Applications of Handshake lemma

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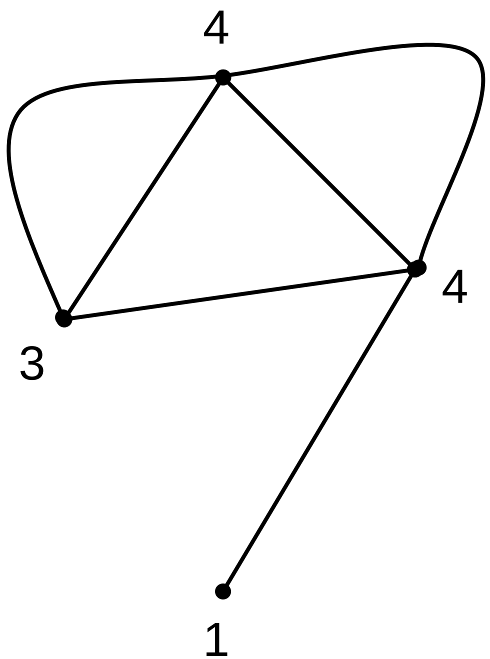
握手定理（回顾）

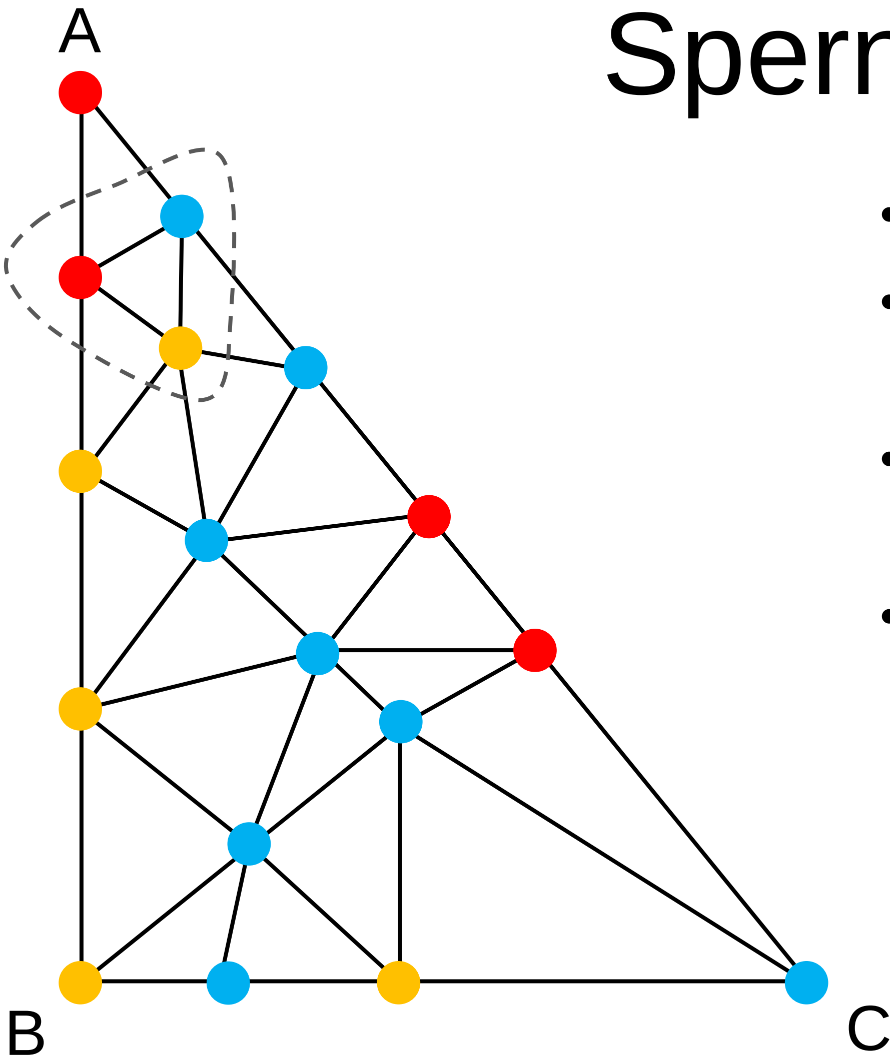
- **握手定理：** 给定无向图 $G=(V,E)$，有 $\sum_{v\in V}\deg_G(v)=2|E|$

- **推论：** 无向图中，度数为奇数的点一定是有偶数多个。



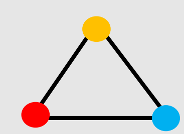
$1+3+4+4=12=2\times 6$

Sperner 引理

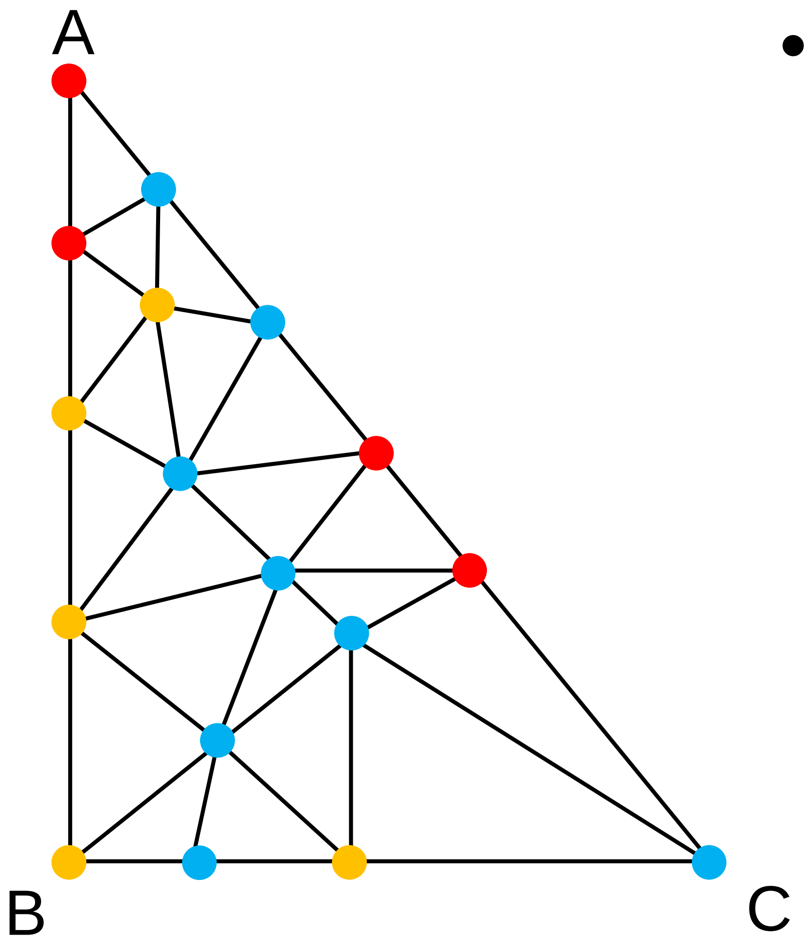


- 已知平面上的一个三角形**ABC**。
- 任意划分成若干小的不重叠三角形。
- 用、、依次对**A**、**B**、**C**三个顶点着色。
- 对其余顶点着色：
  - **BC**边上的点用色或色
  - **AB**边上的点用色或色
  - **AC**边上的点用色或色
  - 其它内部顶点：任意着色

Sperner 引理（平面）:必然存在一个三个顶点不同色的三角形



Sperner's lemma的证明



- 构造图$G = (V, E)$

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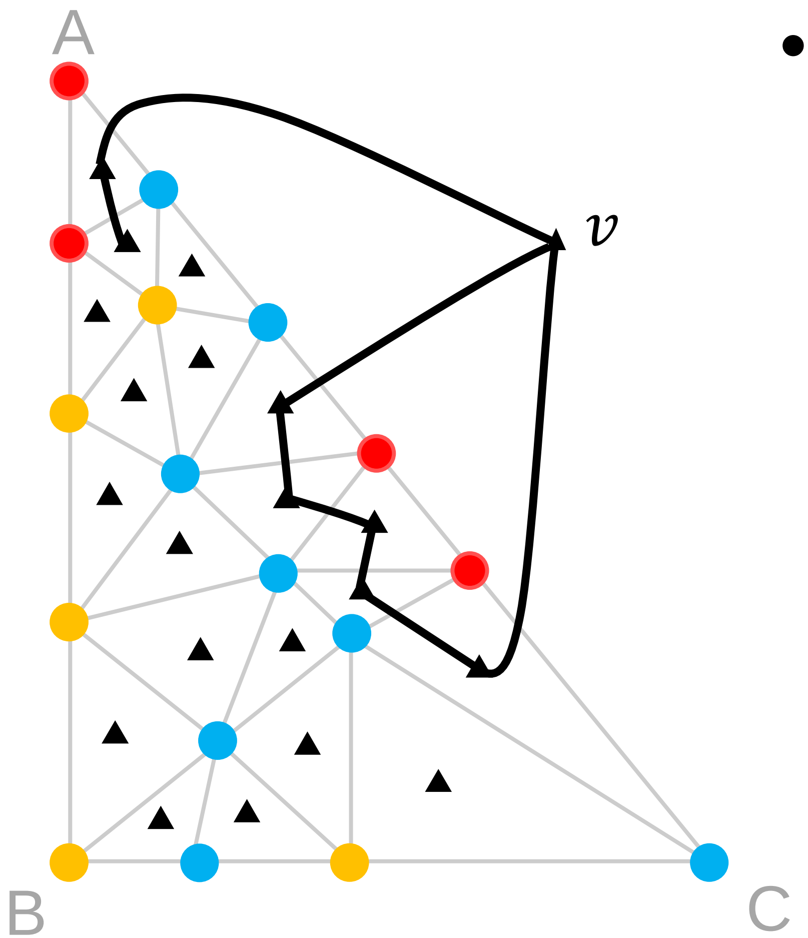
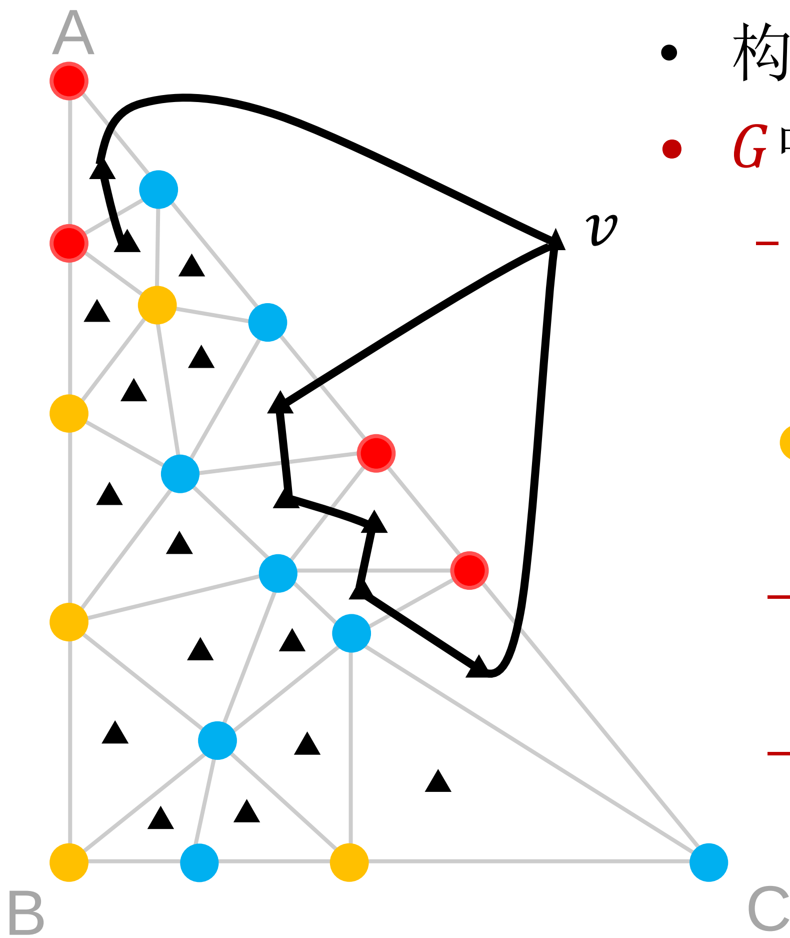
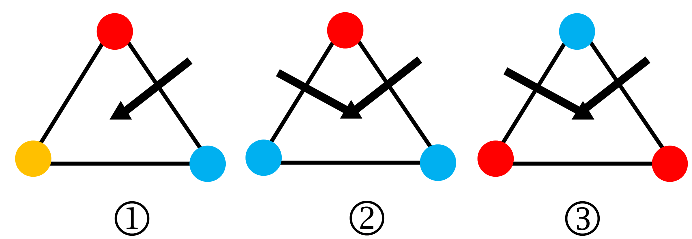


- 构造图$G = (V, E)$
  - V：每个闭合的连续平面（小三角形）抽象为一个点，外面的开放平面也抽象为一个点，用▲表示，取名为v。
  - E：两个▲之间有一条边当且仅当原对应平面相邻且邻边顶点着色为●和●。

Sperner’s lemma的证明

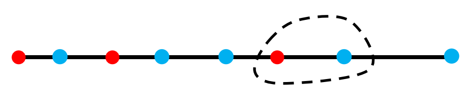
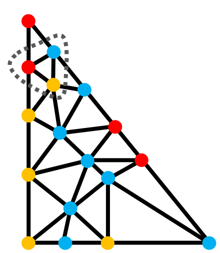


- 构造图$G = (V, E)$
- G中顶点的度数：
  - V在ABC内(非v)度数非0的情况:

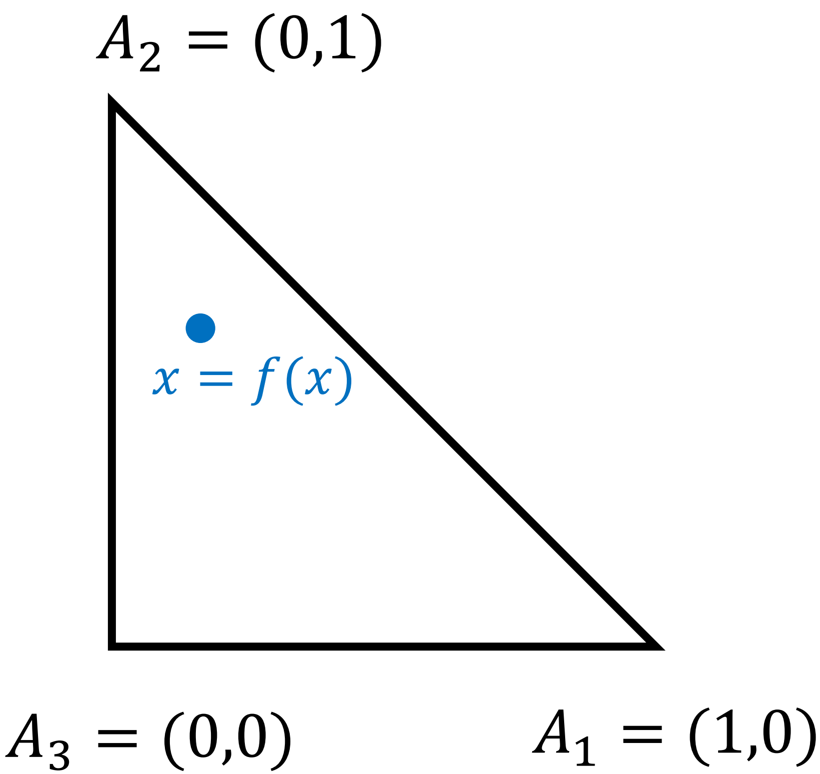


  - V在ABC内(非v)的点在其余情况下度数均为0。
  - V在ABC外的点（点v）的度数:就是AC边上的颜色改变次数，易证其必为**奇数**。
  - 根据**握手定理**， G中必还有度数为奇的点，即情况①必发生。

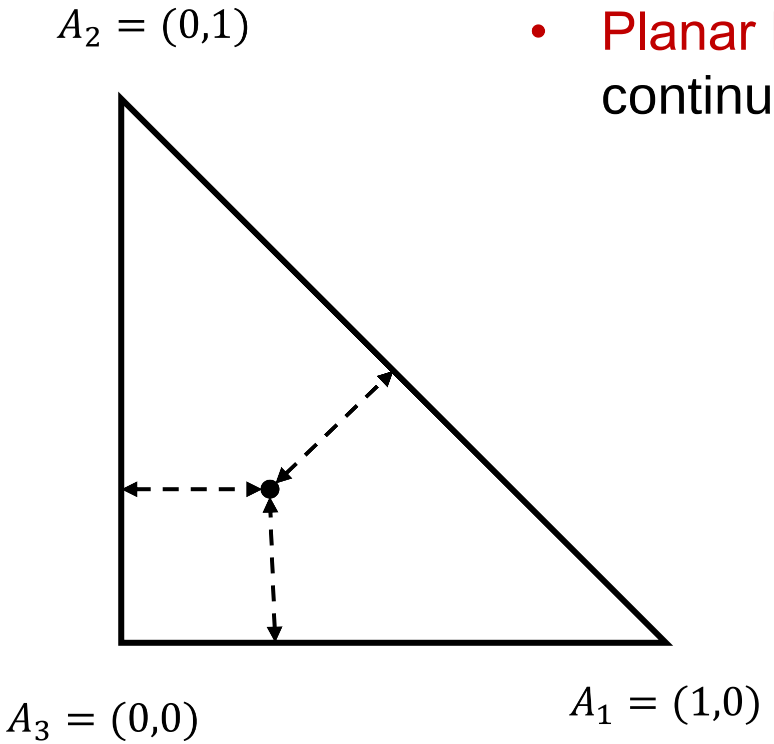
引理的一般形式

- Spernner's lemma (Sperner, 1928)：对任意n维单形体(n-simplex)进行分割并用$n+1$种颜色去着色，则任何合适的单形体分割着色方案下，都必有一个包含所有不同颜色的单元。
- 例:
  - $n=0$
    
  - $n=1$
    
  - $n=2$
    
  - $n=3$ 四面体
  - ……

Application(1) – Fix point



- Δ is a triangle in the plane.
- $f: \Delta \to \Delta$ is continuous: $\forall a \in \Delta, \forall \epsilon > 0, \exists \delta > 0$ $(b \in \Delta \wedge dist(a, b) \leq \delta \to dist(f(a), f(b)) \leq \epsilon.)$
- Planar Brouwer's fixed point theorem: Every continuous function $f: \Delta \to \Delta$ has a fixed point.



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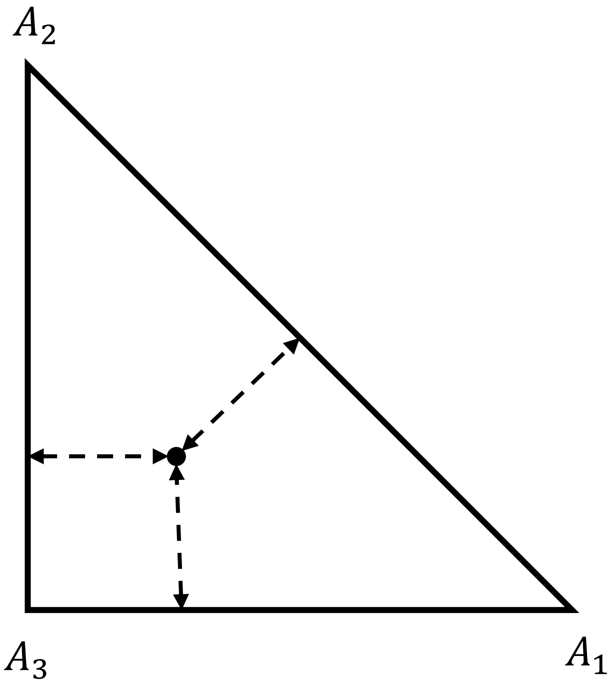

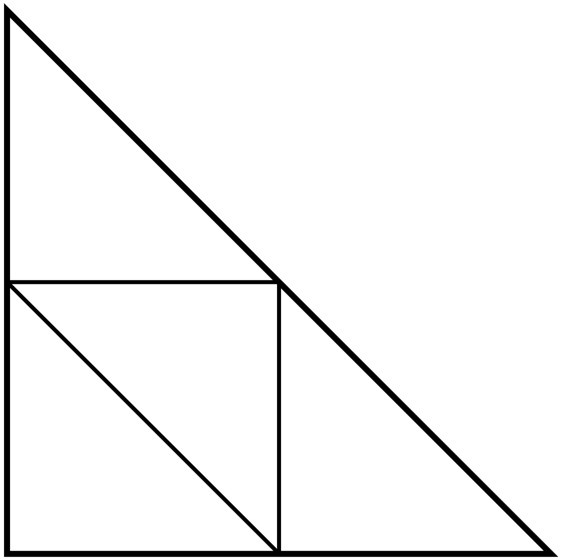

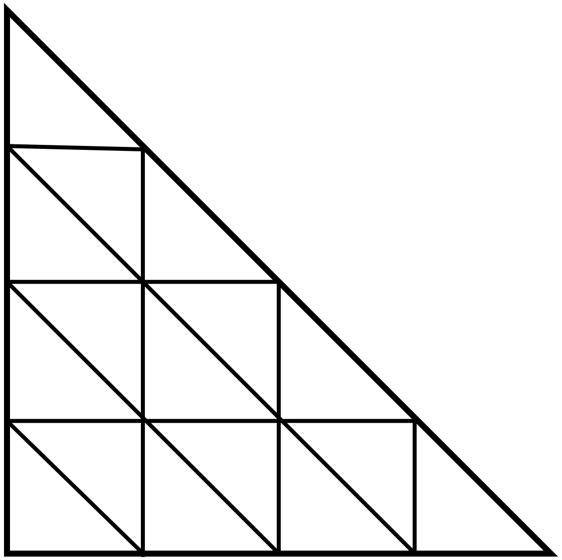

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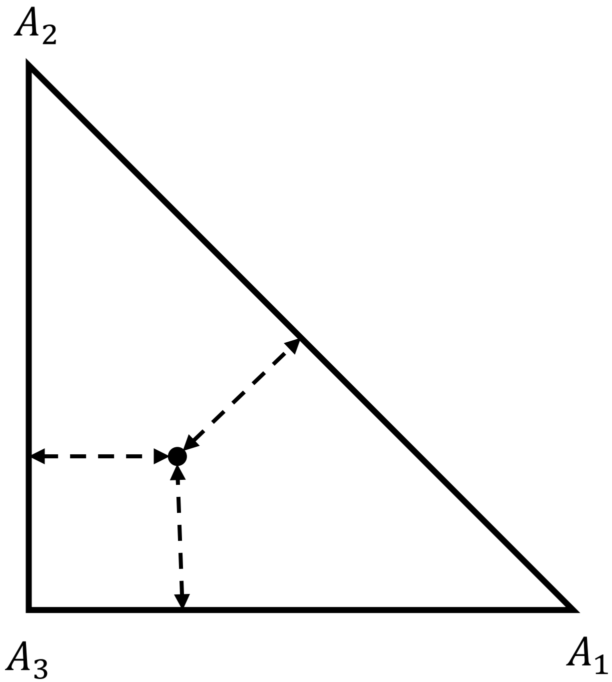
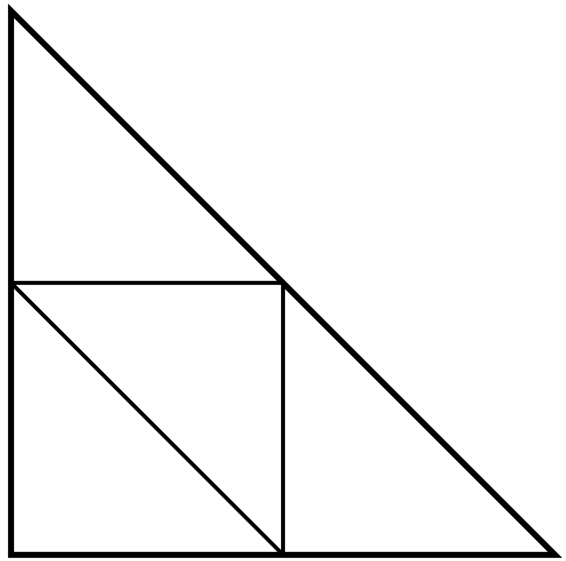
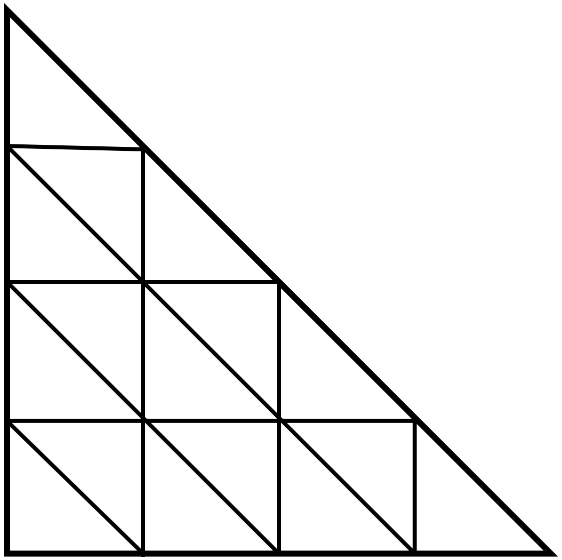
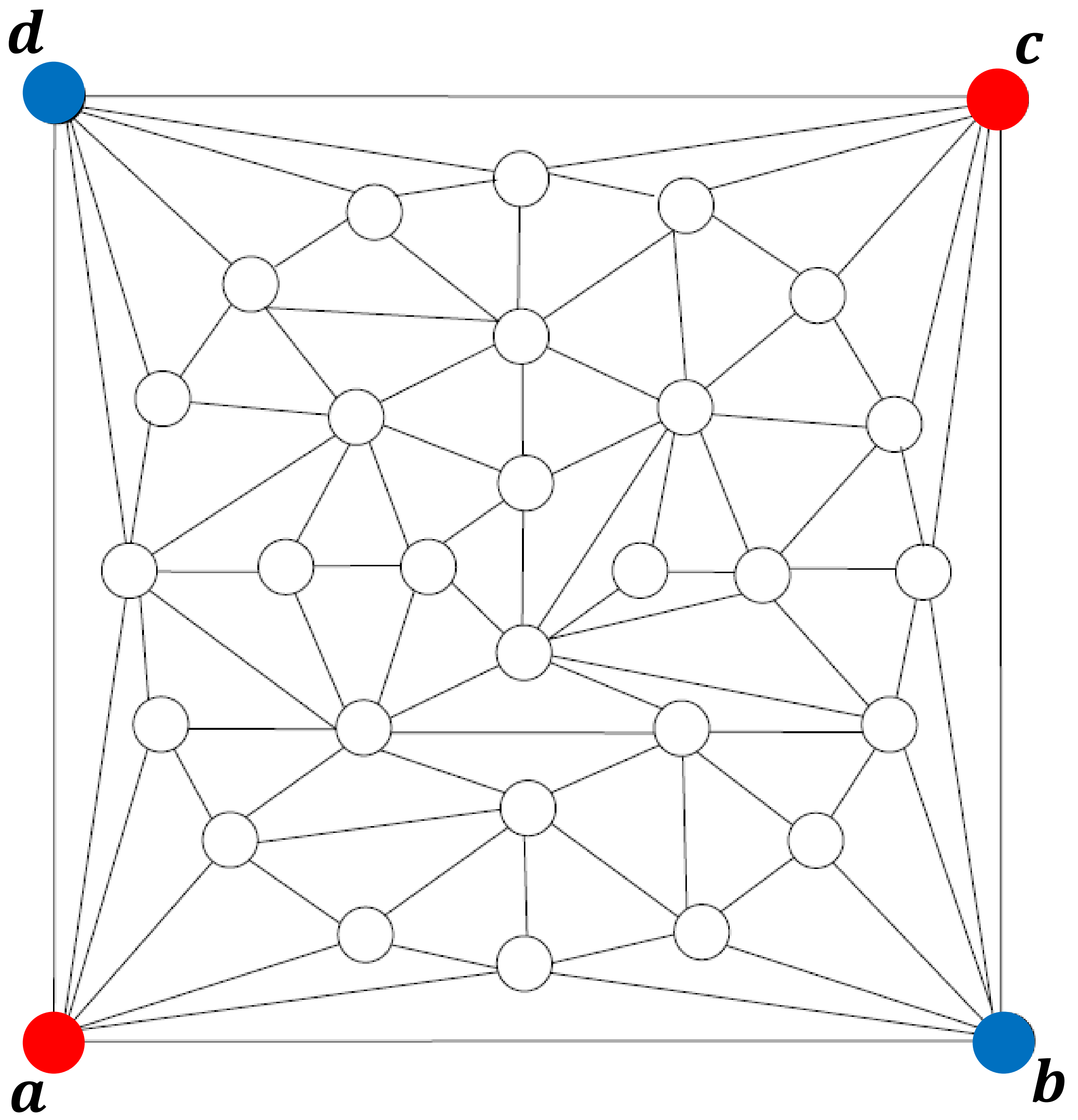






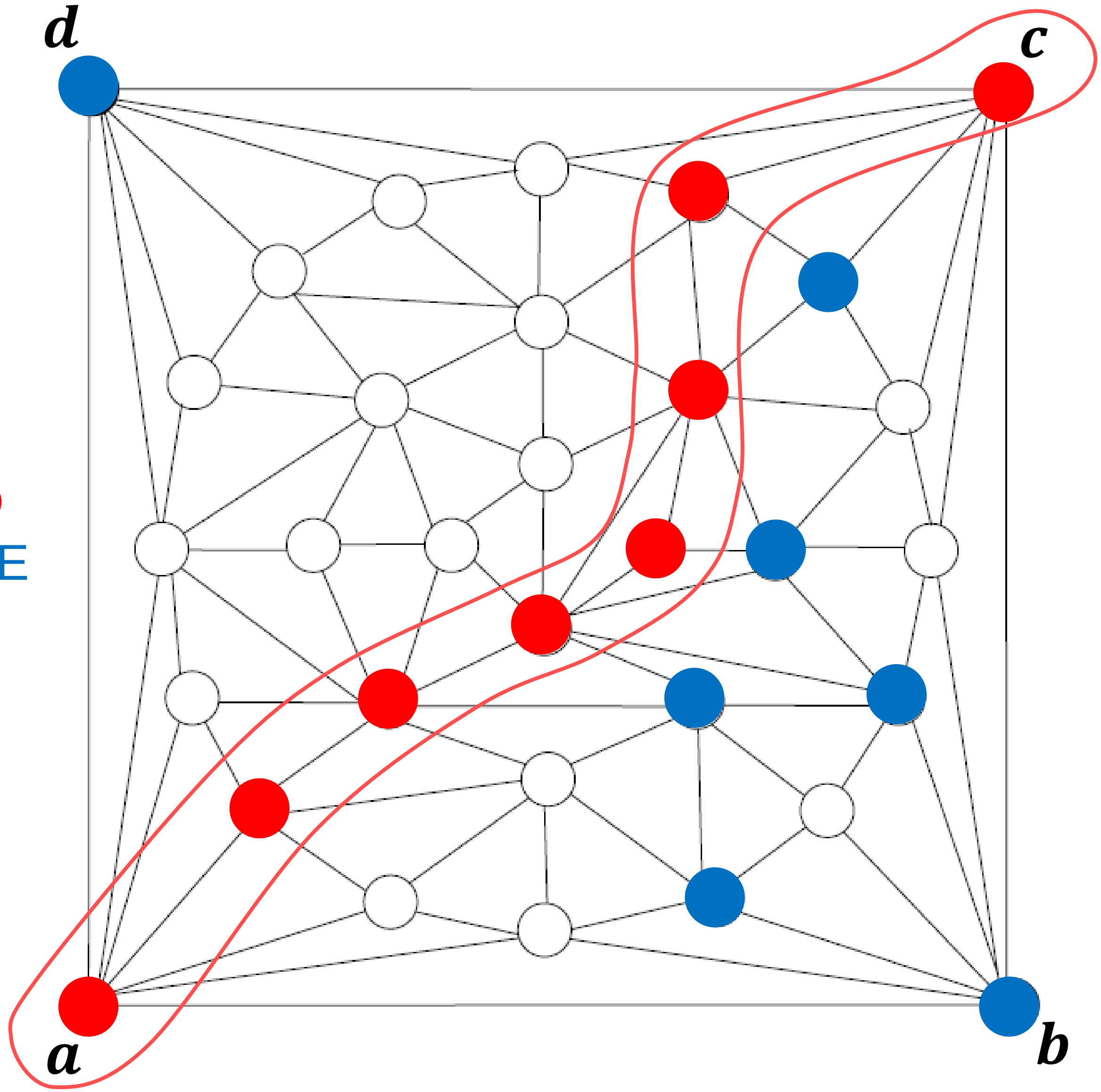
Application(2) – HEX game



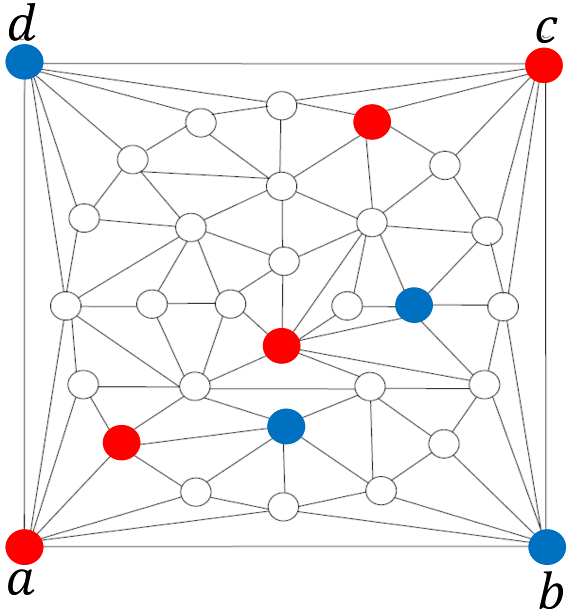
Player1: RED
Player2: BLUE

Proposition: To any given type (outer face is a square, all inner faces are triangles), there must be a winner.

Player1: RED
Player2: BLUE



RED wins!

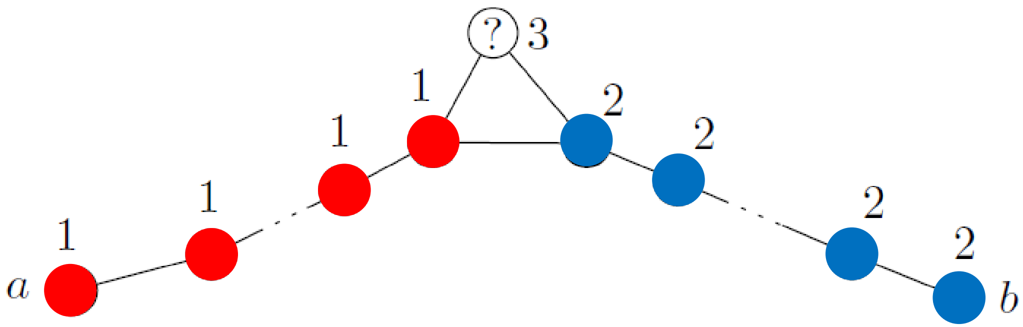


- R = the set of vertices marked red.
- B = the set of vertices marked blue.

Labelling vertex v :

- 1： if the vertex belongs to R and there is an all-red path from a to v.
- 2： if the vertex belongs to B and there is an all-blue path from b to v.
- 3: otherwise.

Observation: If the proposition fails, then both c and d must be labelled by 3.

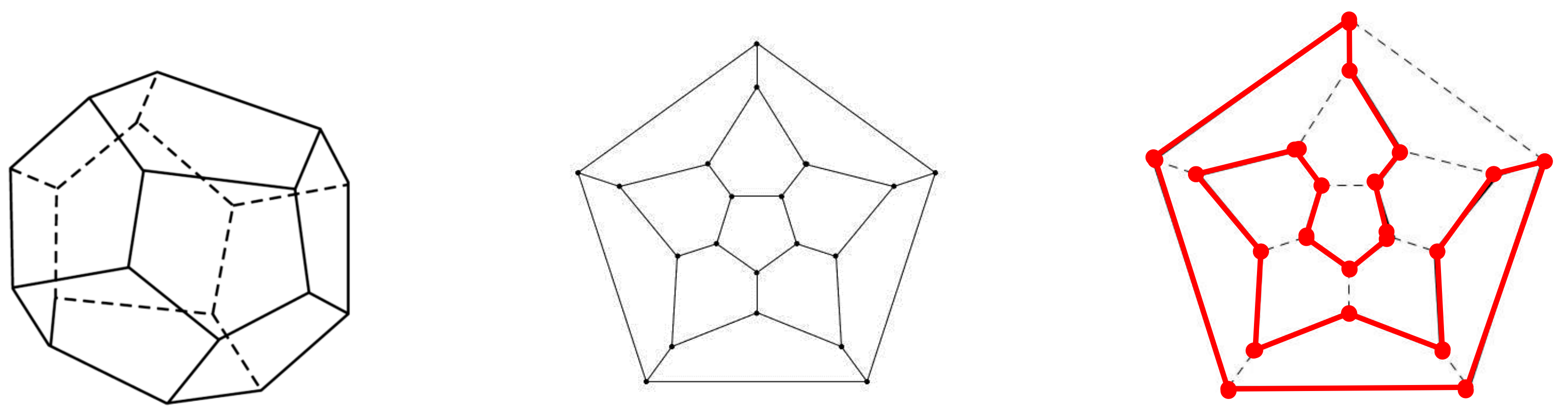


Observation: If the proposition fails, then both c and d must be labelled by 3.

- A triangle contains three different labels {1, 2, 3} will lead to contradiction.
- If both c and d have label 3, then there must be such an triangle.
- Either c is labeled 1 or d is labeled 2.
- Either RED wins or BLUE wins.

哈密顿回路

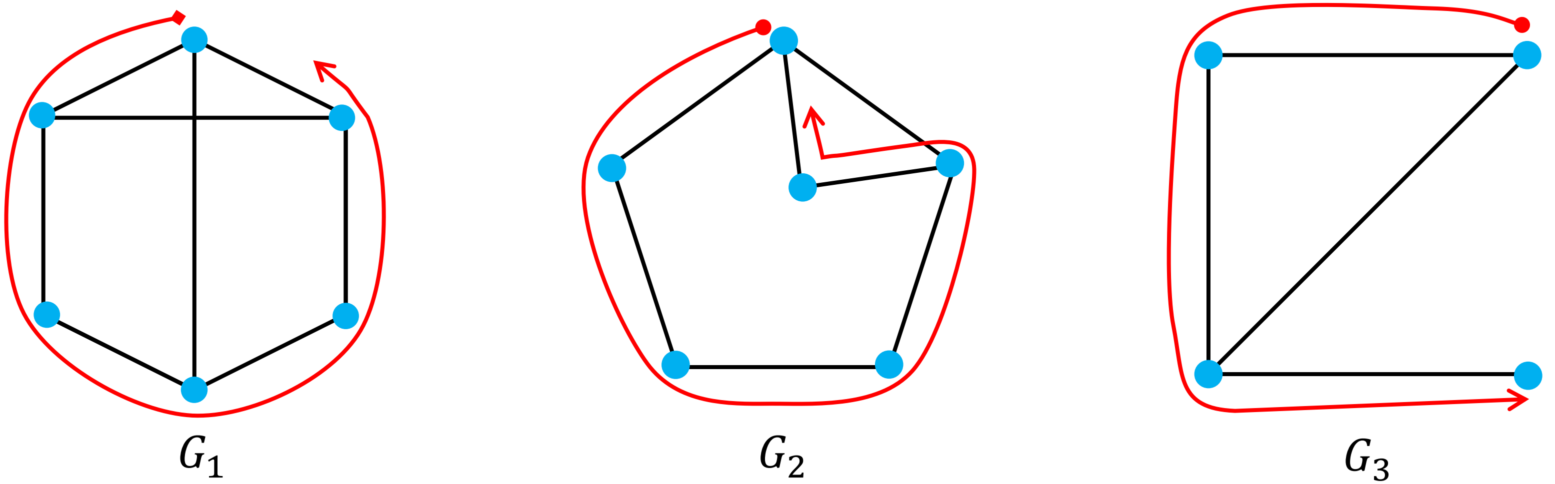
- 19世纪英国数学家哈密顿(Sir William Hamilton) 提出的问题：正凸12面体，把20个顶点比作世界上20个城市，30条棱表示这些城市间的交通路线。问题：能否周游世界，即从某个城市出发，经过每城一次且只一次最后返回出发地。



哈密顿图

- **哈密顿回路***(Hamiltonian cycle)*：如果一个环经过图上所有点正好一次，则此环被称为哈密顿环。
- **哈密顿图***(Hamiltonian graph)*：含有哈密顿环的图，被称为哈密顿图。
- **哈密顿路径***(Hamiltonian path)*：如果一条路径经过图上所有点正好一次，则此路径被称为哈密顿路径。
- 仅考虑简单图（无环、无重边）。

- 例：

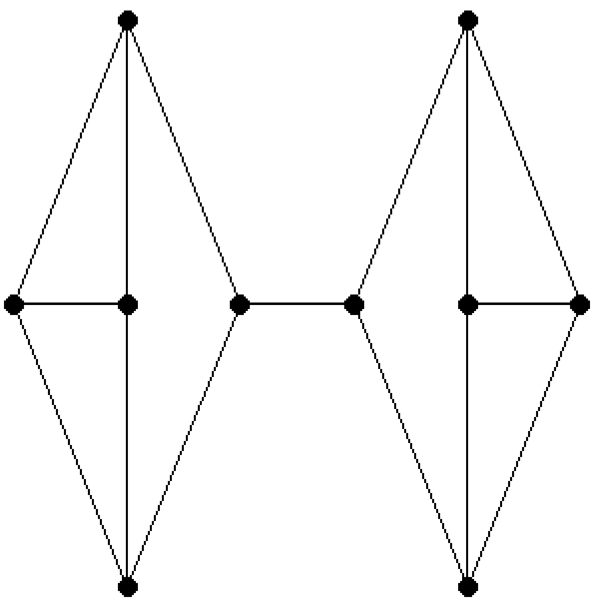


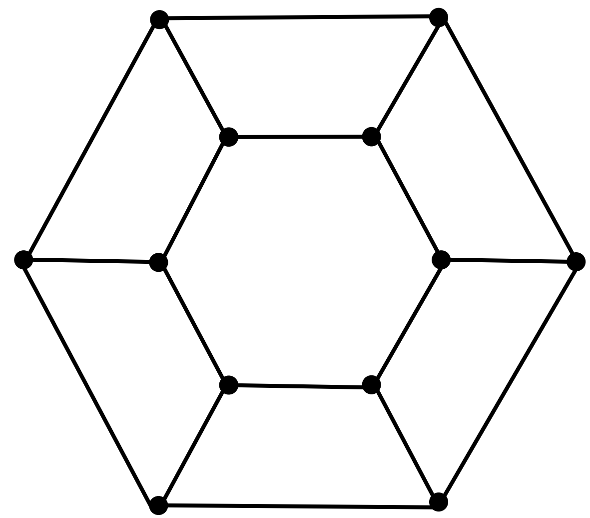
- G_1, G_2, G_3 都含有哈密顿路径
- 仅G_1, G_2 含哈密顿回路，是哈密顿图

哈密顿回路 → 哈密顿路径，反之不成立。

握手定理与哈密顿图

- 定理(Smith)：对3-正则图，包含图上任意边e的哈密顿回路必有偶数条。





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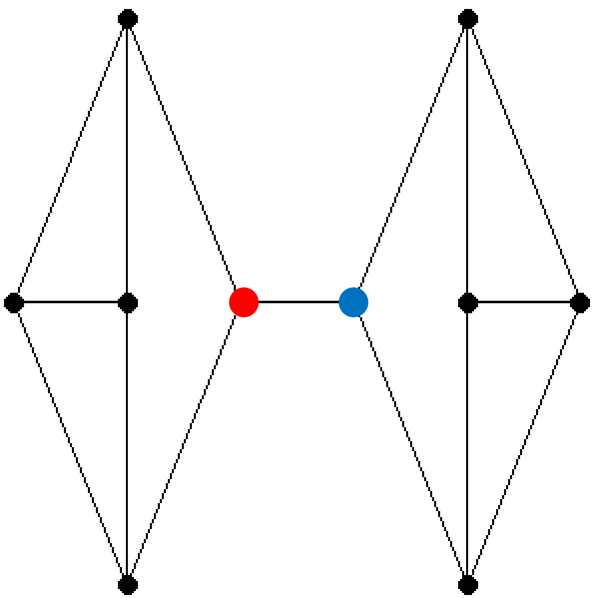

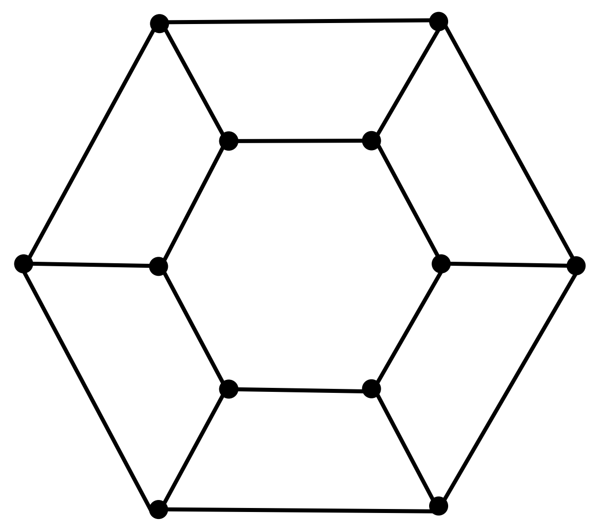

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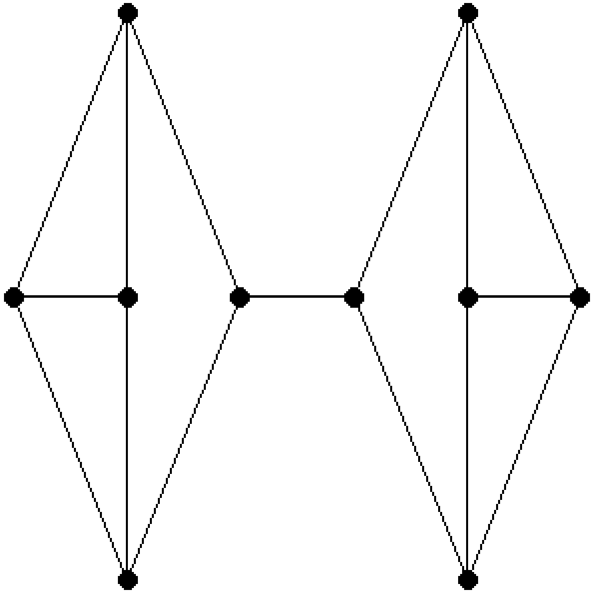

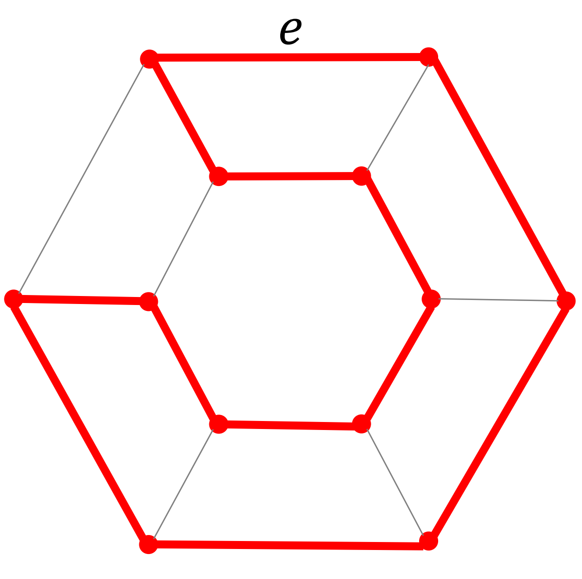

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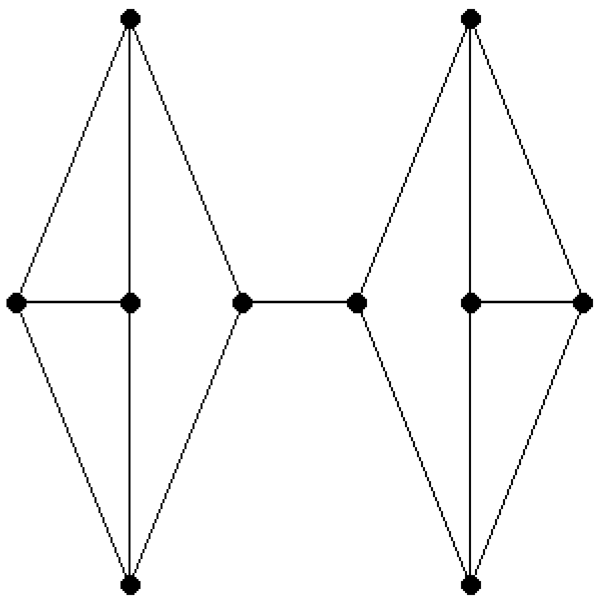

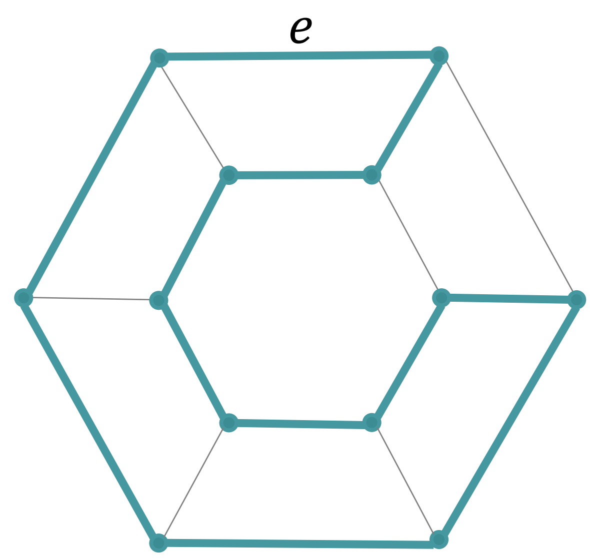

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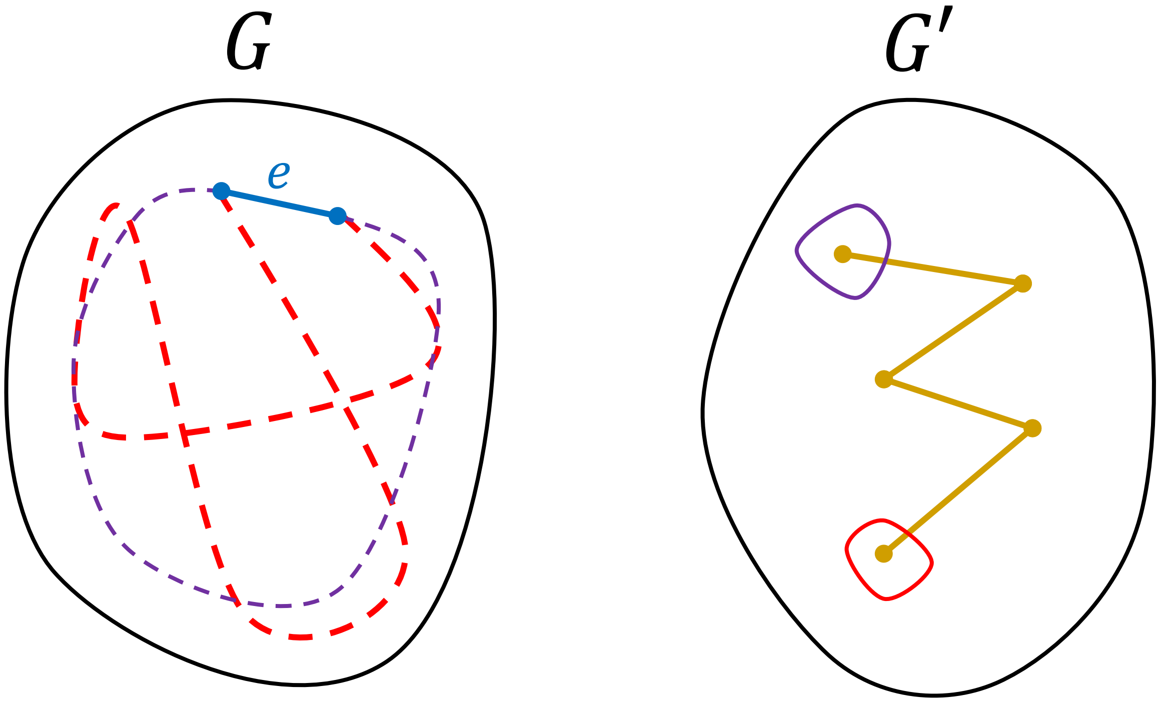

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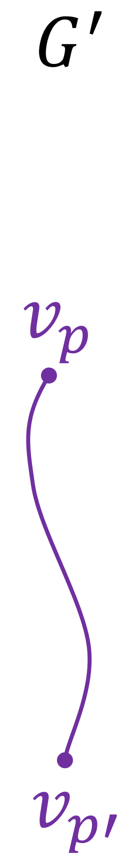
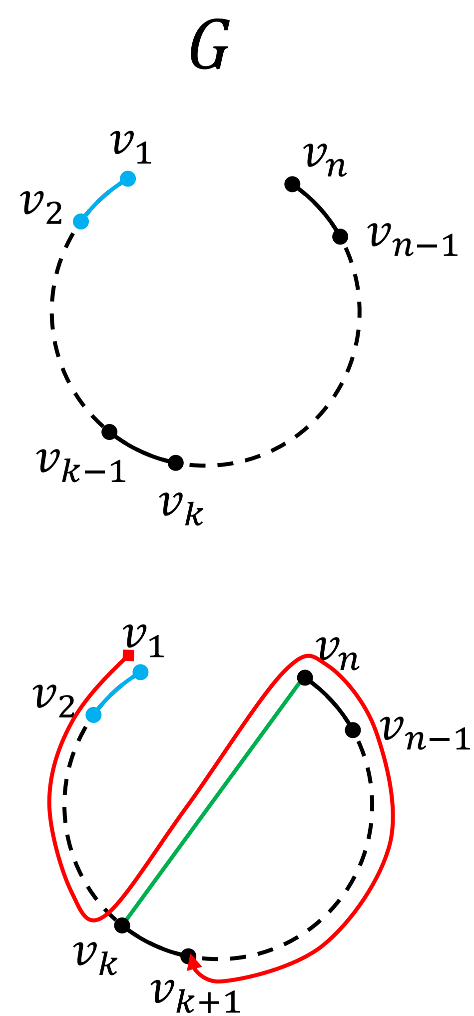
- 定理(Smith)：对3-正则图，包含图上任意边e的哈密顿回路必有偶数条。
- 证明：(Thomason 1978)
  - 图G是3-正则图，$e = \{v_1, v_2\}$是一条固定的边，不失一般性，假设原图中有含有e的哈密顿回路。
  - 构造图$G' = (V', E')$
    - V'中的每一点，代表一条从v_1开始，以e为第一条边的哈密顿路径（由前提假设知V'非空）
    - 构造E'：

- 构造E'
  - $v_p \in V'$ 代表哈密顿路径P；
  - v_n在G中度数为3，故必存在$1 < k < n-1$满足$\{v_k, v_n\} \in E(G)$；
  - $P' = v_1 v_2 \ldots v_k v_n v_{n-1} \ldots v_{k+1}$是哈密顿路径。$v_{p'} \in V'$；
  - $\{v_p, v_{p'}\} \in E'$。



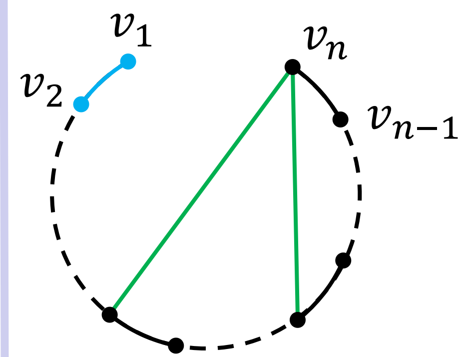


G' G

- E'中任意v_p的度数至多为2：

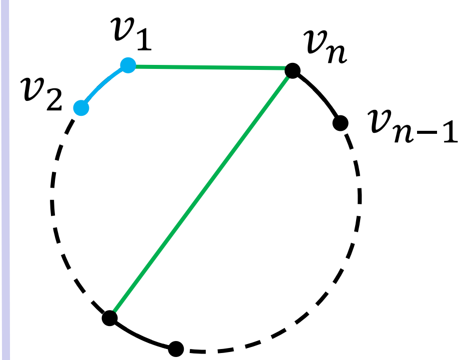
- $deg(v_p) = 1$当且仅当原始用到的哈密顿*路径*实际上是图G中一个哈密顿*回路*。

$deg(v_p) = 2$



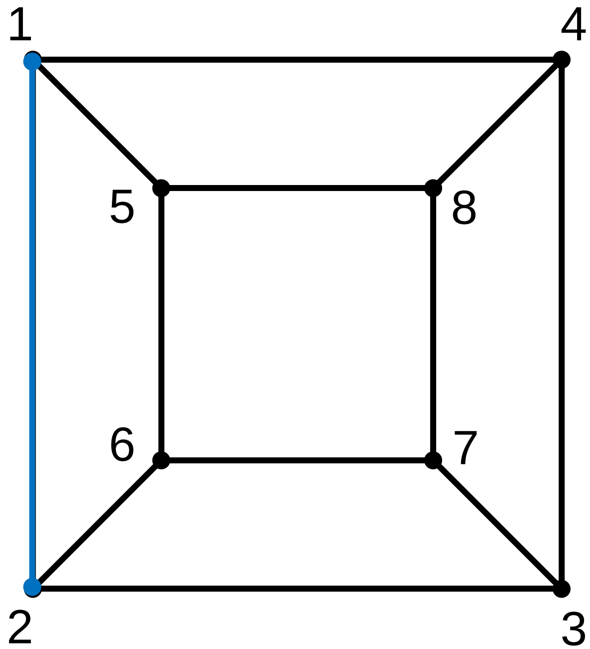
- 根据握手定理，度数为奇数的点必有偶数个，故必存在另一点$deg(v_q) = 1$。

$deg(v_p) = 1$



$deg(v_p) = 1$

- v_q对应图G中的另一条哈密顿回路，得证。



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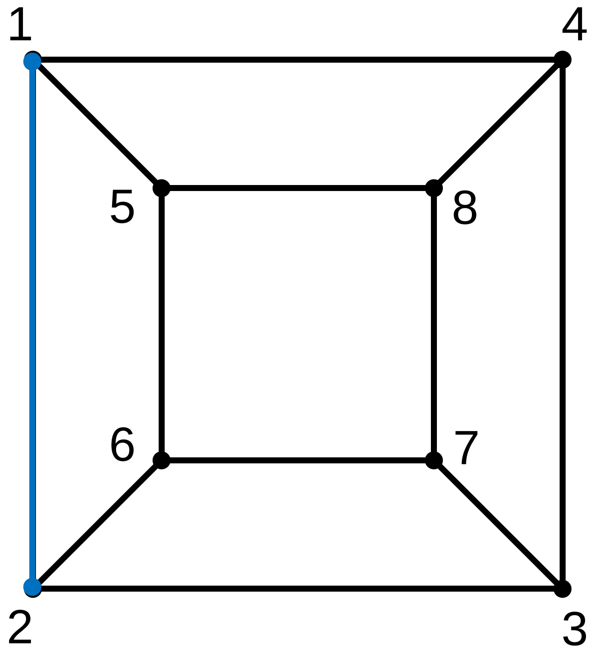

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