

Proposition 1. *Suppose that f is a integrable function on \mathbb{R} with $\|f\|_\infty < 1$. Then*

$$\int_{\mathbb{R}^2} \log [1 + f(x)f(y)] \, dx dy \geq \int_{\mathbb{R}^2} \log [1 - f(x)f(y)] \, dx dy.$$

Proof. Define

$$F(x, y) = \log [1 + f(x)f(y)] - \log [1 - f(x)f(y)].$$

Since $\|f\|_\infty < 1$, by Taylor expansion, we have

$$F(x, y) = 2 \sum_{k=0}^{\infty} \frac{[f(x)f(y)]^{2k+1}}{2k+1}.$$

Due to the fact that $\|f\|_\infty < 1$ and the integrability of f , we can integrate the above series term by term. The desired inequality is then a consequence of

$$\int_{\mathbb{R}^2} [f(x)f(y)]^{2k+1} \, dx dy = \left\{ \int_{\mathbb{R}} [f(x)]^{2k+1} \, dx \right\}^2 \geq 0 \quad \text{for each } k \geq 0.$$

It is obvious that the equality holds if and only if the above integral vanishes for each $k \geq 0$, which is equivalent to $f = 0$ almost everywhere. \square

Applying this proposition to a piecewise constant function with finite pieces, we obtain the following result.

Corollary 1. *Suppose that a_1, \dots, a_n are located in $(-1, 1)$. Then*

$$\prod_{1 \leq i, j \leq n} \frac{1 + a_i a_j}{1 - a_i a_j} \geq 1.$$

Question 1. *When do the equalities hold?*