# The Chain Rule

**Example.** Let $f(x, y) = e^x \sin(xy)$. One can imagine this describes the temperature of the plane at each point $(x, y)$. Now imagine a bug moving in the plane with parametrization $C(t) = (t^2, t^3)$. Then the temperature the bug is feeling at time $t$ is

$$f(C(t)) = e^{t^2} \sin\bigl(t^5\bigr).$$

Of course, we could compute the derivative of this directly, but there's another way.

**Chain Rule.** Let $f$ be differentiable on an open set $U$ and let $C(t)$ be a differentiable curve contained in $U$. Then

$$\frac{d}{dt} f \circ C(t) = (\operatorname{grad} f)(C(t)) \cdot C'(t).$$

Suppose we're in the two-variable case and $C(t) = (x(t), y(t))$. We could rewrite the chain rule as

$$\frac{d}{dt} f \circ C(t) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

where the partial derivatives are of course evaluated at $(x(t), y(t))$.

**Example.** Let $f(x, y, z) = x^2 yz$ and $C(t) = (x(t), y(t), z(t)) = (e^t, t, t^2)$. Then

$$\begin{aligned}
(f \circ C)'(t) &= (D_1 f)x'(t) + (D_2 f)y'(t) + (D_3 f)z'(t) \\
&= 2xyze^t + x^2 z + x^2 y(2t) \\
&= 2e^{2t}t^3 + e^{2t}t^2 + 2e^{2t}t^2.
\end{aligned}$$