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P1100 - Mechanics

Chapter 1 - Kinematics

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3- kinematics of a point: Motion in cartesian coordinates

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1- Introduction

- The object of mechanics is to **describe and explain motion**. Its principles have been established based on experimental results, so it is an **experimental science**.
- Mechanics is marked by several prominent figures, but its history is dominated by Newton and Einstein. Newton laid the foundation of classical mechanics, and Einstein discovered relativistic mechanics.
- It is important to note that Newtonian mechanics is an excellent approximation of relativistic mechanics when working at low speeds.

We will limit our study to the mechanics of the point by dividing the discipline of "mechanics" into two sub-disciplines:

- Kinematics: Study the motion of a particle without knowing the cause:

$$\vec{r}(t), \overrightarrow{v}(t), \vec{a}(t)$$

- Dynamics: Study the motion of a particle and understand the cause. It may be a force : $\vec{r}(t), \overrightarrow{v}(t), \vec{a}(t) + \vec{F}$

By the motion of a body, we mean the change in its position with respect to time in a system of fixed axes called a reference frame. This body can be treated as a particle or a material point if its dimensions are very small compared to other lengths involved in the study.

2- Unit systems and dimensions

Relation entre différents systèmes d'unités:

Fundamental quantities

Unit System	Length	Mass	Time	Force
International	m	kg	s	N
CGS	cm	g	s	dyne

	SI		CGS
Force **Work** **Power** **Pressure**	1 newton = 1 kg m s^{-2} 1 joule = 1Nm 1 watt = 1 j s^{-1} 1 pascal = 1 kg $m^{-1}s^{-2}$		1 dyne = 1 g cm s^{-2} = 10^{-5} N 1 erg = 1 dyne.cm = 10^{-7} J 1 erg.s^{-1} = 1 g cm^2s^{-3} = 10^{-7} W 1 barye = 1 dyne cm^{-2} = 10^{-1} Pa 1 bar = 10^5 Pa = 10^6 dynes/cm^2

Dimensions: mass $\rightarrow M$, $Length \rightarrow L$, $Time \rightarrow T$

Application I-1:

Evaluate the dimensions of pressure $P = \frac{F}{S} = FS^{-1}$.

Solution: We have to write the dimensional equation (or dimensional formula) of F and S.

$S = L^2 \quad \Rightarrow \quad S^{-1} = L^{-2}$

$F = M\,a\,, \qquad a = \frac{V}{T} \quad and\, V = \frac{L}{T} \;\; so \;\; a = \frac{L}{T^2}, \qquad\qquad F = MLT^{-2} \;\; and\, finally\, P = ML^{-1}T^{-2}$

Application I-2:

Find the relationship between dyne and Newton.

Solution: We have to write the dimensional equation of force, we find : $F = MLT^{-2}$.

$$1N = x\ dyn$$

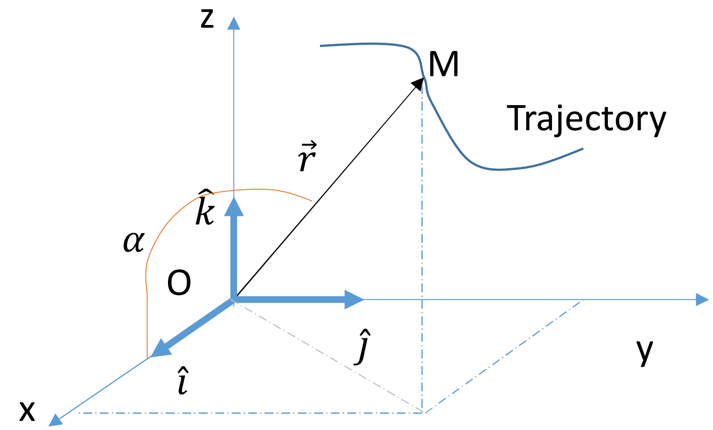
$$x = \frac{N}{dyn} = \frac{kg}{g}\frac{\mathrm{m}}{cm}\left(\frac{s}{s}\right)^{-2}$$

$$= 1000 \times 100 = 10^5$$

$$1N = 10^5\ dyn$$

I-3- kinematics of a point: Motion in cartesian coordinates : stationary frame (x,y,z)

I-3-1- Three dimensional motion:



a- Position vector

Let $\vec{r}$ be the position vector such that que $\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$

The trajectory of a particle can be the path left by a plane in the sky for example.

$\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$

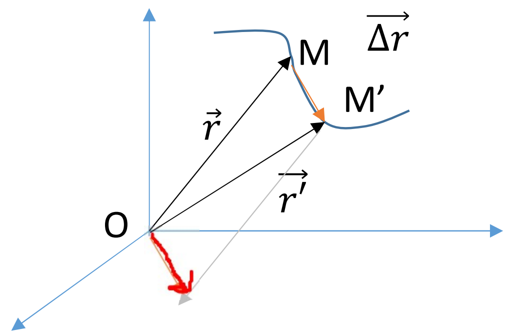
Its magnitude is $r = \sqrt{x^2 + y^2 + z^2}$

Its direction with x'ox is $\cos\alpha = \frac{x}{\sqrt{x^2+y^2+z^2}}$ *; α =($\overrightarrow{Ox}$, $\vec{r}$); we say that cos α is the cosine direction (x) of $\vec{r}$.*

Distance:

Consider two neighboring points M and M′ occupied by the particle M at times t and t+dt in such a way that the arc MM′ can be identified with the vector $\overrightarrow{MM'}$.

$$d\vec{r} = \overrightarrow{MM'} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$



Path length = Curvilinear abscissa $= \overset{\frown}{MM'}$

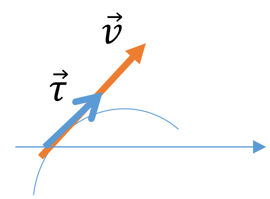
The distance travelled between t_1 and t_2 is $\boxed{S = \int_{t_1}^{t_2} ds = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt}$

b- <u>Instantaneous velocity vector</u>

We define the displacement vector as $\overrightarrow{\Delta r} = \overrightarrow{r'} - \vec{r}$

The instantaneous velocity i $\vec{v} = \boxed{\frac{\overrightarrow{dr}}{dt} = \dot{x}(t)\vec{\imath} + \dot{y}(t)\vec{\jmath} + \dot{z}(t)\vec{k}}$ its magnitude is $v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$

Its direction is always tangent to the trajectory :$\vec{v} = v\,\vec{\tau} \rightarrow \vec{\tau} = \frac{\vec{v}}{v} = \frac{\acute{x}(t)\vec{\imath} + \acute{y}(t)\vec{\jmath} + \acute{z}(t)\vec{k}}{v}$



C- <u>Instantaneous acceleration vector</u>

$$\boxed{\vec{a} = \frac{\overrightarrow{dv}}{dt} = \frac{d^2\vec{r}}{dt^2} = \ddot{x}(t)\vec{\imath} + \ddot{y}(t)\vec{\jmath} + \ddot{z}(t)\vec{k}}$$

Application I-3:

The instantaneous position of a box is given by the following coordinates: x=0.5sin(2t), y=0.5cos(2t), and z=−0.2t. Determine its position and the amplitude of its velocity and acceleration at t=0.75s.

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

Solution $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k} = 0.5\sin 2t\,\hat{\imath} + 0.5\cos 2t\,\hat{\jmath} - 0.2t\,\hat{k}$

$$= 0.499\,\hat{\imath} + 0.035\,\hat{\jmath} - 0.15\,\hat{k} \rightarrow r = 0.522\,m$$

$$\vec{v} = \frac{\overrightarrow{dr}}{dt} = \dot{x}(t)\hat{\imath} + \dot{y}(t)\hat{\jmath} + \dot{z}(t)\hat{k}$$

$$= 0.5 \times 2 \times \cos 2t\,\hat{\imath} + 0.5 \times 2 \times (-\sin 2t)\hat{\jmath} - 0.2\,\hat{k}$$

$$= \cos 2t\,\hat{\imath} - \sin 2t\,\hat{\jmath} - 0.2\,\hat{k}$$

$$\rightarrow v = 1.02\,m/s$$

$$\vec{a} = \frac{\overrightarrow{dv}}{dt} = \ddot{x}(t)\hat{\imath} + \ddot{y}(t)\hat{\jmath} + \ddot{z}(t)\hat{k}$$

$$= -2\sin 2t\,\hat{\imath} - 2\cos 2t\,\hat{\jmath}$$

$$\rightarrow a = 2\,m/s^2$$

I-2-- Rectilinear motion

A motion is rectilinear if the particle moves along a straight line. The acceleration of the particle is reduced to a single component along the direction of motion $\vec{a} = \dot{v}\vec{i}$, and the velocity becomes $\vec{v} = v\,\vec{i}$ thus, the velocity maintains the same direction defined by the unit vector $\vec{i}$.

a- Uniform rectilinear motion: $v = \frac{dx}{dt} = cst$, $\int_{x_0}^{x} dx = \int_0^t v_x dt \rightarrow \boxed{x = v_x t + x_0}$

b- Uniformly accelerated rectilinear motion where the acceleration is constant: $a = \frac{dv_x}{dt} = cst \rightarrow$ $\int_{v_{0x}}^{v_x} dv_x = \int_o^t a dt \rightarrow v_x = at + v_{0x}$ **and** $\boxed{x = \frac{1}{2}at^2 + v_{0x}t + x_0}$

***c- Erratic motion**:* When the motion of a particle over time is erratic, it is difficult to find a mathematical function to describe its position, velocity, or acceleration, we usually use graphs to describe the motion.

$v = \frac{ds}{dt}$, $a = \frac{dv}{dt}$, **so** $\boxed{v dv = a ds}$ **Note that a is the tangential acceleration**

I-3-3- Planar motion: projectile

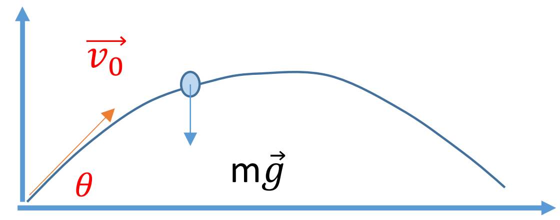
- A 2D motion can be described in cartesian coordinates as following:

$$\vec{r} = \overrightarrow{OM}(t) = \mathrm{x}\hat{i} + \mathrm{y}\hat{j}$$

$$\vec{v}(t) = \dot{x}\hat{i} + \dot{y}\hat{j}$$

$$\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$$

Projectile: An example of 2D motion



Without air resistance:

$$\sum \overrightarrow{F_{ext}} = m\vec{a} \rightarrow \mathrm{m}\vec{g} = m\vec{a} \rightarrow \vec{a} = \vec{g} \quad (1)$$

$$\vec{a} = \begin{cases} a_x = 0 \\ a_y = -g \end{cases} \rightarrow \vec{v} = \begin{cases} v_x = cst = v_{0x} \\ v_y = -gt + v_{0y} \end{cases} \rightarrow \vec{r} = \begin{cases} x = v_{0x}\, t + x_0 \\ y = -\frac{1}{2} g t^2 + v_{0y} t + y_0 \end{cases}$$

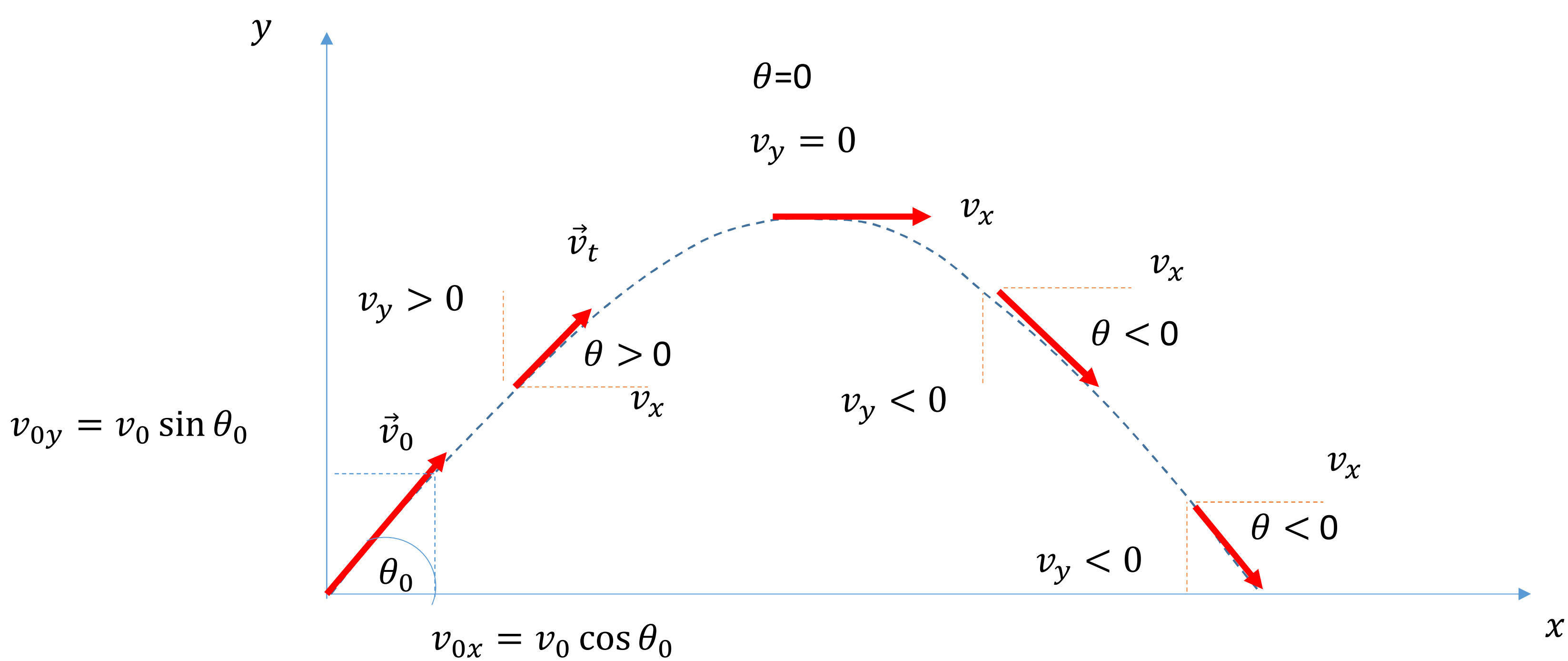
$$\begin{cases} Along\ x\text{: } uniform\ motion \\ Along\ y\text{: } uniformly\ accelerated\ motion \end{cases}$$

$$v_{0x} = v_0 \cos\theta$$
$$v_{0y} = v_0 \sin\theta$$

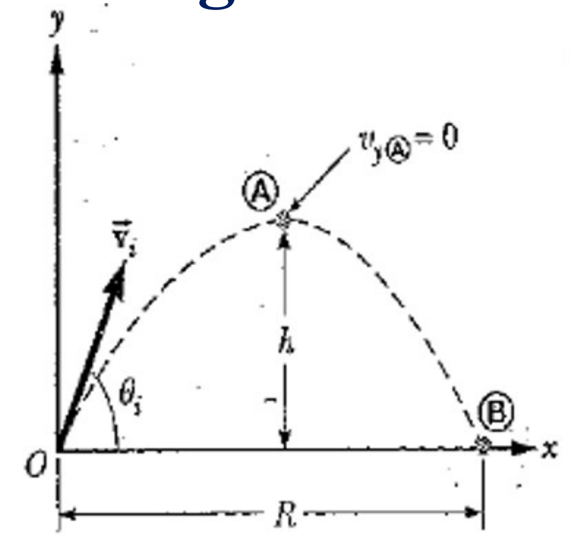
We can find the equation of the trajectory (y=y(x)):

$$y = -\frac{g(x - x_0)^2}{2v_0^2 cos^2\theta} + \tan\theta(x - x_0) + y_0$$

Parabolic path


y
θ=0
$v_y = 0$
v_x
$\vec{v}_t$
v_x
$v_y > 0$
$\theta < 0$
$\theta > 0$
v_x
$v_y < 0$
$v_{0y} = v_0 \sin\theta_0$
$\vec{v}_0$
v_x
$\theta < 0$
$v_y < 0$
θ_0
x
$v_{0x} = v_0 \cos\theta_0$

The horizontal range and the maximum height



Two points are particularly interesting to analyze:

- The peak A of the trajectory, with a maximum height h
- Point B with a horizontal range R

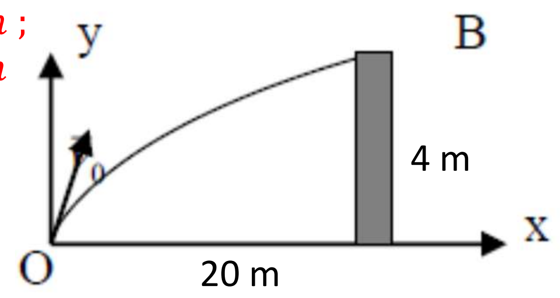
We can calculate h after noting that the y-component of the velocity is zero at the peak.

- $v_{yA} = 0 = v_0 sin\theta - gt_A \Rightarrow \; t_A = \frac{v_0 sin\theta}{g}$
- If we substitute t_A in the y-component and remplace $y = y_A$ with h we obtain : $h = \frac{v_0^2 sin_\theta^2}{2g}$ (note that $y_0 = 0$)
- The range R corresponds to the horizontal position of the projectile at twice the time it takes to reach its peak, that is, at time $t_B = 2t_A = 2\,\frac{v_0 sin\theta}{g}$
- We can write R in the following form :R $= \frac{v_0^2 sin2\theta}{g}$ (note that $x_0 = 0$)

Application I-7:

Messi throws a ball from the origin O of a coordinate system (Oxy) with an initial velocity v0, making an angle θ with the horizontal. He intends to make the ball pass just above the top B of a wall with a height of 4m. The base of the wall is located 20m away from point O. With what velocity should he throw the ball if point B represents the maximum of the ball's trajectory?

$v_{y_B} = 0$;
$x_B = 20\,m$;
$y_B = 4\,m$



Solution:

Eq 0

At B: $v_y = 0 \Rightarrow 0 = -gt + v_0 \sin\theta \Rightarrow t_B = \dfrac{v_o \sin\theta}{g}$.

Eq 1

$$x = (v_0 \cos\theta)\, t = (v_0 \cos\theta) \times \frac{v_o \sin\theta}{g} \Rightarrow \boxed{20g = v_0^2 \sin\theta \cos\theta}$$

2 unknowns: v_0 & θ

Eq 2

$$y = -\frac{1}{2} g t^2 + v_0 \sin\theta\, t \quad \rightarrow 4 = -\frac{1}{2} g \left(\frac{v_0 \sin\theta}{g}\right)^2 + v_0 \sin\theta \left(\frac{v_0 \sin\theta}{g}\right) \quad \rightarrow \boxed{8g = {v_0}^2 sin^2\theta}$$

$$\frac{8}{20} = \frac{\sin\theta}{\cos\theta} \rightarrow \theta = 21{,}8° \qquad , \rightarrow {v_0}^2 = \frac{8g}{sin^2\theta} \rightarrow v_0 = 23{,}5\, m/s$$

Application I-9:

The speed of a particle was constant $v_0 = 5\ m/s$ for $t < 0$ then it experiences an acceleration of $a = -3v^2$ (rectilinear motion along x).

1) Express the velocity and calculate the speed after 5 s.
2) Find the position x. Let $x = 0$ for $t = 0$.
3) Show that $v = 5e^{-3x}$.

$$\int \frac{dx}{x} = ln(x)$$

Solution

a-

$$a = \frac{dv}{dt} = -3v^2 \quad \to \frac{dv}{v^2} = -3dt \to \quad \int_5^v \frac{dv}{v^2} = \int_0^t -3dt \to \quad \frac{-1}{v}\bigg|_5^v = -3t\bigg|_0^t \to \quad \frac{-1}{v} + \frac{1}{5} = -3t \to \quad 3t + \frac{1}{5} = \frac{1}{v} \to \quad v = \frac{5}{1+15t}$$

b-

$$v = \frac{dx}{dt} = \frac{5}{1+15t} \to \int_0^x dx = \int_0^t \frac{5\,dt}{1+15t} \quad \frac{\times 3}{\times 3} \to \quad x = \frac{1}{3} ln(1+15t)\bigg|_0^t = \frac{1}{3}[ln(1+15t) - ln(1)] = \frac{1}{3} ln(1+15t)$$

c-

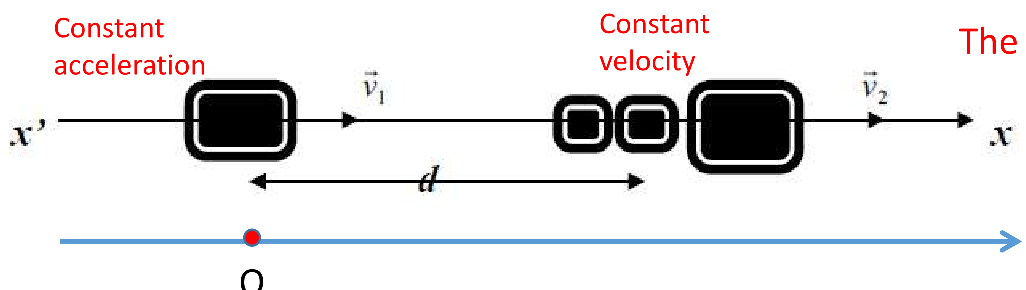
From a: $1 + 15\,t = \frac{5}{v}$

From b: $ln(15t+1) = 3\text{x} \to \quad 15t + 1 = e^{3x} \to \quad v = 5e^{-3x}$

Application I-11:

The driver of a locomotive, moving at speed v_1, suddenly sees in front of him a train traveling on the same track and in the same direction at a lower speed v_2, and which is then at a distance d.

How much braking acceleration is needed to avoid the collision?



The trains collide if $x_1 = x_2$

Solution

A train running on the same track and in the same direction: One-dimensional motion, train at constant speed v_2= constant

$$x_2 = v_2 t + x_0 = v_2 t + d$$

The locomotive with speed v_1 must slow down, resulting in decelerated motion. $x_1 = \frac{1}{2} a\, t^2 + v_1 t + 0 \quad (Note\ that\ a < 0)$

The trains collide if $x_1 = x_2 \quad \rightarrow \quad \frac{1}{2} a\, t^2 + v_1 t = v_2 t + d \rightarrow \quad \frac{1}{2} a\, t^2 + (v_1 - v_2) t - d = 0$

There is no collision if this equation has no solutions, i.e., if $\Delta < 0$

$$(v_1 - v_2)^2 + 2ad < 0 \rightarrow \frac{(v_1 - v_2)^2}{2d} < -a$$