

Homework 7

Problem 1. Which of the following statements about graph G and H are true?

1. G and H are isomorphic if and only if for every map $f : V(G) \rightarrow V(H)$ and for any two vertices $u, v \in V(G)$, we have $\{u, v\} \in E(G) \Leftrightarrow \{f(u), f(v)\} \in E(H)$.
2. G and H are isomorphic if and only if there exists a bijection $f : E(G) \rightarrow E(H)$.
3. If there exists a bijection $f : V(G) \rightarrow V(H)$ such that every vertex $u \in V(G)$ has the same degree as $f(u)$, then G and H are isomorphic.
4. If G and H are isomorphic, then there exists a bijection $f : V(G) \rightarrow V(H)$ such that every vertex $u \in V(G)$ has the same degree as $f(u)$.
5. If G and H are isomorphic, then there exists a bijection $f : E(G) \rightarrow E(H)$.
6. G and H are isomorphic if and only if there exists a map $f : V(G) \rightarrow V(H)$ such that for any two vertices $u, v \in V(G)$, we have $\{u, v\} \in E(G) \Leftrightarrow \{f(u), f(v)\} \in E(H)$.
7. Every graph on n vertices is isomorphic to some graph on the vertex set $\{1, 2, \dots, n\}$.
8. Every graph on $n \geq 1$ vertices is isomorphic to infinitely many graphs.

Solution. 4,5,7,8. □

Problem 2. Two simple graphs $G = (V, E)$ and $G' = (V', E')$. A map $f : V \rightarrow V'$. Now if f satisfies:

- i) It is a bijective function;
- ii) $\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$;

Then we say that graph G and G' are isomorphic to each other. We use $G \cong G'$ to stand for the isomorphism relation.

Consider the following questions:

1. $G = K_n$ (Recall: K_n is a clique with n vertices), $g : V \rightarrow V'$ is a function which only satisfies requirement ii). Prove that G' must contain a subgraph which is a clique with n -vertices.
2. $G = K_{n,m}$ (Recall: $K_{n,m}$ is the so-called complete bipartite graphs), g is the same as in question 1. What will be the simplest G' that is related to G under the new relation.

Solution.

1. Two different vertices in G must be mapped to different vertices in G' . For if $u, v \in V$ are mapped to the same vertex $\omega \in V'$, the edge $\{u, v\} \in E$ cannot be reflected in E' , for G' is a simple graph.
2. G' can be just one edge with two incident vertices.

□

Problem 3. How many graphs on the vertex set $\{1, 2, \dots, 2n\}$ are isomorphic to the graph consisting of n vertex-disjoint edges (i.e. with edge set $\{\{1,2\}, \{3,4\}, \dots, \{2n-1,2n\}\}$)?

Solution. $\frac{(2n \cdot (2n-1))((2n-2) \cdot (2n-3)) \cdots (2 \cdot 1)}{2^n \cdot n!} = (2n-1)(2n-3) \cdots 5 \cdot 3$.

□

Problem 4. Construct an example of a sequence of length n in which each term is some of the numbers $1, 2, \dots, n-1$ and which has an even number of odd terms, and yet the sequence is not a graph score. Show why it is not a graph score.

Solution. E.g. $(1, 1, 3, 3, 4)$. Use the Score theorem to prove that it cannot be a graph score. □