Towards a Self-Similar Analysis of the Turbulent Taylor-Sedov Blast Wave Problem

J. Tiberius Morán-López[1], Daniel W. Zaide[2], James P. Holloway[1], and Oleg Schilling[3]

[1]Department of Nuclear Engineering and Radiological Sciences, [2]Department of Aerospace Engineering, [3]Lawrence Livermore National Laboratory

Introduction

The Taylor-Sedov blast wave problem describes the gas dynamic evolution originating from a large and instantaneous energy release from a point source in radially-symmetric systems. We extend investigations on the Taylor-Sedov problem by incorporating turbulence effects through a K–ϵ model. Turbulence plays significant roles in shock wave applications such as stellar convection zones, supernovae, and high energy density laser studies. Self-Similar motions are a type of dimensional analysis that applies when the blast wave is sufficiently far from the energy source and provide information behind the shock only. To investigate the shock front, it is necessary to revert to original space-time coordinates.

Nomenclature

ρ, Ω	Material Density
v, V	Velocity
E, K	Mean and Turbulent Kinetic Energies
p, P	Material Pressure
ϵ	Dissipation Rate
γ	Polytropic Index
ν_t	Turbulent Viscosity
τ	Reynolds Stress
Π	Pressure Dilatation
r, t	Space and Time Coordinates
ξ	Similarity Variable, $r/R(t)$
d	Geometry (1,2,3), $\lambda = 2/(2+d)$
M_T	Turbulent Mach Number
C, σ, α	Turbulent Constants

Governing Equations

K-ϵ Turbulence Model

By Reynolds-Favre averaging the Navier-Stokes equations and utilizing a gradient diffusion closure approximation, the K-ϵ turbulence model is

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_j}(\overline{\rho}\,\widetilde{v}_j) = 0,$$

$$\frac{\partial}{\partial t}(\overline{\rho}\,\widetilde{v}_i) + \frac{\partial}{\partial x_j}(\overline{\rho}\,\widetilde{v}_i\,\widetilde{v}_j) + \frac{\partial \overline{p}}{\partial x_i} = \frac{\partial \tau_{ij}}{\partial x_j},$$

$$\frac{\partial}{\partial t}\left(\overline{\rho}\,\widetilde{E}\right) + \frac{\partial}{\partial x_j}\left[(\overline{\rho}\,\widetilde{E} + \overline{p})\widetilde{v}_j\right] = \frac{\nu_t}{\sigma_\rho\overline{\rho}}\frac{\partial\overline{p}}{\partial x_j}\frac{\partial\overline{\rho}}{\partial x_j} + \overline{\rho}\,\epsilon - \Pi + \frac{\partial}{\partial x_j}\left(\frac{\mu_t}{\sigma_E}\frac{\partial\widetilde{E}}{\partial x_j}\right),$$

$$\frac{\partial}{\partial t}(\overline{\rho}\,K) + \frac{\partial}{\partial x_j}(\overline{\rho}\,K\,\widetilde{v}_j) = -\frac{\nu_t}{\sigma_\rho\overline{\rho}}\frac{\partial\overline{p}}{\partial x_j}\frac{\partial\overline{\rho}}{\partial x_j} + \tau_{ij}\frac{\partial\widetilde{v}_i}{\partial x_j} - \overline{\rho}\,\epsilon + \Pi + \frac{\partial}{\partial x_j}\left(\frac{\mu_t}{\sigma_K}\frac{\partial K}{\partial x_j}\right),$$

$$\frac{\partial}{\partial t}(\overline{\rho}\,\epsilon) + \frac{\partial}{\partial x_j}(\overline{\rho}\,\epsilon\,\widetilde{v}_j) = \frac{\epsilon}{K}\left[-C_{\epsilon0}\frac{\nu_t}{\sigma_\rho\overline{\rho}}\frac{\partial\overline{p}}{\partial x_j}\frac{\partial\overline{\rho}}{\partial x_j}\right] + \frac{\epsilon}{K}\left[C_{\epsilon1}\,\tau_{ij}\frac{\partial\widetilde{v}_i}{\partial x_j} - C_{\epsilon2}\,\overline{\rho}\,\epsilon + C_{\epsilon3}\,\Pi\right] + \frac{\partial}{\partial x_j}\left(\frac{\mu_t}{\sigma_\epsilon}\frac{\partial\epsilon}{\partial x_j}\right).$$

Self-Similarity Transformation

For self-similalr motions we have $\frac{E_0 t^2}{\rho_0 r^{2+d}} = q$, where E_0 and ρ_0 represent the explosion energy and initial gas densities, respectively; $d = 1,2,3$ for planar, cylindrical, and spherical geometries. Solving for the time-dependent shock front location gives $R(t) = R_0 t^\lambda$, where $R_0 = (1/q)(E_0/\rho_0)^{\frac{1}{2+d}}$, q is a constant determined when reverting back to dimensional space–time coordinates, and $\xi = r/R(t)$, is the similarity and independent variable.
The space and time derivative transformations are

$$\frac{\partial f}{\partial r} = \frac{\xi}{r}\frac{\mathrm{d}f}{\mathrm{d}\xi}\ ,\quad \frac{\partial f}{\partial t} = -\lambda\frac{\xi}{t}\frac{\mathrm{d}f}{\mathrm{d}\xi},$$

with operators defined as

$$F_\xi = [V - \xi]\frac{\mathrm{d}}{\mathrm{d}\xi}\ ,\quad D_\xi = \frac{\mathrm{d}}{\mathrm{d}\xi} + \frac{d-1}{\xi}.$$

Turbulent Density, Velocity, and Energy Equations

In constructing a self-similar turbulent model first consider the original equations. Substituting similarity transformations into the Reynolds–Favre averaged form gives

$$\{F_\xi + D_\xi[V]\}\,\Omega = 0,$$

$$\left\{F_\xi + D_\xi[V] - \frac{d}{2}\right\}\Omega V + \frac{\mathrm{d}P}{\mathrm{d}\xi} = D_\xi[\tau]\ ,$$

$$\{F_\xi + D_\xi[V] - d\}\,\Omega\,E + D_\xi\,[P\,V] = \frac{C_\mu}{\sigma_\rho}\frac{K^2}{\Omega\,\epsilon}\frac{\mathrm{d}\Omega}{\mathrm{d}\xi}\frac{\mathrm{d}P}{\mathrm{d}\xi} + \Omega\,\epsilon - \Pi + D_\xi\left[\frac{C_\mu}{\sigma_E}\Omega\frac{K^2}{\epsilon}\frac{\mathrm{d}E}{\mathrm{d}\xi}\right],$$

with Reynolds stress and pressure–dilatation,

$$\tau = \frac{2}{3}\Omega\,K\left[1 - C_\mu\frac{K}{\epsilon}\left(2\frac{\mathrm{d}}{\mathrm{d}\xi} - \frac{d-1}{\xi}\right)V\right],$$

$$\Pi = \Omega\left[\alpha_2\,\tau\frac{\mathrm{d}V}{\mathrm{d}\xi} + \alpha_3\,\epsilon M_t + \frac{16}{3}\alpha_4\,M_t\,K\,D_\xi\,[V]\right]M_t.$$

Turbulent Kinetic Energy and Dissipation Rate Equations

Similarly turbulent kinetic energy and dissipation rate equations are

$$\{F_\xi + D_\xi[V] - d\}\,\Omega K = -\frac{C_\mu K^2}{\sigma_\rho\,\Omega\epsilon}\frac{\mathrm{d}\Omega}{\mathrm{d}\xi}\frac{\mathrm{d}P}{\mathrm{d}\xi} + \tau\frac{\mathrm{d}V}{\mathrm{d}\xi} - \Omega\epsilon + \Pi + D_\xi\left[\frac{C_\mu}{\sigma_K}\Omega\frac{K^2}{\epsilon}\frac{\mathrm{d}K}{\mathrm{d}\xi}\right],$$

$$\{F_\xi + D_\xi[V] - (4+3d)\}\,\Omega\epsilon = -\frac{\epsilon}{K}\frac{C_{\epsilon2}C_\mu}{\sigma_\rho}\frac{K^2}{\Omega\epsilon}\frac{\mathrm{d}\Omega}{\mathrm{d}\xi}\frac{\mathrm{d}P}{\mathrm{d}\xi} + \frac{\epsilon}{K}\left[C_{\epsilon0}\tau\frac{\mathrm{d}V}{\mathrm{d}\xi} - C_{\epsilon1}\Omega\epsilon + C_{\epsilon3}\Pi\right] + D_\xi\left[\frac{C_\mu}{\sigma_\epsilon}\Omega\frac{K^2}{\epsilon}\frac{\mathrm{d}\epsilon}{\mathrm{d}\xi}\right].$$

Boundary Conditions and Large-Eddy Turnover Time

Consider a system in which turbulence is nonexistent in the cold region and develops sufficiently fast, but not instantaneously, after the shock. From the Rankine-Hugoniot criterium, boundary conditions at the shock front, $\xi = 1$, are

$$\Omega(1) = \frac{\gamma+1}{\gamma-1}\quad V(1) = \frac{2}{\gamma+1}\quad P(1) = \frac{2}{\gamma+1}$$

$$K(1) = \epsilon(1) = 0.$$

Similar to conditions at the shock front, a large-eddy turnover time is obtained by solving a system with constant solution and considering a zero-velocity field prior to the shock. This gives

$$\frac{\mathrm{K}}{\epsilon} = \frac{\mathrm{C}_{\epsilon1} - C_{\epsilon3}}{4 + 3d - C_{\epsilon3}}.$$

We rearrange our system to eliminate necessitating a second boundary condition for density. At the origin, $\xi = 0$, symmetry imposes the conditions

$$V(0) = 0\quad \frac{\mathrm{d}P(0)}{\mathrm{d}\xi} = \frac{\mathrm{d}K(0)}{\mathrm{d}\xi} = \frac{\mathrm{d}\epsilon(0)}{\mathrm{d}\xi} = 0.$$