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I-6- Tangential and normal acceleration:

If the particle moves along a known trajectory, it will be more convenient to use the normal and tangential components of acceleration. Let's examine the acceleration vector in more details. $\vec{v} = v\vec{\tau}$

$$\vec{a} = a_\tau \hat{\tau} + a_n \hat{n}$$

$$\vec{a} = \frac{d(v\hat{\tau})}{dt} = v'\vec{\tau} + v \frac{d(\hat{\tau})}{dt} = v'\hat{\tau} + v \frac{d(\hat{\tau})}{dt} = v'\hat{\tau} + v \left(\frac{d(\hat{\tau})}{d\alpha} \right) \frac{d\alpha}{dt}$$

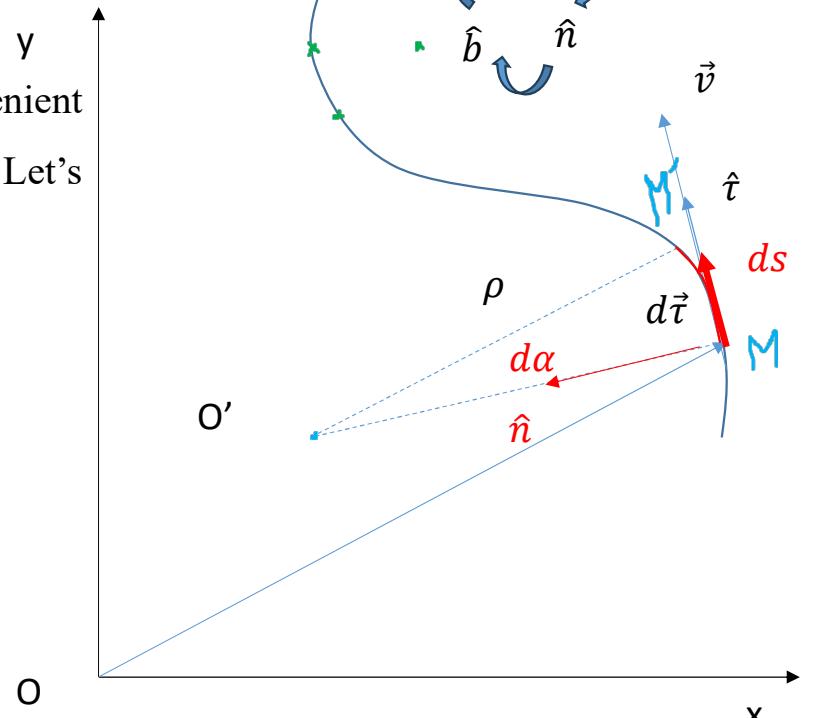
$$\text{Arc } ds = \rho d\alpha \rightarrow \frac{ds}{dt} = \rho \frac{d\alpha}{dt} \rightarrow v = \rho \frac{d\alpha}{dt} \rightarrow \frac{d\alpha}{dt} = \frac{v}{\rho}$$

$$\vec{a} = v'\hat{\tau} + v \frac{v}{\rho} \hat{n} = v'\vec{\tau} + \frac{v^2}{\rho} \hat{n} = a_\tau \hat{\tau} + a_n \hat{n}$$

$$a = \sqrt{a_\tau^2 + a_n^2}$$

$$\text{direction: } \tan \alpha = \frac{a_n}{a_\tau}$$

In the particular case where the equation of the trajectory is known as $y=f(x)$, the radius of curvature at any point on the trajectory is given by the following equation:



$$\rho = \left| \frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2}}{\frac{d^2y}{dx^2}} \right|^2$$

Applications : I-17- A skier descends at a constant speed $v = 6 \text{ m/s}$ a parabolic slope with the equation $y = \frac{1}{20}x^2$. Determine his velocity and acceleration vectors when he reaches point A (10m, 5m).

Solution

Constant speed $v = 6 \text{ m/s}$

$$\vec{v} = v \vec{\tau} = 6 \vec{\tau}$$

* Polar coordinates: we have a symmetry about the y-axis.

* We know the equation of the trajectory \rightarrow use the normal and tangential components

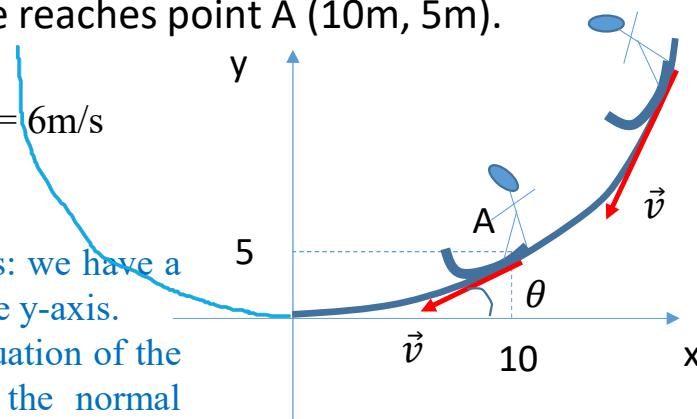
Direction of \vec{v} : We need to calculate the slope or direction vector at Point A.

$$\text{slope: } \tan \theta = \frac{dy}{dx} = \frac{1}{10}x \Big|_{x=10} = 1 \rightarrow \theta = 45^\circ$$

The acceleration $\vec{a} = \vec{a}_t + \vec{a}_n$

$$a_t = v' = 0$$

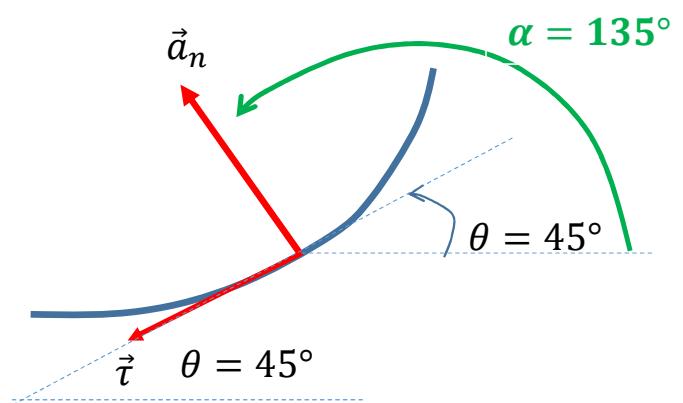
$$a_n = \frac{v^2}{\rho} = ?$$



$$\rho = \left| \frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2}}{\frac{d^2y}{dx^2}} \right| = \left| \frac{\left[1 + \left(\frac{1}{10}x \right)^2 \right]^{3/2}}{\frac{1}{10}} \right| \Big|_{x=10} = 28,3 \text{ m}$$

$$a_n = \frac{36}{28,3} = 1,27 \text{ m/s}^2$$

Direction of \vec{a}_n ? $\alpha = 45^\circ + 90^\circ = 135^\circ$



Cartes.- cylind- polar- normal
and tg

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I-7- Circular motion

From the equations of motion in polar coordinates, we can derive those for circular motion where r is constant.

$$\vec{r} = r\hat{e}_r \rightarrow \vec{v} = \frac{d\vec{r}}{dt} = r \frac{d\hat{e}_r}{dt} = r(\vec{\omega} \wedge \hat{e}_r) \rightarrow \begin{cases} \vec{v} = \vec{\omega} \wedge \vec{r} \\ \vec{v} = \omega r \hat{e}_t \end{cases}$$

$$\vec{v} = \vec{\omega} \wedge \vec{r}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(\omega r \hat{e}_t)}{dt} = r \frac{d\omega}{dt} \hat{e}_t + r\omega \frac{d\hat{e}_t}{dt} \rightarrow$$

$$\vec{a} = \alpha r \hat{e}_t - r\omega^2 \hat{e}_r$$

$$a_t = \alpha r \quad \text{and} \quad a_n = \omega^2 r$$

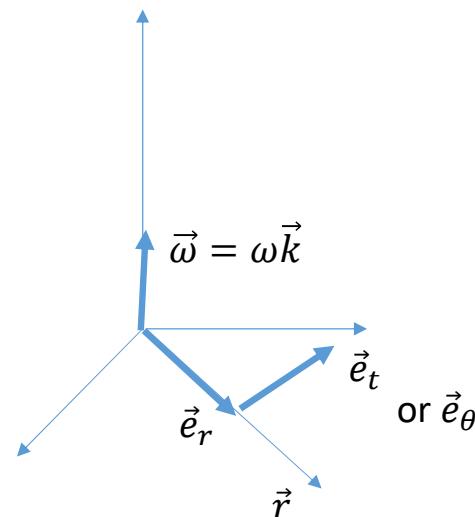
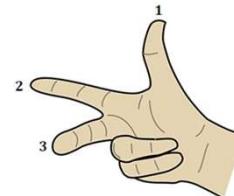
$$\vec{a}_t = \vec{\alpha} \wedge \vec{r}$$

$$\hat{e}'_r = \theta' \hat{e}_\theta = \omega(\hat{k} \wedge \hat{e}_r) = \vec{\omega} \wedge \hat{e}_r$$

$$\hat{e}'_\theta = -\theta' \hat{e}_r = -\omega \hat{e}_r = -\omega(\hat{e}_\theta \wedge \hat{k}) = \omega(\hat{k} \wedge \hat{e}_\theta) = \vec{\omega} \wedge \hat{e}_\theta$$

Orthonormal base:

$$\begin{aligned} \vec{e}_r & \leftarrow \hat{e}_\theta \wedge \hat{k} \\ \vec{k} & \leftarrow \hat{e}_\theta \wedge \hat{e}_r \\ \vec{e}_\theta & = \hat{k} \wedge \hat{e}_r \\ \hat{e}_r & = \hat{e}_\theta \wedge \hat{k} \end{aligned}$$



where $\omega = \frac{d\theta}{dt}$ is the angular speed of rotation

and $\alpha = \frac{d\omega}{dt}$ the angular acceleration of rotation.

By eliminating time between the two equations, we obtain a differential relation between velocity, acceleration, and angular position :

$$\alpha d\theta = \omega d\omega$$

The circular trajectory is closed, so the object returns to the starting point after a time T called the period, such that: $T = \frac{2\pi}{\omega}$

The circular motion can be **uniformly accelerated** if the angular acceleration of the particle remains constant during the rotation. The equations of motion become:

$$\omega = \alpha t + \omega_0 \rightarrow \quad \theta = \frac{1}{2} \alpha t^2 + \omega_0 t + \theta_0 \rightarrow \quad \omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0)$$

Circular motion is **uniform** when the angular velocity is constant, i.e., the tangential acceleration is zero:

$$v = r\omega \quad a_n = r\omega^2 \quad \vec{a}_n = -\frac{v^2}{r} \vec{e}_r$$

Or $\vec{v} = \vec{\omega} \times \vec{r}$

Application I-18- A car is moving on a circular and horizontal road with a radius $R = 90\text{m}$. It starts from rest. It is given that the rate of change of its velocity is $2,1 \text{ m/s}^2$.

a- Determine the time required for its acceleration to reach $2,4 \text{ m/s}^2$.

b- What is its speed at that moment?

Solution

Circular motion $\rightarrow \vec{a} = \vec{a}_t + \vec{a}_n$

Rate of change of speed $\rightarrow \frac{dv}{dt} = a_\tau = 2,1 \text{ m/s}^2$

$$dv = 2,1dt \rightarrow v = 2,1t$$

$$a_n = \frac{v^2}{R} = \frac{(2,1t)^2}{90} = 0,05t^2$$

$$a = \sqrt{a_\tau^2 + a_n^2} = \sqrt{2,1^2 + (0,05t^2)^2} = 2,4 \text{ m/s}^2, \quad t = 4,87 \text{ s}$$

b) $v = 2,1t = 10,23 \text{ m/s}$

Application I-20

Starting from rest, a car moves on a horizontal circular road with a radius $R=50m$ at a speed $v=0.2t^2$. Determine:

a- The acceleration at time $t=3s$.

b- Deduce its angular speed and angular acceleration.

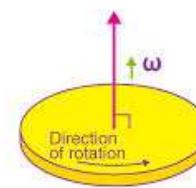
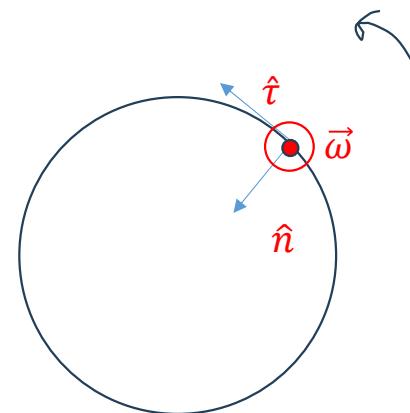
Solution

$$a) \vec{a} = \vec{a}_\tau + \vec{a}_n = \dot{v} \vec{e}_\theta - \frac{v^2}{r} \vec{e}_r = 0,4 t \vec{e}_\theta - \frac{(0,2t^2)^2}{50} \vec{e}_r$$

$$a = \sqrt{0,0042 + 1,44} = 1,2 \text{ m/s}^2$$

$$b) \vec{v} = \vec{\omega} \wedge \vec{r} = \omega R \sin \frac{\pi}{2} \vec{e}_\theta \rightarrow \omega = \frac{v}{R} = \frac{0,2t^2}{R} = 0,03 \text{ rd/s}$$

$$\vec{a}_t = \vec{\alpha} \wedge \vec{r} = \alpha R \vec{e}_\theta \rightarrow \alpha = \frac{a_t}{R} = \frac{0,4t}{R} = 0,024 \text{ rd/s}^2$$



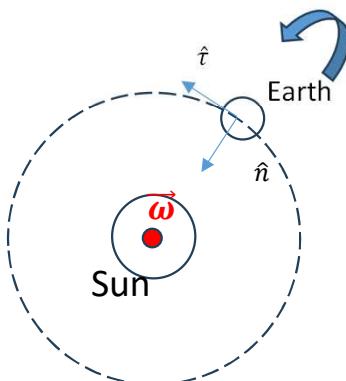
Application I - 21:

The Earth describes around the sun, with uniform motion, an orbit assimilated to a circle of radius $R=1.5 \times 10^8$ km, in $T=365$ days.

a- Determine, with respect to the Copernican reference frame, the translational velocity of the Earth's center as well as its acceleration.

b- Calculate the velocity and acceleration due to the rotation of the Earth on itself for a point on its surface located at the equator and for a point at a latitude of 40° . A

The radius of Earth is taken as 6400 km.



a) Circular motion

$$\vec{v} = \vec{\omega} \wedge \vec{r} \rightarrow v = \omega R = \frac{2\pi}{T} R$$

$$v = \omega R = \frac{2\pi}{365 \times 24 \times 3600} 1,5 \times 10^8 \text{ Km} = 29,88 \text{ Km/s}$$

$$v = cst \rightarrow a_t = 0,$$

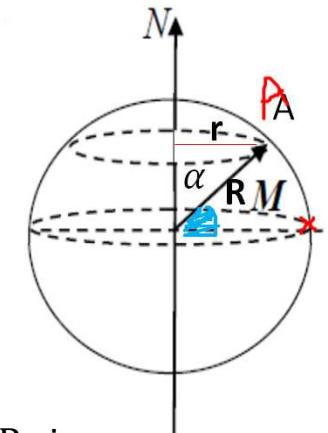
$$a = a_n = \frac{v^2}{R} = \frac{(30 \cdot 10^3)^2}{1,5 \times 10^{11}} = 5,95 \times 10^{-3} \text{ m/s}^2$$

b) A point on the surface of the Earth describes a circle of $r=R\sin\alpha$

$$a = \omega^2 r = \left(\frac{2\pi}{T'}\right)^2 R \sin \alpha$$

$$T' = 1 \text{ day} = 24h = 86400 \text{ s}$$

$$\vec{v} = \vec{\omega} \wedge \vec{r} = v \hat{t}$$



$$\text{For } \alpha = 90^\circ : \quad v = \omega R \sin \alpha = \frac{2\pi}{T'} R \sin \alpha$$

$$(\text{point M}) \quad v = 465,4 \text{ m/s} \text{ and } a = 3,4 \cdot 10^{-2} \text{ m/s}^2$$

For $\alpha = 90^\circ - 40^\circ$ (point A)

$$v = \omega R \sin \alpha = 356,5 \text{ m/s} \text{ and } a = 2,6 \cdot 10^{-2} \text{ m/s}^2$$