كثير مهم

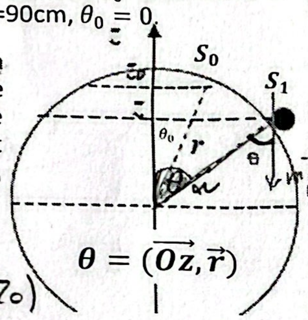
System : m

**II-6-** A particle of mass $m$ is launched from the point $S_0$ (with an elevation ($z_0 = r\cos\theta_0$) on a sphere with center $O$ and radius $r$ with an initial velocity $v_0$ (tangent to the sphere and in the vertical plane passing through $O$); it slides without friction on the sphere and then takes off, leaving the sphere at a point $S_1$. Let $g$ denote the acceleration due to gravity.

a- Express the reaction $R$ of the support on the particle as a function of its elevation $z = r\cos\theta$ at any given moment, and the parameters $m$, $r$, $g$, $v_0$, and $Z_0$.

b- Show that if $v_0 > V$, the particle leaves the sphere right from the start at $S_0$. **Determine V.** Note: $g = 10\,m/s^2$, r=90cm, $\theta_0 = 0$.

c- Calculate the path traveled by the particle on the sphere if it is released at $S_0$ with a velocity $v_0 = \frac{V}{2}$.




$\theta = (\overrightarrow{Oz}, \vec{r})$

$R = R(z, m, g, v_0, z_0)$

$$\sum \vec{F}_{ext} = m\vec{a} \rightarrow m\vec{g} + \vec{R} = m(\vec{a}_t + \vec{a}_n)$$

$$\begin{cases} \vec{n} : mg\cos\theta - R = ma_n = m\dfrac{v^2}{r} & \rightarrow R = m\left(g\cos\theta - \dfrac{v^2}{r}\right) \\ \vec{\tau} : \cancel{m}g\sin\theta = \cancel{m}a_t = \cancel{m}\dfrac{dv}{dt} \end{cases}$$

$vt = at(\theta)$

$$v\,dv = a_t\,ds = g\sin\theta\, r\,d\theta$$

$$\int_{v_0}^{v} v\,dv = \int_{\theta_0}^{\theta} g\,r\sin\theta\,d\theta$$

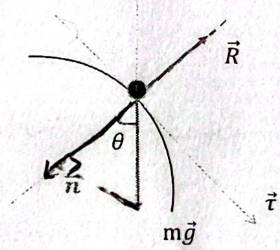
$$\frac{1}{2}(v^2 - v_0^2) = -gr(\cos\theta - \cos\theta_0)$$ We have to eliminate $\theta$

$\cos\theta = z/r$, $\cos\theta_0 = z_0/r$,

$$R = m\left[g\frac{z}{r} - \frac{1}{r}\left(-2gr\left(\frac{z}{r} - \frac{z_0}{r}\right) + v_0^2\right)\right]$$

$$R = \frac{3mgz}{r} - \frac{2mgz_0}{r} - \frac{mv_0^2}{r}$$

$$R = \frac{m}{r}(3gz - 2gz_0 - v_0^2)$$






b) if $v_0 > V$ ; $V = ??$

the particule remain in contact with the surface of the sphere $R > 0$

If the particule leaves the sphere $R \leq 0$ at $S_0$

$R_0 = \frac{m}{r}(3gz_0 - 2gz_0 - v_0^2) \leq 0$

$gz_0 \leq v_0^2 \Rightarrow \sqrt{gz_0} \leq v_0 \Rightarrow V = \sqrt{gz_0} = 3\,m/s$

$V \leq v_0$

$$R = \frac{m}{r}(3gz - 2gz_0 - v_0^2)$$

if $v_0 > V$

b) At the beginning at $s_0, z = z_0$ $R_0 = \frac{m}{r}(gz_0 - v_0^2)$

The particle remains in contact with the surface of the sphere when $R > 0$;
The particle escapes from the surface if $R \leq 0$,

$$R_0 = \frac{m}{r}(gz_0 - v_0^2) \leq 0$$

$$gz_0 - v_0^2 \leq 0$$

$$gz_0 \leq v_0^2$$

$$v_0^2 \geq gz_0 = V^2$$

So for $v_0 \geq \sqrt{gz_0}$ the particle leaves the sphere at the beginning at $S_0$. The lower limit of $v_0$ is

$$V = \sqrt{gz_0} = 3\,m/s$$

c) Path travelled before leaving = Arc $S_0S_1$

The traveled distance is: Arc $S_0S_1 = r(\theta_1 - \theta_0)$

At point $S_1$ ($z = z_1$), the particle leaves the sphere if $R = 0$

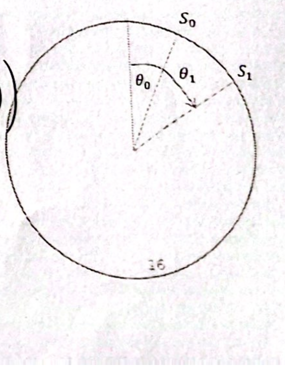
If $v_0 = \frac{V}{2} = \frac{\sqrt{gz_0}}{2}$, $\quad R = \frac{3mgz_1}{r} - \frac{2mgz_0}{r} - \frac{mgz_0}{4r}$

$$R = \frac{3mg}{r}\left(z_1 - \frac{3}{4}z_0\right) = 0$$

$$z_1 = \frac{3z_0}{4} = r\cos\theta_1 \rightarrow \theta_1 = \cos^{-1}\left(\frac{3z_0}{4r}\right)$$

The traveled distance is
Arc $S_0S_1 = r(\theta_1 - \theta_0)$

$$= r\left(\cos^{-1}\left(\frac{3z_0}{4r}\right) - \cos^{-1}\left(\frac{z_0}{r}\right)\right)$$



## 2.6 *Two-body problem and space dynamics*

In this paragraph, we study the most important applications of classical mechanics: the motion of an object subjected to a gravitational force proportional to $1/r^2$; the explanation of **planetary motion** and Kepler's laws, and the study of the motion of ballistic missiles, satellites, and interplanetary probes.

### 2-6-1- The law of universal gravitation

Two arbitrary particles with masses $m_1$ and $m_2$, separated by a distance $r$, exert on each other an attractive force acting along the line connecting them, with a magnitude given by: $F = G\frac{m_1 m_2}{r^2}$ where $G$ represents a universal constant. In the MKSA system: G = 6,67.10^-11 $Nm^2Kg^{-2}$

origine at $m_1$

origine at $m_2$

**Properties :**

1- The vector $\vec{h} = \overrightarrow{OP} \wedge \vec{v}$ (which is referred to as the areal velocity) is a constant of motion.

= vector بيساوي

$h = \frac{area}{time} \rightarrow speed!$

$$\vec{h} = \overrightarrow{OP} \wedge \vec{v} = r\vec{e}_r \wedge (r'\vec{e}_r + r\theta'\vec{e}_\theta) = r^2\theta'\vec{k} = \overline{cst}$$

$$\frac{d\vec{h}}{dt} = \frac{d}{dt}(\overrightarrow{OP} \wedge \vec{v})$$

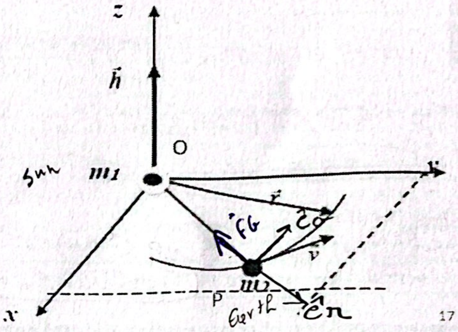
$$= \underbrace{\frac{d\overrightarrow{OP}}{dt}}_{v} \wedge \vec{v} + \overrightarrow{OP} \wedge \frac{d\vec{v}}{dt} = 0 + \overrightarrow{OP} \wedge \vec{a}$$

$\vec{a} \rightarrow -e_r$ $= 0$

$$\frac{d\vec{h}}{dt} = \frac{d\overrightarrow{OP}}{dt} \wedge \vec{v} + \overrightarrow{OP} \wedge \frac{d\vec{v}}{dt} = \vec{v} \wedge \vec{v} + \overrightarrow{OP} \wedge \frac{d\vec{v}}{dt} = 0$$

$$\vec{h} = \overline{cst}$$

2- The motion occurs in a plane passing through $O$. Indeed, since $\vec{h}$ is constant, $\overrightarrow{OP}$ remains perpendicular to a fixed direction during the motion, and thus lies in a plane. To study this motion, it is most appropriate to switch to polar coordinates $(\vec{e}_r, \vec{e}_\theta)$.

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$\vec{h} = r^2\dot{\theta}\,\vec{k}$

$h = r.v = cst$

3- The motion follows the law of areas (Kepler's second law), meaning that the radius vector sweeps out equal areas in equal intervals of time.

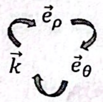
$$\vec{h} = \overrightarrow{OP} \wedge \vec{v} = r\vec{e}_r \wedge (r'\vec{e}_r + r\theta'\vec{e}_\theta)$$

$$= r^2\theta'\vec{k}$$

On the other hand $\vec{h} = h\vec{k} = r^2\frac{d\theta}{dt}\vec{k}$

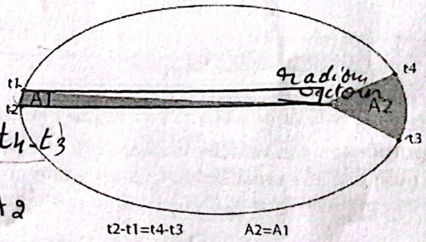
$ds = r d\theta$

Orthonormal base:



$\boxed{hdt = r^2 d\theta}$

ellipse




if $\Delta t = t_2 - t_1$

$= \Delta t' = t_4 - t_3$

then $A_1 = A_2$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$= \frac{dr}{dt}\hat{e}_r + r\frac{d\theta}{dt}\hat{e}_\theta \rightarrow d\vec{r} =$$

$$\Rightarrow d\vec{r} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta$$

$(\Delta r \wedge v)$

$$dA = \frac{1}{2}\overbrace{r.rd\theta}^{d\alpha} = \frac{1}{2}r^2 d\theta$$

$$dA = \frac{1}{2}hdt \rightarrow \frac{dA}{dt} = \frac{1}{2}h = cste$$

Area ال نفس

$E_t = KE + PE$

### 4- The total energy and angular momentum of the particle are conserved:

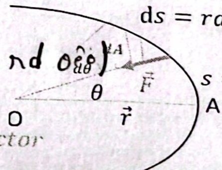
We say that the central force (conservative) $\vec{f} = -\frac{C}{r^2}\vec{e}_r$ (= GMm) applied to the particle $m$ derives from a potential $U(r)$, or in other words, the integral of this force gives a scalar function called potential energy, such that:

$U(r) =$

$$U(r) = -\int \vec{f}(r)d\vec{r} + k \rightarrow U(r) = -\int -\frac{C}{r^2}\vec{e}_r\,d\vec{r} + k$$

Potential Energie

$$= C\int \frac{dr}{r^2} + k = -\frac{C}{r} + k$$




$$U(r) = \int \frac{GMm}{r^2}\hat{e}_r \cdot (dr\hat{e}_r + r\,d\theta\,\hat{e}_\theta)$$

$$= \int \frac{GMm}{r^2}dr$$

$d\vec{r}$: elementary displacement vector

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$U_r = -\frac{GMm}{r} + K$ initial cond

we have an attractive potential

if $r \rightarrow \infty \rightarrow U \rightarrow 0$

$0 + 0 = K$

$U_r = -\frac{GMm}{r}$

$F_G$ is Conservative force, it derive from a potentiel Energy

$$U(r) = -\frac{C}{r} + k$$

We assume that $U(r) \to 0 \; when \; r \to \infty, so \; k = 0$

$C = Gm_1m_2 > 0$, and the potential is attractive

to prove mathemical that tot Energy is Conserve

The **fundamental relation of dynamics** on the particule m :

$$m(r'' - r\theta'^2)\vec{e}_r + m(r\theta'' + 2r'\theta')\vec{e}_\theta = -\frac{C}{r^2}\vec{e}_r$$

$$\begin{cases} m(r'' - r\theta'^2) = -\frac{C}{r^2} & \times r' \\ m(r\theta'' + 2r'\theta') = 0 & \times r\theta' \end{cases}$$

2 differential eq. of motion

$$m(r'r'' - rr'\theta'^2 + r\theta' r\theta'' + 2r\theta' r'\theta') = -\frac{C}{r^2}r'$$

$$m(r'r'' + rr'\theta'^2 + r^2\theta' \theta'') = -\frac{C}{r^2}r'$$

$$m(r'r'' + rr'\theta'^2 + r^2\theta' \theta'') + \frac{C}{r^2}r' = 0$$

The integration of this equation gives:

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$$\underbrace{\frac{1}{2}m(r'^2 + r^2\theta'^2)}_{Kinetique} \underbrace{-\frac{C}{r}}_{Potentiel} = cte \to \boxed{E_c + U(r) = E_m = cst}$$

this represents the law of **conservation of total energy.**

**The angular momentum is conserved**

$$\vec{J} = \vec{r} \wedge m\vec{v} = m\vec{r} \wedge \vec{v} = m\,\vec{h} = \overrightarrow{cst}$$

$$\overrightarrow{OP} \wedge V = h$$