

函数的渐进比较： O 符号

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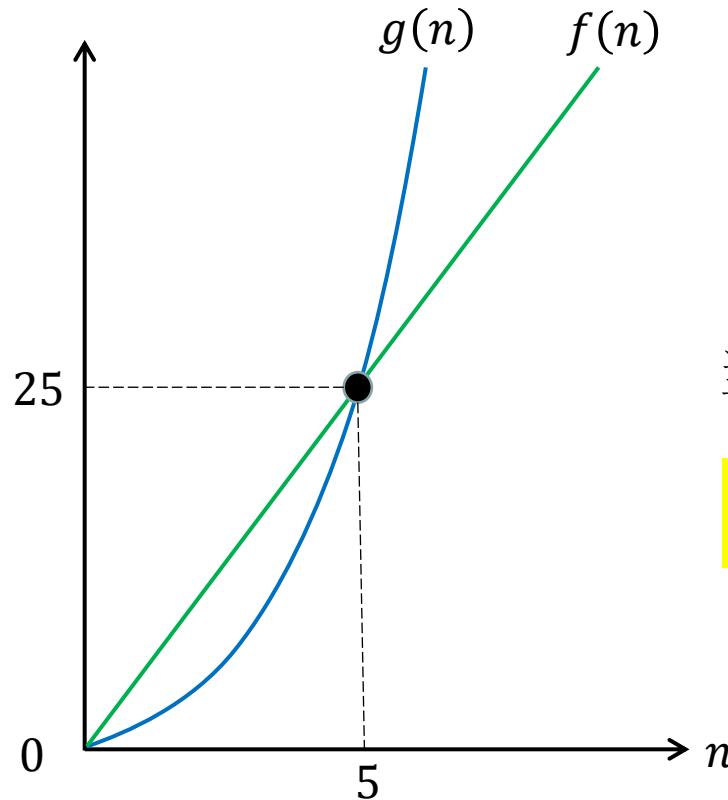
- 在实际应用中，计算某个问题的精确解可能非常困难。此时，一个可行的方案是：不直接寻找精确解，转而寻找可接受的**估值(estimate)**。
- 精确解困难的原因：
 - 物理设备的差异
 - 数据量激增
 - 精确计算公式复杂
 - 网络环境、自然环境
 -

- 1 Bit = Binary Digit
- 8 Bits = 1 Byte
- 1024 Bytes = 1 Kilobyte
- 1024 Kilobytes = 1 Megabyte
- 1024 Megabytes = 1 Gigabyte
- 1024 Gigabytes = 1 Terabyte
- 1024 Terabytes = 1 Petabyte
- 1024 Petabytes = 1 Exabyte
- 1024 Exabytes = 1 Zettabyte
- 1024 Zettabytes = 1 Yottabyte
- 1024 Yottabytes = 1 Brontobyte
- 1024 Brontobytes = 1 Geopbyte



函数的比较

- 比较: $f(n) = 5n$ 以及 $g(n) = n^2$, 其中 $n \in N$ 为自然数。



当 $n \rightarrow \infty$ 时, $f(n)$ 的增长不快于 $g(n)$

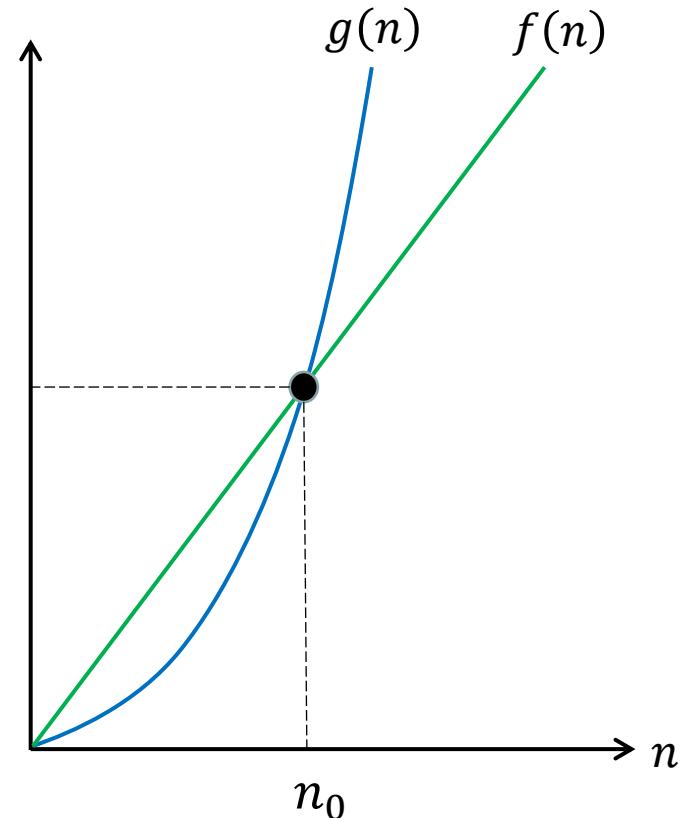
$$f(n) = O(g(n))$$

函数的渐进比较(Asymptotic comparison)

定义: $f, g: N \rightarrow R$ 是两个从自然数到实数的单变量方程

$$f(n) = O(g(n))$$

表示存在常数 n_0 和 C , 使得对所有 $n \geq n_0$, 不等式 $|f(n)| \leq C \cdot g(n)$ 成立。



直观: f 的增长不比 g 快很多。即: $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} \neq \infty$

一些例子

- $1000000 = O(1)$
- $(7n^2 + 6n + 1)(n^3 + 4) = O(n^5)$
- $\binom{n}{2} = n(n - 1)/2 = \frac{1}{2}n^2 + O(n) = O(n^2)$
- $0 < \alpha \leq \beta \Rightarrow n^\alpha = O(n^\beta)$
- $\forall C > 0, a > 1 \ n^C = O(a^n)$
- $\forall C > 0, \alpha > 0 \ (\ln n)^C = O(n^\alpha)$
- 在函数的渐进比较中，部分常用的其他符号如下：

函数的渐进比较

符号	定义	含义
$f(n) = o(g(n))$	$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$	f 的增长远远慢于 g
$f(n) = \Omega(g(n))$	$g(n) = O(f(n))$	f 的增长至少和 g 一样快
$f(n) = \Theta(g(n))$	$f(n) = O(g(n))$ 且 $f(n) = \Omega(g(n))$	f 和 g 几乎是同一数量级
$f(n) \sim g(n)$	$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1$	$f(n)$ 和 $g(n)$ 几乎是一样的

定义: $f, g: N \rightarrow R$ 是两个从自然数到实数的单变量方程
 $f(n) = O(g(n))$

表示存在常数 n_0 和 C , 使得对所有 $n \geq n_0$, 不等式 $|f(n)| \leq C \cdot g(n)$ 成立。

调和级数

- 调和级数(Harmonic number):

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{i=1}^n \frac{1}{n}$$

调和级数估值

- 估计调和级数的值：用数列对调和级数的加项做分类。
- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \dots, \frac{1}{15}, \frac{1}{16}, \frac{1}{17}, \dots$

The diagram shows the harmonic series $H = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{15} + \frac{1}{16} + \dots$ partitioned into five groups labeled G_1, G_2, G_3, G_4, G_5 . The first group G_1 contains the term $\frac{1}{2^1}$. The second group G_2 contains the terms $\frac{1}{2^2}, \frac{1}{2^1}$. The third group G_3 contains the terms $\frac{1}{2^3}, \frac{1}{2^2}$. The fourth group G_4 contains the terms $\frac{1}{2^4}, \frac{1}{2^3}$. The fifth group G_5 contains the terms $\frac{1}{2^5}, \frac{1}{2^4}$. Brackets above the terms indicate the grouping: $(1, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{3})$, $(\frac{1}{3}, \frac{1}{4})$, $(\frac{1}{4}, \dots, \frac{1}{15})$, and $(\frac{1}{15}, \dots)$.

$$\begin{aligned}G_k &= \left\{ \frac{1}{i} \mid \frac{1}{2^k} < \frac{1}{i} \leq \frac{1}{2^{k-1}} \right\} \\&= \left\{ \frac{1}{2^{k-1}}, \frac{1}{2^{k-2}}, \frac{1}{2^{k-3}}, \dots, \frac{1}{2^1} \right\}\end{aligned}$$

每一个 G_k 中的调和级数加项和：

$$\sum_{x \in G_k} x \leq |G_k| \max G_k$$

$$= 2^{k-1} \cdot \frac{1}{2^{k-1}}$$

$$= 1$$

$$\sum_{x \in G_k} x \geq |G_k| \min G_k$$

$$> 2^{k-1} \cdot \frac{1}{2^k}$$

$$= \frac{1}{2}$$

$$\underbrace{1}_{G_1}, \quad \underbrace{\frac{1}{2}, \frac{1}{3}}_{G_2}, \quad \underbrace{\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}}_{G_3}, \quad \underbrace{\frac{1}{8}, \frac{1}{9}, \dots, \frac{1}{15}}_{G_4}, \dots$$

$$G_k = \left\{ \frac{1}{i} \mid \frac{1}{2^k} < \frac{1}{i} \leq \frac{1}{2^{k-1}} \right\}$$

$$= \left\{ \frac{1}{2^{k-1}}, \frac{1}{2^{k-2}}, \frac{1}{2^{k-3}}, \dots, \frac{1}{2^{k-1}} \right\}$$

$$= \left\{ \frac{1}{2^{k-1}}, \frac{1}{2^{k-1}+1}, \frac{1}{2^{k-1}+2}, \dots, \frac{1}{2^k} \right\}$$

$$|G_k| = 2^{k-1}$$

$$\underbrace{1}_{G_1}, \quad \underbrace{\frac{1}{2}, \frac{1}{3}}_{G_2}, \quad \underbrace{\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}}_{G_3}, \quad \underbrace{\frac{1}{8}, \frac{1}{9}, \dots, \frac{1}{15}}_{G_4}, \dots$$

$$\begin{aligned} G_k &= \left\{ \frac{1}{i} \mid \frac{1}{2^k} < \frac{1}{i} \leq \frac{1}{2^{k-1}} \right\} \\ &= \left\{ \frac{1}{2^{k-1}}, \frac{1}{2^{k-2}}, \frac{1}{2^{k-3}}, \dots, \frac{1}{2^{k-1}} \right\} \\ &= \left\{ \frac{1}{2^{k-1}}, \frac{1}{2^{k-1}+1}, \frac{1}{2^{k-1}+2}, \dots, \frac{1}{2^{k-1}} \right\} \end{aligned}$$

$$|G_k| = 2^{k-1}$$

每一个 G_k 中的调和级数加项和：

$$\begin{aligned} \sum_{x \in G_k} x &\leq |G_k| \max G_k \\ &= 2^{k-1} \cdot \frac{1}{2^{k-1}} \\ &= 1 \end{aligned}$$

$$\frac{1}{2} < \sum_{x \in G_k} x \leq 1$$

$$= \frac{1}{2}$$

$$H_n = \underbrace{1 + \frac{1}{2} + \frac{1}{3}}_{G_1} + \frac{1}{4} + \cdots + \underbrace{\frac{1}{n-2} + \frac{1}{n-1}}_{\text{dashed line}} + \frac{1}{n},$$

G_2

$G_{\textcolor{teal}{t}}$

$$G_k = \left\{ \frac{1}{i} \mid \frac{1}{2^k} < \frac{1}{i} \leq \frac{1}{2^{k-1}} \right\}$$

$$\frac{1}{2} < \sum_{x \in G_k} x \leq \textcolor{red}{1}$$

$$2^{k-1} \leq i < 2^k$$

$$k = \lfloor \log_2 i \rfloor + 1 \quad \text{故 } \textcolor{teal}{t} = \lfloor \log_2 n \rfloor + 1$$

$$H_n \leq \textcolor{teal}{t} \cdot \textcolor{red}{1} \leq \log_2 n + 1$$

$$H_n > (\textcolor{teal}{t} - 1) \cdot \frac{1}{2} \geq \frac{1}{2} \lfloor \log_2 n \rfloor$$

$$H_n = 1 + \underbrace{\frac{1}{2} + \frac{1}{3}}_{G_0} + \frac{1}{4} + \cdots + \underbrace{\frac{1}{n-2} + \frac{1}{n-1}}_{G_t} + \frac{1}{n}$$

$$G_k = \left\{ \frac{1}{i} \mid \frac{1}{2^k} < \frac{1}{i} \leq \frac{1}{2^{k-1}} \right\}$$

$$\frac{1}{2} < \sum_{x \in G_k} x \leq 1$$

$$2^{k-1} \leq i < 2^k$$

$$k = \lfloor \log_2 i \rfloor + 1 \quad \text{故 } t = \lfloor \log_2 n \rfloor + 1$$

$$\left. \begin{array}{l} H_n \leq \textcolor{red}{t} \cdot \textcolor{red}{1} \leq \log_2 n + 1 \\ H_n > (\textcolor{red}{t} - 1) \cdot \frac{1}{2} \geq \frac{1}{2} \lfloor \log_2 n \rfloor \end{array} \right\} \quad \begin{aligned} H_n &= \Theta(\log_2 n) \\ &= \Theta(\ln n) \end{aligned}$$