Review: Search problem formulation

- Initial state
- Actions
- Transition model
- Goal state
- Path cost

- What is the optimal solution?
- What is the state space?

Review: Tree search

- Initialize the **fringe** using the **starting state**
- While the fringe is not empty
  - Choose a fringe node to expand according to **search strategy**
  - If the node contains the **goal state**, return solution
  - Else **expand** the node and add its children to the fringe

Search strategies

- A **search strategy** is defined by picking the order of node expansion
- Strategies are evaluated along the following dimensions:
  - **Completeness:** does it always find a solution if one exists?
  - **Optimality:** does it always find a least-cost solution?
  - **Time complexity:** number of nodes generated
  - **Space complexity:** maximum number of nodes in memory
- Time and space complexity are measured in terms of
  - *b:* maximum branching factor of the search tree
  - *d:* depth of the optimal solution
  - *m*: maximum length of any path in the state space (may be infinite)

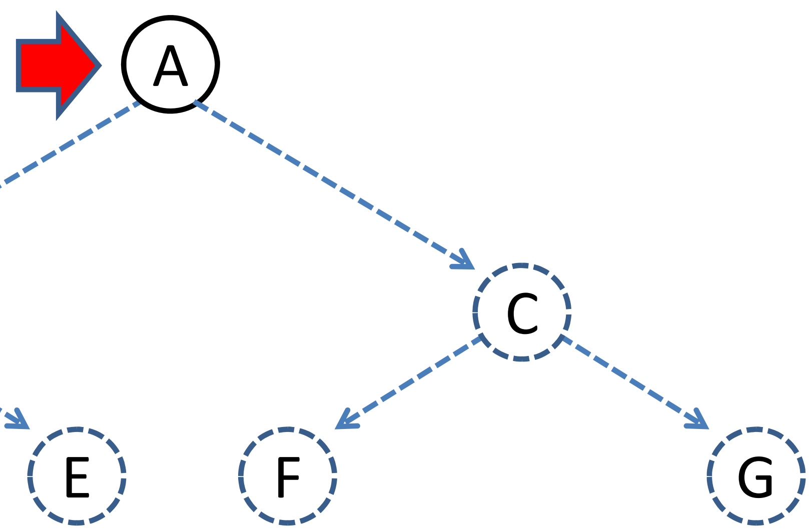
Uninformed search strategies

- Uninformed search strategies use only the information available in the problem definition

- Breadth-first search
- Uniform-cost search
- Depth-first search
- Iterative deepening search

Breadth-first search

- Expand shallowest unexpanded node
- Implementation:
  - *fringe* is a FIFO queue, i.e., new successors go at end



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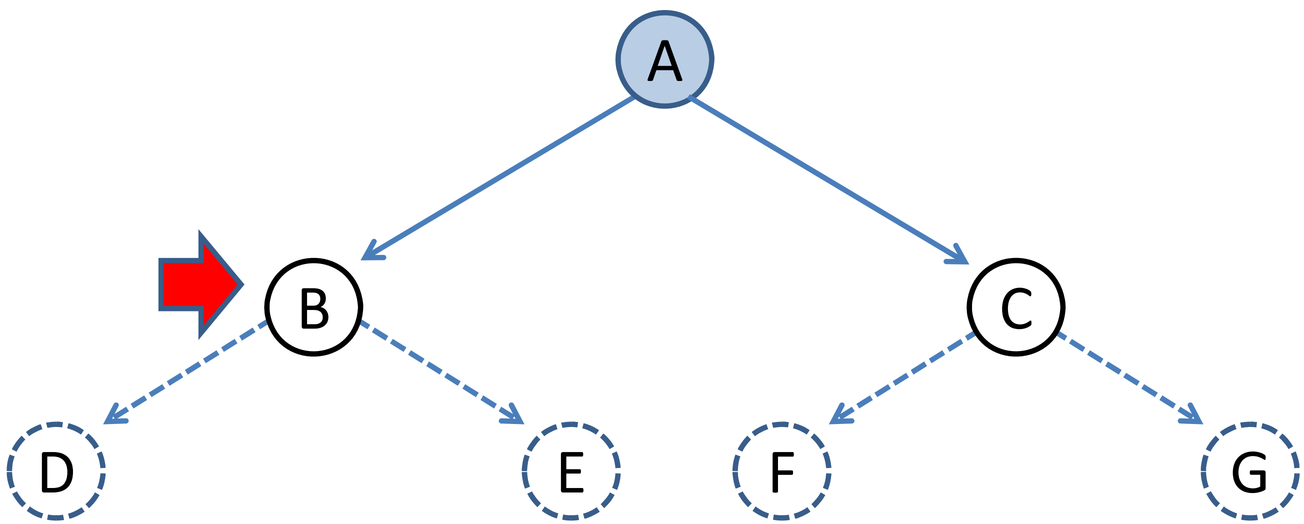

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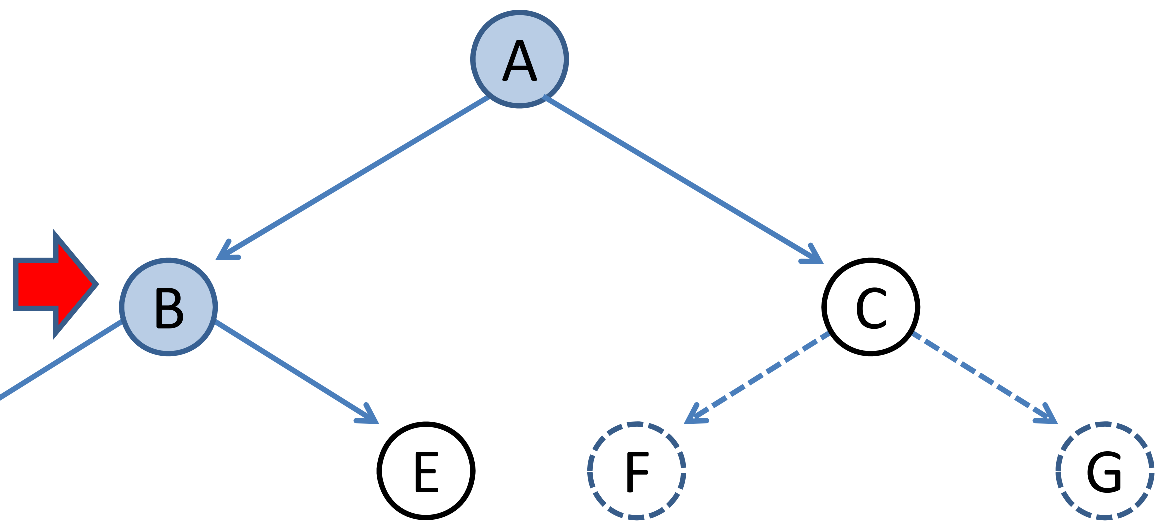

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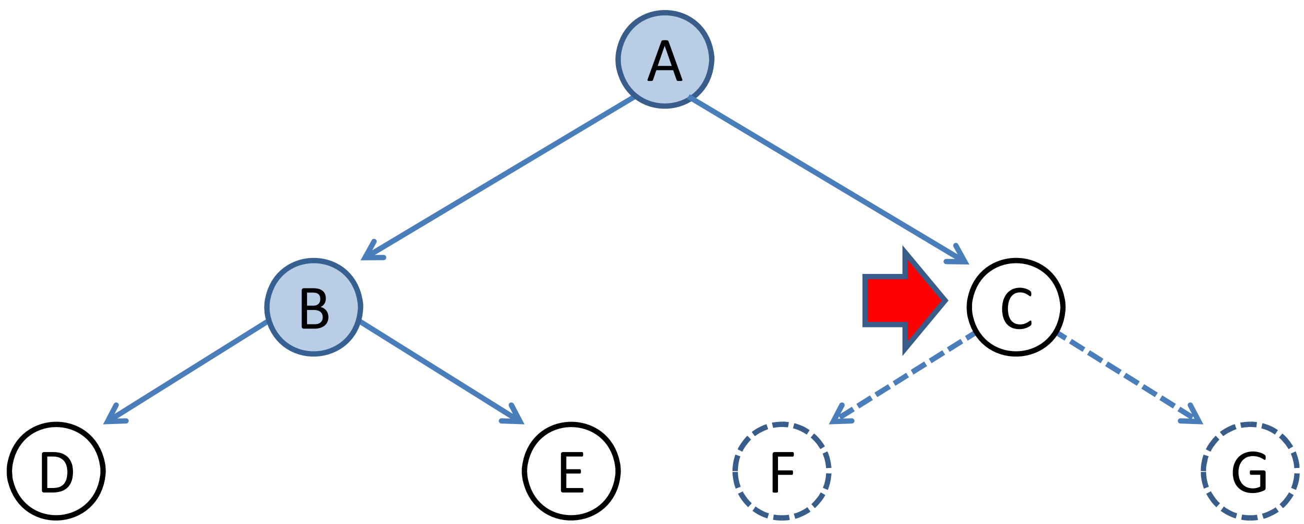

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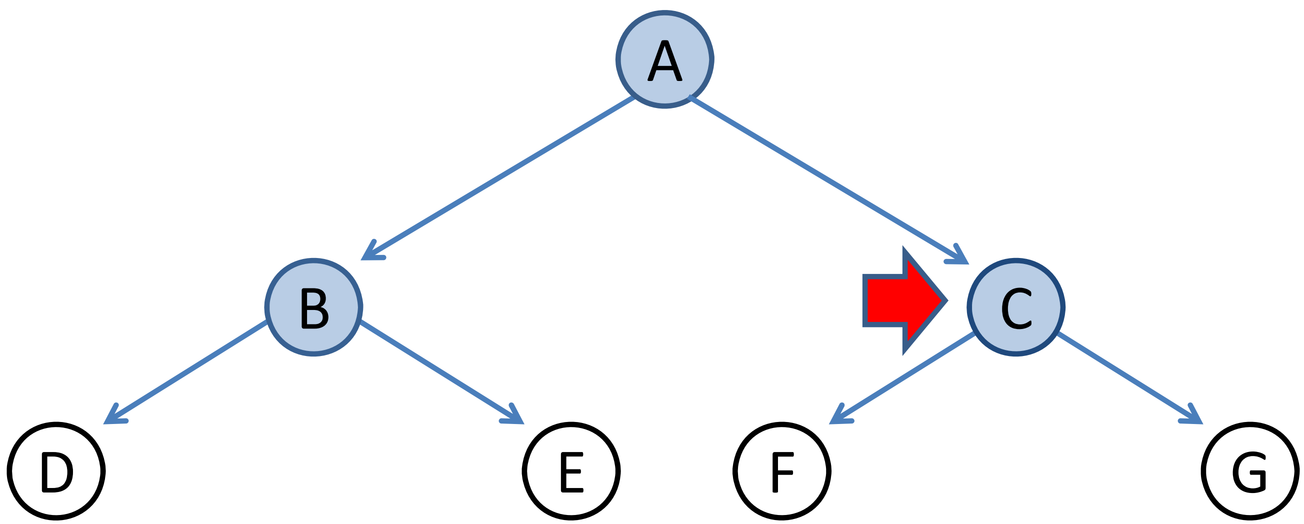


Properties of breadth-first search

- **Complete?**

  Yes (if branching factor *b* is finite)
- **Optimal?**

  Yes – if cost = 1 per step
- **Time?**

  Number of nodes in a *b*-ary tree of depth *d*: $O(b^d)$

  (*d* is the depth of the optimal solution)
- **Space?**

  $O(b^d)$

- Space is the bigger problem (more than time)

Uniform-cost search

- Expand least-cost unexpanded node
- Implementation: *fringe* is a queue ordered by path cost (priority queue)
- Equivalent to breadth-first if step costs all equal

- **Complete?**

  Yes, if step cost is greater than some positive constant ε
- **Optimal?**

  Yes – nodes expanded in increasing order of path cost
- **Time?**

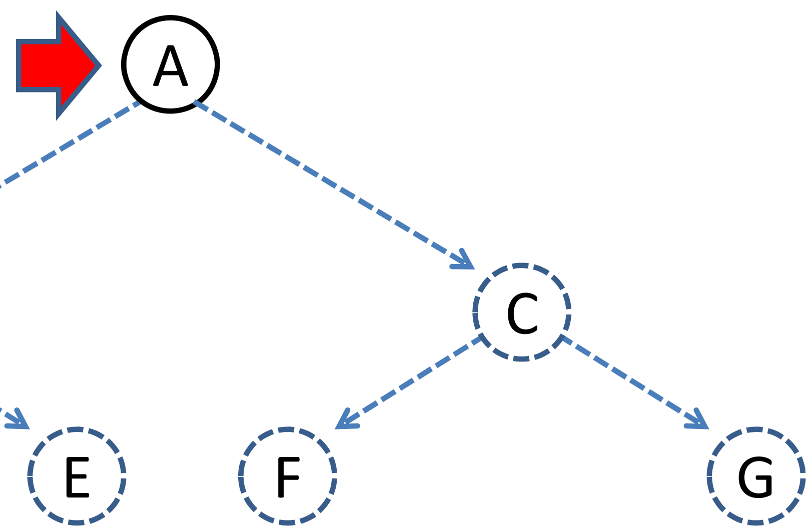
  Number of nodes with path cost ≤ cost of optimal solution (C^*), $O(b^{C^*/\varepsilon})$

  This can be greater than $O(b^d)$: the search can explore long paths consisting of small steps before exploring shorter paths consisting of larger steps
- **Space?**

  $O(b^{C^*/\varepsilon})$

Depth-first search

- Expand deepest unexpanded node
- Implementation:
  - *fringe* = LIFO queue, i.e., put successors at front



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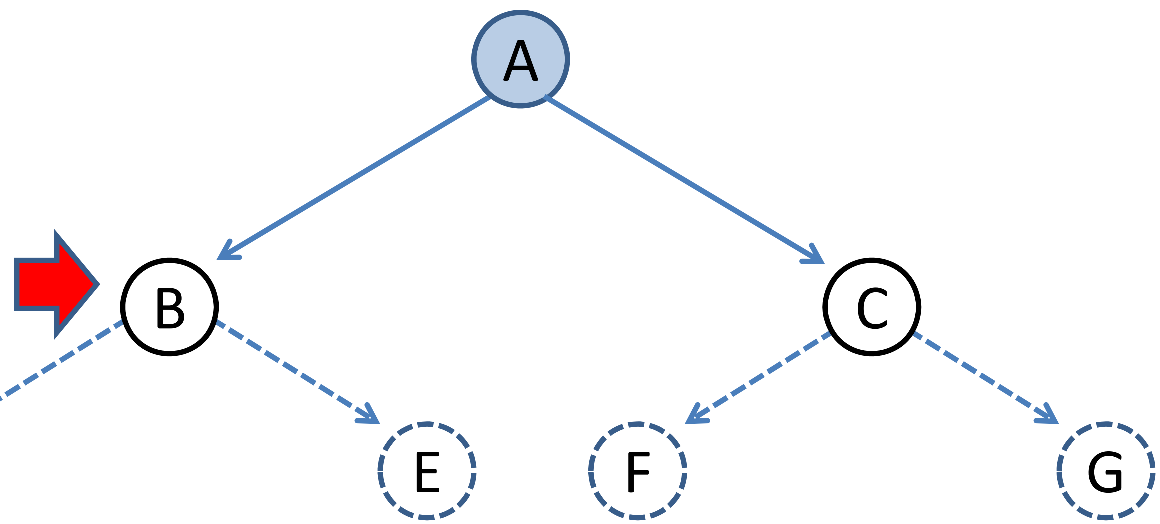

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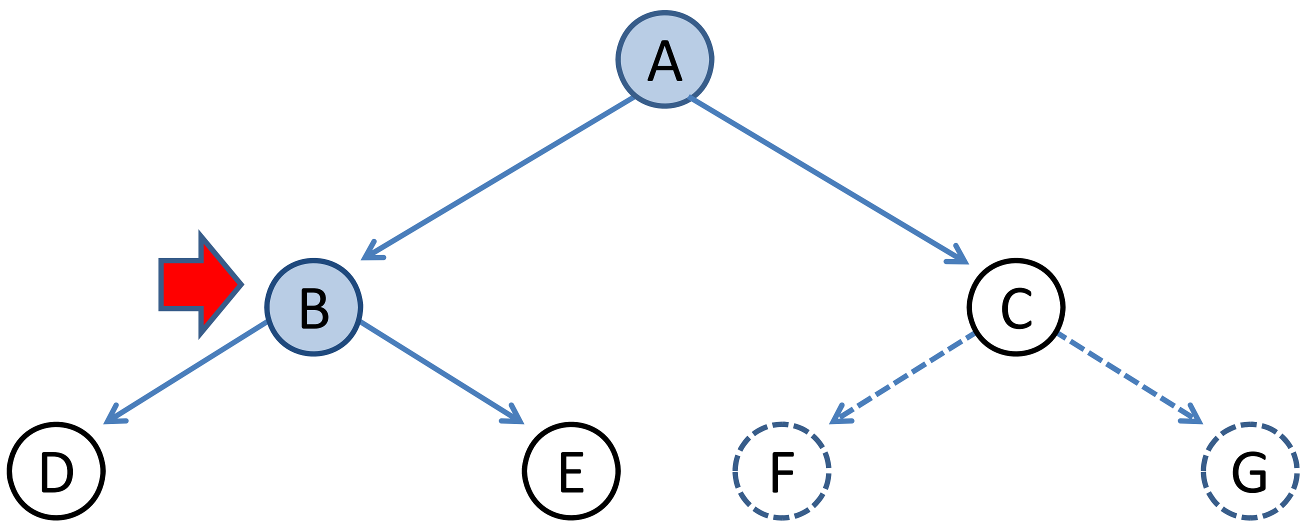

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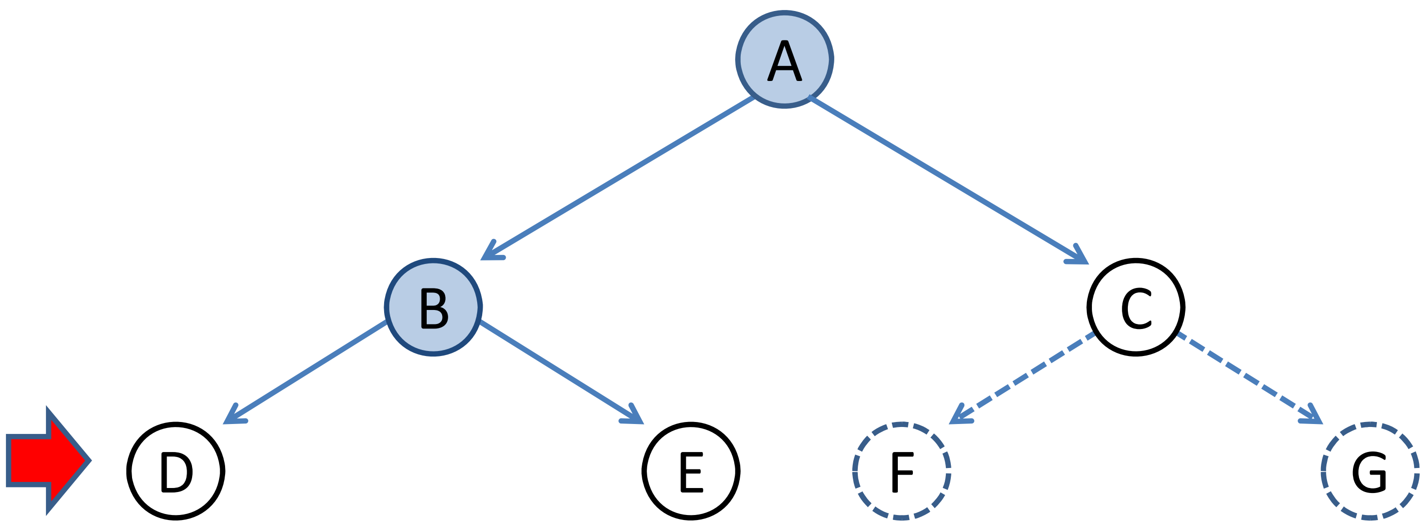

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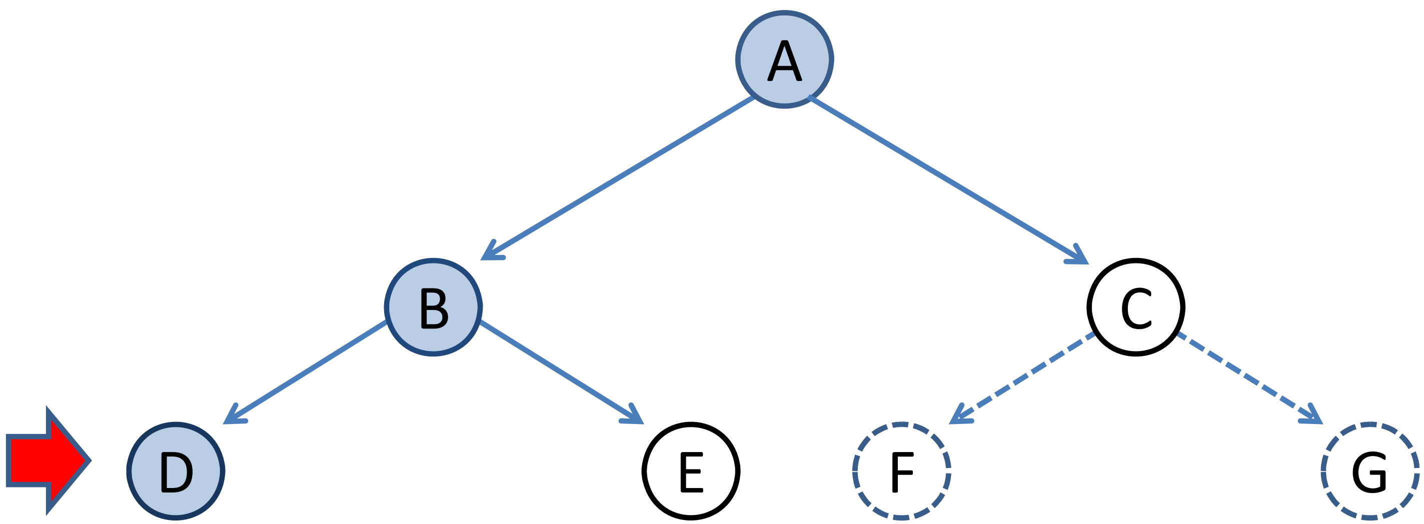

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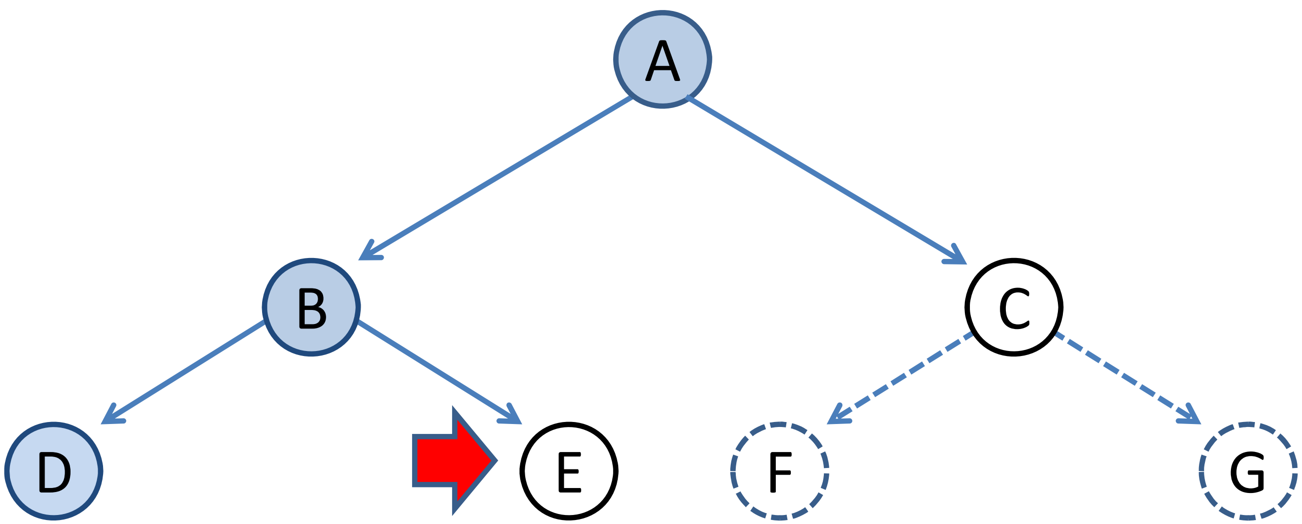

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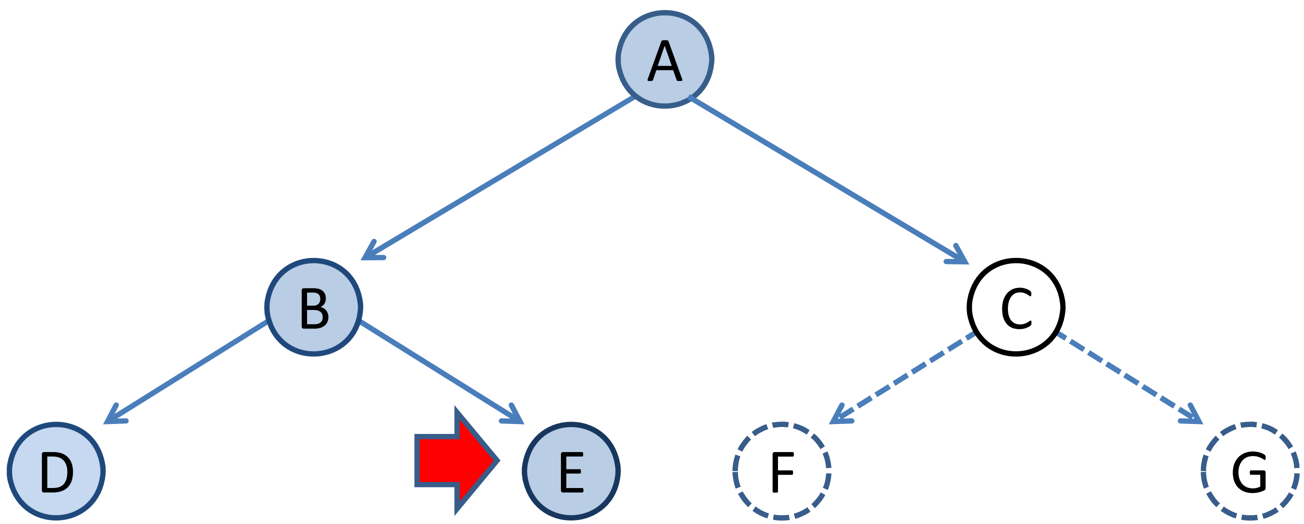

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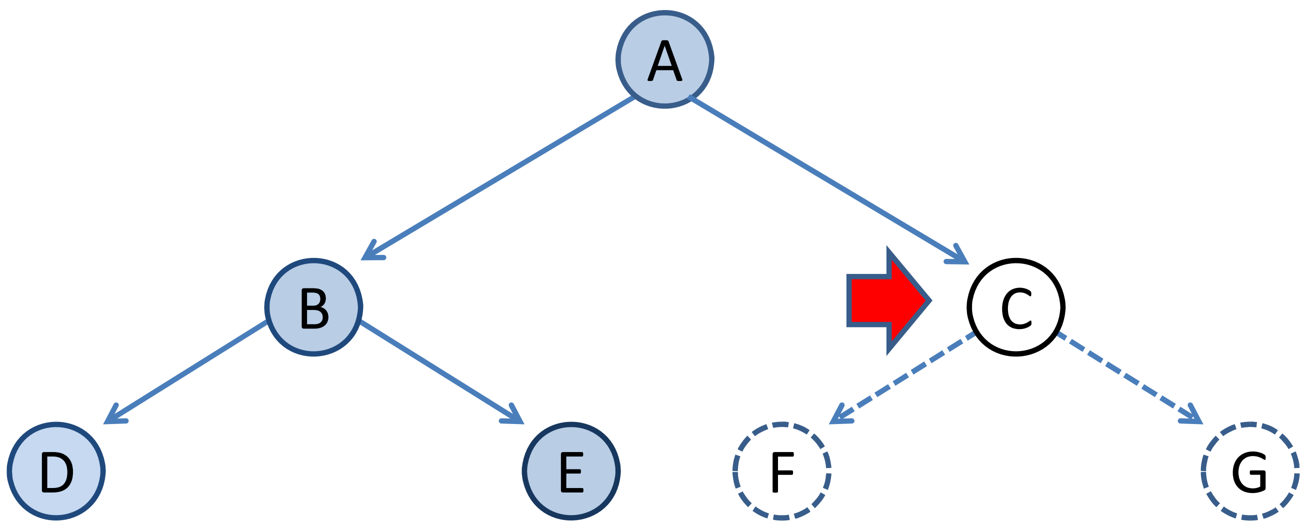

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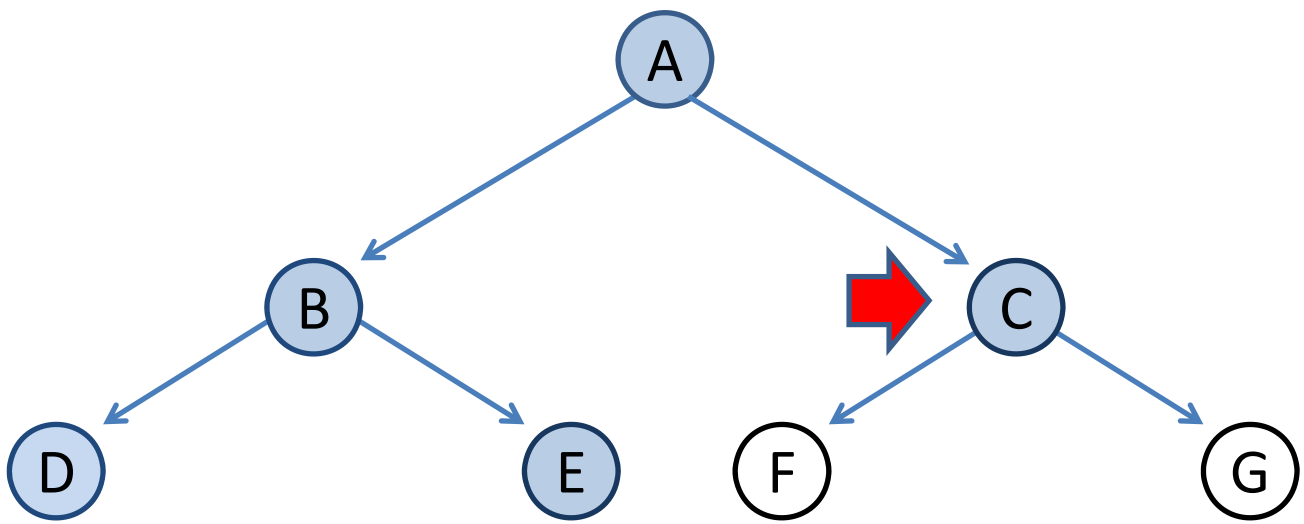


Properties of depth-first search

- **Complete?**

  Fails in infinite-depth spaces, spaces with loops

  Modify to avoid repeated states along path

  → complete in finite spaces
- **Optimal?**

  No – returns the first solution it finds
- **Time?**

  Could be the time to reach a solution at maximum depth *m:* $O(b^m)$

  Terrible if m is much larger than d

  But if there are lots of solutions, may be much faster than BFS
- **Space?**

  $O(bm)$, i.e., linear space!

Iterative deepening search

- Use DFS as a subroutine
  1. Check the root
  2. Do a DFS searching for a path of length 1
  3. If there is no path of length 1, do a DFS searching for a path of length 2
  4. If there is no path of length 2, do a DFS searching for a path of length 3…

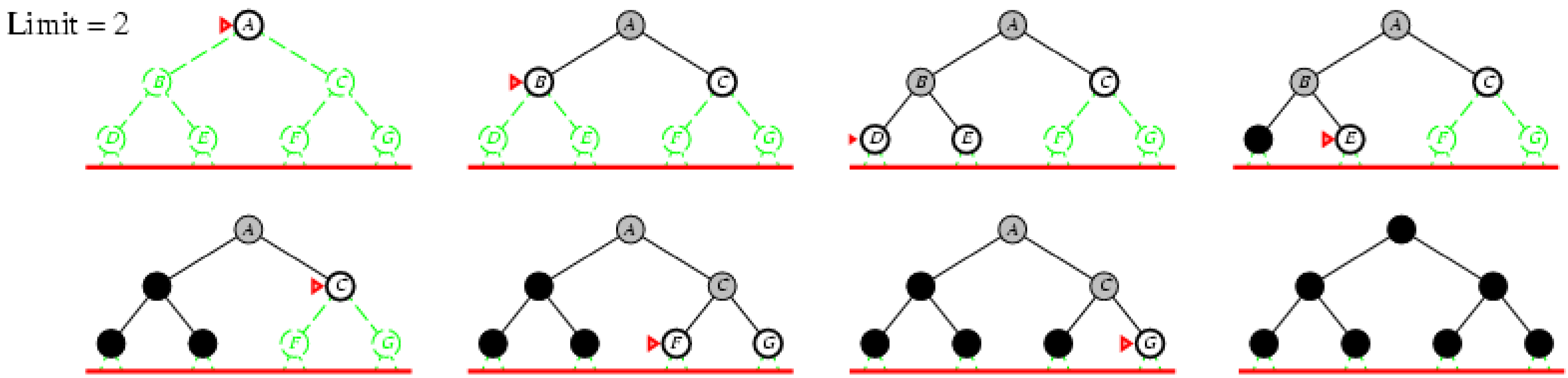
Iterative deepening search

Limit = 0

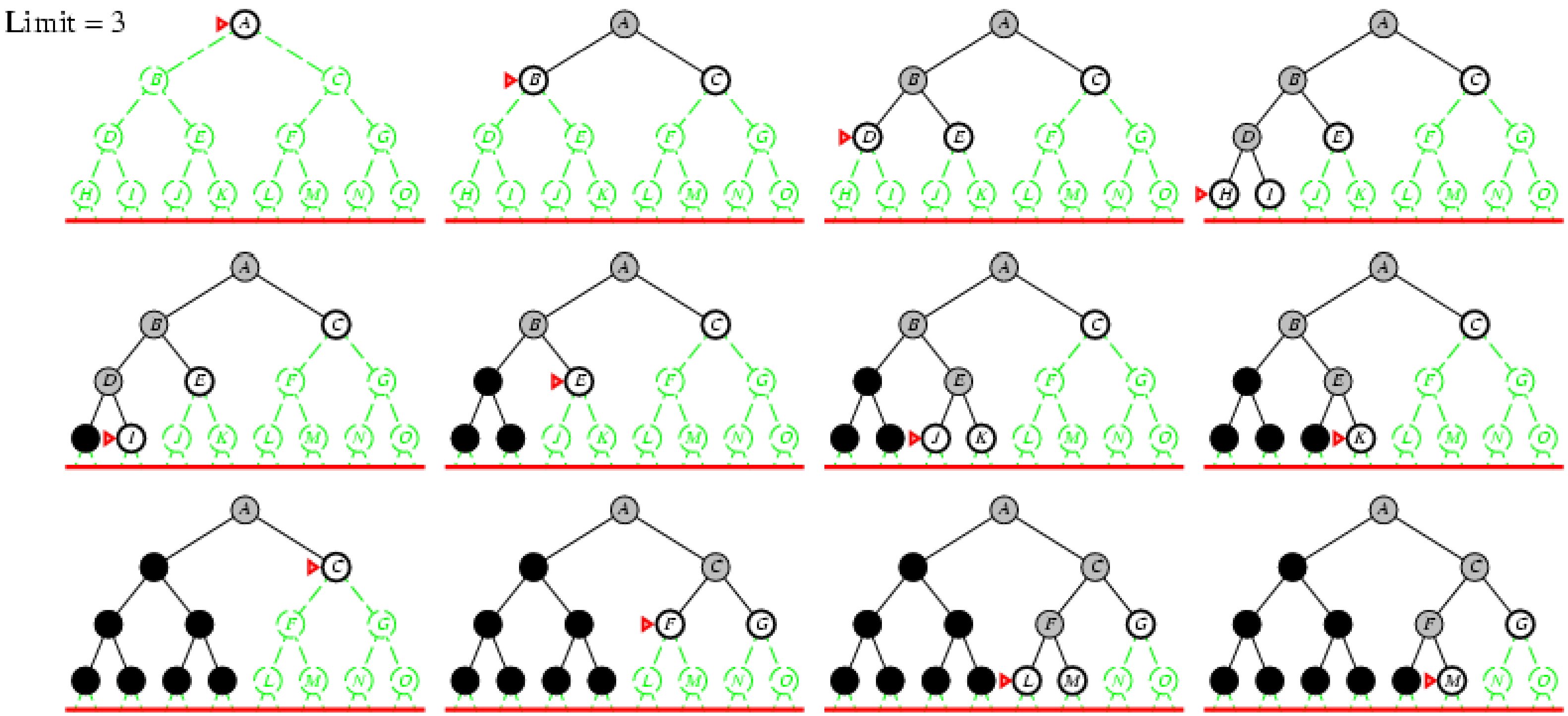
Iterative deepening search

Limit = 1

Iterative deepening search



Iterative deepening search



Properties of iterative deepening search

- **Complete?**

  Yes

- **Optimal?**

  Yes, if step cost = 1

- **Time?**

  $(d+1)b^0 + d\, b^1 + (d-1)b^2 + \ldots + b^d = O(b^d)$

- **Space?**

  $O(bd)$

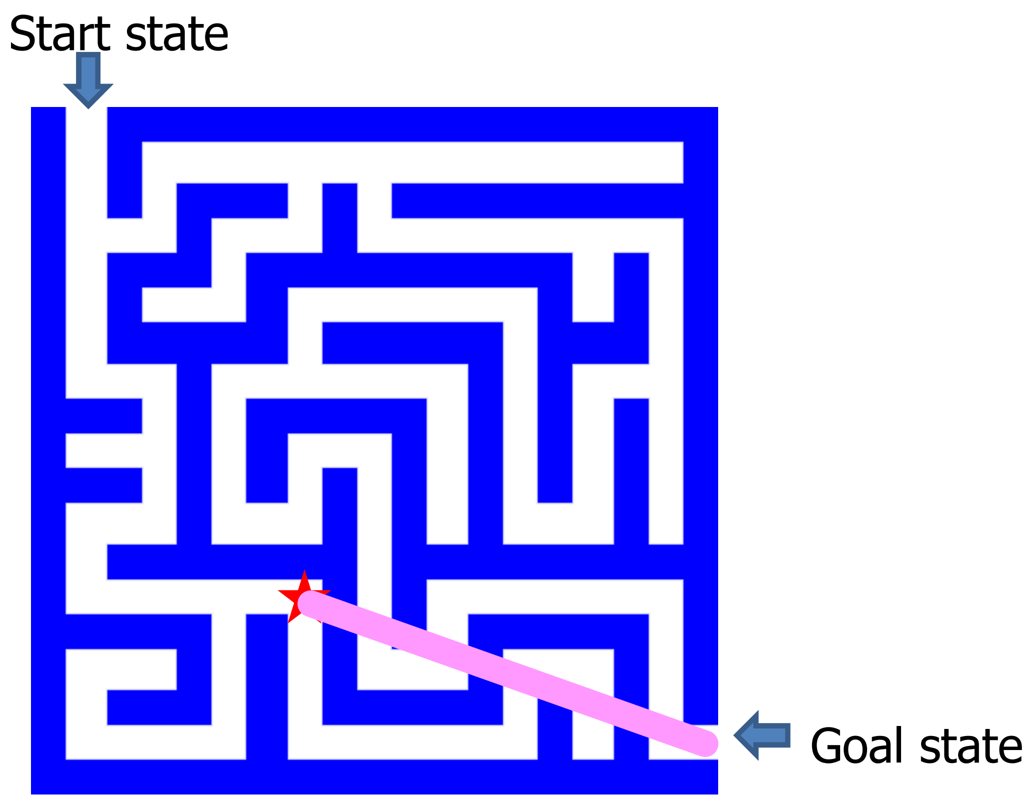
Informed search

- Idea: give the algorithm "hints" about the desirability of different states
  - Use an *evaluation function* to rank nodes and select the most promising one for expansion

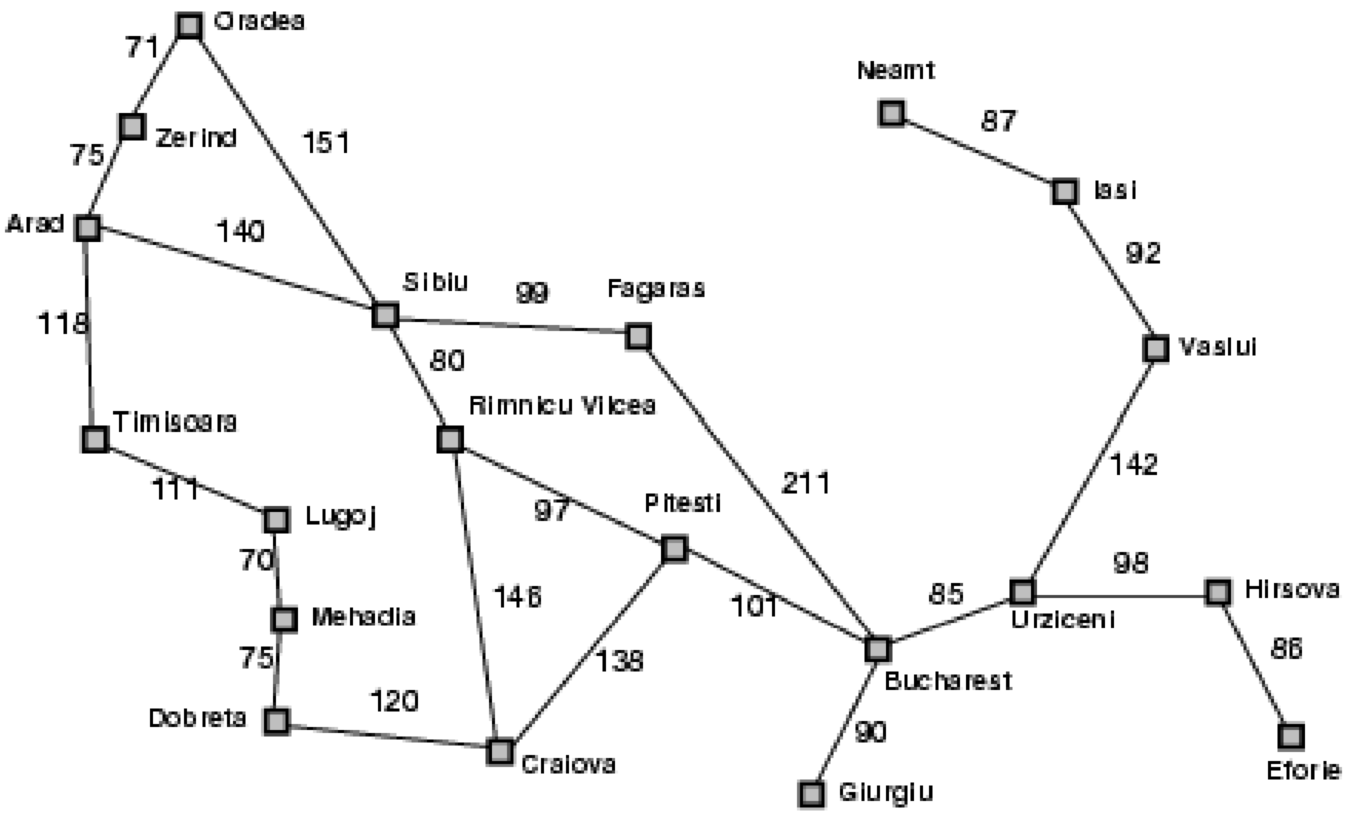
- Greedy best-first search
- A* search

Heuristic function

- Heuristic function $h(n)$ estimates the cost of reaching goal from node n
- Example:



Heuristic for the Romania problem



Straight–line distance to Bucharest

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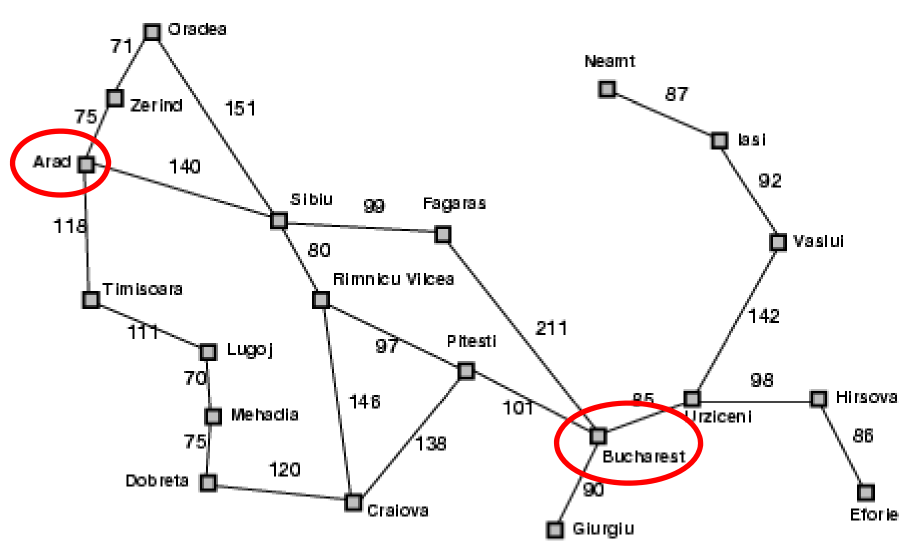
Greedy best-first search

- Expand the node that has the lowest value of the heuristic function $h(n)$

Greedy best-first search example



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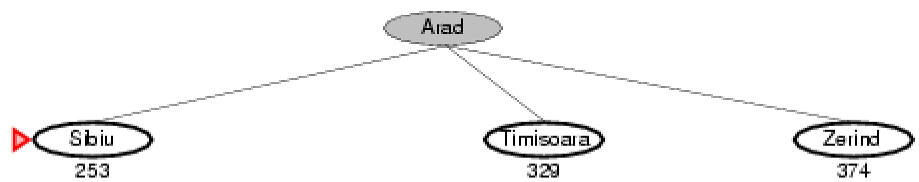

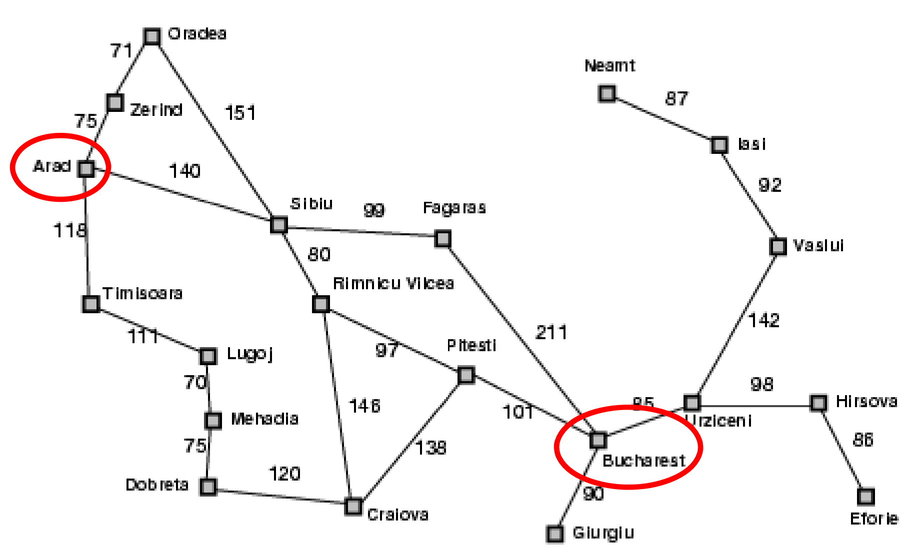

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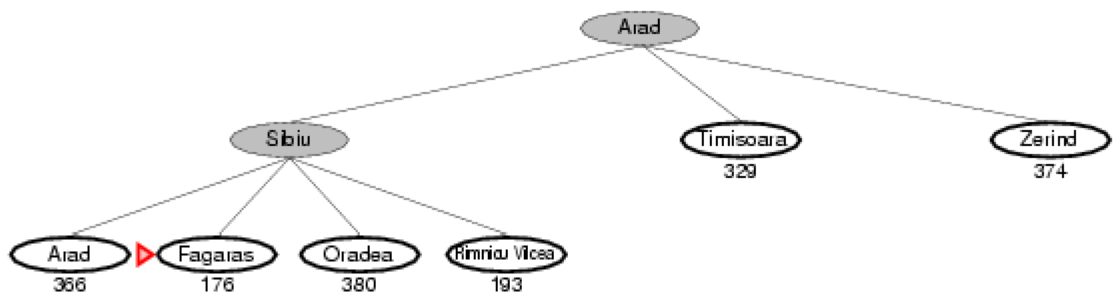

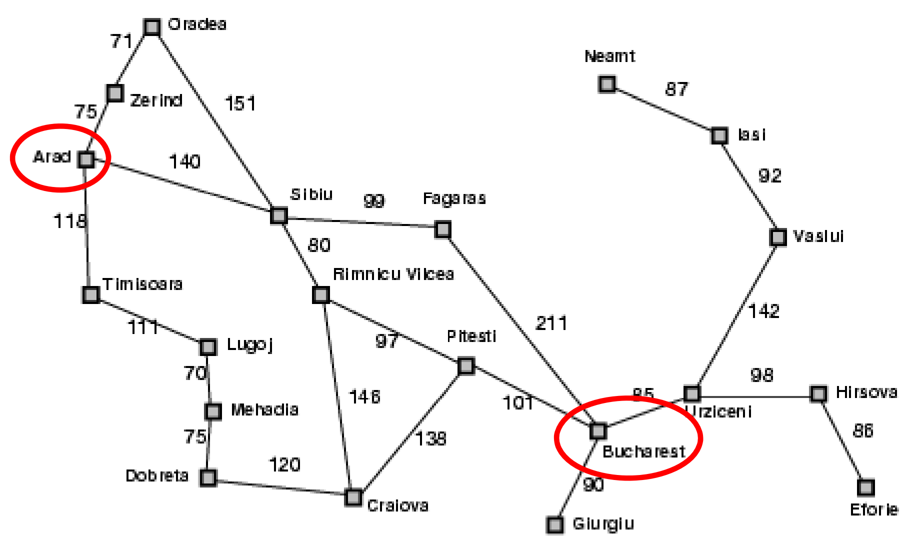

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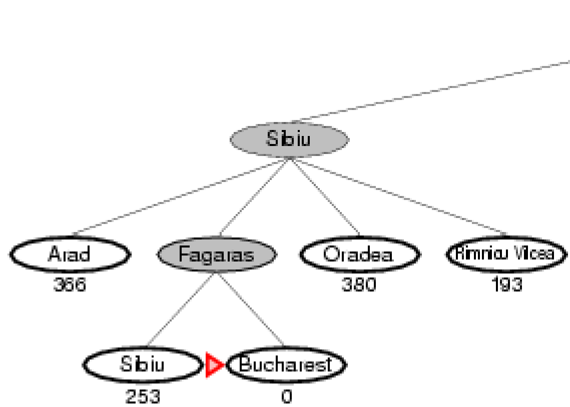

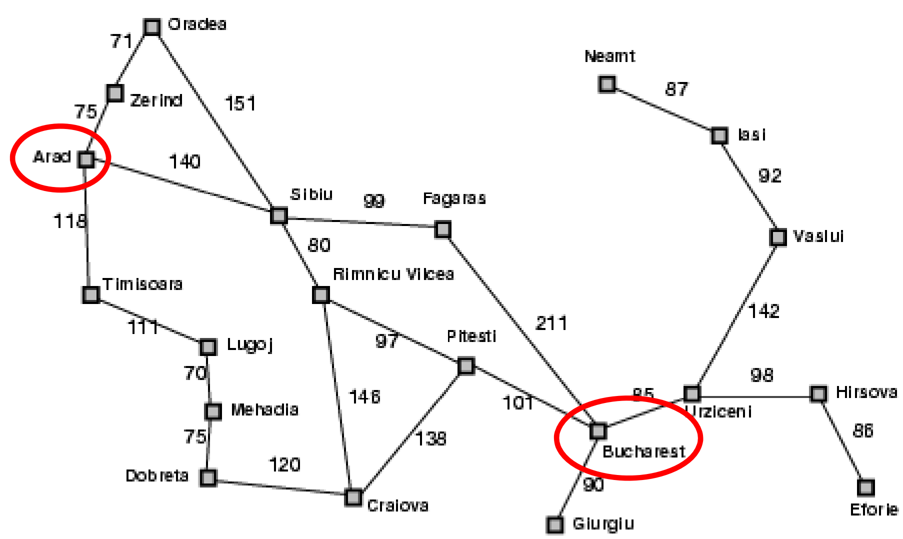

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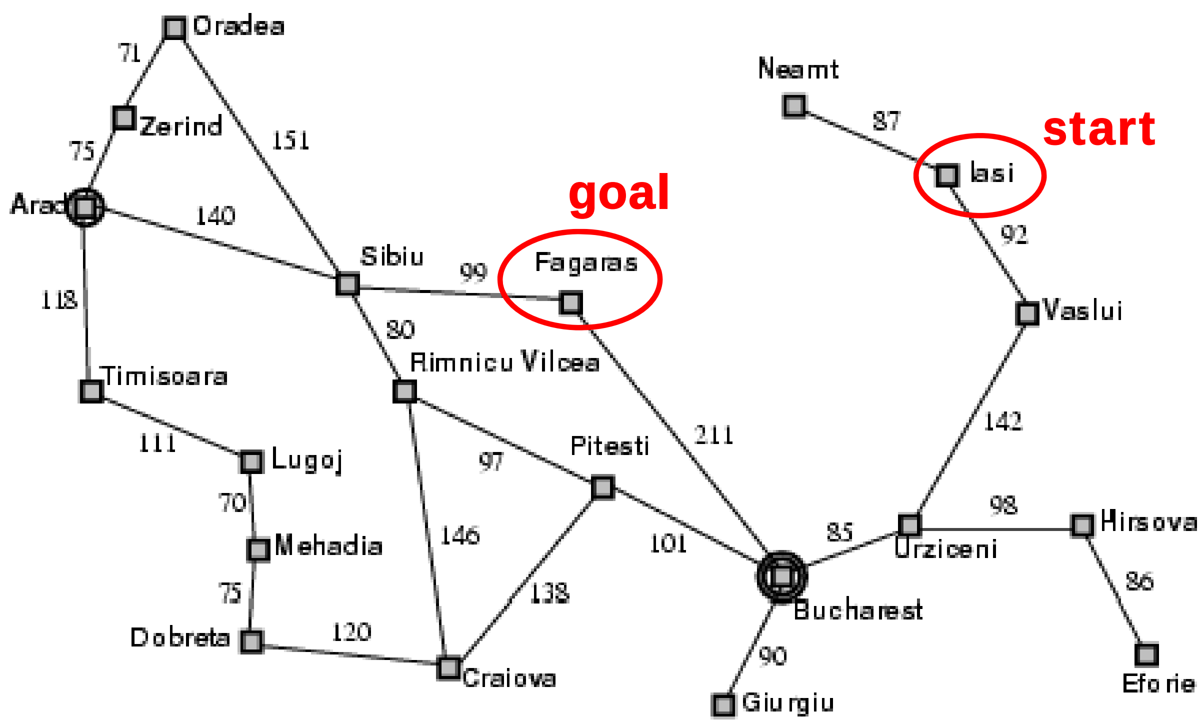
Properties of greedy best-first search

- **Complete?**

  No – can get stuck in loops



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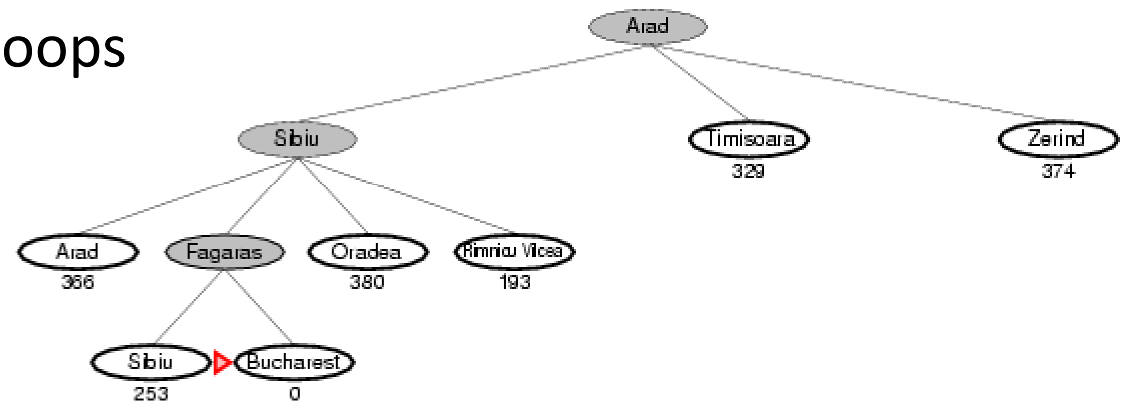
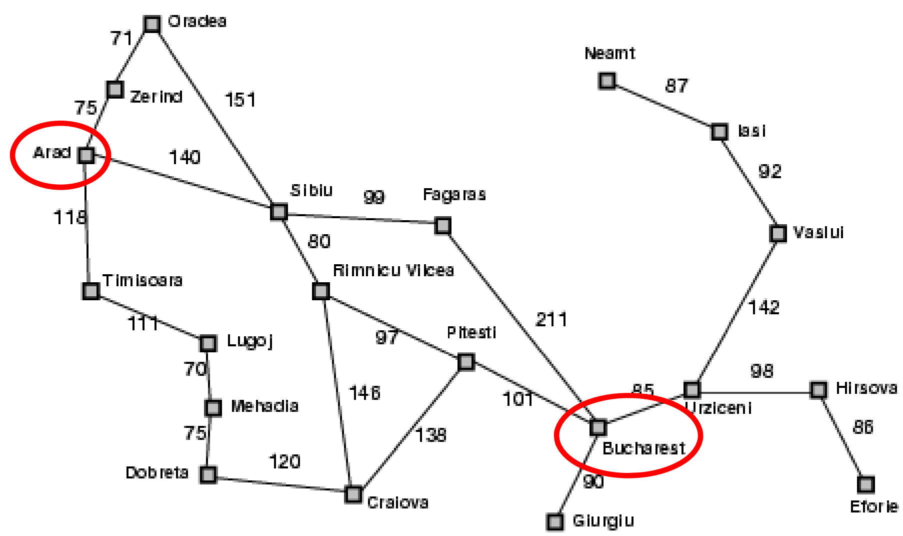
- **Complete?**

  No – can get stuck in loops

- **Optimal?**

  No





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Properties of greedy best-first search

- **Complete?**

  No – can get stuck in loops

- **Optimal?**

  No

- **Time?**

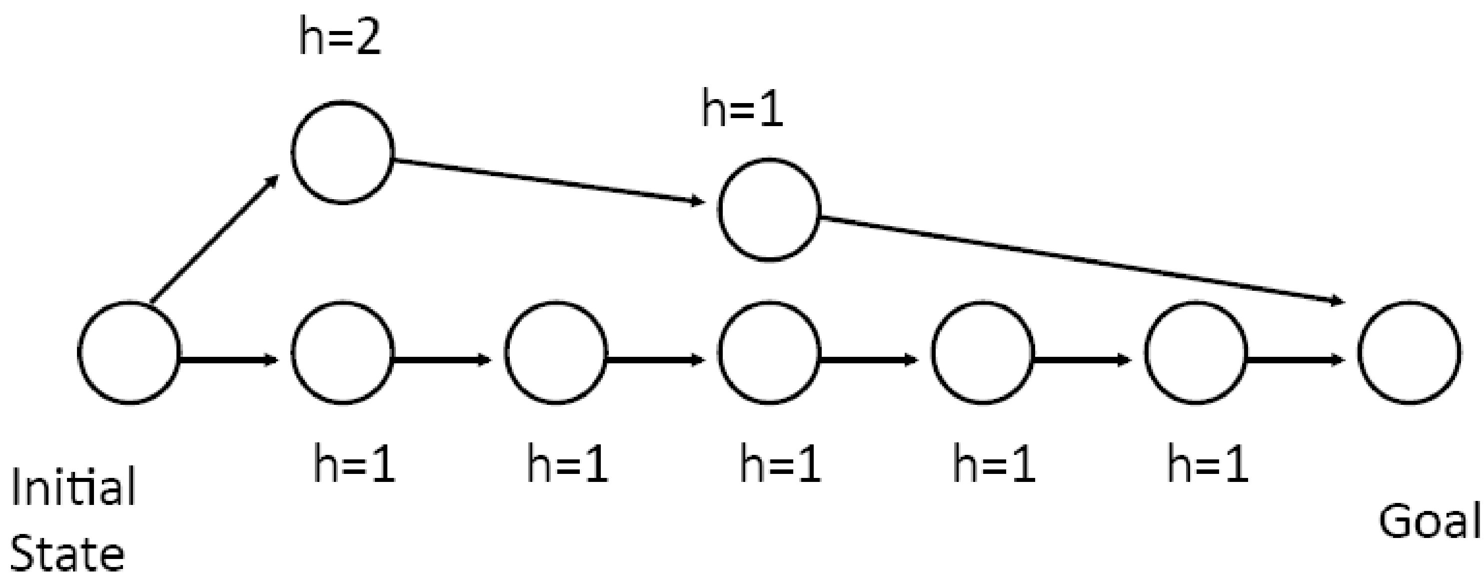
  Worst case: $O(b^m)$

  Best case: $O(bd)$ – If $h(n)$ is 100% accurate

- **Space?**

  Worst case: $O(b^m)$

How can we fix the greedy problem?



A^* search

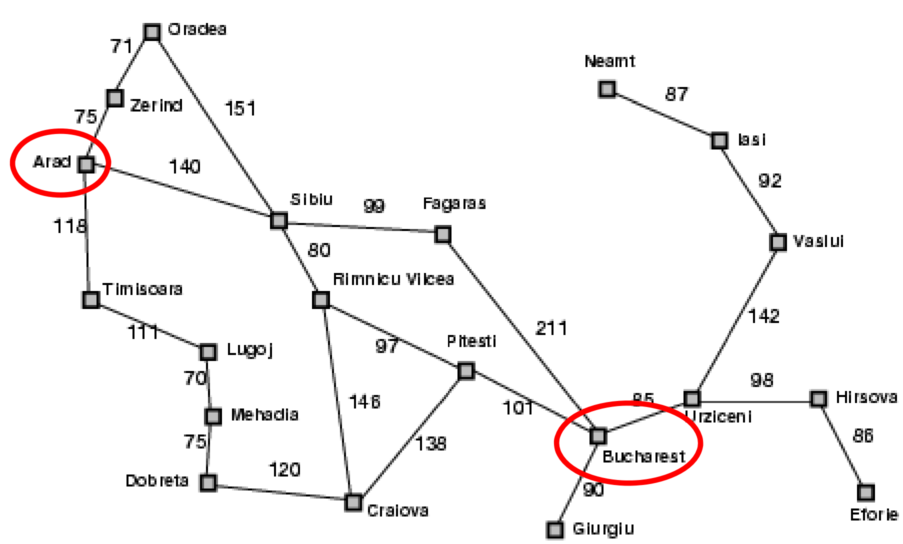
- Idea: avoid expanding paths that are already expensive
- The evaluation function $f(n)$ is the estimated total cost of the path through node n to the goal:

$$f(n) = g(n) + h(n)$$

$g(n)$: cost so far to reach n (path cost)

$h(n)$: estimated cost from n to goal (heuristic)

A* search example



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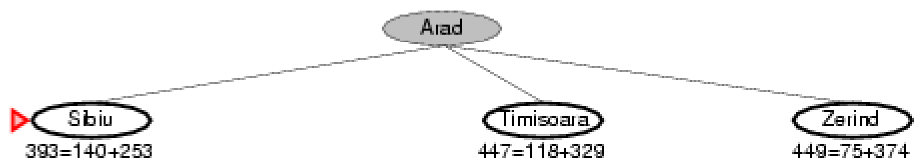

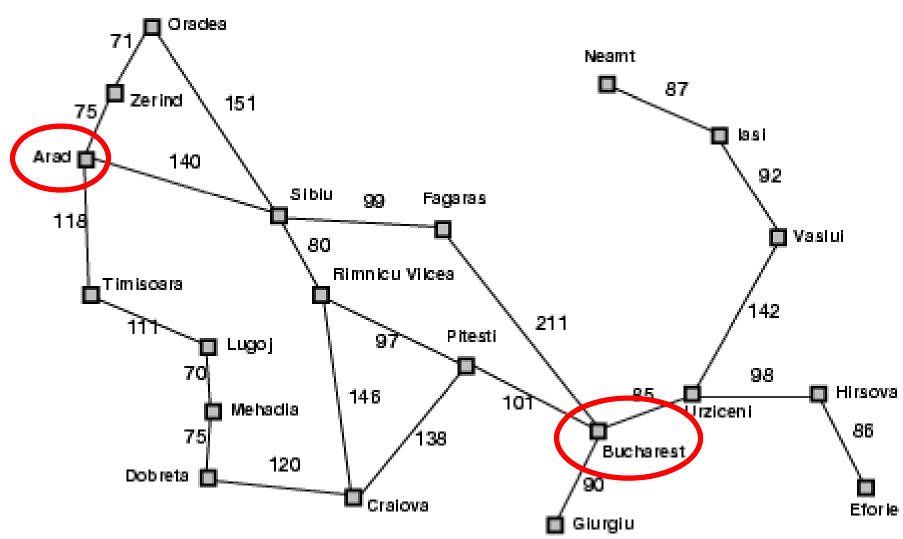

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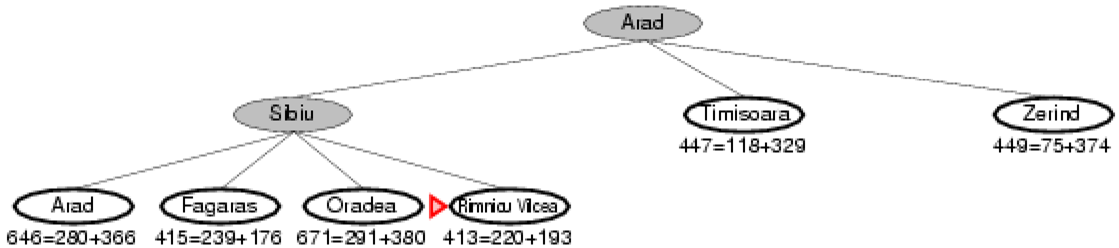

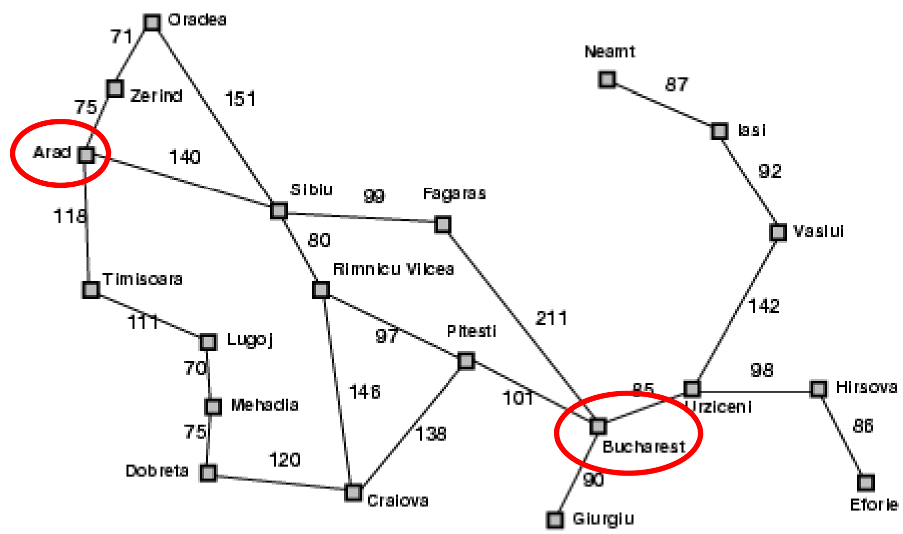

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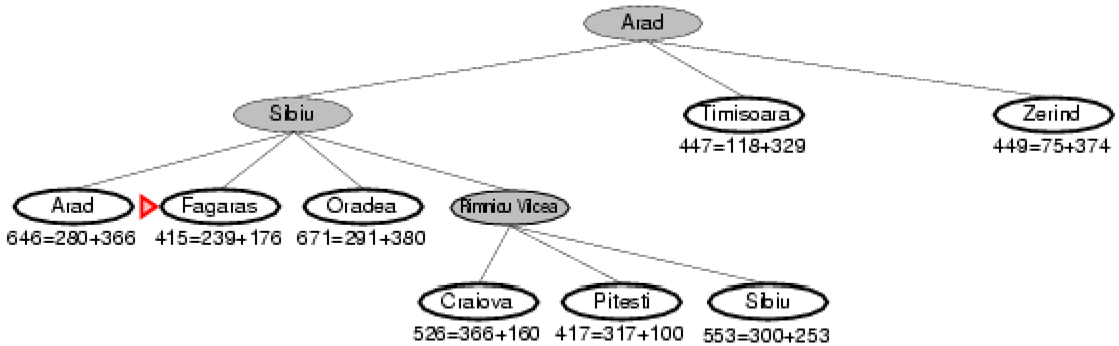

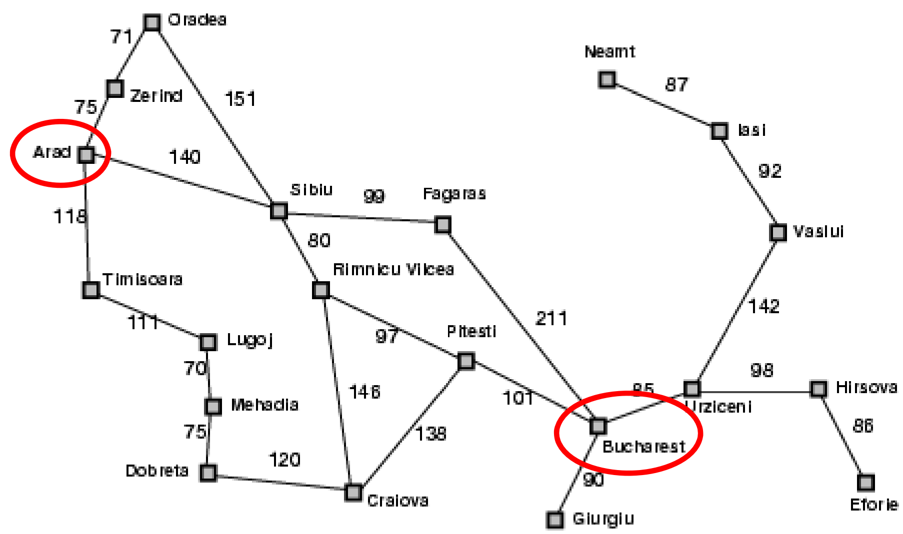

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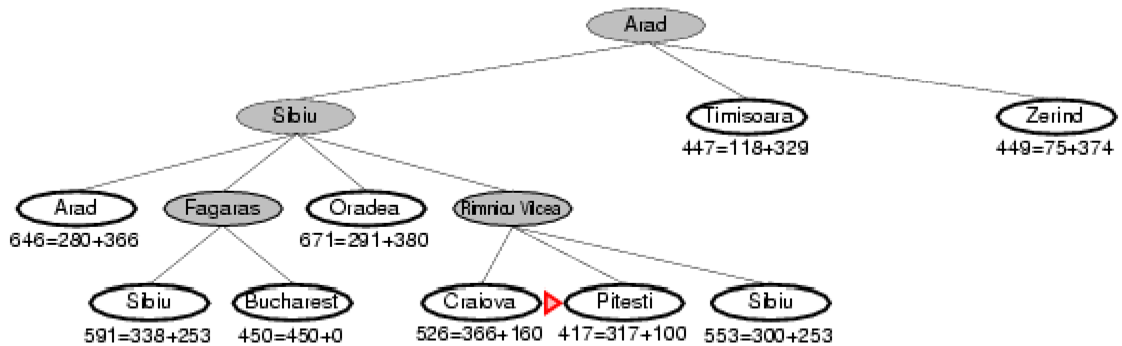

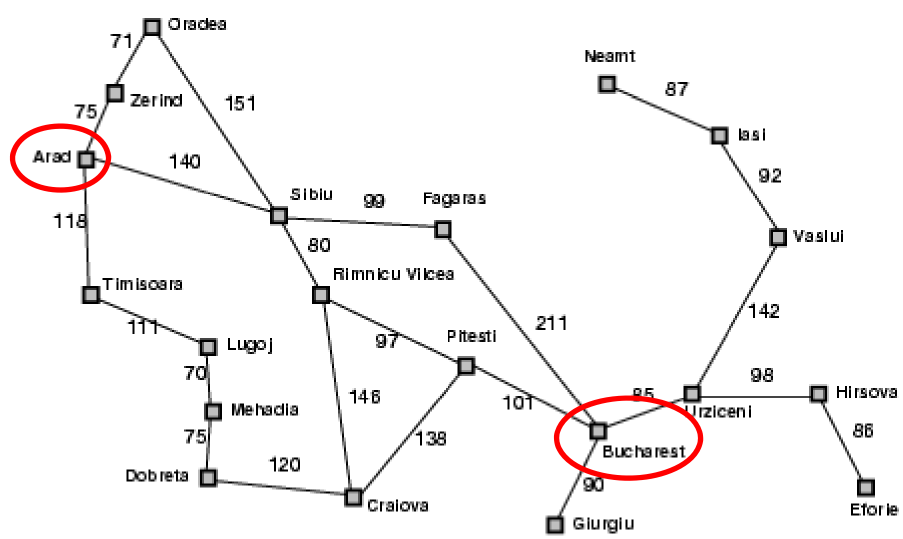

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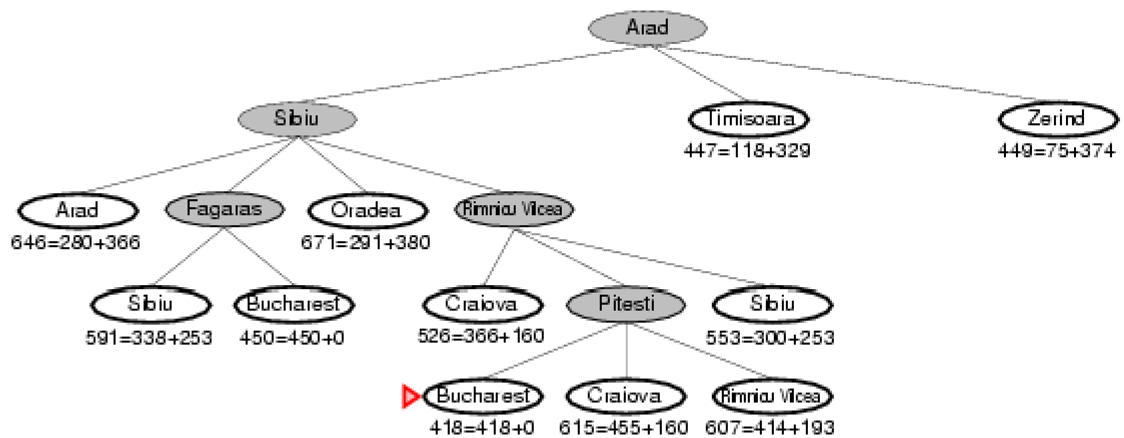

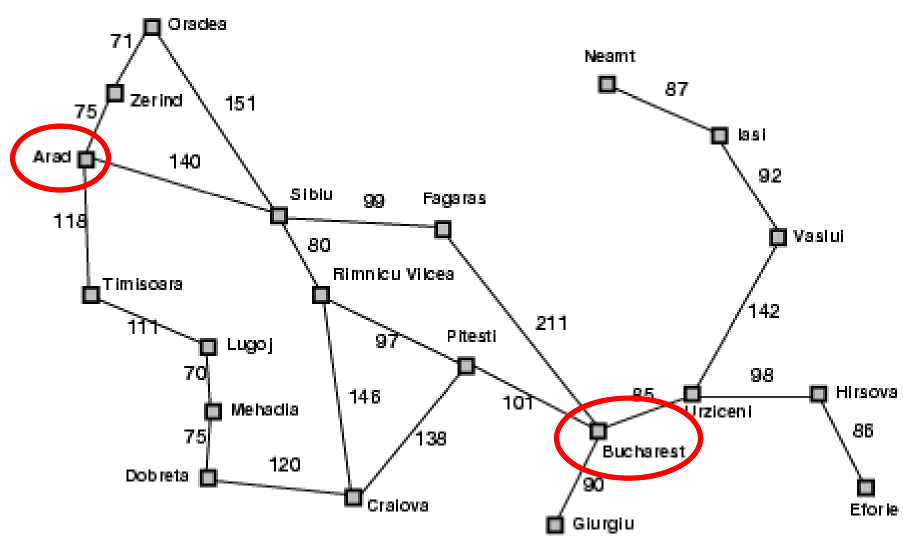

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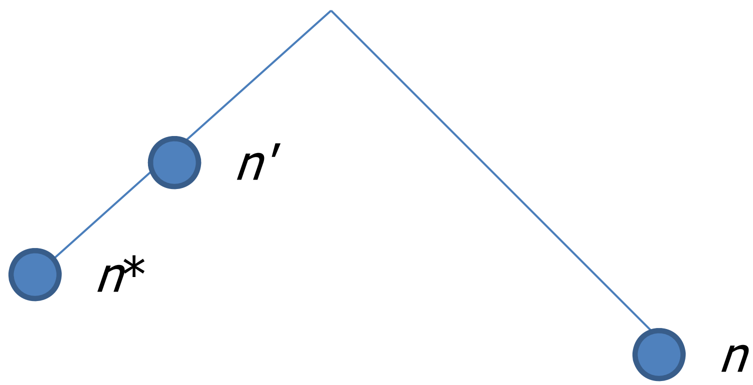
Zerind	374

Admissible heuristics

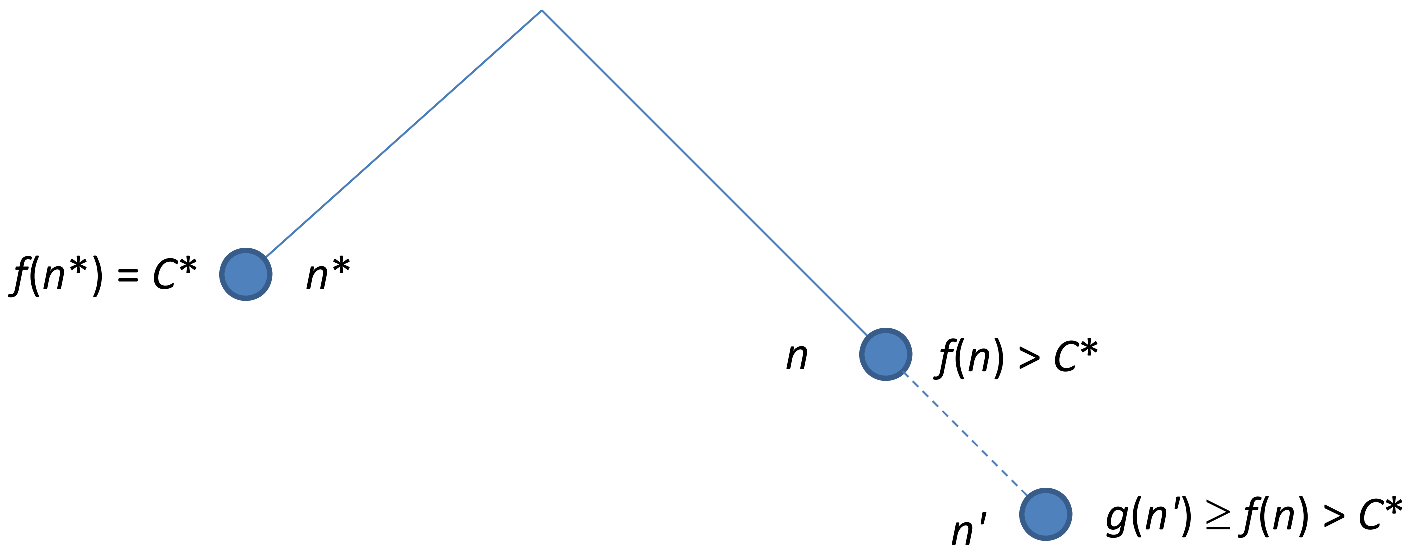
- A heuristic $h(n)$ is admissible if for every node n, $h(n) \leq h^*(n)$, where $h^*(n)$ is the true cost to reach the goal state from n
- An admissible heuristic never overestimates the cost to reach the goal, i.e., it is optimistic
- Example: straight line distance never overestimates the actual road distance
- Theorem: If $h(n)$ is admissible, A^* is optimal

Optimality of A*



- Proof by contradiction
  - Let n^* be an optimal goal state, i.e., $f(n^*) = C^*$
  - Suppose a solution node n with $f(n) > C^*$ is about to be expanded
  - Let n' be a node in the fringe that is on the path to n^*
  - We have $f(n') = g(n') + h(n') \leq C^*$
  - But then, n' should be expanded before n – a contradiction

Optimality of A*



- In other words:
  - Suppose A* terminates its search at n^*
  - It has found a path whose *actual cost* $f(n^*) = g(n^*)$ is lower than the *estimated cost* $f(n)$ of any path going through any fringe node
  - Since $f(n)$ is an *optimistic* estimate, there is no way n can have a successor goal state n' with $g(n') < C^*$

Optimality of A*

- A* is optimally efficient – no other tree-based algorithm that uses the same heuristic can expand fewer nodes and still be guaranteed to find the optimal solution
  - Any algorithm that does not expand all nodes with $f(n) < C^*$ risks missing the optimal solution

Properties of A*

- **Complete?**

  Yes – unless there are infinitely many nodes with $f(n) \leq C^*$
- **Optimal?**

  Yes
- **Time?**

  Number of nodes for which $f(n) \leq C^*$ (exponential)
- **Space?**

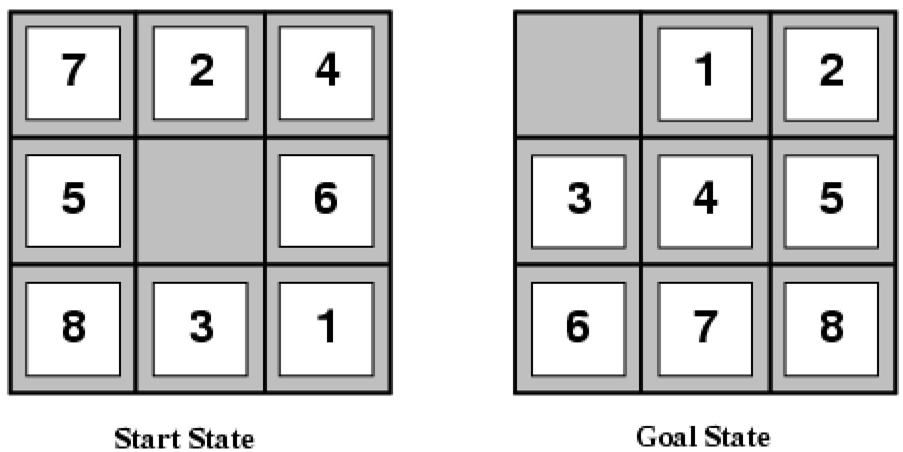
  Exponential

Designing heuristic functions

- Heuristics for the 8-puzzle

  $h_1(n)$ = number of misplaced tiles

  $h_2(n)$ = total Manhattan distance (number of squares from desired location of each tile)



$$h_1(\text{start}) = 8$$

$$h_2(\text{start}) = 3+1+2+2+2+3+3+2 = 18$$

- Are h_1 and h_2 admissible?

Heuristics from relaxed problems

- A problem with fewer restrictions on the actions is called a relaxed problem
- The cost of an optimal solution to a relaxed problem is an admissible heuristic for the original problem
- If the rules of the 8-puzzle are relaxed so that a tile can move anywhere, then $h_1(n)$ gives the shortest solution
- If the rules are relaxed so that a tile can move to any adjacent square, then $h_2(n)$ gives the shortest solution

Dominance

- If h_1 and h_2 are both admissible heuristics and $h_2(n) \geq h_1(n)$ for all n, (both admissible) then h_2 dominates h_1
- Which one is better for search?
  - A* search expands every node with $f(n) < C^*$ or $h(n) < C^* - g(n)$
  - Therefore, A* search with h_1 will expand more nodes

Dominance

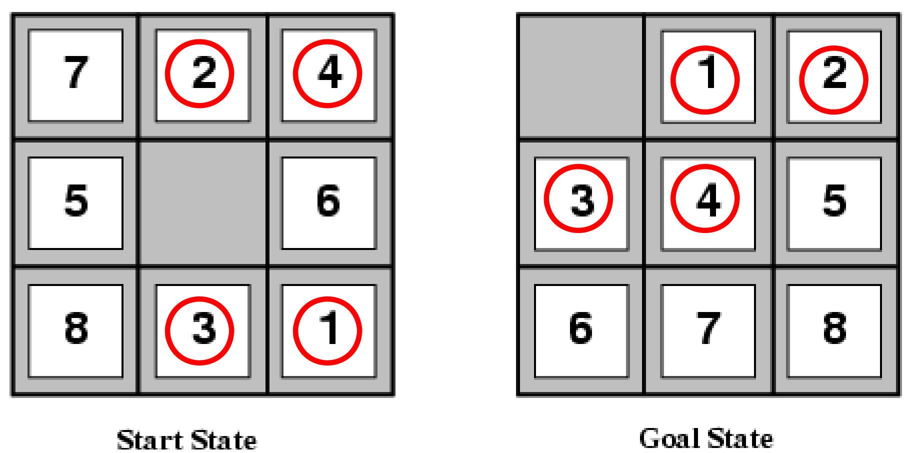
- Typical search costs for the 8-puzzle (average number of nodes expanded for different solution depths):

- d=12 IDS = 3,644,035 nodes
  $A^*(h_1)$ = 227 nodes
  $A^*(h_2)$ = 73 nodes

- d=24 IDS ≈ 54,000,000,000 nodes
  $A^*(h_1)$ = 39,135 nodes
  $A^*(h_2)$ = 1,641 nodes

Heuristics from subproblems

- Let $h_3(n)$ be the cost of getting a subset of tiles (say, 1,2,3,4) into their correct positions
- Can precompute and save the exact solution cost for every possible subproblem instance – *pattern database*



Combining heuristics

- Suppose we have a collection of admissible heuristics $h_1(n)$, $h_2(n)$, …, $h_m(n)$, but none of them dominates the others
- How can we combine them?

$$h(n) = \max\{h_1(n), h_2(n), \ldots, h_m(n)\}$$

Memory-bounded search

- The memory usage of A* can still be exorbitant
- How to make A* more memory-efficient while maintaining completeness and optimality?

- Iterative deepening A* search
- Recursive best-first search, SMA*
  - Forget some subtrees but remember the best f-value in these subtrees and regenerate them later if necessary

- Problems: memory-bounded strategies can be complicated to implement, suffer from "thrashing"

Comparison of search strategies

Algorithm	Complete?	Optimal?	Time complexity	Space complexity
BFS	Yes	If all step costs are equal	$O(b^d)$	$O(b^d)$
UCS	Yes	Yes	Number of nodes with $g(n) \leq C^*$	
DFS	No	No	$O(b^m)$	$O(bm)$
IDS	Yes	If all step costs are equal	$O(b^d)$	$O(bd)$
Greedy	No	No	Worst case: $O(b^m)$ Best case: $O(bd)$	
A*	Yes	Yes	Number of nodes with $g(n)+h(n) \leq C^*$	