HANDOUT: AXIOM OF CHOICE

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1. Axiomatic System

Nowadays, the so-called ZFC axiomatic system is considered the standard foundation for mathematics. Here ZF means Zermelo-Fraenkel set theory, and C stands for ”Choice”, i.e. the axiom of choice.

The ZF system was proposed in the early twentieth century in order to formulate a theory of sets free of paradoxes such as Russell's paradox. Zermelo and Fraenkel expressed their axioms in the formal language of set theory, but we can translate them into ordinary language.

Axiom (Zermelo-Fraenkel)**.**

- Axiom 1 Two sets are equal if and only if they have the same elements.
- Axiom 2 There is a set with no elements called the empty set.
- Axiom 3 For any sets A and B there is a set whose only elements are A and B. (Note that taking $A = B$, this gives you the possibility of forming the singleton set $\{A\}$.)
- Axiom 4 For any set A there's a set whose elements are the elements of the elements of A. (If $A = \{B, C\}$ then this set is what's called the union $B \cup C$ of B and C, whose elements are all the elements of B together with the elements of C. For example if $B = \{1, 2\}$ and $C = \{3, 4\}$, then $B \cup C = \{1, 2, 3, 4\}$.)
- Axiom 5 For any set A there is a set, called the power set of A, whose elements are the subsets of A. (A subset of A is a set whose elements are all contained in A.)
- Axiom 6 For any function f defined on a set A, the values $f(a)$, where a is an element of A, also form a set.
- Axiom 7 Every non-empty set A has a member whose elements are all different from the elements of A. (This axiom insures that if you take an element B of A and then an element C of B, and so on, this process stops after finitely many steps.) There is an infinite set such that if A is in this set, the set $A \cup \{A\}$ is also in the set.
- Axiom 8 There is an infinite set such that if A is in this set, the set $A \cup \{A\}$ is also in the set.

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Axiom of Choice, sometimes referred as Axiom 9, states that every family of nonempty sets has a choice function. A choice function on a collection $\mathscr{H}$ of nonempty sets is a map f with domain $\mathscr{H}$ such that

$$f(X) \in X, \forall X \in \mathscr{H}.$$

As a very simple example, let $\mathscr{H}$ be the collection of nonempty subsets of $\{0,1\}$, i.e., $\mathscr{H} = \{\{0\},\{1\},\{0,1\}\}$. Then $\mathscr{H}$ has the two distinct choice functions f_1 and f_2 given by:

$$f_1(\{0\}) = 0,\ f_1(\{1\}) = 1,\ f_1(\{0,1\}) = 0;$$
$$f_2(\{0\}) = 0,\ f_2(\{1\}) = 1,\ f_2(\{0,1\}) = 1.$$

A more interesting example of a choice function is provided by taking $\mathscr{H}$ to be the set of (unordered) pairs of real numbers and the function to be that assigning to each pair its least element. A different choice function is obtained by assigning to each pair its greatest element.

Why do we have to treat such a seemingly innocuous principle so seriously? At least in Fudan, you will find the following results are all due to the axiom of choice:

- Existence of Lebesgue unmeasurable set. Although the Lebesgue integral is an excellent generalization of Riemann integral, there is a problem that not all subsets of $\mathbb{R}$ are measurable due to the axiom of choice.
- Hahn-Banach theorem. It seems trivial to find a tangent line of a point on the unit ball of $\mathbb{R}^n$, but the existence of such a ”line” in the infinite-dimension case relies on the Hahn-Banach theorem, which also relies on the axiom of choice in its proof.
- Banach-Tarski paradox. This is probably the most counter-intuitive application of the axiom of choice: in $\mathbb{R}^3$ it is possible to decompose a ball into several pieces which can be reassembled by rigid motions to form two balls of the same size as the original.

And in the filed of algebra, the axiom of choice, and its equivalent statements, are widely used in the proofs. In this Abstract Algebra course, you will find that the axiom of choice, or more specifically, Zorn’s Lemma, is used in the proof of the existence of algebraic closure.

2. Zorn’s Lemma

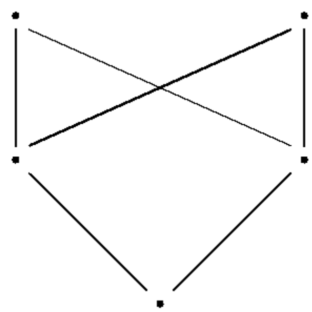
There are many equivalent statements of the axiom of choice, among which the most useful one may be Zorn’s Lemma, which involves the order relation on a set.

Definition 2.1. An order relation $\leq$ on a set X is a binary relation satisfying

i) $a \leq a$ (reflexity);
ii) if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity);
iii) if $a \leq b$ and $b \leq a$, then $a = b$ (anti-symmetry).

A partially ordered set (poset) is a non-empty set X on which some order relation $\leq$ is given. A totally ordered set is a poset in which any two elements a and b either $a \leq b$ or $b \leq a$.

For example, the set of natural numbers $\mathbb{N}$ with the usual order relation forms a poset. We can also impose a different order relation on $\mathbb{N}$ where $a \leq b$ means a divides b. Notice that in the latter case $\mathbb{N}$ is not totally ordered, as we cannot compare coprime numbers in this order relation. In such a case, there exist more than one "ascending chains" (i.e. a sequence of elements $a_1 \leq a_2 \leq \cdots$) in the poset, which can be illustrated by the picture below. In fact, each ascending chain is just a totally ordered subset of the poset, and therefore a totally ordered set has only one ascending chain.



To state Zorn's lemma, we still need some more terminology. An upper bound for a subset Y of a poset X is an element $a \in X$ satisfying $b \leq a$ for any $b \in Y$. An element $m \in X$ is called maximal if for any $x \in X$, $m \leq x$ only implies $m = x$.

Theorem 2.2 (Zorn's Lemma)**.** If in a non-empty poset X every totally ordered subset has an upper bound, then X contains at least one maximal element.

The statement suggests that there may exist many maximal elements. This is because the upper bounds for different totally ordered subsets, or different ascending chains, usually cannot be compared in the given order relation $\leq$.

The proof asserting equivalence can be found in standard textbooks of set theory, for example [Mos06]. Here we give a proof of axiom of choice using Zorn's lemma.

Proof of Axiom of Choice from Zorn's Lemma:

Let X be a nonempty set. We set

$$S = \{f : f \text{ is a choice function on } A \subset \mathscr{P}X\}$$

Here $\mathscr{P}X$ denotes the power set of X.

- S is nonempty, as $f_a : \{a\} \mapsto a$ is a choice function on $\{a\}$ for all $a \in X$.
- We now define the partial order to be $f_1 \leq f_2$ iff $Dom(f_1) \subset Dom(f_2)$. This is a partial order since $\subset$ is always a partial order on $\mathscr{P}X$.

- Let T be a totally ordered subset in S. Let F be the union on the functions in T, i.e.

$$F = \bigcup_{f \in T} f.$$

  It's easy to verify that F is a well-defined function and therefore a choice function. This is an upper bound for T.

Now by Zorn's Lemma we conclude that there exists a maximal element f_{max} in S. We claim that the domain of f_{max} must be the whole of X. If not, pick $a_0 \in X - Dom(f_{max})$ and set

$$\tilde{f} = f_{max} \cup f_{a_0}.$$

Then $\tilde{f}$ extends f_{max} and we've come to a contradiction. □

As another example explaining how to use Zorn's lemma, we now prove

Theorem 2.3. Every nontrivial ring R with unity contains a maximal ideal.

Proof: Let S be the set consisting of all proper ideals in R.

- S is nonempty, since the trivial ideal $\{0\} \in S$.
- The partial order is given by the inclusion of sets.
- Let T be a totally ordered subset in S. If T is empty, then $\{0\}$ is an upper bound for T. Otherwise we set

$$I = \bigcup_{J \in T} J.$$

  We claim that I is an ideal.
  a) For any $x, y \in I$, we may assume that $x \in J_x$, $y \in J_y$. Since T is totally ordered, we may assume further that $J_x \subset J_y$. Hence $x + y \in J_y \subset I$.
  b) For any $x \in J_x \subset I$, $r \in R$, $rx \in J_x \subset I$.

  To show I is an upper bound for T, it suffices to show that I is still a proper ideal. If $I = R$, then there exists an element $J \in T$, such that $1 \in J$. This implies that $J = R$, which contradicts to the fact that J is proper.

Now by Zorn's Lemma we conclude that there exists a maximal element I_{max} in S. By definition it is an maximal ideal. □

As a remark of this section, we propose another equivalent form of the axiom of choice, called the well-ordering theorem. There is a quote from Jerry Bona: "The Axiom of Choice is obviously true, the Well-ordering theorem is obviously false; and who can tell about Zorn's Lemma?"

Theorem 2.4 (Well-ordering Theorem)**.** Every set can be well-ordered. Here, a set X is well-ordered by a strict total order if every non-empty subset of X has a least element under the ordering.

3. Appendix: Russel's Paradox

This section is aimed at showing why we need axiomatic system. We begin with naïve set theory. Contrary to the axiomatic systems, it is an informal collection of assumptions about sets, formulated in natural language. For example, there's a union and an intersection for any two sets; we have a set of natural numbers, and from those we can construct the real numbers. The crucial assumption of naïve set theory, however, is the so-called unrestricted comprehension schema stating that:

For any property $P(x)$, there is a set consisting of exactly those x that satisfy P.

At first look, comprehension certainly makes sense: given any property we should be able to talk about the set of all those objects satisfying the property. For example, when we talk about the set of prime numbers —those x that are natural numbers and have exactly two divisors. It turns out, however, that the unrestricted comprehension schema is highly problematic. Betrand Russell discovered the following paradox in 1901:

Paradox (Russel)**.** Take $P(x)$ to be the property that "x does not contain itself". By the unrestricted comprehension schema, there must be a set A consisting of all those sets x that satisfy $P(x)$. That is, A consists of all sets that do not contain themselves.

If A contains itself, then $P(A)$ must hold, which implies that A does not contain itself, a contradiction. If A does not contain itself, then $P(A)$ holds and A contains itself, yet another contradiction.

Using the Axiom 7 in the ZF system (also known as the axiom of foundation), we know that a set contains no infinitely descending (membership) sequence. Therefore, the "set" of all sets that don't contain themselves is actually not a set —it's a proper class. "Proper class" means that the collection defined as above is not a set because it's too big. The question whether this class contains itself doesn't make sense because it contains only sets.

Axiomatic set theory resolves paradoxes by demystifying them. The Zermelo-Fraenkel axioms of set theory give us a better understanding of sets, according to which we can then settle the paradoxes.

References

[Mos06] Yiannis Moschovakis. *Notes on set theory*. Undergraduate Texts in Mathematics. Springer, New York, second edition, 2006.