Curve Integrals

Recall (or accept) from physics that the work (which has the same units as energy) done by a constant force F over a distance D is $W = FD$. This describes the case of the force pointing in the direction of motion. A slightly more general equation is $W = F \cdot D$, where F is the force vector and D is the displacement vector (imagine pushing a box). But this equation still assumes a straight-line displacement and constant force in a fixed direction. What if our trajectory is a curve $C(t)$ and the force is a vector quantity $F(X)$ that depends on position?

If one zooms in close enough on a continuous vector field, it looks constant, and similarly a curve will look like a straight line segment. The work done by the force on a small time interval $(t, t + \Delta t)$ can then be approximated as

$$F(C(t)) \cdot (C(t + \Delta t) - C(t)).$$

We can rewrite this as

$$F(C(t)) \cdot \frac{C(t + \Delta t) - C(t)}{\Delta t} \Delta t.$$

If we add up these small bits of work and let $\Delta t \to 0$, we end up with an integral.

Thus we define the **integral of F along C** from time a to time b as

$$\int_C F = \int_a^b F(C(t)) \cdot \frac{dC}{dt} dt.$$

Example. $F(x, y) = (x^2 y, y^3)$. Find the integral along the straight line from $(0, 0)$ to $(1, 1)$.

We take $C(t) = (t, t)$, $0 \le t \le 1$. $C'(t) = (1, 1)$. Then

$$F(C(t)) = (t^3, t^3).$$

Our integral is then

$$\int_0^1 (t^3, t^3) \cdot (1, 1) dt = \int_0^1 2t^3 dt = 1/2.$$

In 2-space, if we write $F = (f, g)$, $C(t) = (x(t), y(t))$, then the curve integral can be expressed

$$\int_C F = \int_C f dx + g dy.$$

Symbolically, the expression $f dx + g dy = (f, g) \cdot (dx, dy)$. So one can write

$$\int_C F = \int_a^b \left[f(x(t), y(t)) \frac{dx}{dt} + g(x(t), y(t)) \frac{dy}{dt} \right] dt.$$

Remark: The curve integral is independent of the particular parametrization you take. That is, if $C_1(t)$ and $C_2(t)$ trace out the same curve but proceed at different rates, the integral of F over either curve will be the same.

Example. Compute the integral of $F(x, y) = (x^2, xy)$ on the parabola $x = y^2$ from $(1, -1)$ to $(1, 1)$.

We can parametrize our curve as $C(t) = (t^2, t)$, $-1 \leq t \leq 1$. The integral is then

$$\int_C F \cdot dC = \int_{-1}^{1} f(C(t)) \cdot C'(t)dt = \int_{-1}^{1} (t^4, t^3) \cdot (2t + 1)dt = \int_{-1}^{1} (2t^5 + t^3)dt.$$