High Speed Aerodynamics Homework 01

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I. Part 1

Data for this part is found in the module titled "Airfoil Data Files. The data points are on the airfoil surface starting at the trailing edge and going counterclockwise back to the trailing edge. The filename indicates the Mach number at which the data were computed. The data includes Mach numbers from 0.1 to 0.9 for every 0.1 plus M = 0.85. All these data sets were calculated for $Re = 10{,}000{,}000$ and $\alpha = 4$.

1. Problem Statements:

Using the provided NACA 4412 data files for friction and pressure coefficient found on Canvas:

a) *From a reliable source, find the accepted values for C_l, C_d, and $C_{M_{ac}}$ for the NACA 4412 airfoil at a 4° angle of attack.*
b) *Plot C_l, C_d, and $C_{M_{ac}}$ vs. Mach number for the NACA 4412 at $\alpha = 4°$. Include all Mach numbers for which the data is given. In a table, show the drag coefficient due to friction and the drag coefficient due to pressure for all Mach numbers for which there is data.*
c) *Plot the C_p distributions for the NACA 4412 airfoil at $\alpha = 4°$ for the various Mach numbers. Observe the results. Please turn in the plots for only M = 0.1, 0.6, 0.7 and 0.85.*
d) *Based on your plots and calculations, discuss the variation of C_l, C_d, and $C_{M_{ac}}$ as a function of freestream Mach number for this airfoil. Include a comparison of compressible and "incompressible" results as well as a comparison with accepted values. What is the likely cause for the large variations seen between M = 0.6 and M = 0.7? Comment on the relative importance of pressure and friction drag.*

Assume: Compressible, viscous flow (RANS data). Thin airfoil. Dimensionless data.
Given: RANS pressure and friction data files for NACA 4412 at $\alpha = 4°$ and $Re = 10{,}000{,}000$.
Solution:

To obtain reliable values for C_l, C_d, and $C_{M_{ac}}$ at $\alpha = 4°$, the website www.airfoiltools.com was used, setting the highest possible $Re = 1{,}000{,}000$ and $N_{crit} = 7$ to generate plots of these coefficients as a function of the angle of attack. At $\alpha = 4°$, it was found that, roughly, $C_l = 0.8$, $C_d = 0.01$, and $C_{M_{ac}} = -0.10$.

For the code, first, the Canvas data files are downloaded and stored on the MATLAB path, which is variable defined for reference. Some file names are manipulated for code design. The directory of NACA 4412 files is loaded, and data structures to contain the calculated values are initialized. A loop iterates over each file in the file list, reads the name, determines the Mach number and filetype, and finally reads and records the necessary data columns, ensuring the command 'flipud()' to maintain standard practice for tabulating airfoil data.

When both the pressure and friction data for a Mach number has been recorded, the required computations to obtain C_l, the total and components of C_d, and $C_{M_{ac}}$ are done using the equations below, at all Mach numbers provided. The next step is to process the data to extract the force coefficients. The distinction needs to be made in the drag coefficient from both the pressure and friction data, and as such, it is easier to perform separate calculations. The following equations are used to find the x and z body total force coefficients:

$$C_z = \int_{airfoil} -C_p \ dx + \int_{airfoil} C_f \ dz$$

$$C_x = \int_{airfoil} C_p \ dz + \int_{airfoil} C_f \ dx$$

The total force coefficients were calculated and stored so they may be used to find the lift and drag coefficients, which is the simple process of converting from body axes to wind axes using the angle of attack. The following equations were used to convert to wind axes and determine the lift and drag coefficients:

$$C_l = C_z \cdot cos\, \alpha - C_x \cdot sin\, \alpha$$

$$C_d = C_z \cdot sin\, \alpha + C_x \cdot cos\, \alpha$$

These last equations were used to find the drag force due to friction and pressure independently, and are a combination of the last two sets of equations, isolating as needed:

$$C_{d_p} = -\int_{airfoil} C_p \ dx \cdot sin\, \alpha + \int_{airfoil} C_p \ dz \cdot cos\, \alpha$$

$$C_{d_f} = \int_{airfoil} C_f \ dz \cdot sin\, \alpha + \int_{airfoil} C_f \ dx \cdot cos\, \alpha$$

The moment coefficient about the aerodynamic center was found using the following equation, and assumes the aerodynamic center of the NACA 4412 is at the quarter chord, as defined by the thin airfoil theory:

$$C_{M_{ac}} = \int_{airfoil} (x - x_{ac}) \cdot C_p \ dx + \int_{airfoil} z \cdot C_p \, dz$$

Finally, plots were generated for all calculated coefficients vs. the Mach number for $\alpha = 4°$ and $Re = 10{,}000{,}000$. Also, the pressure distribution C_p was graphed across the upper and lower surfaces for each Mach number, careful to denote in the legend which are which. Also, the drag force and the components of the drag force due to pressure and friction are tabulated to observe how each component contributes to total drag as a function of Mach number. These graphs and the table can be seen in the results section below.

Results:

The graphs of aerodynamic coefficients vs the Mach number are seen below:

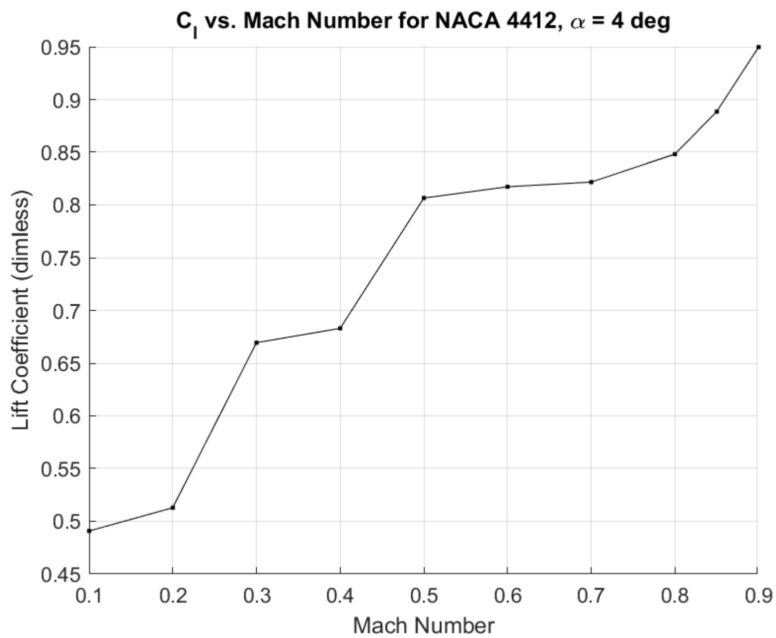


Figure 1.1: Lift coefficient vs. Mach number, $\alpha = 4°$, $Re = 10,000,000$

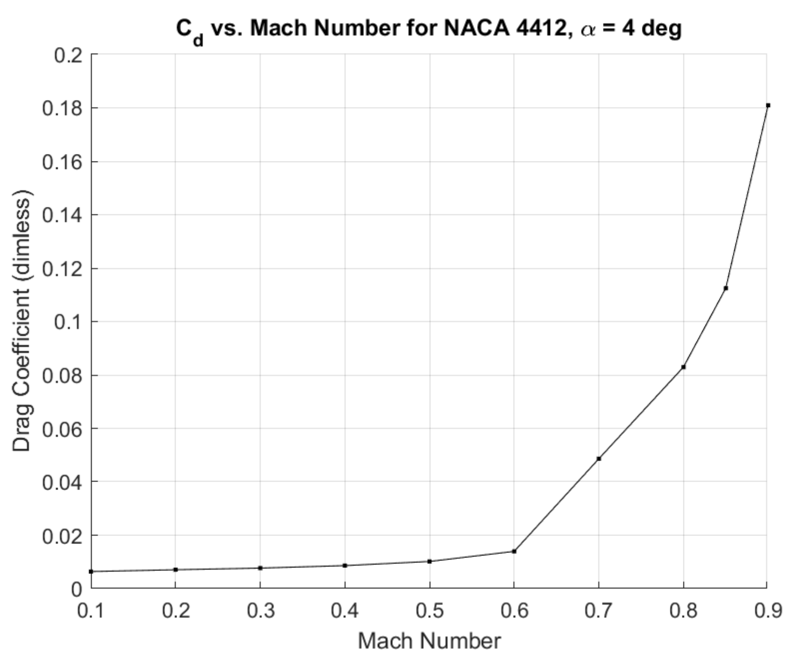


Figure 1.2: Drag coefficient vs. Mach number, $\alpha = 4°$, $Re = 10,000,000$

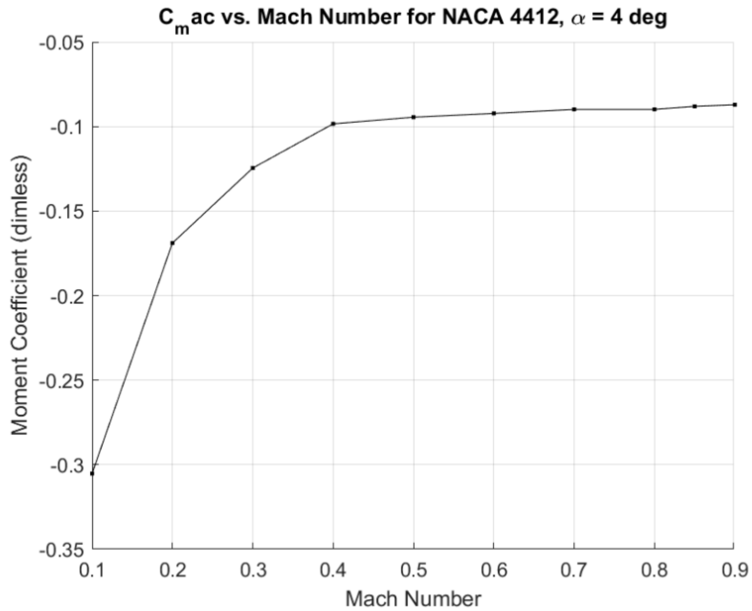


Figure 1.3: Moment coefficient vs. Mach number, $\alpha = 4°$, $Re = 10,000,000$

The table of drag coefficient and drag components at each Mach number is seen below:

M	C_d	C_{d_p}	C_{d_f}
0.10	0.0064	0.0063	0.0001
0.20	0.0071	0.0068	0.0002
0.30	0.0077	0.0073	0.0004
0.40	0.0086	0.0081	0.0005
0.50	0.0102	0.0095	0.0007
0.60	0.0136	0.0131	0.0007
0.70	0.0486	0.0479	0.0009
0.80	0.0829	0.0820	0.0009
0.85	0.1125	0.1113	0.0012
0.90	0.1810	0.1792	0.0018

The graphs of pressure distribution at the requested Mach numbers are seen below:

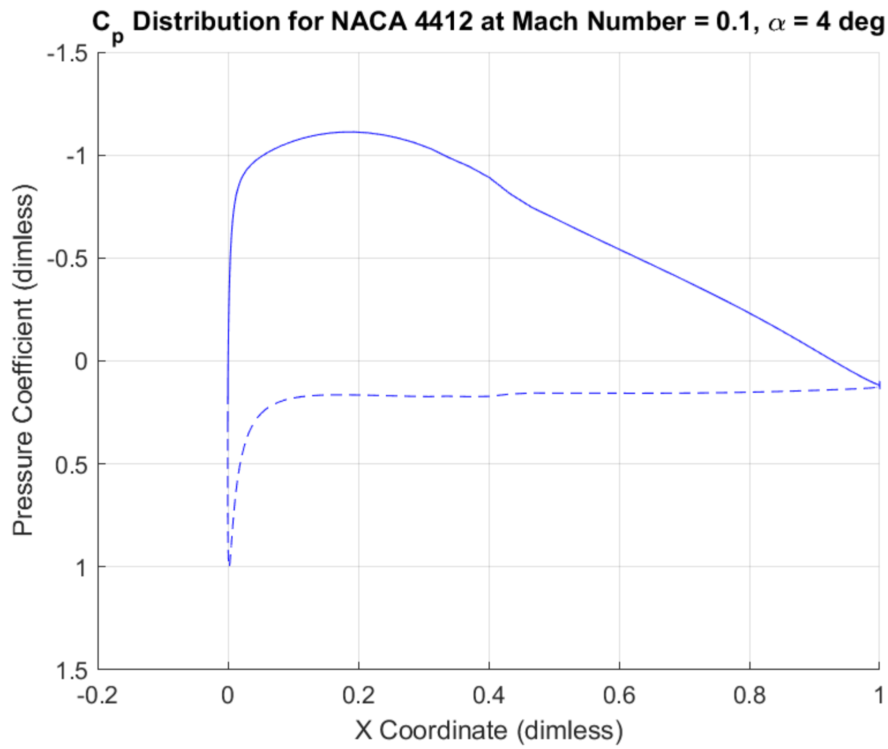


Figure 1.4: Pressure distribution at Mach number = 0.1, $\alpha = 4°$, $Re = 10,000,000$

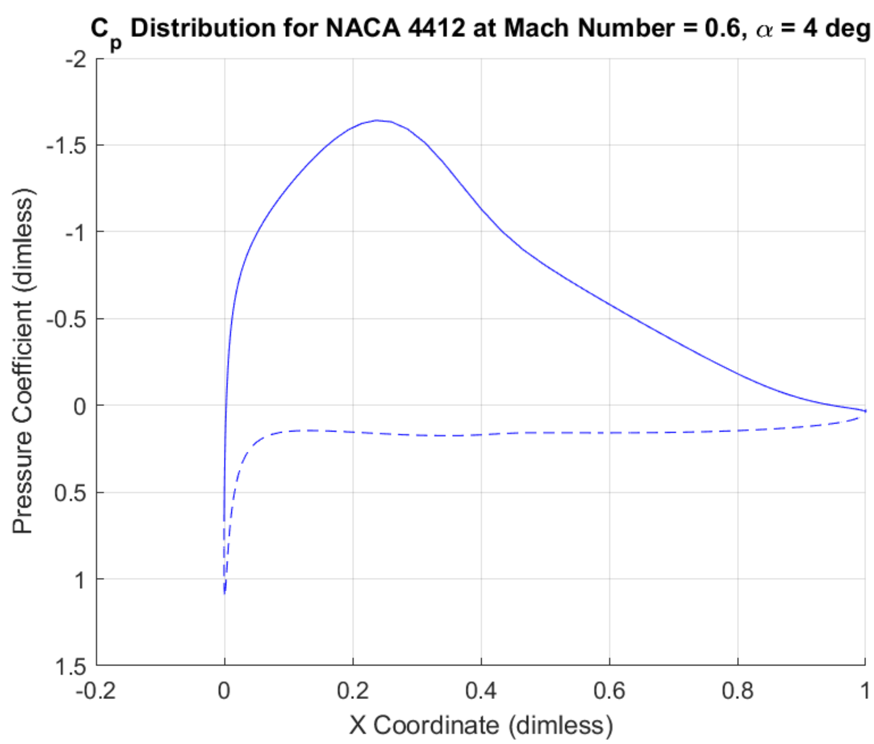


Figure 1.5: Pressure distribution at Mach number = 0.6, $\alpha = 4°$, $Re = 10,000,000$

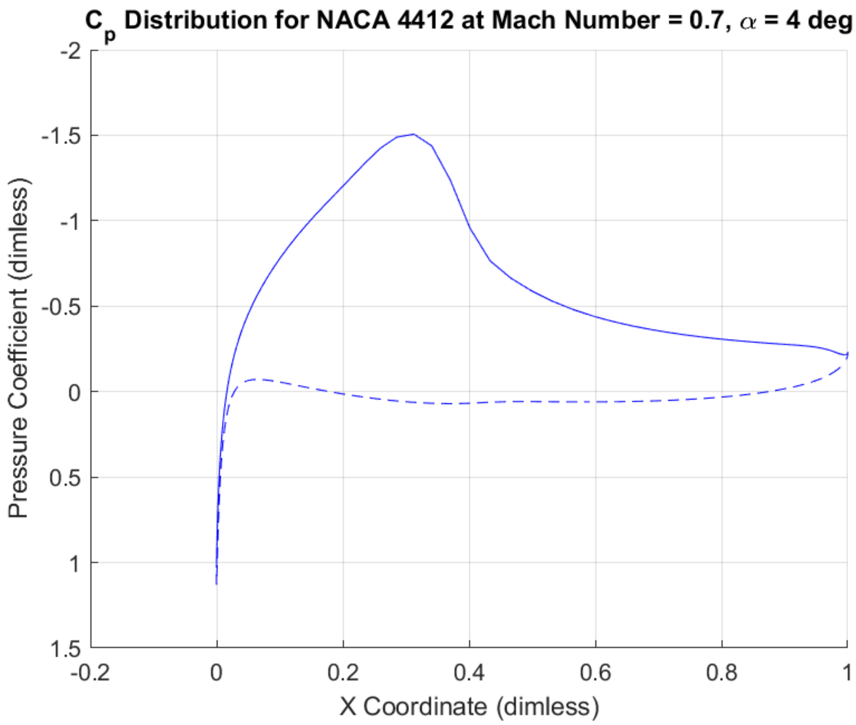


Figure 1.6: Pressure distribution at Mach number = 0.7, α $= 4°$, $Re = 10,000,000$

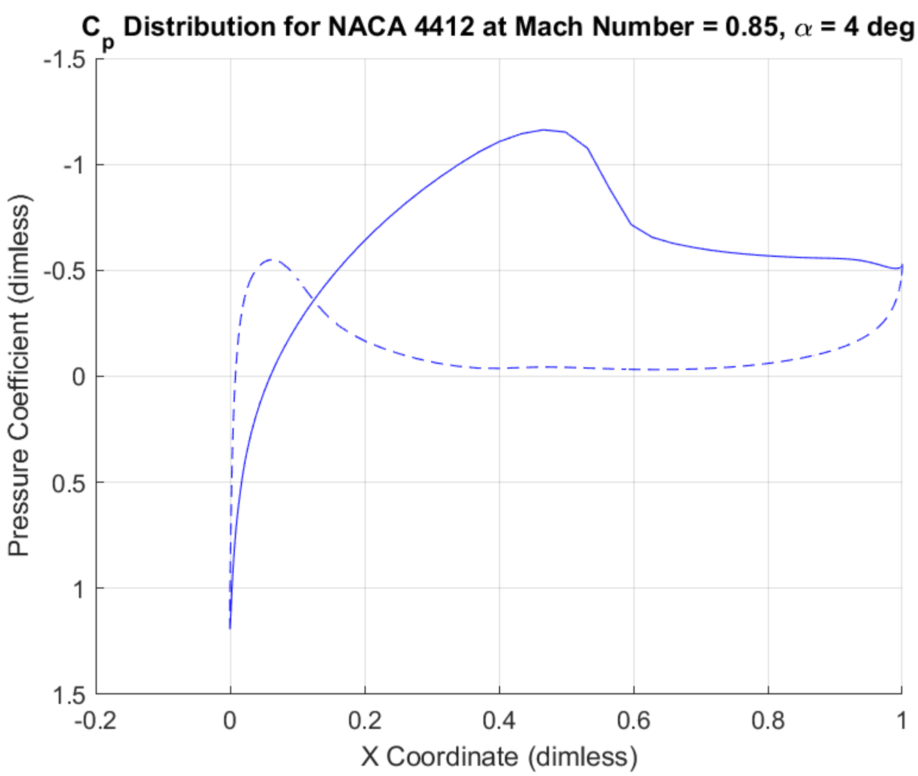


Figure 1.7: Pressure distribution at Mach number = 0.85, $\alpha = 4°$, $Re = 10,000,000$

Discussion:

The ‘accepted values’ that were requested for part (a) were found to be very accurate, if measuring at a Mach number of 0.5. At this Mach number for all three coefficients at a 4-degree AoA, the values estimated from the graphs on www.airfoiltools.com match nearly exactly the values that the RANS data and MATLAB code produced. However, this does not exactly hold up for a higher or lower Mach number, e.g. C_l and $C_{M_{ac}}$ both decrease rapidly as the Mach number falls, and C_d increases rapidly as the Mach number increases.

Overall, the coefficient plots seem to agree with intuition, as we know, an increase in Mach number should cause a buildup of air molecules at the leading edge and cause significantly more drag. The moment coefficient is somewhat more stable at a higher Mach number, which also makes sense upon looking at the pressure distribution and how it changes as a function of Mach number. At lower Mach numbers, i.e. 0.1 – 0.4, the area between the pressure curves seems to be less concentrated toward the leading edge and more distributed around the aerodynamic center, so there will be less lifting force acting across the first quarter-chord from the leading edge, and a larger *negative* pitching moment will be produced. As the Mach number increases, the area representing the lifting force concentrates on the leading edge more, and the negative pitching moment can be seen decreasing in magnitude. This continues until the pitching moment seems to ‘level off’ to the accepted value of -0.1 around a Mach number of 0.5.

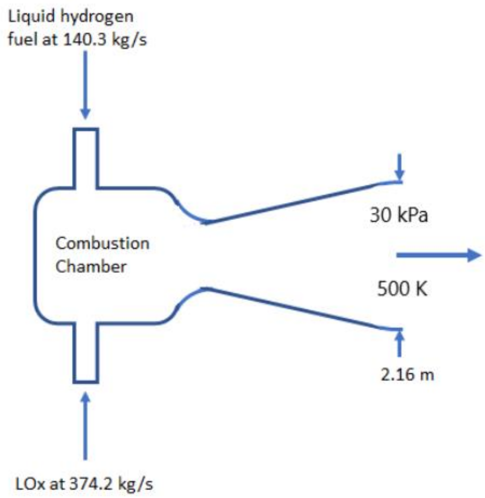
Looking at the pressure distribution between Mach numbers of 0.5 - 0.9, it’s seen that the airfoil loses the traditional pressure distribution shape associated with subsonic speeds around a Mach number of 0.6 or 0.7. It is also around this point that the drag coefficient begins to grow extremely rapidly. The pressure distributions show that the front of the airfoil begins to generate a negative lift, and the C_d table shows that the pressure drag explodes around this point, but friction drag, not so much. These trends indicate an accumulation of air molecules at the leading edge, increasing the pressure on the upper surface near the leading edge, increasing pressure drag, and disrupting traditional ‘incompressible’ flow patterns and assumptions. This could be identified because of the transition from subsonic flow to transonic flow which typically happens around this Mach number, where specific regions of flow are faster than Mach, leading to complex compressible flow patterns and difficult analysis. It is important to note that even in transonic flow, lift generation is not disrupted, which seems counterintuitive when looking at the pressure distribution graphs. However, when looking at Fig. [1.1], it keeps increasing through the transonic region, regardless of the complex flow patterns identified.

A comparison to ‘incompressible’ results is impossible in all reality, as this implies a Mach number of 0. Looking at the equation for Mach number, $M = \frac{u}{a}$, this would mean that the speed of sound, a, approaches infinity. To make a somewhat reasonable approximation, numerical values from low-speed aerodynamics lab two can be used. An approximate Mach number for this test can be found by first calculating the speed of sound using $\gamma = 1.4$, known R_{air}, and the measured temperature used to calculate lab air density. This yields $a = 350.04\, m/s$. The average wind tunnel velocity for the tests run during this lab was $34.8766\, m/s$, meaning the approximate Mach number for this lab is 0.0996. The Reynold number for these tests was approximately 221,991. The lab results from the tests on the NACA 4412 at $\alpha = 4°$ showed approximately that $C_l = 0.7945$ and $C_d = 0.0572$. Moment coefficients were not calculated for this lab. These values are notably higher than what was calculated in this assignment, even for the data at $M = 0.1$.

This is likely to be a result of the major difference in Reynold numbers, instrument & data collection errors, among many other possibilities. Regardless of machine or human errors, the difference is large enough to demonstrate the importance of considering compressibility at higher Mach testing and the impact that differing flow parameters can have on experimental results.

II. Part 2

The figure shows a representation of the space shuttle main engine liquid rocket motor. Use control-volume analysis to solve this problem.



1. **Question 1:**

The rocket fuel (liquid hydrogen at $20\ K$) and the oxidizer (liquid oxygen at $90\ K$) enter the combustion chamber from tanks that can be assumed to be reservoirs ($\vec{V} = 0$ relative to the motor). The specific heat of liquid hydrogen is $9668\ J/kg \cdot K$ and the specific heat of liquid oxygen is $347\ J/kg \cdot K$. Assume the exit gas has a molecular weight of $7.4\ kg/kg \cdot mol$ and $C_p = 4168\,J/kg \cdot K$. Also recall that the universal gas constant $\mathcal{R} = 8314\,J/(kg \cdot mol \cdot K)$. The space shuttle is passing through an altitude of 10,000 ft where the atmospheric pressure is $70\ kPa$.

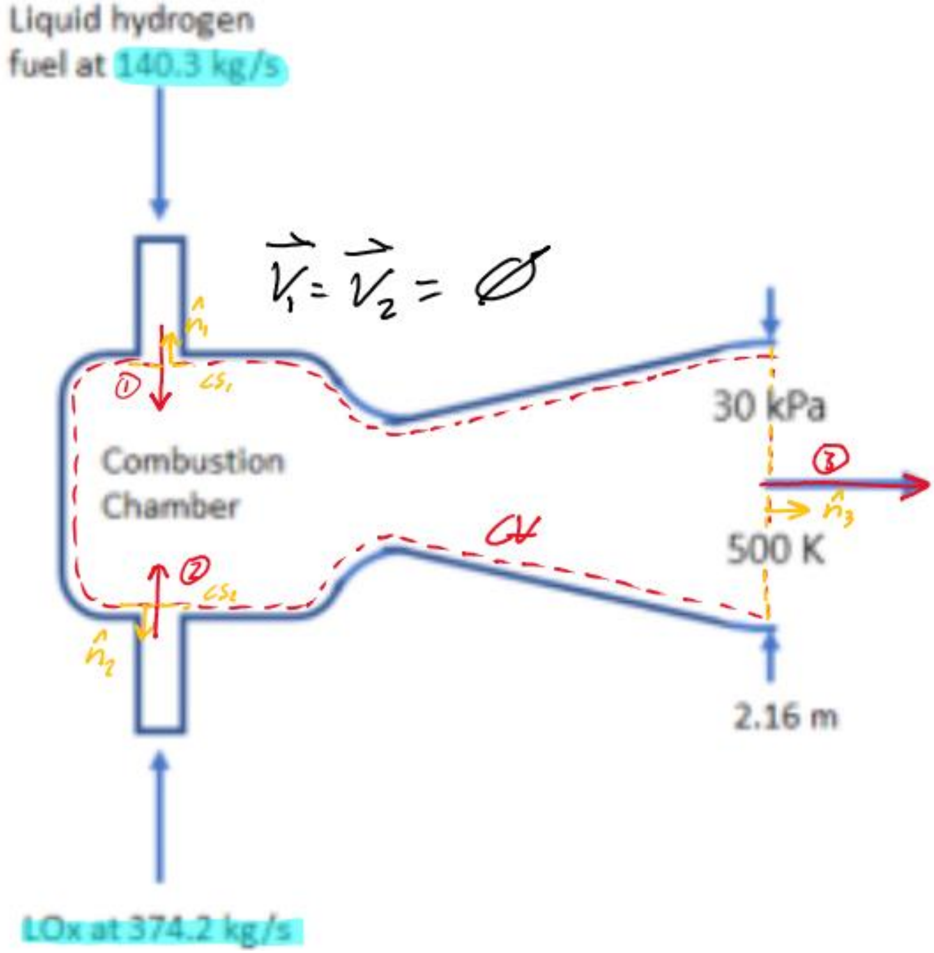
a) *For the conditions shown in the figure, find the velocity (relative to the motor) at the nozzle exit.*
b) *Estimate the thrust force the motor exerts on the shuttle (to which it is attached) for the conditions shown. Be sure to state the assumptions you had to use to find this value.*
c) *Estimate the rate of energy addition to the system due to combustion. You may consider the combustion chamber to be perfectly insulated.*

Assume: Liquid fuel & oxidizer treated as reservoir, steady flow state, calorically perfect exit gas, circular nozzle cross-section, neglect viscous, gravitational, and electromagnetic forces at exit, isolated - no external body forces or work, insulated - adiabatic

Given: $T_H = 20K,\ T_O = 90K,\ \overrightarrow{V_{in}} = 0,\ c_H = 9668\,J/kg \cdot mol,\ c_O = 347\,J/kg \cdot K,\ M = 7.4\,kg/kg \cdot mol,\ C_p = 4168\,J/kg \cdot K,\ \ \mathcal{R} = 8314\,J/(kg \cdot mol \cdot K),\ \ P_{atm} = 70\,kPa,\ \ \dot{m}_1 = 140.3\,kg/s,\ \ \dot{m}_2 = 374.2\,kg/s,\ \ P_3 = 30\,kPa,\ T_3 = 500\,K,\ d_3 = 2.16\,m$

Note: Solution completed by hand. Important calculations are attached in the solution section, and the full pdf will be in the appendix. Explanations can be found in the discussion section.

Solution:



Integral Eqns.

① Mass: $\int_{CS} \rho(\vec{V}\cdot\hat{n})\,dS = \emptyset$

② Momentum: $\int_{CS} \rho\vec{V}(\vec{V}\cdot\hat{n})\,dS + \int_{CS} P\hat{n}\,dS = \vec{F}_{Thrust}$

③ Energy: $\int_{CS} \rho\left(\frac{1}{2}V^2 + h\right)(\vec{V}\cdot\hat{n})\,dS = \dot{Q} + \dot{W}_{ext}$

Simplify

① $\sum \dot{m}_{in} = \dot{m}_1 + \dot{m}_2 = \dot{m}_{out} \rightarrow \sum \dot{m}_{in} = \int_{CS_3} \rho_3(\vec{V}_3\cdot\hat{n}_3)\,dS = \rho_3\vec{V}_3 A_3$

② $F_{Thrust}\cdot\hat{n}_3 = \dot{m}_{out}\vec{V}_3 - \cancel{\sum_{in}\dot{m}\vec{V}} + (P_3 - P_{atm})A_3 \rightarrow \vec{F}_{Thrust} = \dot{m}_{out}\vec{V}_3 + (P_3 - P_{atm})A_3$

③ $\dot{E}_{comb} = \dot{m}_{out}\left(\frac{\vec{V}_3^{\,2}}{2} + h_3\right) - \sum \dot{m}_{in}\left(\frac{\vec{V}_{in}^{\,2}}{2} + h_{in}\right) + \cancel{\dot{Q}} + \cancel{\dot{W}_{ext}} = \dot{m}_{out}\left(\frac{\vec{V}_3^{\,2}}{2} + h_3\right) - \sum \dot{m}_{in}h_{in}$

Mass Conserv. → Exit Velocity

① → $\sum \dot{m}_{in} = \dot{m}_1 + \dot{m}_2 = 140.3 + 374.2 = 514.5\ \mathrm{kg/s}$

→ $\rho_3 \vec{V}_3 A_3$; Also use IGL @ exit: $P_3 = \rho_3 R T_3$

→ $A_3 = \pi/4\, d^2 = 3.664\,\mathrm{m}^2$

→ $d = 2.16\,\mathrm{m}$

→ $R = \dfrac{\bar{R}}{M} = \dfrac{8314}{7.4} = 1123.51\ \dfrac{\mathrm{J}}{\mathrm{kg \cdot K}}$

Dims. $\left\{ \begin{array}{l} \to \bar{R} = 8314\ \dfrac{\mathrm{J}}{\mathrm{kg \cdot mol \cdot K}} \\ \to M = 7.4\ \dfrac{\mathrm{kg}}{\mathrm{kg \cdot mol}} \end{array} \right\} \dfrac{\mathrm{J}}{\cancel{\mathrm{kg \cdot mol}} \cdot \mathrm{K}} \cdot \dfrac{\cancel{\mathrm{kg \cdot mol}}}{\mathrm{kg}} = \dfrac{\mathrm{J}}{\mathrm{kg \cdot K}}$

→ $P_3 = 30E^3\ \mathrm{Pa}$

→ $T_3 = 500\ \mathrm{K}$

$$\Rightarrow \rho_3 = \frac{P_3}{R \cdot T_3} = \frac{30E^3}{1123.51 \cdot 500} = 0.0534\ \frac{\mathrm{kg}}{\mathrm{m}^3}$$

Dims: → $\dfrac{\mathrm{N}}{\mathrm{m}^2} \cdot \dfrac{\mathrm{kg} \cdot \cancel{\mathrm{K}}}{\mathrm{J}} \cdot \dfrac{1}{\cancel{\mathrm{K}}} = \dfrac{\mathrm{kg} \cdot \cancel{\mathrm{m}}}{\cancel{\mathrm{s}^2} \cdot \cancel{\mathrm{m}^2}} \cdot \dfrac{\cancel{\mathrm{s}^2}\, \cancel{\mathrm{kg}}}{\cancel{\mathrm{kg}} \cdot \mathrm{m}^2} = \dfrac{\mathrm{kg}}{\mathrm{m}^3}$

$$\Rightarrow \vec{V}_3 = \frac{\sum \dot{m}_{in}}{\rho_3\, A_3} = \frac{514.5}{0.0534 \cdot 3.664} = \boxed{2629.40\ \mathrm{m/s}}$$

Dims: → $\dfrac{\cancel{\mathrm{kg}}}{\mathrm{s}} \cdot \dfrac{\cancel{\mathrm{m}^3}}{\cancel{\mathrm{kg}}} \cdot \dfrac{1}{\cancel{\mathrm{m}^2}} = \mathrm{m/s}$

/ Momentum Conservation → Thrust

$$② \rightarrow \vec{F}_{Thrust} = \dot{m}_{out}\vec{V}_3 + (P_3 - P_{atm})A_3$$

$$\rightarrow \dot{m}_{out} = \sum_{in} \dot{m} = 574.5 \ \text{kg/s}$$

$$\rightarrow \vec{V}_3 = 2629.40 \ \text{m/s}$$

$$\rightarrow P_3 - P_{atm} = 30E^3 \ \text{Pa} - 70E^3 \ \text{Pa} = -50E^3 \ \text{Pa}$$

$$\rightarrow A_3 = 3.664 \ \text{m}^2$$

$$\Rightarrow \vec{F}_{Thrust} = (514.5 \cdot 2629.4) - (50E^3 \cdot 3.664) = \underline{1.16961E^6 \ N}$$

$$\text{Dims:} \rightarrow \left(\frac{kg}{s} \cdot \frac{m}{s}\right) - \left(\frac{N}{\cancel{m^2}} \cdot \cancel{m^2}\right) = \frac{kg \cdot m}{s^2} = N$$

/ Energy Conservation → Combustion Energy

$$③ \rightarrow \dot{E}_{comb.} = \dot{m}_{out}\left(\frac{\vec{V}_3^2}{2} + h_3\right) - \sum \dot{m}_{in} h_{in}$$

$$\rightarrow h_3 = C_{p_3} T_3 = 4168 \cdot 500 = 2084E^3 \ \frac{J}{kg}$$

$$\rightarrow h_1 = C_1 T_1 = 9668 \cdot 20 = 193.36E^3 \ \frac{J}{kg}$$

$$\rightarrow h_2 = C_2 T_2 = 347 \cdot 90 = 31.23E^3 \ \text{J/kg}$$

$$\text{Dims:} \rightarrow \frac{J}{kg \cdot \cancel{K}} \cdot \cancel{K} = \frac{J}{kg}$$

$$\rightarrow \dot{E}_{comb} = \dot{m}_3\left(\frac{\vec{V}_3^2}{2} + h_3\right) - \dot{m}_1 h_1 - \dot{m}_2 h_2$$

$$= 514.5\left(\frac{2629.4^2}{2} + 2084E^3\right) - 140.3 \cdot 193.36E^3 - 374.2 \cdot 31.23E^6$$

$$\Rightarrow \underline{\dot{E}_{comb} = 2.812E^9 \ W}$$

$$\text{Dims:} \rightarrow \frac{kg}{s} \cdot \frac{m^2}{s^2} = \frac{kg \cdot m^2}{s^2} \cdot \frac{1}{s} = \frac{J}{s} = W$$

Results:

$$\overrightarrow{V_3} = 2629.40\, m/s$$

$$\overrightarrow{F_T} = 1.16961 E^6\, N\, (MN)$$

$$\dot{E_C} = 2.812 E^9\, W\, (GW)$$

Discussion:

The control volume used for this analysis consists of the entirety of the rocket motor and contains three notable control surfaces. Two of these are inlets to the combustion chamber for liquid fuel and oxidizer, and the third is the nozzle exit beyond the combustion chamber. After making the needed assumptions, listed below the problem statements as well as the full written solution in Appendix B, the three necessary conservation equations can be written out in integral form. Steady flow assumption simplifies this greatly.

The mass conservation equation can be simplified greatly because we already know the mass flow rate of each of the inlets, so we do not need to bother too greatly with the reservoir assumptions here and can just replace the inlet terms with numerical mass flow values. The reservoir assumptions will come in later though, so it is important to understand that the velocity is not *exactly* zero, just very close and considered negligible. To compensate for this, the area term is very large, logistically allowing us to still use the mass conservation equation. Thus, the total mass in can be equated to the total mass out. This leaves us with one equation and two unknowns, exit velocity and exit density. The second equation will come from the ideal gas law, and after computing the specific gas constant for the exit gases, this can be used to solve for density. Plugging this back into our mass conservation allows us to finally find the exit velocity.

Next, momentum conservation can be applied to estimate the thrusting force produced by the escaping exhaust gases. First, the general 'external forces' term can be isolated down to just the thrusting force because of our assumptions and the control volume being used. Next, the reservoir assumptions are used to simplify further, as the liquid fuel and oxidizer have miniscule entrance velocity and negligibly contribute to the momentum of the system. This equation can be easily solved for the thrusting force, as we just determined the exit velocity, mass outflow, exit area, and we already know some exit parameters.

Finally, energy conservation is used to estimate the energy addition due to combustion. The reservoir assumptions help simplify this as well, as the inlet velocity terms are both zero. The $\dot{Q}$ term can be neglected as well, as the chamber is stated to be perfectly insulated. Since there are no external forces acting, which was determined in the momentum derivation, the $\dot{W_{ext}}$ can also be neglected. Now, this process is like the last two, except that the enthalpies across all three control surfaces must be calculated first. This is simple, as we know the equation for enthalpy as a function specific heat, all of which were provided in the problem statement. Note that, for liquids, there is no difference between c_p and c_v, and for the exit gases, c_p is used. A nozzle converts potential energy in the form of high pressure to kinetic energy to provide thrust, and as such the specific heat at a constant pressure cannot be used (also because the equation for h is a function of c_p rather than c_v).

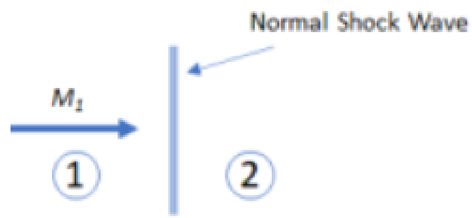
The velocity, thrust, and combustion energy rate of the incompressible, supersonic flow are estimated with relative precision using the three basic conservation equations, a few other easy known equations, and some stable assumptions. These are the equations that generally govern the mechanics of fluids and can be applied more extensively and in more dimensions to characterize other, potentially more complex fluid flows.

2. **Question 2:**

Consider the flow shown in the figure, where a flow traveling a Mach number, M_1, encounters a normal shock wave. The pressure and density ratios across the shock wave are given as:

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2}$$



Flow encountering a normal shock wave.

a) *Write a MATLAB function that returns the pressure, density, and temperature ratios for a given Mach number, M_1, and ratio of specific heats, γ.*
b) *Plot the entropy change, $\Delta s/c_p$, as a function of M_1 for a range of Mach number between 0.4 and 4. Explain how you know that a shock wave is impossible if $M_1 < 1$.*

Assume: Steady flow, 1-dimensional flow, shock wave normal to flow direction, calorically perfect exit gas, adiabatic
Given: $\gamma = 1.4$, pressure and density ratios
Note: Assuming that the statement '… the MATLAB function itself should follow the entire problem solution…' is not implying that we need to code the entirety of the conservation, Mach number, and speed of sound equations derivations, and that we are allowed to start with known equations and the given equations.
Solution:

The given pressure and density ratios are derived from our three conservation equation definitions, the Mach number definition, and the equation for the speed of sound. To find the missing ratio, T_2/T_1, a similar approach to the one used in the question before this can be used: the ideal gas law equation. This can be done because of a key assumption, notably that the air is calorically perfect, thus implying ideal. Consider the ideal gas law:

$$p = \rho R T \qquad [2.2.1]$$

Which can be arranged to isolate temperature as a function of pressure and density. When considering this equation at two different stations, the constant R can be leveraged to equate the states of the air at both stations, forming the following incredibly useful relationship and can be rearranged to derive an expression for the normal shock temperature relation in terms of the pressure and density relations that have been provided in the problem statement:

$$\frac{p_1}{\rho_1 T_1} = R = \frac{p_2}{\rho_2 T_2}$$

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} \cdot \frac{\rho_1}{\rho_2} \quad [2.2.2]$$

Eq. [2.2.2] can either be solved after solving the other two relations, or we can plug in the known relations symbolically and derive a busy expression for the normal shock temperature relation in terms of γ and M_1:

$$\frac{T_2}{T_1} = \frac{\left((\gamma-1)M_1^2+2\right) \cdot \frac{\left(2\gamma(M_1^2-1)\right)}{\gamma+1} + 1}{M_1^2(\gamma+1)}$$

Next, we can derive an expression for the change in entropy using the property ratios returned by the function created in part (a). Evaluating this function in MATLAB for $0.4 \leq M_1 \leq 4$ and $\gamma = 1.4$ allows us to use the modified adiabatic relationship seen below and the computed ratios to determine the change in entropy as a function of Mach number:

$$\Delta s = c_p ln\left(\frac{T_2}{T_1}\right) - R\, ln\left(\frac{p_2}{p_1}\right)$$

$$\frac{\Delta s}{c_p} = ln\left(\frac{T_2}{T_1}\right) - \frac{R}{c_p}\, ln\left(\frac{p_2}{p_1}\right)$$

Where the specific heat at a constant pressure, c_p, is defined as:

$$c_p = \frac{\gamma \cdot R_{air}}{\gamma - 1}$$

Results:

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}$$

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} \cdot \frac{\rho_1}{\rho_2} = \frac{\left((\gamma-1)M_1^2+2\right) \cdot \frac{\left(2\gamma(M_1^2-1)\right)}{\gamma+1} + 1}{M_1^2(\gamma+1)}$$

The graph of $\frac{\Delta s}{c_p}(M_1)$ is seen below:

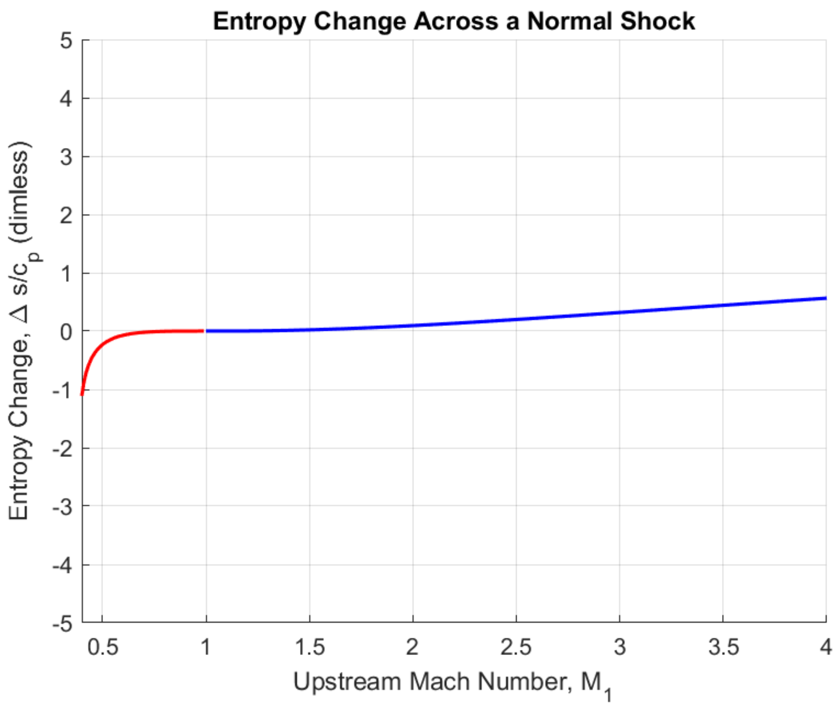


Figure 2.2.1: Graph of $\frac{\Delta s}{c_p}$ for $(0.4 \leq M_1 \leq 4)$

Discussion:

The MATLAB code to return the pressure, density, and temperature ratios can be found in Appendix A, and consists of a MATLAB function that computes the pressure and density ratios using the function parameters and then computes the temperature ratio numerically from those results. This function is then called from a script, passing in γ and M_1 to obtain numerical answers for these adiabatic relations. Then, the script computes the temperature relationship symbolically to obtain the long expression seen above and checks it numerically against the value returned from the function. This, of course, returns equivalent values, following the problem solution both the long way and the short way, as well as verifying a lightweight function that can be used for future homework questions.

Fig. [2.2.1] shows that the computed entropy changes across the normal shock wave. The red line segment indicates the computed value $\Delta s < 0$, and the blue line segment indicates $\Delta s \geq 0$. It is known from the second law of thermodynamics that ds for an adiabatic process, where $dq_a = 0$, is the quantity $ds_{irreversable}$, which must be equal to zero for a reversible process or greater than zero for an irreversible process. Δs in this scenario can *never* be a negative quantity, as it would break the second thermodynamic law. Thus, any value for M_1 where Δs is computed to be negative is physically impossible. Looking at the entropy change equation; this makes intuitive sense. For $M_1 = 1$, it is found that the pressure and temperature ratios are all simply 1, which causes the natural log terms to zero out, and $\Delta s = 0$, indicating a reversible process at exactly the speed of sound. For $M_1 < 1$, the pressure and temperature ratios are always going to be less than 1, which, when plugged into the entropy equation natural logs, will return a negative quantity. The universe cannot use engineering judgement to render this as an impossibility and disregard, which means that this scenario can never actually occur to begin with. It is safe to say that when $M_1 < 1$, there exists no normal shock wave to calculate thermodynamic properties across. This makes intuitive sense, as it's rather widely known that subsonic flow does not create supersonic shock waves.

Appendix A: MATLAB Code

```
%% Header
% Author: Zakary Steenhoek
% Date: 23 January 2025
% AEE362 HW01

clc; clear;

%% Find data files for NACA 4412

% Load data from the text files for both airfoils
dataPath = 'C:\Users\zaste\OneDrive\Documents\Software\MATLAB\AEE362\data\NACA4412';
figPath = ['C:\Users\zaste\OneDrive\Documents\Software\MATLAB' ...
        '\AEE362\HW01\figs'];
dataFiles = dir(fullfile(dataPath));

% Get rid of standard listed dirs
dataFiles(1:2)=[];

% To record the data
plot1Data = zeros(10, 6);

%% Math for NACA 4412

% Known variables
alphadeg = 4;
alpha = deg2rad(alphadeg);
itr1 = 1;

% For the number data files
for itr = 1:length(dataFiles)

    % Build path to the file
    filePath = fullfile(dataPath, dataFiles(itr).name);
    fileData = importdata(filePath);

    % Parse the filename to get the mach number
    subs = split(dataFiles(itr).name, '_'); lastSub = subs{end};
    fileInfo = erase(lastSub, '.txt');
    machNum = str2double(append('0.', num2str(sscanf(fileInfo, 'Ma%d'))));

    if contains(fileInfo, 'Cp')
        % Extract x and z columns and ipc column
        xData = flipud(fileData.data(:, 1));
        zData = flipud(fileData.data(:, 3));
        CpData = flipud(fileData.data(:, 4));
    elseif contains(fileInfo, 'Cf')
        CfData = flipud(fileData.data(:, 4));
        continue
    else
        fprintf('Error parsing filename: %s', fileInfo);
    end
```

```
    % Coefficient calculations
    cmac = trapz(xData, (xData-0.25).*CpData) ...
         + trapz(zData, zData.*CpData);
    cfx = trapz(zData, CpData)+trapz(xData, CfData);
    cfz = -trapz(xData, CpData)+trapz(zData, CfData);
    cl = cfz*cos(alpha)-cfx*sin(alpha);
    cd = cfz*sin(alpha)+cfx*cos(alpha);

    cd_p = -trapz(xData, CpData)*sin(alpha)+trapz(zData, CpData)*cos(alpha);
    cd_f = trapz(zData, CfData)*sin(alpha)+trapz(xData, CfData)*cos(alpha);

    plot1Data(itr1, 1) = machNum;
    plot1Data(itr1, 2) = cl;
    plot1Data(itr1, 3) = cd;
    plot1Data(itr1, 4) = cmac;
    plot1Data(itr1, 5) = cd_p;
    plot1Data(itr1, 6) = cd_f;


    % Plots
    n = length(xData);
    figNum = str2double(append('12', num2str(itr1)));
    figure(figNum); clf; hold on; grid on; axis ij;
    strTitle = append(append('C_p Distribution for NACA 4412 at Mach Number = ', ...
        num2str(machNum)), ', \alpha = 4 deg');
    title(strTitle);
    xlabel('X Coordinate (dimless)'); ylabel('Pressure Coefficient (dimless)');
    plot(xData(n/2:n), CpData(n/2:n), 'b', ...
         xData(1:n/2+1), CpData(1:n/2+1), 'b--');
    % xlim([]); ylim([]);
    hold off;
    itr1 = itr1+1;

    q = findobj('type','figure');
    autosave(q, fileInfo, figPath, 'png', false);
end

plot1Data = sort(plot1Data);

%% Plots

figure(111); clf; hold on; grid on;
title('C_l vs. Mach Number for NACA 4412, \alpha = 4 deg');
xlabel('Mach Number'); ylabel('Lift Coefficient (dimless)');
plot(plot1Data(:,1), plot1Data(:,2), 'k.-');
% xlim([]); ylim([]);
hold off;

% q = findobj('type','figure');
% autosave(q, 'C_l-Ma', figPath, 'png', false);

figure(112); clf; hold on; grid on;
title('C_d vs. Mach Number for NACA 4412, \alpha = 4 deg');
xlabel('Mach Number'); ylabel('Drag Coefficient (dimless)');
plot(plot1Data(:,1), plot1Data(:,3), 'k.-');
% xlim([]); ylim([]);
```

```
hold off;

% q = findobj('type','figure');
% autosave(q, 'C_d-Ma', figPath, 'png', false);

figure(113); clf; hold on; grid on;
title('C_mac vs. Mach Number for NACA 4412, \alpha = 4 deg');
xlabel('Mach Number'); ylabel('Moment Coefficient (dimless)');
plot(plot1Data(:,1), plot1Data(:,4), 'k.-');
% xlim([]); ylim([]);
hold off;

% q = findobj('type','figure');
% autosave(q, 'C_mac-Ma', figPath, 'png', false);

%% Header
% Author: Zakary Steenhoek
% Date: 23 January 2025
% AEE362 HW01

clc; clear;

%% Math

figPath = ['C:\Users\zaste\OneDrive\Documents\Software\MATLAB' ...
    '\AEE362\HW01\figs'];


% Constants
M1 = 1.2;
gamma = 1.4;

% Local functions
p = @(RHO,R,T) RHO*R*T;
h = @(Cp,T) Cp*T;

% Test function to check derivation later
[pressure, density, temperature] = property_ratios(M1, gamma);

% Proof
clear gamma M1
syms gamma M1

pressureRatio = 1+((2*gamma)/(gamma+1))*(M1^2-1);
densityRatio = ((gamma+1)*M1^2)/(2+(gamma-1)*M1^2);
temperatureRatio = simplify(pressureRatio*densityRatio^(-1));

M1 = 1.2;
gamma = 1.4;

pressureRatio = 1+((2*gamma)/(gamma+1))*(M1^2-1);
densityRatio = ((gamma+1)*M1^2)/(2+(gamma-1)*M1^2);
temperatureRatio = (((gamma - 1)*M1^2 + 2)*((2*gamma*(M1^2 - 1))/...
  (gamma + 1) + 1))/(M1^2*(gamma + 1));

% fprintf('\nFor M1 = %.1f and gamma = %.1f:\n', M1, gamma);
```

```
% fprintf('Pressure Ratio (p2/p1): %.4f\n', pressureRatio);
% fprintf('Density Ratio (rho2/rho1): %.4f\n', densityRatio);
% fprintf('Temperature Ratio (T2/T1): %.4f\n', temperatureRatio);

%% Part (b)

% More constants
R = 287;
cp = gamma * R / (gamma - 1);

% Mach number range
clear M1
M1 = 0.4:0.01:4;
% M1 = 1.5;

% To store computed values
ds = zeros(size(M1));
dsNeg = zeros(size(M1));

% Calculate entropy change for each M1
for i = 1:length(M1)
    % Invoke function from part (a)
    [P_ratio, ~, T_ratio] = property_ratios(M1(i), gamma);

    % Entropy change
    ds(i) = cp*log(T_ratio) - R*log(P_ratio);

    % Check positive entropy
    if ds(i) < 0
        dsNeg(i) = ds(i);
        ds(i) = NaN;
    else
        dsNeg(i) = NaN;
    end
end

% Plot entropy change
figure(1); clf; grid on; hold on;
plot(M1, ds/cp, 'b', 'LineWidth', 1.5);
plot(M1, dsNeg/cp, 'r', 'LineWidth', 1.5)
xlabel('Upstream Mach Number, M_1');
ylabel('Entropy Change, \Delta s/c_p (dimless)');
title('Entropy Change Across a Normal Shock');
xlim([0.4 4]); ylim([-5 5]);
hold off;

% q = findobj('type','figure');
% autosave(q, 'Ds', figPath, 'png', false);
```

Appendix B: Part 2, Question 1 Full Written Solution

Part 2.

1. The figure shows a representation of the space shuttle main engine liquid rocket motor. Use control-volume analysis for solving this problem, and be sure to show the control volume you used.

The rocket fuel (liquid hydrogen at 20 K) and the oxidizer (liquid oxygen at 90 K) enter the combustion chamber from tanks that can be assumed to be reservoirs ($V = 0$ relative to the motor). The specific heat of liquid hydrogen is [illegible] and the specific heat of liquid oxygen is [illegible]. Assume the exit gas has a molecular weight of [illegible] kg/kg-mol and c_p = [illegible] J/kg·K. Also recall that the universal gas constant $\mathcal{R}$ = 8314 J/(kg-mol K). Note that for supersonic flows, the exit pressure need not be the same as the atmospheric pressure. The space shuttle is passing through an altitude of 10,000 ft where the atmospheric pressure is 70 kPa.

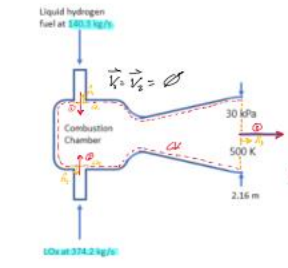
(a) For the conditions shown in the figure, find the velocity (relative to the motor) at the nozzle exit.

(b) Estimate the thrust force the motor exerts on the shuttle (to which it is attached) for the conditions shown. Be sure to state the assumptions you had to use to find this value.

(c) Estimate the rate of energy addition to the system due to combustion. You may consider the combustion chamber to be perfectly insulated. Hint: Recall that the rate of mass flow across a surface entering the combustion chamber is given by

$$\dot{m} = \int_{surface} \rho(\vec{V}\cdot\hat{n})\,dS$$

Notes
- Liquid: $C_p = C_v = C$
- $\to \gamma = \frac{C_p}{C_v} = 1$



Assume

- Assuming $\vec{V}$ of LO_x & LH is zero, so A must be large to maintain mass conserv.
- Assume steady: $\sum \dot{m}_{in} = \sum \dot{m}_{out}$

$$\frac{\partial}{\partial t}\int_{cv} \rho\, d\forall = \varnothing$$

$$\frac{\partial}{\partial t}\int_{cv} \rho\vec{V}(\vec{V}\cdot\hat{n})\, d\forall = \varnothing$$

$$\frac{\partial}{\partial t}\int_{cv} \rho(e + \tfrac{1}{2}V^2)\, d\forall = \varnothing$$

- Calorically Perfect exit gas
- Circular nozzle exit
- Neglect Viscous, gravitational, Electromagnetic forces @ exit
- No body forces
- Adiabatic $\to \dot{Q} = \varnothing$
- Isolated $\to \dot{W}_{ext} = \varnothing$

Integral Eqns.

① Mass: $\int_{CS} \rho(\vec{V}\cdot\hat{n})\, dS = \varnothing$

② Momentum: $\int_{CS} \rho\vec{V}(\vec{V}\cdot\hat{n})\, dS + \int_{CS} P\hat{n}\, dS = \vec{F}_{Thrust}$

③ Energy: $\int_{CS} \rho(\tfrac{1}{2}V^2 + h)(\vec{V}\cdot\hat{n})\, dS = \dot{Q} + \dot{W}_{ext}$

Simplify

① $\sum \dot{m}_{in} = \dot{m}_1 + \dot{m}_2 = \dot{m}_{out} \to \sum \dot{m}_{3_n} = \int_{CS_3} \rho_3(\vec{V}_3\cdot\hat{n}_3)\, dS = \rho_3\vec{V}_3 A_3$

② $F_{Thrust}\cdot\hat{n}_3 = \dot{m}_{out}\vec{V}_3 - \cancel{\sum \dot{m}_{in}\vec{V}} + (P_3 - P_{atm})A_3 \to \vec{F}_{Thrust} = \dot{m}_{out}\vec{V}_3 + (P_3 - P_{atm})A_3$

③ $\dot{E}_{comb} = \dot{m}_{out}\left(\frac{\vec{V}_3^2}{2} + h_3\right) - \sum \dot{m}_{in}\left(\frac{\vec{V}_{in}^2}{2} + h_{in}\right) + \cancel{\dot{Q}} + \cancel{\dot{W}_{ext}} = \dot{m}_{out}\left(\frac{\vec{V}_3^2}{2} + h_3\right) - \sum \dot{m}_{in} h_{in}$

Solve

Mass Conserv. → Exit Velocity

① $\to \sum \dot{m}_{in} = \dot{m}_1 + \dot{m}_2 = 140.3 + 374.2 = 514.5$ kg/s

$\to \rho_3\vec{V}_3 A_3$; Also use IGL @ exit: $P_3 = \rho_3 R T_3$

$\to A_3 = \pi/4\, d^2 = 3.664\, m^2$

$\to d = 2.16$ m

$\to R = \frac{\mathcal{R}}{M} = \frac{8314}{24} = 1123.51\ \frac{J}{kg\cdot K}$

Dims: $\to \mathcal{R} = 8314 \frac{J}{kg\text{-}mol\cdot K}$, $\to M = 24 \frac{kg}{kg\text{-}mol}$; $\frac{J}{kg\text{-}mol\cdot K}\cdot\frac{kg\text{-}mol}{kg} = \frac{J}{kg\cdot K}$

$\to P_3 = 30E^3$ Pa

$\to T_3 = 500$ K

$\Rightarrow \rho_3 = \frac{P_3}{R\cdot T_3} = \frac{30E^3}{1123.51\cdot 500} = 0.0534\ \frac{kg}{m^3}$

Dims: $\to \frac{N}{m^2}\cdot\frac{kg\cdot K}{J}\cdot\frac{1}{K} = \frac{kg\cdot m}{s^2\cdot m^2}\cdot\frac{s^2\, kg}{kg\cdot m^2} = \frac{kg}{m^3}$

$\Rightarrow \vec{V}_3 = \frac{\sum \dot{m}_{in}}{\rho_3 A_3} = \frac{514.5}{0.0534\cdot 3.664} = \boxed{2629.40\ m/s}$

Dims: $\to \frac{kg}{s}\cdot\frac{m^3}{kg}\cdot\frac{1}{m^2} = m/s$

Momentum Conservation → Thrust

② $\to \vec{F}_{Thrust} = \dot{m}_{out}\vec{V}_3 + (P_3 - P_{atm})A_3$

$\to \dot{m}_{out} = \sum_{in} \dot{m} = 514.5$ kg/s

$\to \vec{V}_3 = 2629.40$ m/s

$\to P_3 - P_{atm} = 30E^3$ Pa $- 70E^3$ Pa $= -50E^3$ Pa

$\to A_3 = 3.664\, m^2$

$\Rightarrow \vec{F}_{Thrust} = (514.5\cdot 2629.4) - (50E^3\cdot 3.664) = 1.16961E^6$ N

Dims: $\to \left(\frac{kg}{s}\cdot\frac{m}{s}\right) - \left(\frac{N}{m^2}\cdot m^2\right) = \frac{kg\cdot m}{s^2} = N$

Energy Conservation → Combustion Energy

③ $\to \dot{E}_{comb} = \dot{m}_{out}\left(\frac{\vec{V}_3^2}{2} + h_3\right) - \sum \dot{m}_{in} h_{in}$

$\to h_3 = C_{p_3} T_3 = 4168\cdot 500 = 2084E^3\ \frac{J}{kg}$

$\to h_1 = C_1 T_1 = 9668\cdot 20 = 193.36E^3\ \frac{J}{kg}$

$\to h_2 = C_2 T_2 = 347\cdot 90 = 31.23E^3\ J/kg$

Dims: $\to \frac{J}{kg\cdot K}\cdot K = \frac{J}{kg}$

$\to \dot{E}_{comb} = \dot{m}_3\left(\frac{\vec{V}_3^2}{2} + h_3\right) - \dot{m}_1 h_1 - \dot{m}_2 h_2$

$= 514.5\left(\frac{2629.4^2}{2} + 2084E^3\right) - 140.3\cdot 193.36E^3 - 374.2\cdot 31.23E^3$

$\Rightarrow \dot{E}_{comb} = 2.812E^9$ W

Dims: $\to \frac{kg}{s}\cdot\frac{m^2}{s^2} = \frac{kg\cdot m^2}{s^2}\cdot\frac{1}{s} = \frac{J}{s} = W$