Homework 10

Problem 1. There are n hunters and n rabbits. Each hunter independently selects a rabbit at random, aims a gun at it, and then every hunter shoots his target at the same time. A random variable f is the number of rabbits that survive (assuming that no hunter misses).

1. Formalize the probability space, and what is the probability of every elementary event in this space?
2. Formalize f.
3. What is the expectation of f?

Solution.

1. $\Omega = \{\alpha \mid \alpha : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}\}$, each of probability n^{-n}
2. $f(\alpha) = |\ \{1, 2, \ldots, n\} \backslash \alpha(\{1, 2, \ldots, n\})\ |$.
3. $\mathbf{E}(f) = \left(1 - \frac{1}{n}\right)^n \cdot n \approx \frac{n}{e}$. (Linearity of expectation,).

$\square$

Problem 2. A permutation on the numbers $\{1, 2, \cdots, n\}$ can be represented as a function $\pi : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$. Now the permutation π is chosen uniformly at random from all permutations. Recall that a fixed point of a permutation π is a value for which $\pi(x) = x$. We use $Fix(\pi) = \{x \mid \pi(x) = x\}$ to represent the set of fixed point of π. Find the expectation and variance of the size of $Fix(\pi)$, i.e., $\mathbf{E}[\ |Fix(\pi)|\ ]$, $\mathbf{Var}[\ |Fix(\pi)|\ ]$.

Solution. For given permutation π, to every $i \in \{1, 2, \cdots, n\}$, define random variable x_i as $x_i = 1$ if $i \in Fix(\pi)$, and $x_i = 0$ if $i \notin Fix(\pi)$. Then $|Fix(\pi)| = \sum_i^n x_i$.

To the expectation: $\mathbf{E}[\ |Fix(\pi)|\ ] = \sum_{i=1}^n 1 \cdot \Pr(f(i) = i) = \sum_{i=1}^n 1 \cdot \frac{1}{n} = 1$.

To the variance:

$$
\begin{aligned}
\mathbf{Var}[\ |Fix(\pi)|\ ] &= \mathbf{E}[\ |Fix(\pi)|^2] - \mathbf{E}[\ |Fix(\pi)|\ ]^2 \\
&= \mathbf{E}[(\textstyle\sum_{i=1}^n x_i)^2] - 1 \\
&= \mathbf{E}[\textstyle\sum_{i=1}^n x_i^2] + \mathbf{E}[\textstyle\sum_{1\le i\neq j\le n} x_i x_j] - 1 \\
&= \textstyle\sum_{i=1}^n \mathbf{E}[x_i^2] + \sum_{1\le i\neq j\le n} \mathbf{E}[x_i x_j] - 1 \\
&= \textstyle\sum_{i=1}^n 1\cdot \Pr(x_i = 1) + \sum_{1\le i\neq j\le n} 1\cdot \Pr(x_i = x_j = 1) - 1 \\
&= n\cdot \tfrac{1}{n} + n(n-1)\tfrac{(n-2)!}{n!} - 1 \\
&= 1
\end{aligned}
$$

□

Problem 3. 1. Prove that, for every integer n, there exists a coloring of the edges of the complete graph K_n by two colors so that the total number of monochromatic copies of K_4 is at most $\binom{n}{4}2^{-5}$.

2. Give a randomized algorithm for finding a coloring with at most $\binom{n}{4}2^{-5}$ monochromatic (i.e. single-color) copies of K_4 that runs in expected time polynomial in n.

3. Show how to construct such a coloring deterministically in polynomial time using the method of conditional expectations. (*)

Solution.

1. Coloring every edge in K_4 by red or blue with probability 1/2. The expected value of the total number of monochromatic copies of K_4 is then $2\times\binom{n}{4}\times\left(\frac{1}{2}\right)^6$. Then there must exist some coloring scheme where the total number of monochromatic copies of K_4 is less or equal to $\binom{n}{4}2^{-5}$ (otherwise the expectation would be strictly larger than $\binom{n}{4}2^{-5}$.

2. Color each edge independently and uniformly. Let $p = Pr\left(X \le \binom{n}{4}2^{-5}\right)$ where X is the number of chromatic K_4.

$$
\begin{aligned}
\binom{n}{4}2^{-5} &= \mathbf{E}(X) \\
&= \textstyle\sum_{i\le\binom{n}{4}2^{-5}} i\cdot Pr(X=i) + \sum_{i>\binom{n}{4}2^{-5}} i\cdot Pr(X=i) \\
&\ge p + (1-p)\left(\binom{n}{4}2^{-5}+1\right)
\end{aligned}
$$

which implies $p \ge \frac{32}{\binom{n}{4}}$. The expected number of sampling before finding a suitable coloring is $1/p = \frac{\binom{n}{4}}{32}$. For each sampling, the time needs to count the number of chromatic K_4 is bounded by $\binom{n}{4}$ which is also polynomial. Thus the expected running time of this algorithm is polynomial.

3. Assign colors to each edge of K_n deterministically, in any order $x_1, x_2, \ldots, x_m$ (where x_i is the random variable of the coloring of edge i). Suppose that we have assigned the first k edges. Let $y_1, y_2, \ldots, y_k$ be the corresponding assigned values. We compute the quantities:

   $\mathbf{E}[X \mid x_1 = y_1, x_2 = y_2, \ldots, x_k = y_k, x_{k+1} = \text{red}]$;

   $\mathbf{E}[X \mid x_1 = y_1, x_2 = y_2, \ldots, x_k = y_k, x_{k+1} = \text{blue}]$;

   and then choose the setting with the smaller expectation.

□