

## Points in 3-space and beyond

In the same way that we can specify a point in the plane with two numbers, we can specify a point in space with three numbers  $(x, y, z)$ . In general, in  $n$ -space ( $\mathbb{R}^n$ ), we can specify a point with a list of  $n$  numbers  $(x_1, \dots, x_n)$ .

Given two points in  $\mathbb{R}^3$ , we can define addition on them by adding corresponding coordinates:

$$(a_1, a_2, a_3) + (b_1, b_2, b_3) := (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

In general,

$$(a_1, \dots, a_n) + (b_1, \dots, b_n) := (a_1 + b_1, \dots, a_n + b_n).$$

**Example.** Let  $A = (2, 3)$ ,  $B = (-1, 1)$ . Then  $A + B = (1, 4)$ . The figure looks like a parallelogram.

**Example.** Let  $A = (3, 1)$ ,  $B = (1, 2)$ . Then  $A + B = (4, 3)$ . We obtain a parallelogram again. This is always the case. Starting from the origin  $O = (0, 0)$ , we obtain  $B$  by moving 1 unit right and then 2 units up. We get  $A + B$  by first moving 3 to the right, then 1 up, and then repeating the same movement we did from the origin to  $B$ . In other words, the segment connecting  $O$  to  $B$  and the one connecting  $A$  to  $A + B$  are equal length and parallel. Similarly, the segments from  $O$  to  $A$  and  $B$  to  $B + A = A + B$  will also be equal length and parallel.

We have some not-so-surprising properties of point addition

- $(A + B) + C = A + (B + C)$
- $A + B = B + A$
- $O + A = A + O = A$
- $A + (-A) = O$

where  $O = (0, \dots, 0)$  and  $-A = (-a_1, \dots, -a_n)$ . We note that  $A \mapsto -A$  corresponds to reflection about the origin.

We can also multiply (or *scale*) a point  $A = (a_1, \dots, a_n)$  by a number  $c$ , yielding a point

$$cA = (ca_1, \dots, ca_n).$$

For example, if  $A = (2, -1, 5)$  and  $c = 7$ , then  $cA = (14, -7, 35)$ . We again have some easy properties:

- $c(A + B) = cA + cB$
- $(c_1 + c_2)A = c_1A + c_2A$
- $(c_1c_2)A = c_1(c_2A)$ .

We should comment on the geometric meaning of scaling by a number  $c$ . Let  $A = (1, 2)$  and  $c = 3$ . Then  $cA = (3, 6)$ . We see that the effect of multiplying by 3 is to stretch the point  $A$  away from the origin by a factor of 3. If we set  $c = 1/2$ , this shrinks  $A$  in towards the origin. If we draw a segment from the origin to  $A$ , in the former case, scaling by  $c = 3$  multiplies the length by 3, and scaling by  $c = 1/2$  cuts the length in half.

## Vectors

The discussion above leads us naturally to vectors. Given two points  $A$  and  $B$ , we can define a **located vector** as an ordered pair of points  $(A, B)$ , which is more often written  $\overrightarrow{AB}$ . We think of this as an arrow connecting  $A$  and  $B$ , pointing towards  $B$ . Two located vectors  $\overrightarrow{AB}$  and  $\overrightarrow{CD}$  are said to be **equivalent** if  $B - A = D - C$ . We always have that  $\overrightarrow{AB}$  is equivalent to  $\overrightarrow{O(B - A)}$ . This is actually the unique vector starting at the origin that is equivalent to  $\overrightarrow{AB}$ .

$\overrightarrow{AB}$  and  $\overrightarrow{PQ}$  are said to be **parallel** if for some  $c \neq 0$ , we have  $A - B = c(Q - P)$ . If  $c > 0$  we say the vectors have the *same direction*, and if  $c < 0$ , we say they have the *opposite direction*.

$\overrightarrow{AB}$  and  $\overrightarrow{PQ}$  are said to be **perpendicular** if  $B - A$  and  $Q - P$  are perpendicular in the usual geometric sense.

A located vector starting from the origin is completely determined by its endpoint. So an  $n$ -tuple will be called either a point or a **vector** depending on the context and interpretation.

## The Dot Product

If  $\vec{x} = (x_1, x_2, \dots, x_n)$  and  $\vec{y} = (y_1, y_2, \dots, y_n)$ , their **dot product** is

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n.$$

Note

Some useful properties are

1.  $A \cdot B = B \cdot A$
2.  $A \cdot (B + C) = A \cdot B + A \cdot C = (B + C) \cdot A$
3. If  $x$  is a number,  $(xA) \cdot B = x(A \cdot B)$ ,  $A \cdot (xB) = x(A \cdot B)$
4. If  $A = O$ , then  $A \cdot A = 0$ . Otherwise,  $A \cdot A > 0$ .

Two vectors  $A$  and  $B$  are said to be **perpendicular** or **orthogonal** if  $A \cdot B = 0$ . For the plane and  $\mathbb{R}^3$ , we will see that this definition agrees with our previous and more geometric definition of perpendicular.

The **norm**  $\|A\|$  of a vector  $A = (a_1, \dots, a_n)$  is

$$\|A\| = \sqrt{A \cdot A} = \sqrt{a_1^2 + \dots + a_n^2}.$$