A Non-Local Riemann Solver for Alleviating Numerical Shockwave Anomalies

Daniel W. Zaide

Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI 48109

December 21, 2011

1 Formal Construction

Here we define average Jacobians $\overline{\mathbf{A}}_{j+\frac{1}{2}}$ such that

$$\overline{\mathbf{A}}_{j+\frac{1}{2}}(\mathbf{u}_{j+1}-\mathbf{u}_j)=\mathbf{f}_{j+1}-\mathbf{f}_j$$

and $\overline{\mathbf{A}}_j$ such that $\overline{\mathbf{A}}_{j+\frac{1}{2}}$ such that

$$\overline{\mathbf{A}}_j(\mathbf{u}_{j+1}-\mathbf{u}_{j-1})=\mathbf{f}_{j+1}-\mathbf{f}_{j-1}$$

and define interpolated fluxes

$$\begin{aligned}\mathbf{f}^*_j &= \mathbf{f}_{j-1}+\overline{\mathbf{A}}_j(\mathbf{u}_j-\mathbf{u}_{j-1})\\ &= \mathbf{f}_{j+1}-\overline{\mathbf{A}}_j(\mathbf{u}_{j+1}-\mathbf{u}_j)\end{aligned}$$

Now project $(\mathbf{f}^*_{j+1}-\mathbf{f}^*_j)$ onto the eigenvectors of $\overline{\mathbf{A}}_{j+\frac{1}{2}}$ to obtain

$$\mathbf{f}_{j+\frac{1}{2}}=\frac{1}{2}(\mathbf{f}^*_{j+1}+\mathbf{f}^*_j)-\frac{1}{2}\overline{\mathbf{P}}_{j+\frac{1}{2}}(\mathbf{f}^*_{j+1}-\mathbf{f}^*_j)$$

Where $\overline{\mathbf{P}}=\mathbf{RSL}$ where $\mathbf{S}=\text{sign}(\mathbf{\Lambda})$. The scheme can be written as

$$\begin{aligned}\mathbf{f}_{j+\frac{1}{2}} &= \frac{1}{2}(\mathbf{f}^*_{j+1}+\mathbf{f}^*_j)-\frac{1}{2}\overline{\mathbf{P}}_{j+\frac{1}{2}}(\mathbf{f}^*_{j+1}-\mathbf{f}^*_j)\\
&= \frac{1}{2}(\mathbf{f}_{j+1}-\overline{\mathbf{A}}_j(\mathbf{u}_{j+1}-\mathbf{u}_j)+\mathbf{f}_j+\overline{\mathbf{A}}_{j+1}(\mathbf{u}_{j+1}-\mathbf{u}_j))\\
&\quad -\frac{1}{2}\overline{\mathbf{P}}_{j+\frac{1}{2}}(\mathbf{f}_j+\overline{\mathbf{A}}_{j+1}(\mathbf{u}_{j+1}-\mathbf{u}_j)-\mathbf{f}_{j+1}+\overline{\mathbf{A}}_j(\mathbf{u}_{j+1}-\mathbf{u}_j))\\
&= \frac{1}{2}(\mathbf{f}_j+\mathbf{f}_{j+1})+\frac{1}{2}(\overline{\mathbf{A}}_{j+1}-\overline{\mathbf{A}}_j)(\mathbf{u}_{j+1}-\mathbf{u}_j)-\frac{1}{2}\overline{\mathbf{P}}_{j+\frac{1}{2}}(\mathbf{f}_j-\mathbf{f}_{j+1}+(\overline{\mathbf{A}}_{j+1}+\overline{\mathbf{A}}_j)(\mathbf{u}_{j+1}-\mathbf{u}_j))\\
&= \frac{1}{2}(\mathbf{f}_j+\mathbf{f}_{j+1})-\frac{1}{2}|\overline{\mathbf{A}}|_{j+\frac{1}{2}}(\mathbf{u}_{j+1}-\mathbf{u}_j)\\
&\quad +\frac{1}{2}\left((\mathbf{I}-\overline{\mathbf{P}}_{j+\frac{1}{2}})\overline{\mathbf{A}}_{j+1}-(\mathbf{I}+\overline{\mathbf{P}}_{j+\frac{1}{2}})\overline{\mathbf{A}}_j+2|\overline{\mathbf{A}}|_{j+\frac{1}{2}}\right)(\mathbf{u}_{j+1}-\mathbf{u}_j)\end{aligned}$$

1.1 Other Notes

Two things that come out of this are, that for

$$\mathbf{f}_{j+1/2}(\mathbf{a},\mathbf{a},\mathbf{a},\mathbf{b}) = \mathbf{f}(\mathbf{a}), \qquad \mathbf{f}_{j+1/2}(\mathbf{a},\mathbf{b},\mathbf{b},\mathbf{b}) = \mathbf{f}(\mathbf{b})$$

and that,

$$\mathbf{f}_{j+1/2}(\mathbf{a},\mathbf{a},\mathbf{b},\mathbf{b}) = \mathbf{f}^{Roe}_{j+1/2}(\mathbf{a},\mathbf{b})$$

It should be noted for supersonic flow, $\overline{\mathbf{P}} = \mathrm{sign}(u)\mathbf{I}$ and the extra viscosity terms become

$$\frac{1}{2}\left(\overline{\mathbf{A}}_{j+1}(1-\mathrm{sign}(u_{j+\frac{1}{2}})) - \overline{\mathbf{A}}_j(1+\mathrm{sign}(u_{j+\frac{1}{2}})) + 2|\overline{\mathbf{A}}|_{j+\frac{1}{2}}\right)(\mathbf{u}_{j+1}-\mathbf{u}_j)$$

which selects the upwind $\overline{\mathbf{A}}$ matrix.

To compute $\mathrm{sign}(\mathbf{\Lambda})$, use the standard approximation,

$$\mathrm{sign}(x) = \frac{x}{\sqrt{x^2+\varepsilon^2}}$$

where I choose $\varepsilon^2 = 10^{-8}$.

1.2 Comparison with Karni and Canic

Karni among others, classify these problems as occurring when the characteristics change sign across the shock. It seems like we have this implicitly built in, as in this case, $\overline{\mathbf{P}} \neq \mathbf{I}$ and it turns on in these regions.

2 A General Purpose Entropy Fix

Since this new solver is based off of Roe's Riemann Solver, it also needs an entropy fix. Unlike the original Roe Solver, we require a fix for when the intermediate state is not between the end states. In its current construction, all intermediate shock states on a straight line are stable, whether they result in a monotone solution or not. Here, we propose the addition of an extra term,

$$\mathbf{f}^{entropy} = -\frac{1}{2}|\delta\bar{\Lambda}|_{LR}(\mathbf{u}_R - \mathbf{u}_L)$$

where $\delta\Lambda = \mathrm{diag}(\delta\bar{\lambda}_1,\ldots,\delta\bar{\lambda}_n)$ and

$$\delta\bar{\lambda}_i = \left\{\begin{array}{cl} 0 & \text{if } \bar{\lambda}_i = \bar{u}_i \\ 2\max(0,\bar{\lambda}_i-\lambda_{i,L},\lambda_{i,R}-\bar{\lambda}_i) & \text{if } |\bar{\lambda}_i| < \delta\bar{\lambda}_i \text{ or } \frac{1}{2}(\lambda_{i,R}-\lambda_{i,L}) < \left|\frac{\mathbf{L}_i(\mathbf{f}_R-\mathbf{f}_L)}{\mathbf{L}_i(\mathbf{u}_R-\mathbf{u}_L)}\right| \end{array}\right.$$

where the entropy fix is not applied to entropy waves and we have added an additional condition that if we have an entropy shock and its speed is less than actual wave speed, we should adjust our solver.

3 Taylor Expansion Analysis

Taking

$$\mathbf{f}_j^* = \frac{1}{2}(\mathbf{f}_{j-1} + \mathbf{f}_{j+1}) - \frac{1}{2}\overline{\mathbf{A}}_j(\mathbf{u}_{j+1} - 2\mathbf{u}_j + \mathbf{u}_{j-1})$$

and Taylor expanding gives

$$\mathbf{f}_j^* = \mathbf{f}_j + \frac{h^2}{2}(\mathbf{f}_{xx} - \overline{\mathbf{A}}_j\mathbf{u}_{xx})$$

From $\mathbf{f}_x = \mathbf{A}\mathbf{u}_x$ it follows that $\mathbf{f}_{xx} = \mathbf{A}_x\mathbf{u}_x + \mathbf{A}\mathbf{u}_{xx}$ and

$$\mathbf{f}_j^* = \mathbf{f}_j + \frac{h^2}{2}(\mathbf{A}_x\mathbf{u}_x + (\mathbf{A} - \overline{\mathbf{A}}_j)\mathbf{u}_{xx})$$

So what is $(\mathbf{A} - \overline{\mathbf{A}}_j)$? From the original construction, the Taylor expansion is

$$\begin{aligned}
\overline{\mathbf{A}}_j(\mathbf{u}_{j+1} - \mathbf{u}_{j-1}) &= \mathbf{f}_{j+1} - \mathbf{f}_{j-1} \\
\overline{\mathbf{A}}_j\left(2h\mathbf{u}_x + \frac{h^3}{3}\mathbf{u}_{xxx}\right) &= 2h\mathbf{f}_x + \frac{h^3}{3}\mathbf{f}_{xxx} + O(h^5) \\
\overline{\mathbf{A}}_j\left(2h\mathbf{u}_x + \frac{h^3}{3}\mathbf{u}_{xxx}\right) &= 2h\mathbf{A}\mathbf{u}_x + \frac{h^3}{3}\mathbf{f}_{xxx} + O(h^5) \\
(\overline{\mathbf{A}}_j - \mathbf{A})\mathbf{u}_x &= \frac{h^2}{6}(\mathbf{A} - \overline{\mathbf{A}}_j)\mathbf{u}_{xxx} + \frac{h^2}{6}(\mathbf{A}_x\mathbf{u}_{xx} + 2\mathbf{A}_{xx}\mathbf{u}_x + \mathbf{A}_{xxx}\mathbf{u}) + O(h^4)
\end{aligned}$$

But this seems to only say that the product of $(\mathbf{A} - \overline{\mathbf{A}}_j)$ and $\mathbf{u}_x$ is $O(h^2)$, so alternatively, using the parameter vector, $\mathbf{z}$

$$\mathbf{u}_{j+1} - \mathbf{u}_{j-1} = \overline{\mathbf{B}}_j(\mathbf{z}_{j+1} - \mathbf{z}_{j-1}) \quad \mathbf{f}_{j+1} - \mathbf{f}_{j-1} = \overline{\mathbf{C}}_j(\mathbf{z}_{j+1} - \mathbf{z}_{j-1})$$

where $\overline{\mathbf{B}}_j, \overline{\mathbf{C}}_j$ are linear in $\frac{1}{2}(\mathbf{z}_{j+1} + \mathbf{z}_{j-1})$. Taylor expanding inside $\overline{\mathbf{B}}_j, \overline{\mathbf{C}}_j$ gives

$$\overline{\mathbf{B}}_j = \mathbf{B}(\mathbf{z}) + h^2\mathbf{B}(\mathbf{z}_{xx}) \quad \overline{\mathbf{C}}_j = \mathbf{C}(\mathbf{z}) + h^2\mathbf{C}(\mathbf{z}_{xx})$$

going back to the original Roe matrix, $\overline{\mathbf{A}}_j$ we have that

$$\begin{aligned}
\overline{\mathbf{A}}_j &= \overline{\mathbf{C}}_j(\overline{\mathbf{B}}_j)^{-1} \\
&= (\mathbf{C} + h^2\mathbf{C}_{xx})(\mathbf{B} + h^2\mathbf{B}_{xx})^{-1} \\
&= (\mathbf{C} + h^2\mathbf{C}_{xx})(\mathbf{B}^{-1} - h^2\mathbf{B}^{-1}\mathbf{B}_{xx}\mathbf{B}^{-1}) \\
&= \mathbf{C}\mathbf{B}^{-1} - h^2\mathbf{C}\mathbf{B}^{-1}\mathbf{B}_{xx}\mathbf{B}^{-1} + h^2\mathbf{C}_{xx}\mathbf{B}^{-1} + O(h^4) \\
&= \mathbf{A} - h^2(\mathbf{A}\mathbf{B}_{xx}\mathbf{B}^{-1} - \mathbf{C}_{xx}\mathbf{B}^{-1}) + O(h^4) \\
&= \mathbf{A}(\mathbf{u}) - h^2(\mathbf{A}(\mathbf{z})\mathbf{B}(\mathbf{z}_{xx})\mathbf{B}^{-1}(\mathbf{z}_{xx}) - \mathbf{C}(\mathbf{z}_{xx})\mathbf{B}^{-1}(\mathbf{z})) + O(h^4) \\
&= \mathbf{A}_j + O(h^2)
\end{aligned}$$

Where I have used the identity $(\mathbf{X} + h^2\mathbf{Y})^{-1} = \mathbf{X}^{-1} - h^2\mathbf{X}^{-1}\mathbf{Y}\mathbf{X}^{-1}$ and that $\mathbf{A} = \mathbf{C}\mathbf{B}^{-1}$. Putting this into the original expansion and discretizing $\mathbf{A}_x\mathbf{u}_x$ using a first-order approximation leads to

$$\begin{aligned}
\mathbf{f}_j^* &= \mathbf{f}_j + \frac{h^2}{2}(\mathbf{A}_x\mathbf{u}_x) + O(h^4) \\
&= \mathbf{f}_j + \frac{1}{2}[\mathbf{A}][\mathbf{u}] - \frac{h^2}{4}(\mathbf{A}_{xx}[\mathbf{u}] - [\mathbf{A}]\mathbf{u}_{xx}) + O(h^4)
\end{aligned}$$

Using a second-order centered approximation of the first derivative leads to

$$\mathbf{f}_j^* = \mathbf{f}_j + \frac{1}{8}[\mathbf{A}][\mathbf{u}] + O(h^4)$$

Further, it is clear that, using centered differences,

$$\begin{aligned}
\mathbf{f}_{j+\frac{1}{2}} &= \frac{1}{2}(\mathbf{f}_{j+1}^* + \mathbf{f}_j^*) - \frac{1}{2}\overline{\mathbf{P}}_{j+\frac{1}{2}}(\mathbf{f}_{j+1}^* - \mathbf{f}_j^*) \\
&\approx \frac{1}{2}(\mathbf{f}_{j+1} + \frac{1}{8}[\mathbf{A}]_{j+1}[\mathbf{u}]_{j+1} + \mathbf{f}_j + \frac{1}{8}[\mathbf{A}]_j[\mathbf{u}]_j) - \frac{1}{2}\overline{\mathbf{P}}_{j+\frac{1}{2}}(\mathbf{f}_{j+1} + \frac{1}{8}[\mathbf{A}]_{j+1}[\mathbf{u}]_{j+1} - (\mathbf{f}_j + \frac{1}{8}[\mathbf{A}]_j[\mathbf{u}]_j)) \\
&\approx \frac{1}{2}(\mathbf{f}_{j+1} + \mathbf{f}_j + \frac{1}{8}[\mathbf{A}]_{j+1}[\mathbf{u}]_{j+1} + \frac{1}{8}[\mathbf{A}]_j[\mathbf{u}]_j) - \frac{1}{2}\overline{\mathbf{P}}_{j+\frac{1}{2}}(\mathbf{f}_{j+1} - \mathbf{f}_j + \frac{1}{8}[\mathbf{A}]_{j+1}[\mathbf{u}]_{j+1} - \frac{1}{8}[\mathbf{A}]_j[\mathbf{u}]_j)) \\
&\approx \frac{1}{2}(\mathbf{f}_{j+1} + \mathbf{f}_j) + \frac{1}{16}([\mathbf{A}]_{j+1}[\mathbf{u}]_{j+1} + [\mathbf{A}]_j[\mathbf{u}]_j) \\
&\quad -\frac{1}{2}\overline{\mathbf{P}}_{j+\frac{1}{2}}(\mathbf{f}_{j+1} - \mathbf{f}_j) - \frac{1}{16}\overline{\mathbf{P}}_{j+\frac{1}{2}}([\mathbf{A}]_{j+1}[\mathbf{u}]_{j+1} - [\mathbf{A}]_j[\mathbf{u}]_j)) \\
&\approx \frac{1}{2}(\mathbf{f}_{j+1} + \mathbf{f}_j) - \frac{1}{2}|\overline{\mathbf{A}}|_{j+\frac{1}{2}}(\mathbf{u}_{j+1} - \mathbf{u}_j) + \frac{1}{16}([\mathbf{A}]_{j+1}[\mathbf{u}]_{j+1} + [\mathbf{A}]_j[\mathbf{u}]_j) \\
&\quad -\frac{1}{16}\overline{\mathbf{P}}_{j+\frac{1}{2}}([\mathbf{A}]_{j+1}[\mathbf{u}]_{j+1} - [\mathbf{A}]_j[\mathbf{u}]_j)) \\
&= \mathbf{f}_{j+\frac{1}{2}}^{Roe} + \frac{1}{16}\Big(([\mathbf{A}]_{j+1}[\mathbf{u}]_{j+1} + [\mathbf{A}]_j[\mathbf{u}]_j) - \overline{\mathbf{P}}_{j+\frac{1}{2}}([\mathbf{A}]_{j+1}[\mathbf{u}]_{j+1} - [\mathbf{A}]_j[\mathbf{u}]_j))\Big) + O(h^4)
\end{aligned}$$

4 Stability

Following the work of Barth, the function

$$\mathbf{r}(\mathbf{u}_M; \mathbf{u}_L, \mathbf{u}_R) = \mathbf{f}_{MR} - \mathbf{f}_{LM}$$

has a zero eigenvalue and thus single jacobian matrix, $\det\left[\frac{\partial \mathbf{r}}{\partial \mathbf{u}_M}\right] = 0$ because $\mathbf{u}_M$ is a stationary point. To understand the effect of this zero eigenvalue, consider solving

$$\frac{\partial \mathbf{u}_M}{\partial t} + \mathbf{r}(\mathbf{u}_M; \mathbf{u}_L, \mathbf{u}_R) = 0$$

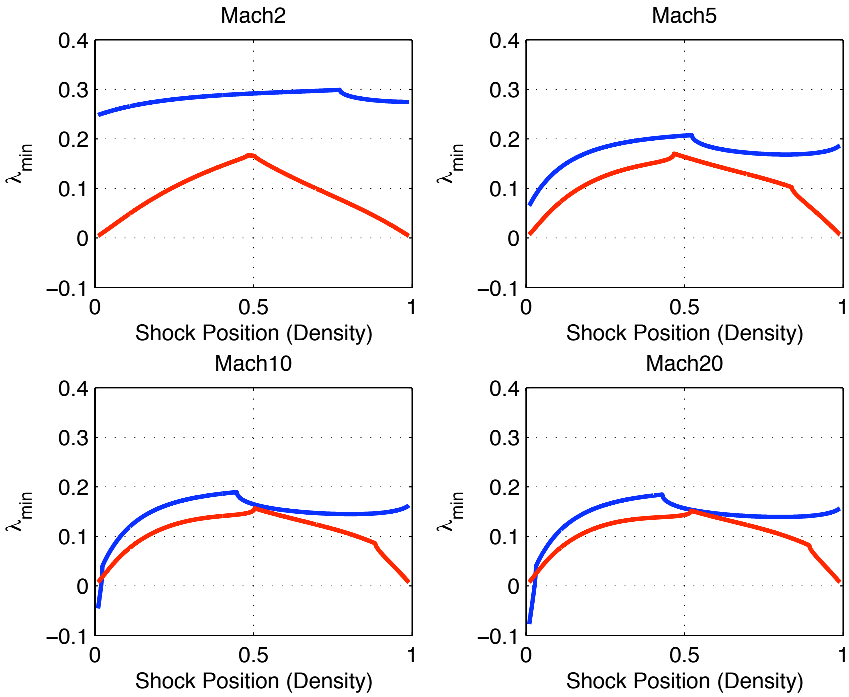
Near a stationary solution, $\mathbf{r}(\mathbf{u}^*; \mathbf{u}_L, \mathbf{u}_R) = 0$, the system can be locally linearized such that the solution error $\mathbf{e} = \mathbf{u}_M - \mathbf{u}^*$ is governed by

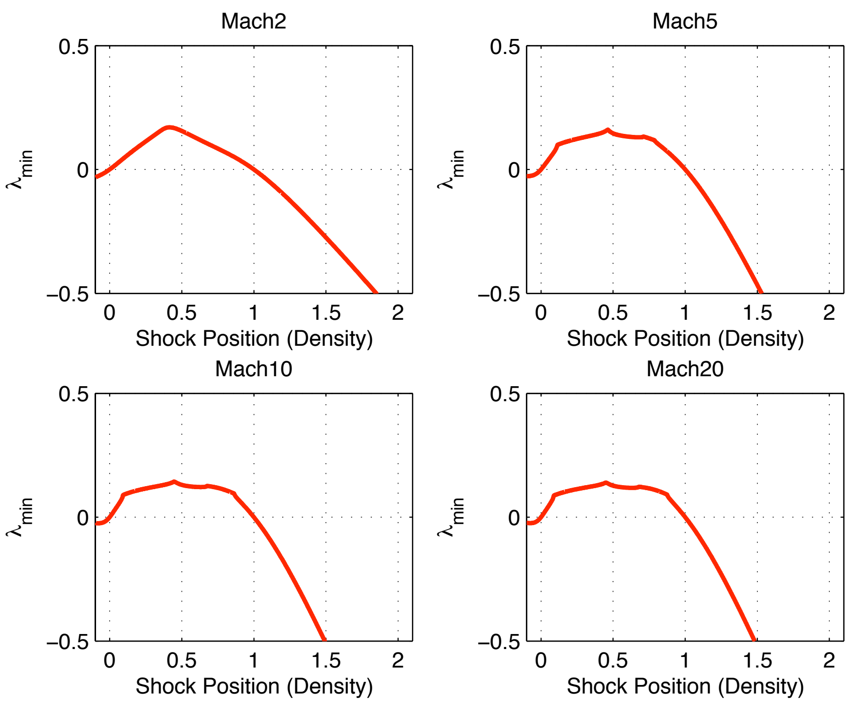
$$\frac{\partial \mathbf{e}}{\partial t} + \frac{\partial \mathbf{r}}{\partial \mathbf{u}_M}(\mathbf{u}^*)\mathbf{e} = 0$$

and the solution can be determined as

$$\mathbf{e}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 e^{-\lambda_2 t} + c_3\mathbf{x}_3 e^{-\lambda_3 t}$$

Stability thus requires $\lambda_i \geq 0$. To avoid the effects of the boundary, look at the eigenvalues of $\frac{\partial \mathbf{r}}{\partial \mathbf{u}}$ for a stationary shock problem, using several extra cells. The results are shown for local (blue) and nonlocal (red) solvers. At Mach 10 and 20, negative eigenvalues are seen for the local solver, but not in our nonlocal variation, where for this work, the initial $\mathbf{u}_M$ is created on the nonphysical Hugoniot for the local solver and on a straight line in state space for the nonlocal solver. If we examine the range outside of $x_S \in [0, 1]$ for the





nonlocal solver, we see that it has at least one negative eigenvalue in all cases.

5 Approximate Forms

One of the biggest issues is that $\overline{\mathbf{P}}_{j+\frac{1}{2}}$ changes significantly when its eigenvalues switch signs (usually when its eigenvalues are small). Normally, since $\text{sign}(\lambda)$ is multiplied by λ in the local Riemann solver, these changes get multiplied by something small and the effects are small, however since we have $\overline{\mathbf{P}}_{j+\frac{1}{2}}\overline{\mathbf{A}}_j$ and $\overline{\mathbf{P}}_{j+\frac{1}{2}}\overline{\mathbf{A}}_{j+1}$ there is no guarantee that this effect is small. One option is to diffuse the sign function using the square root approximation with $\epsilon = O(10^{-4})$ however this limits the effectiveness of the method.

Another issue is the expense in computing extra Roe matrices, so approximate versions of this will attempt to eliminate both problems. First, define $\tilde{\mathbf{A}}_{j+\frac{1}{2}}(\mathbf{u}_{j+2} - \mathbf{u}_{j-1}) = \mathbf{f}_{j+2} - \mathbf{f}_{j-1}$.

$$
\mathbf{f}^{NL}_{j+\frac{1}{2}} = \begin{cases}
\mathbf{f}^{Roe}_{j+\frac{1}{2}} + \frac{1}{2}\left((\mathbf{I} - \overline{\mathbf{P}}_{j+\frac{1}{2}})\overline{\mathbf{A}}_{j+1} - (\mathbf{I} + \overline{\mathbf{P}}_{j+\frac{1}{2}})\overline{\mathbf{A}}_j + 2|\overline{\mathbf{A}}|_{j+\frac{1}{2}}\right)(\mathbf{u}_{j+1} - \mathbf{u}_j) & \text{Original} \\
\frac{1}{2}(\mathbf{f}_{j+1} + \mathbf{f}_j) + \left(\mathbf{A}^-_{j+1} - \mathbf{A}^+_j + \frac{1}{2}|\overline{\mathbf{A}}|_{j+\frac{1}{2}}\right)(\mathbf{u}_{j+1} - \mathbf{u}_j) & (1) \\
\frac{1}{2}(\mathbf{f}^*_{j+1} + \mathbf{f}^*_j) - \frac{1}{2}\overline{\mathbf{P}}_{j+\frac{1}{2}}\tilde{\mathbf{A}}_{j+\frac{1}{2}}(\mathbf{u}_{j+1} - \mathbf{u}_j) & (2) \\
\frac{1}{2}(\mathbf{f}^*_{j+1} + \mathbf{f}^*_j) - \frac{1}{2}|\tilde{\mathbf{A}}|_{j+\frac{1}{2}}(\mathbf{u}_{j+1} - \mathbf{u}_j) & (3) \\
\frac{1}{2}(\mathbf{f}^*_{j+1} + \mathbf{f}^*_j) - \frac{1}{4}\Big(|\overline{\mathbf{A}}|_{j-1,j+2}(\mathbf{u}_{j+2} - \mathbf{u}_{j-1}) - |\overline{\mathbf{A}}|_{j+\frac{1}{2}}(\mathbf{u}_{j+1} - \mathbf{u}_j) & \\
-|\overline{\mathbf{A}}|_{j+1}(\mathbf{u}_{j+2} - 2\mathbf{u}_{j+1} + \mathbf{u}_j) + |\overline{\mathbf{A}}|_j(\mathbf{u}_{j+1} - 2\mathbf{u}_j + \mathbf{u}_{j-1})\Big) & (4) \\
\frac{1}{2}(\mathbf{f}^*_{j+1} + \mathbf{f}^*_j) - \frac{1}{2}\tilde{\mathbf{P}}_{j+\frac{1}{2}}\overline{\mathbf{A}}_{j+\frac{1}{2}}(\mathbf{u}_{j+1} - \mathbf{u}_j) & \text{Does Not Work} \\
\mathbf{f}^{Roe}_{j+\frac{1}{2}} + \frac{1}{2}\left((\mathbf{I} - \overline{\mathbf{P}}_{j+\frac{1}{2}})\mathbf{A}_{j+1} - (\mathbf{I} + \overline{\mathbf{P}}_{j+\frac{1}{2}})\mathbf{A}_j + 2|\overline{\mathbf{A}}|_{j+\frac{1}{2}}\right)(\mathbf{u}_{j+1} - \mathbf{u}_j) & \text{Does Not Work} \\
\mathbf{f}^{Roe}_{j+\frac{1}{2}} + \frac{1}{16}\left(([\mathbf{A}]_{j+1}[\mathbf{u}]_{j+1} + [\mathbf{A}]_j[\mathbf{u}]_j) - \overline{\mathbf{P}}_{j+\frac{1}{2}}([\mathbf{A}]_{j+1}[\mathbf{u}]_{j+1} - [\mathbf{A}]_j[\mathbf{u}]_j)\right) & \text{Does Not Work} \\
\frac{1}{2}(\mathbf{f}^*_{j+1} + \mathbf{f}^*_j) - \frac{1}{2}|\overline{\mathbf{A}}^*|_{j+\frac{1}{2}}(\mathbf{u}^*_{j+1} - \mathbf{u}^*_j) & \text{Does Not Work}
\end{cases}
$$

Approximate algorithm 5. Start by computing waves in each cell, λ_j. For each interface, calculate $\lambda_{j+\frac{1}{2}} = \frac{1}{2}(\lambda_j + \lambda_{j+1})$. If $\lambda_{j-\frac{1}{2}} > 0 > \lambda_{j+\frac{1}{2}}$ then set $\mathbf{f}_{j-\frac{1}{2}} = \frac{1}{2}(\mathbf{f}^*_{j-1} + \mathbf{f}^*_j) - \frac{1}{2}\overline{\mathbf{A}}_j(\mathbf{u}_j - \mathbf{u}_{j-1})$ and $\mathbf{f}_{j+\frac{1}{2}} = \frac{1}{2}(\mathbf{f}^*_{j+1} + \mathbf{f}^*_j) - \frac{1}{2}\overline{\mathbf{A}}_j(\mathbf{u}_{j+1} - \mathbf{u}_j)$.

- Approximations 1 and 5 both allow for stationary shocks. 1 allows for 2-point shocks on the Hugoniot.

6 Burger's Equation

All results at http://www-personal.umich.edu/~zaidedan/main.html

We have the scalar Burgers' equation,

$$u_t + f_x = u_t + \left(\frac{1}{2}u^2\right)_x = u_t + uu_x = 0$$

with interpolated flux

$$\begin{aligned} f_j^* &= \frac{1}{2}(f_{j+1} + f_{j-1}) - \frac{1}{4}(u_{j+1} - 2u_j + u_{j-1}) \\ &= \frac{1}{4}\left[u_{j+1}^2 + u_{j-1}^2 - (u_{j-1} + u_{j+1})(u_{j+1} - 2u_j + u_{j-1})\right] \\ &= \frac{1}{2}[u_j(u_{j+1} + u_{j-1}) - u_{j-1}u_{j+1}] \end{aligned}$$

For the sum of fluxes,

$$f_j^* + f_{j+1}^* = u_j u_{j+1} + \frac{1}{2}(u_{j+1} - u_j)(u_{j+2} - u_{j-1})$$

Examining the flux difference, we get

$$\bar{P}_{j+\frac{1}{2}}(f_{j+1}^* - f_j^*) = \frac{1}{2}\text{sign}(u_{j+1} + u_j)(u_{j+2} + u_{j-1})(u_{j+1} - u_j)$$

which suggests a general formula

$$\mathbf{f}_{j+\frac{1}{2}} = \frac{1}{2}(\mathbf{f}_{j+1}^* + \mathbf{f}_j^*) - \frac{1}{2}\overline{\mathbf{P}}_{j+\frac{1}{2}}\tilde{\mathbf{A}}_{j+\frac{1}{2}}(\mathbf{u}_{j+1} - \mathbf{u}_j)$$

with $\tilde{\mathbf{A}}_{j+\frac{1}{2}}(\mathbf{u}_{j+2} - \mathbf{u}_{j-1}) = \mathbf{f}_{j+2} - \mathbf{f}_{j-1}$.

6.1 An Entropy Fix

Here I'll use the entropy fix that

$$\delta u_{j+\frac{1}{2}} = \max(0, 2(u_{j+1} - u_j))$$

and adjust the flux by $\frac{1}{2}|\delta u_{j+\frac{1}{2}}|(u_{j+1} - u_j)$ if $\frac{1}{2}|u_{j+1} - u_j|(u_{j+1} - u_j) > |f_{j+1}^* - f_j^*|$.

6.2 Stationary Shocks

Setting $\Delta f = 0$ and using the entropy condition gives that $u_L > 0 > u_R$ and that $u_R = -u_L$. Given a middle state $u_M = \alpha u_L + (1-\alpha)u_R = (2\alpha - 1)u_L$, the issue that I have is that since the Roe matrix captures the single wave exactly, we get that $\bar{A}_{LR} = 0$ and the stability analysis suggests that any intermediate state will just sit there using our method, regardless of linearity or not. This is quite interesting. **note?** Computational results show that only a point between u_L and u_R is stable for the standard method, while our method puts all points stable.

6.3 Slowly Moving Shocks

Defining $u_L > 0$ and solving the jump conditions gives $u_R = 2s - u_L$, where $s < u_L$ to satisfy the entropy condition. Putting s slow releases a slowly moving shock. It should be noted as s approaches u_L in speed, our method approaches the standard method. Starting a one-point slowly moving shock where $u_M \notin [u_R, u_L]$, our method adjusts quickly to a monotone slowly moving shock, (although not as quick as the standard method does), however it leads to a slightly (very slight) wider shock than the standard method.

7 Isothermal Euler

In one dimension, these are

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0 \qquad \mathbf{u} = \begin{bmatrix} \rho \\ m \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} m \\ m^2/\rho + p \end{bmatrix}$$

with $m = \rho u$. The equation of state is $p = a^2\rho$ for some constant sound speed, a. The flux jacobian is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ a^2 - u^2 & 2u \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} 1 & 1 \\ u-a & u+a \end{bmatrix} \qquad \mathbf{L} = \mathbf{R}^{-1} = \frac{1}{2a}\begin{bmatrix} (u+a) & -1 \\ -(u-a) & 1 \end{bmatrix}$$

with eigenvalues $\lambda^{\pm} = u \pm a$. The Hugoniot curve can be determined from $s\Delta\mathbf{u} = \Delta\mathbf{f}$ as

$$u_R = u_L \pm a\frac{(\rho_R - \rho_L)}{\sqrt{\rho_R\rho_L}}$$

with shock speed

$$s = u_L \pm a\sqrt{\frac{\rho_R}{\rho_L}}$$

The Roe averaged velocity is

$$u = \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$

After a great deal of math, it can be shown that

$$\begin{aligned} \mathbf{f}^*_{M,\rho} &= \rho_M u_M \\ \mathbf{f}^*_{M,\rho u} &= (\rho_R\rho_L(u_R - u_L) - \rho_M(\rho_R u_R - \rho_L u_L) + \rho_M u_M(\rho_R - \rho_L))\frac{u_R - u_L}{(\sqrt{\rho_L} + \sqrt{\rho_R})^2} \\ &\quad + \rho_M(u_M(u_L + u_R) - u_L u_R + a^2) \end{aligned}$$

7.1 Stationary Shocks

The family of stationary shocks that exists is

$$\rho_R = \rho_L\frac{u_L^2}{a^2}, \qquad u_R = \frac{a^2}{u_L}$$

The entropy condition, $u_L \pm a > u_R \pm a$ leads to $u_L > a$.

Looking at a one-point stationary shock, it can be shown that

$$\mathbf{f}^*_{M,\rho u} = 2\rho_m u_m a + \rho_L(u_L - a)^2 = 2\rho_m u_m a + \rho_R(u_R - a)^2$$

Examining stability now, and scaling $\boldsymbol{\varepsilon} = \mathbf{u}_M \boldsymbol{\varepsilon}$

$$\mathbf{u}_M + \boldsymbol{\varepsilon} = \left[\begin{array}{c} \rho_L \left(\alpha + (1-\alpha)\frac{u_L^2}{a^2} \right) \\ \rho_L u_L \end{array} \right] + \left[\begin{array}{c} \rho_L \boldsymbol{\varepsilon}_\rho \\ \rho_L u_L \boldsymbol{\varepsilon}_m \end{array} \right]$$

which leads to

$$u_{M\boldsymbol{\varepsilon}} = \frac{u_L(1+\boldsymbol{\varepsilon}_m)}{\alpha + (1-\alpha)\frac{u_L^2}{a^2} + \boldsymbol{\varepsilon}_\rho}$$

$$u_{LM\boldsymbol{\varepsilon}} = \frac{u_L + \frac{u_L(1+\boldsymbol{\varepsilon}_m)}{\sqrt{\alpha+(1-\alpha)\frac{u_L^2}{a^2}+\boldsymbol{\varepsilon}_\rho}}}{1 + \sqrt{\alpha + (1-\alpha)\frac{u_L^2}{a^2} + \boldsymbol{\varepsilon}_\rho}} \qquad u_{M\boldsymbol{\varepsilon}R} = \frac{a + \frac{u_L(1+\boldsymbol{\varepsilon}_m)}{\sqrt{\alpha+(1-\alpha)\frac{u_L^2}{a^2}+\boldsymbol{\varepsilon}_\rho}}}{\sqrt{\alpha + (1-\alpha)\frac{u_L^2}{a^2} + \boldsymbol{\varepsilon}_\rho} + \frac{u_L}{a}}$$

Since

$$\overline{\mathbf{A}}_{LR} = \left[\begin{array}{cc} 0 & 1 \\ 0 & 2a \end{array} \right], \qquad \mathbf{Z} = \mathbf{I} - \frac{\Delta t}{2h}(\overline{\mathbf{P}}_{M_\varepsilon R} + \overline{\mathbf{P}}_{LM_\varepsilon})\overline{\mathbf{A}}_{LR} = \left[\begin{array}{cc} 1 & z_{12} \\ 0 & \lambda_z \end{array} \right]$$

where the triangular structure of the matrix results in both eigenvalues lying on the diagonal. Define

$$\lambda_L^+ = u_{LM\boldsymbol{\varepsilon}} + a, \quad \lambda_L^- = u_{LM\boldsymbol{\varepsilon}} - a, \qquad \lambda_R^+ = u_{M\boldsymbol{\varepsilon}R} + a, \quad \lambda_R^- = u_{M\boldsymbol{\varepsilon}R} - a$$

$$\lambda_z = 1 - \frac{\Delta t}{4ah}\left((\lambda_L^- + 2a)|\lambda_L^+| + (\lambda_R^- + 2a)|\lambda_R^+| - (\lambda_L^+ - 2a)|\lambda_L^-| - (\lambda_R^+ - 2a)|\lambda_R^-| \right)$$
$$\lambda_z = 1 - \frac{\Delta t}{4ah}\left((\lambda_L^+)|\lambda_L^+| + (\lambda_R^+)|\lambda_R^+| - (\lambda_L^-)|\lambda_L^-| - (\lambda_R^-)|\lambda_R^-| \right)$$

7.2 Slowly Moving Shocks

Shifting the stationary shock by s leaves the family of moving shocks as

$$\rho_R = \rho_L \frac{(u_L - s)^2}{a^2}, \qquad u_R = s + \frac{a^2}{u_L - s}$$

8 A Non-Dimensional Version

Given the shocktube problem consisting of

$$\mathbf{u}_L = \begin{bmatrix} \rho \\ \rho u \end{bmatrix} \qquad \mathbf{u}_R = \begin{bmatrix} \rho \\ -\rho u \end{bmatrix} \quad \rightarrow \quad \mathbf{f}_L = \begin{bmatrix} \rho u \\ \rho(u^2 + a^2) \end{bmatrix} \qquad \mathbf{f}_R = \begin{bmatrix} -\rho u \\ \rho(u^2 + a^2) \end{bmatrix}$$

for $\rho, u > 0$. This creates two reflecting shocks, similar to the Noh problem without a contact discontinuity. The exact solution is

$$\begin{aligned} s &= \frac{u}{2} \pm \frac{\sqrt{4a^2 + u^2}}{2} \\ &= a\left(\frac{M}{2} \pm \sqrt{\frac{1}{M^2} + \frac{1}{4}}\right) \\ \rho_M &= \rho_L\left(1 + \frac{u^2}{2a^2}\left(1 + \sqrt{\frac{4a^2}{u^2} + 1}\right)\right) \\ &= \rho_L\left(1 + \frac{M^2}{2}\left(1 + \sqrt{\frac{4}{M^2} + 1}\right)\right) \end{aligned}$$

If we place the shocktube interface between cells j and $j+1$, at $j+\frac{1}{2}$, by symmetry we can examine interfaces at $j+\frac{1}{2}, j+\frac{3}{2}, j+\frac{5}{2}$ to see the initial development of the solution. Defining $\nu = \frac{a\Delta t}{\Delta x}$, where $\nu < \frac{1}{M+1}$, we have

$$\mathbf{u}_j^{n+1} - \mathbf{u}_j^n = -\nu(\mathbf{f}_{j+\frac{1}{2}}^n - \mathbf{f}_{j-\frac{1}{2}}^n)$$

At $t = 0$, $\mathbf{f}_{j+\frac{3}{2}} = \mathbf{f}_{j+\frac{5}{2}} = \mathbf{f}_R$ and

$$\mathbf{f}_{j+\frac{1}{2}}^{NL}(\mathbf{u}_L, \mathbf{u}_L, \mathbf{u}_R, \mathbf{u}_R) = \mathbf{f}_{j+\frac{1}{2}}^{Roe}(\mathbf{u}_L, \mathbf{u}_R) = \frac{1}{2}(\mathbf{f}_L + \mathbf{f}_R) - \frac{1}{2}\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} 0 \\ -2\rho u \end{bmatrix} = \begin{bmatrix} 0 \\ \rho(u^2 + ua + a^2) \end{bmatrix}$$

and

$$\mathbf{f}_{j+\frac{3}{2}} - \mathbf{f}_{j+\frac{1}{2}} = \mathbf{f}_R - \mathbf{f}_{j+\frac{1}{2}}^{Roe}(\mathbf{u}_L, \mathbf{u}_R) = \begin{bmatrix} -\rho u \\ -\rho u a \end{bmatrix}$$

so that after one timestep,

$$\mathbf{u}_{j+1}^1 = \begin{bmatrix} \rho(1 + M\nu) \\ -\rho u(1 - \nu) \end{bmatrix}$$

$$\mathbf{f}_{j+1}^1 = \begin{bmatrix} -\rho u(1 - \nu) \\ \rho\left(u^2\left(\frac{(1-\nu)^2}{1+M\nu}\right) + a^2(1 + M\nu)\right) \end{bmatrix}$$

and

$$u_{j+1} = -u\left(\frac{1 - \nu}{1 + M\nu}\right)$$

8.1 The Second Step

Now looking at the second step,

$$
\begin{aligned}
\mathbf{f}^{Roe,1}_{j+\frac{1}{2}}(\mathbf{u}^1_j, \mathbf{u}^1_{j+1}) &= \begin{bmatrix} 0 \\ \rho\left(u^2\left(\frac{(1-\nu)^2}{1+M\nu}\right) + a^2(1+M\nu)\right) \end{bmatrix} - \frac{1}{2}\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} 0 \\ -2\rho u(1-\nu) \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \rho\left(u^2\left(\frac{(1-\nu)^2}{1+M\nu}\right) + a^2(1+M)\right) \end{bmatrix}
\end{aligned}
$$

$$
\tilde{u}_{j+\frac{3}{2}} = -u\frac{(1-\nu)+\sqrt{1+M\nu}}{(1+M\nu)+\sqrt{1+M\nu}}
$$

and for the nonlocal solver, we have that

$$
\overline{\mathbf{P}}_{j+\frac{1}{2}} = \frac{1}{a}\begin{bmatrix} 0 & 1 \\ a^2 & 0 \end{bmatrix} \qquad \tilde{u}_j = u\frac{\sqrt{1+M\nu}-(1-\nu)}{\sqrt{1+M\nu}+(1+M\nu)} \qquad \tilde{u}_{j+1} = -u\frac{\sqrt{1+M\nu}-(1-\nu)}{\sqrt{1+M\nu}+(1+M\nu)}
$$

$$
\begin{aligned}
\mathbf{f}^{NL,1}_{j+\frac{1}{2}} &= \mathbf{f}^{NL}_{j+\frac{1}{2}}\left(\begin{bmatrix} \rho \\ \rho u \end{bmatrix}, \begin{bmatrix} \rho(1+M\nu) \\ \rho u(1-\nu) \end{bmatrix}, \begin{bmatrix} \rho(1+M\nu) \\ -\rho u(1-\nu) \end{bmatrix}\begin{bmatrix} \rho \\ -\rho u \end{bmatrix}\right) \\
&= \mathbf{f}^{Roe,1}_{j+\frac{1}{2}} + \frac{1}{2}\left((\mathbf{I}-\overline{\mathbf{P}}_{j+\frac{1}{2}})\overline{\mathbf{A}}_{j+1} - (\mathbf{I}+\overline{\mathbf{P}}_{j+\frac{1}{2}})\overline{\mathbf{A}}_j + 2|\overline{\mathbf{A}}|_{j+\frac{1}{2}}\right)(\mathbf{u}_{j+1}-\mathbf{u}_j) \\
&= \mathbf{f}^{Roe,1}_{j+\frac{1}{2}} + \frac{1}{2a}\left(\begin{bmatrix} 2\tilde{u}^2_{j+1} - 2a^2 & 0 \\ 0 & 4a\tilde{u}_{j+1} - 2a^2 \end{bmatrix} + 2a|\overline{\mathbf{A}}|_{j+\frac{1}{2}}\right)(\mathbf{u}_{j+1}-\mathbf{u}_j) \\
&= \mathbf{f}^{Roe,1}_{j+\frac{1}{2}} + \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} 0 \\ -2\rho u(1-\nu) \end{bmatrix} + \frac{1}{a}\begin{bmatrix} \tilde{u}^2_{j+1} - a^2 & 0 \\ 0 & 2a\tilde{u}_{j+1} - a^2 \end{bmatrix}\begin{bmatrix} 0 \\ -2\rho u(1-\nu) \end{bmatrix} \\
&= \mathbf{f}^{Roe,1}_{j+\frac{1}{2}} + \begin{bmatrix} 0 \\ 4\rho u^2(1-\nu)\left(\frac{\sqrt{1+M\nu}-(1-\nu)}{\sqrt{1+M\nu}+(1+M\nu)}\right) \end{bmatrix}
\end{aligned}
$$

At the next interface,

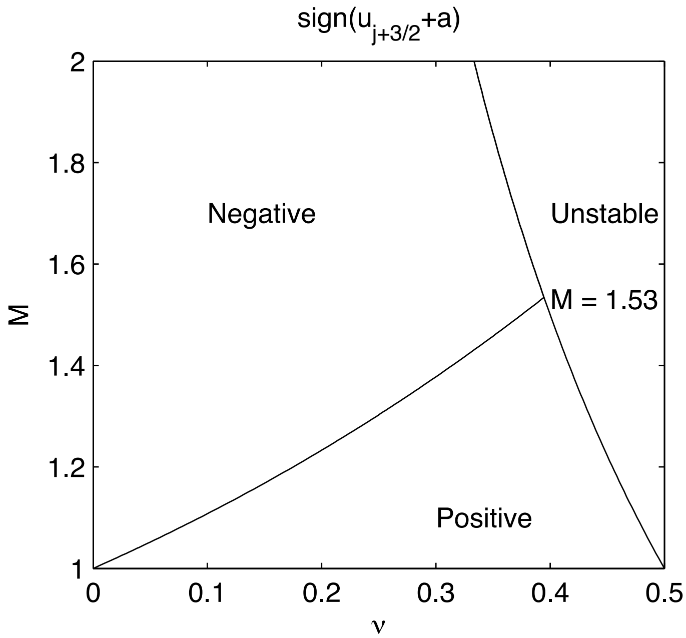
$$
\tilde{u}_{j+\frac{3}{2}} - a = -a\frac{(M+1)}{\sqrt{1+M\nu}} \qquad \tilde{u}_{j+\frac{3}{2}} + a = -a\frac{\left((M+1)-2\sqrt{1+M\nu}\right)}{\sqrt{1+M\nu}}
$$

We know that $\tilde{u}_{j+\frac{3}{2}} - a < 0$ for all ν, M. For $\tilde{u}_{j+\frac{3}{2}} + a$, we note the sign change at $M \approx 1.53$.
This leads to two cases.

8.1.1 $\tilde{u}_{j+\frac{3}{2}} + a < 0$

This is the easier case, since $|\overline{\mathbf{A}}|_{j+\frac{3}{2}} = -\overline{\mathbf{A}}_{j+\frac{3}{2}}$

$$
\begin{aligned}
\mathbf{f}^{Roe,1}_{j+\frac{3}{2}} &= \frac{1}{2}(\mathbf{f}_{j+1}+\mathbf{f}_{j+2}) - \frac{1}{2}|\overline{\mathbf{A}}|_{j+\frac{3}{2}}(\mathbf{u}_{j+2}-\mathbf{u}_{j+1}) \\
&= \frac{1}{2}(\mathbf{f}_{j+1}+\mathbf{f}_{j+2}) + \frac{1}{2}\overline{\mathbf{A}}_{j+\frac{3}{2}}(\mathbf{u}_{j+2}-\mathbf{u}_{j+1}) \\
&= \frac{1}{2}(\mathbf{f}_{j+1}+\mathbf{f}_{j+2}) + \frac{1}{2}(\mathbf{f}_{j+2}-\mathbf{f}_{j+1}) \\
&= \mathbf{f}_{j+2} = \mathbf{f}_R
\end{aligned}
$$



Looking at the next interface, we have that $\overline{\mathbf{P}}_{j+\frac{3}{2}} = -\mathbf{I}$ and $\overline{\mathbf{A}}_{j+2} = -|\overline{\mathbf{A}}|_{j+\frac{3}{2}}$.

$$\begin{aligned}
\mathbf{f}^{NL,1}_{j+\frac{3}{2}} &= \mathbf{f}^{NL}_{j+\frac{3}{2}}\left(\begin{bmatrix}\rho(1+\nu u)\\ \rho u(1-\nu a)\end{bmatrix}, \begin{bmatrix}\rho(1+\nu u)\\ -\rho u(1-\nu a)\end{bmatrix}\begin{bmatrix}\rho\\ -\rho u\end{bmatrix}, \begin{bmatrix}\rho\\ -\rho u\end{bmatrix}\right)\\
&= \mathbf{f}^{Roe,1}_{j+\frac{3}{2}} + \frac{1}{2}\left((\mathbf{I}-\overline{\mathbf{P}}_{j+\frac{3}{2}})\overline{\mathbf{A}}_{j+2} - (\mathbf{I}+\overline{\mathbf{P}}_{j+\frac{3}{2}})\overline{\mathbf{A}}_{j+1} + 2|\overline{\mathbf{A}}|_{j+\frac{3}{2}}\right)(\mathbf{u}_{j+2}-\mathbf{u}_{j+1})\\
&= \mathbf{f}^{Roe,1}_{j+\frac{3}{2}} + \left(\overline{\mathbf{A}}_{j+2} + |\overline{\mathbf{A}}|_{j+\frac{3}{2}}\right)(\mathbf{u}_{j+2}-\mathbf{u}_{j+1})\\
&= \mathbf{f}^{Roe,1}_{j+\frac{3}{2}}
\end{aligned}$$

So after two timesteps,

$$\begin{aligned}
\mathbf{u}^2_{j+1} &= \mathbf{u}^1_{j+1} - \frac{\nu}{a}(\mathbf{f}^1_{j+\frac{3}{2}} - \mathbf{f}^1_{j+\frac{1}{2}})\\
&= \begin{bmatrix}\rho(1+M\nu)\\ -\rho u(1-\nu)\end{bmatrix} - \frac{\nu}{a}\left(\begin{bmatrix}-\rho u\\ \rho(u^2+a^2)\end{bmatrix} - \begin{bmatrix}0\\ \rho\left(u^2\left(\frac{(1-\nu)^2}{1+M\nu}\right)+a^2(M+1)\right)\end{bmatrix}\right)\\
&= \begin{bmatrix}\rho(1+2M\nu)\\ -\rho u\left((1-2\nu)+M\nu\left(1-\frac{(1-\nu)^2}{1+M\nu}\right)\right)\end{bmatrix}
\end{aligned}$$

Using the nonlocal Riemann Solver, we get that

$$\begin{aligned}
\mathbf{u}^{NL,2}_{j+1} &= \mathbf{u}^1_{j+1} - \frac{\nu}{a}(\mathbf{f}^{NL,1}_{j+\frac{3}{2}} - \mathbf{f}^{NL,1}_{j+\frac{1}{2}})\\
&= \mathbf{u}^2_{j+1} + \begin{bmatrix}0\\ \rho u\left(4M\nu(1-\nu)\left(\frac{\sqrt{1+M\nu}-(1-\nu)}{\sqrt{1+M\nu}+(1+M\nu)}\right)\right)\end{bmatrix}
\end{aligned}$$

8.1.2 $\tilde{u}_{j+\frac{3}{2}} + a > 0$

This is the harder case, since the eigenvalues now have different signs. We have

$$\tilde{u}_{j+\frac{3}{2}} = -u\frac{(1-\nu)+\sqrt{1+M\nu}}{(1+M\nu)+\sqrt{1+M\nu}}$$

$$|\overline{\mathbf{A}}|_{j+\frac{3}{2}} \quad = \quad \frac{1}{a}\begin{bmatrix} a^2 - u^2_{j+\frac{3}{2}} & u_{j+\frac{3}{2}} \\ u_{j+\frac{3}{2}}(a^2 - u^2_{j+\frac{3}{2}}) & a^2 + \tilde{u}^2_{j+\frac{3}{2}} \end{bmatrix}$$

$$\begin{aligned}
\mathbf{f}^{Roe,1}_{j+\frac{3}{2}} &= \begin{bmatrix} -\rho u\left(1-\frac{\nu}{2}\right) \\ \rho u^2\left(\frac{1}{2}+\frac{(1-\nu)^2}{2(1+M\nu)}\right)+\rho a^2\left(1+\frac{M\nu}{2}\right) \end{bmatrix} - \frac{1}{2a}\begin{bmatrix} a^2 - u^2_{j+\frac{3}{2}} & u_{j+\frac{3}{2}} \\ u_{j+\frac{3}{2}}(a^2 - u^2_{j+\frac{3}{2}}) & a^2 + \tilde{u}^2_{j+\frac{3}{2}} \end{bmatrix}\begin{bmatrix} -\rho M\nu \\ -\rho u\nu \end{bmatrix} \\
&= \begin{bmatrix} -\rho u\left(\frac{(M+1)\nu\left(M\left(\sqrt{M\nu+1}-\nu\right)+\sqrt{M\nu+1}-1\right)}{2(M\nu+1)^{3/2}}+(1-\nu)\right) \\ \rho u^2\left(\frac{\nu M(M+1)\left((4+\nu(\nu-4)+M\nu)\left(\sqrt{M\nu+1}+1\right)+2M\nu(1-\nu)\right)}{2\left((1+M\nu)+\sqrt{1+M\nu}\right)^3}+\frac{(\nu((M-1)-(1-\nu))+2)}{2(1+M\nu)}\right)+\rho a^2\left(1+\frac{3M\nu}{2}+\frac{-M\nu(M+1)}{2\sqrt{1+M\nu}}\right) \end{bmatrix}
\end{aligned}$$

For the nonlocal solver, we need

$$\tilde{u}_{j+1} = -u\frac{\sqrt{1+M\nu}-(1-\nu)}{\sqrt{1+M\nu}+(1+M\nu)}$$

such that

$$\tilde{u}_{j+1} = \tilde{u}_{j+\frac{3}{2}}\frac{\sqrt{1+M\nu}-(1-\nu)}{\sqrt{1+M\nu}+(1-\nu)} = \tilde{u}_{j+\frac{3}{2}} + u\frac{2(1-\nu)}{\sqrt{1+M\nu}+(1+M\nu)}$$

and

$$\overline{\mathbf{P}}_{j+\frac{3}{2}} = \frac{1}{a}\begin{bmatrix} -\tilde{u}_{j+\frac{3}{2}} & 1 \\ a^2 - \tilde{u}^2_{j+\frac{3}{2}} & \tilde{u}_{j+\frac{3}{2}} \end{bmatrix} \qquad \overline{\mathbf{A}}_{j+2} = \begin{bmatrix} 0 & 1 \\ a^2 - \tilde{u}^2_{j+\frac{3}{2}} & 2\tilde{u}_{j+\frac{3}{2}} \end{bmatrix} \qquad \overline{\mathbf{A}}_{j+1} = \begin{bmatrix} 0 & 1 \\ a^2 - \tilde{u}^2_{j+1} & 2\tilde{u}_{j+1} \end{bmatrix}$$

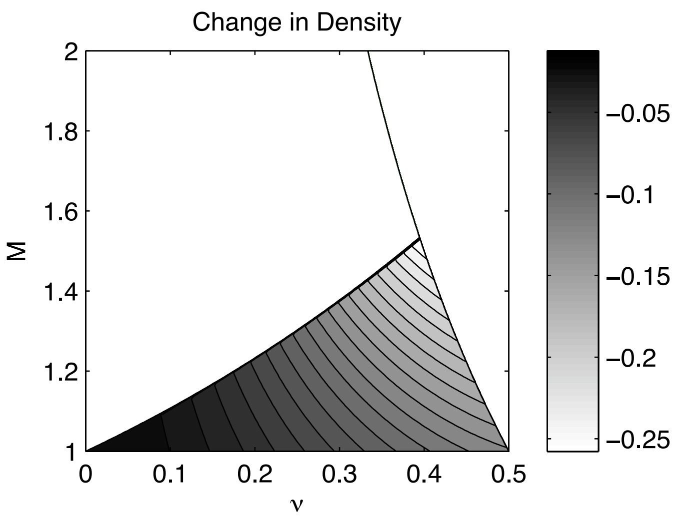
$$\begin{aligned}
\mathbf{f}^{NL,1}_{j+\frac{3}{2}} &= \mathbf{f}^{Roe,1}_{j+\frac{3}{2}} + \frac{1}{2}\left((\mathbf{I}-\overline{\mathbf{P}}_{j+\frac{3}{2}})\overline{\mathbf{A}}_{j+2} - (\mathbf{I}+\overline{\mathbf{P}}_{j+\frac{3}{2}})\overline{\mathbf{A}}_{j+1} + 2|\overline{\mathbf{A}}|_{j+\frac{3}{2}}\right)(\mathbf{u}_{j+2}-\mathbf{u}_{j+1}) \\
&= \mathbf{f}^{Roe,1}_{j+\frac{3}{2}} + \frac{1}{2}\left((\mathbf{I}-\overline{\mathbf{P}}_{j+\frac{3}{2}})\overline{\mathbf{A}}_{j+2} - (\mathbf{I}+\overline{\mathbf{P}}_{j+\frac{3}{2}})\overline{\mathbf{A}}_{j+1} + 2\overline{\mathbf{P}}_{j+\frac{3}{2}}\overline{\mathbf{A}}_{j+\frac{3}{2}}\right)(\mathbf{u}_{j+2}-\mathbf{u}_{j+1}) \\
&= \mathbf{f}^{Roe,1}_{j+\frac{3}{2}} + \frac{1}{2}\left((\mathbf{I}-\overline{\mathbf{P}}_{j+\frac{3}{2}})\overline{\mathbf{A}}_{j+2} - (\mathbf{I}+\overline{\mathbf{P}}_{j+\frac{3}{2}})\overline{\mathbf{A}}_{j+1} + 2\overline{\mathbf{P}}_{j+\frac{3}{2}}\overline{\mathbf{A}}_{j+2}\right)(\mathbf{u}_{j+2}-\mathbf{u}_{j+1}) \\
&= \mathbf{f}^{Roe,1}_{j+\frac{3}{2}} + \frac{1}{2}(\mathbf{I}+\overline{\mathbf{P}}_{j+\frac{3}{2}})(\overline{\mathbf{A}}_{j+2}-\overline{\mathbf{A}}_{j+1})(\mathbf{u}_{j+2}-\mathbf{u}_{j+1}) \\
&= \mathbf{f}^{Roe,1}_{j+\frac{3}{2}} + \frac{1}{2a}\begin{bmatrix} a-\tilde{u}_{j+\frac{3}{2}} & 1 \\ a^2-\tilde{u}^2_{j+\frac{3}{2}} & a+\tilde{u}_{j+\frac{3}{2}} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -(\tilde{u}^2_{j+\frac{3}{2}}-\tilde{u}^2_{j+1}) & 2(\tilde{u}_{j+\frac{3}{2}}-\tilde{u}_{j+1}) \end{bmatrix}\begin{bmatrix} -\rho M\nu \\ -\rho u\nu \end{bmatrix} \\
&= \mathbf{f}^{Roe,1}_{j+\frac{3}{2}} + \rho u\frac{2M\nu(1-\nu)\left((M+1)\sqrt{1+M\nu}+(1+M\nu)\right)}{\left(1+M\nu+\sqrt{1+M\nu}\right)^2}\begin{bmatrix} 1 \\ \frac{a\left(2\sqrt{M\nu+1}-M-1\right)}{\sqrt{M\nu+1}} \end{bmatrix}
\end{aligned}$$

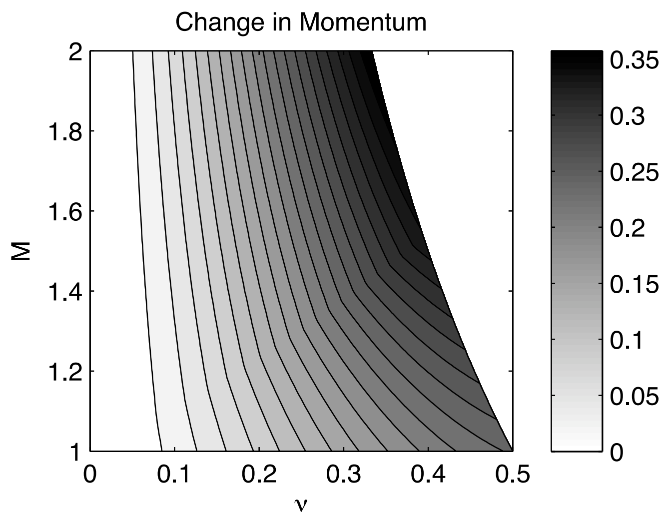
Putting this together,

$$\mathbf{f}^{NL,1}_{j+\frac{1}{2}} \quad = \quad \mathbf{f}^{Roe,1}_{j+\frac{1}{2}} + \begin{bmatrix} 0 \\ 4\rho u^2(1-\nu)\left(\frac{\sqrt{1+M\nu}-(1-\nu)}{\sqrt{1+M\nu}+(1+M\nu)}\right) \end{bmatrix}$$

$$
\begin{aligned}
\mathbf{u}_{j+1}^{NL,2} &= \mathbf{u}_{j+1}^{1} - \frac{\nu}{a}(\mathbf{f}_{j+\frac{3}{2}}^{NL,1} - \mathbf{f}_{j+\frac{1}{2}}^{NL,1}) \\
&= \mathbf{u}_{j+1}^{2} - \left[\begin{array}{c} \rho\frac{2(M\nu)^2(1-\nu)\left((M+1)\sqrt{1+M\nu}+(1+M\nu)\right)}{\left(1+M\nu+\sqrt{1+M\nu}\right)^2} \\ \rho u \left\{ \frac{2M(\nu-1)\nu\left(aM\left(M\left(-3\sqrt{M\nu+1}+M+2\nu\right)+\sqrt{M\nu+1}+1\right)+2\nu\left(M\left(\sqrt{M\nu+1}+\nu\right)+\sqrt{M\nu+1}+1\right)\right)}{\left(M\nu+\sqrt{M\nu+1}+1\right)^2} \right\} \end{array} \right]
\end{aligned}
$$

Putting everything together for both regions, we can observe the difference between the two Riemann Solvers, with density shown on the right and momentum on the left. Note that the changes in density are negative, and the changes in momentum are positive, although since $(\rho u)_{j+1}^2 < 0$ and $\rho_{j+1}^2 > 0$, both density and momentum decrease in magnitude.





9 Euler Equations

$$
\frac{\partial}{\partial t}\left[\begin{array}{c}\rho \\ \rho u \\ E\end{array}\right] + \frac{\partial}{\partial x}\left[\begin{array}{c}\rho u \\ \rho u^2 + p \\ \rho u H\end{array}\right] = \left[\begin{array}{c}0 \\ 0 \\ 0\end{array}\right],
$$

with $H = \frac{E+p}{\rho}$ and an equation of state $p = p(\rho, e)$. For the ideal gas, $p = (\gamma-1)\rho e = (\gamma-1)\left(E - \frac{1}{2}\rho u^2\right)$. This system is also equipped with a speed of sound, $a = \left.\frac{\partial p}{\partial \rho}\right|_s$.

Roe's approximate Riemann solver is described here as

$$
\mathbf{f}(\mathbf{u}_L, \mathbf{u}_R) = \frac{1}{2}\left(\mathbf{f}(\mathbf{u}_L) + \mathbf{f}(\mathbf{u}_R)\right) - \frac{1}{2}\mathbf{R}|\Lambda|\mathbf{L}(\mathbf{u}_R - \mathbf{u}_L)
$$

where $\mathbf{L} = \mathbf{R}^{-1}$,

$$
\mathbf{R} = \left[\begin{array}{c|c|c} 1 & 1 & 1 \\ u-a & u & u+a \\ H-ua & \frac{1}{2}u^2 & H+ua \end{array}\right]
$$

$$
\mathbf{L} = \frac{1}{2a^2}\left[\begin{array}{c|c|c} \frac{\gamma-1}{2}u^2 + ua & -(\gamma-1)u - a & (\gamma-1) \\ 2a^2 - (\gamma-1)u^2 & 2(\gamma-1)u & -2(\gamma-1) \\ \frac{\gamma-1}{2}u^2 - ua & -(\gamma-1)u + a & (\gamma-1) \end{array}\right].
$$

and $\Lambda = \mathrm{diag}(u-a, u, u+a)$ with a and u as density-averaged variables from

$$u = \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \qquad H = \frac{\sqrt{\rho_L}H_L + \sqrt{\rho_R}H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \qquad a = \sqrt{(\gamma-1)\left(H - \frac{1}{2}u^2\right)}$$

Pick a stationary shock as described by Hiro. Start with

$$[\rho_L, u_L, p_L]^T = \left[1, 1, \frac{1}{\gamma M_L^2}\right]^T$$

The right state is

$$\begin{bmatrix} \rho_R \\ u_R \\ p_R \end{bmatrix} = \begin{bmatrix} \frac{(\gamma+1)M_L^2}{(\gamma-1)M_L^2+2} \\ \frac{(\gamma-1)M_L^2+2}{(\gamma+1)M_L^2} \\ \left(1 + \frac{2\gamma}{\gamma+1}(M_L^2-1)\right)\frac{1}{\gamma M_L^2} \end{bmatrix}$$

Intermediate states are

$$\begin{bmatrix} \rho_M \\ u_M \\ p_M \end{bmatrix} = \begin{bmatrix} \rho_L + \varepsilon(\rho_R - \rho_L) \\ u_L + \left(1 - (1-\varepsilon)\left(1 + \varepsilon\frac{M_L^2-1}{1+\frac{\gamma-1}{2}M_L^2}\right)^{-\frac{1}{2}}\left(1 - \varepsilon\frac{M_L^2-1}{\frac{2\gamma}{\gamma-1}M_L^2-1}\right)^{-\frac{1}{2}}\right)(u_R - u_L) \\ p_L + \varepsilon\left(1 + (1-\varepsilon)\frac{\gamma-1}{\gamma+1}\frac{M_L^2-1}{M_L^2}\right)^{-1}(p_R - p_L) \end{bmatrix}$$

10 $\mathbf{f} \rightarrow \mathbf{u}$

How do we go from fluxes to conserved variables.

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{bmatrix} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \frac{\gamma}{\gamma-1} up + \frac{1}{2}\rho u^3 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_2 \\ \frac{3-\gamma}{2}\frac{\mathbf{u}_2^2}{\mathbf{u}_1} + (\gamma-1)\mathbf{u}_3 \\ \frac{\gamma \mathbf{u}_3 \mathbf{u}_2}{\mathbf{u}_1} - \frac{\gamma-1}{2}\frac{\mathbf{u}_2^3}{\mathbf{u}_1^2} \end{bmatrix}$$

where we have that $\mathbf{f}_2 > 0$ and that $\mathbf{f}_1\mathbf{f}_3 > 0$, that is that the second flux component must always be positive, and the first and third must have the same sign. Solving for primitive variables gives

$$\begin{aligned} \rho &= \frac{\mathbf{f}_1(\gamma \mathbf{f}_2 \mp \mathbf{f}_q)}{2\mathbf{f}_3(\gamma-1)} \\ u &= \frac{\gamma \mathbf{f}_2 \pm \mathbf{f}_q}{\mathbf{f}_1(\gamma+1)} \\ p &= \frac{\mathbf{f}_2 \mp \mathbf{f}_q}{\gamma+1} \\ \mathbf{f}_q &= \sqrt{\gamma^2\mathbf{f}_2^2 - 2\mathbf{f}_1\mathbf{f}_3(\gamma^2-1)} \end{aligned}$$

From here we can observe that for both roots,

$$\begin{aligned} \rho u &= \left(\frac{\gamma \mathbf{f}_2 \pm \mathbf{f}_q}{\mathbf{f}_1(\gamma+1)}\right)\left(\frac{\mathbf{f}_1(\gamma \mathbf{f}_2 \mp \mathbf{f}_q)}{2\mathbf{f}_3(\gamma-1)}\right) \\ &= \mathbf{f}_1 \frac{\gamma^2 \mathbf{f}_2 - \mathbf{f}_q^2}{2\mathbf{f}_1\mathbf{f}_3(\gamma^2-1)} \\ &= \mathbf{f}_1 \end{aligned}$$

Note that we need to be careful if $\mathbf{f}_1 = 0$ or $\mathbf{f}_3 = 0$, since this would make one of our variables zero. If both are zero, we must have that $u = 0$, and that $p = \mathbf{f}_2$. If $\mathbf{f}_1 = 0$, all flux terms must be correspondingly zero, (else $\rho = 0$ which is not possible). Lets examine $\mathbf{f}_q$.

$$\begin{aligned} \mathbf{f}_q &= \sqrt{\gamma^2\mathbf{f}_2^2 - 2\mathbf{f}_1\mathbf{f}_3(\gamma^2-1)} \\ \mathbf{f}_q^2 &= \gamma^2(\rho u^2 + p)^2 - \left(\frac{2\gamma}{\gamma-1}\rho u^2 p + \rho^2 u^4\right)(\gamma^2-1) \\ &= \gamma^2(\rho^2u^4 + 2\rho u^2 p + p^2) - \left(\frac{2\gamma}{\gamma-1}\rho u^2 p + \rho^2 u^4\right)(\gamma^2-1) \\ &= \gamma^2\rho^2u^4 + 2\gamma^2\rho u^2 p + \gamma^2 p^2 - 2\gamma(\gamma+1)\rho u^2 p - (\gamma^2-1)\rho^2u^4 \\ &= -2\gamma\rho u^2 p + \gamma^2 p^2 + \rho^2 u^4 \\ \mathbf{f}_q &= |\gamma p - \rho u^2| = \gamma p|1-M^2| \end{aligned}$$

This is good, because $\mathbf{f}_q = 0 \rightarrow M^2 = 1$ which is the sonic point. Because of the absolute value, we have that

$$\mathbf{f}_q = \begin{cases} \gamma p(1-M^2) & \text{Subsonic} \\ 0 & \text{Sonic} \\ \gamma p(M^2-1) & \text{Supersonic} \end{cases}$$

Looking at Mach number, we have that $M^2 = \frac{\rho u^2}{\gamma p}$

$$\begin{aligned} M^2 &= \frac{\rho u^2}{\gamma p} \\ &= \frac{\gamma \mathbf{f}_2 \pm \mathbf{f}_q}{\gamma \mathbf{f}_2 \mp \gamma \mathbf{f}_q} \end{aligned}$$

$$\mathbf{f}_2 = \rho u^2 + p = p\left(\frac{\rho u^2}{p} + 1\right) = p(\gamma M^2 + 1)$$

We have the interpolated fluxes defined as

$$\begin{aligned} \mathbf{f}_j^{*,L} &= \mathbf{f}_{j-1} + \overline{\mathbf{A}}_j(\mathbf{u}_j - \mathbf{u}_{j-1}) \\ \mathbf{f}_j^{*,R} &= \mathbf{f}_{j+1} - \overline{\mathbf{A}}_j(\mathbf{u}_{j+1} - \mathbf{u}_j) \\ \mathbf{f}_j^{*,M} &= \frac{1}{2}(\mathbf{f}_j^{*,L} + \mathbf{f}_j^{*,R}) = \frac{1}{2}(\mathbf{f}_{j+1} + \mathbf{f}_{j-1}) - \frac{1}{2}\overline{\mathbf{A}}_j(\mathbf{u}_{j+1} - 2\mathbf{u}_j + \mathbf{u}_{j-1}) \end{aligned}$$

Choosing $j = 2$,

$$\mathbf{f}_2^{*,L} = \begin{bmatrix} \rho_1 u_1 \\ \rho_1 u_1^2 + p_1 \\ \frac{\gamma}{\gamma-1} u_1 p_1 + \frac{1}{2}\rho_1 u_1^3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2}\bar{u}_2^2(\gamma-3) & (\gamma-3)\bar{u}_2 & \gamma - 1 \\ \frac{\gamma-2}{2}\bar{u}_2^3 - \frac{\bar{a}_2^2 \bar{u}_2}{\gamma-1} & \frac{3-2\gamma}{2}\bar{u}_2^2 + \frac{\bar{a}_2^2}{\gamma-1} & \gamma \bar{u}_2 \end{bmatrix} \begin{bmatrix} \rho_2 - \rho_1 \\ \rho_2 u_2 - \rho_1 u_1 \\ \frac{p_2 - p_1}{\gamma-1} + \frac{1}{2}\rho_2 u_2^2 - \frac{1}{2}\rho_1 u_1^2 \end{bmatrix}$$

writing this as $(\mathbf{f}_2^*)^T = [\mathbf{f}_{2,1}^*, \mathbf{f}_{2,2}^*, \mathbf{f}_{2,3}^*]^T$ we have

$$\begin{aligned} \mathbf{f}_{2,1}^{*,L} &= \mathbf{f}_{2,3}^{*,R} = \rho_2 u_2 \\ \mathbf{f}_{2,2}^{*,L} &= \left(p_2 + \rho_2 u_2^2\right) - \frac{\gamma-3}{2}\left(\rho_1 \left(u_{13} - u_1\right)^2 - \rho_2 \left(u_{13} - u_2\right)^2\right) \\ &= p_2 + \frac{\gamma-1}{2}\rho_2 u_2^2 + \frac{3-\gamma}{2}\rho_1 \left(u_{13} - u_1\right)^2 - \frac{3-\gamma}{2}\rho_2 u_{13}(u_{13} - 2u_2) \\ \mathbf{f}_{2,3}^{*,L} &= \frac{a_{13}^2}{\gamma-1}\left(\rho_2(u_2 - u_{13}) - \rho_1(u_1 - u_{13})\right) + \frac{1}{2}\gamma u_{13}\left(\rho_2(u_2 - u_{13})^2 - \rho_1(u_1 - u_{13})^2\right) \\ &\quad + \frac{1}{2}u_{13}^2\left(\rho_2(3u_2 - 2u_{13}) - \rho_1(3u_1 - 2u_{13})\right) + \frac{\gamma u_{13}}{\gamma-1}(p_2 - p_1) + \frac{\gamma p_1 u_1}{\gamma-1} + \frac{1}{2}\rho_1 u_1^3 \end{aligned}$$

$$\begin{aligned} \mathbf{f}_{2,2}^{*,R} &= \left(p_2 + \rho_2 u_2^2\right) - \frac{\gamma-3}{2}\left(\rho_3 \left(u_{13} - u_3\right)^2 - \rho_2 \left(u_{13} - u_2\right)^2\right) \\ &= p_2 + \frac{\gamma-1}{2}\rho_2 u_2^2 + \frac{3-\gamma}{2}\rho_3 \left(u_{13} - u_3\right)^2 - \frac{3-\gamma}{2}\rho_2 u_{13}(u_{13} - 2u_2) \\ \mathbf{f}_{2,3}^{*,R} &= \frac{a_{13}^2}{\gamma-1}\left(\rho_2(u_2 - u_{13}) - \rho_3(u_3 - u_{13})\right) + \frac{1}{2}\gamma u_{13}\left(\rho_2(u_2 - u_{13})^2 - \rho_3(u_3 - u_{13})^2\right) \\ &\quad + \frac{1}{2}u_{13}^2\left(\rho_2(3u_2 - 2u_{13}) - \rho_3(3u_3 - 2u_{13})\right) + \frac{\gamma u_{13}}{\gamma-1}(p_2 - p_3) + \frac{\gamma p_3 u_3}{\gamma-1} + \frac{1}{2}\rho_3 u_3^3 \end{aligned}$$

To check for physicality, look at $\mathbf{f}_{2,2}^{*,M}$.

$$\begin{aligned} \mathbf{f}_{2,2}^{*,M} &= \left(p_2 + \rho_2 u_2^2\right) + \frac{3-\gamma}{4}\left(\rho_3 \left(u_{13} - u_3\right)^2 - 2\rho_2 \left(u_{13} - u_2\right)^2 + \rho_1 \left(u_{13} - u_1\right)^2\right) \\ &= \left(p_2 + \rho_2 u_2^2\right) + \frac{3-\gamma}{2}\left(\frac{\rho_1 \rho_3 (u_1 - u_3)^2}{(\sqrt{\rho_1} + \sqrt{\rho_3})^2} - \rho_2 \left(u_{13} - u_2\right)^2\right) \end{aligned}$$

This can be made negative with the correct choices of velocities. Now

$$
\begin{aligned}
\mathbf{f}_{2,3}^{*,L} &= \frac{a_{13}^2}{\gamma-1}\left(\rho_2(u_2-u_{13})-\rho_1(u_1-u_{13})\right)+\frac{1}{2}\gamma u_{13}\left(\rho_2(u_2-u_{13})^2-\rho_1(u_1-u_{13})^2\right) \\
&\quad +\frac{1}{2}u_{13}^2\left(\rho_2(3u_2-2u_{13})-\rho_1(3u_1-2u_{13})\right)+\frac{\gamma u_{13}}{\gamma-1}(p_2-p_1)+\frac{\gamma p_1 u_1}{\gamma-1}+\frac{1}{2}\rho_1 u_1^3 \\
\mathbf{f}_{2,3}^{*,R} &= \frac{a_{13}^2}{\gamma-1}\left(\rho_2(u_2-u_{13})-\rho_3(u_3-u_{13})\right)+\frac{1}{2}\gamma u_{13}\left(\rho_2(u_2-u_{13})^2-\rho_3(u_3-u_{13})^2\right) \\
&\quad +\frac{1}{2}u_{13}^2\left(\rho_2(3u_2-2u_{13})-\rho_3(3u_3-2u_{13})\right)+\frac{\gamma u_{13}}{\gamma-1}(p_2-p_3)+\frac{\gamma p_3 u_3}{\gamma-1}+\frac{1}{2}\rho_3 u_3^3 \\
\mathbf{f}_{2,3}^{*,M} &= \frac{a_{13}^2}{2(\gamma-1)}\left(2\rho_2(u_2-u_{13})-\rho_1(u_1-u_{13})-\rho_3(u_3-u_{13})\right) \\
&\quad +\frac{\gamma}{4}u_{13}\left(\rho_2(u_2-u_{13})^2-\rho_1(u_1-u_{13})^2-\rho_3(u_3-u_{13})^2\right)
\end{aligned}
$$

11 Extension

What if we add on a higher order term, Mine was of the idea that

$$\Delta\mathbf{A} = \mathbf{H}\Delta\mathbf{u}$$

and then extend

$$\mathbf{f}_j^* = \mathbf{f}_{j-1} + \overline{\mathbf{A}}_j(\mathbf{u}_j - \mathbf{u}_{j-1}) + \frac{1}{2}\Delta\mathbf{u}^T\mathbf{H}\Delta\mathbf{u}$$

This does require some knowledge of what $\mathbf{H}$ is. Phil's idea appears to be adding on

$$\overline{\mathbf{A}}(\mathbf{u}_{j+1} - 3\mathbf{u}_j + 3\mathbf{u}_{j-1} - \mathbf{u}_{j-2})$$

, of some form.

12 2D Analysis

Something goes wrong with the flux in 2D. Whats going wrong? Again, we have the interpolated fluxes defined, for some direction, as

$$\begin{aligned}
\mathbf{f}_j^{*,L} &= \mathbf{f}_{j-1} + \overline{\mathbf{A}}_j(\mathbf{u}_j - \mathbf{u}_{j-1}) \\
\mathbf{f}_j^{*,R} &= \mathbf{f}_{j+1} - \overline{\mathbf{A}}_j(\mathbf{u}_{j+1} - \mathbf{u}_j) \\
\mathbf{f}_j^{*,M} &= \frac{1}{2}(\mathbf{f}_j^{*,L} + \mathbf{f}_j^{*,R}) = \frac{1}{2}(\mathbf{f}_{j+1} + \mathbf{f}_{j-1}) - \frac{1}{2}\overline{\mathbf{A}}_j(\mathbf{u}_{j+1} - 2\mathbf{u}_j + \mathbf{u}_{j-1})
\end{aligned}$$

Looking solely at the x-direction, we have

$$\mathbf{A} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
\frac{\gamma-3}{2}u^2 + \frac{\gamma-1}{2}v^2 & (3-\gamma)u & (1-\gamma)v & (\gamma-1) \\
-uv & v & u & 0 \\
\left(\frac{\gamma-2}{2}u^2 - \frac{a^2}{\gamma-1}\right)u + \frac{\gamma-2}{2}uv^2 & \frac{3-2\gamma}{2}u^2 + \frac{a^2}{\gamma-1} + \frac{3-2\gamma}{2}v^2 & (1-\gamma)uv & \gamma u
\end{bmatrix}$$

and that

$$\mathbf{f}_1 = \begin{bmatrix} \rho_1 u_1 \\ \rho_1 u_1^2 + p_1 \\ \rho_1 u_1 v_1 \\ \rho_1 u_1 H_1 \end{bmatrix} \qquad (\mathbf{u}_2 - \mathbf{u}_1) = \begin{bmatrix} \rho_2 - \rho_1 \\ \rho_2 u_2 - \rho_1 u_1 \\ \rho_2 v_2 - \rho_1 v_1 \\ E_2 - E_1 \end{bmatrix}$$

where $q^2 = u^2 + v^2, H = \frac{p+E}{\rho}$ and $p = (\gamma-1)\left(E - \frac{1}{2}\rho q^2\right)$. This leads to

$$E_2 - E_1 = \frac{p_2 - p_1}{\gamma - 1} + \frac{1}{2}(\rho_2 q_2^2 - \rho_1 q_1^2)$$

First off, again, we get that

$$\mathbf{f}_2^{*,L} = \mathbf{f}_{2,1D}^{*,L} + \begin{bmatrix}
0 \\
\frac{\gamma-1}{2}\bar{v}_2^2(\rho_2 - \rho_1) + (1-\gamma)\bar{v}_2(\rho_2 v_2 - \rho_1 v_1) + \frac{\gamma-1}{2}(\rho_2 v_2^2 - \rho_1 v_1^2) \\
-\bar{u}_2\bar{v}_2(\rho_2 - \rho_1) + \bar{v}_2(\rho_2 u_2 - \rho_1 u_1) + \bar{u}_2(\rho_2 v_2 - \rho_1 v_1) \\
\frac{\gamma-2}{2}\bar{u}\bar{v}_2^2(\rho_2 - \rho_1) + \frac{3-2\gamma}{2}\bar{v}_2^2(\rho_2 u_2 - \rho_1 u_1) + (1-\gamma)\bar{u}\bar{v}(\rho_2 v_2 - \rho_1 v_1) + \frac{\gamma}{2}\bar{u}_2(\rho_2 v_2^2 - \rho_1 v_1^2)
\end{bmatrix}$$