Bubble Entropy: An Entropy Almost Free of Parameters

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Biomedical Signal Processing Final 25/26
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INTRODUCTION

What is entropy?

Entropy

Thermodynamics

The unavailability of a system's thermal energy for conversion into mechanical work

Information Theory

a measure of the unpredictability

Time Series Analysis

a measure of complexity

Problem with current entropy measures

Strong Dependence on parameters

Small parameter changes → large entropy changes

Results become subjective and dataset-dependent

Sample Entropy

$$\mathrm{SampEn}(m, r) = -\ln\left(\frac{A}{B}\right)$$

where:

- B: number of pairs of vectors X_i^m, X_j^m with distance $\leq r$
- A: number of pairs of vectors X_i^{m+1}, X_j^{m+1} with distance $\leq r$
- Self-matches are **excluded**

Permutation Entropy

$$\mathrm{PE}(m) = -\sum_{\pi} p(\pi)\,\log p(\pi)$$

where:

- $p(\pi)$ is the relative frequency of permutation pattern π

Conditional Permutation Entropy

$$\mathrm{CPE}(m) = \mathrm{PE}(m+1) - \mathrm{PE}(m)$$

Renyi Permutation Entropy

$$\mathrm{RpEn}_{\alpha}(m) = \frac{\log\left(\sum_{j=1}^{m!} p_j^{\alpha}\right)}{(1-\alpha)\log(m)}$$

Bubble Entropy

$$x = \{x_1, x_2, \ldots, x_N\}$$

Bubble Entropy

$$\mathbf{X}_i^{(m)} = (x_i, x_{i+1}, \ldots, x_{i+m-1}), \quad i = 1, \ldots, N - m + 1$$

Bubble Entropy

$$n_i^{(m)} \in \left[0, \frac{m(m-1)}{2}\right]$$

Bubble Entropy

$$p_k^{(m)} = \frac{\#\{i : n_i^{(m)} = k\}}{N - m + 1}$$

Bubble Entropy

$$H_{\text{swaps}}^{(m)} = -\log\left(\sum_{k}\left(p_k^{(m)}\right)^2\right)$$

Bubble Entropy

$$\boxed{\mathrm{bEn}(m) = \frac{H_{\mathrm{swaps}}^{(m+1)} - H_{\mathrm{swaps}}^{(m)}}{\log(m+1)}}$$

$$x = [4,\ 1,\ 3,\ 2,\ 0,\ 5,\ 6] \quad m = 3$$

1. $(4, 1, 3)$ swaps $= 2$
2. $(1, 3, 2)$ swaps $= 1$
3. $(3, 2, 0)$ swaps $= 3$
4. $(2, 0, 5)$ swaps $= 1$
5. $(0, 5, 6)$ swaps $= 0$

$$n^{(3)} = [2, 1, 3, 1, 0]$$

1. $(4, 1, 3, 2)$ swaps $= 4$
2. $(1, 3, 2, 0)$ swaps $= 4$
3. $(3, 2, 0, 5)$ swaps $= 3$
4. $(2, 0, 5, 6)$ swaps $= 1$

$$n^{(4)} = [4, 4, 3, 1]$$

$$k = 0\colon 1/5$$
$$k = 1\colon 2/5$$
$$k = 2\colon 1/5$$
$$k = 3\colon 1/5$$

$$\sum_k (p_k^{(3)})^2 = (1/5)^2 + (2/5)^2 + (1/5)^2 + (1/5)^2 = \frac{7}{25}$$

$$k = 1\colon 1/4$$
$$k = 3\colon 1/4$$
$$k = 4\colon 2/4$$

$$\sum_k (p_k^{(4)})^2 = (1/4)^2 + (1/4)^2 + (2/4)^2 = \frac{6}{16} = \frac{3}{8}$$

$$H_{\text{swaps}}^{(m)} = -\ln\left(\sum_k (p_k^{(m)})^2\right)$$

$$H_{\text{swaps}}^{(3)} = -\ln(7/25) = \ln(25/7) \approx 1.2730$$

$$H_{\text{swaps}}^{(4)} = -\ln(3/8) = \ln(8/3) \approx 0.9808$$

numerator

$$0.9808 - 1.2730 = -0.2922$$

denominator

$$1 + \frac{m(m-1)}{2} = 1 + \frac{3 \cdot 2}{2} = 4$$

$$1 + \frac{(m+1)m}{2} = 1 + \frac{4 \cdot 3}{2} = 7$$

$$\ln(7/4) \approx 0.5596$$

$$bEn(3) \approx \frac{-0.2922}{0.5596} = -0.522$$

Why bubble sort?

Deterministic and interpretable

Measure

Number of swaps needed to sort, not the actual permutations

Parameter Fine Tuning

Only one parameter, with not a lot of importance

Unlike other entropy measures: Bubble Entropy does NOT measure Entropy rate!

Datasets used

1.

Synthetic Signals

Autoregressive (AR) models

White noise input → maximum entropy

Used for stability and convergence analysis

2.

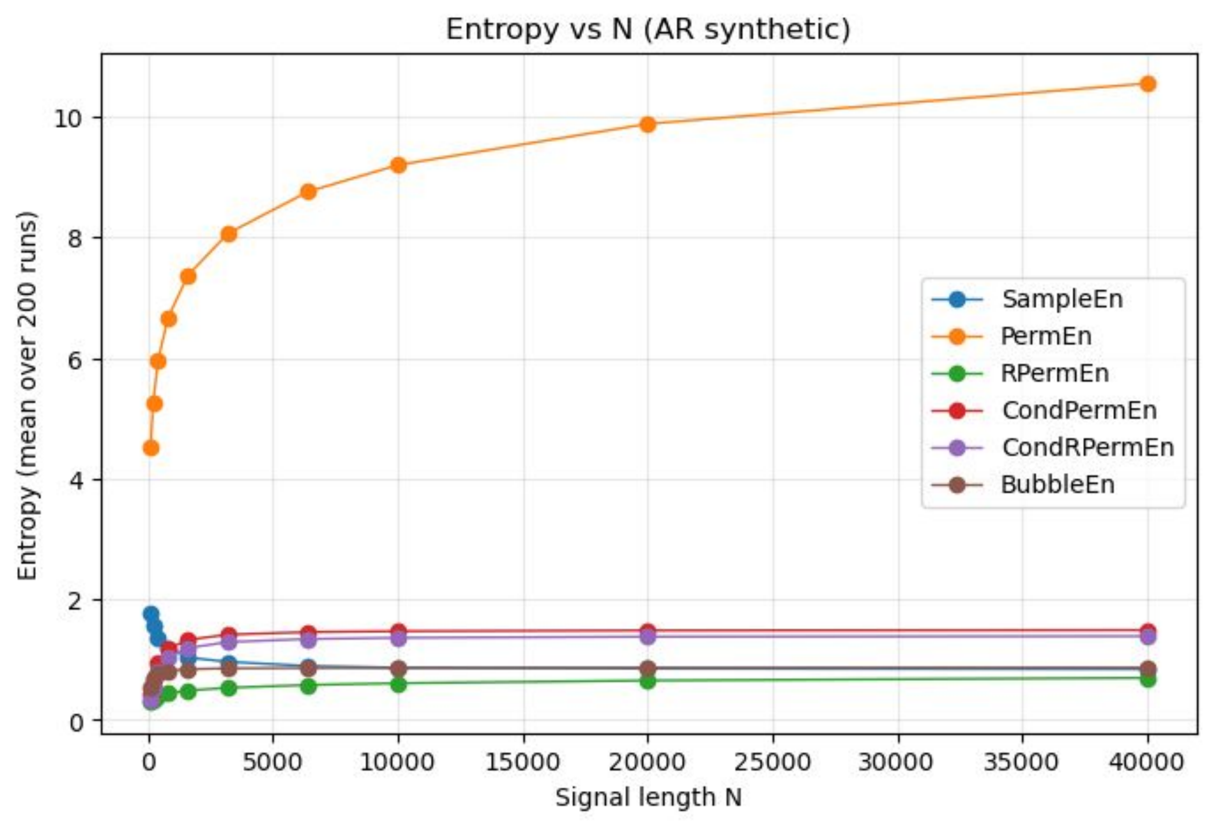
Real Signals (PhysioNet)

NSR: Normal Sinus Rhythm

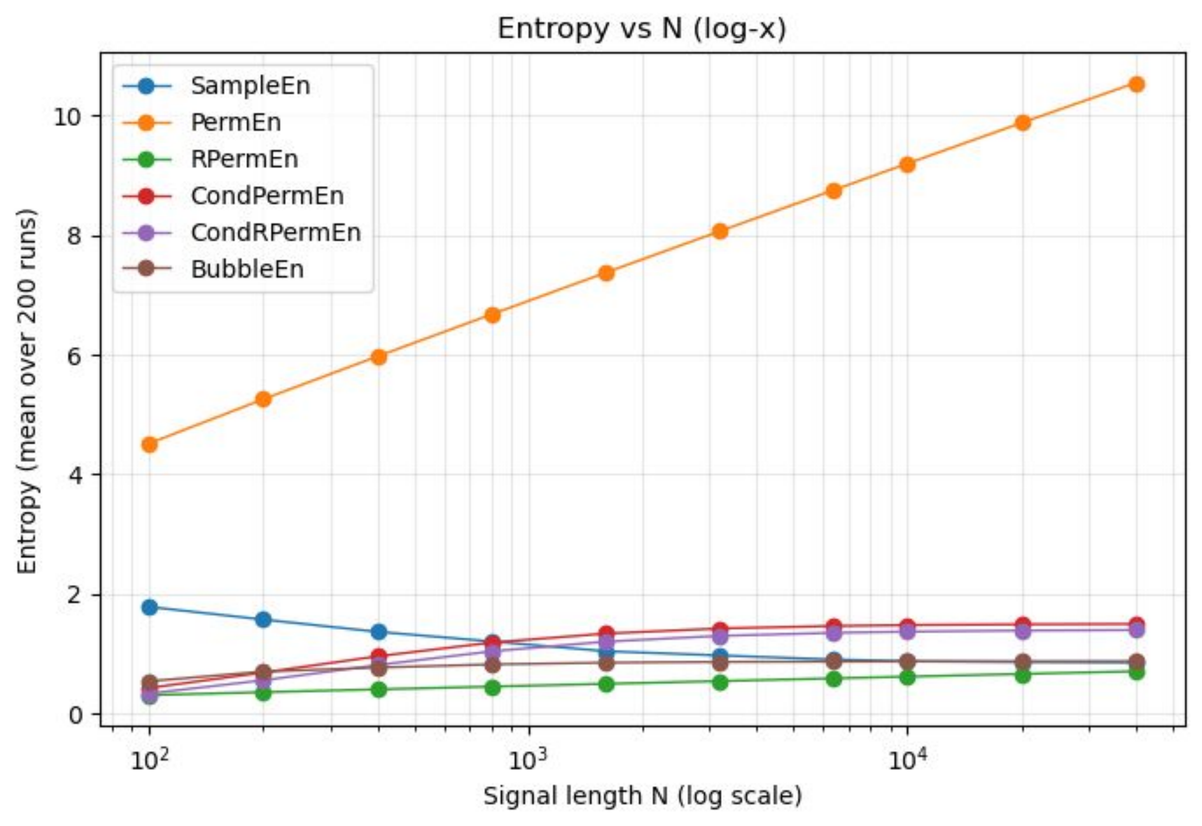
CHF: Congestive Heart Failure patients

Sampling rate of 128Hz

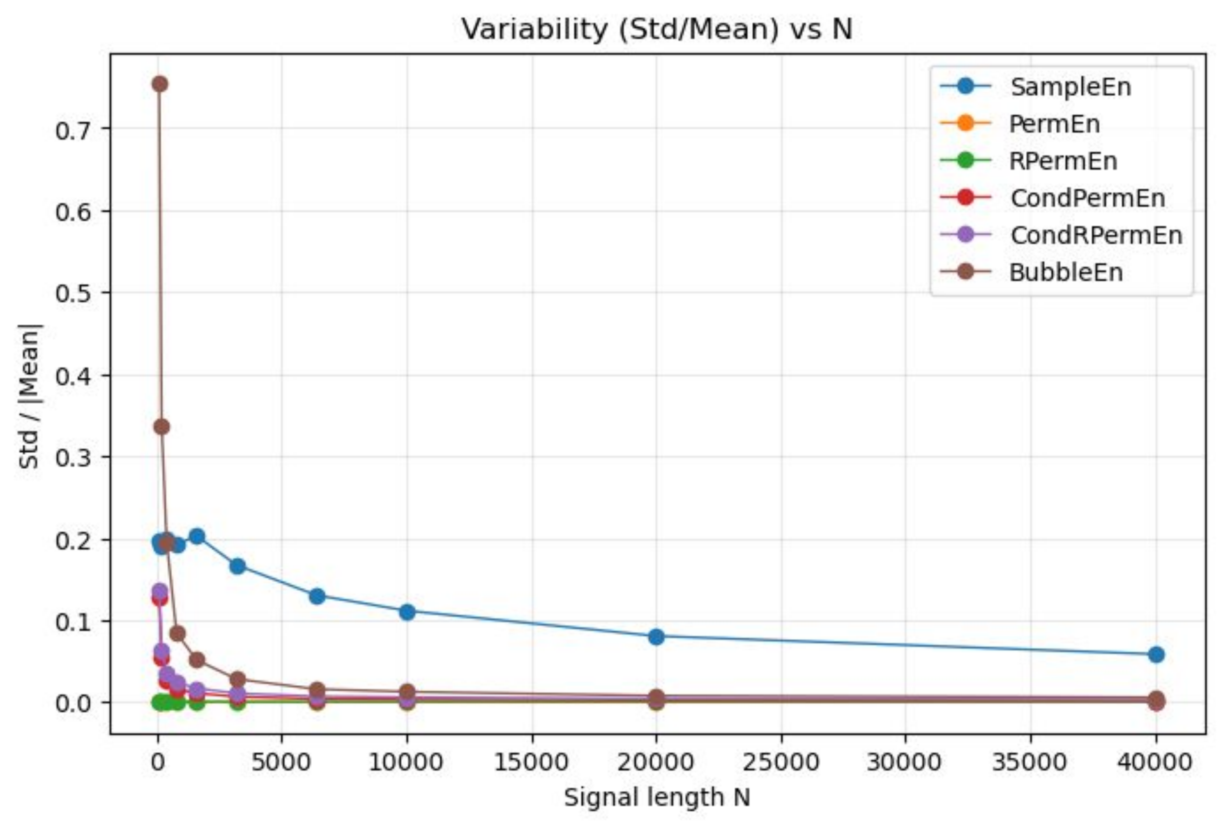
Stability Results



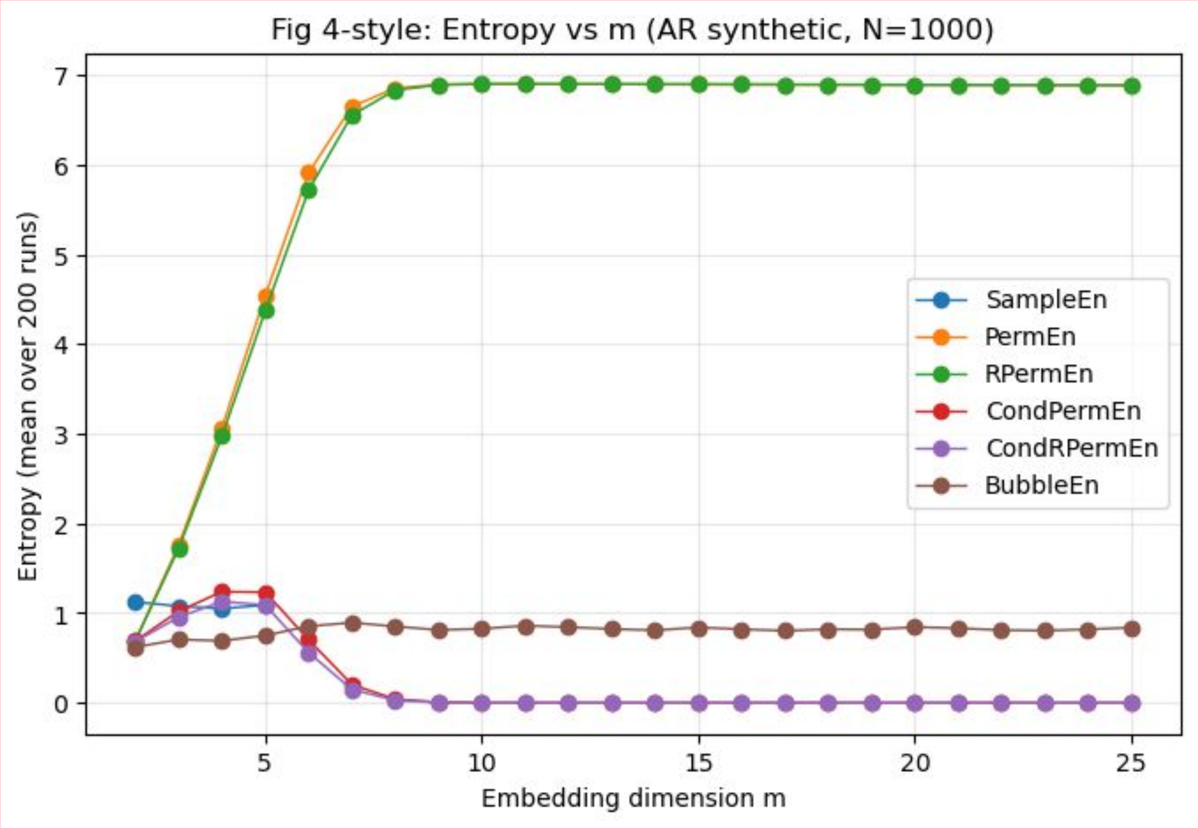
Stability Results



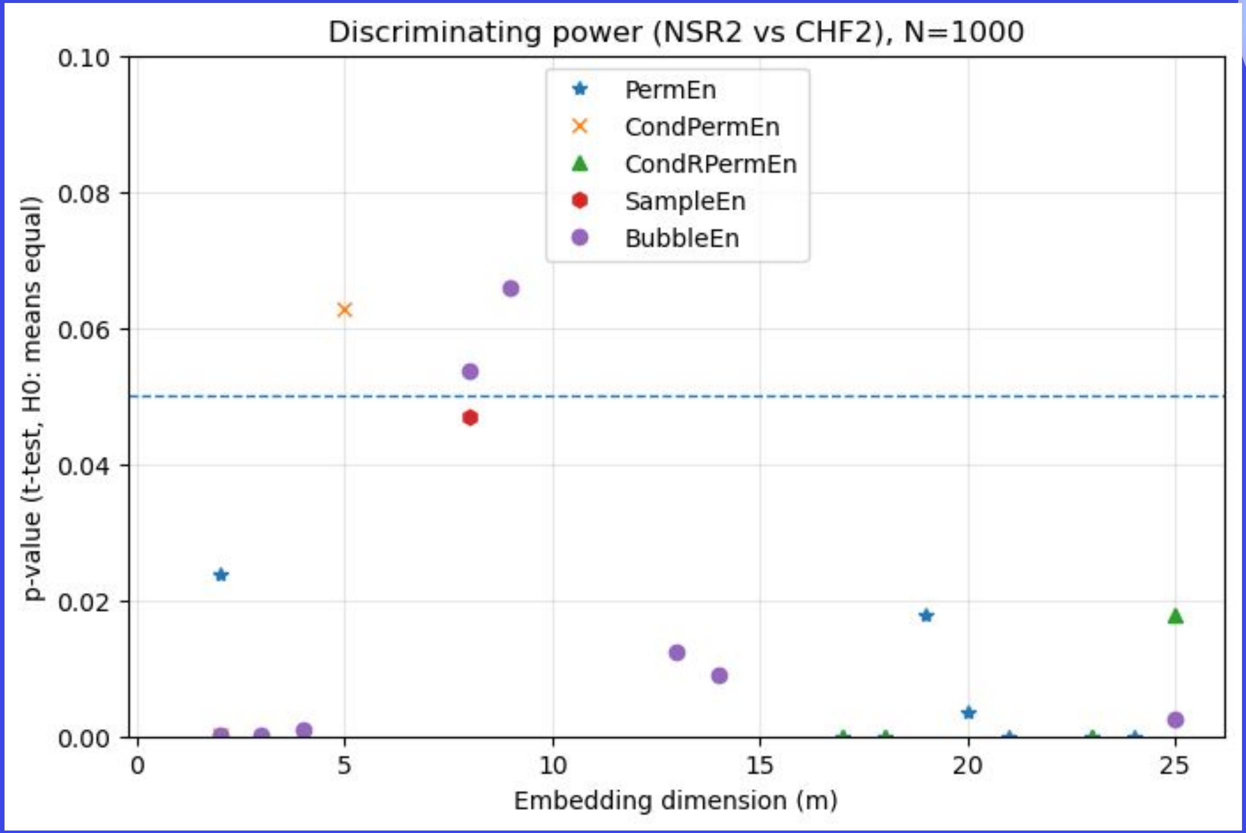
Stability Results



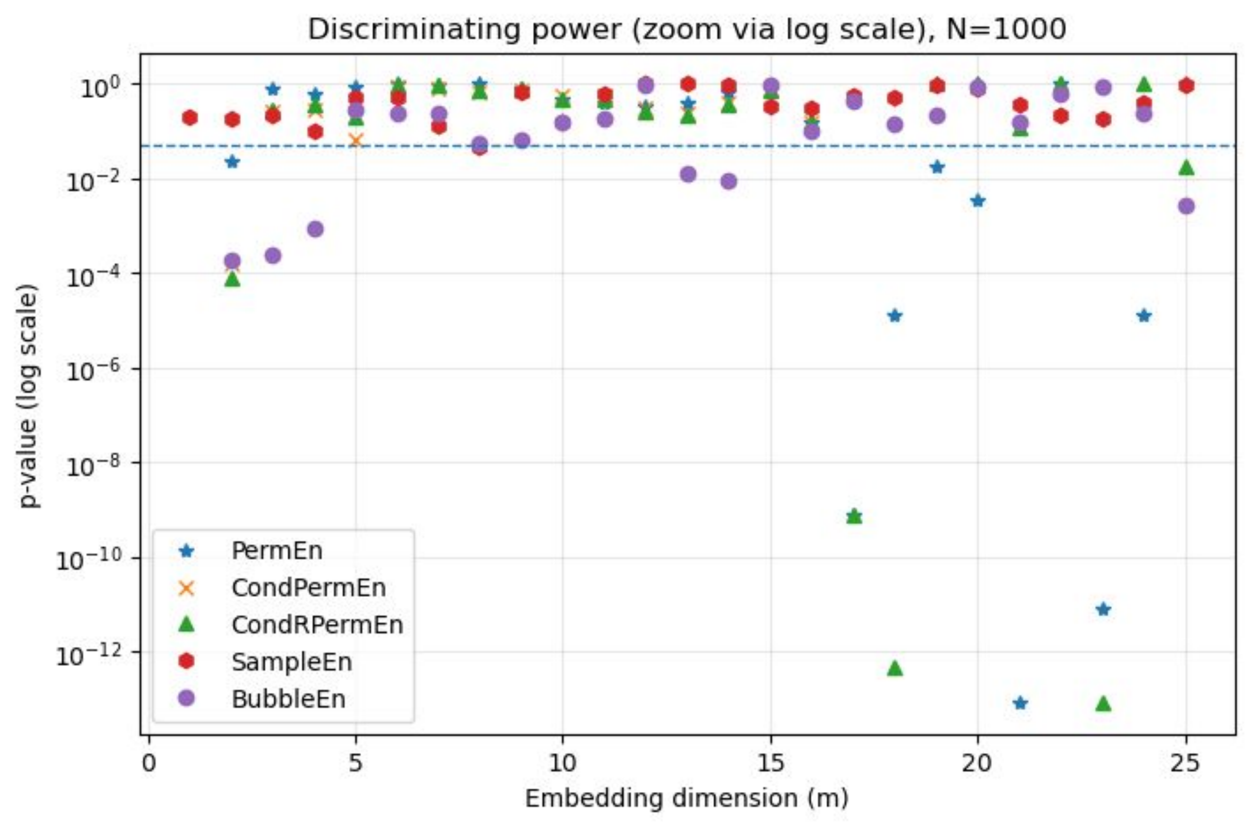
Stability Results



Discriminating Power (NSR vs CHF)



Discriminating Power (NSR vs CHF)
Box plot



Discriminating Power (NSR vs CHF)
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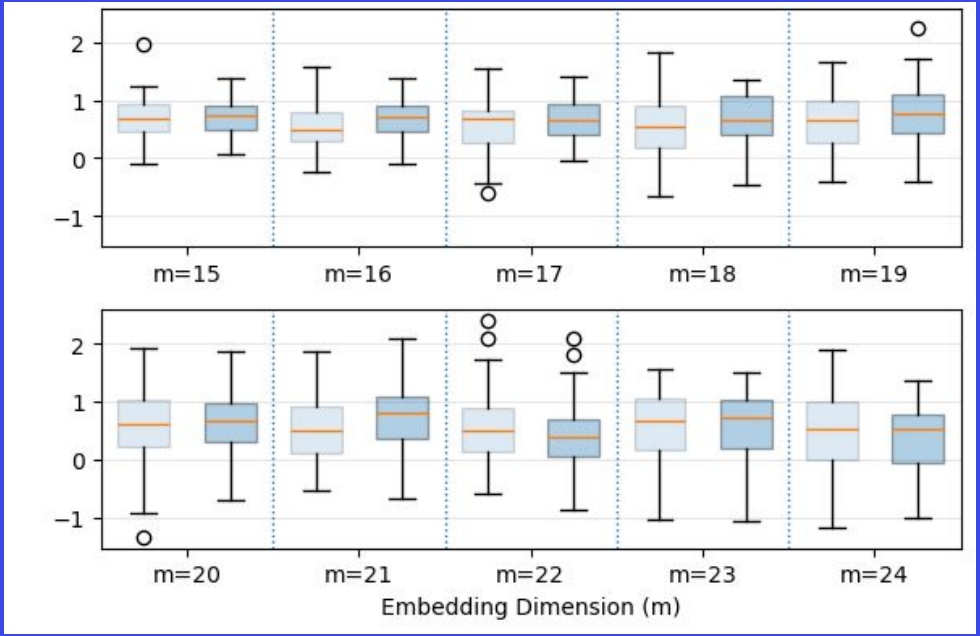


Naive Approach

Bubble sort:

$O(m^2)$ per vector

Optimized Complexity

Exploit overlap between consecutive vectors

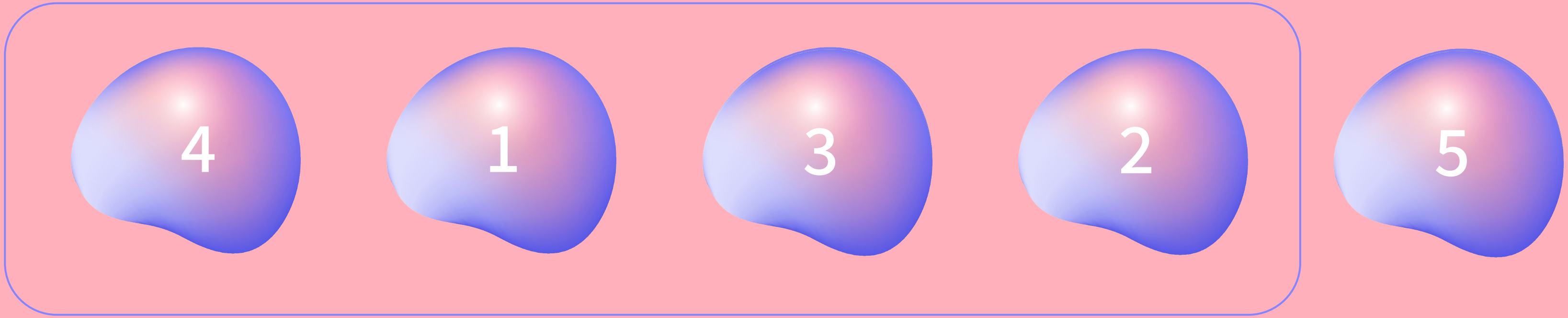
Incremental insertion $\rightarrow O(m)$

$O(mN)$

Bubble Sort: optimized

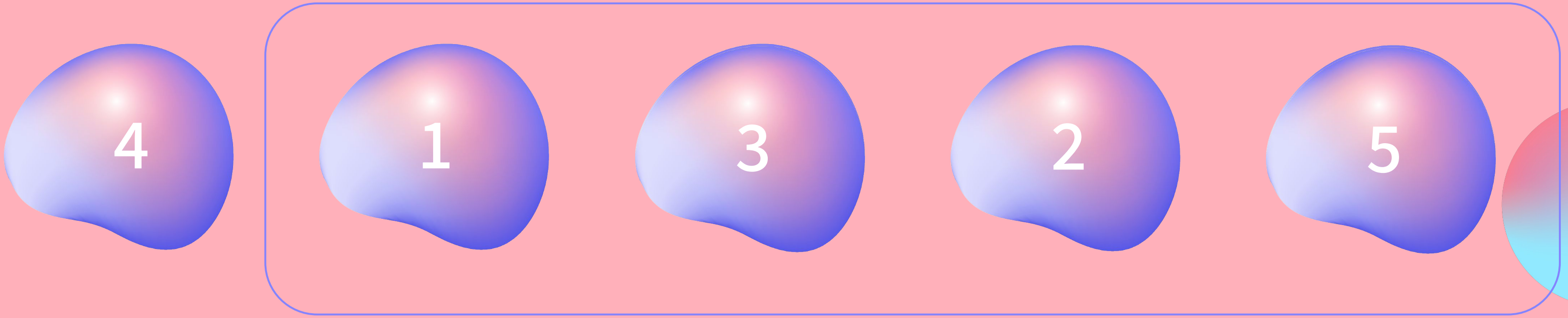
4 1 3 2 5

Bubble Sort: optimized



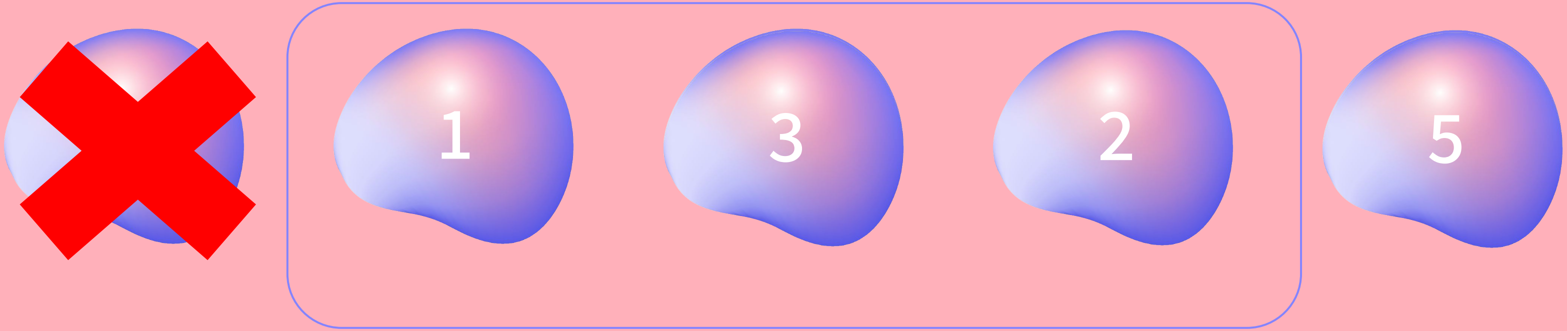
4 inversions

Bubble Sort: optimized



1 inversion

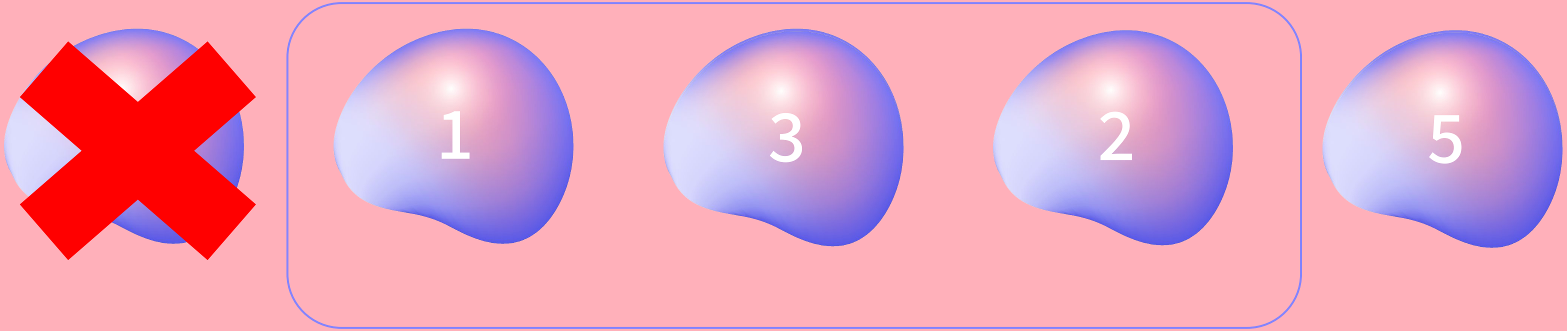
Bubble Sort: optimized



Step 1: Remove 4, and decrease number of inversions by number of inversions involving the 4

O(m)

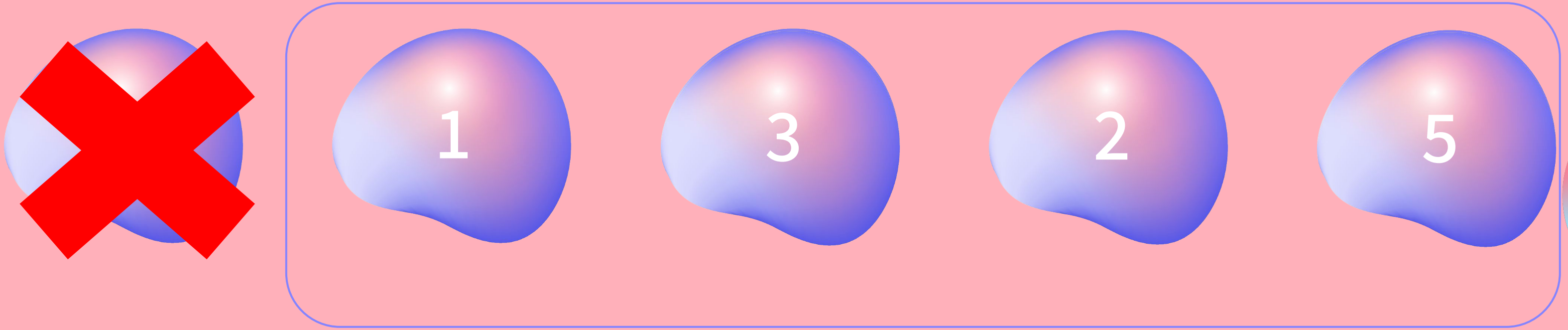
Bubble Sort: optimized



4 - 3 = 1

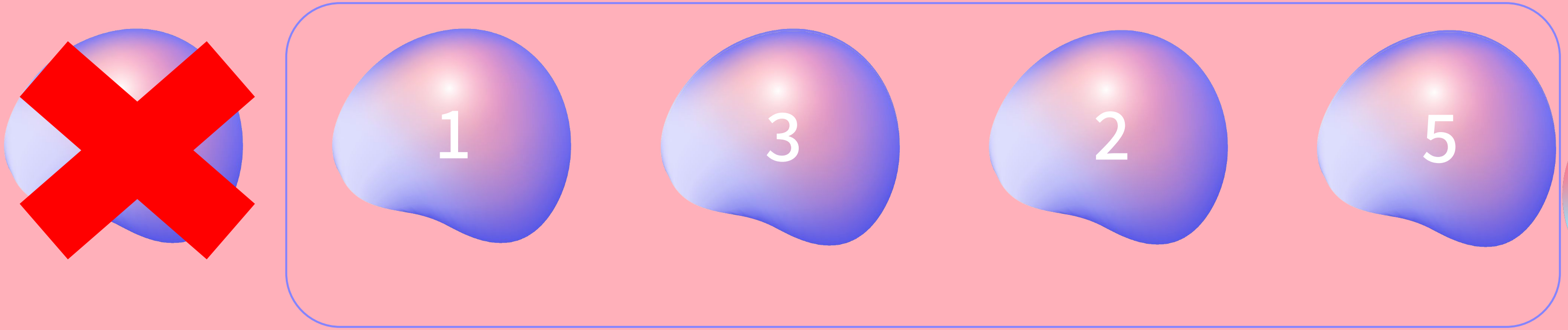
1 inversion left in our list of m-1

Bubble Sort: optimized



Step 2: Add inversions introduced by the new value
O(m)

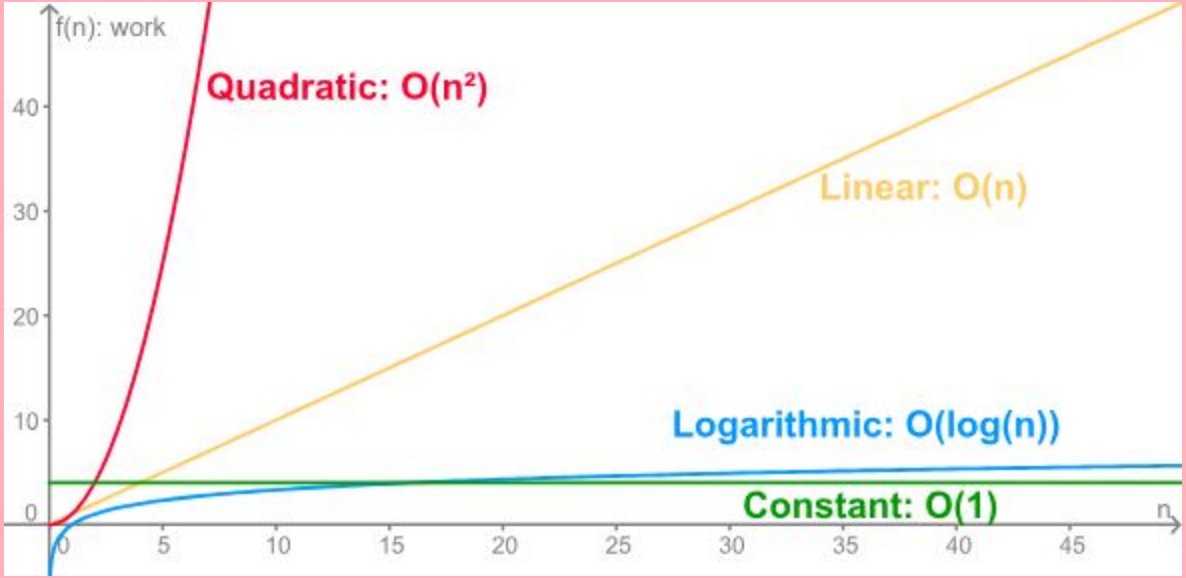
Bubble Sort: optimized



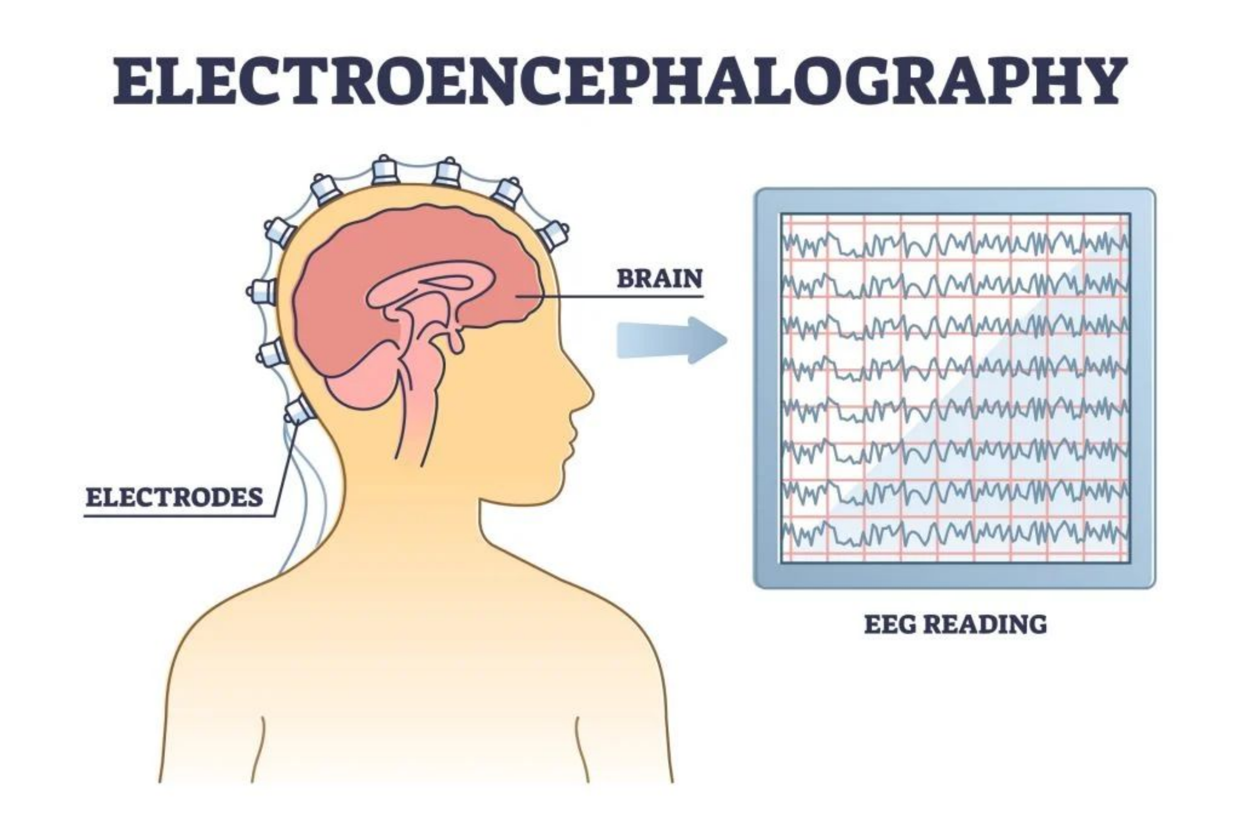
1 + 0 => 1

Bubble Sort: optimized

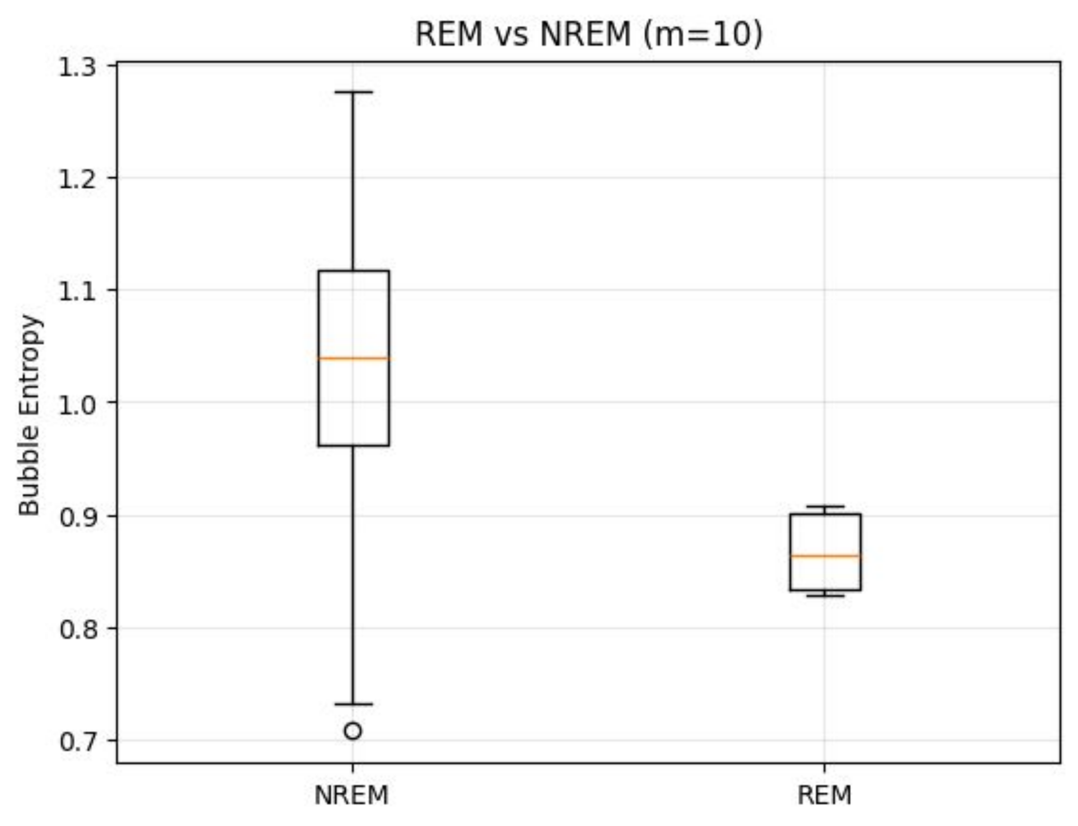
Becomes O(Nm) instead of O(Nm^2)



Trying discrimination on another dataset



Trying discrimination on another dataset



Trying discrimination on another dataset

Best threshold : 1.275

Best accuracy (single subject): 0.95

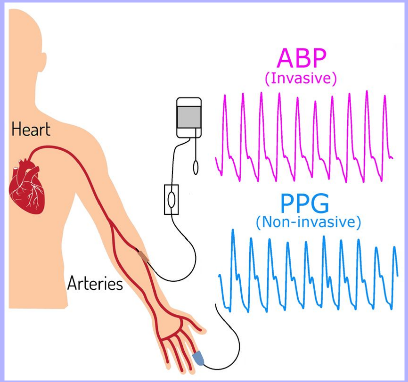
Trying discrimination on another dataset

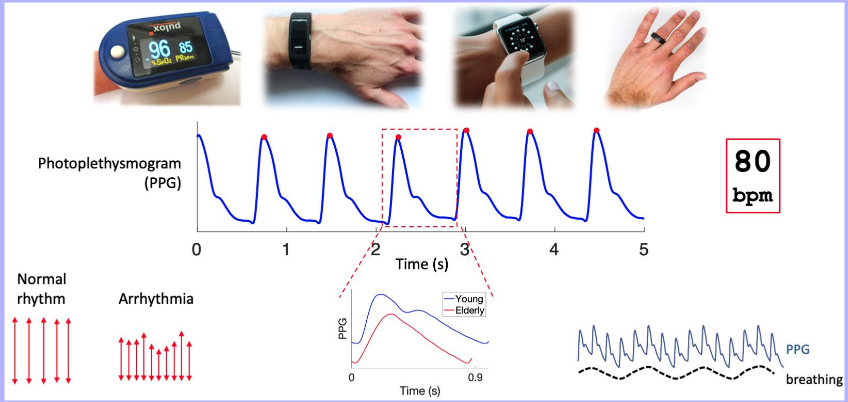
ECG and EEG are very different signal modalities
Bubble Entropy:

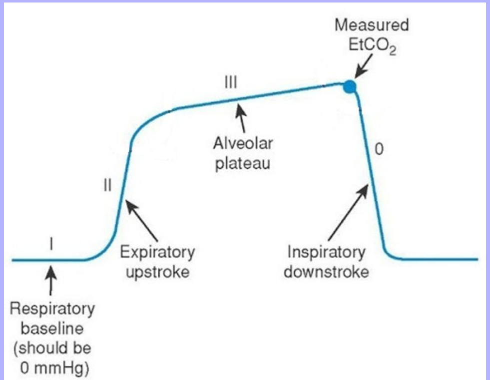
- does not rely on amplitude
- does not rely on scale parameter r
- captures ordinal structure

The method transfers without modification

Other possible applications for Entropy based measures







Blood Pressure (ABP / PPG-derived)

Photoplethysmography (PPG)

Capnography (CO_2 waveform)

Key Advantages of Bubble Entropy

- No threshold parameter r
- Reduced dependence on m
- High stability
- Strong discriminating power
- Interpretable physical meaning (sorting effort)

Limitations & Open Issues

Time delay τ fixed to 1

Focused mainly on HRV

Needs validation on:

- EEG
- Other biomedical signals
- Multiscale settings

Conclusion

Bubble Entropy is a robust, stable, and nearly parameter-free entropy measure