Homework 9

Problem 1. Prove that there exist at most 4^n pairwise non-isomorphic trees on n vertices.

Solution. The code of a tree on n vertices has length $2n$ (prove it). Hence there are at most 4^n distinct codes. As different codes correspond to non-isomorphic trees, this is also the upper bound of the number of pairwise non-isomorphic trees. $\square$

Problem 2. Suppose X and Y are two independent tosses of a fair coin, where we designate 1 for heads and 0 for tails. Let the third random variable $Z = (X + Y) \bmod 2$. Show that X, Y, Z are pairwise-independent to each other, while they are NOT mutually-independent. //(This example is by (by S. Bernstein)

Solution. (by S. Bernstein)

Suppose X and Y are two independent tosses of a fair coin, where we designate 1 for heads and 0 for tails. Let the third random variable $Z = (X + Y) \bmod 2$.

Then jointly the triple $\langle X, Y, Z\rangle$ has the following probability distribution:

$$\langle X, Y, Z\rangle = \begin{cases} \langle 0,0,0\rangle & \text{with probability } 1/4 \\ \langle 0,1,1\rangle & \text{with probability } 1/4 \\ \langle 1,0,1\rangle & \text{with probability } 1/4 \\ \langle 1,1,0\rangle & \text{with probability } 1/4 \end{cases}$$

$i, j, k \in \{0, 1\}$.

It is easy to verify that $Pr(X = i) = Pr(Y = j) = Pr(Z = k) = 1/2$ and $Pr(X = i, Y = j) = Pr(X = i, Z = k) = Pr(Y = j, Z = k) = 1/4$. i.e., X, Y, Z are pairwise independent.

However, $Pr(X = i, Y = j, Z = k) \neq Pr(X = i) \cdot Pr(Y = j) \cdot Pr(Z = k)$. For example, the left side equals $1/4$ for $\langle x, y, z\rangle = \langle 0, 0, 0\rangle$ while the right side equals $1/8$.

In fact, any of $\langle X, Y, Z\rangle$ is completely determined by the first two components. That is as far from independence as random variables can get.

$\square$

Problem 3. A monkey types on a 26 -letter keyboard that has lowercase letters only. Each letter is chosen independently and uniformly at random from the alphabet. If the monkey types 1,000,000 letters. what is the expected number of times the sequence "proof" appears?

Solution. By the linearity of expectation:

$$E[X] = (1/26)^5 \times (1000000 - 4)$$

□

Problem 4. Let (Ω, P) be a finite probability space in which all elementary events have the same probability. Show that if $|\Omega|$ is a prime number then no two nontrivial events (distinct from $\emptyset$ and Ω) can be independent.

Solution.

Suppose $|\Omega| = n$.

Take $P_1 \subseteq \Omega, P_2 \subseteq \Omega$ be two nontrivial events, where $|P_1| = n_1$ and $|P_2| = n_2$. Surely if $P_1 \cap P_2 = \emptyset$ then they are not independent.

Now suppose $|P_1 \cap P_2| = m$, then if they are independent:

$$\frac{m}{n} = Pr(P_1 \cap P_2) = Pr(P_1) \cdot Pr(P_2) = \frac{n_1}{n} \cdot \frac{n_2}{n}$$

then $n = \frac{n_1 \cdot n_2}{m}$ which means n cannot be a prime number. □