Free-arbitrage ESSVI calibration of FX-implied volatility

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Abstract

We describe a robust calibration algorithm of a set of SVI maturity slices, which grants that these slices are free of Butterfly and of Calendar-Spread arbitrage. Given a set of consistent SVI parameters, The numerical implementation is straightforward, robust and quick, yielding an effective solution to the smile problem.

1 ESSVI model

1.1 Introduction

The SSVI model gives a transformation of the log-moneyness $k := log(\frac{K}{F_t})$, with K the strike price of the option and F_t the forward price, and the at-the-money (ATM) implied total variance $\theta_t = t\sigma^2(0,t)$ to a surface $\omega(k,t)$ that approximates the implied total $t\sigma^2(k,t)$. It is assumed that the ATM implied total variance is a function in time of at least class $\mathcal{C}^1$ on $\mathbb{R}^+$. the SSVI parametrization is given by:

$$\omega(k,t) = \frac{1}{2}\big[\theta_t + \rho\psi k + \sqrt{(\psi k + \theta_t)^2 + (1-\rho^2)\theta_t^2}\big]$$

where $|\rho| < 1$ representing the correlation between the stock price and its instantaneous volatility, and $\psi > 0$. We note that the remaining parameters ρ and ψ are not determined by straightforward regression.

Theorem 1. *If the two-dimensional map $w : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies:*

(i) $\omega(\cdot,t)$ *is of class* $C^2(\mathbb{R})$ *for each* $t \geq 0$,

(ii) $\omega(k,t) > 0$ *for all* $(k,t) \in \mathbb{R} \times \mathbb{R}^*$,

(iii) $\omega(k,\cdot)$ *is non-decreasing for each* $k \in \mathbb{R}$,

(iv) for each $(k,t) \in \mathbb{R} \times \mathbb{R}^*$, *the probability density function* $P(k)$ *is non-negative,*

(v) $\omega(k,0) = 0$ *for all* $k \in \mathbb{R}$

Then the corresponding call price surface is free of static arbitrage.

Definition 1. *Let $w : \mathbb{R} \times \mathbb{R}^*_+ \to \mathbb{R}^+$ be a two-dimensional map satisfying Theorem 1 (i)-(ii):*

- *w is said to be free of calendar spread arbitrage if condition (iii) in Theorem 1 holds;*
- *w is said to be free of butterfly arbitrage if condition (iv) in Theorem 1 holds.*

Definition 2. *A volatility surface is said to be free of static arbitrage if and only if both of the following conditions are satisfied:*

1. *It is free of calendar spread arbitrage;*

2. *Each time slice is free of butterfly arbitrage.*

Lemma 1. *A volatility surface ω is free of calendar spread arbitrage if*

$$\frac{\partial}{\partial T}\omega(k,T) \geq 0, \quad \textit{for all } k \in \mathbb{R} \textit{ and } T > 0.$$

Proposition 1. ***Essvi Calendar arbitrage:*** *For two separate maturities $t_1 < t_2$. Let (ρ_1, ψ_1) and (ρ_1, ψ_1) be the parameters corresponding to maturities t_1 and t_2 respectively. There is no calendar spread arbitrage between these slices if and only if the following three inequalities are satisfied*

$$\psi_1 \leq \psi_2$$
$$|\rho_1\psi_1 - \rho_2\psi_2| \leq |\psi_1 - \psi_2|$$

Lemma 2. *the slice of implied volatility $k \to \omega(k,t)$ is free of Butterfly arbitrage if*

$$g(k) := \left(\frac{1 - k\omega'(k)}{2\omega(k)}\right)^2 - \frac{\omega'(k)^2}{4}\left(\frac{1}{\omega(k)} + \frac{1}{4}\right) + \frac{\omega''(k)}{2} > 0 \quad \textit{for all } k \in \mathbb{R}$$

Let (k^*, θ^*) the data point closest to the ATM (Forward). We define $\theta = \theta^* - \rho\phi k^*$ and the parameters

Proposition 2. *ESSVI model is free from butterfly arbitrage, iff:*

$$\psi \leq \frac{4}{1+|\rho|}$$

and

$$\psi \leq \frac{-2\rho k^*}{1+|\rho|} + \sqrt{\frac{(2\rho k^*)^2}{(1+|\rho|)^2} + \frac{4\theta^*}{1+|\rho|}}$$

1.2 Calibration of FX implied volatility using ESSVI

At a given time $t = t_{i+1}$ and given prior paramter ψ_i and ρ_i, we solve the following optimization problem:

$$\begin{cases} \min_{(\psi,\rho)\in\mathbb{R}^2} & f(k,\theta_t,\psi,\rho) \\ & \frac{|\rho\psi - \rho_i\psi_i|}{|\psi - \psi_i|} \leq 1 \\ & \psi_i \leq \psi \leq min\{\psi \leq \frac{-2\rho k^*}{1+|\rho|} + \sqrt{\frac{(2\rho k^*)^2}{(1+|\rho|)^2} + \frac{4\theta^*}{1+|\rho|}}, \frac{4}{1+|\rho|}\} \\ & -1 \leq \rho \leq 1 \\ & \psi \geq 0 \end{cases} \tag{1}$$

1. Calibration of $t_0 = 1day$:
   - We approximate the volatility surface at time t_0 by a quadratic polynomial $\phi(k) = \theta + a_1 k + a_2 k^2$ using a global optimisation called differential evolution optimization without taking in account the nonlinear constraints.
   - solve 1 using SLSQP or Constrained trust region algorithm.
2. Calibration at time t_i: we use the result of optimization algorithm 1 at time $t = t_{i-1}$ as a starting point and solve 1 at time $t = t_i$ using SLSQP or Constrained trust region algorithm.

Figure 1: Volatility interpolation using ESSVI

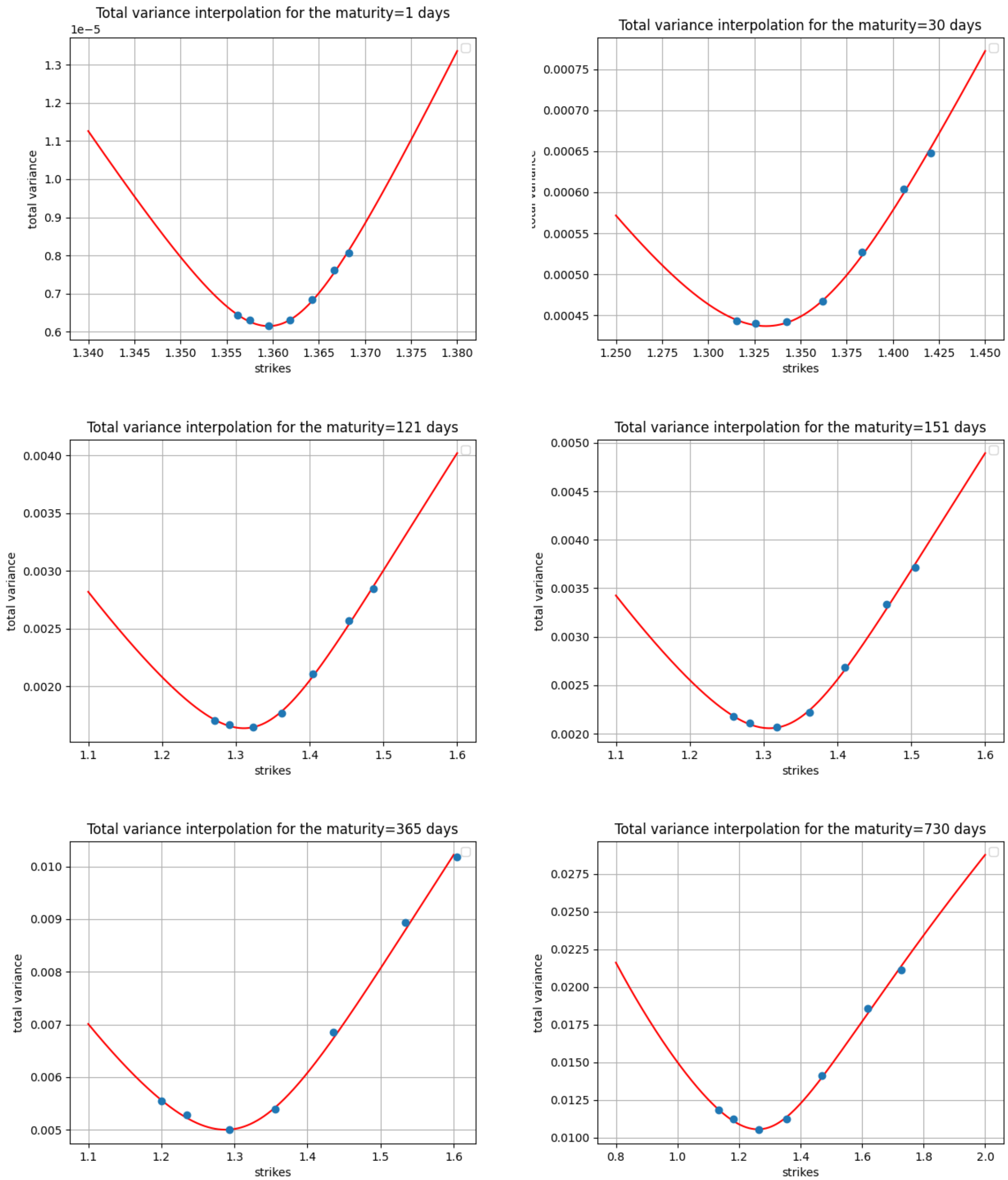
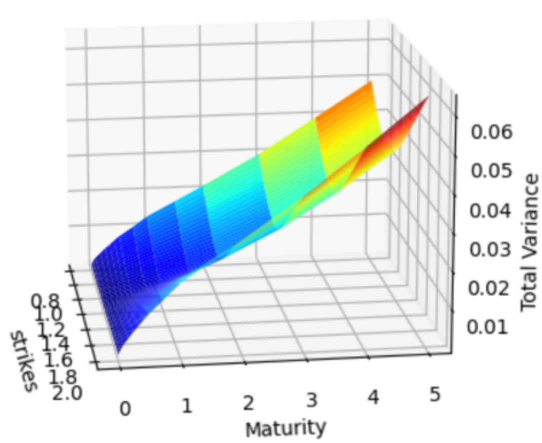


Table 1: USD/CAD market data Mid volatility(in%)

	ATM	25D Call EUR	25D Put EUR	10D Call EUR	10D Put EUR
1months	7.54	8.0055	7.3305	8.5745	7.3145
2months	7.31	7.8365	7.0815	8.4955	7.0855
3months	7.29	7.9145	7.039	8.7	7.07
6months	7.3425	8.106	7.0815	9.084	7.189
9months	7.3625	8.2095	7.095	9.288	7.228
1year	7.34	8.2825	7.07	9.4545	7.274
2years	7.495	8.4095	7.2495	9.633	7.498
3years	7.8875	8.797	7.662	10.039	7.929
4years	8.365	9.2925	8.1575	10.54	8.44
5years	8.79	9.7195	8.582	10.973	8.8585

Figure 2: Implied Volatility Surface using ESSVI



2 Local volatility

2.1 Dupire formula

Local volatility model, which is widely used in the finance industry, is the subject of this section. Within the local volatility framework, the dynamics of the stock price under the risk-neutral measure Q are given by

$$dS_t = S_t\big(\mu dt + \sigma(t)dW_t\big)$$

where the volatility is now a deterministic function of time and the asset price. Given European options market prices, Dupire (1994) derived the local volatility $\sigma_{LV}(K,T)$, as follows:

$$\sigma_{LV}(K,T) = \sqrt{2\frac{\frac{dC}{dT} + q_t C + (r_T - q_T)K + \frac{dC}{dK}}{0.5K^2\frac{d^2C}{dK^2}}} \tag{2}$$

To express local volatility in terms of implied volatility, we need the three derivatives $\frac{dC}{dT}$, $\frac{dC}{dK}$ and $\frac{d^2C}{dK^2}$ that appear in the above equation , but expressed in terms of implied volatility ω.

$$\sigma^2_{LV}(S,T) = \frac{\sigma^2_{IV} + 2\sigma_{IV}T\frac{\partial\sigma_{IV}}{\partial T} + 2(r-q)\sigma_{IV}KT\frac{\partial\sigma_{IV}}{\partial K}}{(1 + d_1(S,K)K\sqrt{T}\frac{\partial\sigma_{IV}}{\partial K})^2 + \sigma_{IV}K^2T\Big[\frac{\partial^2\sigma_{IV}}{\partial K^2} - d_1(S,K)\sqrt{T}\big(\frac{\partial\sigma_{IV}}{\partial K}\big)^2\Big]} \tag{3}$$

Another expression of local volatility in terms of total variance is

$$\sigma^2_{LV}(k,T) = \frac{\frac{\partial\omega}{\partial T}}{1 - \frac{k}{\omega}\frac{\partial\omega}{\partial x} + \frac{1}{4}\Big(-\frac{1}{4} - \frac{1}{\omega} + \frac{k^2}{\omega^2}\Big)\Big(\frac{\partial\omega}{\partial k}\Big)^2 + \frac{1}{2}\frac{\partial^2\omega}{\partial k^2}} \tag{4}$$

To derive a local volatility depending on ESSVI parameterization we will use the last formulation. In fact we have:

$$\frac{\partial\omega}{\partial T} \approx \frac{\omega(k,\theta_{T+h}) - \omega(k,\theta_T)}{h} \quad \text{for a small } h$$

and

$$\begin{aligned}
\frac{\partial\omega}{\partial k} &= 0.5\frac{\rho\psi + \psi(\psi k + \rho\theta)}{\sqrt{(\phi k + \rho\theta)^2 + (1-\rho^2)\theta^2}} \\
\frac{\partial^2\omega}{\partial k^2} &= -0.5\frac{(\phi^2(\rho^2 - 1)\theta^2)}{((\phi k + \rho\theta)^2 + (1-\rho^2)\theta^2)\sqrt{(\phi k + \rho\theta)^2 + (1-\rho^2)\theta^2}}.
\end{aligned}$$

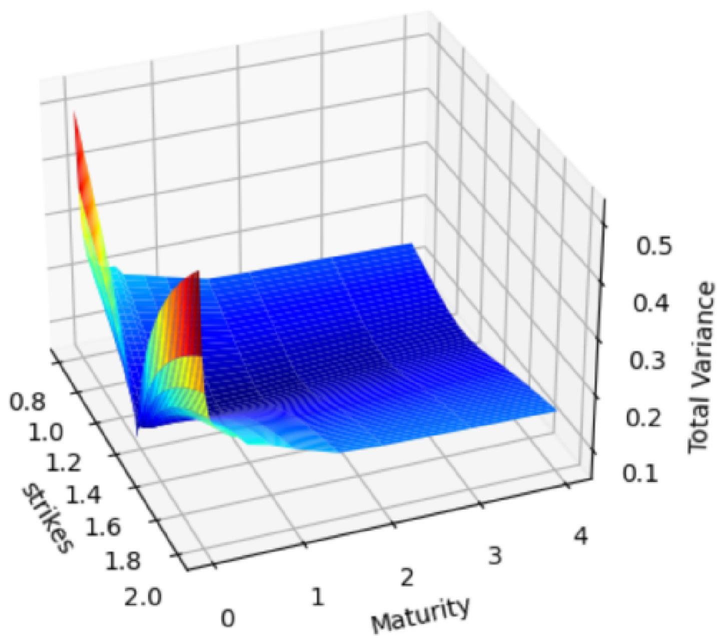
Knowing that the calibration is free arbitrage we are sure and certain that the nominator and the denominator of 4 are positives.

2.2 Arbitrage-free interpolation

Our starting point in this section is a set of SSVI slice parameters $(\theta_i, \psi_i, \rho_i)_{1\le i\le n}$, attached to maturities $0 < T_1 < \cdots < T_n$ and such that:

- Each slice is free of Butterfly arbitrage;
- There is no Calendar Spread arbitrage between any 2 consecutive slices.

Figure 3: Local Volatility Surface using ESSVI



We describe the interpolation scheme between 2 consecutive slices, which we denote by $(\theta_i, \psi_i, \rho_i)$ and$(\theta_{i+1}, \psi_{i+1}, \rho_{i+1})$. For $\lambda \in [0, 1]$ we define the following interpolation scheme:

- $\theta_\lambda = \lambda\theta_i + (1-\lambda)\theta_{i+1}$,
- $\psi_\lambda = \lambda\psi_i + (1-\lambda)\psi_{i+1}$,
- $\rho_\lambda = \lambda\rho_i + (1-\lambda)\rho_{i+1}$,

3 Conclusion

In this paper we studied the ESSVI model as implied volatility model: we started by the analytic part of the ESSVI model and next we established the characterization of the conditions of free arbitrage.
The main result in this paper is the implementation of a robust calibration method for the ESSVI model using SLSQP optimization method that allows an efficient calibration of a constrained optimization problem.