Introduction

First, I'd like to say that mathematics can be very difficult, and everyone has "blind spots" in their understanding. Don't put pressure on yourself to know it all. Just do as much you reasonably can.

Also, I want you to **ask questions**. This is an important part of learning mathematics (and anything else, really). Asking a question is not some sign of weakness or a lack of intelligence. I want you to ask a question even if it feels "stupid". Confusion is a normal state of affairs in mathematics, and it can take a lot of time and effort to clear this confusion up. Quite often, you just need the right shift of perspective to make sense of something, and different people have different ways of understanding any one given thing. This is to say that an explanation that works very well for one person may be very unclear for another.

Points in 3-space and beyond

In the same way that we can specify a point in the plane with two numbers, we can specify a point in space with three numbers (x, y, z). In general, in n-space ($\mathbb{R}^n$), we can specify a point with a list of n numbers $(x_1, \ldots, x_n)$.

Given two points in $\mathbb{R}^3$, we can define addition on them by adding corresponding coordinates:

$$(a_1, a_2, a_3) + (b_1, b_2, b_3) := (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

In general,

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) := (a_1 + b_1, \ldots, a_n + b_n).$$

Example. Let $A = (2, 3)$, $B = (-1, 1)$. Then $A + B = (1, 4)$. The figure looks like a parallelogram.

Example. Let $A = (3, 1)$, $B = (1, 2)$. Then $A + B = (4, 3)$. We obtain a parallelogram again. This is always the case. Starting from the origin $O = (0, 0)$, we obtain B by moving 1 unit right and then 2 units up. We get $A + B$ by first moving 3 to the right, then 1 up, and then repeating the same movement we did from the origin to B. In other words, the segment connecting O to B and the one connecting A to $A+B$ are equal length and parallel. Similarly, the segments from O to A and B to $B + A = A + B$ will also be equal length and parallel.

We have some not-so-surprising properties of point addition

- $(A + B) + C = A + (B + C)$
- $A + B = B + A$
- $O + A = A + O = A$
- $A + (-A) = O$

where $O = (0, \ldots, 0)$ and $-A = (-a_1, \ldots, -a_n)$. We note that $A \mapsto -A$ corresponds to reflection about the origin.

We can also multiply (or *scale*) a point $A = (a_1, \ldots, a_n)$ by a number c, yielding a point

$$cA = (ca_1, \ldots, ca_n).$$

For example, if $A = (2, -1, 5)$ and $c = 7$, then $cA = (14, -7, 35)$. We again have some easy properties:

- $c(A + B) = cA + cB$
- $(c_1 + c_2)A = c_1A + c_2A$
- $(c_1c_2)A = c_1(c_2A)$.

We should comment on the geometric meaning of scaling by a number c. Let $A = (1, 2)$ and $c = 3$. Then $cA = (3, 6)$. We see that the effect of multiplying by 3 is to stretch the point A away from the origin by a factor of 3. If we set $c = 1/2$, this shrinks A in towards the origin. If we draw a segment from the origin to A, in the former case, scaling by $c = 3$ multiplies the length by 3, and scaling by $c = 1/2$ cuts the length in half.

Vectors

The discussion above leads us naturally to vectors. Given two points A and B, we can define a **located vector** as an ordered pair of points (A, B), which is more often written $\overrightarrow{AB}$. We think of this as an arrow connecting A and B, pointing towards B. Two located vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to be **equivalent** if $B - A = D - C$. We always have that $\overrightarrow{AB}$ is equivalent to $\overrightarrow{O(B - A)}$. This is actually the unique vector starting at the origin that is equivalent to $\overrightarrow{AB}$.