Aerodynamics Homework 04

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I. Question 1

The circulation around a vortex should not depend on the specific curve used in the following equation:

$$\Gamma = -\oint \vec{V} \cdot d\vec{s} = -\oint (udx + wdz)$$

A. Question 1.1 Problem Statement:

1. *Using the curve shown in the figure for part a, show that the circulation found using the square curve is equal to the circulation found using the circular curve found in class*

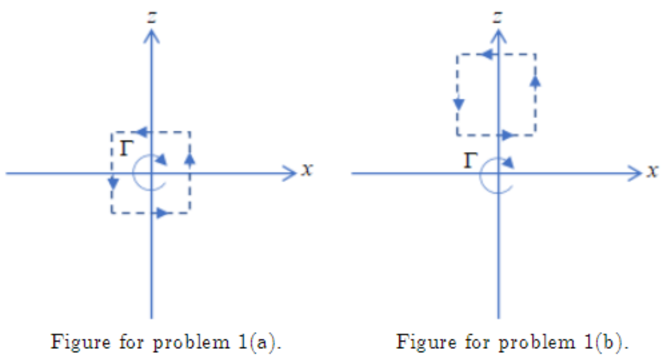

Figure 1

Assume: 2-D analysis, standard convention for defining ϕ, where Γ_v is negative

Given: Provided figures

Solution: To show that circulation is independent of the specific curved used, first consider the definition of the velocity potential ϕ of vortex flow Eq. [1.1.1], and its partial derivatives, Eq. [1.1.2], representing velocity in the radial and tangential directions:

$$\phi = -\frac{\Gamma_v}{2\pi}\theta \tag{1.1.1}$$

$$V_r = \frac{\partial \phi}{\partial r} = 0; V_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\Gamma_v}{2\pi r} \tag{1.1.2}$$

To find whether this is a valid velocity potential, we must determine if it satisfies the Laplace Equation in polar coordinates, Eq. [1.1.3]:

$$\frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0 \tag{1.1.3}$$

Computing the first and second partial derivatives of ϕ w.r.t. r and the second partial derivative of ϕ w.r.t. θ and plugging them into Eq. [1.1.3] shows that this definition of ϕ is valid and allows analysis of the circulation Γ around a surface.

$$\frac{\partial\phi}{\partial \mathrm{r}} = 0;\ \frac{\partial^2\phi}{\partial \mathrm{r}^2} = 0;\ \frac{\partial^2\phi}{\partial\theta^2} = 0$$

$$\frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{1}{r}\cdot 0 + 0 + \frac{1}{r^2}\cdot 0 = 0$$

Consider the equation for circulation Γ, Eq. [1.1.4], where the characteristic velocity is simply the velocity in a vortex, V_θ found in Eq. [1.1.2].

$$\Gamma = -\oint \overrightarrow{V_\theta}\cdot d\vec{s} \qquad [1.1.4]$$

This integration can then be performed around any closed path. The most intuitive path would simply be a circle, as the V_θ found above is defined with consideration for a polar coordinate system. Knowing that $d\vec{s}$ is equivalent to $r \cdot d\theta$, this can be rewritten and evaluated as seen below:

$$\Gamma = -\oint \overrightarrow{V_\theta}\cdot d\vec{s} = -\oint_0^{2\pi} -\frac{\Gamma_v}{2\pi r} r\ d\theta = \Gamma_v \cdot \frac{-2\pi r}{-2\pi r} = \Gamma_v$$

This evaluates simply to $\Gamma = \Gamma_v$, as expected. Now consider a square curve in cartesian coordinates. Converting [1.1.1] to cartesian coordinates yields the following, Eq. [1.1.5]:

$$\phi = -\frac{\Gamma_v}{2\pi}\tan^{-1}\left(\frac{z}{x}\right) \qquad [1.1.5]$$

Since it was already proven that this ϕ is a valid velocity potential, we can go ahead and take the partials to determine the velocity in the x and z directions:

$$u = \frac{\partial\phi}{\partial x} = \frac{\Gamma_v}{2\pi}\left(\frac{z}{x^2+z^2}\right);\ w = \frac{\partial\phi}{\partial z} = -\frac{\Gamma_v}{2\pi}\left(\frac{x}{x^2+z^2}\right)$$

Consider the equation for circulation Γ, Eq. [1.1.6], equivalent to Eq. [1.1.4], which has been converted for use with a function in cartesian coordinates:

$$\Gamma = -\oint (udx + wdz) \qquad [1.1.6]$$

When performing this integration along the closed square seen in Fig.[1.1], it expands to the expression seen in Eq. [1.1.7], where four separate traditional surface integrals are evaluated along each side of the square, equivalent to a single closed surface integral around the entire square. Note that the surfaces 1, 2, 3, and 4 refer to the bottom, right, top, and left sides of the square, respectively.

$$-\oint (udx + wdz) = -\int_1 u\ dx - \int_2 w\ dz - \int_3 u\ dx - \int_4 w\ dz \qquad [1.1.7]$$

Let each side of the square equal the constant $2a$. Since the square lies at the origin, we can define the following parameters for each surface:

$$d\vec{s} = \begin{cases} 1 & -a \le x \le a; \quad z = -a \\ 2 & -a \le z \le a; \quad x = a \\ 3 & \ \ a \le x \le -a; \quad z = a \\ 4 & \ \ a \le z \le -a; \quad x = -a \end{cases}$$

Substituting expressions for u, w, $d\vec{s}$, and known ranges and values yields the following:

$$\Gamma = \frac{\Gamma_v}{2\pi}\left[-\int_{-a}^{a}\left(\frac{-a}{x^2 + a^2}\right)\,dx - \int_{-a}^{a} -\left(\frac{a}{a^2 + z^2}\right)\,dz - \int_{a}^{-a}\left(\frac{a}{x^2 + a^2}\right)\,dx - \int_{a}^{-a} -\left(\frac{-a}{a^2 + z^2}\right)\,dz\right]$$

These integrals can then be computed. Note the similarity of each of the integrands, and knowing the antiderivative of which computes specifically to $arctan\left(\frac{(x,z)}{a}\right)$. Considering the limits of the integrals can also be simplified, the expression evaluates to the following:

$$\Gamma = \frac{\Gamma_v}{2\pi}\left[2\int_{-a}^{a}\left(\frac{a}{x^2 + a^2}\right)\,dx + 2\int_{-a}^{a}\left(\frac{a}{a^2 + z^2}\right)\,dz\right] = \frac{\Gamma_v}{\pi}\left[\arctan\left(\frac{\mathrm{x}}{\mathrm{a}}\right) + \arctan\left(\frac{z}{\mathrm{a}}\right)\right]\Bigg|_{-a}^{a} = \frac{\Gamma_v}{\pi}\cdot\pi = \Gamma_v$$

Just as with the circular curve, the circulation around the vortex is simply equal to the vortex strength.

B. Question 1.2 Problem Statement:

2. *Using the curve shown in the figure for part b, show that the circulation found using the square curve is equal to the circulation that which we found in class for a curve that does not enclose the origin. You may need to recall the following trigonometric identity:*

$$arctan(x) = \frac{\pi}{2} - \frac{1}{x}$$

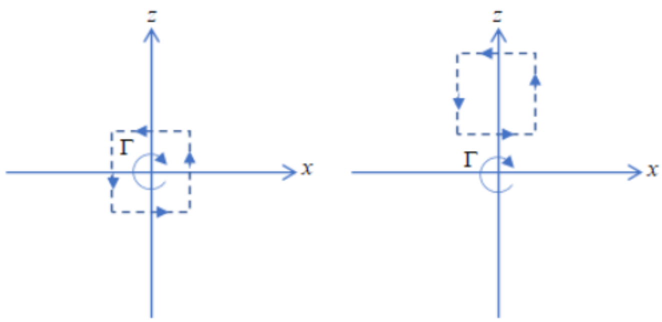

Figure for problem 1(a). Figure for problem 1(b).

Figure 1

Assume: 2-D analysis, standard convention for defining ϕ, where Γ_v is negative

Given: Provided figures

Solution: To show that circulation is independent of the specific curved used, *regardless of whether or not it encloses the point vortex*, first consider the definition of the velocity potential ϕ of vortex flow Eq. [1.2.1], and its partial derivatives, Eq. [1.2.2], representing velocity in the radial and tangential directions:

$$\phi = -\frac{\Gamma_v}{2\pi}\theta \qquad [1.2.1]$$

$$V_r = \frac{\partial\phi}{\partial r} = 0; V_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\Gamma_v}{2\pi r} \qquad [1.2.2]$$

To find whether this is a valid velocity potential, we must determine if it satisfies the Laplace Equation in polar coordinates, Eq. [1.2.3]:

$$\frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \qquad [1.2.3]$$

Computing the first and second partial derivatives of ϕ w.r.t. r and the second partial derivative of ϕ w.r.t. θ and plugging them into Eq. [1.2.3] shows that this definition of ϕ is valid and allows analysis of the circulation Γ around a surface.

$$\frac{\partial\phi}{\partial \mathrm{r}} = 0; \frac{\partial^2\phi}{\partial \mathrm{r}^2} = 0; \frac{\partial^2\phi}{\partial\theta^2} = 0$$

$$\frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{1}{r}\cdot 0 + 0 + \frac{1}{r^2}\cdot 0 = 0$$

Consider the equation for circulation Γ, Eq. [1.2.4], where the characteristic velocity is simply the velocity in a vortex, V_θ found in Eq. [1.2.2].

$$\Gamma = -\oint \overrightarrow{V_\theta} \cdot d\vec{s} \qquad [1.2.4]$$

This integration can then be performed around any closed path. Here, a path similar to the one shown in class will be used, defined in polar coordinates. The area enclosed spans π in the $\widehat{u_\theta}$ direction and r_1 to r_2 in the $\widehat{u_r}$ direction. Knowing that $d\vec{s}$ is equivalent to $r \cdot d\theta$, and the radial velocity is 0, this can be rewritten and evaluated as seen below:

$$\Gamma = -\oint \overrightarrow{V_\theta} \cdot d\vec{s} = -\oint_0^\pi -\frac{\Gamma_v}{2\pi r_1} r_1 \; d\theta - \oint_\pi^0 -\frac{\Gamma_v}{2\pi r_2} r_2 \; d\theta = \frac{\Gamma}{2\pi}[\pi - \pi] = 0$$

This evaluates simply to $\Gamma = 0$, as expected. Now consider a square curve in cartesian coordinates, which does not enclose the origin. Converting [1.2.1] to cartesian coordinates yields the following, Eq. [1.2.5]:

$$\phi = -\frac{\Gamma_v}{2\pi} \tan^{-1}\left(\frac{z}{x}\right) \qquad [1.2.5]$$

Since it was already proven that this ϕ is a valid velocity potential, we can go ahead and take the partials to determine the velocity in the x and z directions:

$$u = \frac{\partial \phi}{\partial x} = \frac{\Gamma_v}{2\pi}\left(\frac{z}{x^2 + z^2}\right); \; w = \frac{\partial \phi}{\partial z} = -\frac{\Gamma_v}{2\pi}\left(\frac{x}{x^2 + z^2}\right)$$

Consider the equation for circulation Γ, Eq. [1.1.6], equivalent to Eq. [1.1.4], which has been converted for use with a function in cartesian coordinates:

$$\Gamma = -\oint (udx + wdz) \qquad [1.2.6]$$

When performing this integration along the closed square seen in Fig.[1], it expands to the expression seen in Eq. [1.2.7], where four separate traditional surface integrals are evaluated along each side of the square, equivalent to a single closed surface integral around the entire square. Note that the surfaces 1, 2, 3, and 4 refer to the bottom, right, top, and left sides of the square, respectively.

$$-\oint (udx + wdz) = -\int_1 u \; dx - \int_2 w \; dz - \int_3 u \; dx - \int_4 w \; dz \qquad [1.2.7]$$

Let each side of the square equal the constant $2a$. Even though the square is no longer centered at the vortex, we can keep the same parameters for the surfaces, provided we shift our reference frame such that the vortex lies a distance greater than a directly below the origin. Conceptually, the problem does not change, however it allows a much easier evaluation. we can still define the following parameters for each surface:

$$d\vec{s} = \begin{cases} 1 & -a \le x \le a; \quad z = -a \\ 2 & -a \le z \le a; \quad x = a \\ 3 & \quad a \le x \le -a; \quad z = a \\ 4 & \quad a \le z \le -a; \quad x = -a \end{cases}$$

Substituting expressions for u, w, $d\vec{s}$, and known ranges and values yields the following. Considering the vortex is fully outside the enclosed surface, the velocities across parallel surfaces are equal and opposite when integrating using the same closed-curve convention:

$$\Gamma = \frac{\Gamma_v}{2\pi}\left[\int_{-a}^{a}\left(\frac{-a}{x^2 + a^2}\right) dx - \int_{-a}^{a}\left(\frac{a}{a^2 + z^2}\right) dz - \int_{a}^{-a}\left(\frac{a}{x^2 + a^2}\right) dx - \int_{a}^{-a} -\left(\frac{-a}{a^2 + z^2}\right) dz\right]$$

These integrals can then be computed. Note the similarity of each of the integrands, and knowing the antiderivative of which computes specifically to $arctan\left(\frac{(x,z)}{a}\right)$, although this is not even necessary, as manipulating the expressions algebraically allows the cancellation of each integral term. Considering the limits of the integrals can also be simplified, the expression evaluates to the following:

$$\Gamma = \frac{\Gamma_v}{2\pi}\left[0\int_{-a}^{a}\left(\frac{a}{x^2 + a^2}\right)\,dx + 0\int_{-a}^{a}\left(\frac{a}{a^2 + z^2}\right)\,dz\right] = \frac{\Gamma_v}{\pi}\cdot 0 = 0$$

Just as with the circular curve, the circulation around a surface that does not enclose the vortex is simply equal to 0.

II.Question 2

The British Royal Air Force cleverly used the Magnus Effect during World War II in the bombing of German dams. The idea was to put backspin on a cylindrical bomb as it was released from a Lancaster bomber so that it would travel a certain distance prior to hitting the target. Assume that the bomb is released from the aircraft with a forward velocity of 390 km/hr. at sea level. The cylinder has a diameter of 120 cm and length of 150 cm. It has a mass of 4500 kg.

A. Question 1 Problem Statements:

1. *What would be the angular velocity* **Ω** *required in order that the cylinder generate lift equal to its weight? Find the instantaneous locations of the stagnation points.*

Assume: 2-D analysis, ideal flow (steady, incompressible, inviscid), $g = 9.81 m/s^2$, $\rho = 1.225 kg/m^3$

***Given*:** $\boldsymbol{V_\infty = 390km/hr, d = 120cm, l = 150cm, M = 4500kg}$

***Solution*:** To derive an equation that can be used to determine lift as a function of Ω, we must first model the potential flow over the body. Consider Eq. [2.1]. This is the equation given in class that models the ideal flow over a cylinder. It takes advantage of the fact that the Laplace Equation is linear, and multiple fundamental solutions can be combined simply through addition and still satisfy the necessary continuity conditions. It is comprised of the velocity potential for steady uniform flow, ϕ_∞, and a doublet, which combines a source and a sink together at a singularity by evaluating the combination of these two fundamental solutions as the distance between them approaches 0.

$$\phi = \frac{m}{2\pi}\frac{\cos\theta}{r} + V_\infty r \cos\theta \quad [2.1]$$

To model the effect of the flow over a cylinder that is spinning, another potential flow term is added. This is the fundamental solution that represents a point vortex, and contributes irrotational vortex potential. This allows us to model the Magnus Effect in an ideal flow state. The final ϕ can be seen below, Eq. [2.2]:

$$\phi = \frac{m}{2\pi}\frac{\cos\theta}{r} + V_\infty r \cos\theta - \frac{\Gamma_v}{2\pi}\theta \quad [2.2]$$

Let us also determine the ψ for this flow. Following the same steps above, starting with the stream function found in class, and adding the point vortex stream function, the final stream function is seen in Eq. [2.3]:

$$\psi = -\frac{m}{2\pi}\frac{\sin\theta}{r} + V_\infty r \sin\theta + \frac{\Gamma_v}{2\pi} ln(r) \quad [2.3]$$

We can then find the velocity in the $\widehat{u_r}$ and $\widehat{u_\theta}$ directions by computing the partial derivatives of ϕ w.r.t. r and θ. These computations result in the following velocity components, seen below in Eq. [2.4] and Eq. [2.5]:

$$u_r = V_\infty \cos\theta - \frac{m}{2\pi}\frac{\cos\theta}{r^2} + 0 \quad [2.4]$$

$$u_\theta = -V_\infty \sin\theta - \frac{m}{2\pi}\frac{\sin\theta}{r^2} - \frac{\Gamma_v}{2\pi r} \quad [2.5]$$

Next, we may consider the reality of the situation we have modeled a flow for and apply some relevant boundary conditions. Knowing that $\psi = 0$ represents the physical boundary of the object, solving $\psi = 0$ for r should give us some symbolic r_o which represents the physical surface of the bomb. This is an important step, as we will need to analyze the pressure over the surface to find a function for lift. However, an issue arises, as the function ψ cannot be solved to find an explicit r_o due to the logarithmic term. This was confirmed using the MATLAB symbolic math toolbox running the code in appendix A. To find r_o, we consider that the velocity in the radial direction on the surface of the cylinder *must* equal zero, as there cannot be flux through a solid surface. Note that simply solving ψ without the term for the point vortex returns the same answer, confirming the assumption made above. Solving $u_r = 0$ and Eq. [2.3] without $\frac{\Gamma}{2\pi} ln(r)$ for r can be seen below:

$$u_{\mathrm{r}} = cos\,\theta \left[V_\infty r^2 - \frac{m}{2\pi}\right] = 0; {r_o}^2 = \frac{m}{2\pi V_\infty}$$

$$\psi = \sin\theta \left[V_\infty r^2 - \frac{m}{2\pi}\right] = 0; {r_o}^2 = \frac{m}{2\pi V_\infty}$$

Considering that $u_{\mathrm{r}} = 0$ at every location on the surface $r = r_o$, we can evaluate the flow over the surface entirely by using only the u_θ term where $r = r_o$. To find the stagnant points, i.e. the locations on the surface where there is no flow velocity, we can solve $u_\theta(r = r_o) = 0$ for values of θ, resulting in Eq. [2.6]:

$$u_\theta = -V_\infty\, sin\,\theta - \frac{m}{2\pi}\frac{sin\,\theta}{{r_o}^2} - \frac{\Gamma_v}{2\pi r_o} = -\sin\theta\left[V_\infty + \frac{m}{2\pi {r_o}^2}\right] - \frac{\Gamma_v}{2\pi r_o} = 0$$

$$u_\theta = -\sin\theta\,[V_\infty + V_\infty] - \frac{\Gamma_v}{2\pi r_o} = -2V_\infty\sin\theta - \frac{\Gamma_v}{2\pi r_o} = 0 \qquad [2.6]$$

The resulting Eq. [2.6] is left in terms of only values that are either known or will be solved for as a function of lift. Next, we consider the Bernoulli Equation, which holds in this ideal flow, Eq. [2.7]

$$P_\infty + \frac{1}{2}\rho V_\infty^2 = P + \frac{1}{2}\rho V^2 \qquad [2.7]$$

From this equation, we can find pressure and the pressure coefficient in the typical way, seen in Eq. [2.8-2.9]:

$$P = P_\infty + \frac{1}{2}\rho(V_\infty^2 - V^2) \qquad [2.8]$$

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 - \frac{V^2}{V_\infty^2} \qquad [2.9]$$

Knowing that V is simply our simplified u_θ, we can substitute Eq. [2.6] into Eq. [2.8-2.9] for V, seen in Eq. [2.9-2.10]:

$$P = P_\infty + \frac{1}{2}\rho\left(V_\infty^2 - \left[-2V_\infty\sin\theta - \frac{\Gamma}{2\pi r_o}\right]^2\right) \qquad [2.10]$$

$$C_p = 1 - \frac{V^2}{V_\infty^2} = 1 - \frac{1}{V_\infty^2}\left[-2V_\infty\sin\theta - \frac{\Gamma_v}{2\pi r_o}\right]^2 \qquad [2.11]$$

To determine the force of lift, we must determine the force in the z direction, Eq. [2.12]:

$$L = F_z = -\int_0^{2\pi} P\hat{n}\cdot\hat{k}\, r_o d\theta = -\int_0^{2\pi} P_\infty + \frac{1}{2}\rho\left(V_\infty^2 - \left[-2V_\infty\sin\theta - \frac{\Gamma}{2\pi r_o}\right]^2\right)\sin\theta\, r_o d\theta \qquad [2.12]$$

This integral is proved to evaluate to Eq. [2.13] under the Kutta-Zhukhousky lift theorem. Details of the integration will be skipped for sanity:

$$L = \rho V_\infty \Gamma \qquad [2.13]$$

The lift L must be equal to the weight of the bomb, $M \cdot g$. Knowing density and free-stream velocity, the required Γ is solved for:

$$L = 4500 \cdot 9.81 = 44145 = 1.225 \cdot 108.33 \cdot \Gamma$$

$$\Gamma = 332.657$$

This Γ is then plugged into Eq. [2.6], isolating the term with gamma to determine what angular velocity Ω is required to achieve this speed at the surface $r = r_o$.

$$\Omega = \frac{\Gamma}{2\pi r_o} = 88.24 rad/s$$

To find the stagnation points, Eq. [2.6] is solved for θ when it is equal to 0:

$$-2V_\infty \sin\theta - \frac{\Gamma_v}{2\pi r_o} = 0;\ -2V_\infty \sin\theta = \frac{\Gamma_v}{2\pi r_o}, \sin\theta = \frac{\Gamma_v}{-2\pi r_o \cdot 2V_\infty}$$

$$\theta = \pm 24.033°$$