High Speed Aerodynamics Homework 04

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February 27, 2025

1. **Questions 1:**

A large reservoir contains air at a pressure of p_0 and temperature T_0. The air is expelled to a large region (having pressure p_b) through a *converging* nozzle.

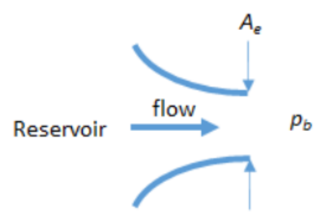

Figure for problem 1.

a) *Show that the mass flow rate through the nozzle can be written as:*

$$\dot{m} = p_0 \sqrt{\frac{\gamma}{RT_0}} A_e M_e \left[1 + \frac{\gamma - 1}{2} M_e^2\right]^{-\frac{\gamma+1}{2(\gamma-1)}}$$

b) *Assume that* $p_0 = \text{constant} = 800\,kPa$. *Plot the quantity* $\dot{m}\sqrt{T_0}/A_e$ *vs. the ratio* p_b/p_0 *for* $0 \leq p_b/p_0 \leq 1$.

c) *Assume that* $p_b = \text{constant} = 100\,kPa$. *Plot the quantity* $\dot{m}\sqrt{T_0}/A_e$ *vs. the ratio* p_b/p_0 *for* $0 < p_b/p_0 \leq 1$.

d) *Explain why the curves in parts (b) and (c) are different from each other.*

Assume:

Quasi-one-dimensional flow. Steady, isentropic flow. Calorically perfect air. $M_e \leq 1$.

Given:

Constant reservoir pressure $p_{0_1} = 800\,kPa$. Constant back pressure $p_{b_2} = 100\,k\text{Pa}$. Pressure ratio $0 \leq \text{p}_\text{b}/\text{p}_0 \leq 1$

Solution:

To derive mass flow rate in the form given, first consider the general mass conservation integral equation:

$$\oiint_S \rho V \, dS = -\frac{\partial}{\partial t} \oiiint_{\mathcal{V}} \rho \, d\mathcal{V}$$

Under steady flow, the RHS partial derivative w.r.t. time will go to zero, as it is assumed that there is no accumulation of mass in the nozzle. The leaves the LHS, which measures mass flux inflow and outflow through the nozzle at any time as $\dot{m}$. Through the nozzle, the mass inflow comes from the reservoir, and the mass outflow is through the exit. Let the control volume be defined such that mass inflow happens through a surface S_1 near the nozzle entrance, the mass outflow happens through a surface S_e at the exit with an area A_e, and no mass flow occurs through the solid boundary of the nozzle. The mass conservation integral then reduces to:

$$\int_{A_1} \rho\left(\overrightarrow{V_1} \cdot \widehat{n_1}\right)\, d\mathrm{A}_1 + \int_{A_e} \rho\left(\overrightarrow{V_e} \cdot \widehat{n_e}\right)\, d\mathrm{A}_\mathrm{e} = 0$$

The normal vector $\hat{n}$ is defined as the unit vector orthogonal to the control volume surface at any point, where positive convention points away from the volume. Thus, the mass inflow dot product term will make the integral across A_1 negative, yielding:

$$\int_{A_1} \rho\left(\overrightarrow{V_1} \cdot \widehat{n_1}\right)\, d\mathrm{A}_1 = \int_{A_e} \rho\left(\overrightarrow{V_e} \cdot \widehat{n_e}\right)\, d\mathrm{A}_\mathrm{e}$$

Evaluating this integral equation gives the known simplified equation for quasi-one-dimensional mass conservation. It is similar in form to one-dimensional flow, where flow properties are function of a single positional variable throughout the CV with a constant cross-section, inversely relating only density and velocity. Quasi-one-dimensional flow additionally allows for variations in cross-sectional areas. The assumptions here are that the flow properties are constant across any cross-sectional area, thus still vary only w.r.t. a single positional variable along each streamline. Net changes in flow parameters in the other dimensions exist, but are assumed to be negligibly small and ignored for this approximation. Thus, general mass conservation through the nozzle:

$$\dot{m} = \rho_1 u_1 A_1 = \rho_e u_e A_e \tag{1.1}$$

Assuming isentropic flow through the nozzle with no heat addition or viscous effects, and assuming perfect gas behavior for air with constant $\gamma = 1.4$ and $R = 287\, J/kg \cdot K$, isentropic relations can be used to express mass flow at the nozzle exit in terms of local properties and thermodynamic properties. First, the Ideal Gas Law for density:

$$p = \rho RT \rightarrow \rho = \frac{p}{RT}$$

And the sound speed relation for velocity:

$$u = M \cdot a = M\sqrt{\gamma RT}$$

So:

$$\dot{m} = \rho_e u_e A_e = \frac{p_e}{RT_e} \cdot M_e\sqrt{\gamma RT_e} \cdot A_e = p_e A_e M_e \sqrt{\frac{\gamma}{RT_e}}$$

This can be expressed more generally under the isentropic assumption in terms of stagnation properties, removing the dependency on the local static exit conditions. Consider the known isentropic stagnation relations for pressure and temperature:

$$\frac{p_0}{p} = \left[1 + \frac{\gamma - 1}{2} M^2\right]^{\frac{\gamma}{\gamma - 1}} \tag{1.2}$$

$$\frac{T_0}{T} = \left[1 + \frac{\gamma - 1}{2} M^2\right] \tag{1.3}$$

Substituting and rearranging:

$$p_e = p_0 \left[1 + \frac{\gamma - 1}{2} {M_e}^2\right]^{-\frac{\gamma}{\gamma-1}}$$

$$T_e = T_0 \left[1 + \frac{\gamma - 1}{2} M_e^2\right]^{-1} \rightarrow T_e^{-1/2} = T_0^{-1/2} \left[1 + \frac{\gamma - 1}{2} M_e^2\right]^{1/2}$$

$$\dot{m} = p_0 A_e M_e \sqrt{\frac{\gamma}{RT_0}} \cdot \left[\left[1 + \frac{\gamma - 1}{2} {M_e}^2\right]^{-\frac{\gamma}{\gamma-1}} \cdot \left[1 + \frac{\gamma - 1}{2} M_e^2\right]^{1/2}\right]$$

$$\dot{m} = p_0 A_e M_e \sqrt{\frac{\gamma}{RT_0}} \cdot \left[\left[1 + \frac{\gamma - 1}{2} {M_e}^2\right]^{-\frac{\gamma}{\gamma-1}+\frac{1}{2}}\right]$$

Thus, the final expression for mass conservation:

$$\dot{m} = p_0 \sqrt{\frac{\gamma}{RT_0}} A_e M_e \left[1 + \frac{\gamma - 1}{2} M_e^2\right]^{-\frac{\gamma+1}{2(\gamma-1)}} \qquad [1.4]$$

This can be rearranged to represent the desired quantity:

$$\frac{\dot{m}\sqrt{T_0}}{A_e} = p_0 \sqrt{\frac{\gamma}{R}} M_e \left[1 + \frac{\gamma - 1}{2} M_e^2\right]^{-\frac{\gamma+1}{2(\gamma-1)}} \qquad [1.5]$$

Where the exit Mach number M_e can be found as a function of backpressure from the total pressure relation:

$$\frac{p_0}{p_b} = \left[1 + \frac{\gamma - 1}{2} {M_e}^2\right]^{\frac{\gamma}{\gamma-1}}$$

Solving for M_e:

$$M_e = \sqrt{\frac{2}{\gamma - 1}\left[\left(\frac{p_b}{p_0}\right)^{-\frac{\gamma-1}{\gamma}} - 1\right]} \qquad [1.6]$$

Note that for a purely converging nozzle, the area-velocity relation restricts the Mach number at the exit to be less than or equal to one, at which point choking occurs. Thus, the final expression for normalized mass rate in terms of total pressure, back pressure ratio, and thermodynamic constants:

$$\frac{\dot{m}\sqrt{T_0}}{A_e} = p_0 \sqrt{\frac{\gamma}{R}} \cdot \sqrt{\frac{2}{\gamma - 1}\left[\left(\frac{p_b}{p_0}\right)^{-\frac{\gamma-1}{\gamma}} - 1\right]} \cdot \left[1 + \frac{\gamma - 1}{2} \cdot \left[\frac{2}{\gamma - 1}\left[\left(\frac{p_b}{p_0}\right)^{-\frac{\gamma-1}{\gamma}} - 1\right]\right]\right]^{-\frac{\gamma+1}{2(\gamma-1)}}$$

This expression can then be computed as a function of $\frac{p_b}{p_0}$ over $0 \leq \mathrm{p_b}/\mathrm{p_0} \leq 1$ while holding constant either p_0 or p_b and plotted to visualize how changes in reservoir pressure and backpressure effect mass rate.

Results:

$$\dot{m} = \rho_e u_e A_e = p_0 \sqrt{\frac{\gamma}{RT_0}} A_e M_e \left[1 + \frac{\gamma - 1}{2} M_e^2\right]^{-\frac{\gamma+1}{2(\gamma-1)}}$$

The plot of normalized mass flux for $p_0 = 800\ kPa$:

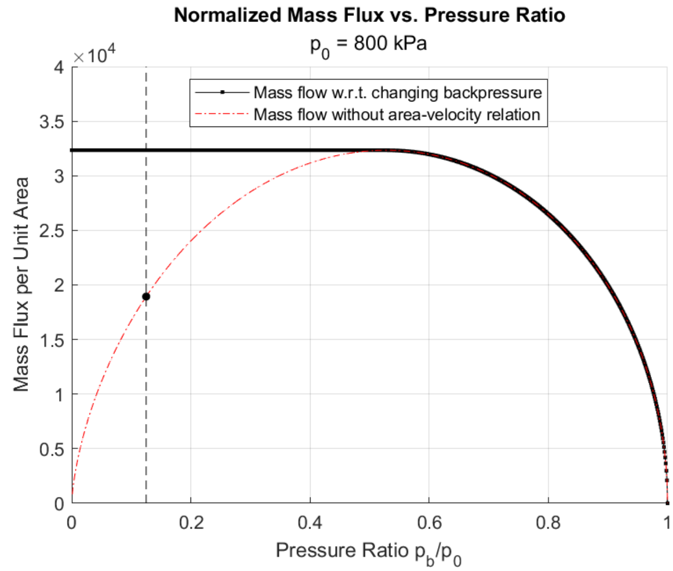

Figure 1.1: Normalized mass flux quantity as a function of backpressure plotted against the backpressure total pressure ratio from $0 \leq p_b \leq p_0$

The plot of normalized mass flux for $p_b = 100\ k$Pa:

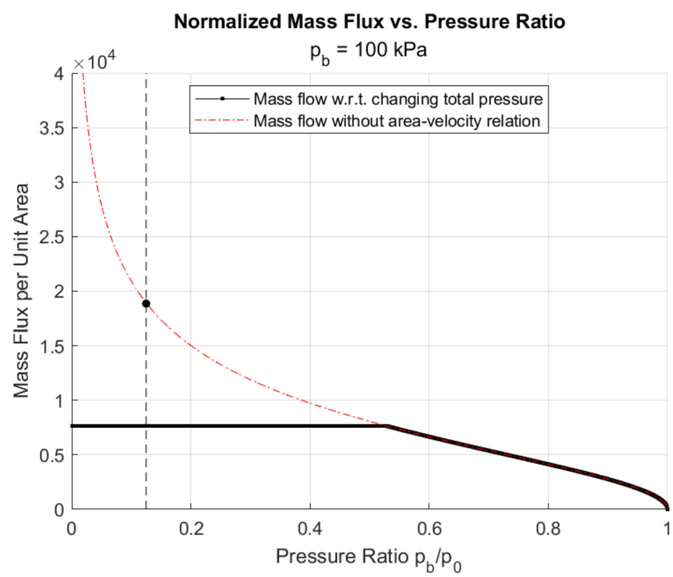

Figure 1.2: Normalized mass flux quantity as a function of total pressure plotted against the backpressure total pressure ratio from $\infty \leq p_0 \leq p_b$

Discussion:

Ideal, isentropic, steady assumptions make it possible to express mass rate through the nozzle in terms of thermodynamic constants, nozzle characteristics, and a measurable pressure ratio. This representation allows the amount of mass flowing through a volume for some continuous ideal gas to vary only as a function of total reservoir pressure and the ambient pressure conditions at the exit. Fixing either to a constant essentially demonstrates how changes to the other impact the amount of mass that can be driven through the nozzle.

Fig. [1.1] demonstrates what happens with a constant high pressure in the reservoir when the backpressure varies from (mathematically) no pressure to the point where the pressures are equalized. Numerically, Fig. [1.1] reaches a maximum norm. mass rate of $3.2335 \cdot 10^4\ \frac{kg\sqrt{°\mathrm{K}}}{s \cdot m^2}$ when the pressure ratio is 0.528. This is the choked condition, where the flow reaches Mach one at the throat and mass flow stops increasing. Through the subsonic range, the mass curve decreases parabolically as the differential pressure drops, eventually attaining zero mass flow at pressure equilibrium.

Fig. [1.2] demonstrates what happens with a constant pressure in the exit environment when the reservoir pressure varies from an infinitely large pressure to the point where the pressures are equalized. Numerically, Fig. [1.2] reaches a maximum norm. mass rate of $0.76550 \cdot 10^4\ \frac{kg\sqrt{°\mathrm{K}}}{s \cdot m^2}$ when the pressure ratio is 0.528. This is still the choked condition, and the mass flow will not increase beyond this. Through the subsonic range, the mass flow decreases much more linearly as the pressure differential drops and eventually attains zero at pressure equilibrium.

It's seen that the choked pressure condition - known as the critical pressure ratio - whereby definition Mach number equals one, is only a function of thermodynamic properties. For perfect air, choking occurs at the critical pressure ratio of 0.528. The equation for maximum normalized choked mass flow is then only a function of total driving reservoir pressure and thermodynamic constants. This realizes the difference between the curves in parts (b) and (c) – for the same fluid, maximum mass flow is parameterized only by the magnitude of static pressure at the critical pressure ratio. Static pressure is defined differently in cases (b) and (c). In (b), it is a constant $p_0 = 800\ kPa$, however in (c), it varies as a function of the pressure ratio and fixed backpressure and is found to be $p_0 = 189.04\ kPa$. This aligns with what we see mathematically - the fixed reservoir pressure at the critical pressure ratio is roughly 4.2 times larger than the variable reservoir pressure at the critical pressure ratio, and the mass flow is scaled by an equal amount. Thus, maximizing normalized mass flow of a perfect gas through a converging nozzle section is a matter of driving fluid from a region with some large total pressure to a region where the local static pressure solves the critical pressure ratio of the fluid.

Figs. [1.1-1.2] also show the theoretical mathematical mass flow described by Eq. [1.5] (the red dotted line) if the area-velocity relation restrictions are not applied. These curves show the unrestricted mathematical effects of changing one pressure versus the other that stem from the way that the pressure ratio is defined, as back pressure over reservoir pressure. Looking specifically at the case described by (b) & Fig. [1.1]: with constant reservoir pressure, the back pressure (and thus the pressure differential) changes linearly w.r.t. the ratio, i.e. there exists a directly proportional relationship. This is unlike (c) & Fig. [1.2], where constant back pressure describes an inversely proportional relationship. Since the Mach curve, restricted or not, is constant between these two cases, the direct/inverse relationship from the pressure ratio is the only one that really matters. In the imaginary 'supersonic' regime in Fig.

[1.1], a constant p_0 is not enough to dominate the equation, and Eq. [1.5] drives mass flow back to zero at high Mach. Conversely in the same imaginary 'supersonic' regime in Fig. [1.2], the inverse p_b/p_0 relation drives $p_0 \rightarrow \infty$ for a small pressure ratio. This mathematically dominates Eq. [1.5] for any Mach, and thus the mass rate asymptotically approaches infinity as the driving reservoir pressure grows to infinity. Neither of these regimes are physically possible as Mach cannot exceed one for a converging nozzle, but it communicates the different curve shape in terms of varying a single pressure.

Another note about the red unrestricted mass flow curves: the pressure ratio for which both cases experience the same differential condition, $p_b/p_0 = 100\ kPa/800\ kPa = 0.125$, is known to be in the theoretical supersonic regime for perfect air according to Eq. [1.6]. For both cases, however, the theoretical mass flows at this point are roughly equal, regardless of the different curve behavior. This point is denoted in the plots by the black marker at the intersection of the theoretical mass curve and the vertical line at $p_b/p_0 = 0.125$, where the normalized mass flow is roughly $1.89 \cdot 10^4\ \frac{kg\sqrt{°\mathrm{K}}}{s \cdot m^2}$. This works to show that, mathematically speaking, Eq. [1.5] holds for some point where the two cases experience the same pressure conditions. This emphasizes the importance of applying area-velocity relations, as the true mass rate under the same conditions is much different than what is predicted and is dictated by the attainable reservoir pressure at the gas critical pressure ratio.

Essentially, the difference between the curves stems from the nature of the relationship between pressures. The directly proportional relationship in case (b) prescribes a higher p_0 at critical pressure while the inverse relationship in case (c) prescribes a lower p_0 at critical pressure. This leads that decreasing back pressure for some p_0 through the subsonic pressure ratio range is more effective than increasing driving pressure for some p_b when maximizing mass rate through a diverging nozzle. The former method preserves a larger total pressure at criticality rather than scaling it down to meet backpressure ratio requirements.

2. **Question 2:**

A pressurized reservoir drives a supersonic wind tunnel shown in the figure. It is desired to have $M = 2.0$ in the test section. Assume that the temperature in the reservoir is held constant at $25\,°C$.

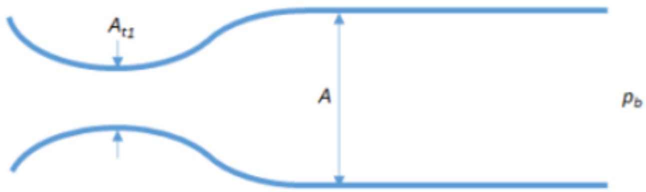

Figure for parts a, b, and d.

a) *If the test section has area $0.5\,m^2$, what should be the area at the throat A_{t1}? If the tunnel exhausts out to the atmosphere ($p_b = 100\,kPa$), what does the pressure in the reservoir have to be for ideal isentropic flow conditions throughout the wind tunnel and past the exit? What is the mass flow rate through the tunnel for this case?*

b) *Because it is not a good idea to have supersonic flow exit to the atmosphere, it is desired to operate the wind tunnel so that there is a shock wave formed just at the exit of the test section. What should be the pressure in the reservoir for this condition? What is the speed of the flow exiting the tunnel? Use the areas and atmospheric conditions from (a). What is the mass flow rate through the tunnel for this case?*

c) *We would like to reduce the speed of the exiting flow even further – to effectively zero – by adding a diffuser to the tunnel test section after the normal shock wave has formed. What should be the pressure in the reservoir for this case? What is the mass flow rate through the tunnel for this case?*

d) *Use the system without a diffuser to answer this problem. Say that the reservoir pump is malfunctioning and the highest pressure that can be attained in the reservoir is 1.5 atm. What is the resulting Mach number in the test section for this case? Explain and show how you obtained your answer. What is the mass flow rate through the tunnel for this case?*

Assume:

Quasi-one-dimensional flow. Steady, isentropic flow. Calorically perfect air. Normal exit shock.

Given:

Test section Mach $M_t = 2.0$. Constant static reservoir temperature $T_R = 25\,°C$. Test section cross-sectional area $A = 0.5\,m^2$. Atmospheric backpressure $P_b = 100\,kPa$. Maximum malfunctioning reservoir pressure $p_{0_{max}} = 1.5\,atm$

Solution:

To meet performance requirements for the supersonic tunnel under the given conditions, compressible quasi-1D conservation laws and thermodynamic relations can be used with the known parameters. This analysis assumes a steady, isentropic flow regime of perfect air such that stagnation properties remain constant:

$$\frac{T_0}{T} = \left[1 + \frac{\gamma - 1}{2} M^2\right]; \quad \frac{p_0}{p} = \left[\frac{T_0}{T}\right]^{\frac{\gamma}{\gamma-1}}; \quad \frac{\rho_0}{\rho} = \left[\frac{T_0}{T}\right]^{\frac{1}{\gamma-1}} \qquad [2.1]$$

The reservoir condition is such that it functions as an infinitely large volume of gas at a constant temperature and pressure with negligible fluid velocity. In an isentropic flow regime, the reservoir conditions exhibit the baseline total fluid properties, since:

$$\mathrm{M}_R = \frac{\mathrm{V}_R}{\mathrm{a}_0} \approx \frac{0}{a_0} \approx 0$$

Then, for constant reservoir temperature T_R and zero reservoir Mach M_R:

$$\frac{T_0}{T_R} = \left[1 + \frac{\gamma - 1}{2} M_R^2\right] = [1 + 0] \rightarrow T_0 = T_R = 298.15\,\mathrm{K}$$

Which is sufficient to prescribe the remaining reservoir relations. Another reference condition occurs at the throat, where the area-velocity and area-Mach number relations govern local fluid behavior:

$$\frac{dA}{A} = (M^2 - 1)\frac{du}{u} \qquad [2.2]$$

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma + 1} \cdot \left(1 + \frac{\gamma - 1}{2} M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} \qquad [2.3]$$

Eq. [2.2] for the area-velocity relation is derived from quasi-1D conservation equations for mass and momentum, Eqs. [2.4-2.5], as well as the isentropic assumption: for any dp there exists a corresponding isentropic $d\rho$, which can be expressed in terms of sound speed:

$$d(\rho u A) = d(\dot{m}) \rightarrow \frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

$$dp = -\rho u\, du \rightarrow \frac{dp}{\rho} = -u\, du = \frac{dp}{d\rho} \cdot \frac{d\rho}{\rho}, \quad \text{where} \quad \frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s = a^2$$

$$a^2 \cdot \frac{d\rho}{\rho} = -u\, du \rightarrow \frac{d\rho}{\rho} = -\frac{u^2 du}{a^2 u} = -M^2 \frac{du}{u}$$

Substituting this expression for $d\rho/\rho$ into the differential mass conservation equation yields Eq. [2.2], which relates changes in area and velocity as a function of the flow regime Mach number. As Mach number approaches zero, Eq. [2.2] evaluates to a constant, describing incompressible flow. At Mach 1, the RHS of Eq. [2.2] is zero, thus changes in area are unimpactful. For $0 < M < 1$, the quantity $(M^2 - 1) < 0$, thus for subsonic flow, decreasing area increases velocity, and vice versa. For $M > 1$, the quantity $(M^2 - 1) > 0$, which conversely says that increasing area speeds up flow. Eq. [2.3] for the area-Mach number relation is similarly derived from isentropic continuity principles, and relates the Mach number at any location to choked sonic conditions at the throat as a function of the local area for any $A \geq A^*$:

$$\dot{m} = \rho^* u^* A^* = \rho u V \rightarrow \frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{u^*}{u} = \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\frac{a^*}{u}; \quad u^* = a^* \text{ for } M^* = 1$$

Using the stagnation density relation at ρ^* and for any ρ, as well as the isentropic Mach relation:

$$\frac{\rho_0}{\rho} = \left[1 + \frac{\gamma - 1}{2}M^2\right]^{\frac{1}{\gamma-1}}; \quad \frac{\rho_0}{\rho^*} = \left[\frac{\gamma + 1}{2}\right]^{\frac{1}{\gamma-1}}; \quad M^{*^2} = \frac{(\gamma + 1)M^2}{2 + (\gamma - 1)M^2}$$

Which simplifies to Eq. [2.3] after substitution. Based on the design conditions, this equation can be used first with the desired local Mach $M_t = 2.0$ and thermodynamic properties of ideal air to determine the area ratio required to drive this magnitude of supersonic acceleration. From this, the local test area solves the required throat area:

$$\frac{A_t}{A^*} = \frac{1}{M_t}\left[\frac{2}{\gamma + 1} \cdot \left(1 + \frac{\gamma - 1}{2}{M_t}^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} = 32.0$$

$$\frac{A_t}{A^*} = 32.0 \rightarrow A^* = \frac{A_t}{32.0} = 0.0156\, m^2$$

When the tunnel exhausts out to the atmosphere after the test section with $dA = 0$, then the flow continues at Mach 2 to the exit condition, where $p_b = 100\, kPa$ and $M_e = 2.0$. For ideal isentropic flow through the entire tunnel, there must exist no discontinuity in fluid parameters, thus $p_b = p_e$. The isentropic assumption relates stagnation pressure to any local pressure and Mach, and it was determined earlier that stagnation pressure occurs in the reservoir. Evaluating the isentropic pressure relation for total pressure at the exit condition:

$$\frac{p_0}{p_e} = \left[1 + \frac{\gamma - 1}{2}{M_e}^2\right]^{\frac{\gamma}{\gamma-1}} = 7.8244 \rightarrow p_0 = 782.44\, kPa$$

This is the required reservoir driving pressure for which isentropic compression & expansion of p_0 to p_b through the CD nozzle regime induces a velocity M_e. The mass flow rate past the reservoir through the tunnel is constant anywhere for steady flow, so evaluating the equation given and derived in question one for mass rate at the sonic condition solves the mass flow through any cross-section in the isentropic flow regime:

$$\dot{m} = p_0\sqrt{\frac{\gamma}{RT_0}}A^* M^* \left[1 + \frac{\gamma - 1}{2}M^{*^2}\right]^{-\frac{\gamma+1}{2(\gamma-1)}}; \quad M^* = 1$$

$$\dot{m} = p_0 A^* \sqrt{\frac{\gamma}{RT_0}\left[1 + \frac{\gamma - 1}{2}\right]^{-\frac{\gamma+1}{\gamma-1}}} = 28.618\, \frac{kg}{s}$$

To induce the exit shock condition, the isentropic regime must end at the tunnel exit, at which point the flow must encounter an instantaneous pressure differential. Shock waves convert fluid velocity into pressure, temperature, and density across at the cost of entropy, thus restricting $p_b > p_e$. Under quasi-1D flow, the supersonic flow streamlines cross the pressure differential perpendicular at every dA, likely inducing the normal strong shock condition. It remains that $dA = 0$ through the test section to the exit, thus flow speed and Mach remain unchanged. The normal shock Mach relation can then be enforced across the exit discontinuity to obtain M_2 as a function of $M_1 = M_t$ and perfect air:

$$M_2 = \sqrt{\frac{2 + (\gamma - 1) \cdot M_t^2}{2\gamma M_t^2 - (\gamma - 1)}} = 0.5774$$

If the normal shock must occur, this will always be the Mach relation for this tunnel configuration with ideal air. The normal shock relations also define postshock pressure for air at some M_1. To ensure a single shock interaction, the postshock static pressure p_2 should be equal to atmospheric backpressure p_b. Thus, the normal shock pressure relation essentially determines what the preshock pressure, $p_1 = p_e$, for this Mach relation needs to be:

$$\frac{p_b}{p_e} = 1 + \frac{2\gamma}{\gamma + 1} \cdot (M_t^2 - 1) = 4.5 \rightarrow p_e = \frac{p_b}{4.5} = 22.22\, kPa$$

Knowing now the required pressure for shocking conditions in the isentropic regime at the exit, the stagnation pressure equation can be used similarly to above, noting that the smaller p_e will demand a lower p_0, as the RHS remains constant for $M_e = 2$:

$$\frac{p_0}{p_e} = \left[1 + \frac{\gamma - 1}{2} {M_e}^2\right]^{\frac{\gamma}{\gamma-1}} = 7.8244 \rightarrow p_0 = 173.88\, kPa$$

The speed of the flow exiting the tunnel can be found from the postshock Mach and temperature, the latter of which can be computed from the preshock temperature in the isentropic regime:

$$\frac{T_0}{T_e} = \left[1 + \frac{\gamma - 1}{2} {M_t}^2\right] \rightarrow T_e = 165.6389\, K$$

$$\frac{T_2}{T_e} = \frac{p_2}{p_1} \cdot \frac{\rho_1}{\rho_2} = 279.5156$$

$$M_2 = \frac{u_2}{a_2} \rightarrow u_2 = M_2\sqrt{\gamma R T_2} = 193.48\, m/s$$

Mass flow varies only as a function of total pressure. The preshock exit condition in the isentropic regime has not changed, and mass must still be conserved across the shock. This can be proven using the area-velocity relation, where it's shown that choking must occur to achieve supersonic acceleration. The choking condition defines *exactly* sonic flow at the throat, from which the minimum required differential pressure is only a function of fluid thermodynamics:

$$\frac{p_0}{p_b} = \left[1 + \frac{\gamma - 1}{2} {M^*}^2\right]^{\frac{\gamma}{\gamma-1}} = 1.8929$$

Since supersonic acceleration is achieved, it's expected that the existing pressure differential in the isentropic regime is of equal or greater magnitude than this critical pressure ratio. Indeed, the existing Δp was found above to be $\frac{p_0}{p_e} = 7.8244$, comfortably exceeding the minimum for choking. Thus, for any supersonic (or sonic) condition in the test section, mass flow is proportional to reservoir pressure. At the throat still, for this reservoir pressure:

$$\dot{m} = p_0 A^* \sqrt{\frac{\gamma}{R T_0}\left[1 + \frac{\gamma - 1}{2}\right]^{-\frac{\gamma+1}{\gamma-1}}} = 6.3695\, \frac{kg}{s}$$

When adding a post-shock diffuser, the flow is known to be subsonic. The area-velocity relation says that this condition requires a $dA > 0$ such that $du < 0$. The specifics of dA aren't important other than the requirement that the flow velocity is effectively reduced to zero. Earlier, the reservoir condition was such that $M \approx 0$. This meets the diffusion requirements; thus, the final exit condition should match atmospheric reservoir conditions, or $p_{0_2} = p_b$.

To find the new driving reservoir pressure, consider that the above scenario still exists as a component of this diffused system. The isentropic regime through the tunnel and the normal shock will maintain the same Mach relationship, $M_1 = 2.0$ and $M_2 = 0.5774$. The postshock stagnation pressure must equal atmospheric pressure, so the static pressure at M_2 is:

$$p_2 = p_b \left[1 + \frac{\gamma - 1}{2} {M_2}^2\right]^{-\frac{\gamma}{\gamma-1}} = 79.781\, kPa$$

Then, the preshock pressure from the normal shock Mach relation:

$$p_e = \frac{p_2}{4.5} = 17.729\, kPa$$

The required reservoir pressure is then:

$$\frac{p_0}{p_e} = \left[1 + \frac{\gamma - 1}{2} {M_e}^2\right]^{\frac{\gamma}{\gamma-1}} = 7.8244 \rightarrow p_0 = 138.72\, kPa$$

Once again, the test section conditions remain unchanged and are at least sonic, so the mass flow at the choked condition for this pressure:

$$\dot{m} = p_0 A^* \sqrt{\frac{\gamma}{RT_0}\left[1 + \frac{\gamma - 1}{2}\right]^{-\frac{\gamma+1}{\gamma-1}}} = 5.074\, \frac{kg}{s}$$

If the reservoir had at most 1.5 atm, then the test section Mach number will depend on whether choking occurs. The information that we have currently is not enough to know this for certain, unless the tunnel has some mechanism to change and optimize the nozzle throat area. Assuming this is not the case, it's possible to find out. With choked flow, the mass rate at this pressure is:

$$\dot{m} = p_0 A^* \sqrt{\frac{\gamma}{RT_0}\left[1 + \frac{\gamma - 1}{2}\right]^{-\frac{\gamma+1}{\gamma-1}}} = 5.486\, \frac{kg}{s}$$

Which must equal the mass rate at any point in the test section and through the exit. The total pressure remains the same prior to the shockwave, so the test area can be used to find the test & exit Mach. This yields 2 real solutions, one subsonic and one supersonic:

$$\dot{m} = p_0 A_e M_e \sqrt{\frac{\gamma}{RT_0}\left[1 + \frac{\gamma - 1}{2} M_e^2\right]^{-\frac{\gamma+1}{\gamma-1}}} = 5.486\, \frac{kg}{s}$$

$$M_e = [5.3143,\ 0.0181]$$

Neither of them is valid. The supersonic solution violates Eq. [2.3] for this tunnel design and throat ratio. The subsonic solution requires much higher back pressure after diffusion through the diverging nozzle section. Since choked flow is impossible here, the solution must be subsonic. This cannot induce a shock to relieve backpressure, so the maximum test section Mach is a function of just the single isentropic pressure relation at p_b:

$$\frac{p_0}{p_e} = \left[1 + \frac{\gamma - 1}{2} {M_e}^2\right]^{\frac{\gamma}{\gamma-1}} \rightarrow M_t = 0.7837$$

Which is subsonic, predicted by the critical pressure ratio. Mass flow then needs to be recomputed at a location where conditions are known:

$$\dot{m} = p_0 A_e M_e \sqrt{\frac{\gamma}{RT_0}\left[1 + \frac{\gamma - 1}{2} M_e^2\right]^{-\frac{\gamma+1}{\gamma-1}}} = 167.942 \; \frac{kg}{s}$$

Results:

For part (a):

$$A^* = 0.0156 \, m^2$$

$$p_0 = 782.44 \, kPa$$

$$\dot{m} = 28.618 \; \frac{kg}{s}$$

For part (b):

$$p_0 = 173.88 \, kPa$$

$$M_2 = 0.5774; \; u_2 = 193.48 \, m/s$$

$$\dot{m} = 28.618 \; \frac{kg}{s}$$

For part (c):

$$p_0 = 138.72 \, kPa$$

$$\dot{m} = 28.618 \; \frac{kg}{s}$$

For part (d):

$$M_t = 0.7837$$

$$\dot{m} = 167.942 \; \frac{kg}{s}$$

Discussion:

This is sufficient to emphasize the importance of the total pressure relation for compressible analysis, if we can assume isentropic conditions. For a perfect gas regime, flow at any Mach is then fully defined only as a function of the static p_s, or stagnation p_0, pressures. Every Mach has a required ratio of pressure such that the isentropic acceleration from p_0 yields p_s, and deceleration from p_s yields p_0.

This is useful particularly for requirement-based design. Knowing a target Mach condition fully defines the minimum pressure ratio you need to sustain. Then, minimizing the driving reservoir pressure is just a matter of reducing static pressure at the exit. It's shown here that a combination of shocking and/or diffusing the exiting air is an effective way of relieving back pressure. This scales down the required driver pressure while maintaining the same pressure ratio. For a pump that draws from the atmosphere, the results show that relieving back pressure is the difference between maintaining 7.8 atm or 1.3 atm.

It also demonstrates flow choking behavior, which occurs at either a very high or very low post-nozzle pressure. The energy associated with an amount of sonic mass must be conserved and converted entropically to eventually meet back pressure. Conversion is through changes in area, so for some pressure and area ratio with choked flow, there are only 2 isentropic static pressure solutions. For in intermediate static pressure, there's not a high enough energy demand to require the maximum mass rate; the high-pressure demand for aggressive recompression, or the low-pressure demand for aggressive supersonic acceleration. Then choked flow cannot be seen, supersonic acceleration isn't possible, and the mass rate will be determined by how much energy is needed to expand to backpressure, which echoes the definition of the isentropic pressure relation.

If the throat area in (d) could be changed to choke any reservoir pressure, then the maximum attainable Mach could be found from linking together pressure ratios across the regions, which becomes a nonlinear system. I tried to find a solution for this scenario in MATLAB, but either I didn't do it right, or the supersonic solution with an exit shock has a test section Mach of 1890. If the reservoir can sustain a vacuum instead of the maximum, there's also a solution for roughly Mach 1.4 with $p_0 = \sim 66.9\ kPa$.

Appendix A: MATLAB Code

```
%% Header
% Author: Zakary Steenhoek
% Date:
% Title:

% Reset
clc; clear;

%% Setup

% Variables
syms
gamma = 1.4;
R = 287;
pb_p0 = 0:0.001:1;

% Equations

% Functions
exitMach = @(PB_Po) sqrt(2/(gamma - 1).*((PB_Po).^(-(gamma - 1)/gamma) - 1));
mdotNorm = @(P0, ME) P0*sqrt(gamma/R).*ME.*(1 + (gamma - 1)/2.*ME.^2).^...
    (-(gamma + 1)/(2*(gamma-1)));

%% Math

% Constant reservoir pressure
p0_dPb = 800E3;
pb_dPb = p0_dPb.*pb_p0;
Me_dPb = exitMach(pb_p0);
MeTh_dPb = Me_dPb; Me_dPb(Me_dPb>1) = 1;
dMassfl_dPb = mdotNorm(p0_dPb,Me_dPb);
dThMassfl_dPb = mdotNorm(p0_dPb,MeTh_dPb);

% Constant exit backpressure
pb_dP0 = 100E3;
p0_dP0 = pb_dP0./(pb_p0(pb_p0>0));
Me_dP0 = exitMach(pb_p0(pb_p0>0));
MeTh_dP0 = Me_dP0; Me_dP0(Me_dP0>1) = 1;
dMassfl_dP0 = mdotNorm(p0_dP0,Me_dP0);
dThMassfl_dP0 = mdotNorm(p0_dP0,MeTh_dP0);
dMassfl_dP0(Me_dP0==1) = dMassfl_dP0(find(dMassfl_dP0(Me_dP0==1),1,"last"));

%% Plots

% Changing backpressure plot
figure(1); clf; hold on; grid on;
title('Normalized Mass Flux vs. Pressure Ratio', 'p_0 = 800 kPa')
xlabel('Pressure Ratio p_b/p_0'); ylabel('Mass Flux per Unit Area');
plot(pb_p0, dMassfl_dPb, 'k.-')
plot(pb_p0, dThMassfl_dPb, 'r-.')
xlin(1/8, 0, 4E4, 'k--');
plot(1/8, dThMassfl_dPb(pb_p0==1/8), 'ko', 'MarkerFaceColor', 'k', MarkerSize=4);
legend('Mass flow w.r.t. changing backpressure', ...
```

```
    'Mass flow without area-velocity relation', ...
    Location='best');
xlim([0 1]); ylim([0 4E4]);
hold off;
% autosave('const_total_press', 'AEE362\HW04\figs');

% Changing reservoir pressure plot
figure(2); clf; hold on; grid on;
title('Normalized Mass Flux vs. Pressure Ratio', 'p_b = 100 kPa')
xlabel('Pressure Ratio p_b/p_0'); ylabel('Mass Flux per Unit Area');
plot(pb_p0(pb_p0>0), dMassfl_dP0, 'k.-')
plot(pb_p0(pb_p0>0), dThMassfl_dP0, 'r-.')
xlin(1/8, 0, 4E4, 'k--');
plot(1/8, dThMassfl_dP0(pb_p0==1/8), 'ko', 'MarkerFaceColor', 'k', MarkerSize=4);
legend('Mass flow w.r.t. changing total pressure', ...
    'Mass flow without area-velocity relation',...
    Location='best');
xlim([0 1]); ylim([0 4E4]);
hold off;
% autosave('const_back_press', 'AEE362\HW04\figs');

%%

%
figure(3); clf; hold on; grid on;
plot(pb_p0, pb_dPb./1E5, 'b-.');
plot(pb_p0, Me_dPb, 'k-');
plot(pb_p0, MeTh_dPb, 'k--');
xlim([0 1]);
% ylim([]);
hold off;

%
figure(4); clf; hold on; grid on;
plot(pb_p0(pb_p0>0), p0_dP0./1E7, 'b-.');
plot(pb_p0(pb_p0>0), Me_dP0, 'k-');
plot(pb_p0(pb_p0>0), MeTh_dP0, 'k--');
xlim([0 1]);
% ylim([0 1E7]);
hold off;

%% Header
% Author: Zakary Steenhoek
% Date:
% Title:

% Reset
clc; clear;

%% Setup

% Variables
syms M G P0 A P
gamma = 1.4;
R = 287;
Mt = 2.0;
```

```
T0 = 25 + 273.15;
At = 0.5;
pb = 100E3;

% Equations
AR = 1/M.*(2/(G+1).*(1 + (G-1)/2).*M.^2).^((G+1)/(2*(G-1)));
mdot = P0*A*M*sqrt((gamma)/(T0*R)*(1+M^2*(gamma-1)/2)^(-(gamma+1)/(gamma-1)));

% Functions
areaRatio = matlabFunction(AR);
massRate = matlabFunction(mdot);

%% Math

% Sonic area for M = 2.0
AtAS = areaRatio(gamma, Mt);
AS = At/AtAS;

% Exit condition
Me = Mt;
pe1 = pb;
PR1 = isentropicRelations(Me, "single",2);
TR1 = isentropicRelations(Me, "single",1);
p01 = pe1*PR1;
massFl1 = massRate(AS,1,p01);

% Shock condition post-shock properties
[PR2, ~, TR2, M2] = normalShockRelations(Mt, gamma);
pe2 = pb/PR2;
p02 = pe2*PR1;
Te = T0/TR1;
T2 = Te*TR2;
u2 = M2*sqrt(gamma*R*T2);
massFl2 = massRate(AS,1,p02);

% Diffused condition post-shock properties
PR3 = isentropicRelations(M2, "single",2);
p2 = pb/PR3;
pe3 = p2/PR2;
p03 = pe3*PR1;
massFl3 = massRate(AS,1,p03);

%% Malfunctioning pump
p04 = pb*1.5;
massFl4 = massRate(AS,1,p04);
clear M; syms M
[M4,param,cond] = solve(massRate(At,M,p04)==massFl4, M, "ReturnConditions",true,"Real",true);
subM4 = M4(2); superM4 = M4(1);
superMassFl4 = massRate(At, superM4, p04)
AtAS4 = areaRatio(gamma, superM4)
AS4 = At/AtAS4


%% Plots

% %
```

```
% subsonAR = areaRatio(gamma, M4);
% % AS = At/AtAS;
%
% absPressureRatio = ((0.2.*M.^2 + 1).^(-3.5)).*((7.*M.^2)./6 - 1/6);
% absPressureRatio = simplify(absPressureRatio);
% % sym P
% assume(M>1)
% assume(P>0)
%
%
% % PRange = 0.66978:0.0000001:0.66980;
% PRange = 0.66979590;
% flag = 0;
%
%
% for i = 1:length(PRange)
%     if ~flag
%
%         [MRange, Param, Cond] = solve(absPressureRatio==PRange, M, ...
%             "ReturnConditions",true);
%
%         if (~isempty(Param) || isempty(MRange))
%             flag = 1;
%             MM = solve(absPressureRatio==PRange, M);
%             MP = PRange(i-1);
%             fprintf('Broke at PR: %1.8f\n', MP)
%             fprintf('Max Machs: %2.5f and %2.5f\n', MM(1), MM(2))
%             break
%         end
%     end
% end
%
% p04 = 0.66979590*pb;
% [PR41, DR4, TR4, M24] = normalShockRelations(MM(1), gamma);
% PR42 = isentropicRelations(MM(1), "single", 2);
% pt = p04/PR42;
% pf = pt*PR41;
%
% % Sonic area for M = MM
% AtAS4 = areaRatio(gamma, MM(1));
% AS4 = At/AtAS4;
% massFl41 = massRate(AS4,1,p04);

%% Header
% Author: Zakary Steenhoek
% Date:
% Title:

% Reset
clc; clear;

%% Setup

% Variables
syms M P0 PE
gamma = 1.4;
```

```
pb = 100E3;


% Equations
shockPR = PE*(1+((2*gamma)./(gamma+1)).*(M.^2-1))^(-1);
machPR = P0*(1+((gamma-1)./2).*M.^2).^(-gamma./(gamma-1));
P11 = matlabFunction(shockPR);
P12 = matlabFunction(machPR);

% Functions
P = @(C) [P11(C(1),pb), P12(C(1),C(2))];
C0 = [1.5;150E3];
options = optimoptions('fsolve','Display','iter');
[C,fval] = fsolve(P,C0,options);
```