# On Non-Conservative Pressure for Wall Heating

Daniel Zaide

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## 1 Governing Equations

The Euler Equations can be written in vector form as

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0 \tag{1}$$

or, expanded as

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2}$$

with the equation of state $p = p(\rho, i)$. For an ideal gas as

$$p = (\gamma - 1)\rho i, \qquad \rho i = \left(E - \frac{1}{2}\rho u^2\right) \qquad H = \frac{E+p}{\rho} \tag{3}$$

Alternative to the energy equation, we can also define a convective equation for pressure of the form

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\left(p_{\text{nc}} - (\gamma-1)\left[E - \frac{1}{2}\rho u^2\right]\right) = 0 \tag{4}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)p_{\text{nc}} + (p + (\gamma-1)p_{\text{nc}})\frac{\partial u}{\partial x} = 0 \tag{5}$$

We can note that from pressure convexity that the actual pressure at the interpolated state is always greater than the linearly interpolated pressure. Thus the pressure from this convective equation will always be lower.

## 2 Basic Method

The first order method in space and time is

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \frac{\Delta t}{\Delta x}(\mathbf{f}(\mathbf{u})_{j+\frac{1}{2}} - \mathbf{f}(\mathbf{u})_{j-\frac{1}{2}}) \tag{6}$$

Within the finite volume context, the choice of $\mathbf{f}(\mathbf{u})_{j+\frac{1}{2}} = \mathbf{f}(\mathbf{u}_j, \mathbf{u}_{j+1})$ will affect the behaviour at the wall. Until we sort out the first-order in space and time method, higher-order methods will not be examined.

## 3 The Linearised Riemann Solver

The linearised Riemann Solver is used to determine the flux as

$$\mathbf{f}(\mathbf{u}_L, \mathbf{u}_R) = \frac{1}{2}(\mathbf{f}(\mathbf{u}_L) + \mathbf{f}(\mathbf{u}_R)) - \frac{1}{2}\mathbf{R}|\Lambda|\mathbf{L}(\mathbf{u}_R - \mathbf{u}_L) \tag{7}$$

where

$$\mathbf{R} = \left[\begin{array}{c|c|c} 1 & 1 & 1 \\ u-a & u & u+a \\ \frac{a^2}{\gamma-1} - ua + \frac{1}{2}u^2 & \frac{1}{2}u^2 & \frac{a^2}{\gamma-1} - ua + \frac{1}{2}u^2 \end{array}\right] \tag{8}$$

$$\mathbf{L} = \frac{1}{2a^2}\left[\begin{array}{c|c|c} \frac{\gamma-1}{2}u^2 + ua & -(\gamma-1)u - a & (\gamma-1) \\ 2a^2 - (\gamma-1)u^2 & 2(\gamma-1)u & -2(\gamma-1) \\ \frac{\gamma-1}{2}u^2 - ua & -(\gamma-1)u + a & (\gamma-1) \end{array}\right]. \tag{9}$$

and $\Lambda = \mathrm{diag}(u-a, u, u+a)$ with $a$ and $u$ as density-averaged variables from

$$u = \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \qquad H = \frac{\sqrt{\rho_L}H_L + \sqrt{\rho_R}H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \qquad a = \sqrt{(\gamma-1)\left(H - \frac{1}{2}u^2\right)} \tag{10}$$

# 4 Determining the Non-Conservative Pressure

Define the non-conservative pressure by $p_{\text{nc}}$.The non-conservative pressure can be determined from the waves at each step. Non-conserved pressure should always be lower, as indicated by Roe [Cite some unpublished note]. The following diagram describes the method used. This

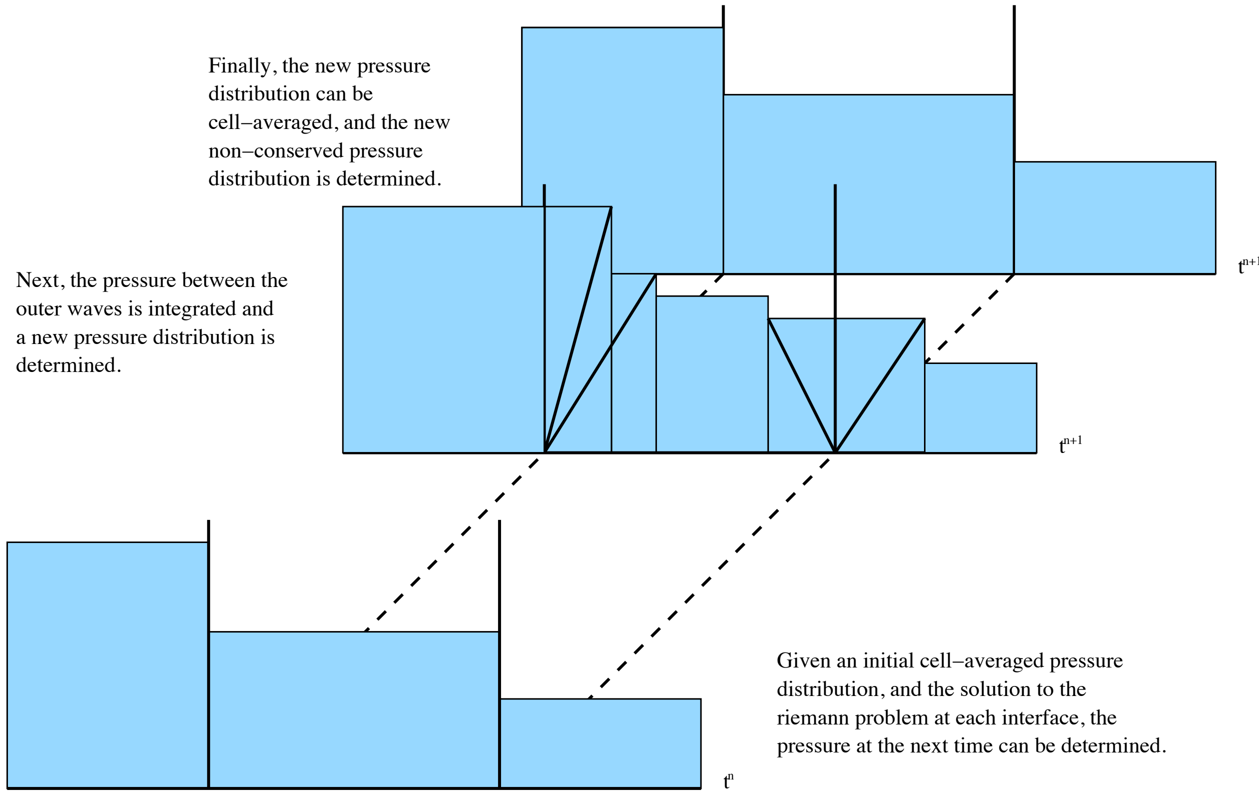


Figure 1: Method for determining new cell-averaged non-conservative pressure

can be expressed as a sum. In the case of the waves in the diagram above, this would be

$$p^{n+1}_{\text{nc},M} = \frac{1}{\Delta x}\left(p_L \Delta t(u-a)_{LM} + p^*_{LM}\Delta t((u+a)_{LM} - (u-a)_{LM}) + \right. \tag{11}$$

$$p_M[\Delta x - \Delta t(u+a)_{LM} + \Delta t(u-a)_{MR}] - p^*_{MR}\Delta t(u-a)_{MR}) \tag{12}$$

The intermediate pressure is $p^* = \frac{1}{2}(p_{\text{nc}L} + p_{\text{nc}R}) - \frac{1}{2}\sqrt{\rho_L\rho_R}a(u_R - u_L)$. **NOTE:** I am unsure whether I should be using non-conservative or conservative pressure in the calculation of $p^*$.

# 5 Updating the Linearised Riemann Solver

It seems that there is some ambiguity how this should be done. If the wave speeds are calculated using non-conservative pressure, $H_{\text{nc}} \leq H \quad \rightarrow \quad a_{\text{nc}} < a$ and the spreading of the waves is less, as seen in the diagram. Following the work of Roe, within $\mathbf{R}|\Lambda|\mathbf{L}\Delta\mathbf{u}$, an additional wave of strength

$$\alpha_4 = \frac{2}{(\gamma-1)u^2}\Delta(p - p_{\text{nc}}) \tag{13}$$

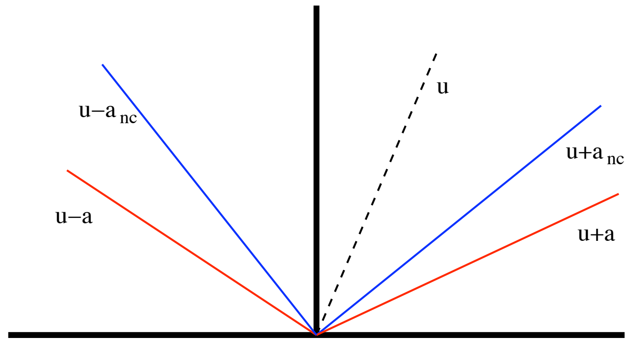


Figure 2: Wave speeds from conservative and non-conservative pressure

with an additional contribution to the flux as in

$$\mathbf{f}(\mathbf{u}_L, \mathbf{u}_R) = \frac{1}{2}(\mathbf{f}(\mathbf{u}_L) + \mathbf{f}(\mathbf{u}_R)) - \frac{1}{2}\mathbf{R}|\Lambda|\mathbf{L}(\mathbf{u}_R - \mathbf{u}_L) - \frac{1}{2}|u| \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{1}{(\gamma - 1)} \Delta(p - p_{\mathrm{nc}}) \quad (14)$$

Here, for consistency, we still maintain that $\mathbf{f}(\mathbf{u}, \mathbf{u}) = \mathbf{f}(\mathbf{u})$.

In this work, I will compare the two choices denoting $\mathbf{f}_{nc1}$ using $a$ and $\mathbf{f}_{nc2}$ using $a_{nc}$.

# 6 Relaxation

As in the work of Roe, nonconservative pressure relaxation can be applied using the exact solution to the relaxation ODE $(p_{\mathrm{nc}})_t = \frac{p - p_{\mathrm{nc}}}{\tau}$, giving

$$p_{nc}^{\mathrm{relaxed}} = p + (p_{nc} - p)e^{-\frac{\Delta t}{\tau}} \quad (15)$$

for a chosen relaxation time $\tau$. We can do this after each timestep.

# 7 Diffusion

Since the relaxation system does not have a diffusion regime, we can implement a diffusion type method. Define a scalar diffusive flux of the form

$$f_{d,i+\frac{1}{2}} = -\frac{k_{i+\frac{1}{2}}}{\Delta x}(u_{i+1} - u_i) \quad (16)$$

and modify the energy equation to the form $E_t + \ldots = \frac{k}{\gamma - 1}(p_{\mathrm{nc}} - p)_{xx}$. For stability, we require that

$$0 \leq \frac{k_i \Delta t}{\Delta x^2} \leq \frac{1}{2} \quad (17)$$

# 8 Noh Problem

The Noh problem consists of $x \in [0,1]$ and $[\rho, u, p](x,0) = [1, 1, p_0]$ with gamma-law equation of state, and $\gamma = 5/3$. The Exact solution is

$$[\rho, u, p](x,t) = \begin{cases} 1, 1, p_0 & x \leq (1 - St) \\ \rho_1, 0, p_1 & x \geq (1 - St) \end{cases} \tag{18}$$

for

$$\rho_1 = (1 + 1/S) \tag{19}$$
$$p_1 = p_0(1 + \gamma M_0^2(1 + S)) \tag{20}$$
$$S = 1/4(\gamma - 3 + \sqrt{(\gamma+1)^2 + 16/M_0^2}) \tag{21}$$

We will examine the solution for the Mach numbers $2, 5, 10, 10^6$ with $M_0^2 = 1/\gamma p_0$. The solution at time $t = 1.0s$ for CFL numbers $0.1, 0.5, 0.9$ will be looked at.
**NOTE:** Only plots for $M_0 = 10^6$ are shown. The others are available upon request.

## 8.1 Standard Solution

Here we present the solution with no games played, no non-conservative pressure, no relaxation, diffusion or any of that. This will serve as a baseline for comparison. SInce we are interested in first improving the density error at the wall, the wall density is shown in the corresponding table.

| Wall Density | | | | |
|---|---|---|---|---|
| $M_0\backslash$ CFL | 0.1 | 0.5 | 0.9 | Exact |
| 2 | 2.7169 | 2.7484 | 2.7748 | 3.0000 |
| 5 | 3.3778 | 3.4192 | 3.4646 | 3.7572 |
| 10 | 3.5336 | 3.5767 | 3.6304 | 3.9343 |
| $10^6$ | 3.5914 | 3.6378 | 3.6835 | 4.0000 |

Looking at the solutions, we notice that the pressure is correct but that there is a significant density error at the wall ($x = 1$). Increasing the CFL number does seem to reduce this error, a phenomenon explained in Roe's note.

Also of interest to Prof. Roe is a plot of conserved pressure against density at the wall. This will tell us how density and pressure progress. Also plotted along side is a straight line connecting the starting point to the exact, correct finishing point. What we can see is that for the most part, the pressure is less at a given density for the higher CFL numbers, which expels less mass from the wall area, thus resulting in a higher wall density. With non-conservative pressure, we hope to reduce the amount of mass initially expelled. We also show a zoomed in version of this plot to show how the solution approaches its final wall value.

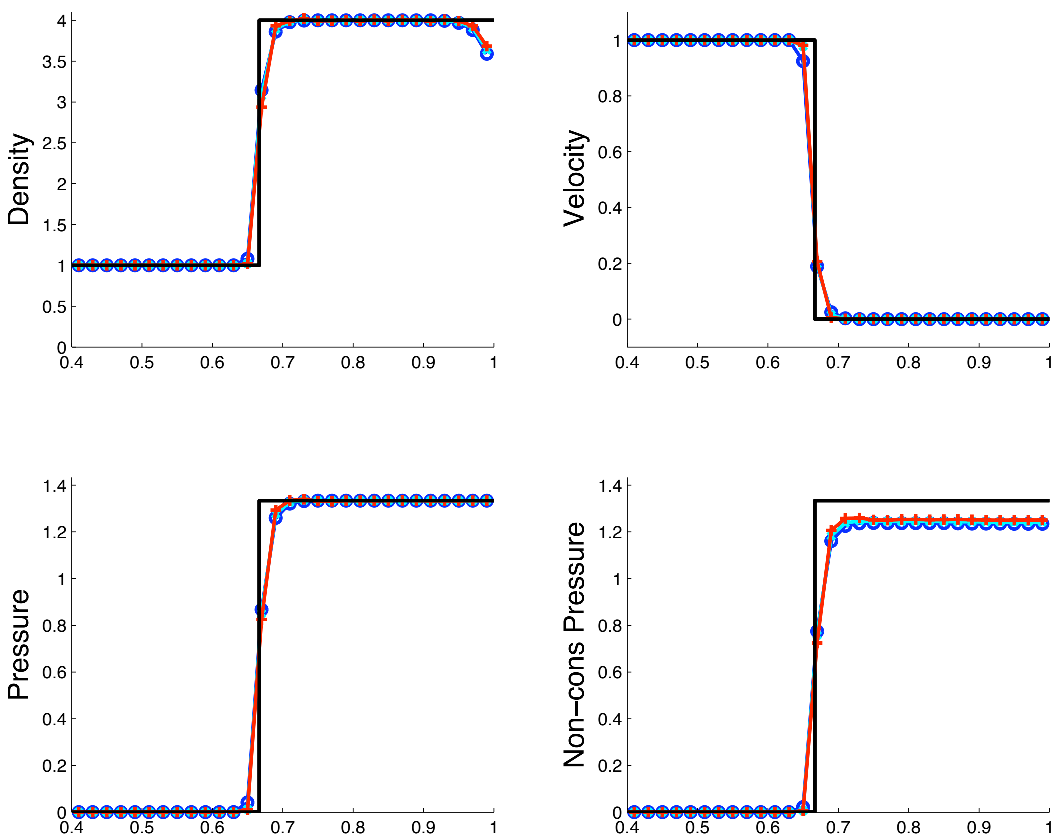


Figure 3: Standard Solutions for CFL 0.1 (blue ∘), 0.5 (cyan ×), 0.9 (red +).

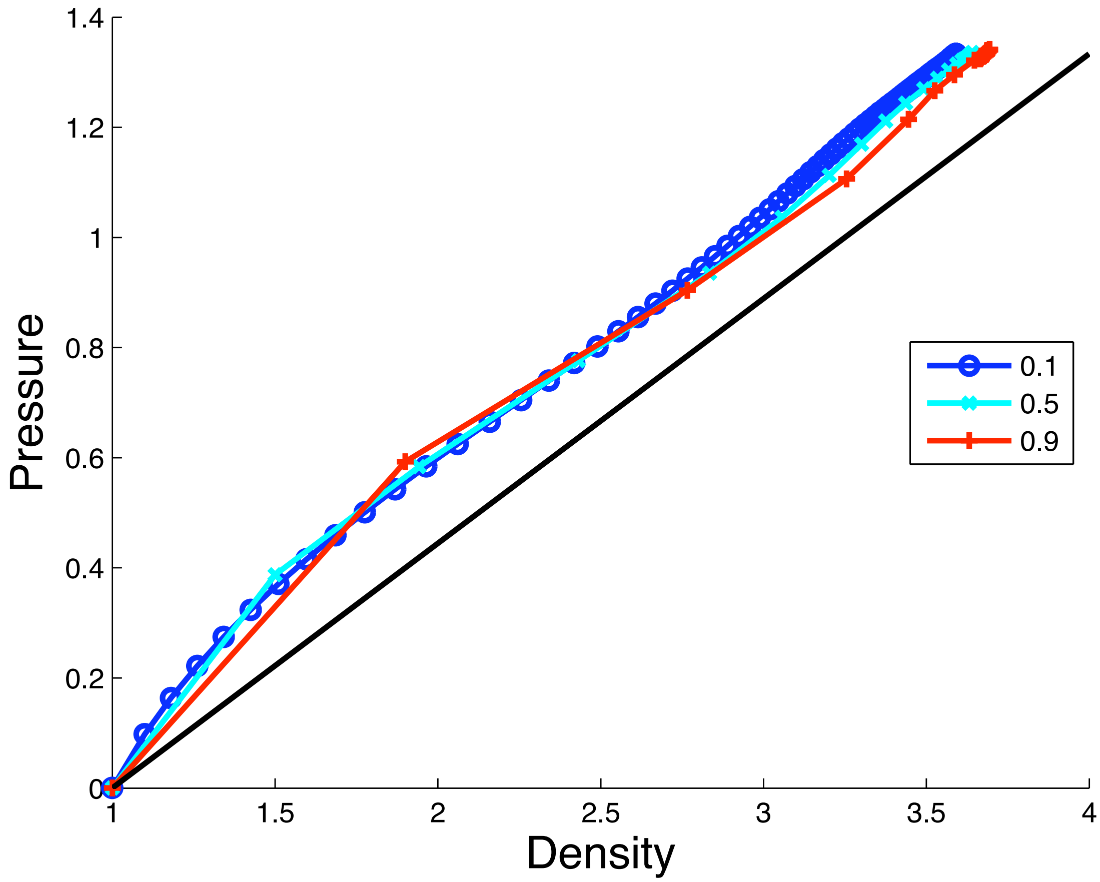


Figure 4: Pressure vs density. for CFL 0.1 (blue $\circ$), 0.5 (cyan $\times$), 0.9 (red $+$).

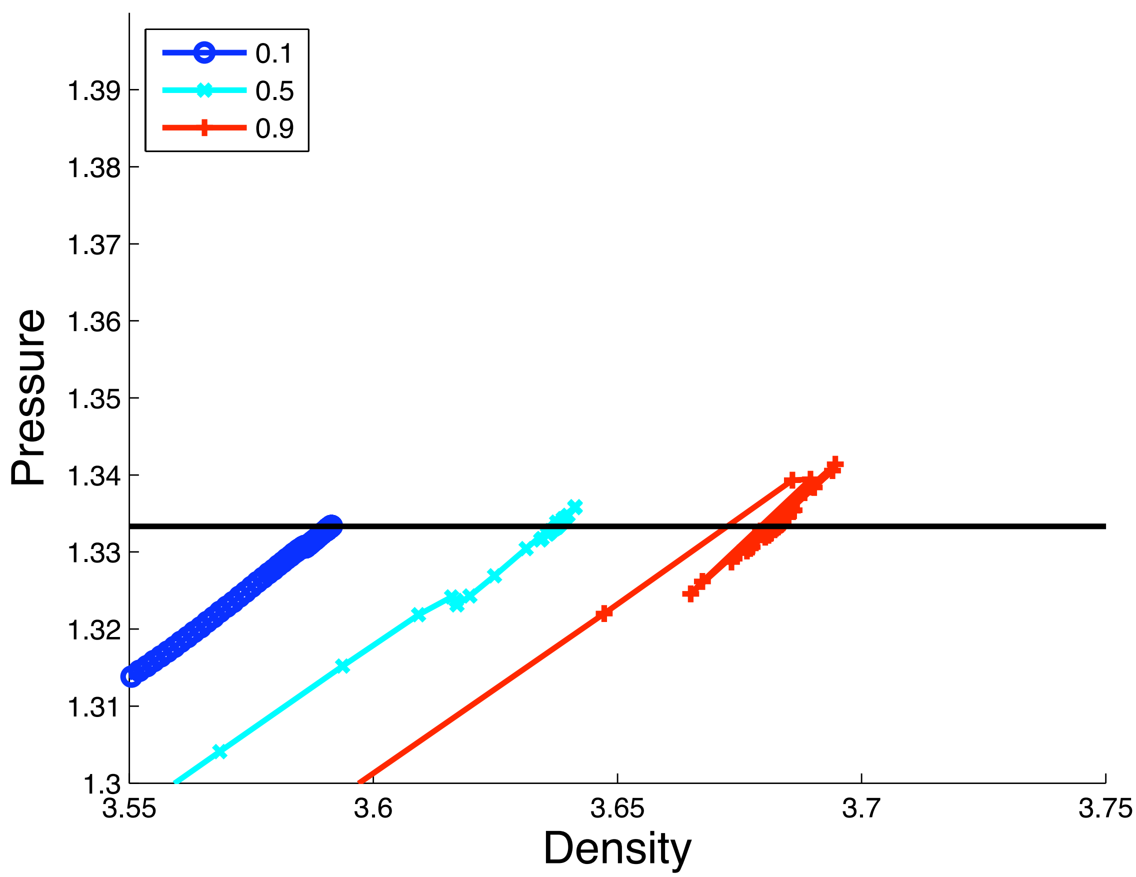


Figure 5: Pressure vs density. for CFL 0.1 (blue ○), 0.5 (cyan ×), 0.9 (red +).

# 9 Relaxation Solutions

## 9.1 First non-conservative flux, $\mathbf{f}_{nc1}$

Here, $a_{roe}$ is calculated using conserved pressure. Here, we explore an infinitely large relaxation time (no relaxation), as well as two other relaxation times which gave different results. With a relaxation time greater than $\Delta t$, we found that it relaxed too fast and left non-conservative pressure identical to conservative pressure. This non-conservative flux actually made the wall density worse than before, likely because the pressure had decreased but the wave speeds had not. We observe that the use of non-conservative pressure results in this time an overshoot in wall pressure and a much more oscillatory path to steady state.

### 9.1.1 $\Delta t/\tau = 0$

| Wall Density | | | | |
|---|---|---|---|---|
| $M_0\backslash$ CFL | 0.1 | 0.5 | 0.9 | Exact |
| 2 | 2.7282 | 2.7602 | 2.7908 | 3.0000 |
| 5 | 3.3638 | 3.4033 | 3.4418 | 3.7572 |
| 10 | 3.5120 | 3.5524 | 3.5936 | 3.9343 |
| $10^6$ | 3.5670 | 3.6085 | 3.6494 | 4.0000 |

### 9.1.2 $\Delta t/\tau = 1/10$

For this case and the following case, the plots look VERY similar and as such, plots are not included.

| Wall Density | | | | |
|---|---|---|---|---|
| $M_0\backslash$ CFL | 0.1 | 0.5 | 0.9 | Exact |
| 2 | 2.7014 | 2.7269 | 2.7556 | 3.0000 |
| 5 | 3.3478 | 3.3705 | 3.4095 | 3.7572 |
| 10 | 3.4998 | 3.5205 | 3.5628 | 3.9343 |
| $10^6$ | 3.5563 | 3.5766 | 3.6168 | 4.0000 |

### 9.1.3 $\Delta t/\tau = 1$

| Wall Density | | | | |
|---|---|---|---|---|
| $M_0\backslash$ CFL | 0.1 | 0.5 | 0.9 | Exact |
| 2 | 2.7151 | 2.7415 | 2.7670 | 3.0000 |
| 5 | 3.3748 | 3.4055 | 3.4436 | 3.7572 |
| 10 | 3.5303 | 3.5606 | 3.6047 | 3.9343 |
| $10^6$ | 3.5879 | 3.6193 | 3.6561 | 4.0000 |

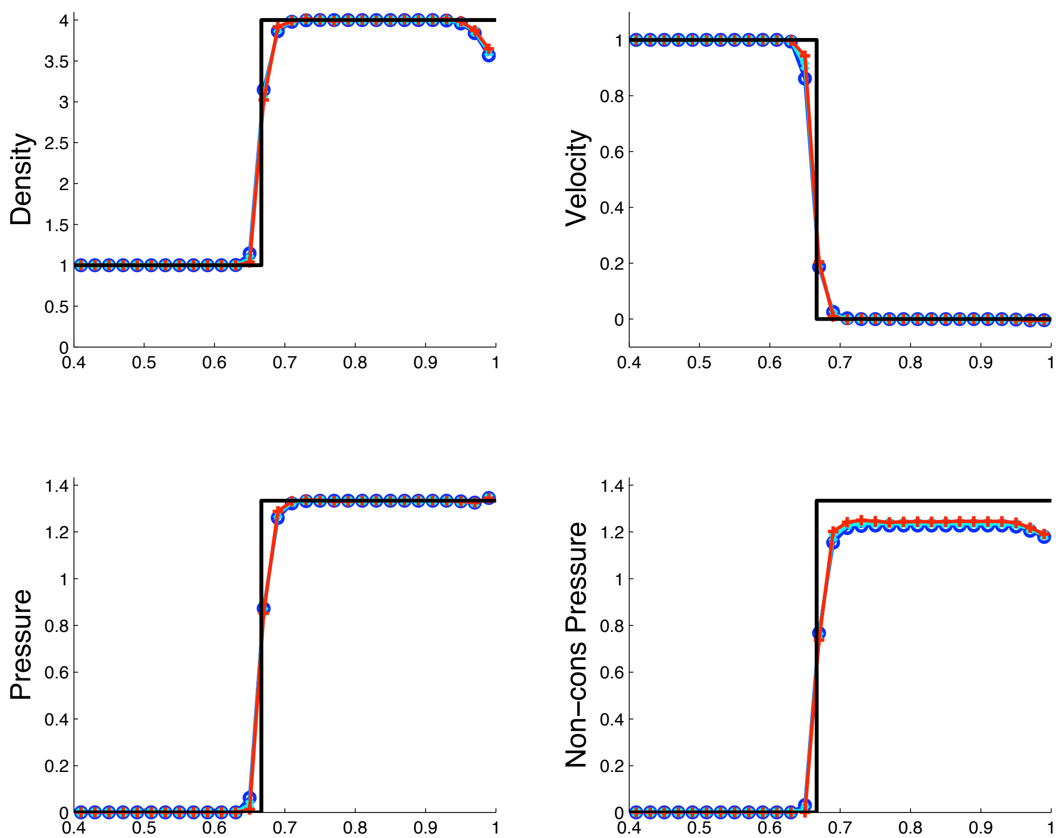


Figure 6: $\mathbf{f}_{nc1}$ Solutions for CFL 0.1 (blue ∘), 0.5 (cyan ×), 0.9 (red +), $\Delta t/\tau = 0$

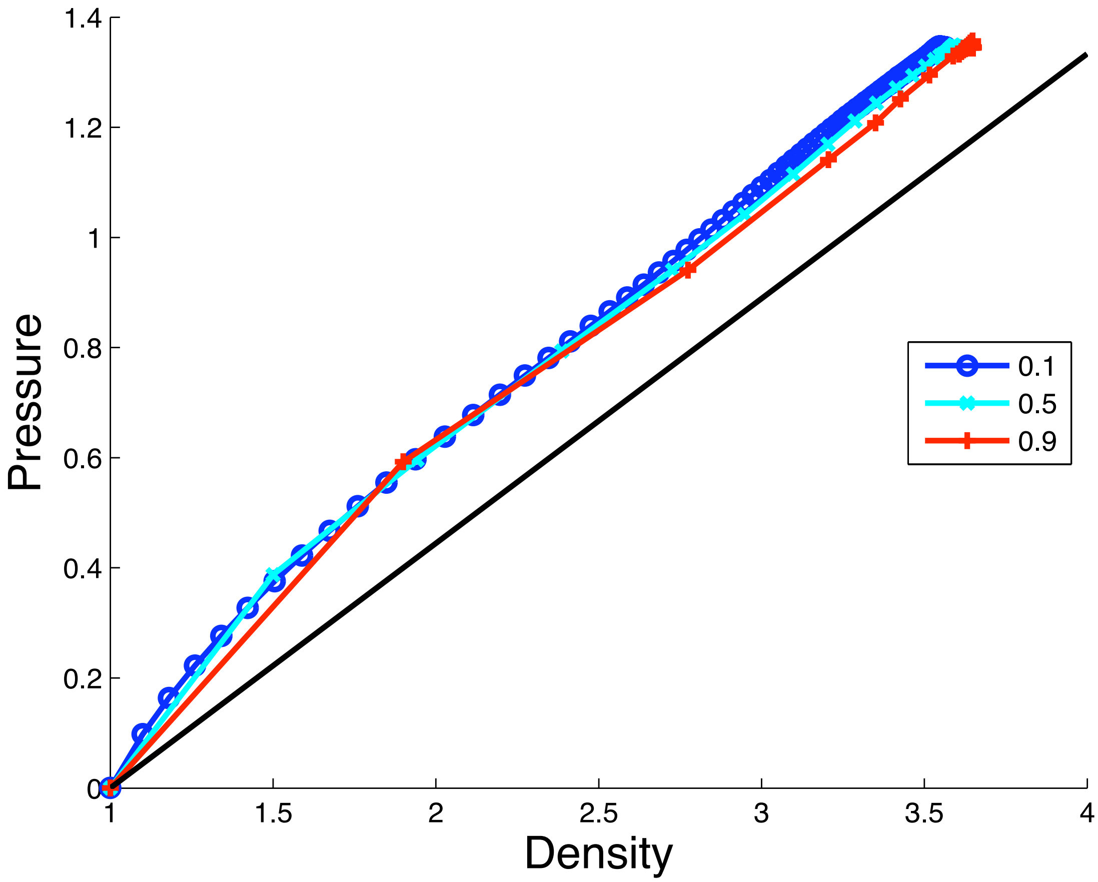


Figure 7: Pressure vs density. for CFL 0.1 (blue $\circ$), 0.5 (cyan $\times$), 0.9 (red $+$), $\Delta t/\tau = 0$

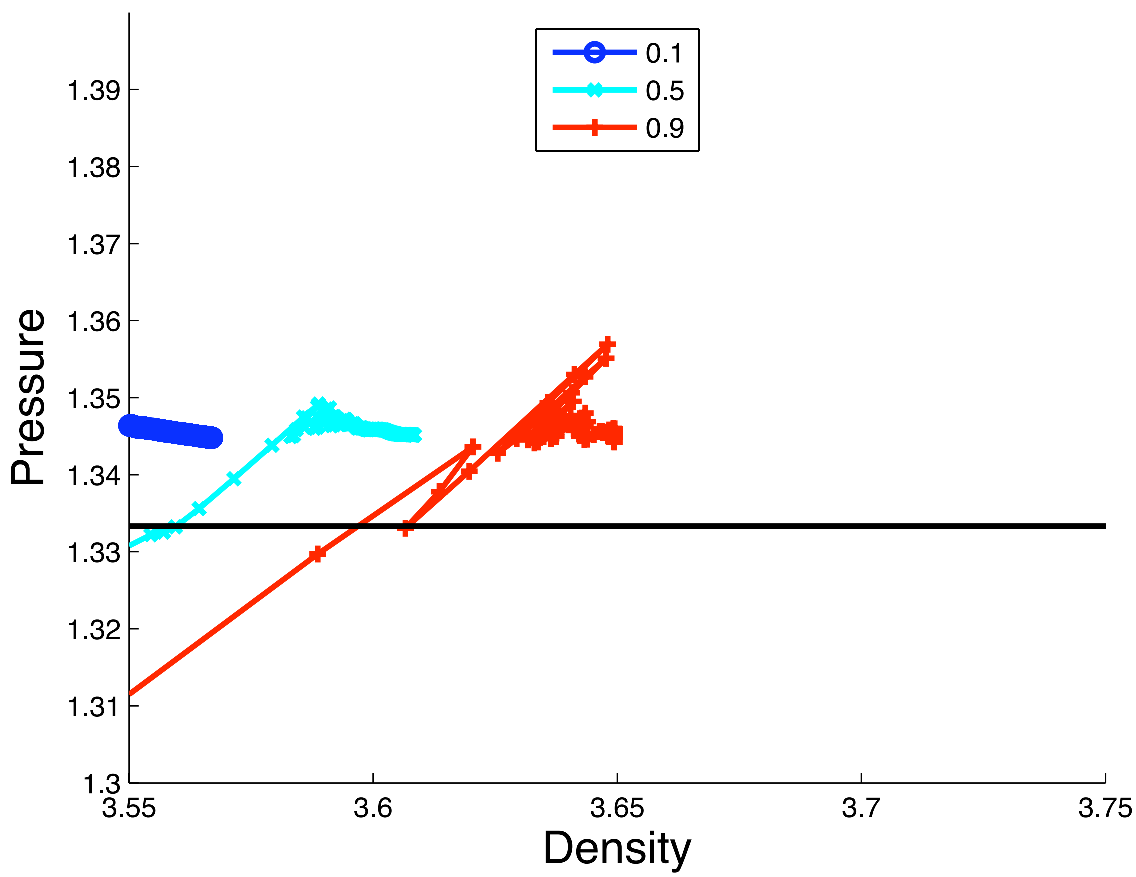


Figure 8: Pressure vs density. for CFL 0.1 (blue $\circ$), 0.5 (cyan $\times$), 0.9 (red $+$), $\Delta t/\tau = 0$

## 9.2 Second non-conservative flux, $\mathbf{f}_{nc2}$

Here, $a_{roe}$ is calculated using non-conserved pressure, so that the spread of the waves is smaller than in the previous flux. In this case, the wall density is higher than using either of the other fluxes. This is a positive result, however the new density is not nearly high enough to call this a success. Examining the zoomed in pressure against density plot, steady state is reached much more slowly near the wall, and again, there is a pressure overshoot.

### 9.2.1 $\Delta t/\tau = 0$

| Wall Density | | | | |
|---|---|---|---|---|
| $M_0\backslash$ CFL | 0.1 | 0.5 | 0.9 | Exact |
| 2 | 2.7693 | 2.8032 | 2.8370 | 3.0000 |
| 5 | 3.4213 | 3.4630 | 3.5058 | 3.7572 |
| 10 | 3.5725 | 3.6158 | 3.6601 | 3.9343 |
| $10^6$ | 3.6284 | 3.6727 | 3.7173 | 4.0000 |

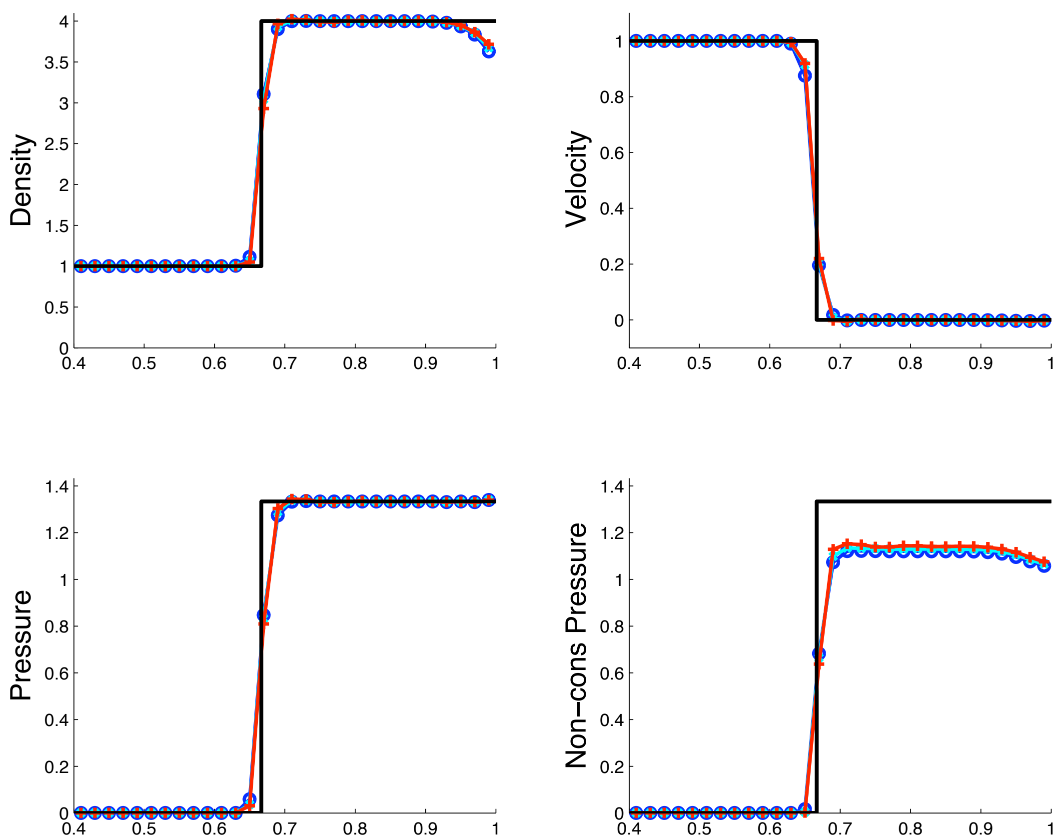


Figure 9: $\mathbf{f}_{nc2}$ Solutions for CFL 0.1 (blue $\circ$), 0.5 (cyan $\times$), 0.9 (red $+$), $\Delta t/\tau = 0$

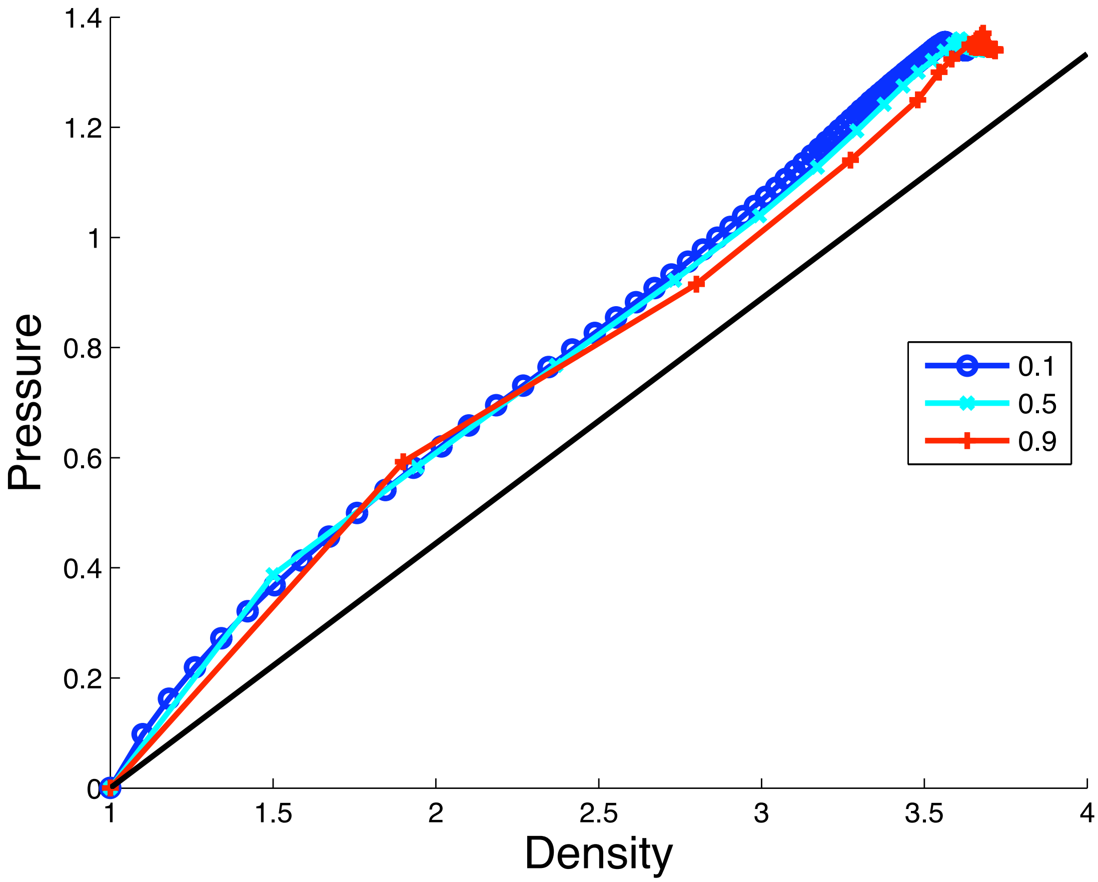


Figure 10: Pressure vs density. for CFL 0.1 (blue $\circ$), 0.5 (cyan $\times$), 0.9 (red $+$), $\Delta t/\tau = 0$

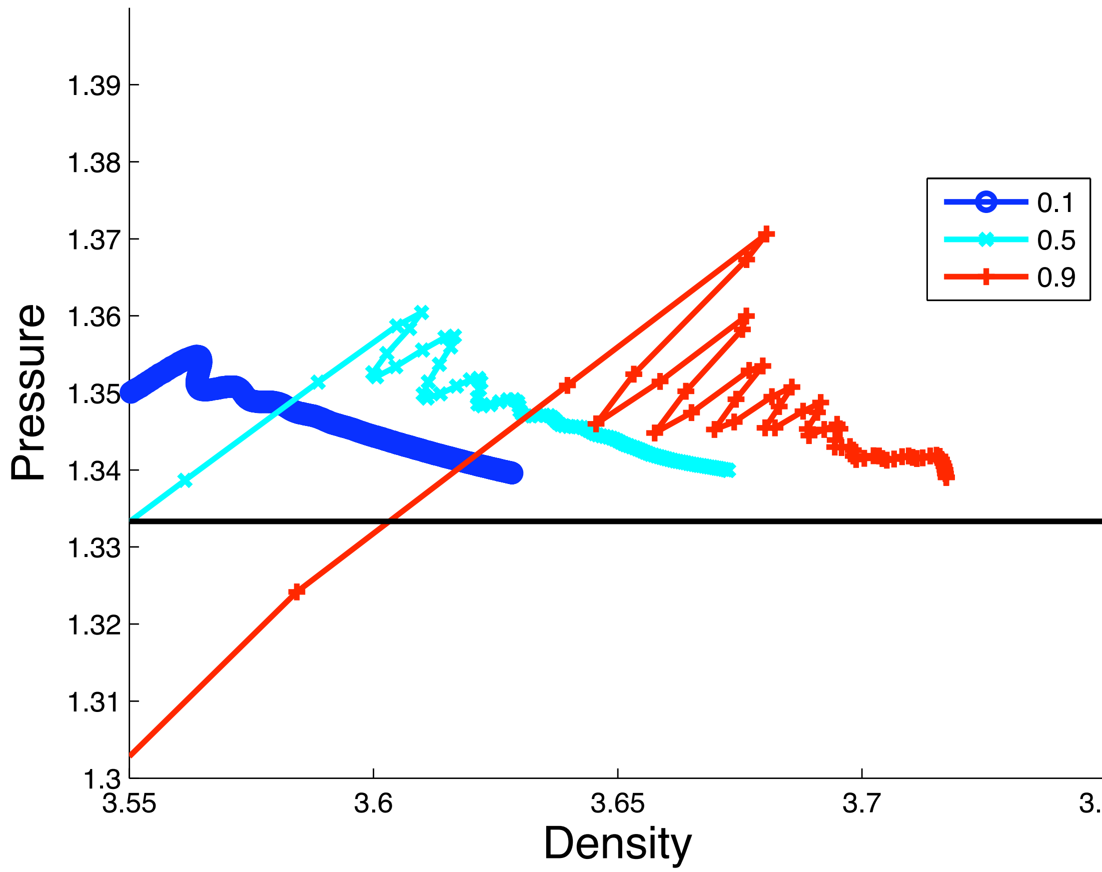


Figure 11: Pressure vs density. for CFL 0.1 (blue $\circ$), 0.5 (cyan $\times$), 0.9 (red $+$), $\Delta t/\tau = 0$

### 9.2.2 $\Delta t/\tau = 1/10$

Again, for this case and the following case, the plots look VERY similar and as such, plots are not included.

| Wall Density | | | | |
|---|---|---|---|---|
| $M_0\backslash$ CFL | 0.1 | 0.5 | 0.9 | Exact |
| 2 | 2.7060 | 2.7302 | 2.7616 | 3.0000 |
| 5 | 3.3552 | 3.3748 | 3.4151 | 3.7572 |
| 10 | 3.5076 | 3.5250 | 3.5676 | 3.9343 |
| $10^6$ | 3.5643 | 3.5808 | 3.6267 | 4.0000 |

### 9.2.3 $\Delta t/\tau = 1$

| Wall Density | | | | |
|---|---|---|---|---|
| $M_0\backslash$ CFL | 0.1 | 0.5 | 0.9 | Exact |
| 2 | 2.7157 | 2.7439 | 2.7713 | 3.0000 |
| 5 | 3.3757 | 3.4084 | 3.4491 | 3.7572 |
| 10 | 3.5312 | 3.5642 | 3.6073 | 3.9343 |
| $10^6$ | 3.5889 | 3.6220 | 3.6705 | 4.0000 |

## 9.3 Combined Results

Combining these results, we can really see the effect of the non-conservative pressure. These are all results for CFL 0.5. We can observe that the major effect of non-conservative pressure occurs after the first few steps. In fact, the standard method seems to give overall the best results, since there is no overshoot in pressure.

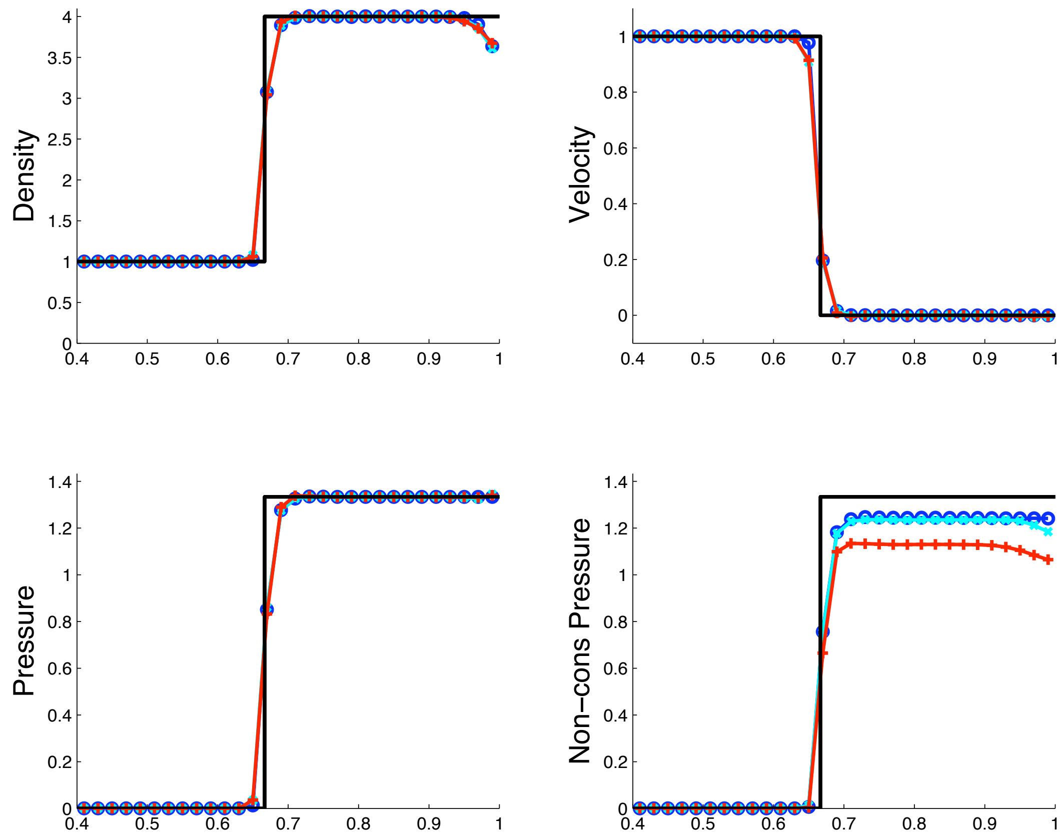


Figure 12: Solutions for Standard (blue $\circ$), $\mathbf{f}_{nc1}$ (cyan $\times$), $\mathbf{f}_{nc2}$ (red $+$), $\Delta t/\tau = 0$

# 10 Diffusion Results

Here, diffusion is only applied to the energy equation, and not equally applied to the non-conservative pressure transport, as Roe had suggested in conversations. This is simply because it works better this way. Here I have chosen

$$\frac{k_i \Delta t}{\Delta x^2} = \frac{1}{2} \tag{22}$$

There is no relaxation applied, obviously there are many combinations and games I can play, but this is preliminary work and I have other things to do.

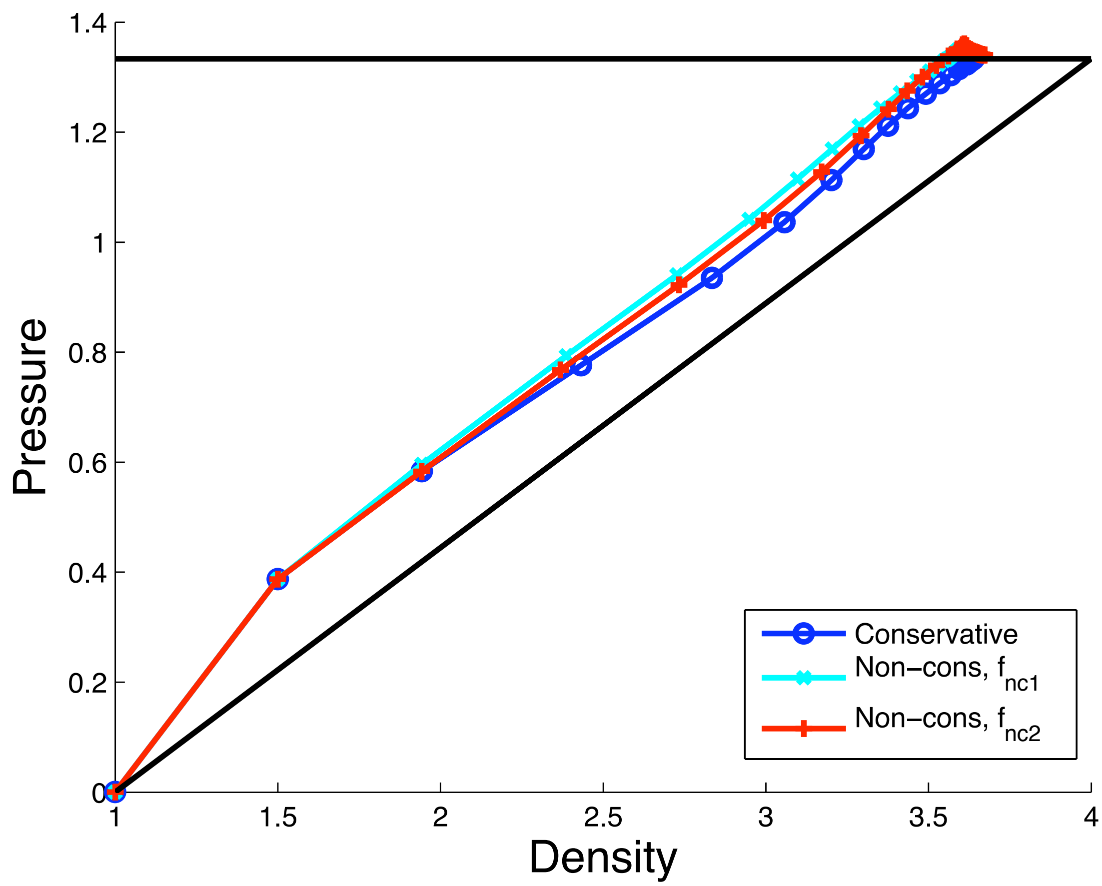


Figure 13: Pressure vs density. for CFL 0.1 (blue ∘), 0.5 (cyan ×), 0.9 (red +), $\Delta t/\tau = 0$

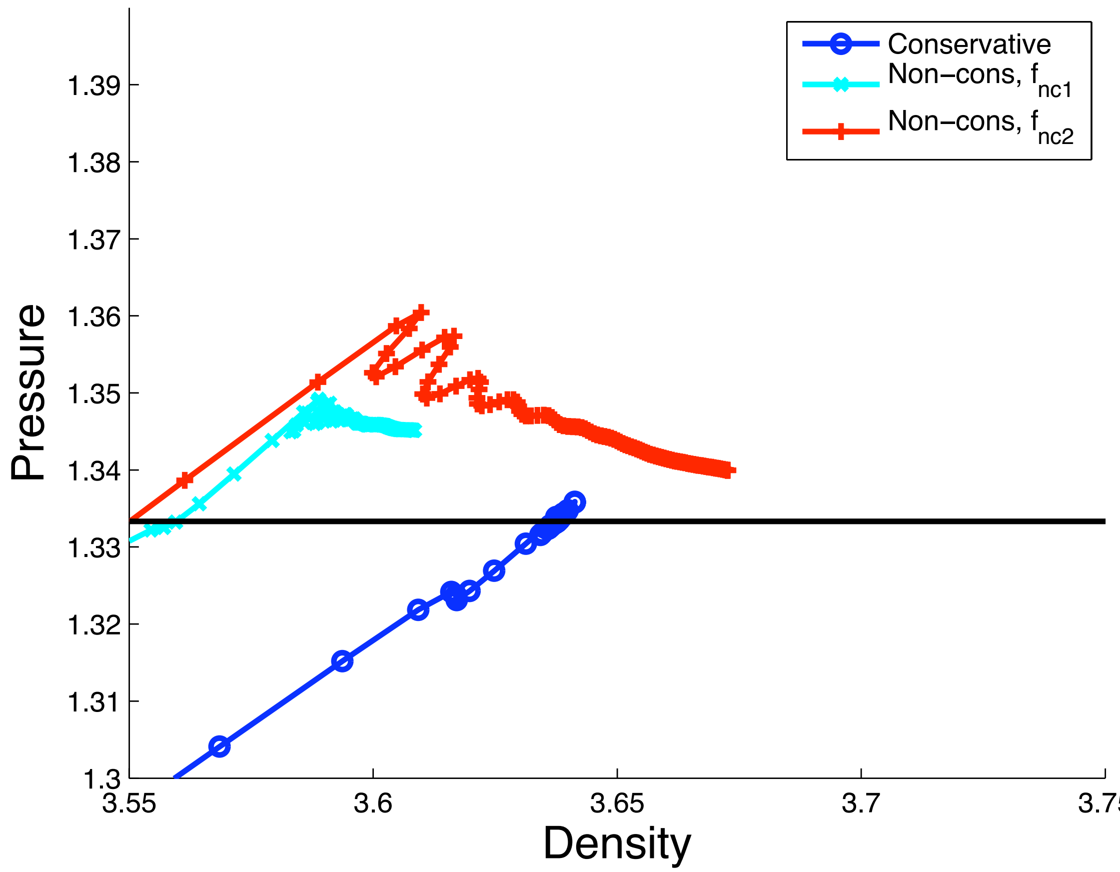


Figure 14: Pressure vs density. for CFL 0.1 (blue ∘), 0.5 (cyan ×), 0.9 (red +), $\Delta t/\tau = 0$

Only the standard solutions are shown. Using non-conservative pressure fluxes has given me some issues, and displays very similar trends when I can convince it to work.

## 10.1 Standard Solution

If we just apply diffusion to the normal method (and forgo conservation) we get much closer to the expected wall value at a cost of significant velocity errors and pressure errors. In the pressure against density plot, we see a much different path resulting in a higher wall density.

| Wall Density | | | | |
|---|---|---|---|---|
| $M_0\backslash$ CFL | 0.1 | 0.5 | 0.9 | Exact |
| 2 | 2.9951 | 2.8878 | 2.8731 | 3.0000 |
| 5 | 3.7995 | 3.579 | 3.5437 | 3.7572 |
| 10 | 3.9897 | 3.7394 | 3.6977 | 3.9343 |
| $10^6$ | 4.0602 | 3.7975 | 3.7534 | 4.0000 |

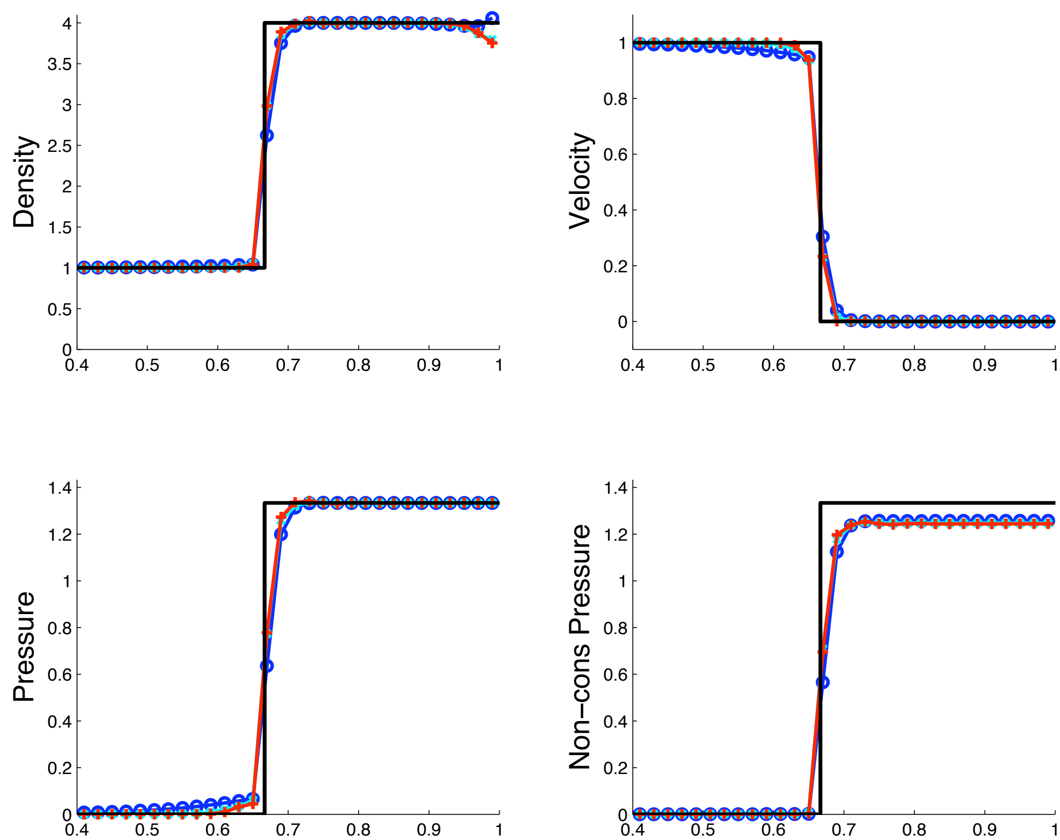


Figure 15: Standard Solutions for CFL 0.1 (blue $\circ$), 0.5 (cyan $\times$), 0.9 (red +).

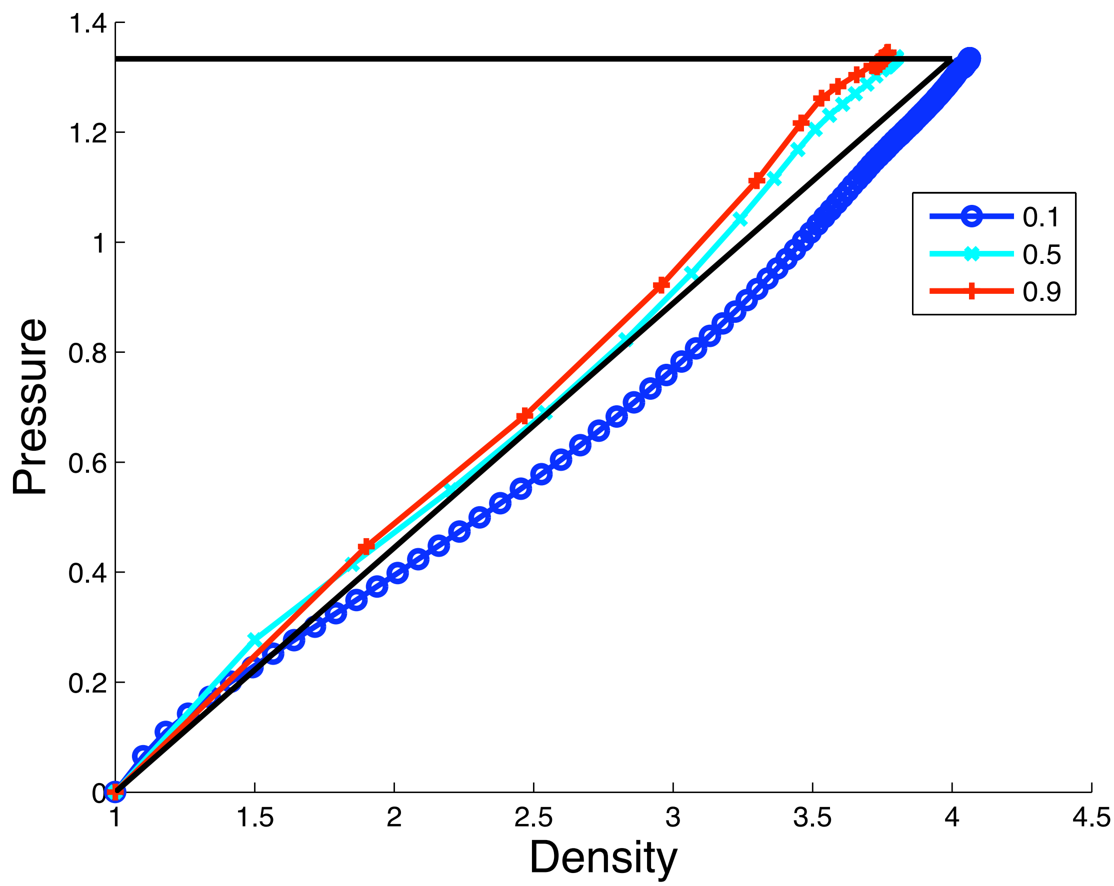


Figure 16: Pressure vs density. for CFL 0.1 (blue ○), 0.5 (cyan ×), 0.9 (red +).