Inequalities, absolute value, distance, etc.

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be points. The distance between them is

$$d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

In the case where $n = 1$ (i.e. when x and y are just numbers), we see the distance is the absolute value of the difference $x - y$ (recall that $\sqrt{x^2} = |x|$).

If a and b are *nonnegative* and $a \leq b$, then $\sqrt{a} \leq \sqrt{b}$. Conversely, if $\sqrt{a} \leq \sqrt{b}$, then $a \leq b$. In particular, we have $|x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2}$. In the same way, $|y| \leq \sqrt{x^2 + y^2}$, and of course, this works for more than just two variables.

Limits

Let $f : X \to Y$ be a function taking points in X to points in Y. In Calc I, X and Y were usually both $\mathbb{R}$. Now we allow X to be $\mathbb{R}^2$ or $\mathbb{R}^3$.

Recall the idea of a function having a limit at a point $x_0 \in X$. Intuitively, this means there is some value L such that when x gets closer to x_0, $f(x)$ gets closer to this value L. Before we get lost in technicalities, let's look at some examples of evaluating limits of multivariable functions.

Example. Compute

$$\lim_{(x,y)\to(1,\pi)} f(x, y) = \frac{x}{y} + \cos(xy).$$

In this case, $f(x, y)$ is continuous at $(1, \pi)$, so evaluating this limit amounts to plugging in $(1, \pi)$, so the limit is $1/\pi + \cos(\pi) = 1/\pi - 1$.

Definition 0.1. A function $f(x, y)$ defined on a region R is **continuous** at a point $(x_0, y_0) \in R$ if, for every positive number ϵ, one can find a corresponding positive number δ_ϵ such that $d((x_0, y_0), (x, y)) < \delta_\epsilon$ (where $(x, y) \in R$) guarantees $d(f(x, y), f(x_0, y_0)) < \epsilon$.

If $f(x, y)$ is real-valued (as most of our functions will be), $d(f(x, y), f(x_0, y_0))$ is the same thing as $|f(x, y) - f(x_0, y_0)|$.

Definition 0.2. We say that the **limit** of $f(x, y)$ as (x, y) approaches (x_0, y_0) equals L if, for every positive number ϵ, one can find a corresponding positive number δ_ϵ such that $0 < d((x_0, y_0), (x, y)) < \delta_\epsilon$ guarantees $d(f(x, y), L) < \epsilon$. One writes

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L.$$

Note in that in the definition of limit, we require that $(x, y) \neq (x_0, y_0)$ (this is equivalent to saying $0 < d((x_0, y_0), (x, y))$).

Perhaps this nonsense is best understood through some examples. Consider the function

$$f(x, y) = \frac{x^2 y^2}{x^2 + y^2}.$$

This function is not defined at $(0, 0)$, but does it still have a limit as $(x, y) \to (0, 0)$?

One would like to get away from the madness of using deltas and epsilons to demonstrate that a function has a limit at a given point. Fortunately, by establishing a few properties and formulas, one can then wield these properties and formulas to compute a wide variety of limits.

Proposition 0.3. (Limit properties)
Suppose for the below that $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$ and $\lim_{(x,y)\to(x_0,y_0)} g(x, y) = M$. Here, f and g are real-valued.

- $\lim_{(x,y)\to(x_0,y_0)} x = x_0$
- $\lim_{(x,y)\to(x_0,y_0)} y = y_0$
- $\lim_{(x,y)\to(x_0,y_0)} f(x, y) \pm g(x, y) = M \pm L$
- $\lim_{(x,y)\to(x_0,y_0)} f(x, y)g(x, y) = LM$
- $\lim_{(x,y)\to(x_0,y_0)} f(x, y)/g(x, y) = L/M$ provided $M \neq 0$.

Showing that these properties hold is a little technical, so we'll take them on faith.