Homework 3

Problem 1. Count the number of linear extensions for the following partial ordering:

X is a disjoint union of sets $X_1, X_2, \ldots, X_k$ of sizes $r_1, r_2, \ldots, r_k$, respectively. Each X_i is linearly ordered by $\preceq$, and no two elements from the different X are comparable.

Solution. $\binom{r_1+r_2+\cdots+r_k}{r_1,r_2,\cdots,r_k}$. □

Problem 2. Given a set X with $|X| = n$, determine the number of ordered set pairs $\langle A, B\rangle$ where $A \subseteq B \subseteq X$.

Solution. As a set B of size b has exactly 2^b subsets A, the number of such ordered pairs is

$$\sum_{0\le b\le n} 2^b \binom{n}{b} = (1+2)^n = 3^n.$$

□

Problem 3. Count the permutations with exactly k fixed points. (Remark: π is a permutation of the set {1,2,..., n}. Call an index i with $\pi(i) = i$, a fixed point of the permutation π.)
[Note: you can use $D(m)$ (i.e., the number of permutations on m items with no fixed points) directly.]

Solution. First choose the points that are fixed. It will have $\binom{n}{k}$ possible choices.

The rest is counting the number of permutation without a fixed point, which is $D(n-k)$.

In all, the answer is $\binom{n}{k} \cdot D(n-k)$. □

Problem 4. How many ways are there to seat n married couples at a round table with $2n$ chairs in such a way that the couples never sit next to each other?
(Note: It is not necessary to simplify the calculation results.)

Solution.(hint)

A_i = {the ways of seating in which the ith couple is adjacent.}

- $|A_i| = (2n-1)! \cdot 2^1/(2n-1)$;
- $|A_i \cap A_j| = (2n-2)! \cdot 2^2/(2n-2)$ for $i \neq j$;
- $\cdots$;
- $|A_{i1} \cap \cdots \cap A_{ik}| = (2n-k)! \cdot 2^k/(2n-k)$ for different A_{ij}s.

The final result should be $(2n)!/(2n) - |A_1 \cup A_2 \cup \cdots \cup A_k|$. Then by PIE....

$\square$