Moving Mesh Stuff

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1 Hyperbolic Relaxation Descritizations

Given the system with arbitrary parameters κ, τ and monitor function $w = w(x)$ as

$$x_t = \kappa p_\xi \tag{1}$$

$$p_t = \frac{w x_\xi - p}{\tau} \tag{2}$$

We wish to take one step of this system. In essence, we wish to advance as far towards steady state without recalculation of $w(x)$, since this is really $w(\overline{\mathbf{u}}(x))$, where to get the cell averages on the new cells requires some sort of interpolation or other complex method. Here we have that p is centered in the cells where x is on the nodes. Here we will use the general descritization, using k, m as parameters in our method.

$$x_j^{n+1} = x_j^n + m(x_{j-1} - 2x_j + x_{j+1}) + \kappa \frac{\Delta t}{\Delta \xi}(p^n_{j+\frac{1}{2}} - p^n_{j-\frac{1}{2}}) \tag{3}$$

$$p_j^{n+1} = p_j^n + k(p_{j-1} - 2p_j + p_{j+1}) + \frac{\Delta t}{\Delta \xi}\left(\frac{w_j^n}{\tau}(x^n_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}) - p_j^n\right) \tag{4}$$

Use $k, m = \frac{1}{2}$, corresponding to Lax-Friedrichs. Given $x^0, p^0 = w^0(\Delta x^0)$, perform one update to get both x^1, p^1 and use x^1, p^1 to get x^2, since getting p^2 requires recalculating w^1 and we want to avoid this. Choose k, m to provide the best stability. We get an equation for g as

$$(g - 1 + 4k\sin^2\frac{\theta}{2} + \sigma)(g - 1 + 4m\sin^2\frac{\theta}{2}) + 4\nu^2\sin^2\frac{\theta}{2} = 0 \tag{5}$$

$$(g-1)^2 + \left(4(k+m)\sin^2\frac{\theta}{2} + \sigma\right)(g-1) + 4\nu^2\sin^2\frac{\theta}{2} + 4m\sin^2\frac{\theta}{2}\left(4k\sin^2\frac{\theta}{2} + \sigma\right) = 0 \tag{6}$$

$$g_{1,2} = 1 - 2(k+m)\sin^2\frac{\theta}{2} - \frac{\sigma}{2} \pm 2\sqrt{\left((k+m)\sin^2\frac{\theta}{2} + \sigma\right)^2 - \sin^2\frac{\theta}{2}\left(\nu^2 + \sigma m + 4km\sin^2\frac{\theta}{2}\right)} \tag{7}$$

2 Hyperbolic Relaxation

Given the system with arbitrary parameters κ, τ and monitor function $w = w(x)$ as

$$x_t = \kappa p_\xi \tag{8}$$

$$p_t = \frac{w x_\xi - p}{\tau} \tag{9}$$

which we can write as

$$\begin{bmatrix} x \\ p \end{bmatrix}_t = \begin{bmatrix} 0 & \kappa \\ \frac{w}{\tau} & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}_\xi - \begin{bmatrix} 0 \\ \frac{p}{\tau} \end{bmatrix} \tag{10}$$

Note that at steady state as $x_t, p_t \to 0$ the system reduces to $p_\xi = 0, p = \frac{w}{\tau} x_\xi$ giving $(w x_\xi)_\xi$, the initial elliptic equation. This system *does not appear to be in conservation form.* The system can also be written as

$$\mathbf{u}_t = \mathbf{A}\mathbf{u}_\xi - \mathbf{g}(\mathbf{u}) \tag{11}$$

with decomposition $\mathbf{A} = R\Lambda R^{-1}$

$$\Lambda = \begin{bmatrix} \sqrt{\frac{\kappa w}{\tau}} & 0 \\ 0 & -\sqrt{\frac{\kappa w}{\tau}} \end{bmatrix} \tag{12}$$

$$R = \begin{bmatrix} \sqrt{\frac{\kappa \tau}{w}} & -\sqrt{\frac{\kappa \tau}{w}} \\ 1 & 1 \end{bmatrix} \tag{13}$$

$$R^{-1} = \frac{1}{2} \begin{bmatrix} \sqrt{\frac{w}{\kappa \tau}} & 1 \\ -\sqrt{\frac{w}{\kappa \tau}} & 1 \end{bmatrix} \tag{14}$$

where $\kappa, \tau, w > 0$. This can be upwinded for stability using centered differences as

$$x_i^{n+1} = x_i + \frac{\kappa \Delta t}{2}(p_{i+1} - p_{i-1}) \tag{15}$$

$$p_i^{n+1} = p_i + \frac{\Delta t}{\tau}\left(\frac{w_i}{2}(x_{i+1} - x_{i-1}) - p_i\right) \tag{16}$$

NB One of the nice things about this is that I can leave the entropy distance as defined on edges, with no need to average or do funny things as defined below in Section 2.

2.1 Initial Conditions

Initially, x is prescribed based on the mesh and p can come from the steady state solution as

$$p(x, 0) = \frac{w}{\tau} x_\xi \tag{17}$$

NB: I'm not sure if I should calculate x_ξ using a centered difference as $\frac{x_{i+1} - x_{i-1}}{2}$ (which seems physically intuitive since there should be no bias in steady state), or I should use a one-sided difference on the grounds that the governing equations are wavelike and I should accommodate for some propagation direction.
NB: Is there anything wrong with choosing $\kappa = 1$ always?

2.2 Boundary Conditions

Defining $\mathbf{v} = R^{-1}u$ reduces the system to

$$\mathbf{v}_t = \Lambda \mathbf{v}_\xi - R^{-1}\mathbf{g} \tag{18}$$

for $R^{-1}\mathbf{g} = [p/2\tau, p/2\tau]^T$. Since we have one outgoing wave at each boundary, we can specify one variable, in this case x to be known, and solve for the other, p. I believe this gives the left (top) and right (bottom) as

$$p_B - p_I \pm \sqrt{\frac{w}{\kappa\tau}}(x_B - x_I) = -\frac{1}{2\tau}(p_B + p_I) \tag{19}$$

for an interior point I and boundary point B. Solving this gives

$$p_B = \left(1 + \frac{1}{2\tau}\right)^{-1} \left(\left(1 - \frac{1}{2\tau}\right) p_I \mp \sqrt{\frac{w}{\kappa\tau}}(x_B - x_I)\right) \tag{20}$$

I can simplify this further by noting that on the left, $x_B < x_I$ and on the right, $x_B > x_I$ such that this becomes

$$p_B = \left(1 + \frac{1}{2\tau}\right)^{-1} \left(\left(1 - \frac{1}{2\tau}\right) p_I + \sqrt{\frac{w}{\kappa\tau}}\Delta x_B\right) \tag{21}$$

It should be noted at steady state, $p = wx_\xi > 0$ $(w > 0, x$ is monotonically increasing), so this may restrict the choice of τ.

NB: Suppose we have a constant monitor function such that, without loss of generality, $w(x) = 1, \tau = 1$ and a uniformly spaced grid $\Delta x =$constant. The steady state solution is that $p = \frac{w}{\tau}x_\xi = 1$. In our boundary condition, though, this reduces to $p_B = \frac{2}{3}\left(\frac{1}{2}p_I + \frac{1}{\kappa}\Delta x\right)$ which is only exact for the right choice of κ, else there is an error.

My tendency would be to use $p_B = 2p_{B-1} - p_{B-2}$ or something to that extent.

3 Entropy Distance Definition

In 1D, the entropy distance I use is defined across a cell interface as

$$D_{i-1/2} = (\mathbf{v}_i - \mathbf{v}_{i-1})^T(\mathbf{u}_i - \mathbf{u}_{i-1}) \tag{22}$$

where we can associate that distance with the interface located at $x_{i-1/2}$. A different question is what is the entropy distance associated in a particular cell. In my work with moving meshes, I complete this by averaging as

$$D_i = \frac{1}{2}(D_{i-1/2} + D_{i+1/2}) \tag{23}$$

This further reduces to, dropping the transpose for notational simplicity, that $\mathbf{ab} = \mathbf{a}^T\mathbf{b} = \mathbf{a}\cdot\mathbf{b}$

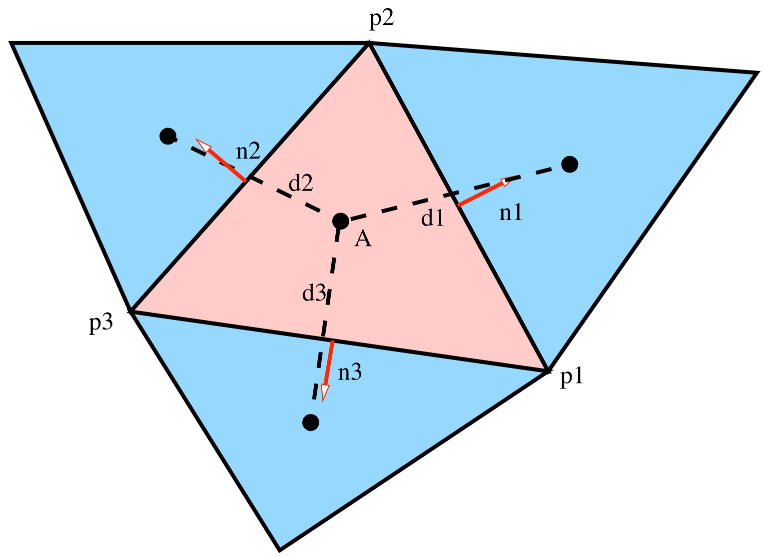
$$D_i = \frac{1}{2}(\mathbf{v}_i\mathbf{u}_i - \mathbf{v}_{i-1}\mathbf{u}_i - \mathbf{v}_i\mathbf{u}_{i-1} + \mathbf{v}_{i-1}\mathbf{u}_{i-1} + \mathbf{v}_{i+1}\mathbf{u}_{i+1} - \mathbf{v}_i\mathbf{u}_{i+1} - \mathbf{v}_{i+1}\mathbf{u}_i + \mathbf{v}_i\mathbf{u}_i) \tag{24}$$

which may or may not simplify to something more physically intuitive. An alternate view may to be to ask what happens if we take the differences between fluxes at the interface and evaluate (with some linearizations?)

$$D_{12} = (\mathbf{F}_{i+1/2} - \mathbf{F}_{i-1/2})^T(\mathbf{f}_{i+1/2} - \mathbf{f}_{i-1/2}) \tag{25}$$

where $\mathbf{f}_{i+1/2}$ is the solution to the conserved variable riemann problem and $\mathbf{F}_{i+1/2}$ is the solution to the entropy variable riemann problem.

4 2D Moving Meshes on Unstructured Grids



4.1 Conservation of Area

Define an initial cell with area A and moved cell with area $\underline{A}$. Define speeds $c^x = x - \underline{x}$ and $c^y = y - \underline{y}$, $\mathbf{c} = [c^x, c^y]^T$ and $c^n = c^x n_x + c^y n_y$ and proceed to write conservation as

$$\begin{aligned}
\int_{\underline{A}} \underline{u}(\underline{x}, \underline{y}) \mathrm{d}\underline{x}\mathrm{d}\underline{y} &= \int_A u(x - c^x, y - c^y) \det \left| \frac{\partial(\underline{x}, \underline{y})}{\partial(x, y)} \right| \mathrm{d}x\mathrm{d}y & (26) \\
&\approx \int_A (u(x, y) - c^x u_x - c^y u_y)(1 - c^x_x - c^y_y) \mathrm{d}x\mathrm{d}y & (27) \\
&\approx \int_A (u(x, y) - c^x u_x - c^y u_y - c^x_x u - c^y_y u) \mathrm{d}x\mathrm{d}y & (28) \\
&= \int_A u(x, y) \mathrm{d}x\mathrm{d}y - \int_A ((c^x u)_x + (c^y u)_y) \mathrm{d}x\mathrm{d}y & (29) \\
&= \int_A u(x, y) \mathrm{d}x\mathrm{d}y - \int_A \nabla \cdot (\mathbf{c}u) \mathrm{d}x\mathrm{d}y & (30) \\
&= \int_A u(x, y) \mathrm{d}x\mathrm{d}y - \oint_{\partial A} (\mathbf{c}u) \cdot d\mathbf{n} & (31) \\
&= \int_A u(x, y) \mathrm{d}x\mathrm{d}y - \sum_{i=1}^{3} (\mathbf{c}u) \cdot \mathbf{n}_i & (32) \\
&= \int_A u(x, y) \mathrm{d}x\mathrm{d}y - \sum_{i=1}^{3} (c^n_i u) & (33) \\
|\underline{A}|\underline{u} &= |A|u - \sum_{i=1}^{3} (c^n_i u_i) & (34)
\end{aligned}$$

for n_i, u_i evaluated at edge midpoints.

4.2 Mesh Redistribution equation

In two dimensions, we wish to solve for gridpoints $\mathbf{z} = (x, y)$ the equation

$$(w\mathbf{z}_\xi)_\xi + (w\mathbf{z}_\eta)_\eta = 0 \tag{35}$$

NB I'm not quite sure how this translates in the unstructured case where there is no global reference coordinate system of (ξ, η).

4.3 Hyperbolic Relaxation

We can now rewrite this as a system of 6 equations as

$$\begin{aligned}
\mathbf{z}_t &= \kappa(\mathbf{p}_\xi + \mathbf{q}_\eta) && (36)\\
\mathbf{p}_t &= \frac{w\mathbf{z}_\xi - \mathbf{p}}{\tau} && (37)\\
\mathbf{q}_t &= \frac{w\mathbf{z}_\eta - \mathbf{q}}{\tau} && (38)
\end{aligned}$$

NB Again, question of dealing with reference coordinate space.

4.4 Hancock ALE

Integrating around a control volume in 2D gives

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n \frac{|A|^n}{|A|^{n+1}} - \frac{\Delta t}{|A|^{n+1}} \left(\sum_{i=1}^{3} (\mathbf{f} - \dot{p_m}\mathbf{u}) \cdot \mathbf{n} \right) \tag{39}$$

where $\dot{p_m}$ is the net velocity of the midpoint, defined as

$$\dot{p_m} = \frac{1}{\Delta t} \left(\frac{1}{2} \sqrt{(x_1^{n+1} + x_2^{n+1})^2 + (y_1^{n+1} + y_2^{n+1})} - \frac{1}{2} \sqrt{(x_1^n + x_2^n)^2 + (y_1^n + y_2^n)} \right) \tag{40}$$

4.5 Monitor function

In 2D with triangular, there are three entropy distances arising from connecting the centroid of a particular element to its three neighbours. Each edge then has its own entropy distance (d1,d2,d3 in the figure). Some possible ideas

- Average the three equally
- Use a weighted average based on the inverse of the distance between two centroids (to put more emphasis on closer centroids). This is geometrical and may go against our goals.
- Use the max or min of the three

I can also envision an anisotropic form arising from using $(\Delta\mathbf{v} \cdot \Delta\mathbf{u})\mathbf{n} \cdot \vec{v}$, dotting the entropy distance in the normal to a face by the velocity across that face such that we get some sort of movement in the flow direction.

4.6 Slope Limiting

Slope limiting can be viewed as defining

$$
\begin{aligned}
u_{\max} &= \max(\overline{u}_i, \overline{u}_{FN_j}) && (41)\\
u_{\min} &= \min(\overline{u}_i, \overline{u}_{FN_j}) && (42)\\
u_{\min} &\leq u_i(x_G, y_G) \leq u_{\max} && (43)
\end{aligned}
$$

where $\overline{u}_i$ is the cell average and $\overline{u}_{FN_j}$ are the first neighbours and x_G, y_G are the locations of flux evaluation on the edges. An Unlimited reconstructed value at the Gauss point can be written as

$$u_G = u(x_c, y_c) + \nabla u \;\; \mathbf{r}_G \tag{44}$$

For my purposes, the Gauss points are just the edge midpoints. Limiting can be done as

$$u_G = u(x_c, y_c) + \phi \nabla u \;\; \mathbf{r}_G \tag{45}$$

for some limiter ϕ. Here is a basic choice:

$$\phi_{G_j} = \begin{cases} \min\left(1, \frac{u_{\max} - \overline{u}_i}{u_{G_j} - \overline{u}_i}\right) & \text{if } u_{G_j} - \overline{u}_i > 0 \\ \min\left(1, \frac{u_{\min} - \overline{u}_i}{u_{G_j} - \overline{u}_i}\right) & \text{if } u_{G_j} - \overline{u}_i < 0 \\ 1 & \text{if } u_{G_j} - \overline{u}_i = 0 \end{cases} \tag{46}$$

$$\phi_i = \min_j \phi_{G_j} \tag{47}$$

5 Analysis of hyperbolic relaxation scheme

Given the system as

$$
\begin{aligned}
x_j^{n+1} &= x_j^n + \kappa \frac{\Delta t}{\Delta \xi}(p_{j+\frac{1}{2}}^n - p_{j-\frac{1}{2}}^n) && (48)\\
p_j^{n+1} &= p_j^n \left(1 - \frac{\Delta t}{\tau}\right) + \frac{\Delta t}{\Delta \xi}\frac{w_j^n}{\tau}(x_{j+\frac{1}{2}}^n - x_{j-\frac{1}{2}}) && (49)
\end{aligned}
$$

Using Fourier Analysis, write this as

$$
\begin{aligned}
(g-1)x &= \kappa \frac{\Delta t}{\Delta \xi} 2i \sin\frac{\theta}{2} p && (50)\\
\left(g - 1 + \frac{\Delta t}{\tau}\right) p &= \frac{w}{\tau}\frac{\Delta t}{\Delta \xi} 2i \sin\frac{\theta}{2} x && (51)
\end{aligned}
$$

Take the determinant of the system of

$$
\det \begin{vmatrix} g-1 & 2i\kappa \frac{\Delta t}{\Delta \xi}\sin\frac{\theta}{2} \\ 2i\frac{w}{\tau}\frac{\Delta t}{\Delta \xi}\sin\frac{\theta}{2} & g - 1 + \frac{\Delta t}{\tau} \end{vmatrix} \tag{52}
$$

Define $\sigma = \frac{\Delta t}{\tau} > 0$ and $\nu = \sqrt{\frac{w\kappa}{\tau}}\frac{\Delta t}{\Delta xi} > 0$ and get the resulting equation obtained by setting the determinant to zero as

$$
(g-1)^2 + \sigma(g-1) + 4\nu^2 \sin^2\frac{\theta}{2} = 0 \tag{53}
$$

This is a quadratic equation with solution

$$
g - 1 = \frac{1}{2}\left[-\sigma \pm \sqrt{\sigma^2 - 16\nu^2 \sin^2\frac{\theta}{2}}\right] \tag{54}
$$

Phil Roe will argue that taking $\theta = 0$ implies that the negative root should be used, but am I not sold as easily on this idea. Rearranging this equation gives

$$
g_{1,2} = 1 - \frac{\sigma}{2}\left[1 \mp \sqrt{1 - 16\frac{\nu^2}{\sigma^2}\sin^2\frac{\theta}{2}}\right] \tag{55}
$$

For stability we need $|g_{1,2}| < 1$.

5.1 g_1

$$
g_1 = 1 - \frac{\sigma}{2}\left[1 + \sqrt{1 - 16\frac{\nu^2}{\sigma^2}\sin^2\frac{\theta}{2}}\right] \tag{56}
$$

5.1.1 Completely Real

In this case, $16\nu^2 < \sigma^2$ and $g_1 < 1$ always, so look for $g_1 > -1$

$$1 - \frac{\sigma}{2}\left(1 + \sqrt{}\right) > -1 \tag{57}$$

$$2 - \frac{\sigma}{2}\left(1 + \sqrt{}\right) > 0 \tag{58}$$

$$\frac{\sigma}{2}\left(1 + \sqrt{}\right) < 2 \tag{59}$$

$$\sqrt{} < \frac{4}{\sigma} - 1 \tag{60}$$

$$1 - 16\frac{\nu^2}{\sigma^2}\sin\frac{\theta}{2} < \frac{16}{\sigma^2} - \frac{8}{\sigma} + 1 \tag{61}$$

$$\nu^2 \sin\frac{\theta}{2} > \frac{\sigma}{2} - 1 \tag{62}$$

$$2(1 + \nu^2) > \sigma \tag{63}$$

This gives

$$4\nu < \sigma < 2(1 + \nu^2) \tag{64}$$

5.2 g_2

$$g_2 = 1 - \frac{\sigma}{2}\left[1 - \sqrt{1 - 16\frac{\nu^2}{\sigma^2}\sin^2\frac{\theta}{2}}\right] \tag{65}$$

5.2.1 Completely Real

To get $g_2 < 1$,

$$1 - \frac{\sigma}{2}\left[1 - \sqrt{}\right] < 1 \tag{66}$$

$$\frac{\sigma}{2}\left[1 - \sqrt{}\right] > 0 \tag{67}$$

$$1 - 16\frac{\nu^2}{\sigma^2}\sin^2\frac{\theta}{2} < 1 \tag{68}$$

$$\frac{\nu^2}{\sigma^2} > 0 \tag{69}$$

So this is fine. Now lets look at $g_2 > -1$

$$1 - \frac{\sigma}{2}\left[1 - \sqrt{}\right] > -1 \tag{70}$$

$$\frac{\sigma}{2}\left(1 - \sqrt{}\right) < 2 \tag{71}$$

$$\sqrt{} > 1 - \frac{4}{\sigma} \tag{72}$$

$$1 - 16\frac{\nu^2}{\sigma^2}\sin\frac{\theta}{2} > \frac{16}{\sigma^2} - \frac{8}{\sigma} + 1 \tag{73}$$

$$\frac{\sigma}{2} - 1 > \nu^2 \sin\frac{\theta}{2} \tag{74}$$

$$\sigma > 2(1 + \nu^2) \tag{75}$$

Which when combined with the first root makes this unstable. So...

5.3 Complex Case

If $16\nu^2 > \sigma^2$, the roots become complex and we get

$$g = 1 - \frac{\sigma}{2} - \frac{i}{2}\sqrt{16\nu^2 \sin^2\frac{\theta}{2} - \sigma^2} \tag{76}$$

$$|g|^2 = 1 - \sigma + 4\nu^2 \sin^2\frac{\theta}{2} \tag{77}$$

$$\sigma > 4\nu^2 \tag{78}$$