Homework 8

Problem 1. Prove that: for any $2k+1$ regular graph G (where $k \geq 1$), any edge e in G lies in an even number of Hamilton cycles.

Solution. Hint: Similar to the Smith's theorem. □

Problem 2. A well-known problem about a tourist climbing a mountain also relates to fixed points. A tourist starts climbing a mountain at 6 in the morning. He reaches the summit at 6pm and spends the night there. At 6 the next morning he starts descending along the same trail and he reaches the starting point at 6 in the evening again. Prove that there is a place on the trail that he passed through at the same time on both days.

Solution. Imagine two tourist on the same day: one ascending, one descending. □

Problem 3. Given a sequence $(d_1, d_2, \ldots, d_n)$ of positive integers (where $n \geq 1$):

(i) There exists a tree with score $(d_1, d_2, \ldots, d_n)$.

(ii) $\sum_{i=1}^{n} d_i = 2n - 2$.

Prove that (i) and (ii) are equivalent.

Solution.

1. $(i) \Rightarrow (ii)$ is obvious.

2. To prove $(ii) \Rightarrow (i)$:

   By induction on the number n.

   For $n = 1, 2$ the implication holds trivially, so let $n > 2$. Suppose the implication holds for any $n-1$ long positive sequence $(d_1, d_2, \ldots, d_{n-1})$ with $\sum_{i=1}^{n-1} d_i = 2(n-1) - 2$.

   For the induction step, consider an length n positive sequence $\ell = (d_1, d_2, \ldots, d_n)$ with $\sum_{i=1}^{n} d_i = 2n - 2$:

   Since the sum of the d_i is smaller than $2n$, there exists an i with $d_i = 1$. w.l.o.g. we assume $d_1 = 1$. With a similar argument we can also

conclude that there must exist some index j such that $d_j \geq 2$. We take $k = \min\{j \mid d_j \geq 2\}$.
Now the sequence $\ell = (d_1, d_2, \ldots, d_k, \ldots, d_n)$= $(1, d_2, \ldots, d_k - 1 + 1, \ldots, d_n)$, we can derive a new sequence $\ell' = (d_2, \ldots, d_k - 1, \ldots, d_n)$. Obviously ℓ' is a $n-1$ length sequence (all positive) with the summation to be $2n - 2 - 1 - 1 = 2(n-1) - 2$. Then according to the induction hypothesis, there exists a tree $\mathcal{T}'$ which corresponds to ℓ'.
Then $\mathcal{T} = (V(\mathcal{T}') \cup \{v_1\}, E(\mathcal{T}') \cup \{v_1, v_k\})$ is the tree which witnesses the validity of the sequence ℓ.

BE CAREFUL: Why is the following 'proof' of the implication (ii)$\Rightarrow$(i) insufficient (or, rather, makes no sense)? We proceed by induction on n. The base case $n = 1$ is easy to check, so let us assume that the implication holds for some $n \geq 1$. We want to prove it for $n+1$.If $D = (d_1, d_2, ..., d_n)$ is a sequence of positive integers with $\sum_{i=1}^{n} d_i = 2n - 2$, then we already know that there exists a tree T on n vertices with D as a score. Add another vertex v to T and connect it to any vertex of T by an edge, obtaining a tree T' on $n+1$ vertices. Let D' be the score of T'. We know that the number of vertices increased by 1, and the sum of degrees of vertices increased by 2 (the new vertex has degree 1 and the degree of one old vertex increased by 1). Hence the sequence D' satisfies condition (ii) and it is a score of a tree, namely of T'. This finishes the inductive step. $\square$