Homework 4

Problem 1. Determine the coefficient of x^3 in $(2+x)^{\frac{3}{2}}/(1-x)$

Solution.

$$\begin{aligned} &= (x+2)^{3/2}(1+x+x^2+\cdots) \\ &= \textstyle\sum_{k=0}^{\infty}\binom{3/2}{k}x^k(2)^{3/2-k}(1+x+x^2+\cdots) \end{aligned}$$

The coefficient of x^3 is $\sum_{k=0}^{3}\binom{3/2}{k}(2)^{3/2-k}$. Then use the Newton formula .... □

Problem 2. Find generating functions for the following sequences (express them in a closed form, without infinite series!):

1. $0, 0, 0, 0, -6, 6, -6, 6, -6, \cdots$.
2. $1, 0, 1, 0, 1, 0, \cdots$.
3. $1, 2, 1, 4, 1, 8 \cdots$

Solution.

□

Problem 3. Let a_n be the number of ordered triples $\langle i, j, k\rangle$ of integer numbers such that $i \geq 0, j \geq 1, k \geq 1$, and $i + 3j + 3k = n$. Find the generating function of the sequence $(a_0, a_1, a_2, \ldots)$ and calculate a formula for a_n.

Solution.

$(1+x+x^2+x^3+\cdots)(x^3+x^6+x^9+\cdots)(x^3+x^6+x^9+\cdots)$
$= \frac{1}{1-x}\frac{x^3}{1-x^3}\frac{x^3}{1-x^3}$
$= \frac{x^6(1+x+x^2)}{(1-x^3)^3} = x^6(1+x+x^2)(1-x^3)^{-3}$.
Then use the generalized binomial theorem. □

Problem 4. If $a(x)$ is the generating function of a sequence $(a_0, a_1, a_2, \ldots)$, please find the generating function of the sequence of partial sums $(a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots)$.

Solution.

Sequence	Generating Function
$(1,1,1,1,\ldots)$	$\frac{1}{1-x}$
$(1,-1,1,-1,\ldots)$	$\frac{1}{1+x}$
$(-6,6,-6,6,\ldots)$	$\frac{-6}{1+x}$
$(0,0,0,0,-6,6,-6,6,\ldots)$	$\frac{-6x^4}{1+x}$
$(1,0,1,0,\ldots)$	$\frac{\frac{1}{1-x}+\frac{1}{1+x}}{2} = \frac{1}{1-x^2}$
$(0,1,0,1,\ldots)$	$\frac{\frac{1}{1-x}-\frac{1}{1+x}}{2} = \frac{x}{1-x^2}$
$(1,2,4,8,\ldots)$	$\frac{1}{1-2x}$
$(2,4,8,\ldots)$	$\frac{\frac{1}{1-2x}-1}{x} = \frac{2}{1-2x}$
$(1,0,2,0,4,0,8,\ldots)$	$\frac{1}{1-2x^2}$
$(1,1,2,1,4,1,8,\ldots)$	$\frac{1}{1-2x^2} + \frac{x}{1-x^2}$
$(1,2,1,4,1,8,\ldots)$	$\frac{\frac{1}{1-2x^2}+\frac{x}{1-x^2}-1}{x} = -\frac{2x^3+2x^2-2x-1}{(1-2x^2)(1-x^2)}$

$$
\begin{aligned}
& a_0 + (a_0+a_1)x + (a_0+a_1+a_2)x^2 + (a_0+a_1+a_2+a_3)x^3 \cdots \\
= \quad & a_0(1+x+x^2+\cdots) \\
& +a_1x(1+x+x^2+\cdots) \\
& +a_2x^2(1+x+x^2+\cdots) \\
= \quad & \frac{a_0}{1-x} + \frac{a_1x}{1-x} + \frac{a_2x^2}{1-x} + \cdots \\
= \quad & \frac{1}{1-x}a(x)
\end{aligned}
$$

□