Notes on the Sobolev (Semi)Norms of Quadratic Functions

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Abstract

This paper studies the H^0 norm and H^1 seminorm of quadratic functions. The (semi)norms are expressed explicitly in terms of the coefficients of the quadratic function under consideration when the underlying domain is an ℓ_p-ball ($1 \le p \le \infty$) in $\mathbb{R}^n$.

Keywords: H^0 Norm · H^1 Seminorm · Quadratic Function · ℓ_p-Ball

1 Motivation and Introduction

Zhang [1] studies the H^1 sminorm of quadratic functions over ℓ_2-balls of $\mathbb{R}^n$, and applies it to derivative-free optimization problems. This paper will investigate the H^0 norm and H^1 seminorm of quadratic functions over ℓ_p-balls ($1 \le p \le \infty$) of $\mathbb{R}^n$. These (semi)norms may be useful for derivative-free optimization when considering trust region methods with ℓ_p trust region.

This paper is organized as follows. Section 2 describes the notations. Section 3 presents the formulae for the Sobolev (semi)norms of quadratic functions over ℓ_p-balls. The formulae are proved in Section 4. Section 5 contains some discussions. Some propositions about the Gamma function are given in Appendix for reference.

2 Notation

In this paper, the following symbols will be used unless otherwise specified.

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- Q is a quadratic function defined by

$$Q(x) = \frac{1}{2} x^{\mathrm{T}} B x + g^{\mathrm{T}} x + c, \quad x \in \mathbb{R}^n, \tag{2.1}$$

where $B \in \mathbb{R}^{n\times n}$ is a symmetric matrix, $g \in \mathbb{R}^n$ is a vector and $c \in \mathbb{R}$ is a scalar.

- $D \in \mathbb{R}^{n\times n}$ is the diagonal of B.

- $p \geq 1$ is a positive constant or $p = \infty$.

- $\|\cdot\|_p$ is the ℓ_p-norm on $\mathbb{R}^n$, and $\mathcal{B}_p^r$ is the ℓ_p-ball centering 0 with radius $r > 0$, i.e.,

$$\mathcal{B}_p^r = \{x \in \mathbb{R}^n;\ \|x\|_p \leq r\}. \tag{2.2}$$

Besides, V_p is the volume of $\mathcal{B}_p^1$.

- Given $x \in \mathbb{R}^n$ and $i \in \{1,\ 2,\ ...,\ n\}$, x_i is the i-th coordinate of x.

- $\Gamma(\cdot)$ is the Gamma function, and $\beta(\cdot,\cdot)$ is the Beta function [2].

- γ is the Euler constant.

- The following symbols are used for the Sobolev (semi)norms [3] of a function f over a domain $\Omega \in \mathbb{R}^n$:

$$\|f\|_{H^0(\Omega)} \equiv \left[\int_\Omega |f(x)|^2 \,\mathrm{d}x\right]^{1/2}, \tag{2.3}$$

$$|f|_{H^1(\Omega)} \equiv \left[\int_\Omega \|\nabla f(x)\|_2^2 \,\mathrm{d}x\right]^{1/2}. \tag{2.4}$$

3 Formulae for the Sobolev (Semi)Norms

The H^0 norm and H^1 seminorm of Q over $\mathcal{B}_p^r$ can be expressed explicitly in terms of its coefficients B, g and c. We present the formulae in Theorem 3.1. Notice that the case $p = \infty$ can also be covered by these formulae, as will be pointed out in Remark 3.2. The proof of Theorem 3.1 is in Section 4.

Theorem 3.1.

$$\|Q\|_{H^0(\mathcal{B}_p^r)}^2 = \frac{I}{2}\|B\|_{\mathrm{F}}^2 + \frac{I}{4}\mathrm{Tr}^2 B + \frac{J-3I}{4}\|D\|_{\mathrm{F}}^2 + K(c\,\mathrm{Tr}B + \|g\|_2^2) + Vc^2, \tag{3.1}$$

and

$$|Q|^2_{H^1(\mathcal{B}^r_p)} = K\|B\|^2_{\mathrm{F}} + V\|g\|^2, \tag{3.2}$$

where

$$I = \frac{[\Gamma(\frac{3}{p})]^2\Gamma(\frac{n}{p})}{[\Gamma(\frac{1}{p})]^2\Gamma(\frac{n+4}{p})} \cdot \frac{n}{n+4} \cdot V_p\, r^{n+4}, \tag{3.3}$$

$$J = \frac{\Gamma(\frac{5}{p})\Gamma(\frac{n}{p})}{\Gamma(\frac{1}{p})\Gamma(\frac{n+4}{p})} \cdot \frac{n}{n+4} \cdot V_p\, r^{n+4}, \tag{3.4}$$

$$K = \frac{\Gamma(\frac{3}{p})\Gamma(\frac{n}{p})}{\Gamma(\frac{1}{p})\Gamma(\frac{n+2}{p})} \cdot \frac{n}{n+2} \cdot V_p\, r^{n+2}, \tag{3.5}$$

$$V = V_p\, r^n. \tag{3.6}$$

Remark 3.2. *Theorem 3.1 covers the case $p = \infty$ as well. When $p = \infty$, we interpret (3.3 – 3.5) in the sense of limit, i.e.,*

$$I = \lim_{p\to\infty} \frac{[\Gamma(\frac{3}{p})]^2\Gamma(\frac{n}{p})}{[\Gamma(\frac{1}{p})]^2\Gamma(\frac{n+4}{p})} \cdot \frac{n}{n+4} \cdot V_\infty\, r^{n+4}, \tag{3.3$'$}$$

$$J = \lim_{p\to\infty} \frac{\Gamma(\frac{5}{p})\Gamma(\frac{n}{p})}{\Gamma(\frac{1}{p})\Gamma(\frac{n+4}{p})} \cdot \frac{n}{n+4} \cdot V_\infty\, r^{n+4}, \tag{3.4$'$}$$

$$K = \lim_{p\to\infty} \frac{\Gamma(\frac{3}{p})\Gamma(\frac{n}{p})}{\Gamma(\frac{1}{p})\Gamma(\frac{n+2}{p})} \cdot \frac{n}{n+2} \cdot V_\infty\, r^{n+2}. \tag{3.5$'$}$$

The limits above can be calculated easily with the help of proposition A.1.

As illustrations of Theorem 3.1, we present the (semi)norms with $p = 2$ as follows.

Corollary 3.3.

$$\|Q\|^2_{H^0(\mathcal{B}^r_2)} = V_2\, r^n \left[\frac{r^4\left(2\|B\|^2_{\mathrm{F}} + \mathrm{Tr}^2 B\right)}{4(n+2)(n+4)} + \frac{r^2(c\mathrm{Tr}B + \|g\|^2_2)}{n+2} + c^2\right]; \tag{3.6}$$

$$|Q|^2_{H^1(\mathcal{B}^r_2)} = V_2\, r^n \left[\frac{r^2}{n+2}\|B\|^2_{\mathrm{F}} + \|g\|^2_2\right]. \tag{3.7}$$

Notice that the formula (3.7) has been proved in Zhang [1].

4 Proofs of Main Results

We assume $n \geq 2$ henceforth, because everything is trivial when $n = 1$.

4.1 Lemmas

For Simplicity, we first prove some lemmas.

First we investigate the integrals of some monomials, which will be presented in the following two lemmas.

Lemma 4.1. *Suppose* $1 \le i, j, k, l \le n$.

a. *The integrals of* x_i *and* $x_i x_j x_k$ *over* $\mathcal{B}_p^r$ *are* 0.

b. *The integrals of* $x_i x_j$ *and* $x_i x_j x_k^2$ *over* $\mathcal{B}_p^r$ *are* 0*, provided that* $i \neq j$.

c. *The integral of* $x_i x_j x_k x_l$ *over* $\mathcal{B}_p^r$ *is* 0*, provided that* i*,* j *and* k *are pairwise different.*

Lemma 4.1 is trivial so we omit the proof.

Lemma 4.2. *Suppose* $1 \le i < j \le n$*, and* k_1*,* k_2 *are even natural numbers. Then*

$$\int_{\mathcal{B}_p^r} x_i^{k_1} x_j^{k_2} \,\mathrm{d}x = \frac{\Gamma(\frac{k_1+1}{p})\Gamma(\frac{k_2+1}{p})\Gamma(\frac{n}{p})}{\Gamma(\frac{1}{p})\Gamma(\frac{1}{p})\Gamma(\frac{n+k_1+k_2}{p})} \cdot \frac{n}{n+k_1+k_2} \cdot V_p\, r^{n+k_1+k_2}. \tag{4.1}$$

Remark 4.3. *Lemma 4.2 covers the case* $p = \infty$ *as well. When* $p = \infty$*, we interpret (4.1) in the sense of limit, i.e.,*

$$\int_{\mathcal{B}_\infty^r} x_i^{k_1} x_j^{k_2} \,\mathrm{d}x = \lim_{p\to\infty} \frac{\Gamma(\frac{k_1+1}{p})\Gamma(\frac{k_2+1}{p})\Gamma(\frac{n}{p})}{\Gamma(\frac{1}{p})\Gamma(\frac{1}{p})\Gamma(\frac{n+k_1+k_2}{p})} \cdot \frac{n}{n+k_1+k_2} \cdot V_\infty\, r^{n+k_1+k_2}. \tag{4.1$'$}$$

Proof. Without loss of generality, we assume $r = 1$. Suppose additionally $n \ge 3$ since things are trivial if $n = 2$. Denote the volume of the unit ℓ_p-ball in $\mathbb{R}^{n-2}$ by $V_{p,n-2}$.

We fist justify (4.1) for $p \in [1, \infty)$, and then show its validity for $p = \infty$ in the sense of (4.1′).

When $p \in [1, \infty)$,

$$\begin{aligned}
&\int_{B_p^1} x_i^{k_1} x_j^{k_2} \,\mathrm{d}x && (4.2)\\
=\ &\int_{|u|^p+|v|^p\le 1} u^{k_1} v^{k_2} \,\mathrm{d}u\,\mathrm{d}v \int_{w\in\mathbb{R}^{n-2}, \|w\|_p \le (1-|u|^p-|v|^p)^{\frac{1}{p}}} \mathrm{d}w && (4.3)\\
=\ &V_{p,n-2} \int_{|u|^p+|v|^p\le 1} u^{k_1} v^{k_2} (1-|u|^p-|v|^p)^{\frac{n-2}{p}} \,\mathrm{d}u\,\mathrm{d}v && (4.4)\\
=\ &4V_{p,n-2} \int_{u^p+v^p\le 1,\ u,v\ge 0} u^{k_1} v^{k_2} (1-u^p-v^p)^{\frac{n-2}{p}} \,\mathrm{d}u\,\mathrm{d}v. && (4.5)
\end{aligned}$$

Consider the transformation

$$u^{\frac{p}{2}} = \rho \cos\theta, \tag{4.6}$$

$$v^{\frac{p}{2}} = \rho \sin\theta. \tag{4.7}$$

Then we have

$$\begin{aligned}
&\int_{B_p^1} x_i^{k_1} x_j^{k_2}\,\mathrm{d}x && (4.8)\\
= \;& \frac{16}{p^2} V_{p,n-2} \int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{2k_1+2}{p}-1} (\sin\theta)^{\frac{2k_2+2}{p}-1}\,\mathrm{d}\theta \int_0^1 \rho^{\frac{2k_1+2k_2+4}{p}-1} (1-\rho^2)^{\frac{n-2}{p}}\,\mathrm{d}\rho && (4.9)\\
= \;& \frac{4}{p^2} V_{p,n-2}\, \beta\Big(\frac{k_1+1}{p}, \frac{k_2+1}{p}\Big)\, \beta\Big(\frac{k_1+k_2+2}{p}, \frac{n-2}{p}+1\Big). && (4.10)
\end{aligned}$$

By setting k_1 and k_2 to 0, we obtain

$$V_p = \frac{4}{p^2} V_{p,n-2}\, \beta\Big(\frac{1}{p}, \frac{1}{p}\Big)\, \beta\Big(\frac{2}{p}, \frac{n-2}{p}+1\Big). \tag{4.11}$$

Hence

$$\begin{aligned}
&\int_{B_p^1} x_i^{k_1} x_j^{k_2}\,\mathrm{d}x && (4.12)\\
= \;& \frac{\beta\Big(\frac{k_1+1}{p}, \frac{k_2+1}{p}\Big)\, \beta\Big(\frac{k_1+k_2+2}{p}, \frac{n-2}{p}+1\Big)}{\beta\Big(\frac{1}{p}, \frac{1}{p}\Big)\, \beta\Big(\frac{2}{p}, \frac{n-2}{p}+1\Big)} V_p && (4.13)\\
= \;& \frac{\Gamma\Big(\frac{k_1+1}{p}\Big)\Gamma\Big(\frac{k_2+1}{p}\Big)\Gamma\Big(\frac{k_1+k_2+2}{p}\Big)\Gamma\Big(\frac{n-2}{p}+1\Big)\Gamma\Big(\frac{2}{p}\Big)\Gamma\Big(\frac{n}{p}+1\Big)}{\Gamma\Big(\frac{1}{p}\Big)\Gamma\Big(\frac{1}{p}\Big)\Gamma\Big(\frac{2}{p}\Big)\Gamma\Big(\frac{n-2}{p}+1\Big)\Gamma\Big(\frac{k_1+k_2+2}{p}\Big)\Gamma\Big(\frac{n+k_1+k_2}{p}+1\Big)} V_p && (4.14)\\
= \;& \frac{\Gamma\Big(\frac{k_1+1}{p}\Big)\Gamma\Big(\frac{k_2+1}{p}\Big)\Gamma\Big(\frac{n}{p}\Big)}{\Gamma\Big(\frac{1}{p}\Big)\Gamma\Big(\frac{1}{p}\Big)\Gamma\Big(\frac{n+k_1+k_2}{p}\Big)} \cdot \frac{n}{n+k_1+k_2} V_p. && (4.15)
\end{aligned}$$

Thus (4.1) holds for finite p.

Now consider infinite p. According to Lebesgue's Dominated Convergence Theorem, we have

$$\int_{\mathcal{B}_p^1} x_i^{k_1} x_j^{k_2}\,\mathrm{d}x \to \int_{\mathcal{B}_\infty^1} x_i^{k_1} x_j^{k_2}\,\mathrm{d}x \quad \text{and} \quad V_p \to V_\infty \quad (p \to \infty). \tag{4.16}$$

which together with (4.1) implies the convergence of $\frac{\Gamma\left(\frac{k_1+1}{p}\right)\Gamma\left(\frac{k_2+1}{p}\right)\Gamma\left(\frac{n}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{n+k_1+k_2}{p}\right)}$ when p tends to infinity and the validity of (4.1′). Thus (4.1) holds for infinite p in the sense of limit. ∎

Remark 4.4. *Via straightforward calculus, we can show that*

$$\int_{\mathcal{B}_\infty^r} x_i^{k_1} x_j^{k_2} \,\mathrm{d}x = \frac{V_\infty \, r^{n+k_1+k_2}}{(k_1+1)(k_2+1)}, \tag{4.17}$$

which is the same with (4.1′) according to Proposition A.1.

Now consider some more integrals which will be used in the computation of the (semi)norms.

Lemma 4.5. *Denote* $\int_{\mathcal{B}_p^r} x_1^2 x_2^2 \,\mathrm{d}x$, $\int_{\mathcal{B}_p^r} x_1^4 \,\mathrm{d}x$ *and* $\int_{\mathcal{B}_p^r} x_1^2 \,\mathrm{d}x$ *by* I, J *and* K*. Then we have*

a.

$$\int_{\mathcal{B}_p^r} g^{\mathrm{T}} x \,\mathrm{d}x = 0; \tag{4.18}$$

$$\int_{\mathcal{B}_p^r} g^{\mathrm{T}} B x \,\mathrm{d}x = 0; \tag{4.19}$$

$$\int_{\mathcal{B}_p^r} (g^{\mathrm{T}} x)(x^{\mathrm{T}} B x) \,\mathrm{d}x = 0. \tag{4.20}$$

b.

$$\int_{\mathcal{B}_p^r} x^{\mathrm{T}} B x \,\mathrm{d}x = K \mathrm{Tr} B; \tag{4.21}$$

$$\int_{\mathcal{B}_p^r} x^{\mathrm{T}} B^2 x \,\mathrm{d}x = K \|B\|_{\mathrm{F}}^2; \tag{4.22}$$

$$\int_{\mathcal{B}_p^r} (x^{\mathrm{T}} B x)^2 \,\mathrm{d}x = I(2\|B\|_{\mathrm{F}}^2 + \mathrm{Tr}^2 B) + (J-3I)\|D\|_{\mathrm{F}}^2; \tag{4.23}$$

$$\int_{\mathcal{B}_p^r} (g^{\mathrm{T}} x)^2 \,\mathrm{d}x = K \|g\|_2^2. \tag{4.24}$$

Proof. (4.18 – 4.20) follow directly from Lemma 4.1. As for (4.21 – 4.24), we only verify (4.23) as an example, because the others are similar and much easier.

Denote the (i,j) entry of B by B_{ij}. According to Lemma 4.1,

$$\begin{aligned}
&\int_{\mathcal{B}_p^r} (x^{\mathrm{T}} B x)^2 \,\mathrm{d}x \\
={}& \int_{\mathcal{B}_p^r} \sum_{i,j,k,l} (x_i B_{ij} x_j)(x_k B_{kl} x_l) \,\mathrm{d}x \\
={}& \int_{\mathcal{B}_p^r} \left(\sum_i B_{ii}^2 x_i^4 + 2 \sum_{i \neq j} B_{ij}^2 x_i^2 x_j^2 + \sum_{i \neq j} B_{ii} B_{jj} x_i^2 x_j^2 \right) \,\mathrm{d}x \\
={}& J \|D\|_{\mathrm{F}}^2 + 2I(\|B\|_{\mathrm{F}}^2 - \|D\|_{\mathrm{F}}^2) + I(\mathrm{Tr}^2 B - \|D\|_{\mathrm{F}}^2) \\
={}& I(2\|B\|_{\mathrm{F}}^2 + \mathrm{Tr}^2 B) + (J - 3I)\|D\|_{\mathrm{F}}^2.
\end{aligned} \tag{4.25}$$

■

4.2 Proofs

Now we give the proofs of our main results.

With Lemma 4.2 and 4.5, the proof of Theorem 3.1 is nearly completed. We present it as follows.

Proof. According to Lemma 4.5,

$$\begin{aligned}
&\|Q\|_{H^0(\mathcal{B}_p^r)}^2 \\
={}& \int_{\mathcal{B}_p^r} \left[\frac{1}{2} x^{\mathrm{T}} B x + g^{\mathrm{T}} x + c \right]^2 \,\mathrm{d}x \\
={}& \int_{\mathcal{B}_p^r} \left[\frac{1}{4} (x^{\mathrm{T}} B x)^2 + (g^{\mathrm{T}} x)^2 + c x^{\mathrm{T}} B x + c^2 \right] \,\mathrm{d}x \\
={}& \frac{1}{4} I(2\|B\|_{\mathrm{F}}^2 + \mathrm{Tr}^2 B) + \frac{1}{4}(J - 3I)\|D\|_{\mathrm{F}}^2 \\
&+ K(\|g\|_2^2 + c\mathrm{Tr}B) + V_p r^n c^2,
\end{aligned} \tag{4.26}$$

and

$$\begin{aligned}
&|Q|_{H^1(\mathcal{B}_p^r)}^2 \\
={}& \int_{\mathcal{B}_p^r} \|Bx + g\|_2^2 \,\mathrm{d}x \\
={}& \int_{\mathcal{B}_p^r} \left[x^{\mathrm{T}} B^2 x + \|g\|_2^2 \right] \,\mathrm{d}x \\
={}& K\|B\|_{\mathrm{F}}^2 + V_p r^n \|g\|_2^2.
\end{aligned} \tag{4.27}$$

Now apply Lemma 4.2.

Since Lemma 4.2 holds for infinite p in the sense of limit, so does Theorem 3.1. ∎

From Theorem 3.1, Corollary 3.3 follows directly.

5 Discussions

5.1 The Invariance Under Orthogonal Transformations

Denote the space of all quadratic functions on $\mathbb{R}^n$ by $\mathcal{Q}$. A functional $\mathcal{F}$ on $\mathcal{Q}$ is said to be invariant under orthogonal transformations provided that

$$\mathcal{F}(Q \circ T) = \mathcal{F}(Q). \tag{5.1}$$

for any $Q \in \mathcal{Q}$ and any orthogonal transformation T on $\mathbb{R}^n$.

Proposition 5.1. *Consider* $\|\cdot\|_{H^0(\mathcal{B}_p^r)}$ *and* $|\cdot|_{H^1(\mathcal{B}_p^r)}$ *as functionals on* $\mathcal{Q}$*. Then*

a. $\|\cdot\|_{H^0(\mathcal{B}_p^r)}$ *is invariant under orthogonal transformations if and only if* $p = 2$*;*

b. $|\cdot|_{H^1(\mathcal{B}_p^r)}$ *is invariant under orthogonal transformations for any* p*.*

Proof. According to Theorem 3.1, $\|\cdot\|_{H^0(\mathcal{B}_p^r)}$ is invariant under orthogonal transformations if and only if

$$J = 3I, \tag{5.2}$$

since $\|B\|_{\mathrm{F}}$, $\mathrm{Tr}B$ and $\|g\|_2$ are invariant while $\|D\|_{\mathrm{F}}$ is not. (5.2) is equivalent to

$$\frac{\Gamma\left(\frac{5}{p}\right)\Gamma\left(\frac{1}{p}\right)}{\left[\Gamma\left(\frac{3}{p}\right)\right]^2} = 3, \tag{5.3}$$

which assumes at most one solution, since the left hand side of it is strictly decreasing for $p \in (0, \infty)$ according to Proposition A.2. Obviously $p = 2$ is a solution to (5.3), and hence the only one. Thus the invariance of $\|\cdot\|_{H^0(\mathcal{B}_p^r)}$ holds if and only if $p = 2$.

The invariance of $|\cdot|_{H^1(\mathcal{B}_p^r)}$ is easy to check in light of Theorem 3.1. ∎

5.2 The Sobolev (Semi)Norms over ℓ_p-Ellipsoids

By ℓ_p-ellipsoid we mean a set of the form

$$\{x \in \mathbb{R}^n;\ \|A(x - x_0)\|_p \leq r\}, \tag{5.4}$$

where $A \in \mathbb{R}^{n\times n}$ is a nonsingular matrix. The Sobolev (semi)norms of Q over an ℓ_p-ellipsoid can be deduced from Theorem 3.1 easily.

5.3 The Weighted Sobolev (Semi)Norms

It may be meaningful to consider weighted Sobolev (semi)norms defined by

$$\|Q\|_{H^0(\mathcal{B}_p^r,w)} = \left[\int_{\mathcal{B}_p^r} w(x)|f(x)|^2\,\mathrm{d}x\right]^{1/2}, \tag{5.5}$$

$$|Q|_{H^1(\mathcal{B}_p^r,w)} = \left[\int_{\mathcal{B}_p^r} w(x)\|\nabla f(x)\|_2^2\,\mathrm{d}x\right]^{1/2}, \tag{5.6}$$

where w is a nonnegative bounded function defined on $\mathcal{B}_p^r$, playing the role of a weight. They can be calculated via the procedure presented in Section 4, and the main labor is integrating related monomials with respect to the weight.

As an example, we show without proof $\|Q\|_{H^0(\mathcal{B}_2^r,w)}$ and $|Q|_{H^1(\mathcal{B}_2^r,w)}$ with the weight

$$w(x) = r^2 - \|x\|_2^2. \tag{5.7}$$

Proposition 5.2.

$$\|Q\|^2_{H^0(\mathcal{B}_2^r,w)} = \frac{2V_2\,r^{n+2}}{n+2}\left[\frac{r^4\left(2\|B\|_{\mathrm{F}}^2 + \mathrm{Tr}^2 B\right)}{4(n+4)(n+6)} + \frac{r^2(c\mathrm{Tr}B + \|g\|_2^2)}{n+4} + c^2\right]; \tag{5.8}$$

$$|Q|^2_{H^1(\mathcal{B}_2^r,w)} = \frac{2V_2\,r^{n+2}}{n+2}\left[\frac{r^2}{n+4}\|B\|_{\mathrm{F}}^2 + \|g\|_2^2\right]. \tag{5.9}$$

Appendix

A Some Propositions about the Gamma Function

Proposition A.1. *Suppose λ is a positive constant. It holds*

$$\lim_{t\to 0^+}\frac{\Gamma(t)}{\Gamma(\lambda t)} = \lambda. \tag{A.1}$$

Proof. According to the Weierstrass product of Gamma function [2], we have

$$\Gamma(t) = \frac{\mathrm{e}^{-\gamma t}}{t} \prod_{k=1}^{\infty} \left[\left(1 + \frac{t}{k}\right)^{-1} \mathrm{e}^{\frac{t}{k}} \right]. \tag{A.2}$$

Hence it suffices to show that

$$\lim_{t \to 0^+} \frac{\prod_{k=1}^{\infty}(1+\frac{t}{k})^{-1}\mathrm{e}^{\frac{t}{k}}}{\prod_{k=1}^{\infty}(1+\frac{\lambda t}{k})^{-1}\mathrm{e}^{\frac{\lambda t}{k}}} = 1, \tag{A.3}$$

which is true according to the following deduction.

When $t \in (0, \frac{1}{2(1+|1-\lambda|)})$,

$$\begin{aligned}
&\left| \log \frac{\prod_{k=1}^{\infty}(1+\frac{t}{k})^{-1}\mathrm{e}^{\frac{t}{k}}}{\prod_{k=1}^{\infty}(1+\frac{\lambda t}{k})^{-1}\mathrm{e}^{\frac{\lambda t}{k}}} \right| \\
=& \left| \sum_{k=1}^{\infty} \left[\frac{t}{k} - \log\left(1+\frac{t}{k}\right) \right] - \sum_{k=1}^{\infty} \left[\frac{\lambda t}{k} - \log\left(1+\frac{\lambda t}{k}\right) \right] \right| \\
\leq& \sum_{k=1}^{\infty} \left| \frac{(1-\lambda)t}{k} - \log\left[1+\frac{(1-\lambda)t}{k+\lambda t}\right] \right| \\
=& \sum_{k=1}^{\infty} \left| \frac{(1-\lambda)t}{k} - \frac{(1-\lambda)t}{k+\lambda t} + \frac{1}{2(1+\xi_k)^2} \cdot \frac{(1-\lambda)^2 t^2}{(k+\lambda t)^2} \right| \\
\leq& \sum_{k=1}^{\infty} \left[\frac{|1-\lambda|\lambda t^2}{k^2} + \frac{(1-\lambda)^2 t^2}{2(1/2)^2 k^2} \right] \\
=& \frac{\pi^2}{6}(\lambda + 2|1-\lambda|)|1-\lambda| t^2,
\end{aligned} \tag{A.4}$$

where ξ_k is due to Taylor expansion with Lagrange remainder, and $\xi_k \geq -\frac{1}{2}$ when $t \in (0, \frac{1}{2(1+|1-\lambda|)})$. ■

Proposition A.2. *Suppose λ and μ are positive constants satisfying $\lambda + \mu \geq 2$. Then the function*

$$\phi(t) = \frac{\Gamma(\lambda t)\Gamma(\mu t)}{[\Gamma(t)]^{\lambda+\mu}} \tag{A.5}$$

is monotonically increasing over $(0, \infty)$, and the monotonicity is strict unless $\lambda = 1 = \mu$.

Proof. We suppose $\lambda = 1 = \mu$ does not happen and prove the strict monotonicity for $\log \phi(t)$ by inspecting its derivative. To do this, consider the *digamma* function

$$\psi(t) = \frac{\mathrm{d}}{\mathrm{d}t} \log \Gamma(t) \tag{A.6}$$

and its partial fraction expansion [2]

$$\psi(t) = -\gamma - \frac{1}{t} + \sum_{k=1}^{\infty} \frac{t}{k(t+k)}. \tag{A.7}$$

For any $t > 0$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \log \phi(t) \\
&= \lambda\psi(\lambda t) + \mu\psi(\mu t) - (\lambda+\mu)\psi(t) \\
&= \lambda \left[-\gamma - \frac{1}{\lambda t} + \sum_{k=1}^{\infty} \frac{\lambda t}{k(\lambda t + k)} \right] + \mu \left[-\gamma - \frac{1}{\mu t} + \sum_{k=1}^{\infty} \frac{\mu t}{k(\mu t + k)} \right] \\
&\quad - (\lambda+\mu) \left[-\gamma - \frac{1}{t} + \sum_{k=1}^{\infty} \frac{t}{k(t+k)} \right] \\
&= \frac{1}{t}(\lambda+\mu-2) + \sum_{k=1}^{\infty} \frac{t}{k} \left(\frac{\lambda^2}{\lambda t + k} + \frac{\mu^2}{\mu t + k} - \frac{\lambda+\mu}{t+k} \right) \\
&= \frac{1}{t}(\lambda+\mu-2) + \sum_{k=1}^{\infty} \frac{kt[(\lambda^2+\mu^2) - (\lambda+\mu)] + t^2\lambda\mu(\lambda+\mu-2)}{(\lambda t + k)(\mu t + k)(t+k)} \\
&> 0,
\end{aligned} \tag{A.8}$$

the last inequality being true because

$$\lambda^2 + \mu^2 > \lambda + \mu \tag{A.9}$$

due to our assumptions on λ and μ. ■

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