Homework 6

Problem 1. Fill in the blanks with either true (✓) or false (×)

$f(n)$	$g(n)$	$f = O(g)$	$f = \Omega(g)$	$f = \Theta(g)$
$2n^3 + 3n$	$100n^2 + 2n + 100$	×	✓	×
$50n + \log n$	$10n + \log\log n$	✓	✓	✓
$50n \log n$	$10n \log\log n$	×	✓	×
$\log n$	$\log^2 n$	✓	×	×
$n!$	5^n	×	✓	×

Problem 2. 1. Find two functions $f(x)$ and $g(x)$ such that $f(x) \neq O(g(x))$ and $g(x) \neq O(f(x))$.

2. Furthermore, we say a function $h : \mathbb{N} \to \mathbb{R}$ is monotonically increasing if it satisfies the property '$x \leq y \implies h(x) \leq h(y)$'.
   Find two monotonically increasing functions $f(x)$ and $g(x)$ such that $f(x) \neq O(g(x))$ and $g(x) \neq O(f(x))$.

(Please give the detailed proof that your functions satisfy the requirements.)

Solution.

1. $\begin{cases} f(n) = |\sin(\frac{\pi}{2}n)|; \\ g(n) = |\cos(\frac{\pi}{2}n)|. \end{cases}$

2. $\begin{cases} f(n) = n^{\sin(\frac{\pi}{2}n)+\frac{\pi}{2}n}; \\ g(n) = n^{\cos(\frac{\pi}{2}n)+\frac{\pi}{2}n}. \end{cases}$

The detailed proof are omitted. Just stick to the definition of $O(-)$. □

Problem 3. Order the index (from (1) to (9)) of the following functions according to their growth rate, and express this ordering using the asymptotic notation O:
(1) $n \ln n$ (2) $(\ln\ln n)^{\ln n}$ (3) $(\ln n)^{\ln\ln n}$ (4) $n \cdot e^{\sqrt{\ln n}}$
(5) $(\ln n)^{\ln n}$ (6) $n \cdot 2^{\ln\ln n}$ (7) $n^{1+\frac{1}{\ln\ln n}}$ (8) $n^{1+\frac{1}{\ln n}}$ (9) n^2

Solution. Define $\preceq$ as '$i \preceq j$ iff $f_i = \mathcal{O}(f_j)$', and we use $i \equiv j$ iff $f_i = \Theta(f_j)$) . Then $3 \preceq 8 \preceq 6 \preceq 1 \preceq 4 \preceq 7 \preceq 9 \preceq 2 \preceq 5$.

$\square$

Problem 4. Check that if we have $f_1(n) = \mathcal{O}(g_1(n))$ and $f_2(n) = \mathcal{O}(g_2(n))$ then $f_1(n) + f_2(n) = \mathcal{O}(g_1(n) + g_2(n))$ and $f_1(n)f_2(n) = \mathcal{O}(g_1(n)g_2(n))$.

Solution. Yes. (You need to give the detail by definition). $\square$

Problem 5. Prove that for $n = 1, 2, \ldots,$ we have

$$2\sqrt{n+1} - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1.$$

Solution. Proof by induction. $\square$