The Chain Rule

Example. Let $f(x,y) = e^x \sin(xy)$. One can imagine this describes the temperature of the plane at each point (x,y). Now imagine a bug moving in the plane with parametrization $C(t) = (t^2, t^3)$. Then the temperature the bug is feeling at time t is

$$f(C(t)) = e^{t^2} \sin\big(t^5\big).$$

Of course, we could compute the derivative of this directly, but there's another way.

Chain Rule. Let f be differentiable on an open set U and let $C(t)$ be a differentiable curve contained in U. Then

$$\frac{d}{dt} f \circ C(t) = (\operatorname{grad} f)(C(t)) \cdot C'(t).$$

Suppose we're in the two-variable case and $C(t) = (x(t), y(t))$. We could rewrite the chain rule as

$$\frac{d}{dt} f \circ C(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

where the partial derivatives are of course evaluated at $(x(t), y(t))$.

Example. Let $f(x,y,z) = x^2yz$ and $C(t) = (x(t), y(t), z(t)) = (e^t, t, t^2)$. Then

$$\begin{aligned}(f \circ C)'(t) &= (D_1 f)x'(t) + (D_2 f)y'(t) + (D_3 f)z'(t)\\ &= 2xyze^t + x^2 z + x^2 y(2t)\\ &= 2e^{2t}t^3 + e^{2t}t^2 + 2e^{2t}t^2.\end{aligned}$$

There are situations where we need only use the standard single variable chain rule.

Example. Let $f(x,y,z) = \sin\big(x^2 - 3yz + xz\big)$. Then

$$\frac{\partial f}{\partial x} = \cos\big(x^2 - 3yz + xz\big)(2x + z).$$

Tangent Plane

Let $f(x,y,z)$ be a function on $\mathbb{R}^3$. Imagine that f models the temperature at each point of the space and that we have a bug moving along a curve $B(t) = (x(t), y(t), z(t))$ in space. Assume the bug started at a point with a comfortable temperature k and so decides to stick to points with temperature k. That is, the bug is moving on the level surface

$$f(x,y,z) = k.$$

That is, we have for all t that

$$f(B(t)) = k.$$

Applying chain rule, we have

$$(\operatorname{grad} f)(B(t)) \cdot B'(t) = 0.$$

So the gradient of f is perpendicular to the path of the bug at every point.

In general, if we fix a point P on a level surface $f(x, y, z) = k$ and look at all differentiable curves passing through P at, say, $t = 0$, the above computation shows that all such curves will be be perpendicular to grad $f(P)$ at $t = 0$ (see the following figure from Lang). Thus, in

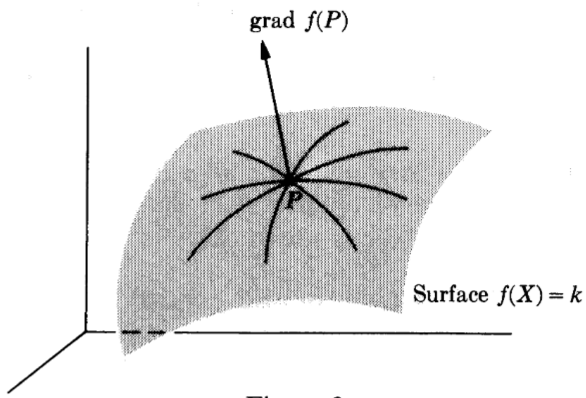

Figure 2

a very real sense, grad $f(P)$ is perpendicular to the surface $f(x, y, z) = k$ itself. This leads to the following definition.

Definition. The **tangent plane** to $f(X) = k$ at P is the plane through P, perpendicular to grad $f(P)$.

Example. Find the tangent plane to $x^2 + y^2 + z^2 = 3$ at the point $(1, 1, 1)$.

Note that this a level surface of the function $f(x, y, z) = x^2 + y^2 + z^3$ (corresponding to $f = 3$). So our normal vector is $N = (2x, 2y, 2z)|_{(1,1,1)} = (2, 2, 2)$. The plane equation is then

$$(2, 2, 2) \cdot (x - 1, y - 1, z - 1) = 0,$$

or

$$x + y + z = 3.$$

We can use the same techniques to find tangent lines to curves in $\mathbb{R}^2$.

Example. Find the tangent line to $x^2y + y^3 = 10$ at $P = (1, 2)$.

We set $f(x, y,) = x^2y + y^3$. Then $\boldsymbol{\nabla}(f)(P) = (4, 13)$. Using the plane equation, our line is described by

$$4x + 13y = 30.$$

If a surface is described as the graph of $z = g(x, y)$, we can still use the techniques above by noting that the surface described by $z = g(x, y)$ is a level surface of the three-variable function $w(x, y, z,) = g(x, y) - z$. In fact, one sees that the surface is precisely the level set corresponding to $w = 0$. So a normal vector to the surface is given by

$$\operatorname{grad} w = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, -1 \right).$$