

# Homework 1

Problem 1. Let  $(X, \leq_1)$ ,  $(Y, \leq_2)$  be (partially) ordered sets. We say that they are isomorphic if there exists a bijection  $f : X \rightarrow Y$  such that for every  $x, y \in X$ , we have  $x \leq_1 y$  if and only if  $f(x) \leq_2 f(y)$ .

1. Draw Hasse diagrams for all nonisomorphic 3-element posets.
2. Prove that any two  $n$ -element linearly ordered sets are isomorphic.
3. Prove that  $(\mathbb{N}, \leq)$  and  $(\mathbb{Q}, \leq)$  are NOT isomorphic. ( where  $\mathbb{N}$  is the set of natural numbers,  $\mathbb{Q}$  is the set of rational numbers,  $\leq$  is the usual ‘less or equal to’ between numbers).

Solution.

1. Omitted
2. Hint: Always map the minimal/least element in one structure to the other.
3. Suppose there is such an isomorphism function  $f : \mathbb{N} \rightarrow \mathbb{Q}$ .  $f(0) = a$ ,  $f(1) = b$ . We have  $a < b$  for  $0 < 1$ . Then consider  $f^{-1}(\frac{a+b}{2})$ .  $\square$

Problem 2. Prove or disprove: If a partially ordered set  $(X, \leq)$  has a single minimal element, then it is a smallest element as well.

Solution. Wrong. Consider  $(\{a\}, \langle a, a \rangle) \cup (\mathbb{Z}, \leq)$ .  $\square$

Problem 3. Let  $(X, \leq)$  and  $(X', \leq')$  be partially ordered sets. A mapping  $f : X \rightarrow X'$  is called an embedding of  $(X, \leq)$  into  $(X', \leq')$  if the following conditions hold:

- $f$  is an injective mapping;
- $f(x) \leq' f(y)$  if and only if  $x \leq y$ .

Now consider the following problem

- 1) Describe an embedding of the set  $\{1, 2\} \times \mathbb{N}$  with the lexicographic ordering into the ordered set  $(\mathbb{Q}, \leq)$ .
- 2) Solve the analog of a) with the set  $\mathbb{N} \times \mathbb{N}$  (ordered lexicographically) instead of  $\{1, 2\} \times \mathbb{N}$ .

Solution.

1)  $f(i, n) = i + y$  where  $i \in \{1, 2\}$  and  $y = \frac{n}{n+1}$ .

2) Similarly.

□

Problem 4. Prove the following strengthening of the Erdős-Szekeres Lemma: Let  $\kappa, \ell$  be natural numbers. Then every sequence of real numbers of length  $\kappa\ell + 1$  contains an nondecreasing subsequence of length  $\kappa + 1$  or a decreasing subsequence of length  $\ell + 1$ .

Solution. Hint:  $\alpha(P) \cdot \omega(P) > \kappa\ell$ . Then similar to the proof of Erdős-Szekeres Lemma: either  $\omega(P) > \kappa$ , which implies the existence of a nondecreasing subsequence of length  $\kappa + 1$ , or  $\omega(P) \leq \kappa$ , then  $\alpha(P) > \ell$ , which implies the other case. □