

Curve Integrals

Recall (or accept) from physics that the work (which has the same units as energy) done by a constant force F over a distance D is $W = FD$. This describes the case of the force pointing in the direction of motion. A slightly more general equation is $W = F \cdot D$, where F is the force vector and D is the displacement vector (imagine pushing a box). But this equation still assumes a straight-line displacement and constant force in a fixed direction. What if our trajectory is a curve $C(t)$ and the force is a vector quantity $F(X)$ that depends on position?

If one zooms in close enough on a continuous vector field, it looks constant, and similarly a curve will look like a straight line segment. The work done by the force on a small time interval $(t, t + \Delta t)$ can then be approximated as

$$F(C(t)) \cdot (C(t + \Delta t) - C(t)).$$

We can rewrite this as

$$F(C(t)) \cdot \frac{C(t + \Delta t) - C(t)}{\Delta t} \Delta t.$$

If we add up these small bits of work and let $\Delta t \rightarrow 0$, we end up with an integral.

Thus we define the **integral of F along C** from time a to time b as

$$\int_C F = \int_a^b F(C(t)) \cdot \frac{dC}{dt} dt.$$

Example. $F(x, y) = (x^2 y, y^3)$. Find the integral along the straight line from $(0, 0)$ to $(1, 1)$.

We take $C(t) = (t, t)$, $0 \leq t \leq 1$. $C'(t) = (1, 1)$. Then

$$F(C(t)) = (t^3, t^3).$$

Our integral is then

$$\int_0^1 (t^3, t^3) \cdot (1, 1) dt = \int_0^1 2t^3 dt = 1/2.$$

In 2-space, if we write $F = (f, g)$, $C(t) = (x(t), y(t))$, then the curve integral can be expressed

$$\int_C F = \int_C f dx + g dy.$$

Symbolically, the expression $f dx + g dy = (f, g) \cdot (dx, dy)$. So one can write

$$\int_C F = \int_a^b \left[f(x(t), y(t)) \frac{dx}{dt} + g(x(t), y(t)) \frac{dy}{dt} \right] dt.$$

Remark: The curve integral is independent of the particular parametrization you take. That is, if $C_1(t)$ and $C_2(t)$ trace out the same curve but proceed at different rates, the integral of F over either curve will be the same.

Example. Compute the integral of $F(x, y) = (x^2, xy)$ on the parabola $x = y^2$ from $(1, -1)$ to $(1, 1)$.

We can parametrize our curve as $C(t) = (t^2, t)$, $-1 \leq t \leq 1$. The integral is then

$$\int_C F \cdot dC = \int_{-1}^1 f(C(t)) \cdot C'(t) dt = \int_{-1}^1 (t^4, t^3) \cdot (2t, 1) dt = \int_{-1}^1 (2t^5 + t^3) dt.$$

Example. Let

$$G(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

Integrate G on the circle of radius 3 centered at the origin from $(3, 0)$ to $(3\sqrt{3}/2, 3/2)$.

We can parametrize the curve C as $C(t) = (3 \cos t, 3 \sin t)$ where $0 \leq t \leq \pi/6$, so that $C'(t) = 3(-\sin t, \cos t)$. Now,

$$G(C(t)) = \left(\frac{-3 \sin t}{9}, \frac{3 \cos t}{9} \right) = \frac{1}{3}(-\sin t, \cos t).$$

So the curve integral is

$$\int_0^{\pi/6} G(C(t)) \cdot C'(t) dt = \int_0^{\pi/6} \frac{1}{3}(-\sin t, \cos t) \cdot [3(-\sin t, \cos t)] dt = \pi/6.$$

Notice that $\pi/6$ is also the change in angle of the parametrized particle over the course of its journey. This is not a coincidence.

An Aside on Differential Forms

A function $f(x, y, z)$ has gradient

$$\text{grad } f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).$$

The **total differential** of f is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

You can view this as a purely symbolic thing, perhaps a fancier way of writing the gradient. However, there is something meaningful about this expression. If we think of dx, dy, dz as small changes in x, y , and z , then this expression gives a way of approximating the corresponding change in f . Remember that differentiability means that for H in some small enough neighborhood of the origin, we can write

$$f(X + H) - f(X) = \text{grad } f(X) \cdot H + \|H\|g(H)$$

where $g(H) \rightarrow 0$ as $H \rightarrow 0$. The left hand side is the change in f , say Δf , going from X to $X + H$. H itself is the change in input, which we could write $H = (\Delta x, \Delta y, \Delta z)$, where we imagine these are small changes in x, y , and z . Dropping the “error” term $\|H\|g(H)$, this reads

$$\Delta f \approx \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + \frac{\partial f}{\partial z} \Delta z.$$

Back to Curve Integrals

Given $x = r \cos \theta$ and $y = r \sin \theta$, we can form their total differentials

$$dx = \cos \theta dr - r \sin \theta d\theta, \quad dy = \sin \theta dr + r \cos \theta d\theta.$$

This is equivalent to

$$dx = \cos \theta dr - y d\theta, \quad dy = \sin \theta dr + x d\theta.$$

We can rewrite this as

$$dx = \frac{x}{\sqrt{x^2 + y^2}} dr - y d\theta, \quad dy = \frac{y}{\sqrt{x^2 + y^2}} dr + x d\theta.$$

In turn, we can say

$$-y dx = \frac{-xy}{\sqrt{x^2 + y^2}} dr - y^2 d\theta, \quad dy = \frac{xy}{\sqrt{x^2 + y^2}} dr + x^2 d\theta.$$

Finally, adding these two equations together and solving for $d\theta$ yields,

$$d\theta = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy.$$

This differential form is telling us that integral of the right hand side above will *always give the change in angle of the position vector over the course of traversing the curve.*

A **path** C is a sequence of curves C_1, \dots, C_m where each C_i is defined on an interval $[a_i, b_i]$ and if we write $P_i = C_i(a_i)$ and $Q_i = C_i(b_i)$, then $P_{i+1} = Q_i$. In other words, one curve ends where the next one starts. The integral along such a path C is defined as

$$\int_C F := \int_{C_1} F + \dots + \int_{C_m} F.$$

A **closed path** is one such that $Q_m = P_1$ (we close the loop).

Example. Evaluate the integral of $F = (x^2, xy)$ along the closed path that goes along $y = x^2$ from $(0, 0)$ to $(1, 1)$, then along the line $y = x$ from $(1, 1)$ back to $(0, 0)$.

We can parametrize this path as two curves C_1 and C_2 . Where $C_1(t) = (t, t^2)$, $0 \leq t \leq 1$ and $C_2(t) = (1 - t, 1 - t)$, $0 \leq t \leq 1$. The integral then becomes

$$\int_C F = \int_0^1 (t^2, t^3) \cdot (1, 2t) dt + \int_0^1 ((1 - t)^2, (1 - t)^2) \cdot (-1, -1) dt = -\frac{1}{3} + \frac{2}{5}.$$

The Reverse Path

For a curve C defined for $a \leq t \leq b$, the **reverse curve** C^- is defined by

$$C^- = C(a + b - t), a \leq t \leq b.$$

Lemma. As one might expect, we have

$$\int_{C^-} F = - \int_C F.$$

This comes from using $u = a + b - t$, $du = -dt$.

Example. Integrate $F(x, y) = (x^2, xy)$ along $y = x$ from $(1, 1)$ to $(0, 0)$. We can use the lemma and instead integrate along the reverse curve and flip the sign of the result. So let $C(t) = (t, t)$, $0 \leq t \leq 1$. Then what we're after is

$$-\int_0^1 2t^2 dt = -2/3.$$

The **reverse path** of C_1, \dots, C_m is C_m^-, \dots, C_1^- .

Path Integrals and Potentials

Theorem. Let F be a vector field on an open set U and suppose $F = \text{grad } \varphi$ for some φ on U . Let C be a path from P to Q . Then

$$\int_C F = \varphi(Q) - \varphi(P).$$

In particular, there is no dependence on the path itself, only on the endpoints.

Proof. Let $g(t) = \varphi(C(t))$. Then $g'(t) = \text{grad } \varphi(C(t)) \cdot C'(t)$. Then we have

$$\int_C F = \int_a^b F(C(t)) \cdot C'(t) dt = \int_a^b \text{grad } \varphi(C(t)) \cdot C'(t) dt.$$

This last integral is the integral of $g'(t)$ from a to b , so this becomes

$$\int_C F = \int_a^b g'(t) dt = g(t) \Big|_a^b = \varphi(C(b)) - \varphi(C(a)) = \varphi(Q) - \varphi(P).$$

□

Corollary. The integral around any closed loop of a conservative vector field is 0. This is because such an integral will end up evaluating to $\varphi(P) - \varphi(P) = 0$.

Notice this gives a test: if you can find a closed loop C with $\int_C F \neq 0$, then F is certainly not conservative.