**(1)** Let $f_1(x_1, x_2) = x_1^2 - x_2^2$, $f_2(x_1, x_2) = 2x_1x_2$. Represent the level sets associated with $f_1(x_1, x_2) = 12$ and $f_2(x_1, x_2) = 16$ on the same figure using Python. Indicate on the figure, the points $\mathbf{x} = [x_1, x_2]^T$ for which $\mathbf{f}(\mathbf{x}) = [f_1(x_1, x_2), f_2(x_1, x_2)]^T = [12, 16]^T$. Para encontrar los puntos de intersección:

$$\begin{aligned} x_1^2 - x_2^2 &= 12, \\ 2x_1x_2 &= 16. \end{aligned}$$

Reemplazando $x_1 = \frac{8}{x_2}$, y sustituyendo en la primera ecuación, se encuentra:

$$x_2^4 + 12x_2^2 - 64 = 0,$$

y la solución es: $x_2 = \pm 2$, y $x_1 = \pm 4$.

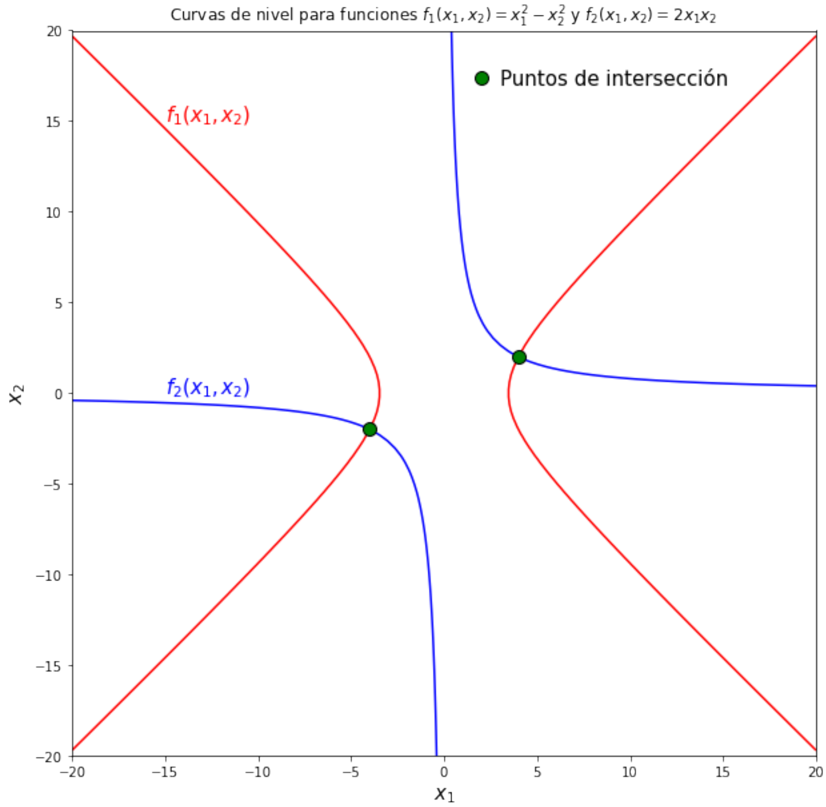


Figura 1: Curvas de nivel en el dominio de -20 a 20.

**(2)** Consider the function $f(\mathbf{x}) = (\mathbf{a}^T\mathbf{x})(\mathbf{b}^T\mathbf{x})$, where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{x}$ are n-dimensional vectors.

Compute the gradient $\nabla f(\mathbf{x})$

El gradiente de una función n-dimensional está definido como:

$$\nabla f(\mathbf{x}) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right]^T,$$

así, desarrollando para un elemento:

$$\frac{\partial f}{\partial x_1}\left[(\mathbf{a}^T\mathbf{x})(\mathbf{b}^T\mathbf{x})\right] = (\mathbf{b}^T\mathbf{x})\frac{\partial f}{\partial x_1}(\mathbf{a}^T\mathbf{x}) + (\mathbf{a}^T\mathbf{x})\frac{\partial f}{\partial x_1}(\mathbf{b}^T\mathbf{x}), = (\mathbf{b}^T\mathbf{x})a_1 + (\mathbf{a}^T\mathbf{x})b_1.$$

Así, se obtiene:

$$\nabla f(\mathbf{x}) = \begin{bmatrix} (\mathbf{b}^T\mathbf{x})a_1 + (\mathbf{a}^T\mathbf{x})b_1 & (\mathbf{b}^T\mathbf{x})a_2 + (\mathbf{a}^T\mathbf{x})b_2 & \ldots & (\mathbf{b}^T\mathbf{x})a_n + (\mathbf{a}^T\mathbf{x})b_n \end{bmatrix}.$$

and the Hessian $\nabla^2 f(\mathbf{x})$

El Hessiano está definido como:

$$\nabla^2 f(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \dots & \vdots \end{bmatrix},$$

desarrollando para la primer columna:

$$\begin{aligned} \frac{\partial^2 f}{\partial x_1^2} \left[ (\mathbf{a}^T \mathbf{x})(\mathbf{b}^T \mathbf{x}) \right] &= 2a_1 b_1, \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} \left[ (\mathbf{a}^T \mathbf{x})(\mathbf{b}^T \mathbf{x}) \right] &= a_1 b_2 + b_1 a_2, \\ &\vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} \left[ (\mathbf{a}^T \mathbf{x})(\mathbf{b}^T \mathbf{x}) \right] &= a_1 b_n + b_1 a_n. \end{aligned}$$

por lo tanto:

$$\nabla^2 f(\mathbf{x}) = \begin{bmatrix} 2a_1 b_1 & a_1 b_2 + b_1 a_2 & \dots & a_1 b_n + b_1 a_n \\ a_1 b_2 + b_1 a_2 & 2a_2 b_2 & \dots & a_2 b_n + b_2 a_n \\ \vdots & \ddots & \ddots & \vdots \\ a_1 b_n + b_1 a_n & \dots & \dots & 2a_n b_n \end{bmatrix}.$$

**(3)** Compute the gradient of

$$f(\theta) = \frac{1}{2} \sum_{i=1}^{n} \left[ g(\mathbf{x}_i) - g(\mathbf{A}\mathbf{x}_i + \mathbf{b}) \right]^2$$

with respect to $\theta$, where $\theta = [a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2]^T$, $\mathbf{x}\ _i \epsilon \mathbb{R}^2, \mathbf{A} \epsilon \mathbb{R}^{2\times 2}, \mathbf{b} \epsilon \mathbb{R}^2$ are defined as follows

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \\ b &= [b_1, b_2]^T \end{aligned}$$

and g: $\mathbb{R}^2 \to \mathbb{R}\ \epsilon\ C^1$.

El gradiente se verá de la forma:

$$\nabla f(\theta) = \begin{bmatrix} \frac{\partial f}{\partial a_{11}} & \frac{\partial f}{\partial a_{12}} & \frac{\partial f}{\partial a_{21}} & \frac{\partial f}{\partial a_{22}} & \frac{\partial f}{\partial b_1} & \frac{\partial f}{\partial b_2} \end{bmatrix},$$

expandiendo la suma:

$$\begin{aligned} f(\theta) &= \frac{1}{2} \left[ g(\mathbf{x}_1) - g(\mathbf{A}\mathbf{x}_i + b) \right]^2 + \sum_{i=2}^{n} \dots, \\ &= \frac{1}{2} \left[ g\left( \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} \right) - g\left( \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) \right]^2 + \sum_{i=2}^{n} \dots, \\ &= \frac{1}{2} \left[ g\left( \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} \right) - g\left( \begin{bmatrix} a_{11}x_{11} + a_{12}x_{21} + b_1 \\ a_{21}x_{11} + a_{22}x_{21} + b_2 \end{bmatrix} \right) \right]^2 + \sum_{i=2}^{n} \dots, \end{aligned}$$

denotando $u_{11} = a_{11}x_{11} + a_{12}x_{21} + b_1$, y $u_{21} = a_{21}x_{11} + a_{22}x_{21} + b_2$, y aplicando el primer término del gradiente para el primer término de la suma:

$$\begin{aligned} \frac{\partial f}{\partial a_{11}} \left[ g\left( \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} \right) - g\left( \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} \right) \right] &= -\frac{\partial f}{\partial g} \frac{\partial g}{\partial u_{11}} x_{11}, \\ &= -x_{11} \left( \frac{\partial f}{\partial g} \frac{\partial g}{\partial u_{11}} \right). \end{aligned}$$

Generalizando el primer término del gradiente:

$$\frac{\partial f}{\partial a_{11}} f(\theta) = \sum_{i=1}^{n} x_{1i} \left( \frac{\partial f}{\partial g} \frac{\partial g}{\partial u_{1i}} \right).$$

Calculando los términos restantes, se obtiene:

$$\nabla f(\theta) = \left[\sum_{i=1}^{n} x_{1i}\left(\frac{\partial f}{\partial g}\frac{\partial g}{\partial u_{1i}}\right), \sum_{i=1}^{n} x_{2i}\left(\frac{\partial f}{\partial g}\frac{\partial g}{\partial u_{1i}}\right), \sum_{i=1}^{n} x_{1i}\left(\frac{\partial f}{\partial g}\frac{\partial g}{\partial u_{2i}}\right), \sum_{i=1}^{n} x_{2i}\left(\frac{\partial f}{\partial g}\frac{\partial g}{\partial u_{2i}}\right), \sum_{i=1}^{n}\left(\frac{\partial f}{\partial g}\frac{\partial g}{\partial u_{1i}}\right), \sum_{i=1}^{n}\left(\frac{\partial f}{\partial g}\frac{\partial g}{\partial u_{2i}}\right)\right].$$

**(4)**Let f(r, $\theta$) be $\mathbb{R}^2 \to \mathbb{R}$ with $r = \sqrt{x^2+y^2}$ and $\theta$ = arctan$\frac{y}{z}$. Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}, \\
&= \left(\frac{x}{\sqrt{x^2+y^2}}\right)\frac{\partial f}{\partial r} + \left(\frac{-y}{x^2(1+\frac{y^2}{x^2})}\right)\frac{\partial f}{\partial \theta}, \\
\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}, \\
&= \left(\frac{y}{\sqrt{x^2+y^2}}\right)\frac{\partial f}{\partial r} + \left(\frac{-1}{x(1+\frac{y^2}{x^2})}\right)\frac{\partial f}{\partial \theta}.
\end{aligned}$$

**(5)** The directional derivative $\frac{\partial f}{\partial v}(x_0, y_0, z_0)$ of a differentiable function $f$ are $\frac{3}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}}$ in the directions of vectors $[0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^T$, $[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}]^T$ and $[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0]^T$. Compute $\nabla f(x_0, y_0, z_0)$.

$$\frac{\partial f}{\partial x_0}0 + \frac{\partial f}{\partial y_0}\frac{1}{\sqrt{2}} + \frac{\partial f}{\partial z_0}\frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}, \tag{1}$$

$$\frac{\partial f}{\partial x_0}\frac{1}{\sqrt{2}} + \frac{\partial f}{\partial y_0}0 + \frac{\partial f}{\partial z_0}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}, \tag{2}$$

$$\frac{\partial f}{\partial x_0}\frac{1}{\sqrt{2}} + \frac{\partial f}{\partial y_0}\frac{1}{\sqrt{2}} + \frac{\partial f}{\partial z_0}0 = -\frac{1}{\sqrt{2}}. \tag{3}$$

Es decir,

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial f}{\partial x_0} \\ \frac{\partial f}{\partial y_0} \\ \frac{\partial f}{\partial z_0} \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix},$$

resolviendo el sistema:

$$\begin{pmatrix} \frac{\partial f}{\partial x_0} \\ \frac{\partial f}{\partial y_0} \\ \frac{\partial f}{\partial z_0} \end{pmatrix} = \begin{pmatrix} -1{,}5 \\ 0{,}5 \\ 2{,}5 \end{pmatrix}.$$

**(6)** Show that the level curves of the function f(x,y) = $x^2+y^2$ are orthogonal to the level curves of g(x,y) = $\frac{y}{x}$ for all (x,y).
El gradiente de la función es ortogonal a las curvas de nivel, así calculando ambos gradientes:

$$\begin{aligned}
\nabla f(x,y) &= \begin{bmatrix} 2x & 2y \end{bmatrix}, \\
\nabla g(x,y) &= \begin{bmatrix} -\frac{y}{x^2} & \frac{1}{x} \end{bmatrix}.
\end{aligned}$$

Calculando el producto punto:

$$\nabla f(x,y) \cdot \nabla g(x,y) = -\frac{2y}{x} + \frac{2y}{x},$$

por lo tanto, son ortogonales.

**(7)** Let f, g, h be differentiable functions, with $f$:$\mathbb{R}^n \to \mathbb{R}^3$, g: $\mathbb{R}^n \to \mathbb{R}^3$ and h: $\mathbb{R}^n \to \mathbb{R}$

$$h(\mathbf{x}) = f(\mathbf{x})^T g(\mathbf{x}) \tag{4}$$

show that

$$Dh(\mathbf{x}) = f(\mathbf{x})^T Dg(\mathbf{x}) + g(\mathbf{x})^T Df(\mathbf{x}) \tag{5}$$

Lo que se pide calcular es el gradiente de la función h(**x**),

$$\nabla h(\mathbf{x}) = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} & \cdots & \frac{\partial h}{\partial x_n} \end{bmatrix},$$

desarrollando h($\mathbf{x}$):

$$
\begin{aligned}
h(\mathbf{x}) &= \begin{bmatrix} f_1(x_1,\ldots,x_n) & f_2(x_1,\ldots,x_n) & f_3(x_1,\ldots,x_n) \end{bmatrix} \begin{bmatrix} g_1(x_1,\ldots,x_n) \\ g_2(x_1,\ldots,x_n) \\ g_3(x_1,\ldots,x_n) \end{bmatrix}, \\
&= f_1(x_1,\ldots,x_n)g_1(x_1,\ldots,x_n) + f_2(x_1,\ldots,x_n)g_2(x_1,\ldots,x_n) + f_3(x_1,\ldots,x_n)g_3(x_1,\ldots,x_n).
\end{aligned}
$$

Calculando la primer derivada:

$$
\frac{\partial h}{\partial x_1} = \left[ g_1 \tfrac{\partial f_1}{\partial x_1} + f_1 \tfrac{\partial g_1}{\partial x_1} + g_2 \tfrac{\partial f_2}{\partial x_1} + f_2 \tfrac{\partial g_2}{\partial x_1} + g_3 \tfrac{\partial f_3}{\partial x_1} + f_3 \tfrac{\partial g_3}{\partial x_1} \right],
$$

ahora, desarrollándo la ec. (5):

$$
\begin{aligned}
Dh(\mathbf{x}) &= \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix} \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \ddots & \frac{\partial g_2}{\partial x_n} \\ \frac{\partial g_3}{\partial x_1} & \cdots & \frac{\partial g_3}{\partial x_n} \end{bmatrix} + \begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix} \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \ddots & \frac{\partial f_2}{\partial x_n} \\ \frac{\partial f_3}{\partial x_1} & \cdots & \frac{\partial f_3}{\partial x_n} \end{bmatrix} \\
&= \left[ f_1 \tfrac{\partial g_1}{\partial x_1} + f_2 \tfrac{\partial g_2}{\partial x_1} + f_3 \tfrac{\partial g_3}{\partial x_1}, \cdots \right] + \left[ g_1 \tfrac{\partial f_1}{\partial x_1} + g_2 \tfrac{\partial f_2}{\partial x_1} + g_3 \tfrac{\partial f_3}{\partial x_1}, \cdots \right].
\end{aligned}
$$

Sumando los primeros términos de la ecuación anterior, coinciden con los encontrados al desarrollar la función, de esta manera, al seguir desarrollando más términos se encuentra que las ecuaciones (4) y (5) son iguales.

**(8)** Consider the **induced matrix norm**

$$
\|A\|_p = \max_{\mathbf{x}\neq 0} \frac{\|\mathbf{Ax}\|_p}{\|\mathbf{x}\|_p}
$$

where $\|\cdot\|_p$ is the $l_p$ norm, ie

$$
\|\mathbf{x}\|_p = (|x_1|^p + |x_2|^p + \ldots |x_n|^p +)^{\frac{1}{p}}
$$

Show that

$$
\|\mathbf{AB}\|_p \leq \|\mathbf{A}\|_p \|\mathbf{B}\|_p
$$

$$
\begin{aligned}
\|\mathbf{AB}\|_p &= \max_{\mathbf{x}\neq 0} \frac{\|(\mathbf{AB})\mathbf{x}\|_p}{\|\mathbf{x}\|_p}, \\
&\leq \max_{\mathbf{x}\neq 0} \frac{\|\mathbf{A}\|_p \|\mathbf{Bx}\|_p}{\|\mathbf{x}\|_p}, \\
&\leq \max_{\mathbf{x}\neq 0} \frac{\|\mathbf{A}\|_p \|\mathbf{B}\|_p \|\mathbf{x}\|_p}{\|\mathbf{x}\|_p}, \\
&\leq \|\mathbf{A}\|_p \|\mathbf{B}\|_p .
\end{aligned}
$$