Let us see how Green's theorem works in the case of a rectangular region. Let $R = [a, b] \times [c, d]$ be a rectangular region, and let C denote the boundary curve, which is a rectangle, oriented counterclockwise. Let $F = \langle M, N \rangle$ be a smooth vector field in the plane, meaning $M(x, y)$ and $N(x, y)$ are smooth real-valued functions. The integral of F around C is

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C M dx + N dy = \oint_C M dx + \oint_C N dy.$$

Let us inspect $\oint_C M dx$. If we parametrize C and evaluate this integral, we notice that the vertical segments contribute nothing, since $dx = 0$ when we traverse a vertical segment. So the integral reduces to

$$\oint_C M dx = \int_a^b M(x, c) dx - \int_a^b M(x, d) dx = \int_a^b (M(x, c) - M(x, d))\, dx.$$

Now, for a fixed x, $M(x, y)$ is a function of y which we can differentiate with respect to y, as $\partial M(x, y)/\partial y$. The fundamental theorem of calculus tells us that

$$M(x, d) - M(x, c) = \int_c^d \frac{\partial M(x, y)}{\partial y} dy.$$

Substituting this back into our prior integral, noting the difference is sign, we obtain

$$\oint_C M dx = \int_a^b \int_c^d -\frac{\partial M(x, y)}{\partial y} dy dx = \int_R -\frac{\partial M}{\partial y} dA.$$

The exact same work will show that

$$\int_C N dy = \int_R \frac{\partial N}{\partial x} dA,$$

where this time there's no negative sign since the orientations agree now.