Homework 11

Problem 1. Use the Lovasz local lemma to show that if

$$4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} \leq 1$$

then it is possible to color the edges of K_n with two colors so that it has no monochromatic (i.e. single color) K_k subgraph.

Solution. E_i: the $i-$th K_k is monochromatic. $Pr(E_i) = 2^{1-\binom{k}{2}}$. Consider the dependency graph, for any different E_i and E_j, they are adjacent if the corresponding K_k share at least one edge. Thus the degree of the dependency graph is bounded by $\binom{k}{2}\binom{n}{k-2}$.

According to the Lovasz local lemma, it is possible that none of the E_i happens under the given inequality. □

Problem 2. What is the expected number of trees with k vertices in $G \in \mathcal{G}(n, p)$?

Solution. By Cayley's formula and the linearity of expectation, it is $\binom{n}{k}k^{k-2}p^{k-1}$ □

Problem 3. suppose that $p = \frac{d}{n}$ where $d = o(n^{1/3})$. Show that $G \in \mathcal{G}(n, p)$ almost surely has no copies of K_4.

Solution. The probability that there exists some 4-vertices sub-graph which induces a K_4 is upper bounded by $p = \binom{n}{4}p^6$. When $d = o(n^{1/3})$, $\lim_{n\to\infty} p = 0$. □

Problem 4. Show that if almost all $G \in \mathcal{G}(n, p)$ have a graph property $\mathcal{P}_1$ and almost all $G \in \mathcal{G}(n, p)$ have a graph property $\mathcal{P}_2$, then almost all $G \in \mathcal{G}(n, p)$ have both properties.

Solution. The portion of the graphs have both properties equals 1 minus the portion of the graphs which does not have property $\mathcal{P}_1$ or $\mathcal{P}_2$. However the portion of the graph does not have property $\mathcal{P}_1$ or $\mathcal{P}_2$ is bounded by the sum of the portion of the graphs does not have property $\mathcal{P}_1$ and the the portion of the graphs does not have property $\mathcal{P}_2$, which both tend to 0 as n approaches ∞. The claim in the question then follows. □

Problem 5. Consider $\mathbf{G}(n, p)$ with $p = \frac{1}{3n}$.

Use the second moment method to show that with high probability there exists a simple path of length 10.

Solution. $I_\ell = \begin{cases} 1 & \ell \text{ is a length 10 simple path} \\ 0 & \text{otherwise} \end{cases}$
X is the number of simple path, then $X = \sum_\ell I_\ell$.
The expectation would be $E(X) = \frac{1}{2}(n)_{11} \times p^{10}$. Thus

$$(E(X))^2 = \frac{1}{4}[(n)_{11}]^2 \times p^{20} \qquad (\star)$$

(Note $(\star)$ is about $\Theta(n^2)$.)
Then to calculate $E(X^2)$.

$$E(X^2) = E\left[(\sum_\ell I_\ell)^2\right] = E\left[\sum_\ell I_\ell \sum_{\ell'} I_{\ell'}\right] = \sum_{\ell,\ell'} E(I_\ell I_{\ell'}) \qquad (\Delta)$$

ℓ and ℓ' will be independent to each other unless they have common vertices or edges. We use $k = |\ell \cap \ell'|$ to stand for the number of common edges between ℓ and ℓ', and $s = |\ell \cap \ell'|$ to stand for the number of common vertices used by ℓ and ℓ'. Obviously $(0 \leq k \leq 10) \wedge (0 \leq s \leq 11) \wedge (k \geq 1 \rightarrow s \geq k + 1)$. (Δ) can be divided into the following subcases:

1. $k = 0$
   (a) $s = 0$: $\sum_{\ell,\ell'} E(I_\ell I_{\ell'}) = \frac{1}{8}(n)_{22} \times p^{20} \leq (E(X))^2$;
   (b) $1 \leq s \leq 11$:
   for each s, $\sum_{\ell,\ell'} E(I_\ell I_{\ell'}) = c \cdot (n)_{22-s} \times (p)^{20} = \mathbf{o}((E(X))^2)$, where c is a constant number.
2. $1 \leq k \leq 10$ ($2 \leq s \leq 11$)
   The general formula of each of these cases (constant many) would be

$$\begin{aligned} \sum_{\ell,\ell'} E(I_\ell I_{\ell'}) &= d \cdot (n)_{22-s} \times p^{20-k} \\ &\leq d \cdot (n)_{22-s} \times p^{20-(s-1)} \\ &= d \cdot (n)_{22-s} \times p^{21-s} \\ &= \mathbf{o}((E(X))^2) \end{aligned}$$

Combining the above results we get that $Var(X) = E(X^2) - (E(X))^2 = \mathbf{o}((E(X))^2)$.

$\square$