On Linear Equations of State for the Euler Equations

1 Governing Equations

The Euler Equations can be written in vector form as

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0 \tag{1}$$

or, expanded as

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(e+p) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2}$$

with the equation of state for p. To get a linear equation of state, lets try

$$p = (\gamma - 1)e$$

2 Flux Jacobian Structure

With

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}} = \mathbf{R}\Lambda\mathbf{L}$$

and defining $h = \frac{(e+p)}{\rho} = \frac{\gamma e}{\rho}$, we get that

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 & 2u & \gamma - 1 \\ -hu & h & \gamma u \end{bmatrix}$$

and the decomposition as

$$\mathbf{R} = \begin{bmatrix} 1 & 1 & 1 \\ u - \frac{2(\gamma-1)h}{2a+(\gamma-1)u} & u & u + \frac{2(\gamma-1)h}{2a-(\gamma-1)u} \\ h - \frac{2(\gamma-1)uh}{2a+(\gamma-1)u} & 0 & h + \frac{2(\gamma-1)uh}{2a-(\gamma-1)u} \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} \frac{u(2a+(\gamma-1)u)^2}{8ah(\gamma-1)} & -\frac{(2a+(\gamma-1)u)^2}{8ah(\gamma-1)} & \frac{2(2a+(\gamma-1)u)}{8ah} \\ 1 - \frac{u^2}{h} & \frac{u}{h} & -\frac{1}{h} \\ -\frac{u(2a-(\gamma-1)u)^2}{8ah(\gamma-1)} & \frac{(2a-(\gamma-1)u)^2}{8ah(\gamma-1)} & \frac{2(2a-(\gamma-1)u)}{8ah} \end{bmatrix}$$

with eigenvalues

$$\lambda_{1,2,3} = \frac{(\gamma+1)}{2}u - a, u, \frac{(\gamma+1)}{2}u + a$$

for a speed of sound

$$a = \sqrt{(\gamma-1)\left(h + (\gamma-1)\frac{u^2}{4}\right)}$$

We have that the wavestrengths

$$\mathbf{L}\mathrm{d}\mathbf{u} = \begin{cases} \frac{2a+(\gamma-1)u}{8ah(\gamma-1)} & [u(2a+(\gamma-1)u)\mathrm{d}\rho - (2a+(\gamma-1)u)\mathrm{d}m + 2(\gamma-1)\mathrm{d}e] \\ \frac{1}{h} & \left[(h-u^2)\mathrm{d}\rho + u\mathrm{d}m - \mathrm{d}e\right] \\ \frac{2a-(\gamma-1)u}{8ah(\gamma-1)} & [-u(2a-(\gamma-1)u)\mathrm{d}\rho + (2a-(\gamma-1)u)\mathrm{d}m + 2(\gamma-1)\mathrm{d}e] \end{cases}$$

3 Expansion Waves

From $l_j\mathrm{d}\mathbf{u} = 0$ across wave $\lambda_k, j \neq k$, we get for a contact that $\mathrm{d}u, \mathrm{d}p = 0$, and across $\lambda_{1,3}$ that

$$\left(\frac{(\gamma-1)}{2}u \pm a\right)\mathrm{d}u = \frac{\mathrm{d}p}{\rho}$$

and

$$\mathrm{d}e = h\mathrm{d}\rho + \rho u\mathrm{d}u$$

We also have from our definitions that

$$(\gamma-1)\mathrm{d}e = \mathrm{d}p$$

and

$$\gamma\mathrm{d}e = \rho\mathrm{d}h + h\mathrm{d}\rho$$

which gives

$$\mathrm{d}h = \frac{\mathrm{d}p}{\rho} + u\mathrm{d}u$$

Now, looking at the speed of sound,

$$\frac{a^2}{\gamma-1} = h + \frac{(\gamma-1)}{4}u^2$$

$$\frac{2a}{\gamma-1}\mathrm{d}a = \mathrm{d}h + \frac{\gamma-1}{2}u\mathrm{d}u$$

$$\frac{2a}{\gamma-1}\mathrm{d}a = \frac{\mathrm{d}p}{\rho} + u\mathrm{d}u + \frac{\gamma-1}{2}u\mathrm{d}u$$

$$\frac{2a}{\gamma-1}\mathrm{d}a = \left(\frac{(\gamma-1)}{2}u \pm a\right) + u\mathrm{d}u + \frac{\gamma-1}{2}u\mathrm{d}u$$

$$\frac{2a}{\gamma-1}\mathrm{d}a = (\gamma u \pm a)\mathrm{d}u$$

and finally

$$\frac{\mathrm{d}a}{\mathrm{d}u} = \frac{(\gamma-1)(\gamma u \pm a)}{2a}$$

This equation is transcendental, and serves as part of the Poisson Curve. To take the next step, combine both equations for de and eliminate to get

$$\gamma(h\mathrm{d}\rho + \rho u\mathrm{d}u) = \rho\mathrm{d}h + h\mathrm{d}\rho$$

$$(\gamma-1)h\mathrm{d}\rho = \rho\mathrm{d}h - \gamma\rho u\mathrm{d}u$$

$$(\gamma-1)\frac{\mathrm{d}\rho}{\rho} = \frac{\mathrm{d}h}{h} - \frac{\gamma u\mathrm{d}u}{h}$$

$$(\gamma-1)\frac{\mathrm{d}\rho}{\rho} = \frac{\mathrm{d}h}{h} - \frac{\gamma u\mathrm{d}u}{\frac{a^2}{\gamma-1} - \frac{\gamma-1}{4}u^2}$$

This can be integrated from $\mathbf{u}_0$ to $\mathbf{u}^*$ to get

$$\left(\frac{\rho^*}{\rho_0}\right)^{\gamma-1} = \frac{h^*}{h_0}\exp\left[\int_{u^*}^{u_0} \frac{\gamma u\mathrm{d}u}{\frac{a(u)^2}{\gamma-1} - \frac{\gamma-1}{4}u^2}\right]$$

where we can get $a(u)$ numerically from the ODE. This can be rearranged to give

$$\left(\frac{\rho^*}{\rho_0}\right)^{\gamma} = \frac{p^*}{p_0}\exp\left[\int_{u^*}^{u_0} \frac{\gamma u\mathrm{d}u}{\frac{a(u)^2}{\gamma-1} - \frac{\gamma-1}{4}u^2}\right]$$

and

$$p^* = p\left(\frac{\frac{a^{*2}}{\gamma-1} - \frac{\gamma-1}{4}u^{*2}}{\frac{a^2}{\gamma-1} - \frac{\gamma-1}{4}u^2}\right)^{\frac{\gamma}{(\gamma-1)}}\left(\exp\left[-\int_{u_0}^{u^*} \frac{\gamma u\mathrm{d}u}{\frac{a(u)^2}{\gamma-1} - \frac{\gamma-1}{4}u^2}\right]\right)^{\frac{1}{\gamma-1}}$$

4 Shockwaves

Define $[\ ] = (\ )^* - (\ )$ and write the jump equations with shock speed S as

$$[\rho u] = S[\rho]$$

$$[p + \rho u^2] = S[\rho u]$$

$$[\gamma u p] = S[p]$$

Cross multiplying to eliminate S we get

$$[\rho u]^2 = [\rho][p + \rho u^2]$$

$$[\rho][\gamma u p] = [p][\rho u]$$

$$[p][p + \rho u^2] = [\rho u][\gamma u p]$$

Taking the first and third equations and solving for ρ^* and then equating, we get

$$\rho = -\frac{[p]}{u[u]}, \frac{[p]^2}{[u](\gamma[up] - u[p])}$$

Taking the last two equations and solving for ρ^* and then equating, we get

$$[p]^2 = \rho[u](\gamma[up] - u[p])$$

Solving for p^* gives

$$p^* = p + \rho[u]\left(\frac{(\gamma u^* - u)}{2} \pm \sqrt{\frac{(\gamma u^* - u)^2}{4} + (\gamma - 1)h}\right)$$

5 Exact Riemann Solver

Putting it together, for a shock we have that

$$\rho^* = \rho\left[\frac{u[p] - \gamma[up]}{u^*[p] - \gamma[up]}\right]$$

and for an expansion we have

$$\left(\frac{\rho^*}{\rho_0}\right)^\gamma = \frac{p^*}{p_0}\exp\left[\int_{u_0}^{u^*} \frac{\gamma u \mathrm{d}u}{\frac{a(u)^2}{\gamma-1} - \frac{\gamma-1}{4}u^2}\right]$$

It should also be noted that

$$p'(u) = \frac{\gamma - 1}{2}\rho u \pm a$$

and

$$p''(u) = \gamma\rho\left(1 \pm \frac{(\gamma-1)u}{2a}\right)$$

5.1 Inside a Left Expansion

To get the solution within a left expansion, look at

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \xi = \left(\frac{\gamma+1}{2}\right)u - a$$

Plugging this into our ODE gives

$$\frac{\mathrm{d}\xi}{\mathrm{d}u} = \frac{\gamma+1}{2} - \left(\frac{\gamma-1}{2}\right)\left(\frac{\gamma u}{\frac{\gamma+1}{2}u - \xi}\right)$$

and

$$\frac{\mathrm{d}\xi}{\mathrm{d}a} = \frac{\gamma+1}{\gamma-1}\left(\frac{a}{\frac{2\gamma}{\gamma+1}(\xi + a) - a}\right) - 1$$

with initial condition

$$\xi_0 = \frac{\gamma+1}{2}u_L - a_L$$

5.2 Inside a Right Expansion

To get the solution within a right expansion, look at

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \xi = \left(\frac{\gamma+1}{2}\right)u + a$$

Plugging this into our ODE gives

$$\frac{\mathrm{d}\xi}{\mathrm{d}u} = \frac{\gamma+1}{2} + \left(\frac{\gamma-1}{2}\right)\left(\frac{\gamma u}{\frac{\gamma+1}{2}u - \xi}\right)$$

and

$$\frac{\mathrm{d}\xi}{\mathrm{d}a} = \frac{\gamma+1}{\gamma-1}\left(\frac{a}{\frac{2\gamma}{\gamma+1}(\xi - a) - a}\right) + 1$$

with initial condition

$$\xi_0 = \frac{\gamma+1}{2}u_R + a_R$$

6 Other thoughts

The Mach number, u_0/a_0, is weird in this system, since

$$M_0^2 = \frac{4\rho_0 u_0^2}{4\gamma p_0 + (\gamma-1)^2\rho_0 u_0^2}$$

thus as $p_0 \to 0$,

$$M_0^2 = \frac{4}{(\gamma-1)^2}$$

which is 9 for $\gamma = 5/3$. It is easy to see that this system satisfies homogeneity,

$$\mathbf{f}(\mathbf{u}) = \mathbf{A}(\mathbf{u})\mathbf{u}$$

Of note is the behavior as $p \propto e \propto h, \to 0$. Here, we get that $a \to \frac{(\gamma-1)}{2}|u|$,

$$\mathbf{R}(h \to 0, u > 0) = \begin{bmatrix} 1 & 1 & 1 \\ u & u & \gamma u \\ 0 & 0 & (\gamma-1)u^2 \end{bmatrix} \qquad \mathbf{R}(h \to 0, u < 0) = \begin{bmatrix} 1 & 1 & 1 \\ (2-\gamma)u & u & u \\ -(\gamma-1)u^2 & 0 & 0 \end{bmatrix}$$

and

$$\lambda_{1,2,3}(h \to 0, u < 0) = u, u, \gamma u \qquad \lambda_{1,2,3}(h \to 0, u > 0) = \gamma u, u, u$$

7 Pressure Transport

If we plug the EOS in, we can observe that

$$\frac{\partial e}{\partial t} + \frac{\partial(\gamma u e)}{\partial x} = 0$$

and identically that

$$\frac{\partial p}{\partial t} + \frac{\partial(\gamma u p)}{\partial x} = 0$$

8 Noh Problem

Here, the Noh problem can again be represented with a right wall and initial conditions ρ_0, u_0, p_0, $u_0 > 0$ such that

$$\rho_1 = \rho_0 \left(1 + \frac{u_0}{S}\right)$$

$$p_1 = p_0\left(1 + \frac{\gamma u_0}{S}\right)$$

$$\frac{S}{u_0} = \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{\gamma p_0}{\rho_0 u_0^2}} = \frac{1}{2} - \sqrt{\frac{1}{M_0^2} + \frac{\gamma}{4}(2-\gamma)}$$

Start with $\rho_0 = 1, u_0 = -1$ and $p_0 = 4/15$ for $M_0^2 = 9/5$. This gives $p_1 = 8/5$ and $\rho_1 = 4$.

S	M_0^2	p_0	ρ_1	p_1
1	$\frac{9}{19}$	$\frac{6}{5}$	2	$\frac{16}{5}$
$\frac{1}{2}$	$\frac{36}{31}$	$\frac{9}{20}$	3	$\frac{39}{20}$
$\frac{1}{3}$	$\frac{9}{5}$	$\frac{4}{15}$	4	$\frac{8}{5}$
$\frac{1}{4}$	$\frac{144}{61}$	$\frac{3}{16}$	5	$\frac{23}{16}$
$\frac{1}{10}$	$\frac{900}{199}$	$\frac{33}{500}$	11	$\frac{583}{500}$

9 Roe-type Riemann Solvers

Define Roe-averaged variables as

$$\tilde{\rho} = \sqrt{\rho_L \rho_R}$$

$$\tilde{u} = \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$

$$\tilde{h} = \frac{\sqrt{\rho_L}h_L + \sqrt{\rho_R}h_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$

$$\tilde{a}^2 = (\gamma - 1)\tilde{h} + \frac{(\gamma-1)^2}{4}\tilde{u}^2$$

and let

$$\hat{\mathbf{f}}(\mathbf{u}_L, \mathbf{u}_R) = \frac{1}{2}\left(\mathbf{f}_L + \mathbf{f}_R\right) - \frac{1}{2}\tilde{\mathbf{R}}\tilde{\Lambda}\tilde{\mathbf{L}}\Delta\mathbf{u}$$

with variables and matrices defined as above.

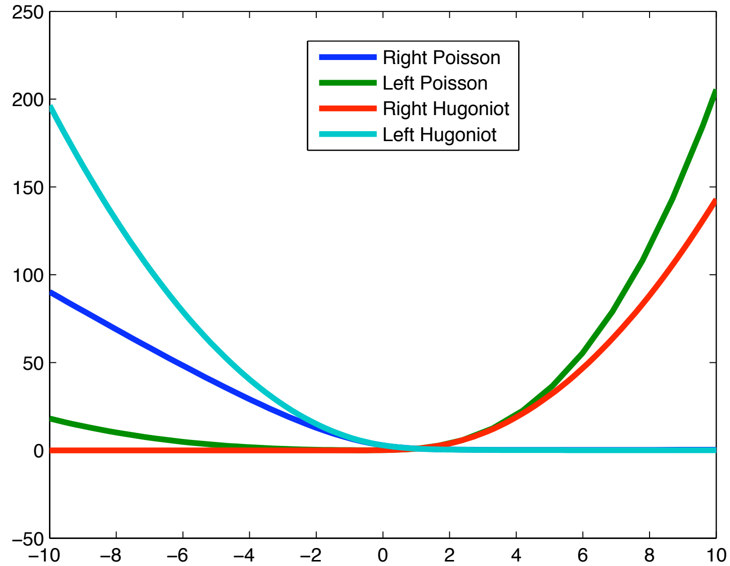

Figure 1: Left and Right moving Hugoniot and Poisson Curves through $(u, p) = (1, 1)$