# A Crude Fix

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Write the Godunov Flux as $[0, P, 0]^T$. Define $\mu = u_0 \frac{\Delta t}{\Delta x}$. From conservation of mass, we have

$$\begin{aligned} \rho_1 &= \rho_0 - \frac{\Delta t}{\Delta x}(0 - \rho_0 u_0) \\ &= \rho_0(1 + \mu) \end{aligned}$$

Conservation of momentum gives

$$\begin{aligned} \rho_1 u_1 &= \rho_0 u_0 - \frac{\Delta t}{\Delta x}(P - p_0 - \rho_0 u_0^2) \\ &= \rho_0 u_0 (1 + \mu) + \frac{\mu}{u_0}(p_0 - P) \end{aligned}$$

and the result that

$$\rho_1 u_1^2 = \frac{1}{\rho_0(1+\mu)} \left( \rho_0^2 u_0^2 (1+\mu)^2 + 2\rho_0 \mu (1+\mu)(p_0 - P) + \frac{\mu^2}{u_0^2}(p_0 - P)^2 \right)$$

And finally conservation of energy gives

$$\begin{aligned} \frac{p_1}{\gamma - 1} + \frac{1}{2}\rho_1 u_1^2 &= \frac{p_0}{\gamma - 1} + \frac{1}{2}\rho_0 u_0^2 - \frac{\Delta t}{\Delta x}\left( 0 - u_0 \left[ \frac{\gamma p_0}{\gamma - 1} + \frac{1}{2}\rho_0 u_0^2 \right] \right) \\ &= \frac{p_0}{\gamma - 1} + \frac{1}{2}\rho_0 u_0^2 + \mu \left[ \frac{\gamma p_0}{\gamma - 1} + \frac{1}{2}\rho_0 u_0^2 \right] \end{aligned}$$

The equation for shock speed, $S$, is

$$\rho_0 S^2 - \frac{\gamma - 3}{2}\rho_0 u_0 S - \frac{\gamma - 1}{2}\rho_0 u_0^2 = \gamma p_0$$

Use this to rearrange energy conservation into

$$\begin{aligned} p_1 &= p_0 - \frac{\gamma - 1}{2}\rho_1 u_1^2 + \frac{\gamma - 1}{2}\rho_0 u_0^2 (1+\mu) + \mu \left( \rho_0 S^2 - \frac{\gamma - 3}{2}\rho_0 u_0 S - \frac{\gamma - 1}{2}\rho_0 u_0^2 \right) \\ &= p_0 - \frac{\gamma - 1}{2}\rho_1 u_1^2 + \frac{\gamma - 1}{2}\rho_0 u_0^2 + \mu \rho_0 S^2 - \frac{\gamma - 3}{2}\mu \rho_0 u_0 S \\ &= p_0 - \frac{\gamma - 1}{2}\rho_1 u_1^2 + \mu \rho_0 S(u_0 + S) - \frac{\gamma - 1}{2}\mu \rho_0 u_0 S \\ &= p_0 + \mu \rho_0 S(u_0 + S) - \frac{\gamma - 1}{2}\left( \mu \rho_0 u_0 (u_0 + S) + 2\mu(p_0 - P) + \frac{\mu^2}{1+\mu}\frac{(p_0 - P)^2}{\rho_0 u_0^2} \right) \end{aligned}$$

To eliminate this extra term, solve the quadratic

$$\rho_0 u_0 (u_0 + S) + 2(p_0 - P) + \frac{\mu}{1+\mu}\frac{(p_0 - P)^2}{\rho_0 u_0^2} = 0$$

I get that the solution is

$$P - p_0 = \left(1 + \frac{1}{\mu}\right)\rho_0 u_0^2 \left( 1 \pm \sqrt{\frac{1 - \frac{\mu S}{u_0}}{1 + \mu}} \right)$$

Since $p_e = p_0 + \rho_0 u_0(u_0 + S)$, we get that when $\mu = \mu_{\max} = u_0/S \rightarrow P = p_e$. To choose the correct root, examine the momentum equation.

$$\begin{aligned}
\rho_1 u_1 &= \rho_0 u_0(1+\mu) + \frac{\mu}{u_0}(p_0 - P) \\
&= \rho_0 u_0(1+\mu) - \frac{\mu}{u_0}\left(1 + \frac{1}{\mu}\right)\rho_0 u_0^2 \left(1 \pm \sqrt{\frac{1 - \frac{\mu S}{u_0}}{1+\mu}}\right) \\
&= \rho_0 u_0 \left((1+\mu) - (1+\mu)\left(1 \pm \sqrt{\frac{1 - \frac{\mu S}{u_0}}{1+\mu}}\right)\right) \\
&= \rho_0 u_0(1+\mu)\left(- \pm \sqrt{\frac{1 - \frac{\mu S}{u_0}}{1+\mu}}\right)
\end{aligned}$$

Examining the limit as $\mu \rightarrow 0$, we get that

$$\rho_1 u_1 = -\rho_0 u_0 \left(\pm 1\right)$$

Since there should be no change in momentum if $\mu \propto \Delta t \rightarrow 0$, the correct root is the negative root.

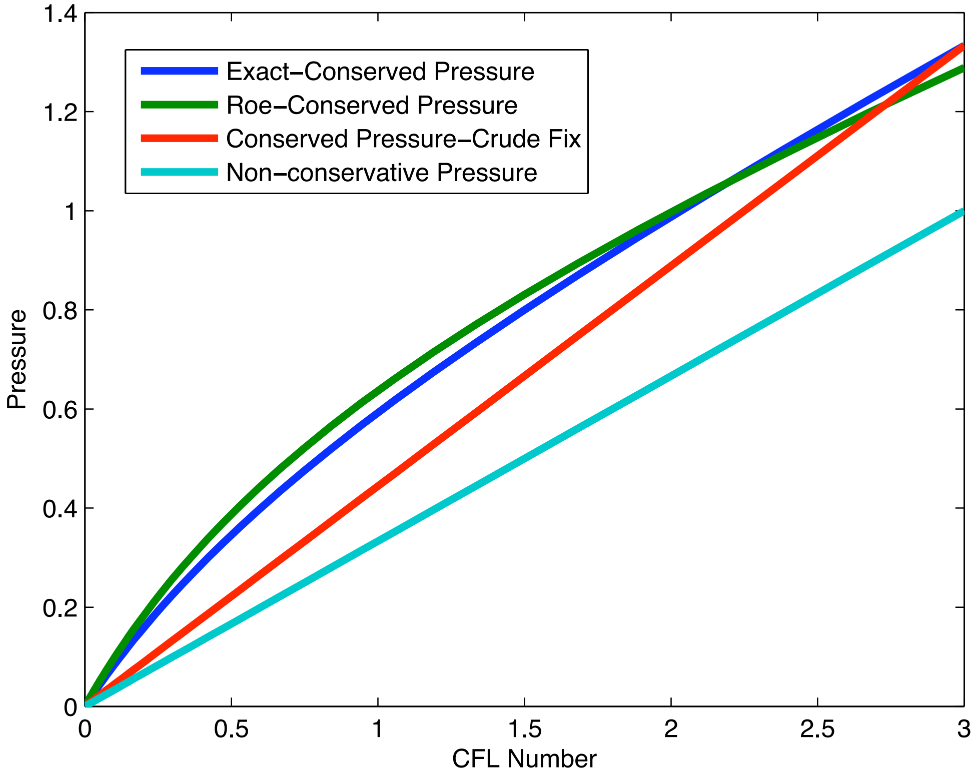


Figure 1: CFL vs wall pressure after one timestep comparing the crude fix to pressures achieved in other ways.

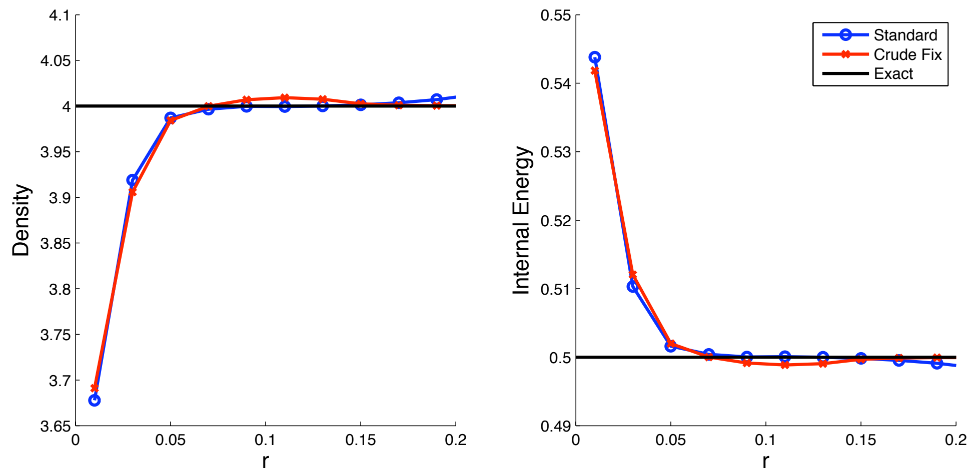


Figure 2: Density and internal energy using crude pressure in wall flux on first timestep

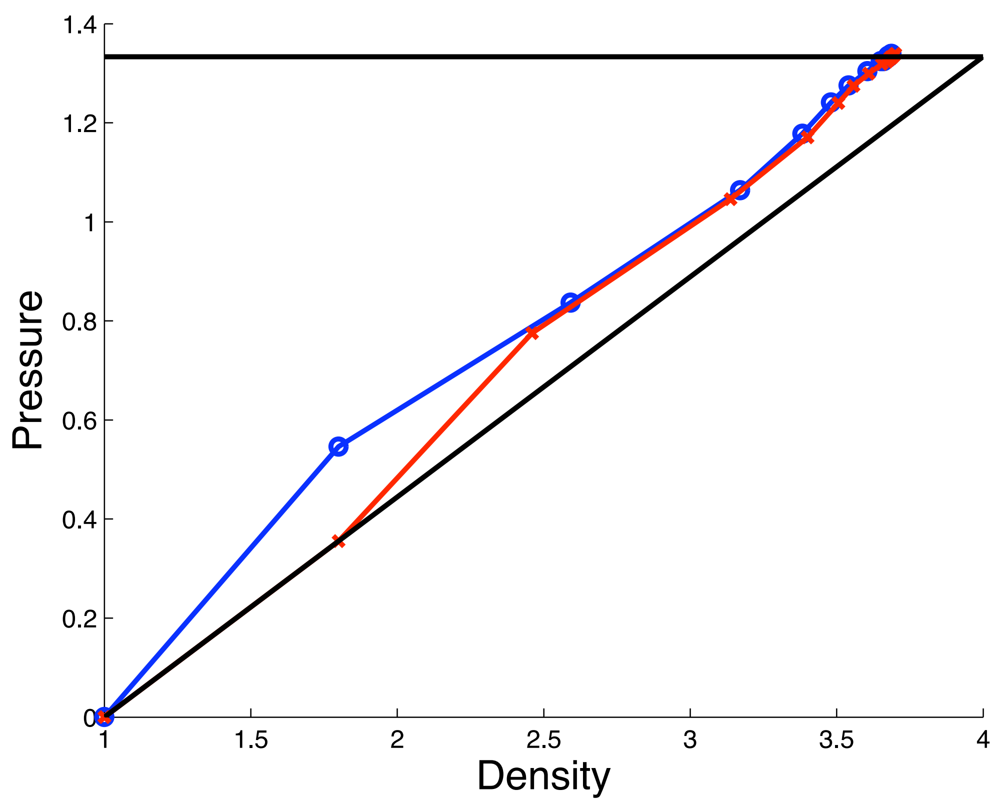


Figure 3: Wall pressure vs wall density using crude pressure in wall flux on first timestep