

The Chain Rule

Example. Let $f(x, y) = e^x \sin(xy)$. One can imagine this describes the temperature of the plane at each point (x, y) . Now imagine a bug moving in the plane with parametrization $C(t) = (t^2, t^3)$. Then the temperature the bug is feeling at time t is

$$f(C(t)) = e^{t^2} \sin(t^5).$$

Of course, we could compute the derivative of this directly, but there's another way.

Chain Rule. Let f be differentiable on an open set U and let $C(t)$ be a differentiable curve contained in U . Then

$$\frac{d}{dt} f \circ C(t) = (\text{grad } f)(C(t)) \cdot C'(t).$$

Suppose we're in the two-variable case and $C(t) = (x(t), y(t))$. We could rewrite the chain rule as

$$\frac{d}{dt} f \circ C(t) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

where the partial derivatives are of course evaluated at $(x(t), y(t))$.

Example. Let $f(x, y, z) = x^2 yz$ and $C(t) = (x(t), y(t), z(t)) = (e^t, t, t^2)$. Then

$$\begin{aligned} (f \circ C)'(t) &= (D_1 f)x'(t) + (D_2 f)y'(t) + (D_3 f)z'(t) \\ &= 2xyz e^t + x^2 z + x^2 y(2t) \\ &= 2e^{2t} t^3 + e^{2t} t^2 + 2e^{2t} t^2. \end{aligned}$$

There are situations where we need only use the standard single variable chain rule.

Example. Let $f(x, y, z) = \sin(x^2 - 3yz + xz)$. Then

$$\frac{\partial f}{\partial x} = \cos(x^2 - 3yz + xz)(2x + z).$$

Tangent Plane

Let $f(x, y, z)$ be a function on \mathbb{R}^3 . Imagine that f models the temperature at each point of the space and that we have a bug moving along a curve $B(t) = (x(t), y(t), z(t))$ in space. Assume the bug started at a point with a comfortable temperature k and so decides to stick to points with temperature k . That is, the bug is moving on the level surface

$$f(x, y, z) = k.$$

That is, we have for all t that

$$f(B(t)) = k.$$

Applying chain rule, we have

$$(\text{grad } f)(B(t)) \cdot B'(t) = 0.$$

So the gradient of f is perpendicular to the path of the bug at every point.