On Godunov-Type Lagrangian Methods

Daniel W. Zaide and Philip L. Roe

University of Michigan, Department of Aerospace Engineering

Introduction

- Computational fluid dynamics tends to adopt different viewpoints in different circumstances.
- When following short, intense transients, codes are often written so as to track particular fluid parcels (Lagrangian methods).
- For long-duration flows, stationary Eulerian coordinates are more convenient.
- Most Eulerian methods compute the passage of information between neighboring cells by solving a Riemann problem (Godunov-type methods).
- Lagrangian methods perform distinctly better when it comes to preservation of sharp material interfaces but Godunov-type methods are better at propagating shocks.
- The goal is then to combine the two methods to achieve the best of both.

Governing Equations

- The Euler Equations can be written in Lagrangian-Eulerian form as

$$\mathbf{u}_t + \mathbf{F}^i(\mathbf{u})_{x_i} = 0$$

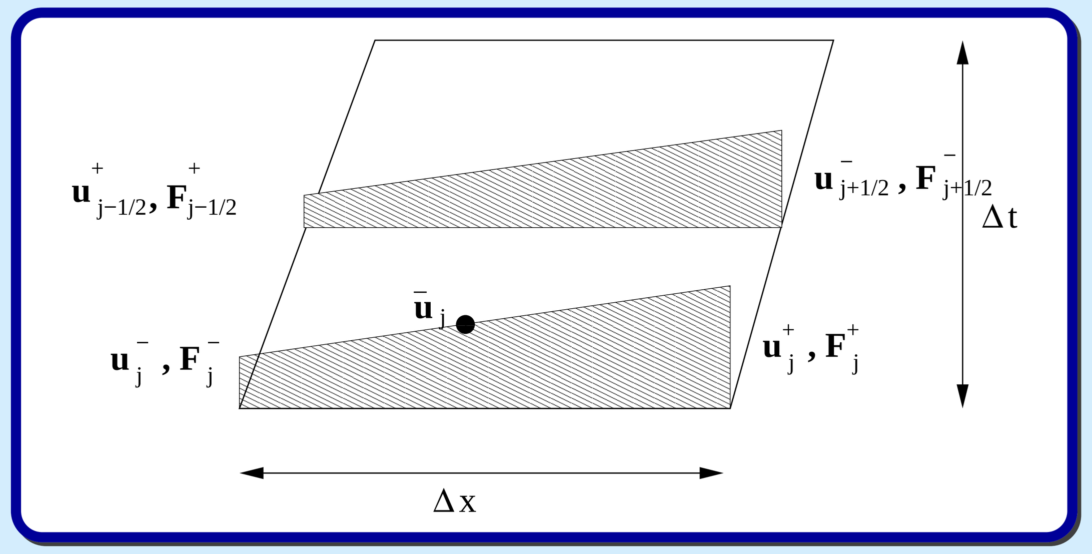
or, expanded as

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\mathbf{v}\\ E\end{bmatrix} + \nabla\cdot\begin{bmatrix}\rho(\mathbf{v}-\mathbf{v}_G)\\ \rho\mathbf{v}(\mathbf{v}-\mathbf{v}_G)+\mathbf{I}p\\ (\mathbf{v}-\mathbf{v}_G)(E+p)\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

for some choice of grid velocity, $\mathbf{v}_G$.
- For $\mathbf{v}_G = \mathbf{v}$, the method is Lagrangian.
- Assume an equation of state $p = p(\rho, \varepsilon)$.
- For an ideal gas the relations are

$$p = (\gamma-1)\rho\varepsilon, \qquad \rho\varepsilon = \left(E - \frac{1}{2}\rho\mathbf{v}^T\mathbf{v}\right).$$

Time Stepping

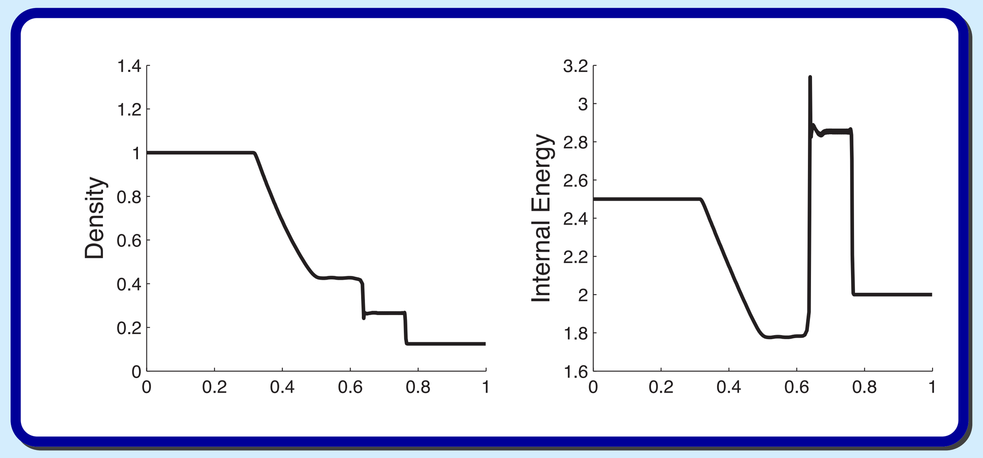


- Second-order in space and time is achieved with a generalized Hancock-type scheme.
- Define the new nodal positions $x_i^{n+1} = x_i^n + \Delta t \dot{u}_i^n$ with an interface speed $\dot{u}_i^n$.
- By integrating the conservation laws around the control volume, the timestepping scheme obtained is

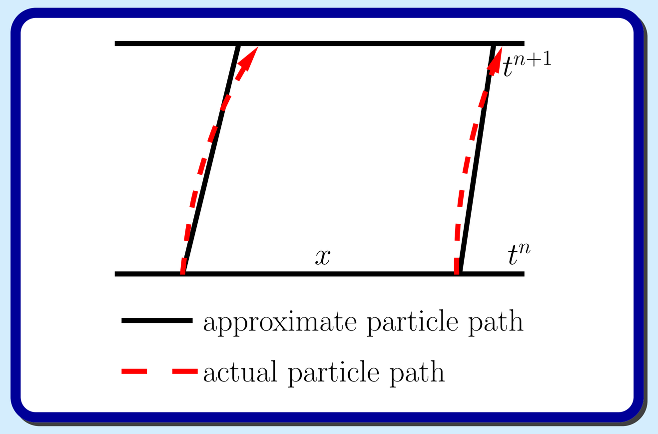
$$\mathbf{u}_i^{n+1}\Delta x_i^{n+1} = \mathbf{u}_i^n \Delta x_i^n - \Delta t\left(\mathbf{F}^{n+\frac{1}{2}}_{i+\frac{1}{2}} - \mathbf{F}^{n+\frac{1}{2}}_{i-\frac{1}{2}}\right).$$

Godunov Philosophy on Lagrangian Grids

- Allowing cell boundaries to follow particle paths results in a common defect among all schemes.



- Here, second-order results for Sod's problem are shown.
- There is a distinct internal energy spike at the contact discontinuity.
- These errors are dismissed in literature as "start-up" errors.
- They arise more plausibly from an inaccurate tracing of the particle paths.



- The error is most severe for paths that pass close to the origin and have high curvature.
- Since mass is assumed to be conserved between paths, the density and internal energy are thrown off.

A Possible Solution

- Propose that all terms be retained in the fluxes, even if the grid is nominally Lagrangian, and call this the **"Leaky Lagrangian"** method.
- There is essentially a "vent" to allow mass to flow between cells when it builds up, mitigating density and internal energy spikes.

Future Research

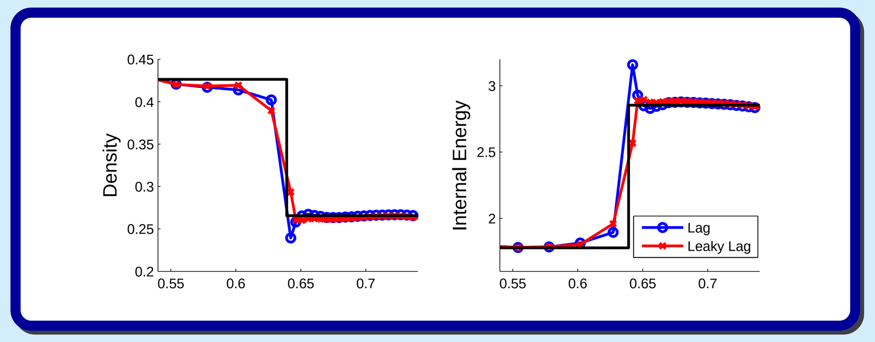
- Develop wave-specific limiting methods to improve sharpness of contact discontinuities.
- Utilize level set methods to predict the new position of material interfaces.
- Transition to multiple dimensions.

Preliminary Results

All results with $\Delta x = 0.01$, CFL $= 0.8$, and an initially uniform grid.

Sod's Problem

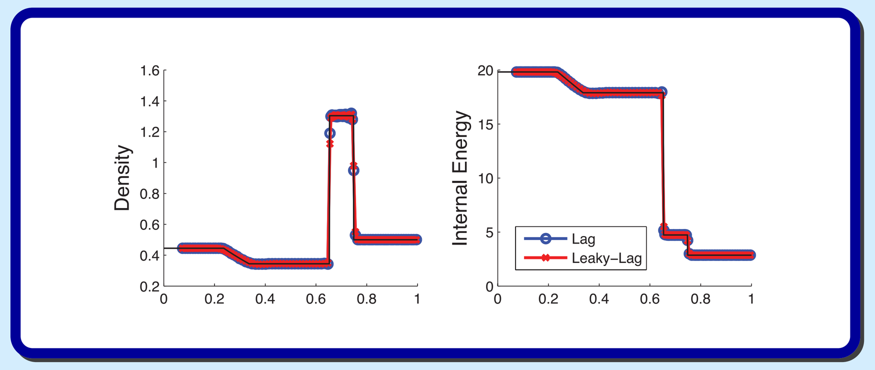
- Shocktube with $[\rho, u, p]_L = [1.0, 0.0, 1.0]$ and $[\rho, u, p]_R = [0.125, 0.0, 0.1]$ and $t = 0.15$.



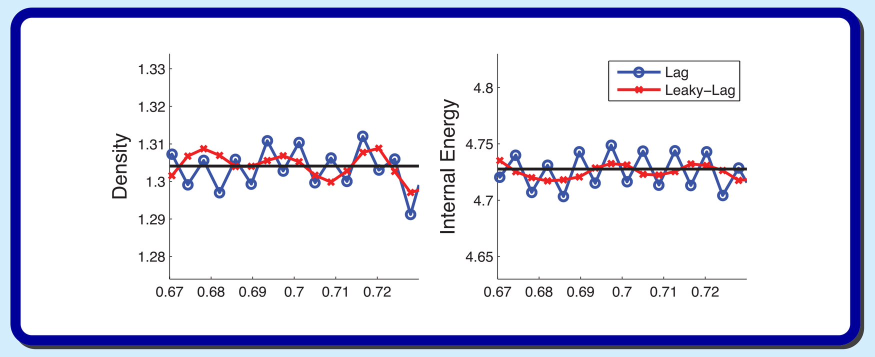
- Zooming in around the contact, the use of the Leaky Lagrangian method has decreased the density and internal energy spikes significantly.

Lax's Problem

- Shocktube with $[\rho, u, p]_L = [0.445, 0.698, 3.528]$ and $[\rho, u, p]_R = [0.5, 0, 0.571]$ and $t = 0.10$.



- In the above figure, both methods give comparable results.



- Zooming in between the contact and shock, there is a build-up of oscillations in the Lagrangian method, while the "venting" of the Leaky Lagrangian diminishes these errors.