Differentiation

Imagine a bug that moves with constant speed on a circular path of radius r around the origin. The angle of the bug's position vector with the $+x$ axis can be written as

$$\theta = \omega t + a.$$

Assume $a = 0$, so that the bug is on the $+x$ axis at time 0. Then the position vector of the bug is

$$X(t) = (r\cos(\omega t), r\sin(\omega t)).$$

Now imagine the bug lives in $\mathbb{R}^3$ with

$$X(t) = (\cos(t), \sin(t), t).$$

This lifts the circular path into a helix.

In general, a **parametrized curve** $X : I \to \mathbb{R}^n$ is a vector-valued function that maps points from an interval I into n-space. In the examples above, I is the entire real line $\mathbb{R}$ (which we consider to be an interval). We can write $X(t)$ as its individual coordinate functions

$$X(t) = (x_1(t), \ldots, x_n(t)).$$

Just as with ordinary real-valued function, we can take derivatives by looking at the limit

$$\lim_{h \to 0} \frac{X(t+h) - X(t)}{h}.$$

Here, dividing by h really means scaling the vector by $1/h$. Writing out components, this is simply

$$\lim_{h \to 0} \frac{(x_1(t+h) - x_1(t), \ldots, x_n(t+h) - x_n(t))}{h}.$$

If the individual components are all differentiable, we obtain a new vector-valued function

$$X'(t) = (x_1'(t), \ldots, x_n'(t)).$$

$X'(t)$ is called the **derivative** or **velocity** of $X(t)$.

So for the example $X(t) = (\cos(t), \sin(t), t)$, we have

$$X'(t) = (-\sin(t), \cos(t), 1).$$

The velocity is parallel to the direction of instantaneous motion.

Example. Find a parametric equation of the tangent line to the curve $X(t) = (\sin t, \cos t)$ at $t = \pi/3$.

We need two pieces of information: a point on the line, and a direction vector of the line. These are supplied by $X(\pi/3)$ and $X'(\pi/3)$ respectively. The tangent line $L(t)$ can thus be written

$$
\begin{aligned}
L(s)|_{t=\pi/3} &= X(\pi/3) + sX'(\pi/3) \\
&= \left(\frac{\sqrt{3}}{2} + \frac{1}{2}s,\ \frac{1}{2} - \frac{\sqrt{3}}{2}s \right).
\end{aligned}
$$

We used the parameter s for the line to avoid confusion with the already defined $X(t)$ above.

The **speed** of the curve $X(t)$, denoted $v(t)$, is defined to be

$$v(t) = \|X'(t)\|.$$