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Equations for binaries with mass loss or kick

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ABSTRACT

In Hwang & Zakamska (2025), we present an in-depth observational and theoretical investigation of wide binaries whose orbital configurations evolve due to mass loss and the associated velocity recoil of one or both of the components of the binary (envisioned as being due to the late stages of stellar evolution). In this online-only supplementary document we have collected some useful equations for the two-body problem in one place. Most of it is textbook material, from lower undergraduate to graduate level, to provide an explanation for the orbital calculations in https://github.com/zakamska/binaries. The three cases considered here are mass loss in impulse regime, mass loss in adiabatic regime and velocity kick without mass loss. If you find this document useful, please cite the paper by Hwang & Zakamska (2025). For more complex cases including the Hamiltonian formulation of the adiabatic mass loss with kick and the full numerical treatment of timescales between impulsive and adiabatic, see the complete paper (Hwang & Zakamska 2025).

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1. EQUIVALENT TWO-BODY PROBLEM

We define the notation for the two-body problem. We have a first body of mass m_1, position $\vec{r}_1$, velocity $\vec{v}_1 = d\vec{r}_1/dt$, and acceleration $\vec{a}_1 = d\vec{v}_1/dt$, and a second object of mass m_2, position $\vec{r}_2$, velocity $\vec{v}_2 = d\vec{r}_2/dt$, and acceleration $\vec{a}_2 = d\vec{v}_2/dt$. The origin of the coordinate system is placed at the system's barycenter $\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/(m_1 + m_2)$. We will also use total mass $m = m_1 + m_2$ and reduced mass $\mu = m_1 m_2/(m_1 + m_2)$.

The two-body problem can be described using an equivalent single-object formulation. The relative separation is $\vec{r} = \vec{r}_1 - \vec{r}_2$ and the relative velocity is $\vec{v} = \vec{v}_1 - \vec{v}_2$, and their magnitudes are $r = |\vec{r}|$ and $v = |\vec{v}|$. For these coordinates, the equation of motion is that of a test particle in the potential of a body with $m_1 + m_2$:

$$\ddot{\vec{r}} = -\frac{G(m_1+m_2)}{r^2}\vec{e}_r = -\frac{Gm}{r^2}\vec{e}_r. \tag{1}$$

To find the physical positions and velocities of the objects from the equivalent formulation:

$$\vec{v}_1 = \frac{m_2}{m_1+m_2}\vec{v};\ \vec{v}_2 = -\frac{m_1}{m_1+m_2}\vec{v};\ \vec{r}_1 = \frac{m_2}{m_1+m_2}\vec{r};\ \vec{v}_2 = -\frac{m_1}{m_1+m_2}\vec{r}, \tag{2}$$

and vice versa:

$$\vec{v} = \frac{m_1+m_2}{m_2}\vec{v}_1 = -\frac{m_1+m_2}{m_1}\vec{v}_2. \tag{3}$$

The orbital solution to the two-body problem is a series of conic sections,

$$r = \frac{a(1-e^2)}{1+e\cos(\phi-\omega)} = \frac{a(1-e^2)}{1+e\cos\varphi} = a(1-e\cos u). \tag{4}$$

where ϕ is the orbital angle that changes with time, ω is the longitude of pericenter, $\varphi = \phi - \omega$ is the true anomaly and u is the eccentric anomaly.

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Other relationships between anomalies:

$$\cos u = \frac{\cos\varphi + e}{1 + e\cos\varphi},\ \sin u = \frac{\sqrt{1-e^2}\sin\varphi}{1+e\cos\varphi},\ \cos\varphi = \frac{\cos u - e}{1 - e\cos u},\ \sin\varphi = \frac{\sqrt{1-e^2}\sin u}{1 - e\cos u}. \tag{5}$$

Mean anomaly M is linear in time:

$$M \equiv u - e\sin u. \tag{6}$$

Velocities:

$$v_\varphi = \sqrt{\frac{G(m_1+m_2)}{a}}\frac{\sqrt{1-e^2}}{1-e\cos u};\ v_r = \sqrt{\frac{G(m_1+m_2)}{a}}\frac{e\sin u}{1-e\cos u};\ v = \sqrt{\frac{G(m_1+m_2)}{a}}\sqrt{\frac{1+e\cos u}{1-e\cos u}}. \tag{7}$$

Given various orbital parameters, the eccentric anomaly can be computed as:

$$\cos u = \frac{1}{e}\frac{\frac{v^2 a}{Gm} - 1}{\frac{v^2 a}{Gm} + 1};\ \cos u = \frac{1}{e}\left(1 - \sqrt{\frac{Gm}{a}}\frac{\sqrt{1-e^2}}{v_\varphi}\right);\ \cos u = \frac{a-r}{ae}. \tag{8}$$

Eccentric anomaly u is $\in [0, \pi]$ if $v_r > 0$. Otherwise after solving one of these equations we should compute $2\pi - u$ instead.

2. USEFUL ORBIT AVERAGES

Orbit averages $\langle f\rangle$ of periodic functions are equivalent to time-averaging the quantity, or alternatively to averaging the quantity over the mean anomaly:

$$\langle f\rangle = \int_0^{2\pi}\frac{\mathrm{d}M}{2\pi}f(M) = \int_{-\pi}^{\pi}\frac{\mathrm{d}u}{2\pi}(1 - e\cos u)f(u). \tag{9}$$

Thus functions that are odd on $-\pi$ to π naturally average out to 0.

For the scalars associated with r:

$$\langle r^2\rangle = a^2\left(1 + \frac{3}{2}e^2\right);\ \langle r\rangle = a\left(1 + \frac{1}{2}e^2\right);\ \langle r^{-1}\rangle = 1/a;\ \langle r^{-2}\rangle = \frac{1}{a^2\sqrt{1-e^2}};\ \langle r^{-3}\rangle = \frac{1}{a^3(1-e^2)^{3/2}}. \tag{10}$$

For vectors, we use Cartesian coordinate system where x is pointing to the pericenter, z is pointing along the angular momentum:

$$\langle\vec{r}\rangle = \langle(r\cos\varphi, r\sin\varphi, 0)\rangle = (\langle r\cos\varphi\rangle, 0, 0). \tag{11}$$

It makes sense that the orbital average of r_y should be zero since the orbit is symmetric for the flip around the x axis, and mathematically r is an even function along the orbit, whereas $\sin\varphi$ is odd. To compute the orbit average of r_x, use eq. (5), (4) and (6):

$$\langle r\cos\varphi\rangle = \int_0^{2\pi}\frac{\mathrm{d}M}{2\pi}a(1-e\cos u)\cdot\frac{\cos u - e}{1 - e\cos u} = \int_0^{2\pi}\frac{\mathrm{d}u}{2\pi}a(1-e\cos u)^2\cdot\frac{\cos u - e}{1-e\cos u} = -3ea/2. \tag{12}$$

It makes sense that this value is negative since positive r_x values are toward the pericenter, and the particle spends significantly more time closer to the apocenter.

Similarly,

$$\langle\vec{e}_r\rangle = \langle(\cos\varphi, \sin\varphi, 0)\rangle = (\langle\cos\varphi\rangle, 0, 0); \tag{13}$$

$$1 + e\cos\varphi = \frac{a(1-e^2)}{r};\quad 1 + e\langle\cos\varphi\rangle = a(1-e^2)\langle\frac{1}{r}\rangle = (1-e^2);\quad \langle\cos\varphi\rangle = -e; \tag{14}$$

$$\langle\vec{e}_r\rangle = (-e, 0, 0) \text{ and } \langle\vec{e}_\varphi\rangle = \langle(-\sin\varphi, \cos\varphi, 0)\rangle = (0, -e, 0). \tag{15}$$

Velocity orbital average is 0 as expected:

$$\langle\vec{v}\rangle = \langle(v_r\cos\varphi - v_\varphi\sin\varphi, v_r\sin\varphi + v_\varphi\cos\varphi, 0)\rangle, \tag{16}$$

$$\langle v_r\cos\varphi\rangle = \langle v_\varphi\sin\varphi\rangle = 0; \tag{17}$$

$$\langle v_r\sin\varphi\rangle = \sqrt{\frac{Gm}{a}}\sqrt{1-e^2}\left(\frac{1-\sqrt{1-e^2}}{e}\right);\quad \langle v_\varphi\cos\varphi\rangle = \sqrt{\frac{Gm}{a}}\sqrt{1-e^2}\left(\frac{-1+\sqrt{1-e^2}}{e}\right). \tag{18}$$

3. CONSERVED QUANTITIES FOR THE EQUIVALENT FORMULATION

The orbital period is

$$P = 2\pi\sqrt{\frac{a^3}{G(m_1+m_2)}}. \tag{19}$$

Energy is the same as the physical energy:

$$E = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} - G\frac{m_1 m_2}{r} = \frac{m_1 m_2}{m_1+m_2}\frac{v^2}{2} - G\frac{m_1 m_2}{r} = -G\frac{m_1 m_2}{2a}. \tag{20}$$

Therefore, the semi-major axis can be found as

$$a = -G\frac{m_1 m_2}{2E}. \tag{21}$$

Angular momentum vector:

$$\vec{L} = \vec{r} \times \vec{v} = \vec{e}_z r v_\varphi;\ |\vec{L}| = \sqrt{G(m_1+m_2)a}\sqrt{1-e^2}. \tag{22}$$

This vector is related to the actual physical angular momentum of the system via

$$\vec{L}_{\text{physical}} = m_1\vec{r}_1 \times \vec{v}_1 + m_2\vec{r}_2 \times \vec{v}_2 = m_1\left(\frac{m_2}{m_1+m_2}\right)^2 \vec{r} \times \vec{v} + m_2\left(\frac{m_1}{m_1+m_2}\right)^2 \vec{r} \times \vec{v} = \frac{m_1 m_2}{m_1+m_2}\vec{L} = \mu\vec{L}. \tag{23}$$

The eccentricity can be found as

$$e = \sqrt{1 - \frac{L^2}{G(m_1+m_2)a}}. \tag{24}$$

Runge-Lenz vector:

$$\vec{R} = \vec{v} \times \vec{L} - G(m_1+m_2)\vec{e}_r. \tag{25}$$

For a Keplerian orbit, this vector is a constant pointing to the pericenter. Setting up a system with $\vec{e}_r$, $\vec{e}_\varphi$ and $\vec{e}_z$, we find that its absolute value is

$$R = \sqrt{(v_\varphi L - G(m_1+m_2))^2 + (v_r L)^2} = G(m_1+m_2)e. \tag{26}$$

Here we can plug in v_φ and v_r as a function of u to demonstrate the equality.

Providing m_1, m_2, $\vec{L}$ and $\vec{R}$ completely defines the orbit in space, and given any of r, v or v_φ one can calculate eccentric anomaly and position on the orbit.

4. JUST MASS LOSS, NO KICK: IMPULSE LIMIT

Mass m_2 instantaneously turns into m_2' while conserving its velocity. The total mass has then changed

$$\text{from } m = m_1 + m_2 \text{ to } m' = m_1 + m_2'. \tag{27}$$

We were in the center of momentum of the binary before the mass loss, but after the mass loss the binary barycenter moves with

$$\vec{v}_0 = \frac{m_1\vec{v}_1 + m_2'\vec{v}_2}{m_1+m_2'} = \vec{v}\frac{m_1(m_2-m_2')}{(m_1+m_2)(m_1+m_2')}. \tag{28}$$

Let's move into this system. In this system, the stars have $\vec{v}_i' = \vec{v}_i - \vec{v}_0$ $(i=1,2)$, but combining these velocities for the equivalent formulation in the new frame we find:

$$\vec{v}' = \frac{m_1+m_2'}{m_2'}\vec{v}_1' = \frac{m_1+m_2'}{m_2'}\left(\vec{v}_1 - \frac{m_1\vec{v}_1 + m_2'\vec{v}_2}{m_1+m_2'}\right) = \vec{v}_1 - \vec{v}_2 = \vec{v}\frac{m_2}{m_1+m_2} + \vec{v}\frac{m_1}{m_1+m_2} = \vec{v}. \tag{29}$$

Therefore, when the mass is shed with no kick, even though there is an overall COM recoil, neither $\vec{r}$ nor $\vec{v}$ change in the equivalent formulation. And since $\vec{r}$ is the same, the components v_φ and v_r are also conserved during mass loss (because $v_r = \vec{v}\cdot\vec{r}/r$).

Therefore, we find:

$$E' = \frac{m_1 m_2'}{m_1 + m_2'} \frac{v^2}{2} - G\frac{m_1 m_2'}{r} = -G\frac{m_1 m_2'}{2a'}, \tag{30}$$

where we can cancel $m_1 m_2'$ on both sides of the equation. Now we plug in v^2 from the energy equation before mass loss:

$$\frac{v^2}{2} = Gm\left(\frac{1}{r} - \frac{1}{2a}\right). \tag{31}$$

Now solving the post-mass-loss energy equation for a', we find:

$$a' = \frac{1}{\frac{m}{m'}\frac{1}{a} + \frac{2}{r}\left(1 - \frac{m}{m'}\right)}. \tag{32}$$

The orbit is disrupted if $a' < 0$, which occurs if

$$\frac{m'}{m} < 1 - \frac{r}{2a}, \tag{33}$$

so will depend on the position along the orbit. [When we need to compute statistics of disrupted binaries, it's an interesting question whether we can time-average here and how to properly do this. In the paper we don't time average; the mean anomaly of every binary is randomly drawn from the appropriate distribution.]

If the orbit is not disrupted, then its eccentricity can be computed from

$$ma(1-e^2) = m'a'(1-e'^2);\; e' = \sqrt{1 - \frac{ma(1-e^2)}{m'a'}}, \tag{34}$$

since $\vec{r}$ and $\vec{v}$ didn't change due to mass loss in the equivalent formulation and therefore $\vec{L}$ is the same before and after.

5. JUST MASS LOSS, NO KICK: ADIABATIC LIMIT

We break the very small mass loss into a series of tiny mass losses. For each tiny mass loss we can write $m' = m + \mathrm{d}m$ and $a' = a + \mathrm{d}a$ and Taylor-expand eq. (32) to the leading order. We get

$$\mathrm{d}a = a\left(\frac{\mathrm{d}m}{m} - \frac{2a}{r}\frac{\mathrm{d}m}{m}\right). \tag{35}$$

If the mass loss is very slow, we can orbit-average, with $\langle \frac{1}{r} \rangle = 1/a$. In this case, we get $\mathrm{d}a/a = -\mathrm{d}m/m$, and $ma = const$ for adiabatic mass loss. This can also be obtained from the Hamiltonian approach to the adiabatic invariant and Delaunay actions for the equivalent problem. In this case since we already established that L is exactly conserved, we find $e = const$, which can also be obtained from Delaunay actions.

The evolution of the Runge-Lenz vector due to mass loss is:

$$\frac{\mathrm{d}\vec{R}}{\mathrm{d}t} = \frac{\mathrm{d}(\vec{v}\times\vec{L})}{\mathrm{d}t} - Gm\frac{\mathrm{d}\vec{e}_r}{\mathrm{d}t} - G\frac{\mathrm{d}m}{\mathrm{d}t}\vec{e}_r. \tag{36}$$

Now let's think about orbit-averaging this for a slow (adiabatic) change in mass. In this case we need to be taking the first two terms at a constant mass, but it's the orbital change of the Runge-Lenz vector for a Keplerian orbit, which is 0. Therefore,

$$\left\langle \frac{\mathrm{d}\vec{R}}{\mathrm{d}t} \right\rangle = -G\frac{\mathrm{d}m}{\mathrm{d}t}\langle \vec{e}_r \rangle. \tag{37}$$

Let's briefly introduce a Cartesian system with x pointing toward the pericenter, z pointing in the direction of the angular momentum vector, and y to form the right system. In this system, $\vec{e}_r = (\cos\varphi, \sin\varphi, 0)$. Orbit averaging $\sin\varphi$ results in 0 because it's an odd function of all the anomalies on $(-\pi, \pi)$. Orbit-averaging $\cos\varphi$ can be done from $1 + e\cos\varphi = a(1-e^2)/r$, since $\langle 1/r \rangle = 1/a$ (shown in eq. 14). Therefore, $\langle \vec{e}_r \rangle = -e\vec{e}_x = -e\vec{R}/R$, and eq. (37) becomes $\dot{\vec{R}} = \dot{m}\vec{R}/m$. Integrating every component in some fixed coordinate system, we get $\vec{R} = \vec{R}_0 m/m_0$. Therefore, the magnitude of $\vec{R}$ is proportional to mass and therefore the eccentricity remains constant (which we already established above), and the direction to the pericenter remains constant during adiabatic mass loss.

6. KICK, NO MASS LOSS

Mass m_1 was moving with $\vec{v}_1$ and mass m_2 was moving with $\vec{v}_2$. Suddenly mass m_2 is kicked and it's now moving with $\vec{v}_2 + \vec{v}_k$. The COM is moving with

$$\vec{v}_0 = \frac{m_1\vec{v}_1 + m_2(\vec{v}_2 + \vec{v}_k)}{m_1 + m_2} = \frac{m_2\vec{v}_k}{m_1 + m_2}. \tag{38}$$

The velocities in that new frame are $\vec{v}_1' = \vec{v}_1 - \vec{v}_0$ and $\vec{v}_2' = \vec{v}_2 + \vec{v}_k - \vec{v}_0$. Recasting them into the equivalent formulation:

$$\vec{v}' = \frac{m_1 + m_2}{m_2}\vec{v}_1' = \frac{m_1 + m_2}{m_2}(\vec{v}_1 - \vec{v}_0) = \vec{v} - \vec{v}_k; \tag{39}$$

$$\vec{v}' = -\frac{m_1 + m_2}{m_1}\vec{v}_2' = -\frac{m_1 + m_2}{m_1}(\vec{v}_2 + \vec{v}_k - \vec{v}_0) = \vec{v} - \frac{m_1 + m_2}{m_1}\vec{v}_k + \frac{m_2}{m_1}\vec{v}_k = \vec{v} - \vec{v}_k. \tag{40}$$

So mass m_2 gets kicked in the positive $\vec{v}_k$ direction, but because the equivalent formulation follows the direction of m_1, it gets kicked in the opposite direction.

We set up a Cartesian system, with $\vec{e}_z$ along $\vec{L}$ and $\vec{e}_x$ along $\vec{R}$ (if the initial eccentricity is zero, any direction orthogonal to $\vec{e}_z$ can be used), and $\vec{e}_y$ to form the right set. We compute a and e from L and R, then with eccentric anomaly u we know r, v_φ and v_r. In this coordinate system, the orbit is in the $x - y$ plane, and

$$\vec{v} = \vec{e}_x\left(v_r \cos\varphi - v_\varphi \sin\varphi\right) + \vec{e}_y\left(v_r \sin\varphi + v_\varphi \cos\varphi\right); \tag{41}$$

$$\vec{r} = \vec{e}_x r\cos\varphi + \vec{e}_y r\sin\varphi. \tag{42}$$

After the kick is applied, the new velocity is $\vec{v}' = \vec{v} - \vec{v}_k$, the new separation immediately after the kick has not changed, it's still $\vec{r}' = \vec{r}$, and from these two vectors the new $\vec{L}'$, $\vec{R}'$ and u' can be calculated, and that gives us the post-kick solution.

7. A COMPLETE SOLUTION WITH MASS LOSS AND KICK ON ARBITRARY TIMESCALE

In Hwang & Zakamska (2025) we solve the full problem of mass loss and kick on arbitrary timescale (impulsive, adiabatic, and everything in between). To track this problem, we set up a fixed Cartesian coordinate system whose axes will remain fixed in direction during the evolution of the object, but we will allow ourselves to move from one COM frame to another translationally. In this system, we fix the z axis to the direction of $\vec{v}_k$, and this orientation remains the same during the long-term evolution of the mass loss because it's coupled to the internal properties of the star and not to the orbit (this is an assumption of our model that we discuss in the paper). In this frame m_1, m_2, $\vec{L}$ and $\vec{R}$ completely define the orbit and u a location along this orbit.

REFERENCES

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