

## Differentiation

Imagine a bug that moves with constant speed on a circular path of radius  $r$  around the origin. The angle of the bug's position vector with the  $+x$  axis can be written as

$$\theta = \omega t + a.$$

Assume  $a = 0$ , so that the bug is on the  $+x$  axis at time 0. Then the position vector of the bug is

$$X(t) = (r \cos(\omega t), r \sin(\omega t)).$$

Now imagine the bug lives in  $\mathbb{R}^3$  with

$$X(t) = (\cos(t), \sin(t), t).$$

This lifts the circular path into a helix.

In general, a **parametrized curve**  $X : I \rightarrow \mathbb{R}^n$  is a vector-valued function that maps points from an interval  $I$  into  $n$ -space. In the examples above,  $I$  is the entire real line  $\mathbb{R}$  (which we consider to be an interval). We can write  $X(t)$  as its individual coordinate functions

$$X(t) = (x_1(t), \dots, x_n(t)).$$

Just as with ordinary real-valued function, we can take derivatives by looking at the limit

$$\lim_{h \rightarrow 0} \frac{X(t+h) - X(t)}{h}.$$

Here, dividing by  $h$  really means scaling the vector by  $1/h$ . Writing out components, this is simply

$$\lim_{h \rightarrow 0} \frac{(x_1(t+h) - x_1(t), \dots, x_n(t+h) - x_n(t))}{h}.$$

If the individual components are all differentiable, we obtain a new vector-valued function

$$X'(t) = (x'_1(t), \dots, x'_n(t)).$$

$X'(t)$  is called the **derivative** or **velocity** of  $X(t)$ .

So for the example  $X(t) = (\cos(t), \sin(t), t)$ , we have

$$X'(t) = (-\sin(t), \cos(t), 1).$$

The velocity is parallel to the direction of instantaneous motion.

**Example.** Find a parametric equation of the tangent line to the curve  $X(t) = (\sin t, \cos t)$  at  $t = \pi/3$ .

We need two pieces of information: a point on the line, and a direction vector of the line. These are supplied by  $X(\pi/3)$  and  $X'(\pi/3)$  respectively. The tangent line  $L(t)$  can

thus be written

$$\begin{aligned} L(s)|_{t=\pi/3} &= X(\pi/3) + sX'(\pi/3) \\ &= \left( \frac{\sqrt{3}}{2} + \frac{1}{2}s, \frac{1}{2} - \frac{\sqrt{3}}{2}s \right). \end{aligned}$$

We used the parameter  $s$  for the line to avoid confusion with the already defined  $X(t)$  above.

The **speed** of the curve  $X(t)$ , denoted  $v(t)$ , is defined to be

$$v(t) = \|X'(t)\|.$$

**acceleration** is the second derivative  $X''(t)$ .

We note also that differentiation is linear, meaning

$$\frac{d}{dt}(X(t) + Y(t)) = X'(t) + Y'(t)$$

and

$$\frac{d}{dt}cX(t) = cX'(t).$$

We also have a product rule:

$$\frac{d}{dt}X(t) \cdot Y(t) = X'(t) \cdot Y(t) + X(t) \cdot Y'(t).$$

This follows from applying the ordinary product rule. If  $X(t) = (x_1(t), x_2(t))$  and  $Y(t) = (y_1(t), y_2(t))$ , then

$$\begin{aligned} \frac{d}{dt}X(t) \cdot Y(t) &= \frac{d}{dt}(x_1y_1 + x_2y_2) \\ &= x_1'y_1 + x_1y_1' + x_2'y_2 + x_2y_2' \\ &= x_1'y_1 + x_2'y_2 + x_1y_1' + x_2y_2' \\ &= X'(t) \cdot Y(t) + X(t) \cdot Y'(t). \end{aligned}$$

Of course, this same argument works in dimensions higher than 2.

## Length of Curves

If we integrate the speed  $v(t)$  of  $X(t)$  from time  $t = a$  to  $t = b$ , we obtain the distance traveled by  $X(t)$  during the time interval  $[a, b]$ :

$$\text{distance} = \int_a^b v(t)dt.$$

**Example.** Let  $X(t) = (\cos(t), \sin(t))$  describe a particle. What distance does  $X(t)$  traverse from  $t = 0$  to  $t = 1$ ?

We have  $X'(t) = (-\sin(t), \cos(t))$ . Then  $v(t) = \|X'(t)\| = \sqrt{(-\sin(t))^2 + \cos^2(t)} = 1$ . So the distance  $D$  is

$$D = \int_0^1 1dt = 1.$$

Note that distance and displacement are not the same thing. In the example above, if we consider the distance traveled from  $t = 0$  to  $t = 2\pi$ , the particle travels a distance of  $2\pi$ , but the net displacement is 0 since it ends up where it started.