Theorié de Hodge II

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Warning: This draft was only used for a seminar. All the mistakes are due to the translator. Text in turquiose and footnotes are added by the translator for the seminar.

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CHAPTER 1

Filtrations

1.1. Filtered objects

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If not explicitly mentioned, by <filtration> we always mean <decreasing filtration>. In this section, by <complex> we mean <bounded below complex>.

1.4.1. Let $\mathrm{T} : \mathscr{A} \to \mathscr{B}$ be a left exact functor between two abelian categories. We suppose that $\mathscr{A}$ has enough injectives; the derived functors $\mathrm{R}^i\mathrm{T} : \mathscr{A} \to \mathscr{B}$ are therefore defined. An object A in $\mathscr{A}$ is called T-acyclic if $\mathrm{R}^i\mathrm{T}(\mathrm{A}) = 0$ for $i > 0$.

1.4.2. Let (A, F) be a filtered object with finite filtration, and $\mathrm{TF} = \{\mathrm{TF}^p\mathrm{A}\}$ be the filtration of TA (they are subobjects since T is left exact). If $\mathrm{Gr}_\mathrm{F}(\mathrm{A})$ is T-acyclic, then $\mathrm{F}^p(\mathrm{A})$ are T-acyclic as successive extensions of T-acyclic objects. The image of the sequence

$$0 \to \mathrm{F}^{p+1}(\mathrm{A}) \to \mathrm{F}^p(\mathrm{A}) \to \mathrm{Gr}^p(\mathrm{A}) \to 0$$

is therefore exact, and

$$\mathrm{Gr}_{\mathrm{TF}}\mathrm{TA} \cong \mathrm{T}\mathrm{Gr}_\mathrm{F}\mathrm{A}. \tag{1.4.2.1}$$

1.4.3. Let A be an object equipped with two finite filtrations F and W, such that $\mathrm{Gr}_\mathrm{F}\mathrm{Gr}_\mathrm{W}\mathrm{A}$ is T-acyclic. The objects $\mathrm{Gr}_\mathrm{F}\mathrm{A}$ and $\mathrm{Gr}_\mathrm{W}\mathrm{F}$ are then T-acyclic, thus also $\mathrm{F}^q(\mathrm{A}) \cap \mathrm{W}^p(\mathrm{A})$. The sequence

$$0 \to \mathrm{T}(\mathrm{F}^q \cap \mathrm{W}^{p+1}) \to \mathrm{T}(\mathrm{F}^q \cap \mathrm{W}^p) \to \mathrm{T}\left((\mathrm{F}^q \cap \mathrm{W}^p)/(\mathrm{F}^q \cap \mathrm{W}^{p+1})\right) \to 0$$

is then exact, and $T(F^q(Gr_W^p(A)))$ is the image of $T(F^q \cap W^p)$ in $T(Gr_W^p(A))$. The diagram

$$\begin{tikzcd} T(F^q \cap W^p) \arrow[r] \arrow[d, "\cong"] & T(F^q Gr_W^p A) \arrow[r] & TGr_W^p A \arrow[d, "\cong"] \\ TF^q \cap TW^p \arrow[rr] & & Gr_{TW}^p TA \end{tikzcd}$$

shows then that the isomorphism 1.4.2.1 relative to W transforms the filtration $Gr_{TW}(TF^q)$ to the filtration $T(Gr_W(F^q))$. [1]

1.4.4. Let K be a complex of objects in $\mathscr{A}$. The hypercohomology $R^iT(K)$ is calculated as follows:

a) We choose a quasi-isomorphism $i : K \to K'$, such that components of K' are T-acyclic. For example, we can take the total complex associated the the injective Cartan-Eilenberg resolution of K.
b) We set

$$R^iT(K) = H^i(T(K'))$$

It can be verified that $R^iT(K)$ does not depend on the choice of K', functorially depend on K, and any quasi-isomorphism $f : K_1 \to K_2$ induces an isomorphism

$$R^iT(f) : R^iT(K_1) \to R^iT(K_2).$$

1.4.5. Let F be a biregular filtration on K, i.e. the filtration induces a finite filtration on every component of K. A T-acyclic filtered resolution of K is a filtered quasi-isomorphism $i : K \to K'$ mapping K into a bireguler filtered complex (a filtered quasi-isomorphism is a quasi-isomorphism together with an isomorphism on Gr), such that $Gr^p(K'^n)$ is T-acyclic. If K' is such a resolution, K'^n is T-acyclic and the filtered complex $T(K')$ defines a spectral sequence

$$E_1^{pq} = R^{p+q}T(Gr^p(K)) \Rightarrow R^{p+q}T(K).$$

[1] The first vertical isomorphism is just by applying T to the short exact sequence involving $F^q \cap W^p$.

This is independent of the choice of K′. We call it the hypercohomology spectral sequence of the filtered complex K. [2]

This spectual sequence functorially depends on K and a filtered quasi-isomorphism induces an isomorphism of spectral sequences.

The differential d_1 of this spectral sequence is the connecting morphism induced by the short exact sequence

$$0 \to \mathrm{Gr}^{p+1}\mathrm{K} \to \mathrm{F}^p\mathrm{K}/\mathrm{F}^{p+2}\mathrm{K} \to \mathrm{Gr}^p\mathrm{K} \to 0.$$

1.4.6. Let K be a complex. We set by $\tau_{\leq p}(\mathrm{K})$ the sub-complex defined by

$$\tau_{\leq p}(\mathrm{K})^n = \begin{cases} \mathrm{K}^n & \text{for } n < p \\ \mathrm{K}er(d) & \text{for } n = p \\ 0 & \text{for } n > p. \end{cases}$$

The so-called canonical filtration of K is deduced by shifting the trivial filtration G for which $\mathrm{G}^0(\mathrm{K}) = \mathrm{K}$ and $\mathrm{G}^1(\mathrm{K}) = 0$. [3] We then have, for the canonical filtration,

$$\mathrm{E}_1^{pq} = \begin{cases} 0 & \text{if } p + q \neq -p \\ \mathrm{H}^{-p} & \text{if } p + q = -p. \end{cases}$$

A quasi-isomorphism $f : \mathrm{K} \to \mathrm{K}'$ is automatically a filtered quasi-isomorphism for the canonical filtration.

1.4.7. The sub-complex $\sigma_{\geq p}(\mathrm{K})$ of K:

$$\sigma_{\geq p}(\mathrm{K})^n = \begin{cases} 0 & \text{for } n < p \\ \mathrm{K}^n & \text{for } n \geq p \end{cases}$$

defines a biregular filtration, called the stupid filtration of K.

Stupid and canonical filtrations give two different hypercohomology spectral sequences of K.

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[2]This is not the Leray spectral sequence, but later we will see in some special cases they are the same.

[3]The filtration $\tau_{\leq p}$ is increasing. To use the spectral sequence above, we need to revert the filtration to get $\mathrm{T}^p = \tau_{\leq -p}$. Explicitly, $\mathrm{Gr}_{\mathrm{T}}^p(\mathrm{K}) = \mathrm{T}^p/\mathrm{T}^{p+1} = \mathrm{H}^{-p}(\mathrm{K})[p]$ as it is supported on $-p$ place.

Example 1.4.8. Let $f : X \to Y$ be a continuous map of topological spaces and let $\mathscr{F}$ be an abelian sheaf on X. Take a f_*-acyclic resolution $\mathscr{F}^\bullet$ of $\mathscr{F}$. We have $R^i f_* \mathscr{F} \cong \mathscr{H}^i(f_*\mathscr{F}^\bullet)$. Take the functor $T = \Gamma(Y,)$. The hypercohomology spectral sequence of $f_*\mathscr{F}^\bullet$ equipped with the canonical filtrations gives

$$E_1^{pq} = H^{p+q}(Y, Gr^p Rf_*\mathscr{F}) = H^{p+q}(Y, R^{-p} f_*\mathscr{F}[p]) = H^{2p+q}(Y, R^{-p} f_*\mathscr{F}) \Rightarrow H^{p+q}(X, \mathscr{F})$$

which is the Leray spectral sequence for f and $\mathscr{F}$ after a renumbering $E_r^{pq} \mapsto E_{r+1}^{2p+q,-p}$.

1.4.9. Let (K, W, F) be a bifiltered biregular complex. For such a complex, we can associate

a)A spectral sequence

$$_W E_1^{p,n-p} = H^n(Gr_W^p(K)) \Rightarrow H^n(K)$$

with differential $_W d_1$ the connection morphism induced by the short exact sequence

$$0 \to Gr_W^{p+1}(K) \to W^p(K)/W^{p+2}(K) \to Gr_W^p(K) \to 0;$$

b) A spectral sequence similarly defined for the filtration F;

c) An exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Gr_F^{p+1} Gr_W^{q+1} K & \longrightarrow & F^p/F^{p+2}(Gr_W^{q+1} K) & \longrightarrow & Gr_F^p Gr_W^{q+1} K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Gr_F^{p+1}(W^q/W^{q+2}(K))) & \longrightarrow & F^p/F^{p+2}(W^q/W^{q+2}(K)) & \longrightarrow & Gr_F^p(W^q/W^{q+2}(K)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Gr_F^{p+1} Gr_W^q K & \longrightarrow & F^p/F^{p+2} Gr_W^q K & \longrightarrow & Gr_F^p Gr_W^q K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The exterior rows and columns of this square define connecting morphisms

$$
\begin{aligned}
{}_{\mathrm{F,W}}d_1 &: \mathrm{H}^n \mathrm{Gr}_{\mathrm{F}}^p \mathrm{Gr}_{\mathrm{W}}^q \mathrm{K} \to \mathrm{H}^{n+1} \mathrm{Gr}_{\mathrm{F}}^{p+1} \mathrm{Gr}_{\mathrm{W}}^q \mathrm{K} \\
{}_{\mathrm{W,F}}d_1 &: \mathrm{H}^n \mathrm{Gr}_{\mathrm{F}}^p \mathrm{Gr}_{\mathrm{W}}^q \mathrm{K} \to \mathrm{H}^{n+1} \mathrm{Gr}_{\mathrm{F}}^{p} \mathrm{Gr}_{\mathrm{W}}^{q+1} \mathrm{K}
\end{aligned}
$$

These morphisms satisfy

$$
{}_{\mathrm{FW}}d_1 \circ_{\mathrm{WF}} d_1 +_{\mathrm{WF}} d_1 \circ_{\mathrm{FW}} d_1 = 0
$$

$$
{}_{\mathrm{FW}}d_1^2 = 0 \quad {}_{\mathrm{WF}}d_1^2 = 0.
$$

The morphisms ${}_{\mathrm{F,W}}d_1$ are the morphisms d_1 of the spectral sequence $\mathrm{E}^{(q)}$, with the term $\mathrm{E}_1^{(q)p,n-p}$ equal to

$$
\mathrm{E}_1^{p,q,n-p-q} := \mathrm{H}^n(\mathrm{Gr}_{\mathrm{F}}^p \mathrm{Gr}_{\mathrm{W}}^q \mathrm{K}) \Rightarrow \mathrm{H}^n(\mathrm{Gr}_{\mathrm{W}}^q \mathrm{K}) =_{\mathrm{W}} \mathrm{E}_1^{q,n-q}, \tag{1.4.9.1}
$$

defined by the filtered complex $\mathrm{Gr}_{\mathrm{W}}^q(\mathrm{K})$. This spectral sequence is the one induced by F on $\mathrm{H}^\bullet \mathrm{Gr}_{\mathrm{W}}^q \mathrm{K}$. Likewise, ${}_{\mathrm{WF}}d_1$ and d_1 of the spectral sequence with the initial terms

$$
\mathrm{E}_1^{p,q,n-p-q} := \mathrm{H}^n(\mathrm{Gr}_{\mathrm{F}}^p \mathrm{Gr}_{\mathrm{W}}^q \mathrm{K}) \Rightarrow \mathrm{H}^m(\mathrm{Gr}_{\mathrm{F}}^p \mathrm{K}). \tag{1.4.9.2}
$$

$$
\begin{tikzcd}
 & \mathrm{H}^n \mathrm{Gr}_{\mathrm{W}}^q \mathrm{K} \arrow[dr, Rightarrow, "({}_{\mathrm{W}}d_1)"] & \\
\mathrm{H}^n \mathrm{Gr}_{\mathrm{F}}^p \mathrm{Gr}_{w}^q \mathrm{K} \arrow[ur, Rightarrow, "({}_{\mathrm{F,W}}d_1)"] \arrow[dr, Rightarrow, "({}_{\mathrm{W,F}}d_1)"'] & & \mathrm{H}^n \mathrm{K} \\
 & \mathrm{H}^n \mathrm{Gr}_{\mathrm{F}}^p \mathrm{K} \arrow[ur, Rightarrow, "({}_{\mathrm{F}}d_1)"'] &
\end{tikzcd}
$$

This construction is symmetric in F and W, via the isomorphism

$$
\mathrm{Gr}_{\mathrm{F}}^p \mathrm{Gr}_{\mathrm{W}}^q \cong \mathrm{Gr}_{\mathrm{W}}^q \mathrm{Gr}_{\mathrm{F}}^p.
$$

1.4.10.

1.4.11. T-acyclic gives similar spectral sequences. Assume $\mathrm{Gr}_F^p \mathrm{Gr}_W^q(K'^m)$ are T-acyclic, then

$$
\begin{aligned}
{}_W E_1^{q,n-q} &= R^n T(\mathrm{Gr}_W^q K) \Rightarrow R^n T(K) \\
{}_F E_1^{p,n-p} &= R^n T(\mathrm{Gr}_F^p K) \Rightarrow R^n T(K) \\
E_1^{p,q,n-p-q} &= R^n T(\mathrm{Gr}_F^p \mathrm{Gr}_W^q K) \Rightarrow_W E_1^{q,n-q} \ (\text{fix } q) \\
E_1^{p,q,n-p-q} &= R^n T(\mathrm{Gr}_F^p \mathrm{Gr}_W^q K) \Rightarrow_F E_1^{p,n-p} \ (\text{fix } p)
\end{aligned}
$$

CHAPTER 2

Hodge structures

2.1. Pure structures

2.2. Hodge theory

2.2.1. Let X be a compact Kahler manifold. According to the Poincaré lemma, the De Rham complex $\Omega_X^\bullet$ is a resolution of the constant sheaf $\mathbb{C}$. We therefore has an isomorphism

$$H^*(X,\mathbb{C}) \cong \mathbb{H}(X,\Omega_X^\bullet),$$

and the stupid filtration of $\Omega_X^\bullet$ defines a hypercohomology spectral sequence

$$E_1^{pq} = H^q(X,\Omega_X^p) \Rightarrow H^{p+q}(X,\mathbb{C}), \tag{2.2.1.1}$$

giving the Hodge filtration of $H^*(X,\mathbb{C})$.

According to Hodge theory, we have:

(A) A spectral sequence 2.2.1.1 is degenerate: $E_1 = E_\infty$.

(B) The Hodge filtration of $H^n(X,\mathbb{C})$ is opposite to the filtration obtained by complex conjugation.

2.2.2. Let V be a real local system on X, i.e. a sheaf of $\mathbb{R}$-vectors locally isomorphic to $\mathbb{R}^n$. Suppose that there exists a bilinear form on V

$$Q : V \times V \to \mathbb{R}$$

which is locally constant and definite. For connected X, this is the case when V is defined by a representation of a finite quotient of the fundamental group of X.

(A) and (B) above remains valid, for the cohomology with coefficients in $V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$. The spectral sequence

$$E_1^{pq} = H^q(X,\Omega_X^p(V)) \Rightarrow H^{p+q}(X,V_\mathbb{C})$$

deduced from the De Rham resolution $\Omega^\bullet_X(V_\mathbb{C})$ of $V_\mathbb{C}$ is degenerate, and there exists an opposite filtration obtained by complex conjugation.

The vector space $H^n(X,V)$ is therefore equipped with a real Hodge structure of weight n canonically.

2.2.3. We show in [2] that the statements (2.2.1) remain valid for complete non-singular algebraic variety, not necessarily Kahler. The demonstration of *loc. cit.*, based on a reduction to the projective case via Chow's lemma and the resolution of singularities, extends to the framework (2.2.2).

2.2.4. Let $\mathscr{L}$ be an invertible sheaf. Here are two ways to define the class $c_1(\mathscr{L})$.

2.2.4.1. The sheaf $\mathscr{L}$ defines an elemnet c in $H^1(X,\mathscr{O}^*)$. The image via df/f : $\mathscr{O}^* \to \Omega^1$ is in $H^1(X,\Omega^1)$. More precisely, df/f defines a morphism of complexes

$$d\log : \mathscr{O}^*[-1] \to [0 \to \Omega^1_X \to \Omega^2_X \to \cdots] = \sigma_{\geq 1}(\Omega^\bullet_X)$$

The complex on the right is sent into $\Omega^\bullet_X$, hence

$$d\log : \mathscr{O}^*[-1] \to \Omega^\bullet_X$$

and the image of c lies in $\mathbb{H}^2(\Omega^*_X)$. This construction will still be valid for an algebraic variety on an arbitrary field k. For $k = \mathbb{C}$, we moreover have $H^2(X,\mathbb{C}) \xrightarrow{\cong} \mathbb{H}^2(X,\Omega^*_X)$ hence a class $c_1'(\mathscr{L}) \in H^2(X,\mathbb{C})$.

2.2.4.2. The exponential exact sequence

$$0 \to \mathbb{Z}(1) \to \mathscr{O} \to \mathscr{O}^* \to 0$$

defines a homomorphism

$$\partial : H^1(X,\mathscr{O}^*) \to H^2(X,\mathbb{Z}(1))$$

hence a class $\partial c = c_1''(\mathscr{L}) \in H^2(X,\mathbb{Z}(1))$.

If i is chosen [1], this class can be identified with

$$c_1'''(\mathscr{L}) \in H^2(X,\mathbb{Z})$$

and for $\mathscr{L} = \mathscr{O}(D)$, $c_1'''(\mathscr{L})$ is nothing but the integral cohomology class defined by D (orientation defined by i).

[1]The imaginary number, i.e. the orientation.

2.2.5. Equivalence of def

2.2.6. Let X be a non-singular projective variety of pure dimension n. A choice of i defines an orientation of X and an isomorphism $\mathbb{Z}(1) \cong \mathbb{Z}$. The corresponding trace map"

$$\mathrm{H}^{2n}(\mathrm{X}, \mathbb{Z}(n)) \to \mathbb{Z}$$

which in fact does not depend on the choice of i.

According to Hodge, for $i \leq n$, the morphism

$$\mathrm{L}^{n-i} = \bigwedge c_1''(\mathscr{O}(1))^{n-i} : \mathrm{H}^i(\mathrm{X}, \mathbb{Z}) \to \mathrm{H}^{2n-i}(\mathrm{X}, \mathbb{Z}(n-i))$$

is an isomorphism and, combined with Poincaré duality

$$\mathrm{H}^i(\mathrm{X}, \mathbb{Z}) \otimes \mathrm{H}^{2n-i}(\mathrm{X}, \mathbb{Z}(n-i)) \to \mathrm{H}^{2n}(\mathrm{X}, \mathbb{Z}(n-i)) \to \mathbb{Z}(-i)$$

it prodives a polarization on the primitive part $\mathrm{H}^i(\mathrm{X}, \mathbb{Z})_{prim} = \mathrm{K}er(\mathrm{L}^{n-i+1})$.

We deduce from this fact that the rational Hodge structure $\mathrm{H}^i(\mathrm{X}, \mathbb{Q})$ is polarizable.

2.3. Mixed structures

CHAPTER 3

Hodge theory of non-singular algebraic varieties

3.1. Logarithmic poles and residues

Before we continue, we can first illustrate a few examples. Let X be a smooth projective variety and Y be a smooth projective codimension 1 subvariety in X. The long exact sequence of relative cohomology gives

$$\cdots \mathrm{H}^i(\mathrm{X},\mathrm{U}) \to \mathrm{H}^i(\mathrm{X}) \to \mathrm{H}^i(\mathrm{U}) \to \mathrm{H}^{i+1}(\mathrm{X},\mathrm{U}) \to \cdots$$

and in particular we have

$$0 \to \mathrm{H}^i(\mathrm{X})/\mathrm{H}^i(\mathrm{X},\mathrm{U}) \to \mathrm{H}^i(\mathrm{U}) \to \mathrm{K}er\left(\mathrm{H}^{i+1}(\mathrm{X},\mathrm{U}) \to \mathrm{H}^{i+1}(\mathrm{X})\right) \to 0$$

Intuitively, H^i has a weight i part arising as a quotient of $\mathrm{H}^i(\mathrm{X})$ and a weight $i+1$ part coming as a subspace of $\mathrm{H}^{i+1}(\mathrm{X})$. Let we specify to the case $\mathrm{X} = \mathbb{P}^1$, $\mathrm{Y} = \{0,\infty\}$ and $\mathrm{U} = \mathrm{X} - \mathrm{Y} \cong \mathbb{G}_m$, we have

$$0 \to \mathrm{H}^1(\mathrm{U}) \to \mathrm{K}er\left(\mathrm{H}^1(\mathrm{X},\mathrm{U}) \to \mathrm{H}^2(\mathrm{X})\right) \to 0$$

Therefore, $\mathrm{H}^1(\mathbb{G}_m)$ has no weight 1 part and is equal to the weight 2 part. This is a pure Hodge structure of dimension 1 and weight 2, so it must be equal to $\mathbb{Q}(-1)$. The De Rham representative $dz/z = -d\bar{z}/\bar{z}$, so it lies in $\mathrm{F}^1 \cap \overline{\mathrm{F}^1} = \mathrm{V}^{1,1}$.

It is interesting why $\mathrm{H}^k(\mathrm{X},\mathrm{U})$ has a natural pure Hodge structure such that the map $\mathrm{H}^k(\mathrm{X},\mathrm{U}) \to \mathrm{H}^k(\mathrm{X})$ is a morphism of Hodge structures. We first observe that U can be retracted to $\mathrm{V} = \mathrm{X} - \mathrm{Y}_\epsilon$, the complement of a tubular neighborhood Y_ϵ of Y, and therefore $\mathrm{H}^k(\mathrm{X},\mathrm{U}) = \mathrm{H}^k(\mathrm{X},\mathrm{V})$. Then algebraic topology tells us that $\mathrm{H}^k(\mathrm{X},\mathrm{V})$ is the reduced cohomology of the Thom complex $\mathrm{T}h(\mathrm{N_X Y})$ of the normal bundle, and we have a Thom isomorphism [1]

$$\mathrm{H}^{k-2}(\mathrm{Y}) \xrightarrow{\cong} \widetilde{\mathrm{H}^k}(\mathrm{T}h(\mathrm{N_X Y})) \xrightarrow{\cong} \mathrm{H}^k(\mathrm{X},\mathrm{V}) \xrightarrow{\cong} \mathrm{H}^k(\mathrm{X},\mathrm{U}).$$

[1]The normal bundle is of complex dimension 1, and therefore the index is shifted by 2.

Notice that a choice of the imaginary number i determines an orientation on the normal bundle inducing the Thom isomorphism.

Thus a natural candidate for the Hodge structure on $H^k(X,U)$ is that of $H^{k-2}(Y)$, but we need to adjust the weight to make the Gysin map $H^{k-2}(Y) \to H^k(X,U) \to H^k(X)$ to be a map of pure Hodge structures. In fact, we actually need to shift the weights up by (1,1), i.e. replace $H^{k-2}(Y)$ by $H^{k-2}(Y) \otimes \mathbb{Q}(-1)$.

Now we consider the Gysin map at the level of differential forms, to be explicit. Since X and Y are both compact, the Gysin map can be described using Poincaré duality as follows:

$$\begin{tikzcd}
\omega & H^{2n-k}(X) \arrow[r, "\cap[X]"] \arrow[d] & H_k(X) \arrow[r] & H^k(X)^* \arrow[d] & f_\omega = [\eta \to \int_X \omega \wedge \eta]] \arrow[d, mapsto] \\
\arrow[from=1-1, to=1-5, bend left, mapsto] & H^{2n-k}(Y) \arrow[r, "\cap[Y]"] & H_k(Y) \arrow[r] & H^k(Y)^* & f_{\omega_Y} == [\eta \to \int_Y \omega_Y \wedge \eta]
\end{tikzcd}$$

To find the Gysin map $H^{k-2}(Y) \to H^k(X)$, we need to find $\phi(\alpha)$ on X such that

$$\int_Y \omega_Y \wedge \alpha = \int_X \omega \wedge \phi(\alpha).$$

We still specify to the case $X = \mathbb{P}^1$, $Y = \{0, \infty\}$. The only nonzero Gysin map is $H^0(Y) \to H^2(X)$. $\omega \in H^0(X)$ can only be the constant function c, and therefore

$$\int_Y \omega_Y \wedge \alpha = (\alpha(0) + \alpha(\infty))c.$$

Hence $\phi : \alpha \mapsto (\alpha(0) + \alpha(\infty)) \cdot Vol$.

For a more complicated example, consider X and Y connected. We want to figure out the case $k = 2$: $H^0(Y) \to H^2(X)$. Locally on X we can choose a uniformizer (local coordinate) y for Y, unique up to multiplication by a nonvanishing holomorphic function ϕ. Since

$$\frac{d(\phi y)}{\phi y} = d \log \phi + \frac{dy}{y}$$

by summing over a partition of unity we can construct a form η that for any local uniformizer y, it is locally equal to $\frac{1}{2\pi i}\frac{dy}{y} + \theta$, where θ is a smooth (1,0)-form on X. We also fix a Hermitian metric on X and let Y_ϵ be a tubular neighborhood of Y. Let

ω be a closed 2-form on X and $\alpha = c$ a constant function on Y (i.e. a representative of $H^0(Y)$). Then we need to fint $\phi(c)$ on X such that

$$c \cdot \int_Y \omega_Y = \int_X \omega \wedge \phi(c)$$

Now

$$\begin{aligned}\int_X \omega \wedge d\eta = \lim_{\epsilon \to 0} \int_{X - Y_\epsilon} \omega \wedge d\eta &= \lim_{\epsilon \to 0} \int_{\partial Y_\epsilon} (\omega \wedge \eta) \\ &= \lim_{\epsilon \to 0} \int \left(\int_{|y| \sim \epsilon} \frac{1}{2\pi i} \omega \wedge \frac{dy}{y} + \omega \wedge \theta \right) \\ &= \int_Y \omega_Y.\end{aligned}$$

More generally, for any k-form α on Y and any extension $\widetilde{\alpha}$ on X, $\phi(\alpha) = d(\eta \wedge \widetilde{\alpha})$ has the desired property. Moreover $\phi(\alpha)$ is shifted by $(1,1)$, which meets the requirement to be a morphism of Hodge structures.

However, we will encounter a problem here: $\phi(\alpha)$ is usually not a C^∞-form on X, and has singularity along Y. Thus, in order to understand the Gysin map at the level of forms, we need to enlarge the class of forms but preserve the cohomology. This results in the holomorphic differential forms with "logarithmic poles".

For a general smooth variety U, it's hard to have an embedding of the form above. Instead, we must allow Y to be a normal crossings divisor. There is a natural generalization of $\Omega^\bullet_X(\log Y)$ to this case calculating the cohomology of U, however, we can no longer write this complex as an extension of two complexes associated to smooth projective varieties. Instead, it has a natural increasing weight filtration.

3.1.1. Let us recall some classic properties of holomorphic differential forms with “logarithmic poles”. The reader will find demonstrations in [3], II, (3.1) to (3.7), for example.

3.1.2. A divisor Y in a complex smooth analytic variety X is called normal crossing if the inclusion of Y in X is locally isomorphic to the inclusion of the union of coordinate hyperplanes in $\mathbb{C}^n$; this does not imply that Y is a union of smooth divisors. Let Y be a normal crossing divisor in X and let j be the inclusion of $X^* = X - Y$ in X. We define by $\Omega^1_X(\log Y)$ the locally free sub-$\mathscr{O}$-module of $j_*\Omega^1_{X^*}$ generated by

Ω^1_X and dz_i/z_i for z_i the local equation of a local irreducible component of Y. The sheaf $\Omega^p_X(\log Y)$ of differential p-forms on W with logarithmic poles along Y is by definition the locally free subsheaf $\wedge^p\Omega^1_X(\log Y)$ of $j_*\Omega^p_{X^*}$.

PROPOSITION 3.1.3. *(i) A section* α *of* $j_*\Omega^p_{X^*}$ *belongs to* $\Omega^p_X(\log Y)$ *if and only if* α *and* $d\alpha$ *have at least simple poles along the divisor* Y.

(ii) $\Omega^p_X(\log Y)$ *forms the smallest sub-complex of* $j_*\Omega^\bullet_{X^*}$ *which is stable under exterior product, contains* $\Omega^\bullet_X$, *and contains the logarithmic differential* df/f *of any meromorphic local section along* Y *of* $j_*\mathcal{O}^*_{X^*}$.

We call $\Omega^\bullet_X(\log Y)$ the logarithmic De Rham complex of X along Y. According to (3.1.3), (ii), this complex is contravariant on the couple (X, X^*).

3.1.4. Local on X, Y is the union of smooth divisors Y_i, and we denote by Y^n (resp. $\widetilde{Y^n}$) the union (resp. disjoint union) of n-to-n intersection of Y_i. Y^n is a subspace Y^n of X, and $\widetilde{Y^n}$ is the normalization of Y^n. [2] We have $\widetilde{Y^0} = Y^0 = X$ and we denote $\widetilde{Y} = \widetilde{Y^1}$.

We define a two-element set, the orientation of a finite set of n elements E, as being the set of generators of $\wedge^n\mathbb{Z}^E$. For $n \geq 2$, this set is the set of conjugacy class of the alternating group if we give a total order on E.

If, for any point y in $\widetilde{Y^n}$, we associate a set of n local components of Y which contains the image of a neighborhood of y in $\widetilde{Y^n}$ in X. We then defines a local system E_n of n-element set on $\widetilde{Y^n}$. The local system of orientations of these sets is a $\mathbb{Z}/2$-torsor. Via the inclusion $\mathbb{Z}/2$ into $\mathbb{C}^*$, this torsor defines a complex local system ϵ^n of rank 1 on $\widetilde{Y^n}$, equipped with an isomorphism $(\epsilon^n)^{\otimes 2} \cong \mathbb{C}$. We have

$$\epsilon^n \cong \bigwedge^n \mathbb{C}^{E_n},$$

Local on $\widetilde{Y^n}$, ϵ^m is equipped with two opposite isomorphisms $\pm\alpha : \epsilon^n \xrightarrow{\cong} \mathbb{C}$. We set

$$\epsilon^n_{\mathbb{Z}} = \alpha^{-1}((2\pi i)^{-n}\mathbb{Z}).$$

We can regard ϵ^n, equipped with $\epsilon^n_{\mathbb{Z}}$, as a twisted version of $\mathbb{Z}(-n)$ on $\widetilde{Y^n}$.

[2] $Y^n = \bigcup Y_{i_1,\cdots,i_n}$, $\widetilde{Y^n} = \coprod Y_{i_1,\cdots,i_n}$.

We denote by ϵ^n_{X} (resp. by $(\epsilon^n_{\mathrm{X}})_{\mathbb{Z}}$) the direct image of ϵ^n (resp. $\epsilon^n_{\mathbb{Z}}$) under the morphism of $\widetilde{\mathrm{Y}^n}$ into X. We have [3]

$$\epsilon^n_{\mathrm{X}} \cong \bigwedge^n \epsilon^1_{\mathrm{X}} \quad (n \geq 0). \tag{3.1.4.1}$$

If Y is a union of distinct divisors $(\mathrm{Y}_i)_{i\in\mathrm{I}}$, the choice of a total order on I trivializes ϵ^n.

3.1.5. The goal is to define residues along $\mathrm{Y}_{i_1,\cdots,i_n}$. Let $p \in \mathrm{Y}_{i_1,\cdots,i_n}$, then all n components Y_{i_j} pass through p, but maybe more. We first need to choose the local coordinates $(\mathrm{U}, t_1, \cdots, t_n)$ centered at p such that $\mathrm{Y}_{i_j} = \{t_{i(j)} = 0\}$, and that the remaining (if possible) components Y_i are given by $\{t_i = 0\}$. Any local section ω of $\Omega^p_{\mathrm{X}}(\log \mathrm{D})$ can be then written as

$$\omega = \eta \wedge \frac{dt_{i(1)}}{t_{i(1)}} \wedge \cdots \frac{dt_{i(n)}}{t_{i(n)}} + \eta'$$

where η has at most poles along components away from Y_{i_j}, and η' is not divisible by the form $\frac{dt_{i(1)}}{t_{i(1)}} \wedge \cdots \frac{dt_{i(n)}}{t_{i(n)}}$. The restriction of η to $\mathrm{Y}_{i_1,\cdots,i_n}$ is independent of the choice of local coordinates and so the map $\omega \mapsto \eta|\mathrm{Y}_{i_1,\cdots,i_n}$ globalizes to be the residue map. We also notice that

$$d\omega = d\eta \wedge \frac{dt_{i(1)}}{t_{i(1)}} \wedge \cdots \frac{dt_{i(n)}}{t_{i(n)}} + d\eta'$$

which implies that the residue map is compatible with derivatives. Furthermore, if in the local descripstion ω has weight $\leq n$, clearly $\eta|\mathrm{Y}_{i_1,\cdots,i_n}$ is holomorphic.

Denote by $\mathrm{W}_n(\Omega^p_{\mathrm{X}}(\log \mathrm{Y}))$ the submodule of $\Omega^p_{\mathrm{X}}(\log \mathrm{Y})$ consisting of the linear combinations of products

$$\alpha \wedge \frac{dt_{i(1)}}{t_{i(1)}} \wedge \cdots \wedge \frac{dt_{i(m)}}{t_{i(m)}} \quad (m \leq n),$$

with α holomorphic and $t_{i(j)}$ the local equations of the distinct local components Y_j of Y. We call $\mathrm{W}_n(\Omega^\bullet_{\mathrm{X}}(\log \mathrm{Y}))$ the weight filtration of $\Omega^\bullet_{\mathrm{X}}(\log \mathrm{Y})$, which is increasing. We have

$$\mathrm{W}_n(\Omega^p_{\mathrm{X}}(\log \mathrm{Y})) \wedge \mathrm{W}_m(\Omega^q_{\mathrm{X}}(\log \mathrm{Y})) \subset \mathrm{W}_{n+m}(\Omega^{p+q}_{\mathrm{X}}(\log \mathrm{Y})). \tag{3.1.5.1}$$

[3] After pushforward to X, n many ϵ^1_{X} lies over the n-fold points.

Let $i_n : \widetilde{Y^n} \to X$ be the morphism into X, one can verify that the correspondence

$$\alpha \wedge \frac{dt_{i(1)}}{t_{i(1)}} \wedge \cdots \wedge \frac{dt_{i(n)}}{t_{i(n)}} \longmapsto (\alpha | Y_{i(1)} \cap \cdots \cap Y_{i(n)}) \otimes (\text{orientation } i(1) \cdots i(n))$$

defines an isomorphism of complexes

$$\mathrm{R}es : Gr_n^W(\Omega_X^\bullet(\log Y)) \cong i_{n*}\Omega^\bullet_{\widetilde{Y^n}}(\epsilon^n)[-n] \tag{3.1.5.2}$$

(the Poincaré residue).

By definition $Gr_n^W(\Omega_X^\bullet(\log Y))$ is linearly generated by $\alpha \wedge \frac{dt_{i(1)}}{t_{i(1)}} \wedge \cdots \wedge \frac{dt_{i(n)}}{t_{i(n)}}$, then

$$res = \oplus res_{i_1,\cdots,i_n}$$

One can construct its inverse by

$$\rho_{i_1,\cdots,i_n} : \Omega_X^p \to Gr_n^W(\Omega_X^{p+n}(\log Y))$$
$$\beta \mapsto \beta \wedge \frac{dt_{i(1)}}{t_{i(1)}} \wedge \cdots \wedge \frac{dt_{i(n)}}{t_{i(n)}}.$$

This map is well-defined, i.e. if we change the coordinates, the expression differs by a form in $W_{n-1}\Omega_X^{p+n}(\log Y)$. Also, the elements of the form $t_{i(j)} \cdot \beta$ and $dt_{i(j)} \wedge \beta'$ map to zero so that the map ρ induces a map of complexes

$$\Omega^\bullet_{Y_{i_1,\cdots,i_n}}[-n] \to Gr_n^W(\Omega_X^\bullet(\log Y)).$$

We add up over all intersections to get the desired inverse for the residue map. [4]

Proposition. *Let* Y *be a smooth hypersurface of a non-singular compact complex $n-$dimensional manifold* X. *Let* $\eta \in \Gamma(\mathscr{E}_X^{2n-1}(\log Y))$ *such that* $d\eta \in \Gamma(\mathscr{E}_X^{2n})$. *Then*

$$\int_Y res(\eta) = \frac{1}{2\pi i}\int_X d\eta.$$

3.1.6.

[4] We need orientation when gluing.

3.1.7. Every point of X admits a fundamental system of open Stein neighborhoods since X^* is Stein. For an analytic coherent sheaf $\mathscr{F}$ on X^*, we therefore have $R^i j_*\mathscr{F} = 0$ for $i > 0$. The De Rham complex is then a j_*-acyclic resolution of the constant sheaf $\mathbb{C}$. Then

$$H^*(X^*, \mathbb{C}) \xrightarrow{\cong} \mathbb{H}^*(X^*, \Omega^\bullet_{X^*}) \xleftarrow{\cong} \mathbb{H}^*(X, j_*\Omega^\bullet_{X^*}) \tag{3.1.7.1}$$

and Leray spectral sequence for the morphism j is identified with the hypercohomology spectral sequence for $j_*\Omega^\bullet_{X^*}$, corresponding to the filtration τ defined by $\tau_{\leq -n}(j_*\Omega^\bullet_{X^*})$ (1.4.6).

Proposition 3.1.8. *The morphisms of filtered complexes*

$$(\Omega^\bullet_X(\log Y), W) \xleftarrow{\alpha} (\Omega^\bullet_X(\log Y), \tau) \xhookrightarrow{\beta} (j_*\Omega^\bullet_{X^*}, \tau)$$

are filtered quasi-isomorphisms. They define a isomorphism between the Leray spectral sequence for j in complex cohomology, and the hypercohomology spectral sequence of the filtered complex $(\Omega^\bullet_X(\log Y), W)$ *on* X.

According to (1.4.5) and (3.1.7), it suffices to prove the first assertion. One will find in [3], II, (6.9) or in [1], the proof of the fact that β is a quasi-isomorphism, and therefore a filtered quasi-isomorphism.

The calculation of β is local. We take for X a polydisk Δ^n with coordinates $(t_1, \cdots, t_n)$ and that Y is given by $t_1 \cdots t_k = 0$. It suffices to show that

$$H^k(X^*, \mathbb{C}) \cong H^k(\Gamma(\Delta^n, \Omega^\bullet_{\Delta^n}(\log Y))).$$

In fact, if we put

$$R^1_{n,k} = \mathbb{C}\frac{dt_1}{t_1} \oplus \cdots \oplus \mathbb{C}\frac{dt_k}{t_k}$$

$$R^p_{n,k} = \bigwedge^p R^1_{n,k}$$

we shall prove by induction that the natural inclusions $\alpha_{n,k} : R^\bullet_{n,k} \to \Omega^\bullet_{\Delta^n}(\log Y)$ are quasi-isomorpisms. This will complete the proof since the p-th cohomology of $R^\bullet_{n,k}$ is exactly the Koszul complex while R^p is the cohomology of X^*, since it is homotopic to a product of k circles.

$$\begin{tikzcd}
0 \arrow[r] & R^\bullet_{n,k-1} \arrow[r] \arrow[d, "\alpha_{n,k-1}"] & R^\bullet_{n,k} \arrow[r] \arrow[d, "\alpha_{n,k}"] & R^\bullet_{n-1,k-1}[-1] \arrow[r] \arrow[d, "\alpha_{n-1,k-1}"] & 0 \\
0 \arrow[r] & \Omega^\bullet_{\Delta^n}(\log Y_{k-1}) \arrow[r] & \Omega^\bullet_{\Delta^n}(\log Y_k) \arrow[r] & \Omega^\bullet_{\Delta^{n-1}}(\log Y_{k-1})[-1] \arrow[r] & 0
\end{tikzcd}$$

By the holomorphic Poincaré lemma, $\alpha_{n,0}$ is a quasi-isomorphism for all n. The induction follows from the five lemma.

We can also directly calculate the cohomology sheaves of both members: $\Omega^\bullet_X(\log Y)$ are determined by 3.1.5.2, while $j_*\Omega^\bullet_{X^*}$ are $R^i j_*\mathbb{C}$, which can be calculated topologically.

For $n \geq p$, we have

$$W_n(\Omega^p_X(\log Y)) = \Omega^p_X(\log Y),$$

so that α is a morphism from $\Omega^\bullet_X(\log Y)$, equipped with τ, into $\Omega^\bullet_X(\log Y)$, equipped with the associated decreasing filtration of W (1.1.3). [5] According to 3.1.5.2, we have

$$(3.1.8.1) \qquad \mathscr{H}^i(Gr^W_n(\Omega^\bullet_X(\log Y))) = \begin{cases} 0 & \text{for } i \neq n \\ \epsilon^n_X & \text{for } i = n; \end{cases}$$

While for the canonical filtration $Gr^\tau_n(\Omega^\bullet_X(\log Y)) = H^n(\Omega^\bullet_X(\log Y))$, and locally on Y we can take a tubular neighborhood such that $\mathbb{H}^n(V, \Omega^\bullet_X(\log Y)) = H^n(V - V \cap Y, \mathbb{C})$ according to the previous turquoise proof. Since $V - V \cap Y$ is homotopic to the product of circles, they have the same cohomology. We then deduce from the first line of this formula that α is an filtered quasi-isomorphism. This proves (3.1.8).

Accoring to (3.1.7), the isomorphism (3.1.8.1) defines an isomorphism

$$(3.1.8.2) \qquad R^n j_*\mathbb{C} \cong \mathscr{H}^n(j_*\Omega^\bullet_{X^*}) \cong \mathscr{H}^n(\Omega^\bullet_X(\log Y)) \cong \epsilon^n_X.$$

The isomorphism (3.1.4.1) corresponds to the cup product via (3.1.8.2).

PROPOSITION 3.1.9. *The canonical morphism from $R^n j_*\mathbb{Z}$ to $R^n j_*\mathbb{C}$ identifies, via (3.1.8.2), $R^n j_*\mathbb{Z}$ with $(\epsilon^n_X)_\mathbb{Z}$ (3.1.4).*

The question is local on X. We can therefore suppose that X is a open polycylinder D^m, where

$$D = \{z \in \mathbb{C} : |z| < 1\},$$

[5]By definition, a filtered morphism is a morphism $f : A \to B$ satisfying $f(F^iA) \subset F^iB$.

and that $Y = \sum_{k=1}^{l} Y_k$, $Y_k = pr_k^{-1}(0)$. The fiber of $R^n j_* \mathbb{Z}$ at 0 is the integral cohomology of $X^* = D^* \times D^{m-l}$, where

$$D^* = \{z \in \mathbb{C} : 0 < |z| < 1\}.$$

The space X^* is homotopic to a torus; its cohomology is therefore torsion free, and the cup product defines the isomorphism

$$\bigwedge^n (R^1 j_* \mathbb{Z})_0 \xrightarrow{\cong} (R^n j_* \mathbb{Z})_0.$$

It is enough to prove (3.1.9) for $n = 1$.

The integral homology $H_1(X^*)$ is generated by the loop γ_k rotating around the divisor Y_k. We have

$$\oint_{\gamma_k} \frac{dz_k}{z_k} = \pm 2\pi i;$$

the integral cohomology is therefore generated by $\frac{1}{2\pi i}\frac{dz_k}{z_k}$ and (3.1.9) is proved.

3.1.10. Let $\mathscr{F}$ be an analytic coherent sheaf on X^*, given as a restriction of an analytic coherent sheaf $\mathscr{F}'$ on X to X^*. We define the meromorphic direct image $j_*^m \mathscr{F}$ of $\mathscr{F}$ by the inductive limit

$$j_*^m \mathscr{F} = \varinjlim \mathscr{F}'(nY).$$

Local on X, Y is the sum of a finite family $(Y_i)_{i\in I}$ of smooth divisors, and we define the filtration by the order of poles P on $j_*^m \mathscr{O}_X^*$ by the formulae

$$\text{(3.1.10.1)} \qquad P^p(j_*^m \mathscr{O}_{X^*}) = \sum_{n\in A_p} \mathscr{O}_X\Big(\sum (n_i + 1) Y_i\Big)$$

where $A_p = \{(n_i)_{i\in I} : \sum_i n_i \le -p \text{ and } \forall i, n_i \ge 0\}$. This construction can be globalized and provides an exhaustive filtration on $j_*^m \mathscr{O}_{X^*}$ such that $P^p = 0$ for $p > 0$.

We define the filtration by the order of poles on the complex $j_*^m \Omega_{X^*}^\bullet = j_*^m \mathscr{O}_{X^*} \otimes \Omega_X^\bullet$

$$\text{(3.1.10.2)} \qquad P^p j_*^m \Omega_{X^*}^k = P^{p-k}(j_*^m \mathscr{O}_{X^*}) \otimes \Omega_X^k.$$

The filtration P induces the stupid filtration $\sigma_{\ge p}\Omega_X^\bullet(\log Y)$ on the subcomplex $\Omega_X^\bullet(\log Y)$, also known as the Hodge filtration F.

PROPOSITION 3.1.11. *The inclusion morphism*

$$(\Omega^{\bullet}_{\mathrm{X}}(\log \mathrm{Y}), \mathrm{F}) \to (j_*^m \Omega^{\bullet}_{\mathrm{X}^*}, \mathrm{P})$$

is a filtered quasi-isomorphism.

This statement was suggested to me by [4]. A demonstration can be found in [3], II, (3.13).

3.2. Mixed Hodge theory

From now on, we mean by scheme a scheme of finite type over $\mathbb{C}$*, and by sheaf on* S *a sheaf on* S^{an}.

3.2.1. Let X be a smooth separated scheme. According to Nagata [11], X is a Zariski open subset of a complete scheme $\overline{\mathrm{X}}$. According to Hironaka [8], we can assume that X is smooth, and such that $\mathrm{Y} = \overline{\mathrm{X}} — \mathrm{X}$ is a normal crossing divisor.

The reader who wants to avoid the reference of Nagata can suppose that X is quasi-projective. The smooth completion $\overline{\mathrm{X}}$ can then be chosen to be projective, and such that Y is a union of smooth divisors. When we limit ourselves to such compactifications, we only need Hodge theory in the standard form (2.2.1).

3.2.2. According to (3.1.7) and (3.1.8), on a

$$\mathrm{H}^*(\mathrm{X}, \mathbb{C}) \cong \mathbb{H}^*(\overline{\mathrm{X}}, \Omega^*_{\overline{\mathrm{X}}}(\log \mathrm{Y})).$$

We define the Hodge filtration F on the complex $\Omega^*_{\overline{\mathrm{X}}}(\log \mathrm{Y})$ as $\mathrm{F}^p = \sigma_{\geq p}$, the stupid filtration (1.4.5). On $\Omega^*_{\overline{\mathrm{X}}}(\log \mathrm{Y})$, we dispose two filtrations: F and W (3.1.5).

3.2.3. We will have to use the fact that there is a bifiltered resolution i : $\Omega^*_{\overline{\mathrm{X}}}(\log \mathrm{Y}) \to \mathrm{K}^{\bullet}$ such that $\mathrm{G}r^p_{\mathrm{F}} \mathrm{G}r^{\mathrm{W}}_n(\mathrm{K}^j)$ are Γ-acyclic sheaves:

$$\mathrm{H}^i(\overline{\mathrm{X}}, \mathrm{G}r^p_{\mathrm{F}} \mathrm{G}r^{\mathrm{W}}_n(\mathrm{K}^j)) = 0 \text{ for } i > 0.$$

Here are two methods to build $\mathrm{K}^{\bullet}$:

a) We can take the canonical resolution of Godement $\mathscr{C}^{\bullet}(\Omega^*_{\overline{\mathrm{X}}}(\log \mathrm{Y}))$, filtered by $\mathscr{C}^{\bullet}(\mathrm{W}_n(\Omega^*_{\overline{\mathrm{X}}}(\log \mathrm{Y})))$ and $\mathscr{C}^{\bullet}(\mathrm{F}^p(\Omega^*_{\overline{\mathrm{X}}}(\log \mathrm{Y})))$. This is a bifiltered resolution because

$\mathscr{C}^\bullet$ is an exact functor. Briefly, $\mathscr{C}^\bullet(\mathscr{A}^\bullet)$ is a cosimplicial object

$$\mathscr{A}^\bullet \to \mathrm{T}ot\left(f_*f^*\mathscr{A}^\bullet \to f_*f^*f_*f^*\mathscr{A}^\bullet \to \cdots\right)$$

where $f : \mathrm{X}^{disc} \to \mathrm{X}$ with X^{disc} equipped with discrete topology.

b) We can take the d''-resolution of $\Omega^*_{\overline{\mathrm{X}}}(\log \mathrm{Y})$. Let $\Omega^{pq}_{\overline{\mathrm{X}}}$ be the sheaf of C^∞ forms of type (p,q); $\mathrm{K}^\bullet$ is therefore the total complex associated to the double complex $\Omega^p_{\overline{\mathrm{X}}}(\log \mathrm{Y}) \otimes_{\mathscr{O}} \Omega^{0,q}_{\overline{\mathrm{X}}}$ (sub-complex of $j_*\Omega^{\bullet\bullet}_{\overline{\mathrm{X}}}$). This complex is filtered by $\mathrm{F}^p(\Omega^\bullet_{\overline{\mathrm{X}}}(\log \mathrm{Y})) \otimes \Omega^{0,\bullet}_{\overline{\mathrm{X}}}$ and $\mathrm{W}_n(\Omega^\bullet_{\overline{\mathrm{X}}}(\log \mathrm{Y})) \otimes \Omega^{0,\bullet}_{\overline{\mathrm{X}}}$; for the proof that it is a bifiltered resolution, we only use the fact that the sheaf $\mathscr{O}_\infty$ of complex C^∞ functions on $\overline{\mathrm{X}}$ is flat on $\mathscr{O}$ (corollay of the Malgrange preparation theorem, a C^∞ version of Weierstrass preparation theorem). The sheaf $\mathrm{G}r_{\mathrm{F}}\mathrm{G}r_{\mathrm{W}}(\mathrm{K}^\bullet)$ are fine, since they are the sheaf of modules over the soft sheaf $\mathscr{O}_\infty$. [6]

3.2.4. With the notations in (3.2.3), the complex cohomology of X appears as the cohomology of the bifiltered complex $\Gamma(\overline{\mathrm{X}}, \mathrm{K}^*)$. We therefore have two spectral sequences convergent to $\mathrm{H}^*(\mathrm{X},\mathbb{C})$. One writes, with the notations in (3.1.4): [7]

$$(3.2.4.1) \qquad {}_{\mathrm{W}}\mathrm{E}_1^{pq} = \mathrm{H}^{p+q}(\overline{\mathrm{X}}, \epsilon^{-p}_{\overline{\mathrm{X}}}[p]) = \mathrm{H}^{2p+q}(\widetilde{\mathrm{Y}^p}, \epsilon^{-p}) \Rightarrow \mathrm{H}^n(\mathrm{X},\mathbb{C})$$

$$(3.2.4.2) \qquad {}_{\mathrm{F}}\mathrm{E}_1^{pq} = \mathrm{H}^{p+q}(\overline{\mathrm{X}}, \Omega^p_{\overline{\mathrm{X}}}(\log \mathrm{Y})[-p]) = \mathrm{H}^q(\overline{\mathrm{X}}, \Omega^p_{\overline{\mathrm{X}}}(\log \mathrm{Y})) \Rightarrow \mathrm{H}^n(\mathrm{X},\mathbb{C}).$$

The first one, after a renumbering ${}_{\mathrm{W}}\mathrm{E}_1^{pq} \mapsto \mathrm{E}_2^{2p+q,-p}$, is nothing but the Leray spectral sequence of the inlucsion j.

For a general spectral sequence associated to a filtered complex (K,F), we have

$$\begin{aligned}
\mathrm{E}_r^p q &= \mathrm{I}m(\mathrm{Z}_r^{pq} \to \mathrm{K}^{p+q}/\mathrm{B}_r^{pq}) \\
&= \mathrm{Z}_r^{pq}/(\mathrm{B}_r^{pq} \cap \mathrm{Z}_r^{pq}) \\
&= \mathrm{K}er(\mathrm{K}^{p+q}/\mathrm{B}_r^{pq} \to \mathrm{K}^{p+q}/(\mathrm{Z}_r^{pq} + \mathrm{B}_r^{pq})
\end{aligned}$$

The first line idenfities E_r^{pq} as a quotient of a sub-object of K^{p+q}. The term E_r^{pq} is therefore equipped with a filtration F_d induced by F, called the first direct filtration.

---

[6] A sheaf of modules over a soft sheaf of rings is a soft sheaf.

[7] Recall that $\mathrm{G}r_p^{\mathrm{W}}(\Omega^\bullet_{\overline{\mathrm{X}}}(\log \mathrm{Y})) = i_*\Omega^\bullet_{\widetilde{\mathrm{Y}^n}}(\epsilon^p)[-p]$, and for the canonical filtration $\mathrm{T}^p = \mathrm{W}_{-p}$, we have $\mathrm{G}r^p_{\mathrm{T}} = \mathrm{G}r^{\mathrm{W}}_{-p} = i_*\Omega\widetilde{\mathrm{Y}^n}^\bullet(\epsilon^{-p})[p]$. After taking cohomology, since $\Omega^\bullet_{\widetilde{\mathrm{Y}^n}}$ is acyclic, we obtain $\mathrm{E}_1^{pq} = \mathrm{H}^{p+q}(\overline{\mathrm{X}}, \epsilon^{-p}[p])$.

Dually, the third line identifies E_r^{pq} as a sub-object of a quotient of K^{p+q}, and there exists a new filtration F_{d^*} induced by F, called the second direct filtration. For higher pages of the spectral sequences E_{r+1}, it can be regarded as a subquotient of E_r, and there exists an induced filtration, called the recurrence filtration. This is well-defined since on E_0 and E_1, $F_d = F_{d^*}$ (1.3.10).

Reall that a morphism of filtered obejcts is called strictly compatible to the filtrations if $coim(f) \to im(f)$ is an isomorphism of filtered objects. We will need the following facts: if (K, F) is a filtered complex, and F induces a finite filtration on every component, then the followings are equivalent:

(i) The spectral sequence attached to F degenerates ($E_1 = E_\infty$).
(ii) The morphism $d : K^i \to K^{i+1}$ is strictly compatible to the filtrations.

For the proof, see Proposition (1.3.2).

THEOREM 3.2.5.

(i) On the terms ${}_W E_r^{pq}$ *of the spectral sequence (3.2.4.1), the first direct filtration, the second direct filtration and the recurrence filtration coincides with* F.
(ii) The spectral sequence ${}_W E$ *which converges to* $H^n(X, \mathbb{C})$ *induces a filtration* W *of* $H^n(X, \mathbb{Q})$. *Neither* W *nor* F *depends on the choice of compactification* $\overline{X}$ *of* X *or the choice of* $K^\bullet$. [8]
(iii) The filtration $W[n]$ *and* F *defines a mixed Hodge structure on* $H^n(X, \mathbb{Z})$*, which is functorially on* X.

According to (3.1.7), the spectral sequence ${}_W E$ is the Leray spectral sequence for j_* (after a renumbering). It can therefore be obtained by tensoring with $\mathbb{C}$ from a spectral sequence of $\mathbb{Q}$-vector spaces and the first assertion of (ii) is true. [9] The complex conjugation acts on ${}_W E$; it can be calculated via (3.1.10).

The following is devoted to the assertion (i). Mainly it can be divided into the following steps:

[8] This means the filtration W can be reduced to $\mathbb{Q}$.

[9] We have shown that W can be replaced with the canonical filtration. Since the canonical filtration can be put on any complex, the weight filtration can be defined over $\mathbb{Q}$.

(i) We analyse the E_1 page of the spectral sequence associated to weight filtration and figure out the filtration on each component, which is induced by classical Hodge theory.

(ii) We show that this spectral sequence degenerates at E_2 page by showing that the differentials are strictly compatible with the filtrations.

LEMMA 3.2.6. *The hypercohomology spectral sequence of the filtered complex* $Gr_n^W(\Omega^\bullet_{\overline{X}}(\log Y))$ *equipped with the filtration induced by the Hodge filtration, degenerates at the term* E_1.

Let Y^n and $\widetilde{Y^n}$ be in (3.1.4) and let $i_n : \widetilde{Y^n} \to X$. According to (3.1.5.2), we have

$$Gr_n^W(\Omega^\bullet_{\overline{X}}(\log Y)) \sim i_{n*}\Omega^\bullet_{\widetilde{Y^n}}(\epsilon^n)[-n].$$

In addition, the Hodge filtration induces the stupid filtration (1.4.5), so that the spectral sequence (3.2.6) is deduced by the translation on the degree of the classical spectral sequence

$$E_1^{pq} = H^q(\widetilde{Y^n}, \Omega^p_{\widetilde{Y^n}}(\epsilon^n)) \Rightarrow H^{p+q}(\widetilde{Y^n}, \epsilon^n).$$

If $\overline{X}$ is projective and Y is a union of smooth divisors, then $\widetilde{Y^n}$ is projective, ϵ^n is a trivial local system and the classical Hodge theory (2.2.1) provides the degeneration (3.2.6). For the general case, we refer to [2] (see (2.2.2), (2.2.3)). [10]

The Hodge theory also guarantees that the Hodge filtration on $H^k(\widetilde{Y^n}, \epsilon^n)$ is k-opposite to its complex conjugation (the complex conjugation is defined in terms of $\epsilon^n_{\mathbb{Z}}$ (3.1.4)). We here, take into account the translation on the degrees:

LEMMA 3.2.7. *The filtration on*

$$_W E_1^{-n,k+n} = \mathbb{H}^k(\overline{X}, Gr_n^W(\Omega^\bullet_{\overline{X}}(\log Y))) \sim H^{k-n}(\widetilde{Y^n}, \epsilon^n),$$

which is the limit of the spectral sequence (3.2.6), is $(k+n)$*-opposite to its complex conjugation.*

Over the complex numbers we could forget the Tate twists at the cost of neglecting a factor of $2\pi i$. But after descent to $\mathbb{Q}$, after using Leray residue formula repeatedly we find that we must twist n times to guarantee that the Gysin map is a morphism

[10]The inside sheaf can be regarded as a tensor product with a vector space (locally), and we have the degeneracy from (2.2.2). If $\widetilde{Y^n}$ is not projective, use (2.2.3), i.e. via Chow lemma.

of Hodge structures. (Naïvely, since we drop the dimension by n, we should in turn tensor with $\mathbb{Q}$ to adjust the weight to get $k+n$.)

LEMMA 3.2.8. *The differential d_1 of the spectral sequence ${}_{\mathrm{W}}\mathrm{E}$ is strictly compatible with the filtration* F.

On the terms E_1, there is only one filtration induced by F to consider ((1.3.10) and (1.3.13), (iii)) [11], and d_1 is compatible to this filtration ((1.3.13), (i)). This filtration is induced via the limit of the spectral sequence (3.2.6) ((1.4.8), (ii)). According to (3.2.7), the morphism d_1

$$d_1 : \mathbb{H}^k(\overline{\mathrm{X}}, \mathrm{G}r_n^{\mathrm{W}}(\Omega^\bullet_{\overline{\mathrm{X}}}(\log \mathrm{Y}))) \to \mathbb{H}^{k+1}(\overline{\mathrm{X}}, \mathrm{G}r_{n-1}^{\mathrm{W}}(\Omega^\bullet_{\overline{\mathrm{X}}}(\log \mathrm{Y}))),$$

or

$$d_1 : \mathrm{H}^{k-n}(\widetilde{\mathrm{Y}^n}, \epsilon^n)(-n) \to \mathrm{H}^{k-n+2}(\widetilde{\mathrm{Y}^{n-1}}, \epsilon^{n-1})(-n+1) \tag{3.2.8.1}$$

is compatible to the filtration which is $(k+n)$-opposite to its complex conjugate. Explicitly, we have the restriction maps $p_j : \widetilde{\mathrm{Y}^n} \to \widetilde{\mathrm{Y}^{n-1}}$, and it's a good exercise using the residue formula to to show that $d_1 = \oplus(-1)^j(p_j)_!$. In fact, if ω is a k-form, then it's easy to see that $d_1[\omega] = [d\omega]$. We are required to show

$$res_j(d\omega) = (-1)^j(p_j)_!\eta$$

if $res(\omega) = \eta$. Since d_1 commutes with the complex conjugation, d_1 respects the bigraduation (of weight $k+n$) defined by F and $\overline{\mathrm{F}}$, which proved (3.2.8). In addition, the cohomology of the complex E_1 will also be bigraded:

LEMMA 3.2.9. *On ${}_{\mathrm{W}}\mathrm{E}_2^{pq}$, the recurrence filtration* F *is q-opposite to the complex conjugation.*

By recurrence on r we can prove that:

LEMMA 3.2.10. *For $r \geq 0$, the differentials d_r of the spectral sequence ${}_{\mathrm{W}}\mathrm{E}$ are strictly compatible to the recurrence filtration* F. *For $r \geq 2$, d_r are zero.*

[11] Explained in previous turquoise text, the first and the second direct filtration agrees.

For $r = 0$ (resp. $r = 1$), we apply (3.2.6) and (1.4.8), (iii) (resp.(3.2.8)). For $r \geq 2$, it suffices to prove that $d_r = 0$. By recurrence and according to (1.3.16), we can suppose that on the terms ${}_{\mathrm{W}}\mathrm{E}_s$ $(s \geq r+1)$, we have $\mathrm{F}_d = \mathrm{F}_r = \mathrm{F}_{d^*}$, and that ${}_{\mathrm{W}}\mathrm{E}_r =_{\mathrm{W}} \mathrm{E}_2$. According to (1.3.13), (i), d_r is therefore compatible to the filtration F_r.

Sur ${}_{\mathrm{W}}\mathrm{E}_r^{pq} =_{\mathrm{W}} \mathrm{E}_2^{pq}$, la filtration F_r, est y-opposée à sa complexe conjuguée...

Ceci prouve (3.2.10), qui, d'après (1.3.16), implique (3.2.5) (i). [12]

According to (1.3.17), the filtration on ${}_{\mathrm{W}}\mathrm{E}_\infty^{pq}$ induced by the filtration F on $\mathrm{H}^{p+q}(\mathrm{X},\mathbb{C})$ is q-opposite to the complex conjugation. Since $q = -p + (p+q)$ [13], this proves the first assertion of (3.2.5) (iii). i.e. this defines a mixed Hodge structure.

3.2.11. Let's prove (ii) and (iii), which will complete the demonstration.

A. Independence of the choice of $\mathrm{K}^\bullet$.

The filtration F and W on $\mathrm{H}^\bullet(\mathrm{X},\mathbb{C})$ are the limit of the hypercohomology spectral sequence of $\Omega^\bullet_{\overline{\mathrm{X}}}(\log \mathrm{Y})$ for the filtration F and W. These spectral sequences, all do not depend on the choice of $\mathrm{K}^\bullet$.

B. Functoriality.

Let $f : \mathrm{X}_1 \to \mathrm{X}_2$ be a morphism of schemes. Suppose we are given a morphism of smooth compactifications

$$\begin{tikzcd} \mathrm{X}_1 \arrow[r, "f"] \arrow[d, hook, "j_1"] & \mathrm{X}_2 \arrow[d, hook, "j_2"] \\ \overline{\mathrm{X}_1} \arrow[r, "\overline{f}"] & \overline{\mathrm{X}_2} \end{tikzcd} \tag{3.2.11.1}$$

with $\mathrm{Y}_i = \overline{\mathrm{X}_i} - \mathrm{X}_i$ normal crossing divisor. The canonical morphism (see (3.1.3)) $\overline{f}^*\Omega^\bullet_{\overline{\mathrm{X}_2}}(\log \mathrm{Y}_2) \to \Omega^\bullet_{\overline{\mathrm{X}_1}}(\log \mathrm{Y}_1)$ is therefore a morphism of bifiltered complexes; it induces a morphism on the hypercohomology, which is compatible to F and W, and thus $f^* : \mathrm{H}^n(\mathrm{X}_2,\mathbb{Z}) \to \mathrm{H}^n(\mathrm{X}_1,\mathbb{Z})$ is a morphism of mixed Hodge structures, for the structures defined by the compactifications $\overline{\mathrm{X}_i}$.

C. Independence of the compactification.

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[12] After we show d_r is strictly compatible to the filtration, it's homological algebra that the filtrations agree.

[13] If we do not shift W, the weight should be $-p$. Hence the shift equals n.

With the notations in B, if f is an isomorphism, then f^* is a bijective morphism of mixed Hodge structures, therefore an isomorphism (2.3.5).

If $\overline{X_1}$ and $\overline{X_2}$ are two smooth compactifications of X, with $Y_i = \overline{X_i} - X_i$ normal crossing divisors, there exists a third smooth compactification $\bar{X}$, with $Y = \bar{X} - X$ normal crossing divisor, which fits into a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, hook'] \arrow[d, hook] \arrow[dr, hook] & \\ \overline{X_1} & \bar{X} \arrow[l] \arrow[r] & \overline{X_2} \end{tikzcd}$$

Namely, we take for $\bar{X}$ a resolution of singularities for the closure of the image of the diagonal of X in $\overline{X_1} \times \overline{X_2}$. The identity morphism from $H^n(X, \mathbb{Z})$ equipped with the mixed Hodge structure defined by $\overline{X_1}$ to $H^n(X, \mathbb{Z})$ equipped with the structure defined by $\overline{X_2}$, is therefore composed of two isomorphisms.

To complete the demonstration of (ii) and (iii), we remark that for any morphism f fits into a diagram (3.2.11.1): we choose the compactification $\overline{X_1}'$ and $\overline{X_2}$ for X_1 and X_2, then we take for $\overline{X_1}$ a resolution of singularities for the closure of the image of the diagonal of X_1 in $\overline{X_1}' \times \overline{X_2}$.

Definition 3.2.12. The mixed Hodge structure of the cohomology of a smooth separated algebraic variety is the mixed Hodge structure (3.2.5), (iii).

Corollary 3.2.13. *With the notations above:*

(i) The spectral sequence (3.2.4.1) degenerate at E_2*, i.e. the Leray spectral sequence for the inclusion* $j : X^* \hookrightarrow X$ *degenerates at* E_3 *(*$E_3 = E_\infty$*).*

(ii) The spectral sequence (3.2.4.2)

$$_F E_1^{pq} = H^q(\bar{X}, \Omega^p_{\bar{X}})(\log Y) \Rightarrow H^{p+q}(X, \mathbb{C})$$

degenerates at E_1.

(iii) The spectral sequence defined for the sheaf $\Omega^p_X(\log Y)$*, equipped with the filtration* W*:*

$$\begin{aligned} E_1^{-n,k+n} &= H^k(\bar{X}, Gr_n^W(\Omega^p_X(\log Y))) \\ &\sim H^k(\widetilde{Y^n}, \Omega^{p-n}_{\widetilde{Y^n}}(\epsilon^n)) \Rightarrow H^k(\bar{X}, \Omega^p_X(\log Y)) \end{aligned}$$

degenerates at E_2.

The assertion (i) is proved in (3.2.10).

Consider the four spectral sequences in (1.4.12), relative to the bifiltered complex $\Omega^\bullet_{\overline{X}}(\log Y)$. According to (i), we have

$$\sum_n \dim H^n(X,\mathbb{C}) = \sum_{p,q} \dim_W E_2^{pq}.$$

For n fixed, the terms ${}_WE_1^{n,k-n}$ are limits of a spectral sequence which degenerates at E_1 (3.2.6) with the initial terms $E_1^{p,n,k-p-n}$ (notation in (1.4.12)) the initial terms of the spectral sequence in (iii). Since ${}_Wd_1$ is strictly compatible to the filtration F on the limit of the spectral sequence (3.2.7), and is (1.4.12) the limit of a morphism between these spectral sequence, which starts with the differentials of (iii), we have

$$Gr_F^p({}_WE_2^{n.k-n}) \sim H^\bullet(E_1^{p,n-1,k-n-p} \to E_1^{p,n,k-n-p} \to E_1^{p,n+1,k-n-p})$$

and $Gr_F^\bullet({}_WE_2^{\bullet\bullet})$ is the sum of the terms $E_2^{\bullet\bullet\bullet}$ of the spectral sequence (iii). We then have

$$\sum_n \dim H^n(X,\mathbb{C}) = \dim E_2^{\bullet\bullet\bullet}. \tag{3.2.13.1}$$

In addition, we have

$$\sum_n \dim H^n(X,\mathbb{C}) \leq \sum \dim_F E_1^{\bullet\bullet}$$

with equality holds if and only if the spectral sequence (ii) degenerates at E_1, and

$$\sum \dim_F E_1^{\bullet\bullet} \leq \sum \dim E_2^{\bullet\bullet\bullet},$$

with equality holds if and only if the spectral sequence (iii) degenerates at E_2. Compare with (3.2.13.1), we obtain (3.2.13).

COROLLARY 3.2.14. *Let* ω *be a meromorphic differential* p*-form on* $\overline{X}$*, holomorphic on* X*, and with logarithmic poles along Y. Then, the restriction* $\omega|X$ *to* X *is closed, and if the cohomology class of* $\omega|X$ *in* $H^p(X,\mathbb{C})$ *is zero, we have* $\omega = 0$.

This is a special caes of ${}_FE_1^{p0} =_F E_\infty^{p0}$ in (3.2.13), (ii).

COROLLARY 3.2.15.

(i) If X *is a smooth complete algebraic variety, the mixed Hodge structure on* $H^n(X,\mathbb{Z})$ *is the classical Hodge structure of weight* n.

(ii) *The Hodge numbers h^{pq} of the mixed Hodge structure of* $\mathrm{H}^n(\mathrm{X},\mathbb{Z})$ *(*X *smooth algebraic) can only be nonzero for $p \leq n$, $q \leq n$ and $p+q \geq n$.*

The assertion (i) is clear; for (ii), we remark that, for Y a union of smooth divisors, the rational Hodge structre on $\mathrm{Gr}_k^{\mathrm{W}}(\mathrm{H}^n(\mathrm{X},\mathbb{Q}))$ is the quotient of a sub-object of a Hodge structure of weight $n+k$, namely

$$\mathrm{H}^{n-k}(\widetilde{\mathrm{Y}^n},\mathbb{Q}) \otimes \mathbb{Q}(-k).$$

3.2.16. Let X be a separated smooth scheme. We know that X admits a smooth compactification $\overline{\mathrm{X}}$ and that the image of $\mathrm{H}^n(\mathrm{X},\mathbb{Z})$ in $\mathrm{H}^n(\overline{\mathrm{X}},\mathbb{Z})$ independent of the choice of $\overline{\mathrm{X}}$ (cf.[6], (9.1) to (9.4)).

COROLLARY 3.2.17. *Under the assumptions (3.2.16), the image of* $\mathrm{H}^n(\mathrm{X},\mathbb{Q})$ *in* $\mathrm{H}^n(\overline{\mathrm{X}},\mathbb{Q})$ *is* $\mathrm{W}_n(\mathrm{H}^n(\mathrm{X},\mathbb{Q}))$ *(*W *the the weight filtration (3.2.12)).*

We can assume that $\overline{\mathrm{X}} - \mathrm{X}$ is a normal crossing divisor. The assertion follows from the fact that $\mathrm{W}[-n]$ is the limit of the Leray spectral sequence for the inclusion $j : \mathrm{X} \hookrightarrow \overline{\mathrm{X}}$.

COROLLARY 3.2.18. *Let f be a morphism from a smooth proper scheme* Y *to a smooth scheme* X *which admits a smooth compactification* $\overline{\mathrm{X}}$*:*

$$\mathrm{Y} \xrightarrow{f} \mathrm{X} \overset{j}{\hookrightarrow} \overline{\mathrm{X}}.$$

Then, the groups $\mathrm{H}^n(\mathrm{X},\mathbb{Q})$ *and* $\mathrm{H}^n(\overline{\mathrm{X}},\mathbb{Q})$ *have the same image in* $\mathrm{H}^n(\mathrm{Y},\mathbb{Q})$.

Since f^* and $(jf)^*$ are strictly compatible to the weight filtration ((3.2.5), (iii)), it suffices to prove that $\mathrm{Gr}^{\mathrm{W}}(f^*)$ and $\mathrm{Gr}^{\mathrm{W}}(f^* j^*)$ have the same image in $\mathrm{Gr}^{\mathrm{W}}(\mathrm{H}^n(\mathrm{Y},\mathbb{Q}))$. According to (3.2.17) and (3.2.15), $\mathrm{Gr}_n^{\mathrm{W}}(j^*)$is an isomorphism, while $\mathrm{Gr}_m^{\mathrm{W}}(f^*) = 0$ for $m \neq n$ since $\mathrm{Gr}_m^{\mathrm{W}}(\mathrm{H}^n(\mathrm{Y},\mathbb{Q})) = 0$ for $m \neq n$.

REMARK 3.2.19. According to (3.1.11) and (1.4.5), under the assumptions of (3.2.5), the Hodge filtration on $\mathrm{H}^n(\mathrm{X},\mathbb{C})$ is the limit of the hypercohomology spectral sequence for the complex $j_m^*\Omega^\bullet_{\mathrm{X}^*}$, equipped with the filtration by the order of poles (3.1.10): this spectral sequence coincides with (3.2.4.2).

CHAPTER 4

Applications and complements

4.1. Theorem of fixed part

4.2. Theorem of semi-simplicity

4.3. Complements of [2]

4.4. Homomorphisms of abelian schemes