

## Functions of Several Variables

Lang has a very specific definition of function. He requires that the output of  $f$  is a number. The input can be any subset of  $n$ -space.

**Example.**  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by  $f(x, y) = \sqrt{x^2 + y^2}$ . We can interpret  $f$  as a function that tells us our distance to the origin when we're standing at a point  $(x, y)$ .

**Example.**  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  defined by  $f(x, y, z) = x^2 - \sin(xyz) + yz^3$ .

The graph of a function on defined on  $S \subset \mathbb{R}^2$  would have the form

$$\{(x, y, f(x, y)) : (x, y) \in S\}.$$

In this case, the graph sits in  $\mathbb{R}^3$ .

For a fixed number  $c$ , the equation  $f(x, y) = c$  describes a curve in  $\mathbb{R}^2$ . Such a curve is called a **level curve**.

**Question.** What do the level curves of  $f(x, y) = x^2 + y^2$  look like? What about  $f(x, y) = \sqrt{x^2 + y^2}$ .

If  $f(x, y, z)$  is a function of three variables, the equation  $f(x, y, z) = c$  describes a surface, called a **level surface**.

**Question.** What do the level surfaces of  $f(x, y, z) = x^2 + y^2 + z^2$  look like? What about  $f(x, y, z) = 3x^2 + 2y^2 + z$ ?

## Partial Derivatives

First consider a function of two variables  $f(x, y)$ . If we hold one of the variables fixed and allow the other to vary, we obtain a function of one variables, and we can take the derivative as we did in Calc I:

$$\lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

This is the **partial derivative with respect to the first variable** or the **partial derivative with respect to  $x$** . The second partial derivative would be

$$\lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

Notations for this include  $D_1f, D_2f; \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}; f_x, f_y$ . And of course, we can extend these ideas to functions of 3 or more variables.

**Example.** Let  $f(x, y) = x^2y^3$ . To compute  $\partial f / \partial x$ , we treat  $y$  as a constant and differentiate as usual:

$$\frac{\partial f}{\partial x} = 2xy^3.$$

Similarly,

$$\frac{\partial f}{\partial y} = 3x^2y^2.$$

Geometrically, for functions of two variables, taking a partial derivative corresponds to slicing the graph at  $x = a$  or  $y = a$  for a constant  $a$  and then looking at the slope of the tangent.

Note that  $D_i f$  is itself a function that we can evaluate at points.

**Example.** Let  $f(x, y) = \sin(xy)$ . Compute  $D_2f(1, \pi)$ .

$$D_2f(x, y) = \cos(xy)x.$$

So then

$$D_2f(1, \pi) = \cos(\pi) \cdot 1 = -1.$$

Notice that we can use vector notation and write the partial derivative with respect to  $x_i$  as

$$(D_i f)(X) = \lim_{h \rightarrow 0} \frac{f(X + hE_i) - f(X)}{h}.$$

The **gradient** of a function is the vector-valued function

$$\text{grad } f(x, y) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right).$$

One can easily generalize this definition to higher dimensions.

**Example.** Let  $f(x, y, z) = x^2y \sin(yz)$ . Find  $\text{grad } f(1, 1, \pi)$ .

We note that for functions  $f, g$  and any constant  $c$

$$\text{grad}(f + g) = \text{grad } f + \text{grad } g, \quad \text{grad}(cf) = c \text{grad } f.$$