A Personal View of the Numerical Wall Heating Phenomenon and the Noh Problem

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I. Introduction

Over the past year and a half, I have studied the wall heating phenomenon and the Noh problem in great detail. It started in May 2009, when I went off to join my advisor, Professor Phil Roe, on his yearlong sabbatical at the University of Cambridge in Cambridge, England. I had no experience dealing with problems of this type, my past work had all been on entropy-conservative schemes, and using entropy as an indicator for moving meshes, so this was a new experience for me.

My initial foray into wall heating came through Professor Roe's work on Lagrangian schemes. It seems that wall heating was a particularly annoying problem in these types of schemes, stemming from the schemes particularly good ability to preserve material interfaces and contact discontinuities. It was curious to me how their great selling point was actually responsible for this problem. After a brief look at Lagrangian schemes and Arbitrary Lagrangian Eulerian schemes, it became apparent that this research could be investigated in the Eulerian frame - this would be much easier since the grid remains fixed in place. The spikes that occurred in Lagrangian schemes in internal energy and density also occurred in Eulerian schemes, albeit they are much harder to find, since they only occur when the contact discontinuity is stationary. A simple examination of Sod's shocktube problem,[1] as in Figure 1, quickly verified this idea. With this in mind, the work quickly shifted to looking within the Eulerian framework for these errors, and thus the investigation into the Noh problem began.

A. The Noh Problem

At a first glance, the Noh problem would appear to be a fairly trivial problem. First published in 1987 by William Noh,[2] the Noh problem has wreaked havoc on even the best numerical methods[a]. Often referred to as a shock reflection problem, the problem consists of shocktube problem, with two states of equal pressure and density moving into eachother with the same

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[a]It should be noted that one of Noh's former collaborators stated that he thought Noh would have preferred the phrase "Noh issue" rather than problem, as the problem really serves to isolate a pervasive issue.

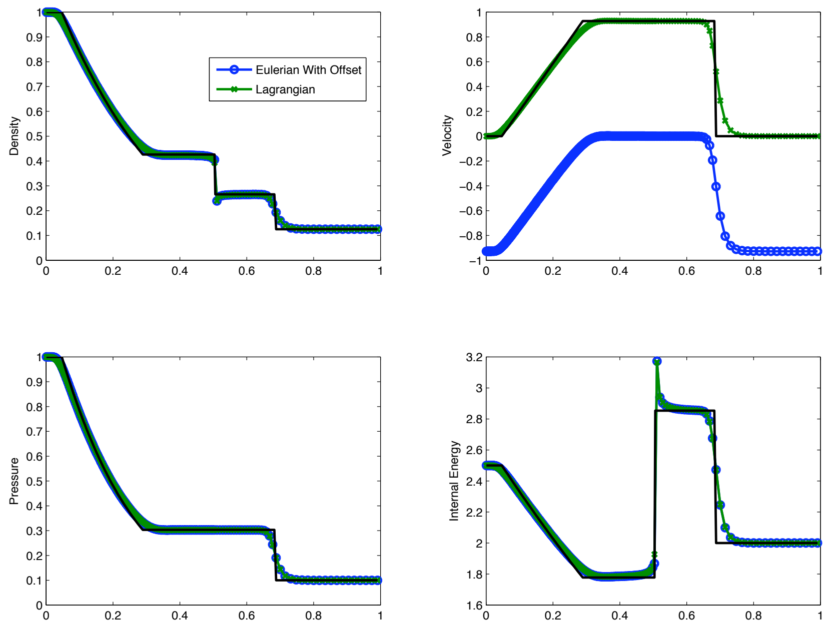

Figure 1. Comparison of Sod's problem for a Lagrangian and Eulerian method. For the Eulerian method, Sod's problem is offset to ensure a stagnation region around the entropy wave. Both methods give similar errors in internal energy.

speed. two shocks are formed in opposite directions, leaving a stagnation region in the middle. With the symmetry of the problem, a wall boundary condition can be introduced at the shocktube interface. This leads to the wall heating problem, reducing computational cost by half, however introducing a potential boundary condition issue. With the wall boundary condition in place, there are then variants of the problem in planar, cylindrical, and spherical symmetry, all of which can be modeled in one dimension with added source terms, and all which have an analytic solution.

The Noh problem is important since all numerical shock-capturing methods leave a defect in density in the stagnation region between shocks. Since density and temperature are inversely proportional, this results in a large overshoot in temperature, hence the term 'wall heating'. This is particularly problematic since this error does not disappear with grid refinement or other similar techniques. While the Noh problem may just be one pervasive instance of this problem, it casts doubt on this reliable and robust class of methods for simulations with strong shockwaves.

A year after Noh's work, a british colleague, Paul Glaister, examined similarity solutions[3] and furthered the knowledge of analytic solutions in this field. Years later, Ralph Menikoff, published work examining the role of shock width, a paper describing the Noh problem as an isolated instance of larger problem in shock interaction. The work concludes that there is a nonuniform convergence of the inviscid limit to the hyperbolic solution, which can be expected around the shock. This results in an entropy error due to shock interaction.[4] Both Menikoff and Noh examine the effect of Von Neumann's artificial viscosity[5] in alleviating these problems.

Gehmeyr et al examined the Noh problem in 1997,[6] looking at shock reflections from a Noh-type problem. Kun Xu and Jishan Hu,[7] investigated the stability of stationary viscous shocks and errors made in the projection stage of the numerical scheme. This work suggests that viscous terms need to play a larger role in upwind schemes rather than treated as an afterthought. They proposed the use of the gas kinetic scheme BGK to physically alleviate problems of this type, and in later work[8] they show results that claim to eliminate the carbuncle phenomenon, a similar issue to the Noh problem. One of the first attempted 'fixes', Fedkiw et al[9] introduce a pressure fix at the boundary to help alleviate wall heating. This isobaric fix, while not eliminating the problem, only alleviates it at boundaries and not at locations of shock interaction.

William Rider looked at wall heating from both an Eulerian and a Lagrangian reference frame[10] and uses the behavior of mass coordinates in converging geometry as reasoning for the significantly worse performance in the Lagrangian frame. In 2002, Toro examined three initial value problems[11] that cause anomalies, inadvertently creating a situation similar to the Noh problem and providing a simple, although incomplete, explanation for the anomaly. More recent work has focused not so much on wall heating, but on schemes that do reasonably well on the Noh problem and have other built-in advantages.[12–15]

II. Governing Equations and Numerical Methods

A. Governing Equations

The governing equations are the Euler Equations, conservation of mass, momentum, and total energy, in one dimension as

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0 \tag{1}$$

or, expanded as

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u H \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2}$$

with $H = \frac{E+p}{\rho}$ and an equation of state $p = p(\rho, e)$. For the ideal gas, $p = (\gamma - 1)\rho e = (\gamma - 1)\left(E - \frac{1}{2}\rho u^2\right)$. This system is also equipped with a speed of sound, $a = \left.\frac{\partial p}{\partial \rho}\right|_s$.

B. Numerical Methods

The first order finite volume method in space and time is

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \frac{\Delta t}{\Delta x}(\mathbf{f}(\mathbf{u})_{j+\frac{1}{2}} - \mathbf{f}(\mathbf{u})_{j-\frac{1}{2}}) \tag{3}$$

The choice of Riemann solver, $\mathbf{f}(\mathbf{u})_{j+\frac{1}{2}} = \mathbf{f}\left(\mathbf{u}_j, \mathbf{u}_{j+1}\right)$, is not extremely relevant, since there is no Riemann solver that can completely eliminate the phenomenon. The preferred choice for this work is Roe's approximate Riemann solver[16] since it is the least dissipative approximate Riemann solver[17] and permits one point stationary shocks.[18] Since all shock capturing methods reduce to first-order around a shock, higher-order methods are not examined; the spurious waves produced by shocks are simply better preserved by these methods.[19]

Roe's approximate Riemann solver is described here as

$$\mathbf{f}(\mathbf{u}_L, \mathbf{u}_R) = \frac{1}{2}\left(\mathbf{f}(\mathbf{u}_L) + \mathbf{f}(\mathbf{u}_R)\right) - \frac{1}{2}\mathbf{R}|\Lambda|\mathbf{L}(\mathbf{u}_R - \mathbf{u}_L) \tag{4}$$

where $\mathbf{L} = \mathbf{R}^{-1}$,

$$\mathbf{R} = \left[\begin{array}{c|c|c} 1 & 1 & 1 \\ u-a & u & u+a \\ H-ua & \frac{1}{2}u^2 & H+ua \end{array}\right] \tag{5}$$

and $\Lambda = \mathrm{diag}(u-a, u, u+a)$ with a and u as density-averaged variables from

$$u = \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \qquad H = \frac{\sqrt{\rho_L}H_L + \sqrt{\rho_R}H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \qquad a = \sqrt{(\gamma-1)\left(H - \frac{1}{2}u^2\right)} \tag{6}$$

III. The Noh Problem

As described in the introduction, the Noh problem consists of a fluid moving into a wall. Mathematically, this can be written as

$$[\rho, u, p](x,t) = \begin{cases} \rho_1, 0, p_1 & x \leq St \\ \rho_0, u_0, p_0 & x \geq St \end{cases} \tag{7}$$

with Mach number $M_0^2 = \rho_0 u_0^2/\gamma p_0$

$$\begin{aligned} \rho_1 &= \rho_0\left(1 - \frac{u_0}{S}\right) && (8) \\ p_1 &= p_0\left(1 + \gamma M_0^2\left(1 - \frac{S}{u_0}\right)\right) && (9) \\ S &= \frac{u_0}{4}\left[(3-\gamma) + \sqrt{(\gamma+1)^2 + \frac{16}{M_0^2}}\right] && (10) \end{aligned}$$

and a wall boundary condition at $x = 0$. Representative numerical solutions are shown in Figure 2 for $u_0 < 0$. For all methods, a density defect occurs at the wall.

IV. A Pressure Overshoot?

One of the suggested mechanisms for wall heating comes from the convexity of the equation of state.[7,11] To analyze this, lets examine the first step of the Noh problem. Any consistent finite-volume method that avoids explicit diffusion will place identical fluxes on all interfaces except for the one at $x = 0$, where the only non-zero flux is the interface pressure in the momentum equation. Solutions therefore depend on just two parameters, one being this pressure p, and the other being the Courant number ν, defined here as $\nu = S\Delta t/\Delta x$ with S being the speed of the reflected shock. Whatever pressure P is used, the conserved variables will change in proportion to ν. Then the pressure in the cells next to the origin

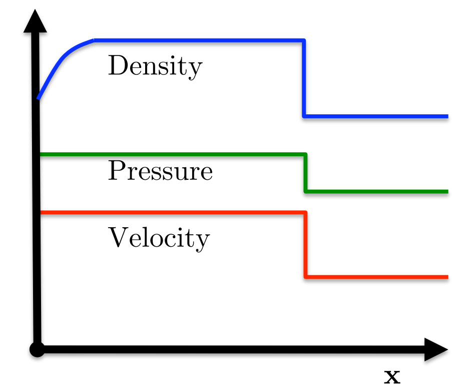

Figure 2. Representative solutions for the Noh Problem.

will be updated using the equation of state. For example, using the exact Riemann solution for an ideal gas, this results in the following expression for the pressure in the first cells as a function of Courant number;

$$p(\nu) = \nu p_1 + (1-\nu)p_0 + \left(\frac{\gamma-1}{2}\right)\frac{\nu(1-\nu)\rho_0\rho_1 u_0^2}{(\nu\rho_1 + (1-\nu)\rho_0)}. \tag{11}$$

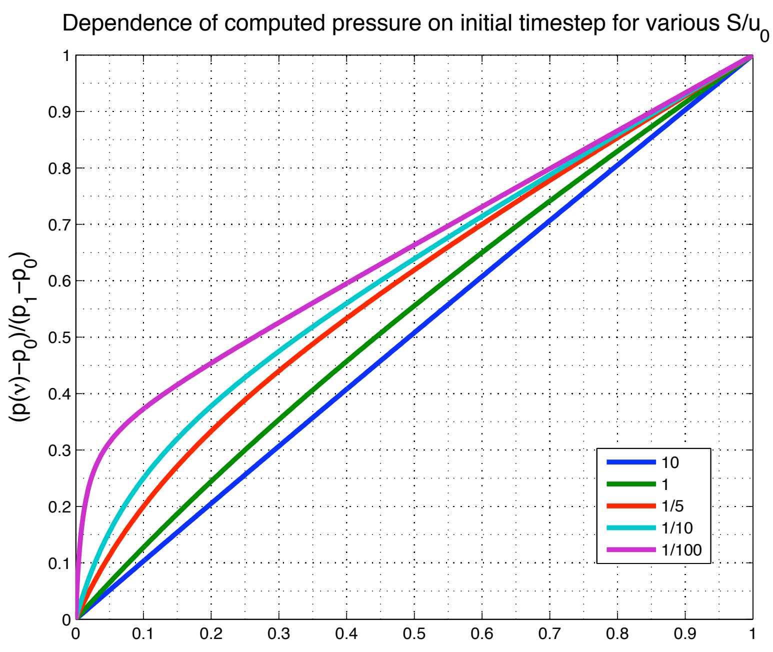

Figure 3. Dependence of computed pressure after the initial timestep for various S/u_0.

The leading terms represent the exact evolution of the pressure. The remaining term, which is always positive, reflecting the convexity of the pressure law, is an error arising from the evaluation of internal energy as the difference in total and kinetic energies. This is best visualized in Figure 3, where Equation 11 has been rearranged as

$$\frac{p(\nu) - p_0}{p_1 - p_0} = \nu + \left(\frac{\gamma - 1}{2}\right)\frac{\nu(1-\nu)}{\nu + S/u_0} \tag{12}$$

and plotted for several ratios of shock speed to freestream speed. Although the error vanishes for ν equal to 0 or 1.0, it cannot be removed by choosing $\nu = 0$ (no progress) or $\nu = 1.0$ (unstable). It is clear from this relation that as the shock moves faster relative to the flow velocity, this nonlinearity disappears. As the shock moves slower, a a larger excess pressure is observed.[b]

At the second and subsequent time steps, this excessive pressure expels mass from the central region. The pressure and velocity reach constant values by emitting acoustic waves, but the density is left too low. This density (temperature) anomaly is traceable to initial errors in pressure, or, equivalently, entropy. This error persists until late times because there is no diffusion mechanism to remove it. Similar effects occur in methods employing artificial viscosity, since this is also a form of mixing.

V. A Possible Cure

With a pressure error in mind, a potential cure can be developed. First, alternative to the energy equation, a convective equation for nonconservative pressure of the form

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\left(p_{\mathrm{nc}} - (\gamma - 1)\left[E - \frac{1}{2}\rho u^2\right]\right) = 0 \tag{13}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)p_{\mathrm{nc}} + (p + (\gamma - 1)p_{\mathrm{nc}})\frac{\partial u}{\partial x} = 0 \tag{14}$$

is introduced. To numerically evaluate this, the method shown in Figure 4 is used. At each step, this integration is performed, and p_{nc} is determined in each cell. In the case of the waves in Figure 4, this would be

$$p_{\mathrm{nc},M}^{n+1} = \frac{1}{\Delta x}\left(p_L \Delta t(u-a)_{LM} + p_{LM}^{*}\Delta t((u+a)_{LM} - (u-a)_{LM}) + \right. \tag{15}$$

$$\left. p_M[\Delta x - \Delta t(u+a)_{LM} + \Delta t(u-a)_{MR}] - p_{MR}^{*}\Delta t(u-a)_{MR}\right) \tag{16}$$

The intermediate pressure is $p^* = \frac{1}{2}(p_{\mathrm{nc}L} + p_{\mathrm{nc}R}) - \frac{1}{2}\sqrt{\rho_L \rho_R}a(u_R - u_L)$. To add the effect of this new equation, following the work of Roe, within $\mathbf{R}|\Lambda|\Delta\mathbf{v}$, an additional wave of strength

$$\Delta\mathbf{v}_4 = \frac{2}{(\gamma-1)u^2}\Delta(p - p_{\mathrm{nc}}) \tag{17}$$

makes an additional contribution to the energy flux as in

$$\mathbf{f}_{\mathrm{nc}}(\mathbf{u}_L, \mathbf{u}_R) = \frac{1}{2}(\mathbf{f}(\mathbf{u}_L) + \mathbf{f}(\mathbf{u}_R)) - \frac{1}{2}\mathbf{R}|\Lambda|\Delta\mathbf{v} - \frac{1}{2}|u|\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\frac{1}{(\gamma-1)}\Delta(p - p_{\mathrm{nc}}) \tag{18}$$

[b]This tends to agree with work on slowly moving shockwaves and other known shock anomalies.

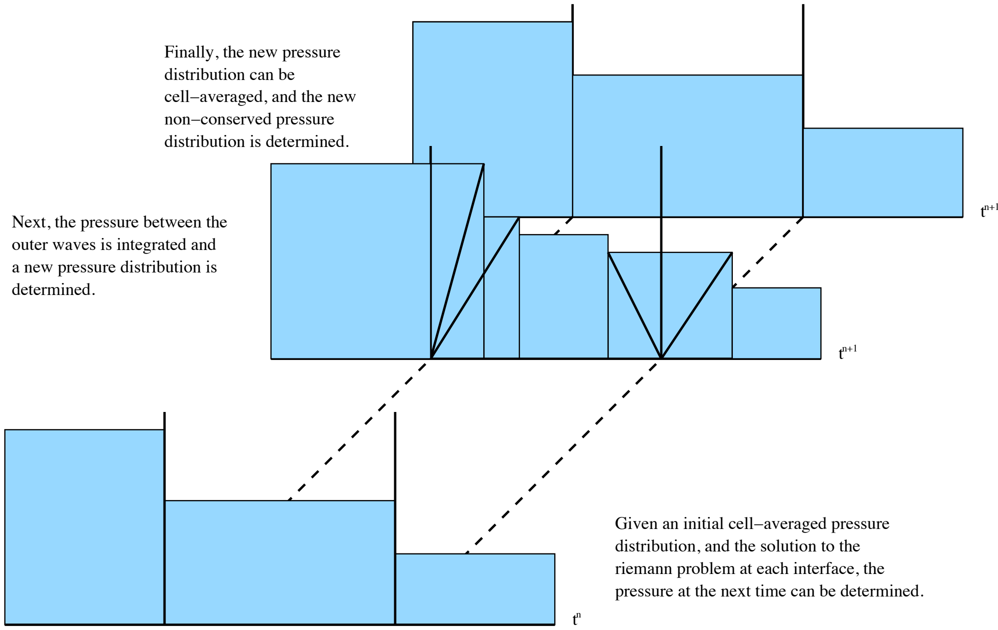

Figure 4. Method for determining new cell-averaged non-conservative pressure

The wave propagation speed comes from $\rho a^2 = p + (\gamma - 1)p_{\text{nc}}$ which leads to the averaging as

$$H = \frac{\sqrt{\rho_L} H_L + \sqrt{\rho_R} H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \tag{19}$$

$$H_{L,R} = \frac{E_{L,R} + p_{\text{nc},L,R}}{\rho_{L,R}} \tag{20}$$

$$a = \sqrt{(\gamma - 1)\left(H - \frac{1}{2}u^2\right)} \tag{21}$$

Here, for consistency, $\mathbf{f}_{\text{nc}}(\mathbf{u}, \mathbf{u}) = \mathbf{f}(\mathbf{u}, \mathbf{u}) = \mathbf{f}(\mathbf{u})$ is maintained. Without the enforcement of consistency, a solution where the incorrect shock speed leads to a overshoot of density forms. To smooth out the introduction of these terms, a relaxation method can be applied using the exact solution to the relaxation ODE $(p_{\text{nc}})_t = \frac{p - p_{\text{nc}}}{\tau}$, giving

$$p_{nc}^{\text{relaxed}} = p + (p_{nc} - p)e^{-\frac{\Delta t}{\tau}} \tag{22}$$

for a chosen relaxation time τ. This can be done after each timestep. Since the relaxation system does not have a diffusion regime, a diffusion type method can be implemented. Define a scalar diffusive flux of the form

$$f_{d,i+\frac{1}{2}} = -\frac{k_{i+\frac{1}{2}}}{\Delta x}((p_{\text{nc}} - p)_{i+1} - (p_{\text{nc}} - p)_i) \tag{23}$$

and modify the energy equation to the form $E_t + (u(E + p))_x = \left(\frac{k}{\gamma - 1}(p_{\text{nc}} - p)_x\right)_x$. For simplicity, define $C_{i+\frac{1}{2}} = \frac{k_{i+\frac{1}{2}}\Delta t}{\Delta x^2}$, with stability requiring that

$$0 \leq C_{i+\frac{1}{2}} \leq \frac{1}{2} \tag{24}$$

We propose the CFL dependent choice of

$$C_{i+\frac{1}{2}} = \text{CFL}\frac{1}{2}\frac{|\Delta \mathbf{v}_{i+\frac{1}{2},2}|}{\sum_j |\Delta \mathbf{v}_{i+\frac{1}{2},j}|} \tag{25}$$

which represents the relative strength of the contact.

A. Physical Interpretations of the Flux

If we ignore the changes of the Linearized Eigenstructure and focus on the additional wave and the effect of diffusion, it turns out that we are changing the energy flux, f_E. This results in

$$\begin{aligned} f_{E,\text{nc}} &= f_E - \frac{|u|}{2(\gamma-1)}\Delta(p-p_{\text{nc}}) - \frac{k}{\Delta x}\Delta(p-p_{\text{nc}}) && (26) \\ &= f_E - \left(\frac{|u|}{2(\gamma-1)} + \frac{k}{\Delta x}\right)\Delta(p-p_{\text{nc}}) && (27) \end{aligned}$$

We have that

$$\frac{|u|}{2(\gamma-1)} + \frac{k}{\Delta x} > 0 \tag{28}$$

and by convexity that

$$(p-p_{\text{nc}}) > 0 \tag{29}$$

From this, a gradient in the difference between conservative and non-conservative pressures results in a reduction of energy flux in the direction of that gradient.

B. Sample Results

We will examine the solution for two cases: $M_0 = 2, 10^6$ with $M_0^2 = 1/\gamma p_0$. The solution at time $t = 1.0s$ for CFL numbers $0.1, 0.5, 0.8$ will be looked at for uniform grid spacing with $\Delta x = 0.02$. In the table below, wall densities are shown. The exact densities are 3 and 4 respectively.

	$M_0 = 2$			$M_0 = 10^6$		
CFL	0.1	0.5	0.8	0.1	0.5	0.8
Flux	No Diffusion					
$\mathbf{f}$	2.7169	2.7484	2.7705	3.5914	3.6378	3.6778
$\mathbf{f}_{\text{nc}}$	2.7406	2.7729	2.7962	3.6415	3.6823	3.7125
Flux	Diffusion					
$\mathbf{f}$	2.7615	2.7910	2.8116	3.6392	3.6781	3.7102
$\mathbf{f}_{\text{nc}}$	2.8107	2.8410	2.8613	3.7394	3.7741	3.7982

From this table, we can see that diffusion improves the wall density calculation. While other choices of diffusion exist, our choice minimizes the effect of diffusion where it is not needed, such as in shock regions.

VI. Isolating Pressure Convexity

If the explanation given above is both correct and complete, then no phenomenon analogous to wall heating should result if the pressure were to depend linearly on the conserved variables. There is no physically meaningful closure of the gasdynamic equations with this property, and the only nonphysical possibility that is dimensionally consistent is to take the pressure as proportional to the total energy rather than the internal energy. In this way the pressure itself becomes a conserved variable. Therefore we set

$$p = (\gamma - 1)E. \tag{30}$$

Under this assumption, the gasdynamic equations remain hyperbolic, with regular shocks, rarefactions, and contacts. We want to see if initial data analogous to the Noh problem results in an error being deposited on the wall.

Other thoughts

1. Total Energy is not galilean invariant, yet how come in the Noh problem the energy shock position corresponds to the exact solution? Menikoff and others are aware of this.[11,20]

2. Clinton Groth mentions that he believes any shock with a finite thickness (viscous, etc) will produce a residue at the wall. The fact that the Euler equations have infinitely thin shocks in theory but finite thickness shocks when discretized is what adds that additional dissipation term to our modified equation, thus perhaps the Noh problem is simply unavoidable. We seek an "exact" solution that is not mathematically realizable from the discrete form, even in the limit as $\Delta x \to 0$, Since this math to show this only applies in the linear case (I think?), this may not hold around shocks, where the solution is not smooth. I am not a mathematician, so I can only speculate.

3. Can I generate a 'start-up' error midway through a simulation? That is, can I recreate a two-point shock that behaves like an initialized shocktube? Coarse grid to fine grid ... right away..

4. What is the effect of the Roe-E-Consistent Flux on wall heating, etc. Do they give shock consistency for a stationary shock. Not a one point shock.

5. What happens if I solve a problem entirely in entropy variables.

6. FCT + upwind. FCT = LW + min-phase

7. Isothermal + temperature convection (aT).

8. double shock reflection with different shock speeds.

9. $A(\tilde{u})$ such that $\tilde{u}$ are linear.

Acknowledgements

The first author is currently supported by the DOE NNSA under the Predictive Science Academic Alliances Program by grant DE-FC52-08NA28616 and by a Natural Sciences and Engineering Research Council of Canada (NSERC) Postgraduate Scholarship. Discussions with Professor Bram Van Leer were helpful in the foundations of this work.

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