)_{MR}) && (15) \end{aligned}$$

The intermediate pressure is $p^* = \frac{1}{2}(p_{\mathrm{nc}L} + p_{\mathrm{nc}R}) - \frac{1}{2}\sqrt{\rho_L \rho_R} a(u_R - u_L)$.

5 Updating the Linearised Riemann Solver

Following the work of Roe, within $\mathbf{R}|\Lambda|\Delta\mathbf{v}$, an additional wave of strength

$$\Delta \mathbf{v}_4 = \frac{2}{(\gamma-1)u^2}\Delta(p - p_{\mathrm{nc}}) \tag{16}$$

makes an additional contribution to the energy flux as in

$$\mathbf{f}_{\mathrm{nc}}(\mathbf{u}_L, \mathbf{u}_R) = \frac{1}{2}(\mathbf{f}(\mathbf{u}_L) + \mathbf{f}(\mathbf{u}_R)) - \frac{1}{2}\mathbf{R}|\Lambda|\Delta\mathbf{v} - \frac{1}{2}|u| \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{1}{(\gamma-1)}\Delta(p - p_{\mathrm{nc}}) \tag{17}$$

The wave propagation speed comes from $\rho a^2 = p + (\gamma - 1)p_{\text{nc}}$ which leads to the averaging as

$$H = \frac{\sqrt{\rho_L}H_L + \sqrt{\rho_R}H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \tag{18}$$

$$H_{L,R} = \frac{E_{L,R} + p_{\text{nc},L,R}}{\rho_{L,R}} \tag{19}$$

$$a = \sqrt{(\gamma - 1)\left(H - \frac{1}{2}u^2\right)} \tag{20}$$

Here, for consistency, we still maintain that $\mathbf{f}_{\text{nc}}(\mathbf{u}, \mathbf{u}) = \mathbf{f}(\mathbf{u}, \mathbf{u}) = \mathbf{f}(\mathbf{u})$. Without the enforcement of consistency, we end up with a solution as below, where the incorrect shock speed leads to a overshoot of density.

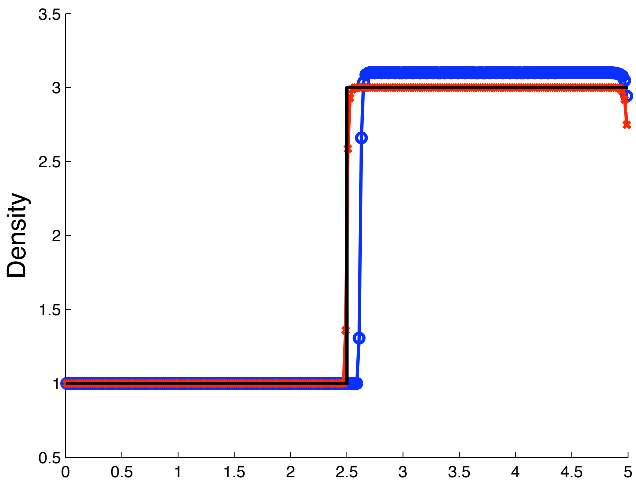

Figure 2: Effect of enforcing consistency of the Riemann Solver. Density result for the Noh Problem, $M_0 = 2$, $t = 5$. Consistent method in red, inconsistent in blue, and exact solution in black.

6 Diffusion

Since the relaxation system does not have a diffusion regime, we can implement a diffusion type method. Define a scalar diffusive flux of the form

$$f_{d,i+\frac{1}{2}} = -\frac{k_{i+\frac{1}{2}}}{\Delta x}((p_{\text{nc}} - p)_{i+1} - (p_{\text{nc}} - p)_i) \tag{21}$$

and modify the energy equation to the form $E_t + (u(E+p))_x = \left(\frac{k}{\gamma-1}(p_{\text{nc}} - p)_x\right)_x$. For simplicity, define $C_{i+\frac{1}{2}} = \frac{k_{i+\frac{1}{2}}\Delta t}{\Delta x^2}$, with stability requiring that

$$0 \leq C_{i+\frac{1}{2}} \leq \frac{1}{2} \tag{22}$$

We propose the CFL dependent choice of

$$C_{i+\frac{1}{2}} = \text{CFL}\frac{1}{2}\frac{|\Delta \mathbf{v}_{i+\frac{1}{2},2}|}{\sum_j |\Delta \mathbf{v}_{i+\frac{1}{2},j}|} \tag{23}$$

which represents the relative strength of the contact.

7 Physical Interpretations of the Flux

If we ignore the changes of the Linearized Eigenstructure and focus on the additional wave and the effect of diffusion, it turns out that we are changing the energy flux, f_E. This results in

$$\begin{aligned} f_{E,\mathrm{nc}} &= f_E - \frac{|u|}{2(\gamma-1)}\Delta(p - p_{\mathrm{nc}}) - \frac{k}{\Delta x}\Delta(p - p_{\mathrm{nc}}) && (24) \\ &= f_E - \left(\frac{|u|}{2(\gamma-1)} + \frac{k}{\Delta x}\right)\Delta(p - p_{\mathrm{nc}}) && (25) \end{aligned}$$

We have that

$$\frac{|u|}{2(\gamma-1)} + \frac{k}{\Delta x} > 0 \tag{26}$$

and by convexity that

$$(p - p_{\mathrm{nc}}) > 0 \tag{27}$$

From this, a gradient in the difference between conservative and non-conservative pressures results in a reduction of energy flux in the direction of that gradient.

8 Noh Problem

The Noh problem consists of $x \in [0, 5]$ and $[\rho, u, p](x, 0) = [1, 1, p_0]$ with gamma-law equation of state, and $\gamma = 5/3$. The Exact solution is

$$[\rho, u, p](x, t) = \begin{cases} 1, 1, p_0 & x \leq (10 - St) \\ \rho_1, 0, p_1 & x \geq (10 - St) \end{cases} \tag{28}$$

for

$$\begin{aligned} \rho_1 &= (1 + 1/S) && (29) \\ p_1 &= p_0(1 + \gamma M_0^2(1 + S)) && (30) \\ S &= 1/4\left(\gamma - 3 + \sqrt{(\gamma+1)^2 + 16/M_0^2}\right) && (31) \end{aligned}$$

We will examine the solution for two cases: $M_0 = 2, 10^6$ with $M_0^2 = 1/\gamma p_0$. The solution at time $t = 1.0s$ for CFL numbers $0.1, 0.5, 0.8$ will be looked at for uniform grid spacing with $\Delta x = 0.02$. In the table below, wall densities are shown. The exact densities are 3 and 4 respectively.

	$M_0 = 2$			$M_0 = 10^6$		
CFL	0.1	0.5	0.8	0.1	0.5	0.8
Flux	No Diffusion					
f	2.7169	2.7484	2.7705	3.5914	3.6378	3.6778
$\mathbf{f}_{\mathrm{nc}}$	2.7406	2.7729	2.7962	3.6415	3.6823	3.7125
Flux	Diffusion					
f	2.7615	2.7910	2.8116	3.6392	3.6781	3.7102
$\mathbf{f}_{\mathrm{nc}}$	2.8107	2.8410	2.8613	3.7394	3.7741	3.7982

From this table, we can see that diffusion improves the wall density calculation. While other choices of diffusion exist, our choice minimizes the effect of diffusion where it is not needed, such as in shock regions.