The Hancock Scheme

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I. Governing Equations

The governing equations are the Euler Equations, however, this methodology also applies to any hyperbolic system of PDEs with an entropy, such as the shallow water, the Navier-Stokes, and the Magnetohydrodynamics equations.[1] All of these can be written in vector form as

$$\mathbf{u}_t + \mathbf{f}^i_{x_i} = \mathbf{u}_t + \mathbf{f}^i_{\mathbf{u}}\mathbf{u}_{x_i} = \mathbf{u}_t + \mathbf{A}^i\mathbf{u}_{x_i} = 0 \tag{1}$$

II. Hancock Approach

We have taken a more straightforward approach, based on a generalization to moving meshes of the Hancock scheme.[2] The Hancock scheme is second-order, monotone, and fully discrete. It is the subject of a recent analysis by Berthon[3] and has been extended to third-order by Suzuki.[4] Here we make a straightforward generalization of the second-order version, and have found it very satisfactory. Although this is a two-step scheme, it requires only a single call to a Riemann solver for each timestep per interface.

II.A. One dimension

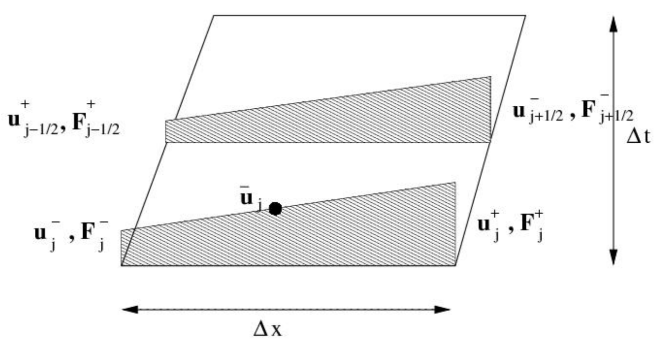

Figure 1. A finite volume cell in a one-dimensional moving mesh.

By integrating the conservation law (1) around the control volume we obtain

$$\mathbf{u}^{n+1}_j = \mathbf{u}^n_j \ - \frac{\Delta t}{\Delta x}\left(\mathbf{f}^{n+\frac{1}{2}}_{i+\frac{1}{2}} - \mathbf{f}^{n+\frac{1}{2}}_{i-\frac{1}{2}}\right). \tag{2}$$

In Hancock's method we take the initial data in cell i be

$$\mathbf{u}_i(x,0) = \bar{\mathbf{u}}_i + \tilde{\mathbf{S}}_i(x - \bar{x}_i), \tag{3}$$

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where the bar denotes cell averaged values and the slopes $\tilde{\mathbf{S}}_i$ are limited. Evaluate $\mathbf{u}_i$ at the left and right cell edges, giving $\mathbf{u}_i^-$ and $\mathbf{u}_i^+$. From these states, evaluate $\mathbf{f}_i^-$ and $\mathbf{f}_i^+$. Then, halfway through the timestep evaluate

$$\mathbf{u}^+_{i-\frac{1}{2}} = \bar{\mathbf{u}}_i - \frac{1}{2}\tilde{\mathbf{S}}\Delta x_i - \frac{1}{2}\frac{\Delta t}{\Delta x_i}(\mathbf{f}_i^+ - \mathbf{f}_i^-) \tag{4}$$

$$\mathbf{u}^-_{i+\frac{1}{2}} = \bar{\mathbf{u}}_i + \frac{1}{2}\tilde{\mathbf{S}}\Delta x_i - \frac{1}{2}\frac{\Delta t}{\Delta x_i}(\mathbf{f}_i^+ - \mathbf{f}_i^-). \tag{5}$$

These are the states to use in the Riemann problem in Equation (2). Note that if we assume constant data $\mathbf{S} = 0$, this reduces to Forward Euler.

II.B. Two dimensions

Define a cell j with edges i and linear conserved variable distribution

$$\mathbf{u}_j = \overline{\mathbf{u}}_j + \tilde{\nabla}\mathbf{u}_j \cdot (\mathbf{r}_i - \mathbf{r}_j) \tag{6}$$

for cell center $\mathbf{r}_j = (x_j, y_j)$, edge midpoint $\mathbf{r}_i = (x_i, y_i)$ and limited gradient $\tilde{\nabla}\mathbf{u}_j$. The Hancock update is then

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \frac{\Delta t}{A_j}\sum_i \left(\mathbf{f}_i^{n+\frac{1}{2}} \cdot \mathbf{n}_i^{n+\frac{1}{2}}\right) \tag{7}$$

where $\mathbf{n}_i = [n_x, n_y]^T$ is the scaled outward edge normal and the flux is calculated via a Riemann solver. The conserved variables at $t^{n+\frac{1}{2}}$ at edge i are

$$\mathbf{u}_i^{n+\frac{1}{2}} = \overline{\mathbf{u}}_j^n + \tilde{\nabla}\mathbf{u}_j^n \cdot (\mathbf{r}_i - \mathbf{r}_j) - \frac{\Delta t}{2A_j}\sum_k \mathbf{f}(\overline{\mathbf{u}}_j^n + \tilde{\nabla}\mathbf{u}_j^n \cdot (\mathbf{r}_k - \mathbf{r}_j)) \cdot \mathbf{n}_k \tag{8}$$

References

[1] Serre, D., Systems of conservation laws, Vol. I, Cambridge University Press, 1999.

[2] van Leer, B., "On the relationship between the upwind-differencing schemes of Godunov, Engquist-Osher, and Roe," SIAM J. Sci. Stat. Comput., Vol. 5, No. 1, 1984, pp. 1–20.

[3] Berthon, C., "Why the MUSCL-Hancock scheme is L1 -stable," Numerische Mathematik, Vol. 104, 2006, pp. 27–46.

[4] Suzuki, Y., Discontinuous Galerkin methods for extended hydrodynamics, Ph.D. thesis, University of Michigan, 2008.