High Speed Aerodynamics Homework 02

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I. Part 1

1. Question 1:

Consider the liquid rocket motor you analyzed in homework assignment #1. The exit pressure is much less than the atmospheric pressure (back pressure), so something must occur in the flow outside the nozzle to equalize pressure. Your classmate speculates that, because we know that the pressure increases across a shock wave, a normal shock might occur just past the nozzle exit.

a) Find the Mach number, velocity, temperature, and pressure in the flow downstream of a normal shock that would occur in this situation.

b) Considering your results, do you think your classmate's hypothesis is correct?

Assume:

From the previous assignment, used to solve conservation equations: Liquid fuel & oxidizer treated as reservoir, steady flow state, calorically perfect exit gas, circular nozzle cross-section, neglect viscous, gravitational, and electromagnetic forces at exit, isolated - no external body forces or work, insulated – adiabatic

For this assignment, used to solve and predict exit shock: Supersonic exit flow (found in HW01), overexpanded flow condition at exit ($p_\infty > p_e$),

Given: $T_H = 20K,\ T_O = 90K,\ \overrightarrow{V_{in}} = 0,\ c_H = 9668\,J/kg\cdot mol,\ c_O = 347\,J/kg\cdot K,\ M = 7.4\,kg/kg\cdot mol,\ C_p = 4168\,J/kg\cdot K,\ \ \mathcal{R} = 8314\,J/(kg\cdot mol\cdot K),\ \ P_{atm} = 70\,kPa,\ \ \dot{m}_1 = 140.3\,kg/s,\ \ \dot{m}_2 = 374.2\,kg/s,\ \ P_3 = 30\,kPa, T_3 = 500\,K, d_3 = 2.16\,m$

Solution:

To find the downstream conditions, the pressure, temperature, and density relations derived in the previous homework assignment can be used, along with the normal shock Mach relation. First, the ratio of specific heats for the exit gas, γ_e, must be determined. This should be close to the ratio of specific heats used for air. To check this, the given specific heat, c_p, and the calculated specific gas constant, R, can be used with known thermodynamic relations:

$$R = c_p - c_v \xrightarrow{yields} \mathrm{c_v = c_p - R}$$

And:

$$\gamma = \frac{c_p}{c_v}$$

Thus:

$$\gamma = \frac{c_p}{c_p - R}$$

From the previous homework:

$$R = \frac{\mathcal{R}}{M} = 1123.51\ \frac{J}{kg \cdot K}$$

The other unknown quantity is given, $c_p = 4168$. Thus:

$$\gamma_e = \frac{4168}{4168 - 1123.51} = 1.369$$

These quantities can then be used to find the speed of sound at the exit, a_∞, using the known speed of sound equation and the temperature at the exit:

$$a_e = \sqrt{\gamma R T_e} = \sqrt{1.369 \cdot 1123.51 \cdot 500} = 876.96\ \frac{m}{s}$$

This can then be used with the exit velocity computed in the previous homework assignment, which is verified as correct in the figure provided in this homework assignment as $V_e = 2629.1\ \frac{m}{s}$, to determine the upstream (exit) Mach number:

$$M_e = \frac{V_e}{a_e} = \frac{2629.1}{876.96} = 3.00$$

Treating the exit conditions as the upstream conditions, i.e., $M_1 = M_e = 3.00$ and $\gamma = \gamma_e = 1.369$, the normal shock relations can then be applied. These relationships yield the following ratios, and are as seen below:

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1) = 10.23$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2} = 4.00$$

$$M_2^2 = \frac{1 + \frac{\gamma - 1}{2}M_1^2}{\gamma M_1^2 - \frac{\gamma - 1}{2}} \xrightarrow{yields} M_2 = 0.4683$$

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} \cdot \frac{\rho_1}{\rho_2} = 2.56$$

Solving these ratios using the exit conditions as the upstream conditions, where $T_1 = 500\,K$, $p_1 = 30\,kPa$, and $\rho_1 = 0.0534\ \frac{kg}{m^3}$, the downstream conditions, which represent the flow conditions after crossing the theoretical, proposed normal shock wave, are found:

$$\frac{p_2}{p_1} = 10.23 \xrightarrow{yields} p_2 = 306.96\,kPa$$

$$\frac{\rho_2}{\rho_1} = 4.00 \xrightarrow{yields} \rho_2 = 0.2139\ \frac{kg}{m^3}$$

$$\frac{T_2}{T_1} = 2.56 \xrightarrow{yields} T_2 = 1{,}277.5\,K$$

$$M_2 = \frac{V_2}{a_e} = 0.4683 \xrightarrow{yields} V_2 = 410.71\ \frac{m}{s}$$

Results:

After solving the normal shock relations for the exhaust gases at the nozzle exit, the following results are found:

$$M_2 = 0.4683$$

$$V_2 = 410.71\ \frac{m}{s}$$

$$T_2 = 1{,}277.5\ K$$

$$p_2 = 306.96\ kPa$$

$$\rho_2 = 0.2139\ \frac{kg}{m^3}$$

Discussion:

It is known that the shock wave is a region that causes a sudden, discontinuous change in flow parameters, characterized by a sudden increase of pressure, temperature, density, and entropy, and a sudden decrease in velocity. A normal shock wave can form in front of a supersonic object where the flow is turned heavily and the shock cannot remain attached to the body, or in the absence of a body, a normal shock can form due to abrupt changes in flow parameters, e.g. at the inlet of a supersonic aircraft, where the flow must be decelerated to subsonic speeds before entering the turbine. Knowing this, the hypothesis that a normal shock occurs just past the nozzle exit holds theoretical validity. After using normal shock relations, tangible flow parameters beyond the proposed shock in this scenario were computed. Firstly, looking at the pressure increase, as this sounds like the basis for my classmate's hypothesis: the calculations show that, while the pressure does increase, it increases to a value that is roughly 4.4 times larger than atmospheric pressure at 10,000 feet. The Mach number decreases as expected to subsonic speeds, and the resulting velocity is roughly half the speed of sound, and 0.156 times the exit velocity. The other relations show similar results, as the values align with normal shock theory; increasing across the shock. Additionally, the entropy relations across a shock wave that used in the previous homework assignment can be used to show that the process is highly irreversible and dissipative, as $\frac{\Delta s}{c_p}(M_1 = 3.00) \approx 0.3$.

Under these conditions, however, the flow has a lot of adjustment to do before it equalizes with the atmospheric pressure. The flow has already lost a lot of usable energy, as shown in the entropy increase, which has been used to compress and heat the molecules. This entropy increase means that less energy remains in the flow for further thermodynamic adjustments – essentially, the flow loses its ability to do efficient work, making it likely that another process or mechanism is needed to bring the exhaust gases and atmospheric gases to thermodynamic equilibrium. It could be assumed that the effect of the rocket passing through the atmosphere is simply an irreversible process, and that with enough time, the exhaust would chaotically dissipate into the atmospheric reservoir, changing the total temperature and pressure by negligible amounts. This would be true, because no matter what happens after the nozzle, the flow must eventually equilibrate. However, the theory fails to address the fact that, locally at the exit, a normal shock is inefficient and unlikely.

Instead, a transitional process would make more sense than a dissipative process. Consider a different case, where the exit gases near or at the boundary of the nozzle would begin to experience compression effects from the ambient atmosphere at a higher static pressure. This would act to 'turn the flow into itself' in a sense and would mimic the effects of supersonic flow rounding a concave corner. From my high-speed aerodynamics class, after studying oblique shock and expansion waves, I would argue to my classmate that, rather than a normal shock forming at the nozzle exit, an oblique shock wave, or a series of oblique shock and expansion fans, would form at the exit. I would not predict the exact form that these oblique shock relations will take, but from what we know about shock relations, the normal shock wave is special case of the oblique wave which experiences the largest entropy generation and discontinuities in other flow parameters. On the other hand, a series of oblique shock relations would act to 'step up' and gradually blend the exit flow with atmospheric conditions, generating less entropy and maintaining kinetic energy and work potential.

I disagree with my classmate, and I argue that a series of oblique shock relations occur at the nozzle exit to gradually adjust the exhaust flow closer to atmospheric conditions, before homogenizing completely with the atmosphere. My hypothesis indicates that the backpressure is relieved through a series of smaller, irreversible thermodynamic jumps rather than a singular abrupt, large, and irreversible thermodynamic jump. This gradual expansion and low entropy generation is more efficient and physically favorable. This preference could be viewed through a more familiar lens: Newton's first law, universal inertia, where there exists an inherent resistance to change. This theory also aligns with my own observations, as pictures and videos of jet and rocket engine nozzles typically showcase cool shock diamonds and Mach discs, which are formed through a complex relationship of oblique shock relations.

2. **Question 2:**

Consider an airfoil, such as the NACA 4412, operating in a wind tunnel with upstream conditions $p_\infty = 101\,kPa$ and $T_\infty = 10°C$.

a) If the wind speed is $30\,m/s$, find the pressure at the stagnation point on the airfoil leading edge. What is the pressure coefficient at the stagnation point? Find the stagnation pressure predicted by Bernoulli's equation. Comment on these results.

b) If the wind speed is $300\,m/s$, find the pressure at the stagnation point on the airfoil leading edge. What is the pressure coefficient at the stagnation point? Find the stagnation pressure predicted by Bernoulli's equation. Comment on these results.

c) Show formally that the isentropic-flow equation for the stagnation pressure reduces to Bernoulli's equation for small Mach number.

Assume: Isentropic flow condition – adiabatic and reversible. Ideal air considerations. Isolated – no external forces, neglecting gravitational, electromagnetic, and viscous forces. Adiabatic – no heat added, or work done. Steady flow.

Given: Upstream conditions $p_\infty = 101\,kPa$ and $T_\infty = 10°C$.

Solution:

To find the pressure at the stagnation point on the airfoil, we must assume isentropic flow conditions. This allows us to find the pressure where the flow velocity is reduced to zero without losses and measure the total flow energy. We also assume the fluid in the tunnel is air, and that it is calorically ideal, which is a safe assumption for subsonic air in a controlled environment.

First, when the freestream wind speed is $30\,m/s$, we can begin by solving for the speed of sound under the given conditions, which depends solely on freestream temperature, converted to kelvin, $T_\infty = 283.15\,K$:

$$a_\infty = \sqrt{\gamma R T_\infty}$$

Where, for ideal air:

$$\gamma = 1.4$$

$$R = 287\,J/kg \cdot K$$

Thus:

$$a_\infty = 337.30\,m/s$$

And:

$$M = \frac{V_\infty}{a_\infty} = 0.0889$$

Under these conditions where the freestream Mach number is small, flow can be treated as incompressible, and the Bernoulli equation can be reliably used to determine stagnation pressure. However, we will first use the total pressure relation for isentropic compressible flow to determine the stagnation pressure, and then we will compare the results obtained. It is expected, however, that the results obtained will be very similar. Consider the equation derived under the isentropic and perfect gas assumptions for total, or stagnation pressure:

$$p_0 = p_\infty \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}$$

This can be solved by plugging in the freestream static pressure, p_∞, the ratio of specific heats for air, γ, used previously to determine a_∞, and the Mach number found above, M:

$$p_0 = 101\, E^3 \cdot \left(1 + \frac{1.4 - 1}{2} 0.0889^2\right)^{\frac{1.4}{1.4 - 1}} = 101{,}560.39\, Pa$$

Now consider the expression for stagnation pressure known from the Bernoulli equation, which can be used under incompressible flow assumptions:

$$p_0 = p_\infty + \frac{1}{2} \rho_\infty V_\infty^2$$

We can find free stream density using the ideal gas law, seen below

$$p = \rho RT \xrightarrow{yields} \rho_\infty = \frac{p_\infty}{RT_\infty} = 1.2429\, kg/m^3$$

Thus:

$$p_0 = 101\, E^3 + \frac{1}{2} 1.2429 \cdot 30^2 = 101{,}559.29\, Pa$$

These pressures are the same within a single thousandth of a percent, thus the difference is imperceivable and fully negligible. Then, the pressure coefficient can be calculated using either value, seen below:

$$C_p = \frac{p_0 - p_\infty}{\frac{1}{2} \rho V_\infty^2} = \frac{101.56 E^3 - 101 E^3}{\frac{1}{2} 1.24 \cdot 30^2} = 1.0$$

This value is expected to be exactly one, as defined under incompressible theory. This is verified, as MATLAB found the difference between pressure coefficients to be less than 0.2%.

Next, the same calculations are performed on the high-speed flow regime, still using isentropic flow and ideal gas assumptions. The upstream conditions are the same, except for flow velocity. This means that the speed of sound we found earlier is the same between the tests, and the new Mach number can be found:

$$M_\infty = \frac{V_\infty}{a_\infty} = 0.89$$

Plugging this Mach number and the same ratio of specific heats for ideal air into our isentropic relation:

$$p_0 = 101\, E^3 \cdot \left(1 + \frac{1.4 - 1}{2} 0.889^2\right)^{\frac{1.4}{1.4 - 1}} = 168{,}881.76\, Pa$$

Note that this is significantly higher than in the low-speed test. Next, we will go against what we know about transonic flow, and will try and use the Bernoulli equation to predict stagnation pressure. It is expected that this will be inaccurate due to the compressibility effects of higher-speed flow:

$$p_0 = 101\, E^3 + \frac{1}{2} 1.2429 \cdot 300^2 = 156{,}928.79\, Pa$$

This value is confirmed to be much different than the one found using isentropic relations. The pressure coefficient is determined using the stagnation pressure found through isentropic relations:

$$C_p = \frac{p_0 - p_\infty}{\frac{1}{2}\rho V_\infty^2} = \frac{168.88E^3 - 101E^3}{\frac{1}{2}1.24 \cdot 300^2} = 1.2137$$

This result shows that the local static pressure at the stagnation point is higher than our free-stream reference, which indicates a build-up of air molecules near the leading edge of the airfoil, causing a high-pressure zone.

To show that the isentropic relationship is a more well-rounded and complete formulation for determining stagnation pressure, it should be shown that the Bernoulli equation for stagnation pressure can be derived from this isentropic relationship using accepted approximations for when the flow is incompressible. This holds significance when talking about incompressible flow and what this assumption really means. Consider the isentropic pressure relationship that we've been using in ratio form:

$$\frac{p_0}{p_\infty} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}$$

The next step in the derivation depends on the desired approximation to use, as derivations commonly use either an approximation of the natural logarithm and exponential functions, or the binomial approximation. Both approximations are well-documented and accepted practice under their respective conditions. (*Homework assignments are typically where I aim simultaneously to suffer and to learn as much as possible. I started this assignment early and can spare the time, so for my own practice, I wrote out both*). First, the natural logarithm and exponential approach:

$$ln\left(\frac{p_0}{p_\infty}\right) = \frac{\gamma}{\gamma - 1} \cdot ln\left(1 + \frac{\gamma - 1}{2} M^2\right)$$

Next, apply the approximation: $ln(1 + x) \approx x$, where x is proportional to the quantity M^2 times the constant $\frac{\gamma-1}{2}$, dependent on the known γ. Internet sources say that this approximation is valid for $x < 0.1$. If the Mach number is small, the square operation will result in a 'small' x. Some MATLAB shows that for $\gamma = 1.4$, the largest possible M for $x \leq 0.1$ is 0.71, but this is qualified down to 0.3 as the typical range for incompressible flow, for which $x = 0.018$. This simplifies the above into:

$$ln\left(\frac{p_0}{p_\infty}\right) = \frac{\gamma}{\gamma - 1} \cdot \frac{\gamma - 1}{2} M^2$$

Simplifying more and exponentiating again:

$$\frac{p_0}{p_\infty} = e^{\frac{\gamma}{2} M^2}$$

Next, applying the approximation: $e^x \approx 1 + x$, where x is proportional to the quantity M^2 times the constant $\frac{\gamma}{2}$, dependent on the known γ. Like above, MATLAB found the smallest possible $M = 0.38$, which is closer to the familiar range for incompressible flow. Thus:

$$\frac{p_0}{p_\infty} = 1 + \frac{\gamma}{2} M^2$$

Next, knowing that $M = \frac{V_\infty}{a_\infty}$, $a_\infty^2 = \gamma RT$, and $RT = \frac{p_\infty}{\rho_\infty}$:

$$p_0 = p_\infty + p_\infty\left[\frac{\gamma}{2} \cdot \frac{V_\infty^2}{\gamma RT} \cdot \left(\frac{\rho_\infty}{p_\infty} \cdot RT\right)\right] \xrightarrow{yields} p_0 = p_\infty + \frac{1}{2}\rho_\infty V_\infty^2$$

This is the long and somewhat more rigorous approach. Alternatively, the binomial approximation can be used, where: $(1+x)^n \approx 1+nx$, where x is the same parameter as described earlier. For the incompressible Mach number assumption, which supersedes the theoretical mathematical max Mach number, this approximation holds for small x. Thus:

$$\frac{p_0}{p_\infty} = 1 + \frac{\gamma}{\gamma - 1} \cdot \frac{\gamma - 1}{2} M^2$$

The same final simplification process shown above can be used now to obtain the same result:

$$p_0 = p_\infty + p_\infty \left[\frac{\gamma}{2} \cdot \frac{V_\infty^2}{\gamma RT} \cdot \left(\frac{\rho_\infty}{p_\infty} \cdot RT\right)\right] \xrightarrow{yields} p_0 = p_\infty + \frac{1}{2}\rho_\infty V_\infty^2$$

Results:

$$p_0(M = 0.0889) = 101.56E^3\ Pa$$

$$p_0(M = 0.889) = 168.88E^3\ Pa$$

$$p_0 = p_\infty \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}} = p_\infty + \frac{1}{2}\rho_\infty V_\infty^2 \quad \text{for } \{0 \le M \le 0.3\}$$

Discussion:

Stagnation pressure calculations at the two different airspeeds is a good way to showcase the effects that compressibility has at higher Mach numbers, and the derivation of the Bernoulli equation from the more complete isentropic pressure relation is good to gain intuition on the incompressible assumption that was made regularly last semester.

The MATLAB for the solution involved creating a function for all isentropic relations for sanity and ease of use later, which is handy to discuss the other isentropic relations at the limit of incompressible flow. Firstly though, at the low-speed flow regime for this problem, it was found that the pressure relation found nearly no difference in stagnation pressure compared to Bernoulli. The isentropic relation for density found $\rho_0 = 1.2478$, which is an increase of just 0.4%. The temperature relation found $T_0 = 283.598$, increasing by 0.16%. At the limit of incompressible flow, $M = 0.3$, the difference in pressure compared to the Bernoulli equation is 0.13%, the density increases by 4.56%, and the temperature increases by 1.88%. In the high-speed flow described in the second part of this question, where $M = 0.889$, the pressure difference is 7.61% compared to Bernoulli's equation, the density increases by 44.37%, and the temperature increases by 15.82%.

These calculations support the general conclusion that flow can be considered incompressible while $M \le 0.3$. In this range, the change in density is less than 5%. This, and the change in other quantities, can be written off as negligible for most approximations for low-speed solutions. However, the full isentropic relations should be used if a higher level of precision is needed, as these draw directly from thermodynamics and more robust assumptions that hold true for a wider range of Mach numbers and situations. The first of these assumptions being the ideal gas law, which typically fails under very high pressure or very low temperature conditions not typically seen in subsonic flow. The other being isentropic flow, which implies adiabatic flow – where no work is done and no heat is added to the system - and implies that all kinetic energy is converted into pressure in a reversible, lossless manner, as to not generate entropy. This assumption typically fails in the presence of shock waves, viscous flow regions, or heat & work addition, none of which are common in subsonic flow around a streamlined body such as the NACA 4412 airfoil.

3. Question 3:

Consider a body with blunt leading edge in a supersonic flow. We have argued that a bow shock must form in front of the body in order for the flow to reach stagnation conditions at the body leading edge:

a) *For a range of Mach numbers, $1 \leq M_1 \leq 5$, plot the variation of $\frac{u_2}{a_1}$ and $\frac{u_2}{a_2}$ equal M_2 on the same plot*

b) *Find the downstream Mach number, M_2, for the limiting case of $M_1 \rightarrow \infty$*

c) *Based on the results, argue that the bow shock must "stand off" the blunt leading edge and not touch the leading edge itself*

Assume: p_1, ρ_1, and T_1 remain constant. M_1 is varied only by changing the upstream air speed. Ideal air. Shock is locally normal w.r.t. the stagnation streamline and normal shock relations apply. Steady, adiabatic, and inviscid flow.

Given: Shock interaction with the stagnation streamline is the extreme oblique shock case for high flow turning, i.e. a normal shock wave.

Solution:

To first find the variation in the ratios of downstream flow velocity to upstream and downstream speeds of sound, it should be stated that the downstream conditions are determined according to the normal shock relations that were used for the previous homework assignment. These relations can be used because the bow shock is locally normal to the stagnation streamline, as stated, which means that we can isolate the upstream conditions of this streamline and compute across the normal shock to determine the stagnation conditions on this streamline after the shock.

Assuming that p_1, ρ_1, and T_1 remain constant, the variations in $\frac{u_2}{a_1}$ and $\frac{u_2}{a_2}$ can be plotted without defining numerical values for p_1, ρ_1, and T_1, but rather in terms of the ratios. First, consider p_1, ρ_1, and T_1 to be normalized and made dimensionless w.r.t. themselves, such that the upstream conditions p_1, ρ_1, and T_1 all equal dimensionless 1. Then, the ratios returned from the normal shock relations MATLAB function created in the previous homework represent the normalized, dimensionless downstream conditions, requiring no further manipulation. The first quantity to be plotted is the ratio of the post-shock flow speed to the upstream speed of sound:

$$\frac{u_2}{a_1}$$

And the second quantity to be plotted is the ratio of post-shock flow speed to the downstream speed of sound:

$$\frac{u_2}{a_2}$$

The first important step to computing these two ratios is to find the downstream flow velocity. While the flow velocity is not directly given from the normal shock relations, we can use control volume analysis and the mass continuity equation, $\dot{m}_1 = \dot{m}_2$, to form an important relationship:

$$\dot{m}_1 = \dot{m}_2 \rightarrow \rho_1 u_1 = \rho_2 u_2 \xrightarrow{yields} \frac{\rho_2}{\rho_1} = \frac{u_1}{u_2}$$

Essentially, the inverse of the density ratio computed in the MATLAB function returns the normalized downstream flow velocity. Next, since the upstream flow temperature is stated to be constant, the a_1 term can be normalized to a constant dimensionless 1. Thus, the first quantity $\frac{u_2}{a_1}$ can be plotted as the inverse density ratio against the Mach number for $1 \leq M_1 \leq 5$. The next quantity holds only one more unknown, the downstream speed of sound. Knowing the speed of sound is only a function of temperature, we focus on the downstream temperature, which is known in its dimensionless form as the temperature ratio. Consider the following:

$$a_2 = \sqrt{\gamma R T_2}$$

$$T_2 = T_1 \cdot \frac{T_2}{T_1}$$

Thus:

$$a_2 = \sqrt{\gamma R \left(T_1 \cdot \frac{T_2}{T_1}\right)} = \sqrt{\gamma R T_1 \cdot \frac{T_2}{T_1}} = \mathrm{a}_1 \cdot \sqrt{\frac{T_2}{T_1}}$$

It was just stated that the a_1 term is constant and normalized to a dimensionless 1, thus the downstream speed of sound can be solved simply as the square root of the temperature ratio returned by the MATLAB normal shock relations function. Thus, quantities to plot can be found as functions of the Mach number range as the following, after applying the normalized values simplifications:

$$\frac{u_2}{a_1} = \rho_2^{-1}$$

$$\frac{u_2}{a_2} = \frac{\rho_2^{-1}}{\sqrt{T_2}}$$

Next, solving the limiting case of $\mathrm{M}_1 \rightarrow \infty$:

$$\lim_{M_1 \to \infty} M_2 = \lim_{M_1 \to \infty} \sqrt{\frac{1 + \frac{\gamma - 1}{2} M_1^2}{\gamma M_1^2 - \frac{\gamma - 1}{2}}} = \sqrt{\frac{0 + \frac{\gamma - 1}{2} \cdot M_1^2}{\gamma \cdot M_1^2 - 0}} = \sqrt{\frac{\frac{\gamma - 1}{2}}{\gamma}} = \sqrt{\frac{\gamma - 1}{2\gamma}}$$

For ideal air, once again, we have $\gamma = 1.4$. Solving this simplified limital equation to find M_2 for M_1 approaching infinity:

$$M_2 = \sqrt{\frac{1.4 - 1}{2 \cdot 1.4}} = \sqrt{\frac{0.4}{2.8}} = \sqrt{0.1428} = 0.3780$$

This convergence to a finite, subsonic Mach number indicates that, no matter how fact the object is travelling through the air, the stagnation streamline has a non-zero flow velocity at a location immediately past the (locally normal) bow shock wave formation. This flow velocity is represented as a Mach number, and is entirely dependent on the thermodynamic properties of the fluid in question, solved in this case for ideal air.

Results:

The plot of postshock flow velocity variation w.r.t. preshock and postshock speeds of sound is shown below:

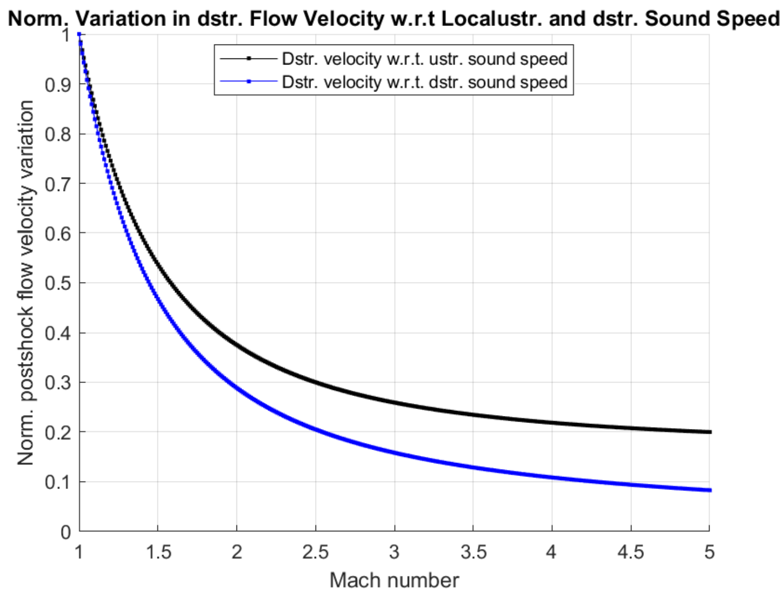


Figure 3.1: Plot of the two quantities; variation in postshock flow speed w.r.t. preshock speed of sound in black, and variation in postshock flow speed w.r.t. postshock speed of sound in blue.

The same plot of postshock flow velocity variation w.r.t. preshock and postshock speeds of sound is shown below, with the limiting case of $M_1 \to \infty$:

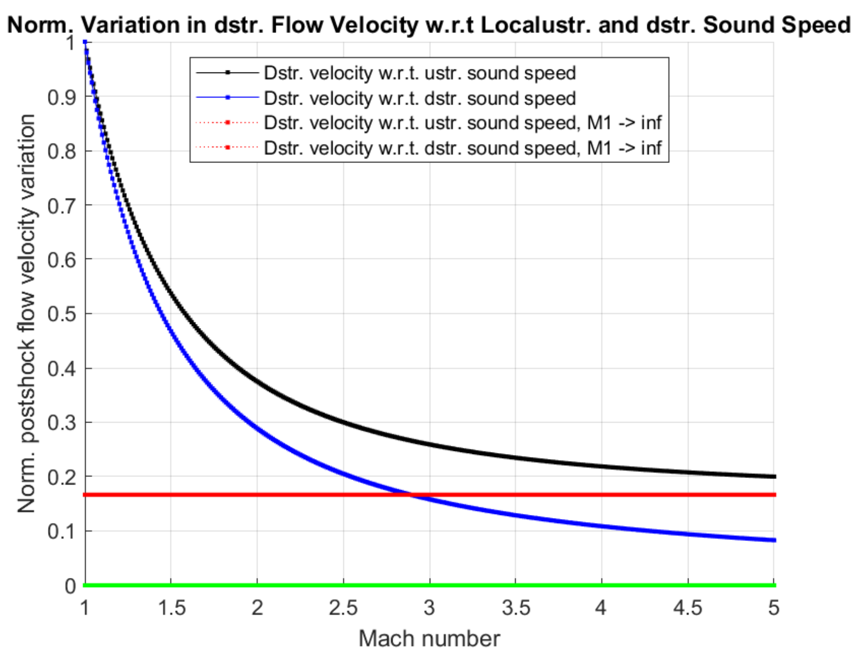


Figure 3.2: Same plot as above in addition to the limiting case as $M_1 \to \infty$ w.r.t. preshock and postshock speeds of sound, in red and green, respectively.

Discussion:

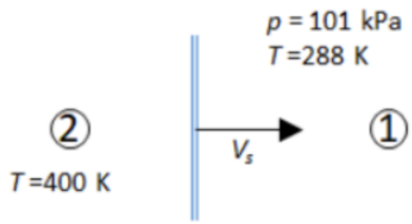
To argue that the bow shock must stand off from the leading surface, consider the known property of the actual stagnation point, where the flow velocity is equal to zero, by definition. From the results obtained in (b), this leads to one concrete conclusion: that the location on the stagnation streamline immediately past the shock wave formation cannot be the same as the location on the stagnation streamline for which the stagnation point occurs. There must be some physical separation between the post-shock flow and the stagnation point, which will occur due to further compression and viscous effects within the layer beneath the shock, at some point between the downstream shock and the leading surface of a body moving supersonically.

This argument is also supported by Figs. [3.1-3.2]. Fig. [3.1] clearly shows that, as Mach number increases, the fluid velocity just past the bow shock decreases non-linearly, and intuition says that the shape of the graph indicates that it asymptotically approaches some steady state value as Mach number grows very large. This aligns with what is known about the normal shock, i.e. the magnitude of discontinuous fluid velocity drop w.r.t. local speeds of sound across the shock is a function of preshock Mach number. Essentially, the postshock Mach number decreases as the preshock Mach number increases, which is known well from previous questions. This is seen to happen non-linearly across the range of Mach numbers tested, which is generally considered as the supersonic flow regime. As flow graduates to a hypersonic state, the postshock Mach number, when calculated w.r.t. both pre and postshock speeds of sound, will tend to jump discontinuously across the shock to the same values. This is shown mathematically in the limit calculation shown in the solution for (b). For the postshock Mach number relative to the preshock speed of sound, the M_2 value approaches the calculated 0.378, and the normal shock density relations show that this condition results in postshock fluid velocity approaching $\frac{1}{6} \approx 0.1667$. The postshock Mach number relative to the postshock speed of sound, M_2, approaches infinity, while fluid velocity relative to the postshock sound speed approaches 0. Here, the normal shock density and temperature relations show that the temperature ratio is *incredibly* large, and knowing how the postshock flow velocity is calculated w.r.t. postshock sound speed, would mathematically drive this quantity to zero, as there is essentially an 'infinity' in the denominator. More on this in a second. Also note the pressure ratio at the limital case, which grows very large as well. This is all used to parameterize the flow conditions beyond the bow shock, as well as generate intuition on what happens to this type of shock as the Mach increases. Recall that it was said that this wave is *not* incident to the blunt leading surface of the projectile because presence of flow velocity immediately beyond the normal shock along the stagnation streamline contradicts the definition of the stagnation point. Does this hold up under limital conditions?

A critically important point to discuss is that, as the preshock Mach grows large, the postshock temperature grows large as well. Intuition expects this from normal shock relations and physical observations. The significance that this holds, however, lies in the postshock speed of sound, proportional to the square root of the postshock temperature. As this temperature grows to infinity alongside the preshock Mach, the postshock sound speed also grows to infinity. This is a characteristic recognized from incompressible theory, where sound would travel instantaneously through an incompressible medium. As preshock Mach increases further, the subsonic layer between the bow shock and the blunt face becomes a region characterized by higher density, infinite pressure, infinite temperature, and incompressibility. This brings us to the theoretical limit of this shock condition, represented by a constant postshock flow velocity and an impossible state of matter. Even here, the shock wave does not lie incident to the stagnation point, and the closest it will get is likely at some ‘sweet spot’ *near* the computed thermodynamic limit, where the postshock fluid is minimally hostile, maintaining some minimally high pressure and temperature that keep the shock wave separated from the stagnation point for only enough time for postshock velocity to dissipate to zero. This last part is speculation, and in reality, there would probably be a transition to an ionized plasma or supercritical fluid state that cannot be predicted with any of our ideal supersonic flow equations, but it’s fun to guess.

4. Question 4:

A shock wave is moving into stationary air, region 1. The temperature and pressure in region 1 are $T_1 = 288\,K$ and $p_1 = 101\,kPa$. After the shock wave passes, the temperature is measured as $T_2 = 400\,K$.



Shock moving into region 1.

a) *Find the Mach number (M_s) and the velocity of the shock wave*
b) *Find the air velocity and the Mach number of the flow behind the shock wave (region 2) relative to a fixed observer*
c) *Find the stagnation temperature in region 2. Explain your result considering the corresponding result for a stationary shock wave*

Assume: Calorically ideal air. Steady, one-dimensional flow. Adiabatic shock wave. No viscous effects.
Given: $T_1 = 288\,K$, $p_1 = 101\,kPa$, $u_1 = 0\,m/s$, $T_2 = 400\,K$, $\gamma = 1.4$, $R = 287\,J/kg \cdot K$
Solution:

To find the Mach number of the shock wave, we can use the normal shock relations that were derived in the previous homework assignment, specifically the one for temperature:

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{2 + (\gamma-1)M_1^2}$$

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} \cdot \frac{\rho_1}{\rho_2} = \frac{\left((\gamma-1)M_1^2 + 2\right) \cdot \frac{\left(2\gamma(M_1^2-1)\right)}{\gamma+1} + 1}{M_1^2(\gamma+1)}$$

Both the temperature before and after the shock were given, so the temperature ratio is a known value, equal to $400/288$:

$$\frac{400}{288} = \frac{\left((1.4-1)M_1^2 + 2\right) \cdot \frac{\left(2(1.4)(M_1^2-1)\right)}{1.4+1} + 1}{M_1^2(1.4+1)} \xrightarrow{yields} M_1 = 1.6013$$

To determine the velocity of the shock wave, the speed of sound w.r.t. the temperature at station 1 can be used:

$$a_1 = \sqrt{\gamma R T_1} = 340.1741\,m/s$$

$$u_1 = a_1 \cdot M_1 = 544.73\,m/s$$

To determine the velocity and Mach number of the flow behind the shock wave, the normal shock velocity relation can be used, which is also known as just the inverse of the density ratio:

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2}$$

This equation will solve for the velocity relative to the shock, however when relative to a stationary frame of reference, the velocity at station one is known as the shock wave velocity that was just found. Solving the normal shock velocity relation for u_2:

$$u_2 = u_1 \left[\frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2}\right]^{-1} = 267.8152\, m/s$$

The Mach number behind the shock wave can be found using the Mach number shock relation or using the temperature in this region, T_2, to determine the speed of sound:

$$a_2 = \sqrt{\gamma R T_2} = 400.89\, m/s$$

$$M_2 = \frac{u_2}{M_2} = 0.668$$

The stagnation temperature can be found using isentropic relations for stagnation temperature:

$$\frac{T_{0_2}}{T_2} = 1 + \frac{\gamma - 1}{2} M_2^2 \xrightarrow{yields} T_{0_2} = 453.44\, K$$

Results:

$$M_1 = 1.6013$$

$$u_1 = 544.73\, m/s$$

$$M_2 = 0.668$$

$$u_2 = 267.8152\, m/s$$

$$T_{0_2} = 453.44\, K$$

Discussion:

The quantities requested could all be found as a function of either the known normal shock relations derived from the last assignment or given in class or found as a function of our known thermodynamic isentropic relations. For the stagnation temperature, however, the result is different than what we’d expect from a stationary shock wave. Typically, the total stagnation temperature remains constant across the shock wave, but this is not seen here. Because energy conservation must still hold, in this stationary time frame, the post-shock stagnation temperature increases because the flow is still moving relative to the stationary frame of reference, and thus the stagnation temperature must increase due to the remaining kinetic energy.

II. Appendix

```
%% Header
% Author: Zakary Steenhoek
% Date: 6 February 2025
% AEE362 HW01

clc; clear;

%% Math

% Variables
R = 8314;
M = 7.4;
cp = 4168;
T_e = 500;
V_e = 2629.1;
p_e = 30E3;
R_e = R/M;
rho_e = p_e/(R_e*T_e);
gamma_e = cp/(cp-R_e);
a_e = sqrt(R_e*gamma_e*T_e);
M_e = V_e/a_e;

[pRatio,rhoRatio,tRatio,M2] = normalShockRelations(M_e, gamma_e, 1);
p_2 = p_e*pRatio
T_2 = T_e*tRatio
rho_2 = rho_e*rhoRatio
V_2 = M2*a_e

%% Header
% Author: Zakary Steenhoek
% Date: 6 February 2025
% AEE362 HW01

clc; clear;

%% Math

% Variables
R = 287;
gamma = 1.4;
T1 = 10+273.15;
V1_1 = 30;
V1_2 = 300;
p1 = 101E3;

% Calculations
rho1 = p1/(R*T1);
a = sqrt(R*gamma*T1);

% Bernoulli equation for total pressure
totalPressure = @(PINF, RHO, VINF) PINF + 0.5*RHO*VINF^2;

%% Low speed flow regime
```

```
% This mach number
M_1 = V1_1/a;

% Stagnation pressure according to isentropic relations
[p0_IL,~,~] = isentropicRelations(M_1, gamma, p1, rho1,T1);

% Stagnation pressure according to the Bernoulli equation
p0_BL = totalPressure(p1, rho1, V1_1);

% Compute pressure coefficient
Cp_IL = (p0_IL-p1)/(0.5*rho1*V1_1^2);
Cp_BL = (p0_BL-p1)/(0.5*rho1*V1_1^2);

% Results
fprintf('In low-speed flow (M = %.4f):\n', M_1);
fprintf(['Stagnation pressure according to isentropic relations ' ...
    '(Pa): %.2f\n'], p0_IL);
fprintf(['Stagnation pressure according to the Bernoulli ' ...
    'equation (Pa): %.2f\n'], p0_BL);
fprintf(['Pressure coefficient using isentropic value: ' ...
    '%.4f\n'], Cp_IL);
fprintf(['Pressure coefficient using Bernoulli value: ' ...
    '%.4f\n\n'], Cp_BL);

%% High speed flow regime

% This mach number
M_2 = V1_2/a;

% Stagnation pressure according to isentropic relations
[p0_IH,~,~] = isentropicRelations(M_2, gamma, p1, rho1,T1);

% Stagnation pressure according to the Bernoulli equation
p0_BH = totalPressure(p1, rho1, V1_2);

% Compute pressure coefficient
Cp_IH = (p0_IH-p1)/(0.5*rho1*V1_2^2);
Cp_BH = (p0_BH-p1)/(0.5*rho1*V1_2^2);

% Results
fprintf('In high-speed flow (M = %.4f):\n', M_2);
fprintf(['Stagnation pressure according to isentropic relations ' ...
    '(Pa): %.2f\n'], p0_IH);
fprintf(['Stagnation pressure according to the Bernoulli ' ...
    'equation (Pa): %.2f\n'], p0_BH);
fprintf(['Pressure coefficient using isentropic value: ' ...
    '%.4f\n'], Cp_IH);
fprintf(['Pressure coefficient using Bernoulli value: ' ...
    '%.4f\n\n'], Cp_BH);

%% Formal Derivation Stuff

% Var
C = (gamma-1)/2;
V = gamma/2;
```

```
M = 0:0.01:1;

% Small ln approx
x = C*M.^2;
y = V*M.^2;
smallM_1 = find(x >= 0.1, 1);
smallM_2 = find(y >= 0.1, 1);
M(smallM_1)
M(smallM_2)
x(31)
y(31)

[p0, rho0, T0] = isentropicRelations(0.31, 1.4, p1, rho1,T1)
Cp = (p0-p1)/(0.5*rho1*300^2)
totalPressure(p1, rho1, 300)
rho0/rho1


%% Header
% Author: Zakary Steenhoek
% Date: 6 February 2025
% AEE362 HW01

clc; clear;

%% Setup

% Variables
R = 287;
gamma = 1.4;

%% Low speed flow regime

% Mach number range
M1 = 1:0.01:5;

% Normal shock relation ratios for Mach vector
[pRatio, rhoRatio, tRatio, M2] = normalShockRelations(M1, gamma);
[pLim, rhoLim, tLim, ~] = normalShockRelations(1E6, gamma);

%% Plots

% Without limiting case
figure(1); clf; hold on; grid on;
title(['Norm. Variation in dstr. Flow Velocity w.r.t Local' ...
    'ustr. and dstr. Sound Speed']);
xlabel('Mach number'); ylabel('Norm. postshock flow velocity variation');
plot(M1, 1./rhoRatio, 'k.-');
plot(M1, 1./(rhoRatio.*sqrt(tRatio)), 'b.-');
xlim([1 5]); ylim([0 1]);
legend('Dstr. velocity w.r.t. ustr. sound speed', ...
       'Dstr. velocity w.r.t. dstr. sound speed', ...
       'Location','best');
hold off;

% q = findobj('type','figure');
```

```
% autosave(q, 'noLims', 'AEE362\HW02\figs');

% With limiting case
figure(2); clf; hold on; grid on;
title(['Norm. Variation in dstr. Flow Velocity w.r.t Local' ...
    'ustr. and dstr. Sound Speed']);
xlabel('Mach number'); ylabel('Norm. postshock flow velocity variation');
plot(M1, 1./rhoRatio, 'k.-');
plot(M1, 1./(rhoRatio.*sqrt(tRatio)), 'b.-');
plot(M1, 1./rhoLim, 'r.:');
plot(M1, 1./(rhoLim.*sqrt(tLim)), 'g.:');
xlim([1 5]); ylim([0 1]);
legend('Dstr. velocity w.r.t. ustr. sound speed', ...
       'Dstr. velocity w.r.t. dstr. sound speed', ...
       'Dstr. velocity w.r.t. ustr. sound speed, M1 -> inf', ...
       'Dstr. velocity w.r.t. dstr. sound speed, M1 -> inf', ...
       'Location','best');
hold off;

% q = findobj('type','figure');
% autosave(q, 'yesLims', 'AEE362\HW02\figs');

%% Header
% Author: Zakary Steenhoek
% Date: 6 February 2025
% AEE362 HW01

clc; clear;

%% Setup
gamma = 1.4;
R = 287;
T1 = 288;
T2 = 400;
p1 = 101E3;
syms M1
a1 = sqrt(gamma*R*T1);
a2 = sqrt(gamma*R*T2);

pressureRatio = 1+((2*gamma)./(gamma+1)).*(M1.^2-1);
densityRatio = ((gamma+1).*M1.^2)./(2+(gamma-1).*M1.^2);
temperatureRatio = pressureRatio.*densityRatio.^(-1);
M_2 = matlabFunction(sqrt((1+((gamma-1)./2).*M1.^2)./(gamma.*M1.^2-(gamma-1)./2)));
T02 = matlabFunction(1+(gamma-1)/2*M1);

M = solve(temperatureRatio == (T2/T1), M1);
V1 = a1*M(1);
dR = matlabFunction(densityRatio);
V2 = V1*(dR(M)).^(-1)
m2 = M_2(M)
M2 = V2/a2
t02 = T2*T02(M2(1))
t01 = T1*T02(M(1))
```