Lec 6

A composition of n with t parts is a sequence $(a_1, \ldots, a_t)$ of positive integers adding to n.

Let S denote the set of all compositions regardless of n. For $(a_1, \ldots, a_t) \in S$ we have $w(a_1, \ldots, a_t) = a_1 + \ldots + a_t$

Claim : $\Phi_S(x) = \frac{1-x}{1-2x} = 1 + \frac{x}{1-2x}$

Proof. We can partition S into sets S_t which denotes the set of compositions with t parts,

By Sum Lemma

$$\Phi_S(x) = \Phi_{S_1}(x) + \Phi_{S_2}(x) + \ldots$$

$S_0 = \{\varepsilon\}$, so $\Phi_{S_0}(x) = x^0 = 1$

For $t \geq 1$, we have $S_t = (\mathbb{Z}_{\geq 1})^t$

By Product Lemma $\Phi_{S_t}(x) = (\Phi_{\mathbb{Z}_{\geq 1}}(x))^t$

$$\Phi_{\mathbb{Z}_{\geq 1}}(x) = x^1 + x^2 + x^3 \ldots$$
$$= \frac{x}{1-x}$$

Therefore $\Phi_S(x) = \Phi_{S_0}(x) + \Phi_{S_1}(x) + \dots$

$$= 1 + (\Phi_{Z_1}(x))^1 + (\Phi_{Z_2}(x))^2 + \dots$$

$$= \frac{1}{1 - \Phi_{Z \geq 1}(x)}$$

$\downarrow$

Note that we can do this because $[x^0]\, \Phi_{Z \geq 1}(x) = 0$

$$= \frac{1}{1 - \frac{x}{1-x}}$$

$$= \frac{1-x}{1-2x}$$

Another proof: (recursive decomposition)

We can decompose any non-empty composition $(a_1, a_2, \dots, a_t)$ into $((a_1, \dots, a_{t-1}), a_t)$

$$S = \{\varepsilon\} \cup S \times Z_{\geq 1}$$

By the sum and product lemma,

$$\Phi_S(x) = \Phi_{\{\varepsilon\}}(x) + \Phi_S(x)\, \Phi_{Z_{\geq 1}}(x)$$
$$= 1 + \Phi_S(x) \frac{x}{1-x}$$

$$\left(1 - \frac{x}{1-x}\right) \Phi_S(x) = 1$$

$$= \frac{1-x}{1-2x}$$

Claim: for $n \in \mathbb{Z}_{\geq 1}$, the number of compositions of n is 2^{n-1}

Proof. The number of compositions of n is

$$[x^n] \Phi_S(x) = [x^n] \left(1 + \frac{x}{1-2x}\right)$$

$$= [x^n] \left(\frac{x}{1-2x}\right) \quad \text{since } n \geq 1$$

$$= [x^{n-1}] \frac{1}{1-2x}$$

$$= [x^n] \left(1 + 2x + (2x)^2 + (2x)^3 + \dots\right)$$

$$= 2^{n-1}$$

Combinatorial Proof: We can think of a composition of n as a way of dividing up a sequence of n ones.
For example we can encode the composition

$(2,1,2)$ of 5 as $(1,1 \mid 1 \mid 1,1)$

A sequence of n ones has $n-1$ commas and we can replace any subset of these commas with dividers to get a composition of n

So there are 2^{n-1} compositions of n