Homework 5

Problem 1. Express the n^{th} term of the sequences given by the following recurrence relations

1. $a_0 = 2, a_1 = 3, a_{n+2} = 3a_n - 2a_{n+1}$ $(n = 0, 1, 2, \ldots)$.
2. $a_0 = 1, a_{n+1} = 2a_n + 3$ $(n = 0, 1, 2, \ldots)$.

Solution.

1. Characteristic function is $x^2 + 2x - 3 = (x+3)(x-1) = 0$.

   Let $f_n = a(-3)^n + b \cdot 1^n$. Then $\left\{ \begin{array}{lcl} 2 & = & a + b \\ 3 & = & -3a + b \end{array} \right. \Rightarrow a = -1/4, b = 9/4.$

   $\therefore$ the n-th term is f_n.

2. Characteristic function for the homogeneous part is $x = 2$. Take $a_n = p2^n + \lambda$

   $a_0 = 1, a_1 = 5$. Now $\left\{ \begin{array}{lcl} 1 & = & p + \lambda \\ 5 & = & 2p + \lambda \end{array} \right. \Rightarrow p = 4, \lambda = -3.$

$\square$

Problem 2. Solve the recurrence relation $a_{n+2} = \sqrt{a_{n+1}a_n}$ with initial conditions $a_0 = 2, a_1 = 8$ and find $\lim_{n\to\infty} a_n$.

Solution. Consider the sequence $b_n = \log_2 a_n$. Then

$$2\log_2 a_{n+2} = \log_2 a_{n+1} + \log_2 a_n$$

i.e. $2b_{n+2} = b_{n+1} + b_n$. $b_0 = 1, b_1 = 3$. One can find $b_n = (-\frac{4}{3})(-\frac{1}{2})^n + \frac{7}{3}$. $\therefore a_n = 2^{(-\frac{4}{3})(-\frac{1}{2})^n + \frac{7}{3}}$. $\lim_{n\to\infty} a_n = 2^{\frac{7}{3}}$. $\square$

Problem 3. Show that for any $n \geq 1$, the number $\frac{1}{2}[(1+\sqrt{2})^n + (1-\sqrt{2})^n]$ is an integer.

[Hint: Derive a recurrence relation for which the given value serves as a solution.]

Solution. Consider $\lambda_1 = (1+\sqrt{2})^n$, $\lambda_2 = (1-\sqrt{2})^n$. They are solutions to the characteristic function $(x-1-\sqrt{2})\cdot(x-1+\sqrt{2}) = x^2-2x-1$.

Thus the original sequence satisfies the recurrence $a_{n+2} = 2a_{n+1} + a_n$, with $a_0 = a_1 = 1$. □

Problem 4. Calculate $\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}(-4)^{-k}$.

[Hint: Let $f(n)$ be the intended summation. Then represent $f(n)$ recursively by utilizing $\binom{n-k}{k} = \binom{n-1-k}{k} + \binom{n-1-k}{k-1}$.]

Solution. Let

$$f(n) = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}(-4)^{-k}$$

$$\begin{aligned}
f(n) &= 1 + \sum_{k=1}^{\lfloor n/2\rfloor}\left[\binom{n-1-k}{k} + \binom{n-1-k}{k-1}\right](-4)^{-k} \\
&= \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-1-k}{k}(-4)^{-k} + \sum_{k=1}^{\lfloor n/2\rfloor}\binom{n-1-k}{k-1}(-4)^{-k} \\
&= \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-1-k}{k}(-4)^{-k} + \sum_{k=0}^{\lfloor \frac{n-2}{2}\rfloor}\binom{n-2-k}{k}(-4)^{-k-1} \\
&= f(n-1) - \tfrac{1}{4}f(n-2)
\end{aligned}$$

Characteristic function $x^2 - x + \frac{1}{4} = 0$. Thus $x = \frac{1}{2}$. We get

$$f(n) = \frac{1}{2^n}(an+b)$$

$f(0) = b = 1, f(1) = \frac{1}{2}(a+b) = 1$, thus $a = b = 1$, therefore $f(n) = \frac{n+1}{2^n}$. □