Prescribing a Linear Stationary Shock Structure

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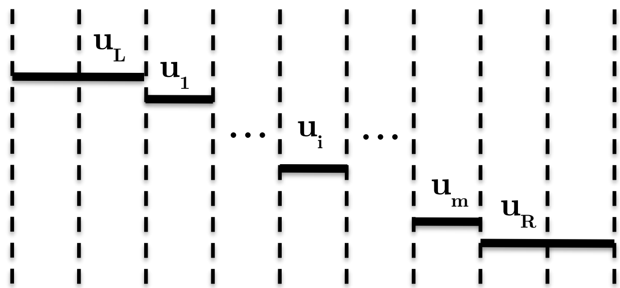
I. Stationary Shockwaves



Suppose we have a 1D system of n equations and we can define an m-point stationary shock, where m is the number of intermediate shock states. Our system requires that all fluxes are equal, such that $[\mathbf{f}] = 0$ gives a shock speed of 0. In this case, we have $\mathbf{f}_L = \mathbf{f}_R = \mathbf{f}$, and with $\mathbf{u}_L$ prescribed, we have $\mathbf{u}_R$. So now we are looking for $\mathbf{u}_i$'s, of which there are m. Thus we have mn unknowns. We have $\mathbf{f} = \mathbf{f_{i,i+1}}$, $i = L, 1, 2, \ldots, m$ the interface fluxes, which are $(m+1)n$ equations. If we write our flux as

$$\mathbf{f}_{i,i+1} = \frac{1}{2}(\mathbf{f}_i + \mathbf{f}_{i+1}) - \frac{1}{2}\mathbf{Q}(\mathbf{u}_{i+1} - \mathbf{u}_i)$$

For some matrix $\mathbf{Q} = \mathbf{Q}(\mathbf{u}_L, \mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{u}_R)$ with positive eigenvalues. At this point it is unclear how local we want to be. Lets start with the Roe Matrix,

$$\mathbf{Q} = |\mathbf{A}|^{\text{Roe}} + \mathbf{Z}$$

where $\mathbf{Z}$ is another matrix.

$$|\mathbf{A}|^{\text{Roe}} = \mathbf{R}|\Lambda|\mathbf{R}^{-1}$$

Lets choose $\mathbf{Z}$ such that we get a prescribed structure. What we want is a stable structure such that

$$\mathbf{u}_i = \alpha_i \mathbf{u}_L + (1 - \alpha_i)\mathbf{u}_R$$

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It is equivalent to require that

$$\mathbf{u}_i = \beta_i \mathbf{u}_{i-1} + (1-\beta_i)\mathbf{u}_{i+1}$$

This gives

$$\begin{aligned}
\mathbf{u}_i - \mathbf{u}_{i-1} &= (1-\beta_i)(\mathbf{u}_{i+1} - \mathbf{u}_{i-1}) \\
\mathbf{u}_{i+1} - \mathbf{u}_i &= -\beta_i(\mathbf{u}_{i+1} - \mathbf{u}_{i-1})
\end{aligned}$$

One reason for doing this is that now if we examine the flux, since we want all fluxes to equal $\mathbf{f}$, we get that for cell i,

$$\begin{aligned}
\mathbf{f} = \frac{1}{2}(\mathbf{f}_i + \mathbf{f}_{i+1}) - \frac{1}{2}\mathbf{Q}_{i,i+1}(\mathbf{u}_{i+1} - \mathbf{u}_i) \\
\mathbf{f} = \frac{1}{2}(\mathbf{f}_{i-1} + \mathbf{f}_i) - \frac{1}{2}\mathbf{Q}_{i-1,i}(\mathbf{u}_i - \mathbf{u}_{i-1})
\end{aligned}$$

with the substitutions above, these become

$$\begin{aligned}
\mathbf{f} &= \frac{1}{2}(\mathbf{f}_i + \mathbf{f}_{i+1}) + \frac{\beta_i}{2}\mathbf{Q}_{i,i+1}(\mathbf{u}_{i+1} - \mathbf{u}_{i-1}) \\
\mathbf{f} &= \frac{1}{2}(\mathbf{f}_{i-1} + \mathbf{f}_i) - \frac{1-\beta_i}{2}\mathbf{Q}_{i-1,i}(\mathbf{u}_{i+1} - \mathbf{u}_{i-1})
\end{aligned}$$

Subtracting these and multiplying by two we get that

$$\mathbf{f}_{i+1} - \mathbf{f}_{i-1} + (\beta_i \mathbf{Q}_{i,i+1} + (1-\beta_i)\mathbf{Q}_{i-1,i})(\mathbf{u}_{i+1} - \mathbf{u}_{i-1}) = 0$$

For a homogeneous PDE, this requirement turns into

$$(\beta_i \mathbf{Q}_{i,i+1} + (1-\beta_i)\mathbf{Q}_{i-1,i})(\mathbf{u}_{i+1} - \mathbf{u}_{i-1}) = -(\mathbf{A}_{i+1}\mathbf{u}_{i+1} - \mathbf{A}_{i-1}\mathbf{u}_{i-1})$$

We want this to hold for all β_i, as well.

A. One-point Stationary Shocks

For a one point stationary shock, this requirement becomes

$$\mathbf{Q}_{mR}(\mathbf{u}_R - \mathbf{u}_m) = \mathbf{Q}_{Lm}(\mathbf{u}_m - \mathbf{u}_L)$$

or

$$(\beta \mathbf{Q}_{mR} + (1-\beta)\mathbf{Q}_{Lm}) = 0$$

for all $\beta \in [0,1]$. From this, it is clear that $\mathbf{Q}$ cannot be diagonal, since it would need to be positive, in which case it would not satisfy the equations. Setting both diagonals to zero is not sufficient either, since that gives at least one negative root. This gives us several solutions.

1. $\mathbf{Q}_{mR} = \mathbf{Q}_{Lm} = 0$. This is a trivial solution, and corresponds to a standard centered flux function, $\mathbf{f}_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{f}_i + \mathbf{f}_{i+1})$, which does not help because it forces $\mathbf{f}_m$ to be $\mathbf{f}_L = \mathbf{f}_R$, so $\mathbf{u}_m$ is either the left or the right state.

2. $\mathbf{Q}_{mR} = \frac{1}{\beta}\mathbf{I}^*$, $\quad \mathbf{Q}_{Lm} = \frac{1}{\beta-1}\mathbf{I}^*$. Where $\mathbf{I}^*$ is any constant matrix, and is not all zeros. This is not valid since it corresponds to one $\mathbf{Q}$ matrix with negative eigenvalues, which puts the dissipation in the wrong direction.

3. $\mathbf{Q}_{mR}^{-1}\mathbf{Q}_{Lm} = \mathbf{Q}_{Lm}\mathbf{Q}_{mR}^{-1} = \frac{\beta}{\beta-1}$ and $\mathbf{Q}_{mR}\mathbf{Q}_{Lm}^{-1} = \mathbf{Q}_{Lm}^{-1}\mathbf{Q}_{mR} = \frac{\beta-1}{\beta}$. The two matrices can be of this form. I'm not sure what this structure says.

II. Linear Hugoniot Curve Equations

Lets look at

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0 \qquad \mathbf{u} = \begin{bmatrix} u \\ a \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} \frac{1}{2}(u^2 + a^2) \\ ua \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} u & a \\ a & u \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{L} = \mathbf{R}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

After a lot of math, the Hugoniot curve is

$$u_R \pm a_R = u_L \pm a_L$$

and the expansion gives $u \pm a =$ constant across an expansion. The shock speed is $S = u_L \pm a_R = u_R \pm a_L$. The Roe-averaged variables are arithmetic averages.

A. One-point Stationary Shock

Given u_L, a_L and using $\mathbf{f}_L = \mathbf{f}_R$ gives $u_R = \pm a_L, u_L = \pm a_R$ (for one case). Now we want to find $\mathbf{u}_m$ such that $\mathbf{u}_m = \beta\mathbf{u}_L + (1-\beta)\mathbf{u}_R$ and

$$(\beta\mathbf{Q}_{mR} + (1-\beta)\mathbf{Q}_{Lm}) = 0$$

Since $\mathbf{L}, \mathbf{R}$ are constant, the system reduces to

$$\beta|\Lambda_{mR}| + (1-\beta)|\Lambda_{Lm}| = 0$$

which is two equations,

$$\beta|\lambda_{mR}^{\pm}| + (1-\beta)|\lambda_{Lm}^{\pm}| = 0$$

where

$$\begin{aligned} \lambda_{mR}^{\pm} &= \beta\lambda_L^{\pm} + (2-\beta)\lambda_R^{\pm} \\ \lambda_{Lm}^{\pm} &= (1+\beta)\lambda_L^{\pm} + (1-\beta)\lambda_R^{\pm} \end{aligned}$$

This, however, simplifies, since across a stationary shock, we have that $\lambda_L^{\pm} = -(\lambda_R^{\pm})$ (if $\lambda_L^{\pm} = (\lambda_R^{\pm})$ then there are no jumps in u or a). Thus

$$\begin{aligned} \lambda_{mR}^{\pm} &= (\beta - (2-\beta))\lambda_L^{\pm} = 2(\beta-1)\lambda_L^{\pm} \\ \lambda_{Lm}^{\pm} &= ((1+\beta) - (1-\beta))\lambda_L^{\pm} = 2\beta\lambda_L^{\pm} \end{aligned}$$

Plugging this in gives

$$\begin{aligned} \beta|2(\beta-1)\lambda_L^{\pm}| + (1-\beta)|2\beta\lambda_L^{\pm}| &= 0 \\ 2\beta(1-\beta)|\lambda_L^{\pm}| + 2(1-\beta)\beta|\lambda_L^{\pm}| &= 0 \end{aligned}$$

III. Isothermal Euler Equations

In one dimension, these are

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0 \qquad \mathbf{u} = \begin{bmatrix} \rho \\ m \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} m \\ m^2/\rho + p \end{bmatrix}$$

with $m = \rho u$. The equation of state is $p = a^2\rho$ for some constant sound speed, a. The flux jacobian is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ a^2 - u^2 & 2u \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} 1 & 1 \\ u-a & u+a \end{bmatrix} \qquad \mathbf{L} = \mathbf{R}^{-1} = \frac{1}{2a}\begin{bmatrix} (u+a) & -1 \\ -(u-a) & 1 \end{bmatrix}$$

with eigenvalues $\lambda^{\pm} = u \pm a$. The Hugoniot curve can be determined from $s[\mathbf{u}] = [\mathbf{f}]$ as

$$u_R = u_L \pm a\frac{(\rho_R - \rho_L)}{\sqrt{\rho_R\rho_L}}$$

with shock speed

$$s = u_L \pm a\sqrt{\frac{\rho_R}{\rho_L}}$$

The Roe averaged velocity is a Roe average, with

$$\begin{aligned}
|\mathbf{A}|^{\mathrm{Roe}} &= \frac{1}{2a}\begin{bmatrix} (a-u)|a+u| + (a+u)|a-u| & |a+u| - |a-u| \\ (|a+u| - |a-u|)(a-u)(a+u) & (a+u)|a+u| + (a-u)|a-u| \end{bmatrix} \\
|\mathbf{A}|^{\mathrm{Roe}} &= \frac{1}{2a}\begin{bmatrix} \lambda^-|\lambda^+| + \lambda^+|\lambda^-| & |\lambda^+| - |\lambda^-| \\ (|\lambda^+| - |\lambda^-|)\lambda^-\lambda^+ & \lambda^+|\lambda^+| + \lambda^-|\lambda^-| \end{bmatrix}
\end{aligned}$$

which makes the requirement for a one-point stationary shock

$$\beta(|\mathbf{A}|^{\mathrm{Roe}}_{mR} + \mathbf{Z}_{mR}) = (\beta - 1)(|\mathbf{A}|^{\mathrm{Roe}}_{Lm} + \mathbf{Z}_{Lm})$$

Let $\mathbf{Z}_{i,i+1} = \mathbf{g}(\mathbf{u}_i, \mathbf{u}_{i+1})$. Now we have 4 unknowns, the $\mathbf{g}$'s and 4 equations. This system can then possibly be solved. In an attempt to simplify this, treat each side as independent

We have that

$$\begin{aligned}
\rho_m &= \beta\rho_L + (1-\beta)\rho_R \\
\rho_m u_m &= \beta\rho_L u_L + (1-\beta)\rho_R u_R
\end{aligned}$$

This leads to

$$u_m = \frac{\beta\rho_L u_L + (1-\beta)\rho_R u_R}{\beta\rho_L + (1-\beta)\rho_R}$$

and

$$\begin{aligned}
u_{Lm} = \frac{1}{2}(u_L + u_m) &= \frac{u_L}{1 + \frac{1-\beta}{\beta}\frac{\rho_R}{\rho_L}} + \frac{\frac{1}{2}(u_L + u_R)}{\frac{\beta}{1-\beta}\frac{\rho_L}{\rho_R} + 1} \\
u_{mR} = \frac{1}{2}(u_m + u_R) &= \frac{\frac{1}{2}(u_L + u_R)}{1 + \frac{1-\beta}{\beta}\frac{\rho_R}{\rho_L}} + \frac{u_R}{\frac{\beta}{1-\beta}\frac{\rho_L}{\rho_R} + 1}
\end{aligned}$$

Euler Equations

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$$

or, expanded as

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho u \\ E\end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix}\rho u \\ \rho u^2 + p \\ \rho u H\end{bmatrix} = \begin{bmatrix}0 \\ 0 \\ 0\end{bmatrix},$$

with $H = \frac{E+p}{\rho}$ and an equation of state $p = p(\rho, e)$. For the ideal gas, $p = (\gamma-1)\rho e = (\gamma-1)\left(E - \frac{1}{2}\rho u^2\right)$. This system is also equipped with a speed of sound, $a = \left.\frac{\partial p}{\partial \rho}\right|_s$.

Roe's approximate Riemann solver is described here as

$$\mathbf{f}(\mathbf{u}_L, \mathbf{u}_R) = \frac{1}{2}\left(\mathbf{f}(\mathbf{u}_L) + \mathbf{f}(\mathbf{u}_R)\right) - \frac{1}{2}\mathbf{R}|\Lambda|\mathbf{L}(\mathbf{u}_R - \mathbf{u}_L)$$

where $\mathbf{L} = \mathbf{R}^{-1}$,

$$\mathbf{R} = \left[\begin{array}{c|c|c} 1 & 1 & 1 \\ u-a & u & u+a \\ \frac{a^2}{\gamma-1} - ua + \frac{1}{2}u^2 & \frac{1}{2}u^2 & \frac{a^2}{\gamma-1} + ua + \frac{1}{2}u^2 \end{array}\right]$$

$$\mathbf{L} = \frac{1}{2a^2}\left[\begin{array}{c|c|c} \frac{\gamma-1}{2}u^2 + ua & -(\gamma-1)u - a & (\gamma-1) \\ 2a^2 - (\gamma-1)u^2 & 2(\gamma-1)u & -2(\gamma-1) \\ \frac{\gamma-1}{2}u^2 - ua & -(\gamma-1)u + a & (\gamma-1) \end{array}\right].$$

and $\Lambda = \mathrm{diag}(u-a, u, u+a)$ with a and u as density-averaged variables from

$$u = \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \qquad H = \frac{\sqrt{\rho_L}H_L + \sqrt{\rho_R}H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \qquad a = \sqrt{(\gamma-1)\left(H - \frac{1}{2}u^2\right)}$$

Pick a stationary shock as described by Hiro. Start with

$$[\rho_L, u_L, p_L]^T = \left[1, 1, \frac{1}{\gamma M_L^2}\right]^T$$

The right state is

$$\begin{bmatrix}\rho_R \\ u_R \\ p_R\end{bmatrix} = \begin{bmatrix} \frac{(\gamma+1)M_L^2}{(\gamma-1)M_L^2+2} \\ \frac{(\gamma-1)M_L^2+2}{(\gamma+1)M_L^2} \\ \left(1 + \frac{2\gamma}{\gamma+1}(M_L^2 - 1)\right)\frac{1}{\gamma M_L^2} \end{bmatrix}$$

Intermediate states are

$$\begin{bmatrix}\rho_M \\ u_M \\ p_M\end{bmatrix} = \begin{bmatrix} \rho_L + \varepsilon(\rho_R - \rho_L) \\ u_L + \left(1 - (1-\varepsilon)\left(1 + \varepsilon\frac{M_L^2-1}{1+\frac{\gamma-1}{2}M_L^2}\right)^{-\frac{1}{2}}\left(1 - \varepsilon\frac{M_L^2-1}{\frac{2\gamma}{\gamma-1}M_L^2-1}\right)^{-\frac{1}{2}}\right)(u_R - u_L) \\ p_L + \varepsilon\left(1 + (1-\varepsilon)\frac{\gamma-1}{\gamma+1}\frac{M_L^2-1}{M_L^2}\right)^{-1}(p_R - p_L) \end{bmatrix}$$

If the shockspeed is nonzero, using the same left states we get that

$$\begin{bmatrix} \rho_R \\ u_R \\ p_R \end{bmatrix} = \begin{bmatrix} \frac{S-1}{S-\frac{2(S-1)}{\gamma+1}+\frac{2}{M_L^2(\gamma+1)(S-1)}-1} \\ \frac{2(S-1)}{\gamma+1} - \frac{2}{M_L^2(\gamma+1)(S-1)} + 1 \\ \frac{2(S-1)^2}{\gamma+1} - \left(\frac{2}{\gamma+1} - \frac{1}{\gamma}\right)\frac{1}{M_L^2} \end{bmatrix}$$

IV. Experimental Math

Given the condition that

$$\frac{\partial r_i^k}{\partial u_j} r_j^k = 0 \qquad |r^k| = 1$$

for the k^{th} integral curve , the non-normalized form of this becomes

$$\frac{1}{|r^k|^2}\left[\frac{\partial r_i^k}{\partial u_j} - \frac{r_i^k}{|r^k|^2}\left(r_\ell^k \frac{\partial r_\ell^k}{\partial u_j}\right)\right] r_j^k = 0$$

If $r_I^k = 1$ for one I and all k, we get that $\frac{\partial r_I^k}{\partial u_j} = 0$ and that

$$r_\ell^k \frac{\partial r_\ell^k}{\partial u_j} r_j^k = 0$$

It can be shown that this results in

$$\frac{\partial r_i^k}{\partial u_j} r_j^k = 0 \qquad i \neq I$$

which in turn satisfies the previous equation, thus

$$\frac{\partial r_i^k}{\partial u_j} r_j^k = 0$$

holds, provided one $r_i^k = 1$.

A. Isothermal Euler

Given the Isothermal Euler equations, start with $p = p(\rho, m)$ and get that

$$A = \begin{bmatrix} 0 & 1 \\ -u^2 + p_\rho & 2u + p_m \end{bmatrix}$$

and that one eigendecomposition is

$$R = \begin{bmatrix} u + \frac{p_m}{2} + a & u + \frac{p_m}{2} - a \\ u^2 - p_\rho & u^2 - p_\rho \end{bmatrix} \quad \Lambda = \begin{bmatrix} u + \frac{p_m}{2} - a & 0 \\ 0 & u + \frac{p_m}{2} + a \end{bmatrix}$$

for $a = \sqrt{\frac{p_m^2}{4} + up_m + p_\rho}$. Now, let $R = [r^1, r^2]$ and we have that

$$\begin{aligned}
\frac{\partial a}{\partial \rho} &= \frac{1}{2a}\left(p_{\rho m}\left(\frac{p_m}{2} + u\right) - \frac{u}{2\rho}p_m + p_{\rho\rho}\right) \\
\frac{\partial a}{\partial m} &= \frac{1}{2a}\left(p_{mm}\left(\frac{p_m}{2} + u\right) + \frac{1}{\rho}p_m + p_{m\rho}\right) \\
\frac{\partial r_1}{\partial \rho} &= -\frac{u}{\rho} + \frac{1}{2}p_{m\rho} \pm \frac{\partial a}{\partial \rho} \\
\frac{\partial r_1}{\partial m} &= \frac{1}{\rho} + \frac{1}{2}p_{mm} \pm \frac{\partial a}{\partial m} \\
\frac{\partial r_2}{\partial \rho} &= -\frac{2u^2}{\rho} - p_{\rho\rho} \\
\frac{\partial r_2}{\partial m} &= \frac{2u}{\rho} - p_{m\rho}
\end{aligned}$$

Putting this all together leads to a very, very complex system for general pressure equations. So lets assume $p = p(\rho)$. This leads to

$$A = \begin{bmatrix} 0 & 1 \\ -u^2 + p_\rho & 2u \end{bmatrix}$$

and that one eigendecomposition is

$$R = \begin{bmatrix} 1 & 1 \\ u-a & u+a \end{bmatrix} \quad \Lambda = \begin{bmatrix} u-a & 0 \\ 0 & u+a \end{bmatrix}$$

for $a = \sqrt{p_\rho}$. Now, let $R = [r^1, r^2]$ and we have that

$$\begin{aligned}
\frac{\partial a}{\partial \rho} &= \frac{p_{\rho\rho}}{2\sqrt{p_\rho}} \\
\frac{\partial r_1}{\partial \rho} &= 0 \\
\frac{\partial r_1}{\partial m} &= 0 \\
\frac{\partial r_2}{\partial \rho} &= -\frac{u}{\rho} \pm \frac{\partial a}{\partial \rho} \\
\frac{\partial r_2}{\partial m} &= \frac{1}{\rho}
\end{aligned}$$

This leads to $a = \frac{C}{\rho}$ and $p = -\frac{C^2}{\rho} = -Ca$. The hugoniot curve is then $u_R - u_L = \pm(a_R - a_L)$ and the expansion curve is the same too.

B. Euler Equations

We get that

$$A = \begin{bmatrix} 0 & 1 & 0 \\ p_\rho - u^2 & 2u + p_m & p_E \\ u(p_\rho - H) & H + up_m & u(1 + p_E) \end{bmatrix}$$

The eigenvalues of this system are

$$\lambda = u, u + \tilde{u} \pm a$$

where

$$\tilde{u} = \frac{1}{2}(p_m + up_E) \qquad a^2 = \tilde{u}^2 + (p_\rho + up_m + Hp_E)$$

and

$$R = \begin{bmatrix} 1 & 1 & 1 \\ u & u + \tilde{u} - a & u + \tilde{u} + a \\ -\frac{p_\rho + up_m}{p_E} & H + u(\tilde{u} - a) & H + u(\tilde{u} + a) \end{bmatrix}$$

1. General EOS

To keep the structure the same as that of an ideal gas, we want that $\tilde{u} = 0$ and that $-\frac{p_\rho + up_m}{p_E} = \frac{1}{2}u^2$. This gives two relations:

$$p_m = -up_E \qquad \frac{1}{2}u^2 p_E = p_\rho$$

which reduce the speed of sound to

$$a^2 = \left(H - \frac{1}{2}u^2\right) p_E$$

and the eigenvectors are

$$R = \begin{bmatrix} 1 & 1 & 1 \\ u & u - a & u + a \\ \frac{1}{2}u^2 & H - ua & H + ua \end{bmatrix}$$

The requirements for linear integral curves for $\lambda = u + a$ are

$$\begin{aligned} \frac{\partial r_2}{\partial u_j} r_j &= \frac{\partial(u+a)}{\partial \rho} + \frac{\partial(u+a)}{\partial m}(u+a) + \frac{\partial(u+a)}{\partial E}(H+ua) \\ \frac{\partial r_3}{\partial u_j} r_j &= \frac{\partial(H+ua)}{\partial \rho} + \frac{\partial(H+ua)}{\partial m}(u+a) + \frac{\partial(H+ua)}{\partial E}(H+ua) \end{aligned}$$

Further, we have that $\frac{\partial u}{\partial \rho} = -\frac{u}{\rho}$, $\frac{\partial u}{\partial m} = \frac{1}{\rho}$, $\frac{\partial H}{\partial \rho} = -\frac{H}{\rho} + \frac{p_\rho}{\rho}$, $\frac{\partial H}{\partial m} = \frac{p_m}{\rho}$, $\frac{\partial H}{\partial E} = \frac{1}{\rho} + \frac{p_E}{\rho}$ which results in

$$\begin{aligned}
\frac{\partial r_2}{\partial u_j} r_j &= \frac{\partial(u+a)}{\partial \rho} + \frac{\partial(u+a)}{\partial m}(u+a) + \frac{\partial(u+a)}{\partial E}(H+ua) \\
&= -\frac{u}{\rho} + \frac{\partial a}{\partial \rho} + \frac{u+a}{\rho} + (u+a)\frac{\partial a}{\partial m} + (H+ua)\frac{\partial a}{\partial E} \\
&= \frac{a}{\rho} + \frac{\partial a}{\partial \rho} + (u+a)\frac{\partial a}{\partial m} + (H+ua)\frac{\partial a}{\partial E} \\
\frac{\partial r_3}{\partial u_j} r_j &= \frac{\partial(H+ua)}{\partial \rho} + \frac{\partial(H+ua)}{\partial m}(u+a) + \frac{\partial(H+ua)}{\partial E}(H+ua) \\
&= \frac{\partial H}{\partial \rho} + u\frac{\partial a}{\partial \rho} - \frac{au}{\rho} + (u+a)\frac{\partial H}{\partial m} + u(u+a)\frac{\partial a}{\partial m} + \frac{au+a^2}{\rho} + (H+ua)\frac{\partial H}{\partial E} + u(H+ua)\frac{\partial a}{\partial E} \\
&= -\frac{H}{\rho} + \frac{p_\rho}{\rho} + u\frac{\partial a}{\partial \rho} - \frac{au}{\rho} + (u+a)\frac{p_m}{\rho} + u(u+a)\frac{\partial a}{\partial m} + \frac{au+a^2}{\rho} + (H+ua)\left(\frac{1}{\rho} + \frac{p_E}{\rho}\right) + u(H+ua)\frac{\partial a}{\partial E} \\
&= u\left(\frac{a}{\rho} + \frac{\partial a}{\partial \rho} + (u+a)\frac{\partial a}{\partial m} + (H+ua)\frac{\partial a}{\partial E}\right) + \frac{p_\rho}{\rho} + (u+a)\frac{p_m}{\rho} + (H+ua)\frac{p_E}{\rho} + \frac{a^2}{\rho} \\
&= u\left(\frac{a}{\rho} + \frac{\partial a}{\partial \rho} + (u+a)\frac{\partial a}{\partial m} + (H+ua)\frac{\partial a}{\partial E}\right) + \frac{p_\rho + up_m + Hp_E}{\rho} + a\frac{p_m}{\rho} + ua\frac{p_E}{\rho} + \frac{a^2}{\rho} \\
&= u\left(\frac{a}{\rho} + \frac{\partial a}{\partial \rho} + (u+a)\frac{\partial a}{\partial m} + (H+ua)\frac{\partial a}{\partial E}\right) + \frac{a}{\rho}(p_m + up_E) + \frac{2a^2}{\rho} \\
&= u\left(\frac{a}{\rho} + \frac{\partial a}{\partial \rho} + (u+a)\frac{\partial a}{\partial m} + (H+ua)\frac{\partial a}{\partial E}\right) + \frac{2a^2}{\rho}
\end{aligned}$$

Combining these two equations gives a rather interesting result: $2a^2/\rho = 0$, or $a = 0$, which in turn says that p is constant. So now, one constraint needs to be loosened to prevent this. I suggest that $\tilde{u} = 0$, which leads to $p_m = -up_E$ and the final term of the first eigenvector as $-\frac{p_\rho + up_m}{p_E} = u^2 - \frac{p_\rho}{p_E}$. The speed of sound is $a^2 = p_\rho + (H - u^2)p_E$. Introducing this changes $\frac{\partial r_3}{\partial u_j} r_j$ to

$$\begin{aligned}
\frac{\partial r_3}{\partial u_j} r_j &= \frac{\partial(H+ua)}{\partial \rho} + \frac{\partial(H+ua)}{\partial m}(u+a) + \frac{\partial(H+ua)}{\partial E}(H+ua) \\
&= \frac{\partial H}{\partial \rho} + u\frac{\partial a}{\partial \rho} - \frac{au}{\rho} + (u+a)\frac{\partial H}{\partial m} + u(u+a)\frac{\partial a}{\partial m} + \frac{au+a^2}{\rho} + (H+ua)\frac{\partial H}{\partial E} + u(H+ua)\frac{\partial a}{\partial E} \\
&= -\frac{H}{\rho} + \frac{p_\rho}{\rho} + u\frac{\partial a}{\partial \rho} - \frac{au}{\rho} + (u+a)\frac{p_m}{\rho} + u(u+a)\frac{\partial a}{\partial m} + \frac{au+a^2}{\rho} + (H+ua)\left(\frac{1}{\rho} + \frac{p_E}{\rho}\right) + u(H+ua)\frac{\partial a}{\partial E} \\
&= u\left(\frac{a}{\rho} + \frac{\partial a}{\partial \rho} + (u+a)\frac{\partial a}{\partial m} + (H+ua)\frac{\partial a}{\partial E}\right) + \frac{p_\rho}{\rho} + (u+a)\frac{p_m}{\rho} + (H+ua)\frac{p_E}{\rho} + \frac{a^2}{\rho} \\
&= u\left(\frac{a}{\rho} + \frac{\partial a}{\partial \rho} + (u+a)\frac{\partial a}{\partial m} + (H+ua)\frac{\partial a}{\partial E}\right) + \frac{p_\rho + up_m + Hp_E}{\rho} + a\frac{p_m}{\rho} + ua\frac{p_E}{\rho} + \frac{a^2}{\rho} \\
&= u\left(\frac{a}{\rho} + \frac{\partial a}{\partial \rho} + (u+a)\frac{\partial a}{\partial m} + (H+ua)\frac{\partial a}{\partial E}\right) + \frac{a}{\rho}(p_m + up_E) + \frac{2a^2}{\rho} \\
&= u\left(\frac{a}{\rho} + \frac{\partial a}{\partial \rho} + (u+a)\frac{\partial a}{\partial m} + (H+ua)\frac{\partial a}{\partial E}\right) + \frac{2a^2}{\rho}
\end{aligned}$$

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