

## Curve Integrals

Recall (or accept) from physics that the work (which has the same units as energy) done by a constant force  $F$  over a distance  $D$  is  $W = FD$ . This describes the case of the force pointing in the direction of motion. A slightly more general equation is  $W = F \cdot D$ , where  $F$  is the force vector and  $D$  is the displacement vector (imagine pushing a box). But this equation still assumes a straight-line displacement and constant force in a fixed direction. What if our trajectory is a curve  $C(t)$  and the force is a vector quantity  $F(X)$  that depends on position?

If one zooms in close enough on a continuous vector field, it looks constant, and similarly a curve will look like a straight line segment. The work done by the force on a small time interval  $(t, t + \Delta t)$  can then be approximated as

$$F(C(t)) \cdot (C(t + \Delta t) - C(t)).$$

We can rewrite this as

$$F(C(t)) \cdot \frac{C(t + \Delta t) - C(t)}{\Delta t} \Delta t.$$

If we add up these small bits of work and let  $\Delta t \rightarrow 0$ , we end up with an integral.

Thus we define the **integral of  $F$  along  $C$**  from time  $a$  to time  $b$  as

$$\int_C F = \int_a^b F(C(t)) \cdot \frac{dC}{dt} dt.$$

**Example.**  $F(x, y) = (x^2 y, y^3)$ . Find the integral along the straight line from  $(0, 0)$  to  $(1, 1)$ .

We take  $C(t) = (t, t)$ ,  $0 \leq t \leq 1$ .  $C'(t) = (1, 1)$ . Then

$$F(C(t)) = (t^3, t^3).$$

Our integral is then

$$\int_0^1 (t^3, t^3) \cdot (1, 1) dt = \int_0^1 2t^3 dt = 1/2.$$

In 2-space, if we write  $F = (f, g)$ ,  $C(t) = (x(t), y(t))$ , then the curve integral can be expressed

$$\int_C F = \int_C f dx + g dy.$$

Symbolically, the expression  $f dx + g dy = (f, g) \cdot (dx, dy)$ . So one can write

$$\int_C F = \int_a^b \left[ f(x(t), y(t)) \frac{dx}{dt} + g(x(t), y(t)) \frac{dy}{dt} \right] dt.$$

**Remark:** The curve integral is independent of the particular parametrization you take. That is, if  $C_1(t)$  and  $C_2(t)$  trace out the same curve but proceed at different rates, the integral of  $F$  over either curve will be the same.