阶乘估值、二项式系数估值

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阶乘估值

- ***n*的阶乘(*n factorial*)**：

$$n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1 = \prod_{i=1}^{n} i.$$

- 对大小为n的集合X，该集合上的置换一共有$n!$个。

极值点估值$(n \geq 2)$

$$n! = \prod_{i=1}^{n} i \leq \prod_{i=1}^{n} n = n^n$$

$$n! = \prod_{i=2}^{n} i \geq \prod_{i=2}^{n} 2 = 2^{n-1}$$

- 对估计的改进：
  - 降低上界
  - 提高下界

极值点估值$(n \geq 2)$

$$n! = \prod_{i=1}^{n} i \leq \left(\prod_{i=1}^{n/2} \frac{n}{2}\right)\left(\prod_{i=n/2+1}^{n} n\right) = \left(\frac{n}{\sqrt{2}}\right)^n \quad \boxed{< n^n}$$

$$n! = \prod_{i=1}^{n} i \geq \prod_{i=n/2+1}^{n} i > \prod_{i=n/2+1}^{n} \frac{n}{2} = \left(\frac{n}{2}\right)^{n/2} = \left(\sqrt{\frac{n}{2}}\right)^n \quad \boxed{> 2^n}$$

$F = \{f \mid f:\{1,2,\cdots n\} \to \{1,2,\cdots,n\}\}$ 中任取一个函数g，g是单射函数的概率是多少?

$$\frac{n!}{n^n} \leq \frac{\left(\frac{n}{\sqrt{2}}\right)^n}{n^n} = 2^{-n/2}$$

高斯估值

$$\left(\sqrt{\frac{n}{2}}\right)^n \le n! \le \left(\frac{n}{\sqrt{2}}\right)^n$$

$$\Uparrow$$

算数-几何均值不等式(Arithmetic-geometric mean inequality)：对任意实数x, y，必有：

$$\sqrt{xy} \le \frac{x+y}{2}$$

$$n! = 1 \cdot 2 \cdot \cdots k \cdots \cdot n$$
$$n! = n \cdot (n-1) \cdot \cdots (n+1-k) \cdots \cdot 1$$

$$n! = \sqrt{n! \cdot n!} = \sqrt{\prod_{i=1}^{n} i(n+1-i)}$$

$$= \prod_{i=1}^{n} \sqrt{i(n+1-i)} \le \prod_{i=1}^{n} \frac{n+1}{2} = \left(\frac{n+1}{2}\right)^n$$

高斯估值

$$\left(\sqrt{\frac{n}{2}}\right)^n \leq n! \leq \left(\frac{n+1}{2}\right)^n$$

$$i(n+1-i) \geq n$$
$$i = 1,2,\ldots,n.$$

$$\Uparrow$$

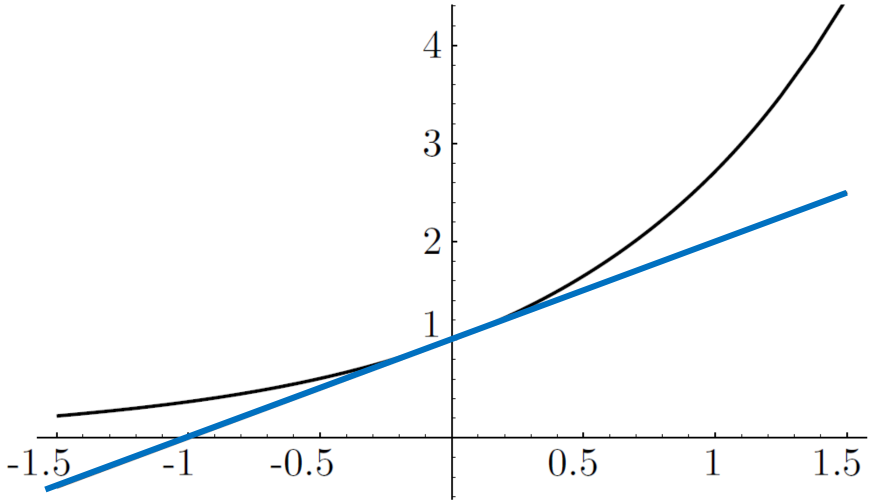
$$n! = 1 \cdot 2 \cdot \cdots k \cdots \cdot n$$
$$n! = n \cdot (n-1) \cdot \cdots (n+1-k) \cdots \cdot 1$$

$$n! = \sqrt{n! \cdot n!} = \sqrt{\prod_{i=1}^{n} i(n+1-i)}$$
$$= \prod_{i=1}^{n} \sqrt{i(n+1-i)}$$
$$\geq \prod_{i=1}^{n} \sqrt{n} = n^{n/2}$$

进一步优化

$$n^{\frac{n}{2}} \leq n! \leq \left(\frac{n+1}{2}\right)^n$$

欧拉数(Euler number) $e = 2.7182\ldots$



$$1 + x \leq e^x$$

进一步优化

$$e\left(\frac{n}{e}\right)^n \le \boxed{n! \ \le en\left(\frac{n}{e}\right)^n} \qquad 1+x \le e^x$$

证明：上界（归纳法）

- $n=1$： $1 \ge 1!$；
- 设 $n=k$ 时结论成立；
- $n=k+1$：

$$n! = n\cdot(n-1)! \le n\cdot e(n-1)\left(\frac{n-1}{e}\right)^{n-1}$$

$$= en\left(\frac{n}{e}\right)^n \cdot e\cdot\left(\frac{n-1}{n}\right)^n$$

而 $e\cdot\left(\frac{n-1}{n}\right)^n = e\cdot\left(1-\frac{1}{n}\right)^n \le e\cdot\left(e^{-1/n}\right)^n = e\cdot e^{-1} = 1$

进一步优化

$$\boxed{e\left(\frac{n}{e}\right)^n \le n!} \le en\left(\frac{n}{e}\right)^n \qquad 1 + x \le e^x$$

证明：下界（归纳法）

- $n = 1$： $1 \le 1!$；
- 设 $n = k$ 时结论成立；
- $n = k + 1$：

$$n! = n \cdot (n-1)! \ge n \cdot e\left(\frac{n-1}{e}\right)^{n-1}$$

$$= e\left(\frac{n}{e}\right)^n \cdot e \cdot \left(\frac{n-1}{n}\right)^{n-1}$$

进一步优化

$$\boxed{e\left(\frac{n}{e}\right)^n \le n!} \le en\left(\frac{n}{e}\right)^n$$

$$1 + x \le e^x$$

$$n! \ge e\left(\frac{n}{e}\right)^n \cdot e \cdot \left(\frac{n-1}{n}\right)^{n-1}$$

而 $e \cdot \left(\frac{n-1}{n}\right)^{n-1} = e \cdot \left(\frac{n}{n-1}\right)^{1-n} = e \cdot \left(1 + \frac{1}{n-1}\right)^{1-n}$

$$= e \cdot \left(\left(1 + \frac{1}{n-1}\right)^{n-1}\right)^{-1}$$

$$\ge e \cdot \left(\left(e^{\frac{1}{n-1}}\right)^{n-1}\right)^{-1} = e \cdot e^{-1} = 1$$

Stirling 公式

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

即 $$\lim_{n\to\infty}\frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{n!} = 1$$

二项式系数估值

$$\binom{n}{k} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k(k-1)\cdot\ldots\cdot 2\cdot 1}$$

$$= \prod_{i=0}^{k-1} \frac{n-i}{k-i}$$

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$= \frac{n!}{k!\cdot(n-k)!}$$

初步估值

由定义显然 $\binom{n}{k} \le n^k$

当 $n \ge k > i \ge 0$ 时 $\dfrac{n-i}{k-i} \ge \dfrac{n}{k}$

故 $\binom{n}{k} = \prod_{i=0}^{k-1} \dfrac{n-i}{k-i} \ge \left(\dfrac{n}{k}\right)^k$

利用二项式定理估值

二项式定理(*Binomial Theorem*):
对任意非负整数n，如下等式成立：

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

对 $n \geq 1, 1 \leq k \leq n$

取 $0 < x < 1$

$$\binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n = (1+x)^n$$

显然 $\binom{n}{0} + \binom{n}{1} x + \cdots + \binom{n}{k} x^k \leq (1+x)^n$

故有 $\frac{1}{x^k}\binom{n}{0} + \frac{1}{x^{k-1}}\binom{n}{1} + \cdots + \binom{n}{k} \leq \frac{(1+x)^n}{x^k}$ ，且$0 < x < 1$

故有 $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} \leq \frac{(1+x)^n}{x^k}$

利用二项式定理估值

$$\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{k}\le\frac{(1+x)^n}{x^k}$$

取 $x=\frac{k}{n}$

$$\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{k}\le\left(1+\frac{k}{n}\right)^n\left(\frac{n}{k}\right)^k$$

$$1+x\le e^x$$

$$\le\left(e^{k/n}\right)^n\left(\frac{n}{k}\right)^k$$

$$=\left(\frac{en}{k}\right)^k$$

$$\binom{n}{k}\le\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{k}\le\left(\frac{en}{k}\right)^k$$