High Speed Aerodynamics Homework 03

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I. Part 1

1. **Question 1:**

A shock tube used for studying chemical kinetics will generally use hydrogen or helium as the driver gas and the reactant diluted in an inert gas such as Argon in the low-pressure region. It is desired to generate a test region with low velocity but very high temperature. For this problem, the location of the diaphragm is at $x = 0$. The gas on one side of the diaphragm ($x < 0$) is helium ($\gamma = 1.66,\ R = 2077\, J/kg \cdot K$) pressurized to $400\, kPa$ and heated to $400\, K$ while the gas on the other side ($x < 0$) is argon ($\gamma = 1.667,\ R = 208\, J/kg \cdot K$) at $20\, kPa$ and $15\, °C$. The diaphragm ruptures at $t = 0\, s$.

a) *Find the location of the normal shock wave, the contact surface, and the expansion region (head and tail) at $t = 1\, ms$. Assume the shock tube is long enough that there have been no reflections by this time*

b) *Plot the gas velocity, the pressure and the temperature as a function of x at $t = 1\, ms$.*

Assume: Calorically ideal argon and helium. 1-D adiabatic flow. Long shock tube.

Given: $\gamma_h = 1.66,\ R_h = 2077\, J/kg \cdot K,\ \gamma_a = 1.667,\ R_a = 208\, J/kg \cdot K,\ p_h = 400 \cdot 10^3\, Pa,\ T_h = 400\, K,\ p_a = 20 \cdot 10^3\, Pa, T_h = 288.15\, K$

Solution:

To derive the necessary relations, consider the mass continuity, momentum, and energy equations for one dimensional flow:

$$\rho_1 u_1 = \rho_2 u_2$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$

$$h_1 + u_1^2/2 = h_2 + u_2^2/2$$

When considering a moving shock made to be stationary in the lab frame, the velocities u_1 and u_2 can be replaced by the 'upstream' velocity, i.e. shock speed W, and the 'downstream' velocity, i.e. the shock speed minus the processed velocity – also known as the contact surface velocity u_p:

$$\rho_1 W = \rho_2 (W - u_p) \quad [1.1]$$

$$p_1 + \rho_1 W^2 = p_2 + \rho_2 (W - u_p)^2 \quad [1.2]$$

$$h_1 + W^2/2 = h_2 + (W - u_p)^2/2 \quad [1.3]$$

These equations govern normal shocks moving into stagnant gas. Eq. [1.1] can be rearranged for shock speed, contact surface speed, and the differential speed:

$$W - u_p = W \cdot \frac{\rho_1}{\rho_2} \tag{1.1.1}$$

$$W = \left(W - u_p\right) \cdot \frac{\rho_2}{\rho_1} \tag{1.1.2}$$

$$u_p = W\left(1 - \frac{\rho_1}{\rho_2}\right) \tag{1.1.3}$$

Eq. [1.1.1] can be used in Eq. [1.2], and then rearranging for shock speed W:

$$p_1 + \rho_1 W^2 = p_2 + \rho_2 W^2 \cdot \left(\frac{\rho_1}{\rho_2}\right)^2$$

$$p_2 - p_1 = W^2\left[\rho_1 - \rho_2\left(\frac{\rho_1}{\rho_2}\right)^2\right] \cdot \frac{\rho_1}{\rho_1} = \rho_1 W^2\left[1 - \frac{\rho_1}{\rho_2}\right]$$

$$W^2 = \frac{p_2 - p_1}{\rho_1\left[1 - \frac{\rho_1}{\rho_2}\right]} = \frac{p_2 - p_1}{\rho_2 - \rho_1} \cdot \frac{\rho_2}{\rho_1} \tag{1.4}$$

Using Eq. [1.1.2] in Eq. [1.4]:

$$\left(W - u_p\right)^2 \cdot \left(\frac{\rho_2}{\rho_1}\right)^2 = \frac{p_2 - p_1}{\rho_2 - \rho_1} \cdot \frac{\rho_2}{\rho_1}$$

$$\left(W - u_p\right)^2 = \frac{p_2 - p_1}{\rho_2 - \rho_1} \cdot \frac{\rho_1}{\rho_2} \tag{1.5}$$

With Eq. [1.4] for shock speed and Eq. [1.5] for velocity differential, substitute into Eq. [1.3], replacing the enthalpy terms h with the equivalent expression $e + \frac{p}{\rho}$:

$$e_1 + \frac{p_1}{\rho_1} + \frac{W^2}{2} = e_2 + \frac{p_2}{\rho_2} + \frac{\left(W - u_p\right)^2}{2}$$

$$e_1 + \frac{p_1}{\rho_1} + \frac{\frac{p_2 - p_1}{\rho_2 - \rho_1} \cdot \frac{\rho_2}{\rho_1}}{2} = e_2 + \frac{p_2}{\rho_2} + \frac{\frac{p_2 - p_1}{\rho_2 - \rho_1} \cdot \frac{\rho_1}{\rho_2}}{2}$$

This can be simplified to obtain Eq. [1.6] for the internal energy differential $e_2 - e_1$, where specific volume v is the inverse of density:

$$e_2 - e_1 = \frac{p_1 + p_2}{2} \cdot (v_1 - v_2) \tag{1.6.1}$$

This is the Hugoniot equation derivation w.r.t. parameters associated with moving shocks instead of stationary ones; they're the same, nothing different is happening thermodynamically. Assuming ideal gases, where $e = c_v T$ and $v = \frac{RT}{p}$, temperature and density relations can be derived in terms of the ratio of pressures $\frac{p_2}{p_1}$ rather than Mach number, and the ratio of specific heats γ:

$$c_v T_2 - c_v T_1 = \frac{p_1 + p_2}{2} \cdot \left(\frac{RT_1}{p_1} - \frac{RT_2}{p_2}\right)$$

Knowing $c_v = \frac{R}{\gamma - 1}$ and simplifying:

$$\frac{RT_2}{\gamma-1}-\frac{RT_1}{\gamma-1}=\frac{p_1+p_2}{2}\cdot\left(\frac{RT_1}{p_1}-\frac{RT_2}{p_2}\right)$$

$$\frac{T_2}{T_1}=\frac{\left(\frac{p_2}{p_1}+\frac{1}{\gamma-1}+1\right)}{\left(\frac{p_1}{p_2}+\frac{1}{\gamma-1}+1\right)}=\frac{p_2}{p_1}\cdot\frac{\gamma p_1-p_2+\gamma p_2}{\gamma p_1-p_1+\gamma p_2}=\frac{p_2}{p_1}\cdot\left[\frac{\frac{\gamma+1}{\gamma-1}+\frac{p_2}{p_1}}{1+\frac{\gamma+1}{\gamma-1}\cdot\frac{p_2}{p_1}}\right] \qquad [1.6.2]$$

Using this in the equation of state for an ideal gas, $\rho=\frac{P}{RT}$:

$$\frac{\rho_2}{\rho_1}=\frac{p_2}{p_1}\cdot\frac{T_1}{T_2}=\frac{1+\frac{\gamma+1}{\gamma-1}\cdot\frac{p_2}{p_1}}{\frac{\gamma+1}{\gamma-1}+\frac{p_2}{p_1}} \qquad [1.6.3]$$

Consider this Mach number to be the shock Mach number:

$$M_s=\frac{\mathrm{W}}{a_1} \qquad [1.7]$$

Where flow speed u is replaced by shock speed W, and a_1 is the upstream speed of sound or the speed of sound in the stationary argon ahead of the shock, given by:

$$a_1=\sqrt{\gamma_a RT_a}=316\,m/s$$

Since we've proven we can use normal shock relations across the moving shock wave, take the normal shock pressure ratio, and for the Mach number in terms of the specific heat of the driven fluid, argon, and the pressure ratio. Initially at $t=0$, the pressure ratio can be assumed to be 10, for a strong shock:

$$\frac{p_2}{p_1}=1+\frac{2\gamma}{\gamma+1}\cdot(M_s^2-1)$$

$$M_s=\sqrt{\frac{\gamma+1}{2\gamma}\cdot\left(\frac{p_2}{p_1}-1\right)+1}=5.29$$

Equating Eq. [1.9] to Eq. [1.7] allows us to solve for shock speed W in terms of the specific heat of the driven fluid, argon, and the pressure ratio:

$$\frac{\mathrm{W}}{a_1}=\sqrt{\frac{\gamma+1}{2\gamma}\cdot\left(\frac{p_2}{p_1}-1\right)+1}$$

$$W=a_1\cdot\sqrt{\frac{\gamma+1}{2\gamma}\cdot\left(\frac{p_2}{p_1}-1\right)+1}=1{,}671.7\,m/s$$

Finally, substituting Eq. [1.10] and Eq. [1.6.2] into Eq. [1.1.3] for contact surface speed:

$$u_p=W\left(1-\frac{\rho_1}{\rho_2}\right) \qquad [1.1.3]$$

$$u_p=W\cdot\left(1-\left(\frac{1+\frac{\gamma+1}{\gamma-1}\cdot\frac{p_2}{p_1}}{\frac{\gamma+1}{\gamma-1}+\frac{p_2}{p_1}}\right)^{-1}\right)$$

$$u_p = \frac{a_1}{\gamma} \cdot \left(\frac{p_2}{p_1} - 1\right) \sqrt{\frac{\frac{2\gamma}{\gamma+1}}{\frac{p_2}{p_1} + \frac{\gamma-1}{\gamma+1}}} = 595.813\, m/s$$

The head of the expansion wave travels at the speed of sound of the fluid it is moving into, i.e. helium:

$$V_h = -a_h = \sqrt{\gamma_h R_h T_h} = 1{,}1744\, m/s$$

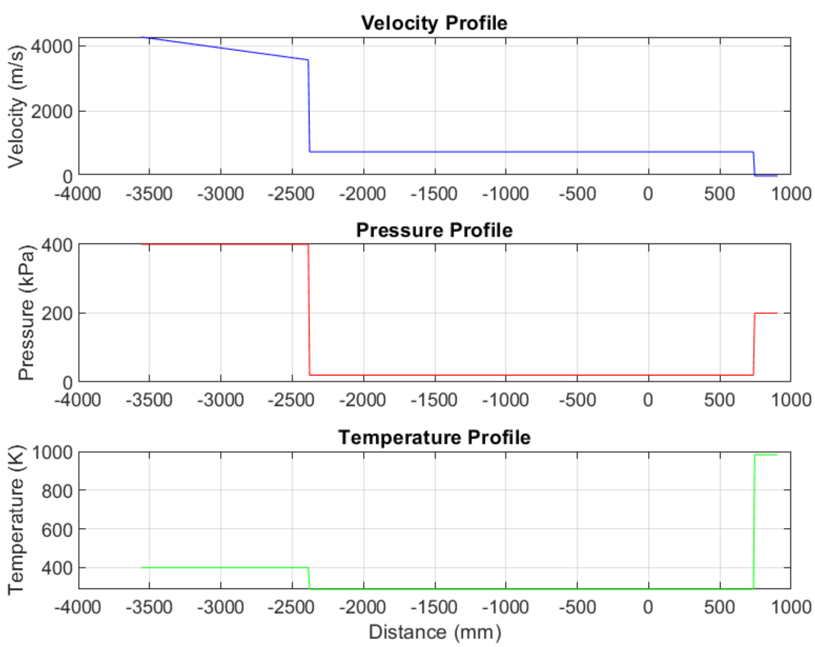
Relative to the head of the expansion wave, the tail moves away from it at the contact surface speed. In the lab frame, the speed of the expansion wave tail is the post-shock velocity minus the local speed of sound:

$$V_t = 462.8879 - 980.0232 = -517.1354\, m/s$$

The locations of these waves at $t = 0.001\, s$ can be determined generally as $V \cdot t$. These results and the plot are in the results section below.

Results:

$$x_{shock} = 1.6717\, m$$
$$x_{contact} = 0.5958\, m$$
$$x_{head} = -1.1744\, m$$
$$x_{tail} = 0.7115\, m$$



Discussion:

Honestly the profile graphs did not come out right. The plot above is kinda hardcoded but it captures the essence of what is happening.

2. **Question 2:**

   Air at $0°C$ and 150 kPa is flowing out of a duct at speed u_1. A valve at the end of the duct is suddenly closed so that the velocity of the air at the closed end must instantaneously reduce from u_1 to zero. This leads to the formation of a normal shock wave that travels back down the duct. Note that region 1 is to the left of the shock wave and region 2 is to the right of the shock wave once the valve is closed.

   a) *Explain why the shock wave shown in the figure must form and travel down the duct in the direction opposite that of the original flow. That is, why is there a shock wave and why is V_s in the opposite direction to u_1?*
   b) *A pressure sensor just in front of the valve (inside the duct) reads $400\ kPa$ after the valve is closed. Find the Mach number of the flow in region 1 relative to the shock wave. Find the Mach number of the flow in region 2 relative to the shock wave.*
   c) *Find the shock velocity, V_s, relative to the duct. Find the original air flow velocity, u_1, relative to the duct.*

Assume: Calorically ideal air. 1-D steady flow. Adiabatic. Valve closes instantly

Given: $\gamma = \gamma_{air} = 1.4$, $R = R_{air} = 287\, J/kg \cdot K$

Solution:

See Discussion below for (a). To find the Mach number in regions 1 and 2 relative to the shock wave, normal shock relations and thermodynamic properties can be used across the shock. The static pressure ratio across the shock is known from the given pressures in region 1 and immediately after the valve is closed in region 2, when the shock wave forms. The Mach number in region 1 is found from normal shock relations as:

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1} \cdot (M_1^2 - 1) \xrightarrow{yields} M_1 = \sqrt{\frac{2}{\gamma - 1}\left(\frac{p_2}{p_1} - 1\right) + 1}$$

$$M_1 = \sqrt{\frac{2}{1.4-1}\left(\frac{400 \cdot 10^3}{150 \cdot 10^3} - 1\right) + 1} = 3.0551$$

The Mach number in region 2 can be found using the familiar normal shock relations:

$$M_2 = \sqrt{\frac{1 + \frac{\gamma - 1}{2} \cdot M_1^2}{\gamma M_1^2 - \frac{\gamma - 1}{2}}}$$

$$M_2 = \sqrt{\frac{1 + \frac{1.4 - 1}{2} \cdot 3.06^2}{\gamma \cdot\ 3.06^2 - \frac{1.4 - 1}{2}}} = 0.472$$

To determine the shock velocity V_s and original flow velocity u_1 relative to the duct, first consider the stationary lab/duct frame. Here, the shock moves backwards, and the air is moving forwards. In the shock frame, the shock does not move – the oncoming velocity relative to the shock is the difference between the flow speed u_1 and the shock speed V_s. Note that also in this frame, the shock velocity is positive, and the oncoming flow velocity is negative, opposite to the duct frame. This is the frame for which the Mach numbers above are calculated. Using mass conservation:

$$\rho_1 \cdot (V_s - u_1) = \rho_2 \cdot (V_s - u_2)$$

$$V_s - u_1 = u_{s_1} = M_1 a_1$$

Where:

$$a_1 = \sqrt{\gamma R T_1} = \sqrt{1.4 \cdot\ 287 \cdot\ 273.15} = 331.2879\, m/s$$

So:

$$u_{s_1} = 3.0551 \cdot 331.2879 = 1{,}012.1\, m/s$$

In the lab frame, the post-shock air does not have velocity. In the shock frame however, the post-shock air does have a velocity, and it's Mach number is known, thus:

$$V_s - u_2 = u_{s_2} = M_2 a_2$$

Where a_2 can be found from the temperature in region 2, computed from normal shock relations:

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} \cdot \frac{\rho_1}{\rho_2} = \frac{p_2}{p_1} \cdot \frac{2 + (\gamma - 1)M_1^2}{(\gamma + 1)M_1^2}$$

$$\frac{T_2}{T_1} = 2.7444 \xrightarrow{yields} T_2 = 749.6269\, K$$

Thus:

$$a_2 = \sqrt{\gamma R T_2} = 548.817\, m/s$$

And:

$$u_{s_2} = 0.472 \cdot 584.817 = 259.0497\, m/s$$

Since the post-shock air does not have velocity in the lab frame:

$$u_{s_2} = V_s - 0 = V_s$$

$$V_s = 259.0497\, m/s$$

And:

$$u_1 = V_s - u_{s_1} = 259.0497 - 1{,}012.1 = -753.0515\, m/s$$

Noting that the sign convention is still flipped, the results present the values w.r.t. the duct.

Results:

$$M_1 = 3.0551$$

$$M_2 = 0.472$$

$$V_s = 259.0497\, m/s$$

$$u_1 = 753.0515\, m/s$$

Discussion:

When the valve is closed, the physical barrier forces the fluid to go from u_1 to zero. The kinetic energy associated with the mass flow of velocity u_1 must be converted into another form to conform to the first law of thermodynamics and continuity. Under the adiabatic flow conditions, consider the familiar energy conservation equation:

$$c_v T_1 + p_1/\rho_1 + u_1^2/2 = c_v T_2 + p_2/\rho_2 + u_2^2/2$$

Or, after simplification:

$$c_p T_1 + u_1^2/2 = c_p T_2 + u_2^2/2$$

This is just to say that the total temperatures T_{0_1} and T_{0_2} must be equal. Thus, there must be an increase in static temperature and pressure from region 1 to 2. This happens instantly initially, as stated in the problem. The process in which flow parameters must change irreversibly and abruptly due to discontinuous conditions induces a shock wave. This phenomenon works to increase pressure, temperature, and density, while decreasing flow velocity, at the cost of entropy generation. The fluid upstream & unprocessed by the shock will be lower pressure, temperature, density, and higher velocity than the air behind. In the scenario presented, the fluid in region 1 must be the upstream condition, with higher velocity and lower pressure than the initially stagnant air in region 2. The shock wave must travel in the direction of low pressure and high velocity, as it 'carries information' about the pressure increase. Here, that will always be the direction opposite of u_1.

As for the relativity discussion, it is important to note that attention must be paid to sign convention. When analyzing in the shock frame, the incoming flow velocity is more intuitively thought of as the sum of the magnitudes of shock velocity and incoming flow, like the relative velocity of two cars driving towards each other. This also clarifies the post shock velocity – it is known that the induced mass velocity behind a moving shock cannot be zero. This must hold true in the shock frame, not necessarily in the lab frame. Thus the computed Mach number in region two, combined with the known temperature ratio and pressure ratio across the shock, *and* the fact that $u_2 = 0$ in the lab frame allows us to find the relative velocity behind the shock as the shock speed itself.

II. Appendix

```
%% Header
% Author: Zakary Steenhoek
% Date:
% Title:

% Reset
clc; clear;

%% Setup

% Variables
% syms T1 T2 p1 p2 gamma
gamma1 = 1.667;  % Specific heat ratio for Argon (Driven gas)
R1 = 208;  % Gas constant for Argon (J/kg*K)
P1 = 20e3;  % Initial pressure of Argon (Pa)
T1 = 288;  % Initial temperature of Argon (K)
gamma4 = 1.66;  % Specific heat ratio for Helium (Driver gas)
R4 = 2077;  % Gas constant for Helium (J/kg*K)
P4 = 400e3;  % Initial pressure of Helium (Pa)
T4 = 400;  % Initial temperature of Helium (K)
t = 1e-3;  % Time in seconds (1 ms)

% Equations

% Functions

%% Math

% Compute Speed of Sound in Region 1 (Pre-shock Argon)
a1 = sqrt(gamma1 * R1 * T1)
% Pr = P4/P1
Pr = 10;

% Solve for M1 using normal shock relations
M1 = sqrt((2/(gamma1-1))*(Pr-1)+1)
W = a1*M1
[PR, DR, TR, M2] = normalShockRelations(M1, gamma1);
T2 = T1*TR
P2 = P1*PR
a2 = sqrt(gamma1 * R1 * T2)
u2 = a2*M2;  % Post-shock velocity

up = a1/gamma1*(Pr-1)*sqrt((((2*gamma1)/(gamma1+1))/(Pr+((gamma1-1)/(gamma1+1))))

% Compute Expansion Fan Velocities
a4 = sqrt(gamma4 * R4 * T4);  % Speed of sound in helium]
nu4 = 2 * a4 / (gamma4 - 1);
exp_head_speed = -a4;  % Expansion fan head speed
exp_tail_speed = u2-a4;  % Expansion fan tail speed


% Compute Positions at t = 1 ms
x_shock = W*t;
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```
x_contact = up*t;
x_exp_head = exp_head_speed * t;
x_exp_tail = exp_tail_speed * t;

%% Plots

% Define x positions for plotting
x_positions = linspace(-x_exp_tail, x_shock, 500);
velocity_profile = arrayfun(@(x) piecewise_velocity(x, x_exp_tail, x_exp_head, x_contact, x_shock, nu4, u2),
x_positions);
pressure_profile = arrayfun(@(x) piecewise_pressure(x, x_exp_head, x_contact, x_shock, P4, P1, P2), x_positions);
temperature_profile = arrayfun(@(x) piecewise_temperature(x, x_exp_head, x_contact, x_shock, T4, T1, T2),
x_positions);

% Plot results
figure;
subplot(3,1,1);
plot(x_positions * 1000, velocity_profile, 'b');
ylabel('Velocity (m/s)');
title('Velocity Profile');
grid on;

subplot(3,1,2);
plot(x_positions * 1000, pressure_profile / 1000, 'r');
ylabel('Pressure (kPa)');
title('Pressure Profile');
grid on;

subplot(3,1,3);
plot(x_positions * 1000, temperature_profile, 'g');
ylabel('Temperature (K)');
xlabel('Distance (mm)');
title('Temperature Profile');
grid on;

% Piecewise function for velocity profile
function v = piecewise_velocity(x, x_exp_tail, x_exp_head, x_contact, x_shock, nu4, u3)
  if x < x_exp_head
    v = (x - x_exp_tail) / (x_exp_head - x_exp_tail) * nu4;
  elseif x >= x_exp_head && x < x_contact
    v = u3;
  else
    v = 0;
  end
end

% Piecewise function for pressure profile
function p = piecewise_pressure(x, x_exp_head, x_contact, x_shock, P4, P1, P2)
  if x < x_exp_head
    p = P4;
  elseif x >= x_exp_head && x < x_contact
    p = P1;
  else
    p = P2;
  end
end
```

```
% Piecewise function for temperature profile
function T = piecewise_temperature(x, x_exp_head, x_contact, x_shock, T4, T1, T2)
    if x < x_exp_head
        T = T4;
    elseif x >= x_exp_head && x < x_contact
        T = T1;
    else
        T = T2;
    end
end

%% Header
% Author: Zakary Steenhoek
% Date:
% Title:

% Reset
clc; clear;

%% Setup

% Variables
syms
gamma = 1.4;
R = 287;
p1 = 150E3;
p2 = 400E3;
T1 = 273.15;

% Equations

% Functions
soundSpd = @(GAMMA,R,T) sqrt(GAMMA*R*T);

%% Math

% Speed of sound
a1 = sqrt(gamma*R*T1);

% Mach numbers
M1 = sqrt((2/(gamma-1))*(p2/p1-1)+1)

% Ratios across the shock
[PR, DR, TR, M2] = normalShockRelations(M1, gamma);
T2 = T1*TR;
a2 = sqrt(gamma*R*T2);



% Shock frame incoming air velocity
us1 = M1*a1
us2 = M2*a2
Vs = us2;
u1 = Vs;
```