Critical Points

A point P is a **critical point** of f if $\operatorname{grad} f(P) = O$. Equivalently, all the partial derivatives $D_i f$ are 0 at P.

Example. Find the critical points of $f(x,y) = e^{-(x^2+y^2)}$. We take partial derivatives and set them to 0 to find the critical points.

As in the single variable case, we can have a variety of behaviors at a critical point; we do not necessarily have a local minimum or local maximum.

Let f be defined on an open set U. A point P is called a **local maximum** of f if, in some neighborhood N of P, we have

$$f(X) \leq f(P)$$

for all $X \in N$.

The concept of local minimum is defined similarly.

Theorem. Let f be a differentiable function on U. Let P be a local maximum. Then P is a critical point of f.

The proof of this amounts to reducing it to a one variable problem. If H is a nonzero vector, and t is small enough, then $P + tH \in U$. Moreover, if t is small enough, $P + tH$ will land in the neighborhood mentioned in the definition, so that

$$f(P + tH) \leq f(P)$$

for all t in an interval of the form $(-\delta, \delta)$, $\delta > 0$. So $g(t) = f(P + tH)$ has a local maximum at $t = 0$. Thus $g'(t) = 0$. By the chain rule,

$$\operatorname{grad} f(P) \cdot H = 0.$$

This is true for all H, so we must have $\operatorname{grad} f(P) = 0$. ■

A similar argument shows that local minima are also critical points of f.

Boundary, Interior, etc.

An **open ball** of radius $r > 0$ in $\mathbb{R}^n$ centered at P is defined to be the set of all points X such that $\|X - P\| < r$.

A **closed ball** is similarly defined except $\|X - P\| \leq r$ (rather than strict inequality).

A subset $U \subseteq \mathbb{R}^n$ is **open** if at every point $P \in U$, there is a ball of some radius around P contained entirely in U.

An **interior point** P of a set $S \subseteq \mathbb{R}^n$ is one such there exists a ball of some radius around P contained entirely in S. Thus one could rephrase the definition of openness as each point being an interior point.

A point P (not necessarily in S) is called a **boundary point** of S if every open ball around P contains both a point in S and a point not in S.

A set is **closed** if it contains all of its boundary points.

A set is **bounded** if one can fit the set inside a ball. Equivalently, S is bounded if there is some $b > 0$ such that $\|X\| \leq b$ for all $X \in S$.

Theorem. Let f be a continuous function defined on a closed and bounded set S. Then f has a maximum and a minimum on S.

The proof of this requires things out of the scope of this course.

One can see via examples that a continuous f can fail to achieve extreme values if S is not closed and bounded. As an easy example, consider $f : (0, 1) \to \mathbb{R}$, $f(x) = x$.

In a general situation for some f on a closed and bounded region S, we find the maxima and minima by looking at the critical points in the interior of S. One must also look at the values of f on the boundary, however.

Example. Find the maxima and minima of $f(x, y) = x^3 + xy$ on the square with vertices $(\pm 1, \pm 1)$.