Functions of Several Variables

Lang has a very specific definition of function. He requires that the output of f is a number. The input can be any subset of n-space.

Example. $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = \sqrt{x^2+y^2}$. We can interpret f as a function that tells us our distance to the origin when we're standing at a point (x,y).

Example. $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x,y,z) = x^2 - \sin(xyz) + yz^3$.

The graph of a function on defined on $S \subset \mathbb{R}^2$ would have the form

$$\{(x, y, f(x,y)) : (x,y) \in S\}.$$

In this case, the graph sits in $\mathbb{R}^3$.

For a fixed number c, the equation $f(x,y) = c$ describes a curve in $\mathbb{R}^2$. Such a curve is called a **level curve**.

Question. What do the level curves of $f(x,y) = x^2 + y^2$ look like? What about $f(x,y) = \sqrt{x^2+y^2}$.

If $f(x,y,z)$ is a function of three variables, the equation $f(x,y,z) = c$ describes a surface, called a **level surface**.

Question. What do the level surfaces of $f(x,y,z) = x^2+y^2+z^2$ look like? What about $f(x,y,z) = 3x^2 + 2y^2 + z$?

Partial Derivatives

First consider a function of two variables $f(x,y)$. If we hold one of the variables fixed and allow the other to vary, we obtain a function of one variables, and we can take the derivative as we did in Calc I:

$$\lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}.$$

This is the **partial derivative with respect to the first variable** or the **partial derivative with respect to x**. The second partial derivative would be

$$\lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

Notations for this include $D_1 f, D_2 f$; $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$; f_x, f_y. And of course, we can extend these ideas to functions of 3 or more variables.

Example. Let $f(x,y) = x^2y^3$. To compute $\partial f/\partial x$, we treat y as a constant and differentiate as usual:

$$\frac{\partial f}{\partial x} = 2xy^3.$$

Similarly,

$$\frac{\partial f}{\partial y} = 3x^2y^2.$$

Geometrically, for functions of two variables, taking a partial derivative corresponds to slicing the graph at $x = a$ or $y = a$ for a constant a and then looking at the slope of the tangent.

Note that $D_i f$ is itself a function that we can evaluate at points.

Example. Let $f(x, y) = \sin(xy)$. Compute $D_2 f(1, \pi)$.

$$D_2 f(x, y) = \cos(xy)x.$$

So then

$$D_2 f(1, \pi) = \cos(\pi) \cdot 1 = -1.$$

Notice that we can use vector notation and write the partial derivative with respect to x_i as

$$(D_i f)(X) = \lim_{h \to 0} \frac{f(X + hE_i) - f(X)}{h}.$$

The **gradient** of a function is the vector-valued function

$$\operatorname{grad} f(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right).$$

One can easily generalize this definition to higher dimensions.

Example. Let $f(x, y, z) = x^2 y \sin(yz)$. Find $\operatorname{grad} f(1, 1, \pi)$.