Points in 3-space and beyond

In the same way that we can specify a point in the plane with two numbers, we can specify a point in space with three numbers (x, y, z). In general, in n-space ($\mathbb{R}^n$), we can specify a point with a list of n numbers $(x_1, \ldots, x_n)$.

Given two points in $\mathbb{R}^3$, we can define addition on them by adding corresponding coordinates:

$$(a_1, a_2, a_3) + (b_1, b_2, b_3) := (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

In general,

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) := (a_1 + b_1, \ldots, a_n + b_n).$$

Example. Let $A = (2, 3)$, $B = (-1, 1)$. Then $A + B = (1, 4)$. The figure looks like a parallelogram.

Example. Let $A = (3, 1)$, $B = (1, 2)$. Then $A + B = (4, 3)$. We obtain a parallelogram again. This is always the case. Starting from the origin $O = (0, 0)$, we obtain B by moving 1 unit right and then 2 units up. We get $A + B$ by first moving 3 to the right, then 1 up, and then repeating the same movement we did from the origin to B. In other words, the segment connecting O to B and the one connecting A to $A+B$ are equal length and parallel. Similarly, the segments from O to A and B to $B + A = A + B$ will also be equal length and parallel.

We have some not-so-surprising properties of point addition

- $(A + B) + C = A + (B + C)$
- $A + B = B + A$
- $O + A = A + O = A$
- $A + (-A) = O$

where $O = (0, \ldots, 0)$ and $-A = (-a_1, \ldots, -a_n)$. We note that $A \mapsto -A$ corresponds to reflection about the origin.

We can also multiply (or *scale*) a point $A = (a_1, \ldots, a_n)$ by a number c, yielding a point

$$cA = (ca_1, \ldots, ca_n).$$

For example, if $A = (2, -1, 5)$ and $c = 7$, then $cA = (14, -7, 35)$. We again have some easy properties:

- $c(A + B) = cA + cB$
- $(c_1 + c_2)A = c_1A + c_2A$
- $(c_1c_2)A = c_1(c_2A)$.

We should comment on the geometric meaning of scaling by a number c. Let $A = (1, 2)$ and $c = 3$. Then $cA = (3, 6)$. We see that the effect of multiplying by 3 is to stretch the point A away from the origin by a factor of 3. If we set $c = 1/2$, this shrinks A in towards the origin. If we draw a segment from the origin to A, in the former case, scaling by $c = 3$ multiplies the length by 3, and scaling by $c = 1/2$ cuts the length in half.

Vectors

The discussion above leads us naturally to vectors. Given two points A and B, we can define a **located vector** as an ordered pair of points (A, B), which is more often written $\overrightarrow{AB}$. We think of this as an arrow connecting A and B, pointing towards B. Two located vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to be **equivalent** if $B - A = D - C$. We always have that $\overrightarrow{AB}$ is equivalent to $\overrightarrow{O(B-A)}$. This is actually the unique vector starting at the origin that is equivalent to $\overrightarrow{AB}$.

$\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are said to be **parallel** if for some $c \neq 0$, we have $A - B = c(Q - P)$. If $c > 0$ we say the vectors have the *same direction*, and if $c < 0$, we say they have the *opposite direction*.

$\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are said to be **perpendicular** if $B - A$ and $Q - P$ are perpendicular in the usual geometric sense.

The Dot Product