Functions of Several Variables

Lang has a very specific definition of function. He requires that the output of f is a number. The input can be any subset of n-space.

Example. $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \sqrt{x^2 + y^2}$. We can interpret f as a function that tells us our distance to the origin when we're standing at a point (x, y).

Example. $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x, y, z) = x^2 - \sin(xyz) + yz^3$.

The graph of a function on defined on $S \subset \mathbb{R}^2$ would have the form

$$\{(x, y, f(x, y)) : (x, y) \in S\}.$$

In this case, the graph sits in $\mathbb{R}^3$.

For a fixed number c, the equation $f(x, y) = c$ describes a curve in $\mathbb{R}^2$. Such a curve is called a **level curve**.

Question. What do the level curves of $f(x, y) = x^2 + y^2$ look like? What about $f(x, y) = \sqrt{x^2 + y^2}$.

If $f(x, y, z)$ is a function of three variables, the equation $f(x, y, z) = c$ describes a surface, called a **level surface**.

Question. What do the level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$ look like? What about $f(x, y, z) = 3x^2 + 2y^2 + z$?

Partial Derivatives

First consider a function of two variables $f(x, y)$. If we hold one of the variables fixed and allow the other to vary, we obtain a function of one variables, and we can take the derivative as we did in Calc I:

$$\lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

This is the **partial derivative with respect to the first variable** or the **partial derivative with respect to x**. The second partial derivative would be

$$\lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

Notations for this include $D_1 f, D_2 f$; $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$; f_x, f_y. And of course, we can extend these ideas to functions of 3 or more variables.

Example. Let $f(x, y) = x^2 y^3$. To compute $\partial f / \partial x$, we treat y as a constant and differentiate as usual:

$$\frac{\partial f}{\partial x} = 2xy^3.$$

Similarly,

$$\frac{\partial f}{\partial y} = 3x^2 y^2.$$

Geometrically, for functions of two variables, taking a partial derivative corresponds to slicing the graph at $x = a$ or $y = a$ for a constant a and then looking at the slope of the tangent.

Note that $D_i f$ is itself a function that we can evaluate at points.

Example. Let $f(x, y) = \sin(xy)$. Compute $D_2 f(1, \pi)$.

$$D_2 f(x, y) = \cos(xy)x.$$

So then

$$D_2 f(1, \pi) = \cos(\pi) \cdot 1 = -1.$$

Notice that we can use vector notation and write the partial derivative with respect to x_i as

$$(D_i f)(X) = \lim_{h \to 0} \frac{f(X + hE_i) - f(X)}{h}.$$

The **gradient** of a function is the vector-valued function

$$\operatorname{grad} f(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right).$$

One can easily generalize this definition to higher dimensions.

Example. Let $f(x, y, z) = x^2 y \sin(yz)$. Find $\operatorname{grad} f(1, 1, \pi)$.

We note that for functions f, g and any constant c

$$\operatorname{grad}(f + g) = \operatorname{grad} f + \operatorname{grad} g, \ \operatorname{grad}(cf) = c \operatorname{grad} f.$$

Continuity, Differentiability, and Gradient

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be points. The distance between them is

$$d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

In the case where $n = 1$ (i.e. when x and y are just numbers), we see the distance is the absolute value of the difference $x - y$ (recall that $\sqrt{x^2} = |x|$).

If a and b are *nonnegative* and $a \leq b$, then $\sqrt{a} \leq \sqrt{b}$. Conversely, if $\sqrt{a} \leq \sqrt{b}$, then $a \leq b$. In particular, we have $|x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2}$. In the same way, $|y| \leq \sqrt{x^2 + y^2}$, and of course, this works for more than just two variables.

Recall the idea of a function being continuous at $x_0 \in X$. Intuitively, this means that as x gets closer to x_0, $f(x)$ gets closer $f(x_0)$.

Definition. A function $f(x, y)$ defined on a region R is **continuous** at a point $(x_0, y_0) \in R$ if, for every positive number ϵ, one can find a corresponding positive number δ_ϵ such that $d((x_0, y_0), (x, y)) < \delta_\epsilon$ (where $(x, y) \in R$) guarantees $d(f(x, y), f(x_0, y_0)) < \epsilon$.

Example. Let's look at a single variable example first. Let $f(x) = 10x$. Let us show that f is continuous at 0. So let $x_0 = 0$. Then $f(x_0) = f(0) = 0$. Now, somebody hands us some $\epsilon > 0$, and we need to respond by finding a $\delta > 0$ such that x being within δ of x_0 guarantees that $f(x)$ is within ϵ of $f(x_0)$. Now,

$$|f(x) - f(x_0)| = |10x| = 10|x|,$$

while

$$|x - x_0| = |x|.$$

Let $\delta = \epsilon/10$. Then if $d(x_0, x) = |x - x_0| < \delta$, we have

$$|f(x) - f(x_0)| = 10|x| = 10|x - x_0| < 10(\epsilon/10) = \epsilon.$$

This shows that f is continuous at $x_0 = 0$.

In this case, the same argument works at any x_0, not just 0.