Points in 2-space, 3-space, and Beyond

In the same way that we can specify a point in the plane with two numbers, we can specify a point in space with three numbers (x, y, z). In general, in n-space ($\mathbb{R}^n$), we can specify a point with a list of n numbers $(x_1, \ldots, x_n)$.

Given two points in $\mathbb{R}^3$, we can define addition on them by adding corresponding coordinates:

$$(a_1, a_2, a_3) + (b_1, b_2, b_3) := (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

In general,

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) := (a_1 + b_1, \ldots, a_n + b_n).$$

Example. Let $A = (2, 3)$, $B = (-1, 1)$. Then $A + B = (1, 4)$. The figure looks like a parallelogram.

Example. Let $A = (3, 1)$, $B = (1, 2)$. Then $A + B = (4, 3)$. We obtain a parallelogram again. This is always the case. Starting from the origin $O = (0, 0)$, we obtain B by moving 1 unit right and then 2 units up. We get $A + B$ by first moving 3 to the right, then 1 up, and then repeating the same movement we did from the origin to B. In other words, the segment connecting O to B and the one connecting A to $A+B$ are equal length and parallel. Similarly, the segments from O to A and B to $B + A = A + B$ will also be equal length and parallel.

We have some not-so-surprising properties of point addition

- $(A + B) + C = A + (B + C)$
- $A + B = B + A$
- $O + A = A + O = A$
- $A + (-A) = O$

where $O = (0, \ldots, 0)$ and $-A = (-a_1, \ldots, -a_n)$. We note that $A \mapsto -A$ corresponds to reflection about the origin.

We can also multiply (or *scale*) a point $A = (a_1, \ldots, a_n)$ by a number c, yielding a point

$$cA = (ca_1, \ldots, ca_n).$$

For example, if $A = (2, -1, 5)$ and $c = 7$, then $cA = (14, -7, 35)$. We again have some easy properties:

- $c(A + B) = cA + cB$
- $(c_1 + c_2)A = c_1A + c_2A$
- $(c_1c_2)A = c_1(c_2A)$.

We should comment on the geometric meaning of scaling by a number c. Let $A = (1, 2)$ and $c = 3$. Then $cA = (3, 6)$. We see that the effect of multiplying by 3 is to stretch the point A away from the origin by a factor of 3. If we set $c = 1/2$, this shrinks A in towards the origin. If we draw a segment from the origin to A, in the former case, scaling by $c = 3$ multiplies the length by 3, and scaling by $c = 1/2$ cuts the length in half.

Vectors

The discussion above leads us naturally to vectors. Given two points A and B, we can define a **located vector** as an ordered pair of points (A, B), which is more often written $\overrightarrow{AB}$. We think of this as an arrow connecting A and B, pointing towards B. Two located vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to be **equivalent** if $B - A = D - C$. We always have that $\overrightarrow{AB}$ is equivalent to $\overrightarrow{O(B-A)}$. This is actually the unique vector starting at the origin that is equivalent to $\overrightarrow{AB}$.

$\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are said to be **parallel** if for some $c \neq 0$, we have $A - B = c(Q - P)$. If $c > 0$ we say the vectors have the *same direction*, and if $c < 0$, we say they have the *opposite direction*.

$\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are said to be **perpendicular** if $B - A$ and $Q - P$ are perpendicular in the usual geometric sense.

A located vector starting from the origin is completely determined by its endpoint. So an n-tuple will be called either a point or a **vector** depending on the context and interpretation.

The Dot Product

If $\vec{x} = (x_1, x_2, \ldots, x_n)$ and $\vec{y} = (y_1, y_2, \ldots, y_n)$, their **dot product** is

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Note

Some useful properties are

1. $A \cdot B = B \cdot A$
2. $A \cdot (B + C) = A \cdot B + A \cdot C = (B + C) \cdot A$
3. If x is a number, $(xA) \cdot B = x(A \cdot B)$, $A \cdot (xB) = x(A \cdot B)$
4. If $A = O$, then $A \cdot A = 0$. Otherwise, $A \cdot A > 0$.

Two vectors A and B are said to be **perpendicular** or **orthogonal** if $A \cdot B = 0$. For the plane and $\mathbb{R}^3$, we will see that this definition agrees with our previous and more geometric definition of perpendicular.

The **norm** or **magnitude** (or **length**) $\|A\|$ of a vector $A = (a_1, \ldots, a_n)$ is

$$\|A\| = \sqrt{A \cdot A} = \sqrt{a_1^2 + \cdots + a_n^2}.$$

Note that $\|-A\| = \|A\|$. More generally, for any number c, we have $\|cA\| = |c|\|A\|$. For two points A, B, the **distance** between them is

$$\|A - B\| = \sqrt{(A - B) \cdot (A - B)}.$$

A vector E is a **unit vector** if $\|E\| = 1$. Dividing a nonzero vector by its norm always yields a unit vector, since

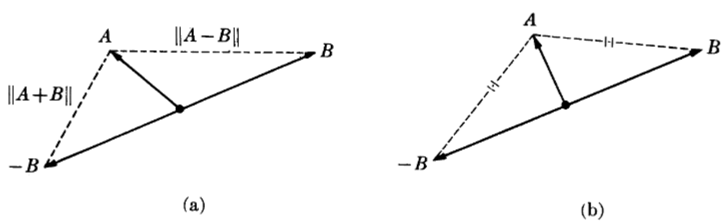
$$\left\| \frac{A}{\|A\|} \right\| = \frac{1}{\|A\|} \|A\| = 1.$$

Two nonzero vectors A and B have the **same direction** if there is some $c > 0$ such that $cA = B$. So, for instance, $A/\|A\|$ is a unit vector in the same direction as A.

Perpendicularity, Angle Between Vectors

We have two notions of "perpendicular" floating around. One says A and B are perpendicular if $A \cdot B = 0$. The other is the more familiar notion of A and B forming a right angle. Suppose that A and B lie in the plane. We can convince ourselves that A and B form a right angle precisely when

$$\|A - B\| = \|A + B\|.$$



If we accept this, then the equivalence of our two definitions of perpendicularity will follow from

$$\|A + B\| = \|A - B\| \iff A \cdot B = 0.$$

To prove this, observe that

$$\begin{aligned}
\|A + B\| = \|A - B\| &\iff \|A + B\|^2 = \|A - B\|^2 \\
&\iff A \cdot A + 2A \cdot B + B \cdot B = A \cdot A - 2A \cdot B + B \cdot B \\
&\iff 4A \cdot B = 0 \\
&\iff A \cdot B = 0.
\end{aligned}$$

Suppose again that we have two nonzero vectors A and B in the plane, located at the origin. If we move along the line through $\overrightarrow{OB}$, there will be some point P on this line such that $\overrightarrow{PA}$ is perpendicular to $\overrightarrow{OB}$. Then $P = cB$ for some number c. Then we have $(A - P) \cdot B = (A - cB) \cdot B = 0$, which is to say

$$A \cdot B - cB \cdot B = 0,$$

so that

$$c = \frac{A \cdot B}{B \cdot B}.$$

Conversely, we see that

$$\left(A - \frac{A \cdot B}{B \cdot B} B\right) \cdot B = A \cdot B - A \cdot B = 0.$$

Thus, this is the unique number c that makes $A - cB$ perpendicular to B. This number c

is called the **component** of A along B. If we do a little plane geometry, we see that

$$\cos\theta = \frac{c\|B\|}{\|A\|},$$

which can be rewritten as

$$\|A\|\|B\|\cos\theta = A \cdot B.$$

Parametric Lines

Given a direction vector A and a point P, the **parametric line** in the direction of A passing through P is given by

$$X(t) = P + tA,$$

where t ranges in $\mathbb{R}$. One can think of this as the position X of a particle or bug as it travels with the passing of time t. $X(t)$ is sometimes called the **position vector** of the particle/bug. The position vector is a vector located at the origin, terminating at the position of the bug.

Given two points P and Q, we can parametrize the line segment between them as

$$X(t) = P + t(Q - P),\ \ 0 \leq t \leq 1.$$

Planes

A plane M in $\mathbb{R}^3$ is determined by two pieces of data: a point P lying on the plane, and a vector N perpendicular to the plane. If we walk from P to some other point X also on the plane, we must have that $X - P$ is perpendicular to N, otherwise X won't lie on the plane M. So the plane is the set of points X satisfying

$$N \cdot (X - P) = 0.$$

Note that this gives a nice interpretation of the equation for a line in the plane $ax+by = c$. (a, b) is a normal vector to the line! If $c = 0$, the equation becomes $(a, b) \cdot (x, y) = 0$, so we have a line through the origin, consisting of all vectors (located at the origin) perpendicular to (a, b). Changing the value of c yields a family of parallel lines.

The Cross Product

If $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ are vectors in $\mathbb{R}^3$, their cross product $A \times B$ is the determinant

$$\begin{vmatrix} E_1 & E_2 & E_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

This is really more of mnemonic device than an actual definition, since the determinant is defined only for matrices with numerical entries.

$A \times B$ is perpendicularto both A and B. We also have *anticommutativity*, meaning $B \times A = -(A \times B)$.

One can verify that $\|A \times B\|^2 = \|A\|^2 \|B\|^2 - (A \cdot B)^2$.

Using our geometric formula for the dot product, we have

$$\begin{aligned}
\|A \times B\|^2 &= \|A\|^2 \|B\|^2 - \|A\|^2 \|B\|^2 \cos^2(\theta) \\
&= \|A\|^2 \|B\|^2 (1 - \cos^2(\theta)) \\
&= \|A\|^2 \|B\|^2 \sin^2(\theta).
\end{aligned}$$

Taking square roots, we have

$$\|A \times B\| = \|A\| \|B\| |\sin(\theta)|.$$

So the magnitude of the cross product is the area of parallelogram spanned by A and B.