Proposition 1. *Suppose that f is a integrable function on $\mathbb{R}$ with $\|f\|_\infty < 1$. Then*

$$\int_{\mathbb{R}^2} \log\left[1 + f(x)f(y)\right] \mathrm{d}x\mathrm{d}y \;\geq\; \int_{\mathbb{R}^2} \log\left[1 - f(x)f(y)\right] \mathrm{d}x\mathrm{d}y,$$

The equality holds if and only if $f = 0$ almost everywhere.

Proof. Define

$$F(x,y) \;=\; \log\left[1 + f(x)f(y)\right] - \log\left[1 - f(x)f(y)\right].$$

Since $\|f\|_\infty < 1$, by Taylor expansion, we have

$$F(x,y) \;=\; 2\sum_{k=0}^{\infty} \frac{[f(x)f(y)]^{2k+1}}{2k+1}.$$

Due to $\|f\|_\infty < 1$ and the integrability of f, we can integrate the above series term by term. The desired inequality is then a consequence of

$$\int_{\mathbb{R}^2} [f(x)f(y)]^{2k+1} \mathrm{d}x\mathrm{d}y \;=\; \left\{\int_{\mathbb{R}} [f(x)]^{2k+1} \mathrm{d}x\right\}^2 \;\geq\; 0 \qquad \text{for each} \qquad k \geq 0.$$

It is obvious that the equality holds if and only if the above integral vanishes for each $k \geq 0$, which is equivalent to $f = 0$ almost everywhere. □

Applying this proposition to a piecewise constant function with finite pieces, we obtain the following result.

Corollary 1. *Suppose that a_1, ..., a_n are located in $(-1, 1)$. Then*

$$\prod_{1 \leq i,j \leq n} \frac{1 + a_i a_j}{1 - a_i a_j} \;\geq\; 1.$$

The equality holds if and only if $a_1 = \cdots = a_n = 0$.