Aerodynamics Homework 07

Zakary Steenhoek

November 14, 2024

I. Part 1

Use the MATLAB program 'liftingline15.m' as a starting point, and modify it as needed to answer the following questions:

A. Question 1 Problem Statements:

1. *A planar wing has a symmetric airfoil and an aspect ratio of 10. Use MATLAB to compare the lift and drag coefficients for taper ratios of 0.1 through 1.0. Plot* C_{D_i}/C_L^2 *vs.* λ*. Plot Oswald efficiency factor vs.* λ*. Which taper ratio seems to be optimum? Why*
2. *Use MATLAB to study the effect of aspect ratio on the Oswald efficiency factor. For the taper ratio found to be optimum in problem 1, and aspect ratios of 4, 6, 8, 10, 15 and 30, plot* C_{D_i}/C_L^2 *vs. aspect ratio. Plot Oswald efficiency factor vs. aspect ratio. Discuss the results.*
3. *Use MATLAB to study the effect of aspect ratio on the three-dimensional lift-curve slope,* $dC_L/d\alpha$ *. For the taper ratio you found to be optimum in problem 1, and aspect ratios 4, 6, 8, 10, 15 and 30, plot* C_L *vs.* α *with* $0 \leq \alpha \leq 12°$*. Do this for an untwisted wing with symmetric airfoil. Find the lift-curve slope for each wing. Discuss your results. Compare your results to the theoretical results for an elliptically loaded wing.*
4. *Use MATLAB to study the effect of geometric twist on the Oswald efficiency factor. For an aspect ratio of 10 and a taper ratio of 1, vary the geometric twist to find the minimum drag. Do this for several angles of attack between 2 and 10 degrees. Plot optimal twist angle vs. CL. Show that you can get nearly ideal loading distribution even for an untapered wing. Explain*.

Assume: Symmetric airfoil, 0-degree twist unless specified, ideal conditions

Given*:* liftingline15.m code

Solution:

While the given MATLAB script is a good starting point, modifications were necessary for each of these problems to produce the desired graphs and results. Details regarding the solution (code) originating from the original script will be mostly omitted, as it was given as is and remains mostly unchanged for each of these questions, and as well to reduce monotony throughout.

For all questions, the α_{ZL} was set to 0 degrees, as the problem statements specifically stated airfoil symmetry. From previous study, we know that a symmetric airfoil produces zero lift at an AoA of zero degrees. As well, for all questions, the number of spanwise stations was set to 1000. This practically guarantees that the code will converge on a solution, although it does impact execution time.

For the first question, geometric twist is set to zero degrees, and α is set to three degrees, as the problem states unimportance, if the wing generates lift. Taper is set to a vector from 0 to 1, incrementing by 0.01. As such, there are 100 different taper ratios to compare, and a smooth curve is obtained. The example code was wrapped in a for loop to perform computations at every taper ratio, and then the aspect ratio with the maximum Oswald efficiency and minimum C_{D_i}/C_L^2 is extracted. For the purposes of solution explanation, the most efficient taper ratio was found to be 0.36, but more details can be found in the results and discussion. Both the Oswald efficiency and the C_{D_i}/C_L^2 curve are plotted vs. taper ratio λ.

For the second question, geometric twist is set to zero degrees, and α is still set to three degrees, as the problem states unimportance, if the wing generates lift. The taper ratio is set to 0.36, and the aspect ratio is a vector of the given values: 4, 6, 8, 10, 15, 30. The given code was wrapped in a for loop to perform calculations at each aspect ratio. Both the Oswald efficiency and the C_{D_i}/C_L^2 curve are plotted vs. aspect ratio.

For the third question, geometric twist is still set to zero degrees, but the AoA varies from 0 to 12 in three-degree increments, as well as the aspect ratio, in the same manner as in question 2. The requires the base code to be wrapped in two for loops, one for the taper, and an outer one for the angle of attack. For each aspect ratio, the lift-curve slope was plotted as a function of angle of attack.

For the last question, the outer loop for angle of attack remained the same, except for the range, which omitted an AoA of 0 and 1 degree. The inner loop is based on the geometric twist, which ranges from -15 degrees to 15 degrees. In retrospect this is overkill, as the positive range will not contain the optimal twist angle for this wing shape/sweep. At each angle of attack, the optimal wing twist is determined by parsing the 3-D data array at each angle of attack and finding the index with the highest Oswald efficiency. This twist angle is plotted against its associated lift coefficient. A right-hand inverted y-axis is included to show the angles of attack associated with each point as well.

Results:

First, the graphs from question 1 are shown below, the first for C_{D_i}/C_L^2 vs. taper ratio, and second for Oswald efficiency vs. taper ratio:

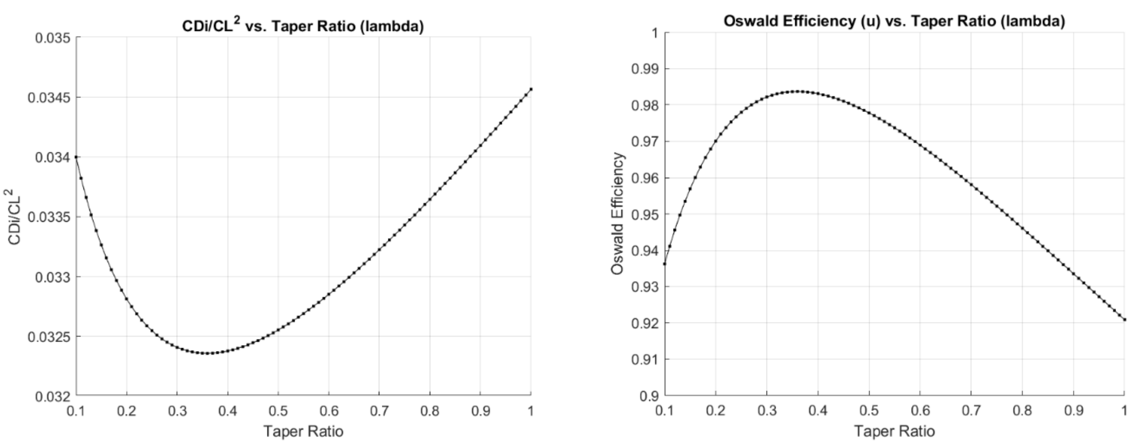


Figure [1.1-1.2]: Note the inverse relation between these graphs. The code in Appendix A includes a snippet to read the ideal taper ratio for both graphs, which returns 0.36 for both curves.

Next, the graphs from question 2 are shown below, the first for C_{D_i}/C_L^2 vs. aspect ratio, and second for Oswald efficiency vs. aspect ratio:

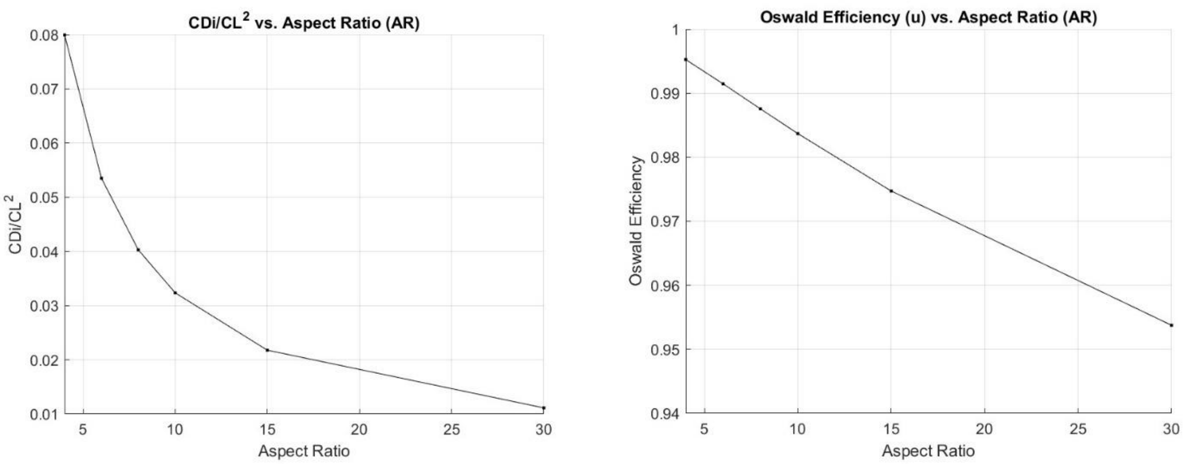


Figure [1.3-1.4]: Notice the opposite inverse relation for these graphs. As the aspect ratio increases, the induced drag by unit lift squared curve decreases, however the Oswald efficiency also decreases.

Next, the lift-curve slope data and graphs from question 3 are shown below. For readability, the lift-curve graph for each aspect ratio has been tiled into a single figure, but the individual graphs can be seen in Appendix B.

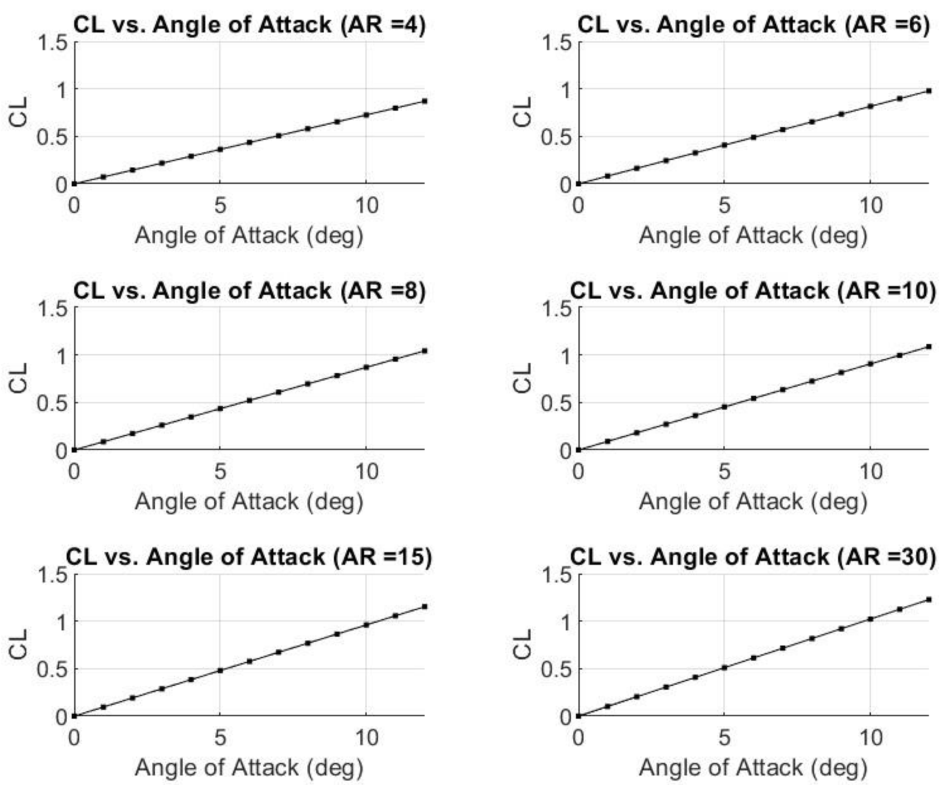


Figure [1.5]: Note the subtle slope increases as the aspect ratio increases, as shown in the table below.

Aspect Ratio	Lift-Curve Slope
4	0.0725
6	0.0815
8	0.0870
10	0.0906
15	0.0960
30	0.1022

Table [1.1]: The lift-curve slope increases as the aspect ratio increases

Next, the graph from question 4 is shown below, which shows the trend of optimal twist at different angles of attack vs. the lift coefficient at that optimal twist:

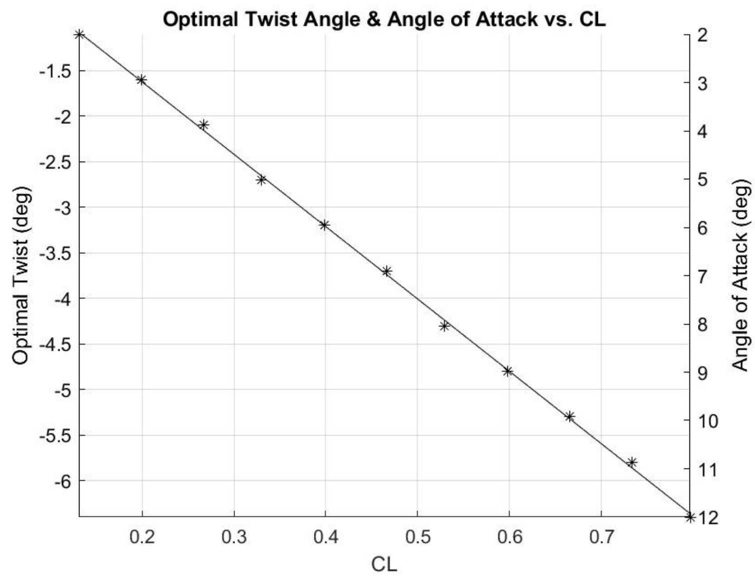


Figure [1.6]: At each angle of attack, the geometric twist with the highest Oswald efficiency is plotted vs. the resulting lift coefficient. The righthand axis is optional for a better understanding of the trend

Angle of Attack (deg)	Optimal Geometric Twist (deg)	Lift Coefficient
2	-1.1	0.1313
3	-1.6	0.1990
4	-2.1	0.2668
5	-2.7	0.3304
6	-3.2	0.3981
7	-3.7	0.4658
8	-4.3	0.5294
9	-4.8	0.5971

10	-5.3	0.6649
11	-5.8	0.7326
12	-6.4	0.7962

Table [1.2]: At each angle of attack, the angle of geometric twist with the highest Oswald efficiency is shown, along with the associated lift coefficient.

Discussion:

In question 1, it was determined that the optimal taper ratio was 0.36. In Fig. [1.1-1.2], it is evident that this is the taper ratio that results in the highest Oswald efficiency and the smallest value for C_{D_i}/C_L^2. The MATLAB code converged on this same value for both curves because of the relationship between Oswald efficiency and the induced drag per unit lift squared. Consider the following known equation for induced drag coefficient:

$$C_{D_i} = \frac{C_L^2}{\pi u AR} \tag{1.1}$$

Where C_L is the lift coefficient determined by aerodynamic coefficients, u is the Oswald efficiency, and AR is the aspect ratio. This can be arranged to form the quantity C_{D_i}/C_L^2, as seen below:

$$\frac{C_{D_i}}{C_L^2} = \frac{1}{\pi u AR} \tag{1.2}$$

This equation shows that the quantity C_{D_i}/C_L^2 is inversely proportional to the Oswald efficiency, and when the aspect ratio is held constant, it is directly inverse to the Oswald efficiency. The opposite holds true if the induced drag equation is rearranged for the Oswald efficiency, as seen below:

$$u = \frac{C_L^2}{\pi C_{D_i} AR} \tag{1.3}$$

As such, it is expected from a mathematical standpoint that these graphs are directly inverse, and as such, will converge on the same taper ratio.

To explain why this taper ratio is optimum, consider the Oswald efficiency, which measures how close a wing's lift distribution is to the ideal elliptical shape. The elliptical shape is considered ideal in theoretical and empirical aerodynamic studies because the reduction in chord and a rounded shape towards the tip of a wing reduces wing tip vortices. This produces the smallest amount of induced drag and results in a less severe and more uniform downwash profile. This also affects lift distribution and results in a more uniform and optimal lift distribution. For the parameters given in the problem statement, a reduction in chord by 64% in this trapezoidal wing planform provides a most nearly ideal wing loading and induced drag.

In question two, it is shown that there is an opposite relationship between Oswald efficiency the C_{D_i}/C_L^2 curve vs the aspect ratio. With the ideal taper found in question 1, wing planforms with smaller aspect ratios have a higher Oswald efficiency than planforms with a larger aspect ratio, as seen in Eq. [1.4]. This tracks in accordance with Eq. [1.3], as aspect ratio has an inverse relationship with Oswald efficiency. This also tracks from an empirical consideration of the ideal elliptical planform, as an increase in aspect ratio would make the planform more of a 'stretched' oval and distribute the lift further away from the root chord, resulting in a wing loading with a more rectangular shape. It is found later in question three that the lift coefficient - which has a direct quadratic relationship with Oswald efficiency - increases as a function of aspect ratio, but these results show that the increase in lift is not enough to counteract the increase in aspect ratio.

On the contrary, a small aspect ratio seems to induce a significant amount of drag, as shown in Fig. [1.3]. The induced drag per square unit lift coefficient is massive for super small aspect ratios, but falls quickly as the aspect ratio increases. This trend continues until the aspect ratio falls between 10 and 15, where the drag decrease slows as the aspect ratio grows very large.

Small aspect ratios generally have a high Oswald efficiency and perform how an ideal wing would, but they generate large amounts of induced drag due to high-strength wing tip vortices. Larger aspect ratios generate less induced drag, but the lift distribution pulls the loading distribution away from the root to form a more rectangular wing loading.

The results for question three, the results show that the lift-curve slope increases as a function of aspect ratio. This makes sense from what we know about the lift-curve slope of a finite airfoil, which differs from the lift-curve slope of an infinite airfoil because of the effects of induced drag, an unavoidable consequence of lift generation. The 3-dimensional lift-curve slope is determined from the following equation, which is derived from the 2-dimensional version from thin airfoil theory, as shown below:

$$\frac{dC_L}{d\alpha} = 2\pi$$

However, for a finite wing, lift depends on the effective angle of attack, seen below:

$$\frac{dC_L}{d\alpha_{eff}} = \frac{dC_L}{d(\alpha - \alpha_i)} = 2\pi$$

Where $\alpha_i = \frac{C_L}{\pi AR}$ from the lifting-line theory. Integrating the above and implementing the lifting-line theory:

$$C_L = 2\pi(\alpha - \alpha_i) = \left(\alpha - \frac{C_L}{\pi AR}\right)$$

Further differentiating, we can find the lift-curve slope for a finite wing:

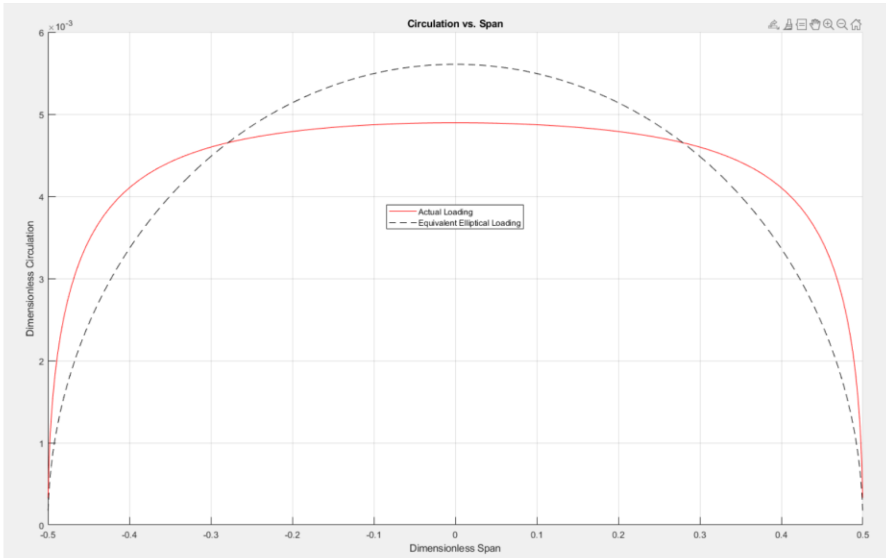
$$\frac{dC_L}{d\alpha} = \frac{2\pi}{1 + \frac{2\pi}{\pi AR}} = \frac{2\pi}{1 + \frac{2}{AR}}$$

The inverse-inverse relationship that the slope has with aspect ratio here explains the increase in lift-curve slope as a function of aspect ratio. The difference in slope from the infinite span described by the thin-airfoil theory is the division by the quantity $1 + \frac{2}{AR}$, which will always be greater than 1, meaning the slope for a finite wing will always be less than that of an infinite wing.

The increase is gradual, but does not seem to be particularly linear, as a quadratic fit matches the trend of the data the best with the following equation:

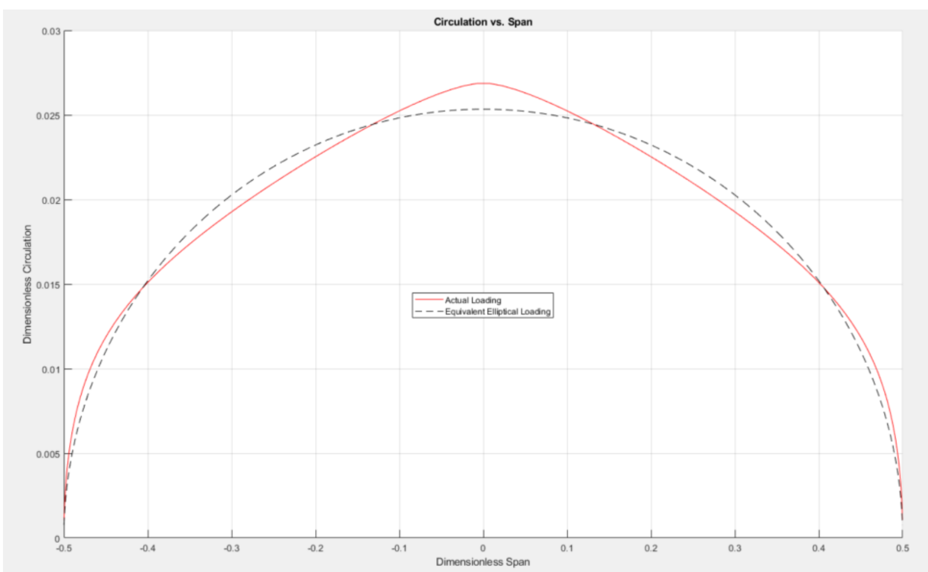
$$\frac{d^2C_l}{d\alpha dAR} - 0.0001x^2 + 0.0035 + 0.0620$$

Finally, for question four, we are asked to determine the effect of geometric twist on Oswald efficiency factor, which as we know, describes how close a wing matches the ideal elliptical loading distribution. The parameters are set such that the planform shape is rectangular with an aspect ratio of 10, which is a relatively average value. With an untwisted wing, this would result in a lower Oswald efficiency, as the loading distribution would have a rectangular shape, as seen in the graph below (generated from the original liftingline15.m script with a couple small modifications):



Notice the flatter lift distribution and the greater circulation slope and value at the tips of the wings, indicating that this wing will generate a fair bit of induced drag. The Oswald efficiency here is 0.920889 and the induced drag coefficient is 0.00965453. The angle of attack here is six degrees.

Referencing Tab. [1.2], we can see that the optimal twist angle for this planform at six degrees is -3.2 degrees. Modifying *only* the twist parameter produces the following loading distribution:



Here, the Oswald efficiency is 0.988331, and the induced drag coefficient is 0.00510388. The wing tips generate less lift, which shifts the loading towards the root and results in a more ideal distribution. The induced drag is also seen to be lower than the untwisted wing, which is to be expected. This distribution shape is common for all the twist angles in the table, which all have a very similar Oswald efficiency.

The lower lift produced at the wingtips at first seems like a downside, however this reduces the load on the control surfaces and the stress on the mechanical actuators involved. It also gives the control surfaces a greater range of control, as the effective relative angle can be increased without reaching local stall. It is also important to note that, as the angle of attack of the aircraft is increased, and the entire wing begins to approach the stall angle, the tips of the wings will be the last portion to lose lift, which lets the control surfaces remain effective even if part or the rest of the wing has lost lift. This is important for safety and maneuverability at critical angles of attack and allows the pilot to potentially execute maneuvers to regain positive lift.

Finally, a note on the negativity of geometric twist: To reduce wingtip loading, the effective angle of attack must be reduced with respect to the root angle of incidence. If the root angle of incidence is 4 degrees, the tip angle of incidence must be lower, i.e. a negative twist. This way, regardless of the angle of attack of the wing, the tip is always at a lower effective angle than the root. From the textbook, this is referred to as wash-out. The opposite is much more rarely seen, and is called wash-in.

II. Part 2

A proposed turboprop airplane is projected to weigh 45000 N and, for reasons of hangar storage, it will be limited to having a wingspan of 12 m. The aircraft will have an untwisted wing of elliptical planform. The design operating conditions are at a velocity of 200 km/hr at an altitude of 10,000 ft.

A. Problem Statements:

1. *Explain (with mathematical justification) why we expect to have an elliptical Γ distribution over the wingspan.*
2. *To avoid stall, the maximum effective angle of attack ($\alpha - \alpha_{ZL} - \alpha_i$) of the wing under these conditions is 10 degrees. Find the minimum wing area that will allow the aircraft to meet these operating conditions. For this wing area, what is the root chord length?*
3. *For the conditions given and the wing area determined in part (b), Find the downwash velocity and the induced angle of attack.*
4. *Find the induced drag and the induced drag coefficient (C_{D_i}).*

Assume: Small disturbances, thin airfoil(s)

Given: W = 45000 N, b = 12 m (s=6m), untwisted elliptical planform, $V_\infty = 200 km/hr$, altitude = 10,000 ft

Solution:

To explain why we expect an elliptical lift Γ distribution, consider the known equation for circulation distribution over an elliptical planform, Eq. [2.1]:

$$\Gamma(y) = \Gamma_0 \sqrt{1 - \left(\frac{2y}{b}\right)^2} \tag{2.1}$$

Quickly plotting this in Desmos online, seen in Fig. [2.1], shows that the distribution is of an elliptical shape. Note that y was replaced by x to obtain a horizontal curve in the x-y plane:

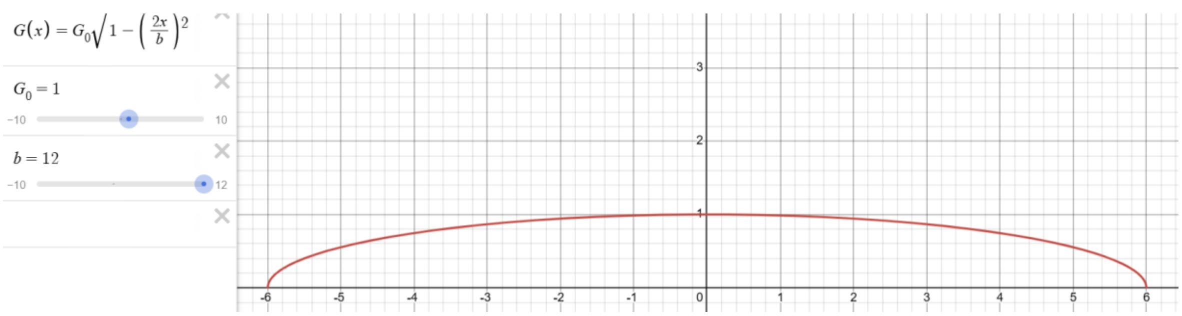


Figure [2.1]: Loading distribution according to Eq. [2.1].

To determine the minimum wing area, first consider the known spanwise lift equation from the Kutta-Joukowski theorem:

$$L = \int_{-s}^{s} \rho_\infty V_\infty \Gamma(y)\, dy$$

Substituting Eq. [2.1], the spanwise Γ distribution for an elliptical wing, Eq. [2.1], forming Eq. [2.2]:

$$L = \int_{-s}^{s} \rho_\infty \, V_\infty \Gamma_0 \sqrt{1 - \left(\frac{y}{s}\right)^2} \; dy \tag{2.2}$$

This integration can be made simpler by the coordinate transform, Eq. [2.3]:

$$y = -s \cos \phi \, , dy = s \sin \phi \; d\phi \tag{2.3}$$

Substituting and simplifying:

$$L = \int_{0}^{\infty} \rho_\infty \, V_\infty \Gamma_0 \sqrt{1 - \left(\frac{2 - s \cos \phi}{b}\right)^2} \; s \sin \phi \; d\phi = \int_{0}^{\infty} \rho_\infty \, V_\infty \Gamma_0 \sqrt{1 - \cos \phi^2} \; s \sin \phi \; d\phi$$

Now, integrating to get Eq. [2.4]:

$$L = \rho_\infty V_\infty \Gamma_0 s \frac{\pi}{2} = \frac{\pi}{4} b \rho_\infty V_\infty \Gamma_0 \tag{2.4}$$

Next, considering the known definition of the lift coefficient, and Eq. [2.4]:

$$C_L = \frac{L}{\frac{1}{2} \rho_\infty V_\infty^2 \mathrm{S}} = \frac{\pi b \Gamma_0}{2 V_\infty S}$$

From here, we can solve for the mid-span circulation as a function of the lift coefficient, Eq. [2.5]:

$$\Gamma_0 = \frac{2 C_L V_\infty S}{\pi b} \tag{2.5}$$

Substituting Eq. [2.4] into Eq. [2.4], we obtain an expression for lift as a function of known parameters, planform area, and lift coefficient, Eq. [2.6]:

$$L = \frac{1}{2} \rho_\infty V_\infty^2 C_L S \tag{2.6}$$

Now, consider the aircraft flying in steady, level, unaccelerated flight (SLUF) conditions, where the aircraft is in Newtonian equilibrium. Under these conditions, lift is equal to weight, and thrust is equal to drag. For this question, we will focus on the former condition, where $L = W$. This reduces Eq. [2.6] to only two unknowns, the lift coefficient, and the planform area. Since we are solving for the planform area, we will focus on resolving the lift coefficient. The problem states ‘minimum wing area’, which implies that we must maximize our other parameters. Density, velocity, and lift are already fully constrained, meaning that we must maximize the lift coefficient to minimize the planform area. To determine the maximum lift coefficient, consider that the problem states the maximum effective angle of attack, i.e. $(\alpha - \alpha_{ZL} - \alpha_i)$, is 10 degrees. We know that a wing will generate maximum lift (and therefore maximum lift coefficient) at its maximum effective angle of attack, so this value will be used here.

The lift-curve slope for a 2-dimensional airfoil and a 3-dimensional wing are not the same, as we know finite wing effects reduce the geometric angle of attack to the effective angle of attack. Wingtip vortices induce drag and create downwash, creating disturbances in the airflow under and around the wing. This was shown in question 1, and an equation was derived which ‘converts’ the lift-curve slope from 2-D to 3-D. All that to say that the equation derived in question one will *not* be used here. That equation converts the geometric AoA to effective AoA by considering an equation for the induced AoA according to lifting-line theory. Here, we have simply been given the effective angle of attack, which already accounts for all the geometric, induced, and zero lift angles. This effective AoA can be used according to the known standard thin airfoil lift-curve slope of 2π/rad, or ~0.11/deg. The lift coefficient is derived below:

$$C_L = 2\pi(\alpha - \alpha_{ZL} - \alpha_i) = \sim 0.11 \cdot 10° = 1.0966$$

Finally, we just need to evaluate ρ_∞ at standard atmospheric conditions at 10,000 feet. From the textbook tables, the density in $slug/ft^2$ at 10,000 feet geometric altitude is 1.7556E-3. Converting this to standard metric for sanity:

$$1.7556E^{-3}\frac{slug}{ft^3} = 0.9048\frac{kg}{m^3}$$

Also converting air speed to standard metric:

$$200\frac{km}{hr} = 55.5556\frac{m}{s}$$

Finally, Eq. [2.6] can be rearranged to solve for S, the wing planform area:

$$S = \frac{L}{\frac{1}{2}\rho_\infty V_\infty^2 C_L} = \frac{\mathrm{W}}{\frac{1}{2}\rho_\infty V_\infty^2 C_L} = \frac{45000}{0.5 \cdot 0.9048 \cdot 55.5556^2 \cdot 1.0966} = 29.388\ m^2$$

For an elliptical planform, the equation for chord as a function of span, Eq. [2.7], is of the same form as the equation for Γ distribution, Eq. [2.1]:

$$c(y) = c_o\sqrt{1 - \left(\frac{y}{s}\right)^2} \tag{2.7}$$

Where c_o is the root chord. The area of the wing can be found by integration:

$$S = \int_{-s}^{s} c_o\sqrt{1 - \left(\frac{y}{s}\right)^2}\, \mathrm{dy} = \frac{\pi c_o s}{2}$$

This can be rearranged to determine c_o:

$$c_o = \frac{2S}{\pi s} = \frac{2 \cdot 29.388}{\pi \cdot 6} = 3.1182\ m$$

Next, to determine the downwash velocity, consider the equation for induced velocity at any spanwise point due to the cumulative effect of all trailing vortices, Eq. [2.8]:

$$\omega(y_1) = \frac{1}{4\pi}\int_{-s}^{s}\frac{d\Gamma/dy}{y - y_1}\, dy \tag{2.8}$$

Since we already know our equation for $\Gamma(y)$, we can differentiate and substitute:

$$\frac{d}{dy}\Gamma(y) = -\Gamma_0\frac{y}{\mathrm{s}\sqrt{s^2 - y^2}}$$

$$\omega_{y1} = -\frac{\Gamma_0}{4\pi \mathrm{s}}\int_{-s}^{s}\frac{\mathrm{y}}{\sqrt{s^2 - y^2}(y - y_1)}\, dy$$

This integral can be evaluated to obtain the following expression for downwash at any point y_1, Eq. [2.9]

$$\omega_{y1} = -\frac{\Gamma_0}{4\pi \mathrm{s}}\left[\pi + y_1\left(\int_{-s}^{s}\frac{1}{\sqrt{s^2 - y^2}(y - y_1)}\, dy\right)\right] \tag{2.9}$$

We know that elliptic loading is symmetric about $y = 0$, which is only possible if the remaining integral term is equal to 0. This means Eq. [2.9] reduces to a constant that represents the downwash velocity across the entire spanwise position on the wing with elliptical circulation distribution, Eq. [2.10]. Note the negative indicating the downwash is in the negative x-direction, implied by the name, but resulting from the nature of the trailing vorticity direction:

$$\omega(y) = -\frac{\Gamma_0}{4\mathrm{s}} \tag{2.10}$$

Since we have already obtained an expression for Γ_0, Eq. [2.5], we can simply substitute to find an expression for the downwash velocity, knowing that $AR = \frac{b^2}{S}$:

$$\omega(y) = -\frac{C_L V_\infty S}{\pi b^2} = -\frac{C_L V_\infty}{\pi \mathrm{AR}} = \frac{1.0966 \cdot 55.5556 \cdot 29.388}{\pi \cdot 12^2} = 3.9576 \, {}^{\mathrm{m}}/_{\mathrm{s}}$$

To find the downwash angle, ε, we can use the geometric relationship between freestream velocity and downwash, Eq. [2.11]:

$$\varepsilon = -\frac{\omega}{V_\infty} \tag{2.11}$$

We can now use Eq. [2.10] and Eq. [2.5] to obtain the following expression, which is used to compute the downwash angle, knowing that $AR = \frac{b^2}{S}$:

$$\varepsilon = \frac{\Gamma_0}{4V_\infty \mathrm{s}} = \frac{2C_L V_\infty S}{4V_\infty s\pi b} = \frac{C_L \mathrm{S}}{\pi b^2} = \frac{C_L}{\pi \mathrm{AR}} = 4.0815°$$

Next, to compute the total vortex drag, i.e. the total induced drag, consider the known spanwise induced drag distribution given by the Kutta-Joukowski theorem:

$$D_i = -\int_{-s}^{s} \rho_\infty \, \omega(y) \Gamma(y) \, dy$$

Substituting from Eq. [2.1] and Eq. [2.10], and re-introducing our coordinate transform, this standard integral can be computed, Eq. [2.12]:

$$D_i = -\int_{-s}^{s} -\frac{\Gamma_0 \rho_\infty}{4\mathrm{s}} \Gamma_0 \sqrt{1 - \left(\frac{y}{\mathrm{s}}\right)^2} \, dy = \frac{\Gamma_0^2 \rho_\infty}{4\mathrm{s}} \int_0^\pi \sqrt{1 - \cos\phi^2} \; s \sin\phi \, d\phi \tag{2.12}$$

$$D_i = \frac{\pi}{8} \Gamma_0^2 \rho_\infty = \frac{\pi \rho_\infty}{8} \left(\frac{2C_L V_\infty S}{\pi b}\right)^2 = \frac{\pi \cdot 0.9048}{8} \left(\frac{2 \cdot 1.0966 \cdot 55.556 \cdot 29.388}{\pi \cdot 12}\right)^2 = 3205.549 \, N$$

Finally, to compute the induced drag coefficient, consider the known transformation from drag force to drag coefficient, Eq. [2.13]:

$$C_{D_i} = \frac{D_i}{\frac{1}{2} \rho_\infty V_\infty^2 S} \tag{2.13}$$

Substituting the above Eq. [2.12] for drag, the beloved Eq. [2.5], and still knowing $AR = \frac{b^2}{S}$:

$$C_{D_i} = \frac{\pi \Gamma_0^2}{4V_\infty^2 S} = \frac{C_L^2}{\pi} \left(\frac{S}{b^2}\right) = \frac{C_L^2}{\pi AR} = 0.07811$$

What an incredibly intuitive process.

Results:

$$S = 29.388\ m^2$$

$$c_o = 3.1182\ m$$

$$\omega = 3.9576\ \mathrm{m}/\mathrm{s}$$

$$\varepsilon = 4.0815°$$

$$D_i = 3205.549\ N$$

$$C_{D_i} = 0.07811$$

Discussion:

I feel as though there is not much to discuss here – the math speaks - except for the fact that all these calculations were made tenfold easier given an elliptical planform and circulation distribution. The textbook provides general techniques for non-elliptical spanwise circulation, which involves some nasty Fourier representations, many A_n coefficient calculations, and some extra parameters, e.g., the lift-curve slope parameter τ, induced drag factor δ, non-unity Oswald efficiency, etc., to obtain the values above. These techniques could have been used here, but I choose life.

I.Appendix A

```
%% Header
% Author: Zakary Steenhoek
% Date: 14 November 2024
% AEE 360 HW07
% Description: program to compute Oswald efficiency factor for different
% wing characteristics. Currently limited to trapezoidal wings.
% Lengths are made dimensionless by span (b) circulation is made
% dimensionless by V_inf*b

clc; clear; clf; %close all;

%% Wing Parameters

% Hardcode these
taper = 0:0.01:1;
AR = 10;
N = 1000;
twist = 0*pi/180;
alphaZL = 0*pi/180;
alpha = 3*pi/180;

% To store values
all_data = zeros(length(taper), 4);

% Loop on taper values
for itr1 = 1:length(taper)
    index = 1:N;
    Gamma = zeros(N,1);

    theta = index.*pi/(N+1);
    ybar = -cos(theta)/2;

    alphag = alpha+twist.*abs(2.*ybar);
    cbar = 2/(AR*(1+taper(itr1)))*(1-2*(1-taper(itr1))*abs(ybar));

    % Init matrix
    Matrix = zeros(N,N);
    b = zeros(N, 1);

    % Loop on N
    for n = 1:N
        for m = 1:N
            if theta(m) == 0
                Matrix(m,n) = 2/(pi*cbar(m))*sin(n*theta(m))+n;
            else
                Matrix(m,n) = 2/(pi*cbar(m))*sin(n*theta(m))+n*sin(n*theta(m))/sin(theta(m));
            end
        end
        b(n,1) = alphag(n)-alphaZL;
    end

    % solve for series coefficents (An's)
    A = Matrix\b;
```

```
    CL = A(1)*pi*AR;
    delta = 0;

    for m = 1:N
        if m ~= 1
            if A(1)~= 0
                delta = delta + m*A(m)^2/A(1)^2;
            else
                delta = 0;
            end
        end
        for n = 1:N
            Gamma(m) = Gamma(m) + 2*A(n)*sin(n*theta(m)); % series assumed for Gamma = 2*sum An*sin(n*theta)
        end
    end

    u = 1/(1+delta);  % Oswald efficiency factor

    CDi = CL^2/(pi*AR*u); % induced drag coefficient
    K = CDi/CL^2;

    % find "equivalent" elliptical distribution
    Gamma0 = 2*CL/(pi*AR); Gamma_e = Gamma0*sqrt(1-4*ybar.^2);

    % Store iteration data
    all_data(itr1, 1) = taper(itr1);
    all_data(itr1, 2) = CDi;
    all_data(itr1, 3) = CL;
    all_data(itr1, 4) = u;
end
cdcl = all_data(:, 2)./(all_data(:, 3).^2);

[max_u, max_ui] = max(all_data(:,4));
[min_cdcl, min_cdcli] = min(cdcl);
[max_CL, max_CLi] = max(all_data(:,3));
[max_CD, max_CDi] = max(all_data(:,2));

%%
clc;
fprintf('Maximum Oswald Efficiency of %g occurs at taper ratio %g \r', max_u, all_data(max_ui,1));
fprintf('Minimum CDi/CL^2 of %g occurs at taper ratio %g \r', min_cdcl, all_data(min_cdcli,1));
fprintf('Maximum lift coefficient of %g occurs at taper ratio %g \r', max_CL, all_data(max_CLi,1));
fprintf('Maximum induced drag of %g occurs at taper ratio %g \r', max_CD, all_data(max_CDi,1));

%% Plot Data

close all;

figure(1); clf; hold on; grid on;
title('CDi/CL^2 vs. Taper Ratio (lambda)');
xlabel('Taper Ratio'); ylabel('CDi/CL^2');
plot(taper, cdcl, 'k.-');
xlim([0.1, 1]); %ylim([0.9, 1]);
hold off;
```

```
figure(2); clf; hold on; grid on;
title('Oswald Efficiency (u) vs. Taper Ratio (lambda)');
xlabel('Taper Ratio'); ylabel('Oswald Efficiency');
plot(taper, all_data(:, 4), 'k.-');
xlim([0.1, 1]); ylim([0.9, 1]);
hold off;


% q = findobj('type','figure');
% figDir = 'C:\Users\zaste\OneDrive\Documents\Software\MATLAB\AEE360\HW07\figures';
% Autosave(q,'HW07P1Q1_', figDir, 'png')

%% Header
% Author: Zakary Steenhoek
% Date: 14 November 2024
% AEE 360 HW07
% Description: program to compute Oswald efficiency factor for different
% wing characteristics. Currently limited to trapezoidal wings.
% Lengths are made dimensionless by span (b) circulation is made
% dimensionless by V_inf*b

clc; clear; clf; %close all;

%% Wing Parameters

% Hardcode these
taper = 0.36;
% AR = [4 6 8 10 15 30];
AR = 4:0.1:30;
N = 100;
twist = 0*pi/180;
alphaZL = 0*pi/180;
alpha = 3*pi/180;

% To store values
all_data = zeros(length(AR), 5);

% Loop on AR values
for itr1 = 1:length(AR)
    index = 1:N;
    Gamma = zeros(N,1);

    theta = index.*pi/(N+1);
    ybar = -cos(theta)/2;

    alphag = alpha+twist.*abs(2.*ybar);
    cbar = 2/(AR(itr1)*(1+taper))*(1-2*(1-taper)*abs(ybar));

    % Init matrix
    Matrix = zeros(N,N);
    b = zeros(N, 1);

    % Loop on N
    for n = 1:N
        for m = 1:N
            if theta(m) == 0
```

```
            Matrix(m,n) = 2/(pi*cbar(m))*sin(n*theta(m))+n;
         else
            Matrix(m,n) = 2/(pi*cbar(m))*sin(n*theta(m))+n*sin(n*theta(m))/sin(theta(m));
         end
      end
      b(n,1) = alphag(n)-alphaZL;
   end

   % solve for series coefficents (An's)
   A = Matrix\b;

   CL = A(1)*pi*AR(itr1);
   delta = 0;

   for m = 1:N
      if m ~= 1
         if A(1)~= 0
            delta = delta + m*A(m)^2/A(1)^2;
         else
            delta = 0;
         end
      end
      for n = 1:N
         Gamma(m) = Gamma(m) + 2*A(n)*sin(n*theta(m)); % series assumed for Gamma = 2*sum An*sin(n*theta)
      end
   end

   u = 1/(1+delta);  % Oswald efficiency factor

   CDi = CL^2/(pi*AR(itr1)*u); % induced drag coefficient
   K = CDi/CL^2;

   % find "equivalent" elliptical distribution
   Gamma0 = 2*CL/(pi*AR(itr1)); Gamma_e = Gamma0*sqrt(1-4*ybar.^2);

   % Store iteration data
   all_data(itr1, 1) = AR(itr1);
   all_data(itr1, 2) = CDi;
   all_data(itr1, 3) = CL;
   all_data(itr1, 4) = u;
   all_data(itr1, 5) = K;
end

[max_u, max_ui] = max(all_data(:,4));
[min_K, min_Ki] = min(all_data(:,5));
[max_CL, max_CLi] = max(all_data(:,3));
[max_CD, max_CDi] = max(all_data(:,2));

%%
clc;
fprintf('Maximum Oswald Efficiency of %g occurs at aspect ratio %g \r', max_u, all_data(max_ui,1));
fprintf('Minimum CDi/CL^2 of %g occurs at aspect ratio %g \r', min_K, all_data(min_Ki,1));
fprintf('Maximum lift coefficient of %g occurs at aspect ratio %g \r', max_CL, all_data(max_CLi,1));
fprintf('Maximum induced drag of %g occurs at aspect ratio %g \r', max_CD, all_data(max_CDi,1));

%% Plot Data
```

```
close all;

figure(1); clf; hold on; grid on;
title('CDi/CL^2 vs. Aspect Ratio (AR)');
xlabel('Aspect Ratio'); ylabel('CDi/CL^2');
plot(AR, all_data(:,5), 'k.-');
xlim([4, 30]); %ylim([0.9, 1]);
hold off;

figure(2); clf; hold on; grid on;
title('Oswald Efficiency (u) vs. Aspect Ratio (AR)');
xlabel('Aspect Ratio'); ylabel('Oswald Efficiency');
plot(AR, all_data(:, 4), 'k.-');
xlim([4, 30]); ylim([0.94, 1]);
hold off;

% q = findobj('type','figure');
% figDir = 'C:\Users\zaste\OneDrive\Documents\Software\MATLAB\AEE360\HW07\figures';
% Autosave(q,'HW07P1Q2_', figDir, 'jpeg');

%% Header
% Author: Zakary Steenhoek
% Date: 14 November 2024
% AEE 360 HW07
% Description: program to compute Oswald efficiency factor for different
% wing characteristics. Currently limited to trapezoidal wings.
% Lengths are made dimensionless by span (b) circulation is made
% dimensionless by V_inf*b

clc; clear; clf; %close all;

%% Wing Parameters

% Hardcode these
taper = 0.36;
AR = [4 6 8 10 15 30];
N = 1000;
twist = 0*pi/180;
alphaZL = 0*pi/180;
alphaDeg = 0:1:12;
alpha = alphaDeg.*pi/180;

% To store values
all_data = zeros(length(AR), 4, length(alpha));
figDir = 'C:\Users\zaste\OneDrive\Documents\Software\MATLAB\AEE360\HW07\figures';

% Loop on AR values
for itr2 = 1:length(alpha)
    for itr1 = 1:length(AR)
        index = 1:N;
        Gamma = zeros(N,1);

        theta = index.*pi/(N+1);
        ybar = -cos(theta)/2;
```

```
        alphag = alpha(itr2)+twist.*abs(2.*ybar);
        cbar = 2/(AR(itr1)*(1+taper))*(1-2*(1-taper)*abs(ybar));

        % Init matrix
        Matrix = zeros(N,N);
        b = zeros(N, 1);

        % Loop on N
        for n = 1:N
            for m = 1:N
                if theta(m) == 0
                    Matrix(m,n) = 2/(pi*cbar(m))*sin(n*theta(m))+n;
                else
                    Matrix(m,n) = 2/(pi*cbar(m))*sin(n*theta(m))+n*sin(n*theta(m))/sin(theta(m));
                end
            end
            b(n,1) = alphag(n)-alphaZL;
        end

        % solve for series coefficents (An's)
        A = Matrix\b;

        CL = A(1)*pi*AR(itr1);
        delta = 0;

        for m = 1:N
            if m ~= 1
                if A(1)~= 0
                    delta = delta + m*A(m)^2/A(1)^2;
                else
                    delta = 0;
                end
            end
            for n = 1:N
                Gamma(m) = Gamma(m) + 2*A(n)*sin(n*theta(m)); % series assumed for Gamma = 2*sum
An*sin(n*theta)
            end
        end

        u = 1/(1+delta);  % Oswald efficiency factor

        CDi = CL^2/(pi*AR(itr1)*u); % induced drag coefficient
        K = CDi/CL^2;

        % find "equivalent" elliptical distribution
        Gamma0 = 2*CL/(pi*AR(itr1)); Gamma_e = Gamma0*sqrt(1-4*ybar.^2);

        % Store iteration data
        all_data(itr1, 1, itr2) = AR(itr1);
        all_data(itr1, 2, itr2) = CDi;
        all_data(itr1, 3, itr2) = CL;
        all_data(itr1, 4, itr2) = u;
    end

end
```

```
%% Plot data

slopesCL = zeros(1,2,length(AR));

close all;

for itr3 = 1:length(AR)
    % Grab the current slope
    coeffs = polyfit(alphaDeg, squeeze(all_data(itr3, 3, :)), 1);
    slopesCL(:,:,itr3) = coeffs;

    % Plot the CL vs AoA data
    figure(itr3); clf; hold on; grid on;
    title(strcat('CL vs. Angle of Attack (AR = ', num2str(AR(itr3)), ')'));
    xlabel('Angle of Attack (deg)'); ylabel('CL');
    scatter(alphaDeg, squeeze(all_data(itr3, 3, :)), 'k*');
    fplot(poly2sym(coeffs), 'k-');
    xlim([0, 12]); ylim([0, 1.5]);
    hold off;
end

tiledlayout(figure(99), 'flow'); clf;

for itr3 = 1:length(AR)
    % Tile plot the CL vs AoA data
    nextplot = nexttile; hold on; grid on;
    title(strcat('CL vs. Angle of Attack (AR = ', num2str(AR(itr3)), ')'));
    xlabel('Angle of Attack (deg)'); ylabel('CL');
    scatter(alphaDeg, squeeze(all_data(itr3, 3, :)), 'k.');
    fplot(poly2sym(slopesCL(:,:,itr3)), 'k-');
    xlim([0, 12]); ylim([0, 1.5]);
    hold off;
end

% q = findobj('type','figure');
% Autosave(q,'HW07P1Q3_', figDir, 'jpeg');

%%

[ddCLda, S] = polyfit(AR, squeeze(slopesCL(:, 1, :)), 2);

figure(100); clf; hold on; grid on;
fplot(poly2sym(ddCLda), 'k.-');
scatter(AR, squeeze(slopesCL(:, 1, :)), 'k*');

%% Header
% Author: Zakary Steenhoek
% Date: 14 November 2024
% AEE 360 HW07
% Description: program to compute Oswald efficiency factor for different
% wing characteristics. Currently limited to trapezoidal wings.
% Lengths are made dimensionless by span (b) circulation is made
% dimensionless by V_inf*b

clc; clear; clf; %close all;
```

```
%% Wing Parameters

% Hardcode these
taper = 1;
AR = 10;
N = 1000;
alphaZL = 0*pi/180;
twistDeg = -15:0.1:15;
twist = twistDeg.*pi/180;
alphaDeg = 2:1:12;
alpha = alphaDeg.*pi/180;

% To store values
all_data = zeros(length(twist), 4, length(alpha));
figDir = 'C:\Users\zaste\OneDrive\Documents\Software\MATLAB\AEE360\HW07\figures';

% Loop on AR values
for itr2 = 1:length(alpha)
    for itr1 = 1:length(twist)
        index = 1:N;
        Gamma = zeros(N,1);

        theta = index.*pi/(N+1);
        ybar = -cos(theta)/2;

        alphag = alpha(itr2)+twist(itr1).*abs(2.*ybar);
        cbar = 2/(AR*(1+taper))*(1-2*(1-taper)*abs(ybar));

        % Init matrix
        Matrix = zeros(N,N);
        b = zeros(N, 1);

        % Loop on N
        for n = 1:N
            for m = 1:N
                if theta(m) == 0
                    Matrix(m,n) = 2/(pi*cbar(m))*sin(n*theta(m))+n;
                else
                    Matrix(m,n) = 2/(pi*cbar(m))*sin(n*theta(m))+n*sin(n*theta(m))/sin(theta(m));
                end
            end
            b(n,1) = alphag(n)-alphaZL;
        end

        % solve for series coefficents (An's)
        A = Matrix\b;

        CL = A(1)*pi*AR;
        delta = 0;

        for m = 1:N
            if m ~= 1
                if A(1)~= 0
                    delta = delta + m*A(m)^2/A(1)^2;
                else
                    delta = 0;
```

```
            end
          end
          for n = 1:N
            Gamma(m) = Gamma(m) + 2*A(n)*sin(n*theta(m)); % series assumed for Gamma = 2*sum An*sin(n*theta)
          end
        end

        u = 1/(1+delta);  % Oswald efficiency factor

        CDi = CL^2/(pi*AR*u); % induced drag coefficient
        K = CDi/CL^2;

        % find "equivalent" elliptical distribution
        Gamma0 = 2*CL/(pi*AR); Gamma_e = Gamma0*sqrt(1-4*ybar.^2);

        % Store iteration data
        all_data(itr1, 1, itr2) = twistDeg(itr1);
        all_data(itr1, 2, itr2) = CDi;
        all_data(itr1, 3, itr2) = CL;
        all_data(itr1, 4, itr2) = u;
    end
end


[max_u, max_ui] = max(all_data(:, 4, :));
plot_data = zeros(length(alpha), 4);


%%
clc;
plot_data(:, 4) = alphaDeg;

for itr3 = 1:length(alphaDeg)
    plot_data(itr3, 1) = all_data(max_ui(1, 1, itr3), 1, itr3);
    plot_data(itr3, 2) = max_u(1, 1, itr3);
    plot_data(itr3, 3) = all_data(max_ui(1, 1, itr3), 3, itr3);

    % fprintf('At alpha = %g: \r', alphaDeg(itr3));
    % fprintf('Maximum Oswald Efficiency of %g occurs at twist angle %g, ', max_u(1, 1, itr3), all_data(max_ui(1, 1, itr3), 1, itr3));
    % fprintf('with a lift coefficient of %g \r\r', all_data(max_ui(1, 1, itr3), 3, itr3));
end


%% Plot Data

coeffs = polyfit(plot_data(:, 3), plot_data(:, 1), 1);

close all;

figure(1); clf; colororder({'k','k'});
hold on; grid on;
title('Optimal Twist Angle & Angle of Attack vs. CL');
xlabel('CL'); ylabel('Optimal Twist (deg)');
plot(plot_data(:, 3), plot_data(:, 1), 'k*');
```

```
fplot(poly2sym(coeffs), 'k-');
xlim([min(plot_data(:, 3)), max(plot_data(:, 3))]);
ylim([min(plot_data(:, 1)), max(plot_data(:, 1))]);
hold off;

yyaxis right; hold on;
ylim([min(plot_data(:, 4)), max(plot_data(:, 4))]); axis ij;
plot(plot_data(:, 4), plot_data(:, 1), 'k*')
ylabel('Angle of Attack (deg)');
hold off;

% q = findobj('type','figure');
% Autosave(q,'HW07P1Q4_', figDir, 'jpeg');
```