Lebanese University
Faculty of Sciences, B3
Tripoli - Lebanon

P1100 - Mechanics

Chapter 1 – Kinematics

Tutorial

Department of Physics
Lebanese University

Exercise 1:

The acceleration of a particle is given by:

$$a = k \cdot \frac{V^{n+1}}{r^n}$$

where k is a constant, and n is an integer. Find the dimension of k.

Solution We have $a = k\ \frac{v^{n+1}}{r^n} \Rightarrow k = \frac{ar^n}{v^{n+1}}$

The equation of dimensions gives : $[k] = \left[\frac{ar^n}{v^{n+1}}\right]$.

As $[a] = LT^{-2}\ (acceleration), [r] = L\ (distance) et\ [v] = LT^{-1}\ (speed)$.

The dimension of k becomes :

$$[k] = \frac{LT^{-2}L^n}{(LT^{-1})^{n+1}} = \frac{L^{n+1}T^{-2}}{L^{n+1}T^{-(n+1)}} = L^0T^{n+1-2} = T^{n-1}$$

The unit in SI of constant k is (s^{n-1})

Exercise 2:

The kinematic viscosity of a liquid is a physical quantity that is given by:

$$\mu = \frac{P \cdot t}{\rho}$$

where P denotes the liquid's pressure, t the time of flow, and ρ the liquid's density.

Find the dimensions of:

a) The pressure P.
b) The kinematic viscosity μ.

Solution

a)
To write the dimension of the pressure we need to know its expression.
The pressure is simply the force over the area of the surface on which it acts.

P=F/S

The force can be given by (2nd law of Newton): F=m.a, with m the mass and a the acceleration.

$$[F] = [m][a] = MLT^{-2}\ and\ [S] = L^2\ so\ [P] = \frac{MLT^{-2}}{L^2} = MT^{-2}L^{-1}$$

b)

$$[\mu] = \frac{[P.t]}{[\rho]} = \frac{MT^{-2}L^{-1}T}{ML^{-3}} = L^2T^{-1}\ with\ [\rho] = [m]/[V] = ML^{-3}$$

Exercise 3: To Skip

In 1900, Max Planck supposed that the energy of an oscillator is quantized as:

$$E = n \cdot h \cdot \nu \text{ or } E = n \cdot h \cdot \frac{c}{\lambda}$$

with n being a dimension-less integer, h the constant of Planck, ν the frequency corresponding to the wavelength λ, and c the speed of light. Find the dimension of h.

Solution

$E = nh\nu \Rightarrow h = \frac{E}{n\nu}$; the dimension of h is then given by : $[h] = \left[\frac{E}{n\nu}\right] = \frac{[E]}{[n][\nu]}$.

We have to find the dimensions of E, n and ν.

n is a number without dimension, i.e. $[n] = 1$.

The frequency is given by $\nu = \frac{1}{T}$ with T the period so $[\nu] = \left[\frac{1}{T}\right] = T^{-1}$.

The energy can be given by: kinetic energy = mass x speed²/2 with speed = distance / time ; so :

[speed] = LT^{-1} and [energy] = ML^2T^{-2} (remark : 2 is without dimension).

Finaly : $[h] = \frac{ML^2T^{-2}}{1 \times T^{-1}} = ML^2T^{-1}$.

Exercise 4: To Skip

The gravitational force of attraction of two particles of respective masses m and m', and distant by r, is given by $F = G \cdot \frac{m \cdot m'}{r^2}$ where G denotes the universal gravitation constant.

Find the dimension, as well as the unit of G in SI.

Solution

In this exercise we have the physical formula that gives the force of attraction due to mass between two particules. We want to find the dimension of constant G.

$$F = G\frac{mm'}{r^2} \quad donc\ G = \frac{Fr^2}{mm'}$$

In the equation of dimension we need to find everything in terms of the length (L in meter), the mass (M in Kg) and the time (T in second).

F is given from the second law of Newton by F=m.a with m the mass and a the acceleration. So, $[F] = MLT^{-2}$

The equation of dimension of G is :

$$[G] = \frac{[F][r]^2}{[m][m']} = \frac{MLT^{-2}L^2}{M^2} = M^{-1}L^3T^{-2}$$

$$Unit\ in\ SI: m^3/Kg.s^2$$

Exercise 5:

The parametric equations that define the position of a particle are:

$$x = 3t^2, y = 4t + 2 \text{ and } z = 6t^3 - 8.$$

Determine at time $t = 2s$:

a) The vectors of velocity and acceleration.

b) The direction cosines of the tangent to the trajectory.

Solution

a.

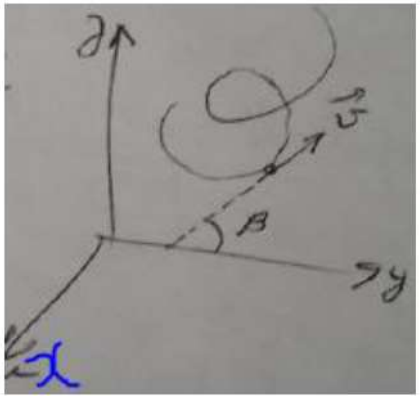
The path of the particle is described by : $x = 3t^2; y = 4t + 2 ; z = 6t^3 - 8$

The velocity vector of the particle can be expressed as : $\vec{v} = \frac{d\vec{r}}{dt} = x'\vec{i} + y'\vec{j} + z'\vec{k}$

$x' = \frac{dx}{dt} = 6t \quad ; \quad y' = \frac{dy}{dt} = 4 \quad and \quad z' = \frac{dz}{dt} = 18t^2$

So at $t = 2s \quad \rightarrow x' = 12 ; y' = 4 ; z' = 72$

$\Rightarrow \vec{v} = 12\vec{i} + 4\vec{j} + 72\vec{k}$



The acceleration vector of the particle can be expressed as : $\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt} = x''\vec{i} + y''\vec{j} + z''\vec{k}$

$\rightarrow x'' = 6; y'' = 0; z'' = 36t$

So at $t = 2s \quad \rightarrow x'' = 6 ; y'' = 0 ; z'' = 72 \quad \Rightarrow \vec{a} = 6\vec{i} + 72\vec{k}$

b.

The direction cosines are the cosines of the angles between the velocity vector and the x, y and z axes. They equal the coordinates of vector $\vec{v}$ over its length v (the speed) :

$$\vec{v}.\vec{\imath} = v \times 1\cos\alpha \Rightarrow \cos\alpha = \frac{\vec{v}.\vec{\imath}}{v} = \frac{\left(12\vec{\imath} + 4\vec{\jmath} + 72\vec{k}\right).\vec{\imath}}{\sqrt{(12^2 + 4^2 + 72^2)}}$$

$\cos\alpha = \frac{12}{73} = 0.16 \ \Rightarrow \ \alpha = 80.55°$; $\cos\beta = \frac{4}{73} = 0.05 \ \Rightarrow \ \beta = 86.86°$

$$\cos\gamma = \frac{72}{73} = 0.99 \ \Rightarrow \ \gamma = 9.95°$$

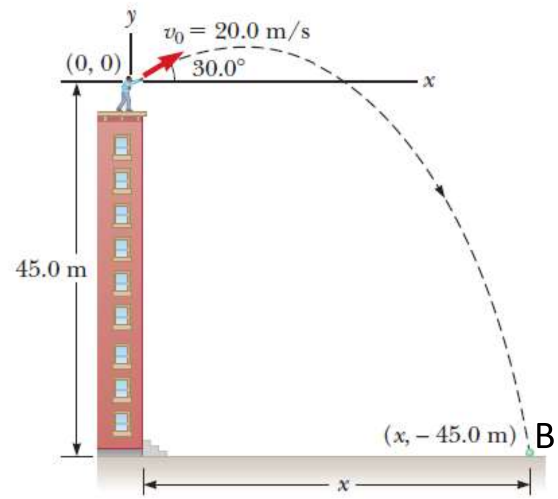
Exercise 6:

A ball is thrown upward from the top of a building at an angle of 30° to the horizontal and with an initial speed of 20 m / s. The point of release is 45 m above the ground.

a) How long does it take for the ball to hit the ground?

b) Find the ball's speed at impact.

c) Find the horizontal distance (x) of the stone from the building to the point on the ground where the stone lands.



Solution

$h = 45\ m, \qquad B(x = x_B,\ y_B = -45m).$

$\longrightarrow$Ball (Projectile): $v_0 = 20\ {}^{m}/_{sec}$ and $\theta_0 = 30^0$: Initial conditions.

$\rightarrow \vec{v_0} = v_{x0}\vec{i} + v_{y0}\vec{j}$.

Acoording to the figure:

$$\begin{cases} v_{x0} = +v_0\cos(\theta_0) \\ v_{y0} = +v_0\sin(\theta_0) \end{cases} \Rightarrow \begin{cases} v_{x0} = +20\cos(30^0) \\ v_{y0} = +20\sin(30^0) \end{cases} \Rightarrow \begin{cases} v_{x0} = +17.3\ {}^{m}/_{sec} \\ v_{y0} = +10\ {}^{m}/_{sec} \end{cases}$$

Summary on Projectile:

Motion along x (constant v_x)

$$a_x = 0$$

$$v_x = constant = v_{0_x}$$

$$x = v_{0x}\mathrm{t} + x_0$$

Motion along y: (constant a_y)

$$a_y = -g$$

$$v_y = -g\mathrm{t} + v_{0_y}$$

$$y = -\frac{1}{2}gt^2 + v_0 \sin\theta t + y_0$$

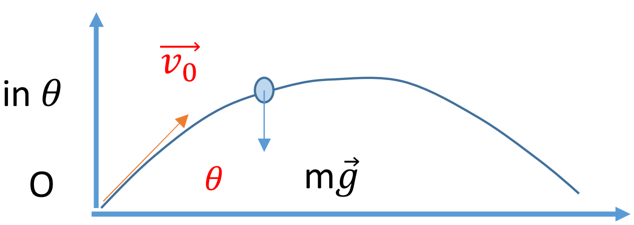
Eq. of the trajectory: $y = -\frac{g(x-x_0)^2}{2v_0^2 cos^2\theta} + \tan\theta(x - x_0) + y_0$

$$h = \frac{v_0^2 sin_\theta^2}{2g} = \max height$$

$$t_{\max height} = \frac{v_0 sin\theta}{g}$$

$$\mathrm{R} = \frac{v_0^2 sin2\theta}{g}$$

$v_{0y} = v_0 \sin\theta$



$v_{0x} = v_0 \cos\theta$

a) $t_B = ?$ $\longrightarrow$ Equations of motion for a projectile are:

$$\begin{cases} x(t) = v_{x0}t + x_0 \\ y(t) = -\frac{1}{2}gt^2 + v_{y0}t + y_0 \end{cases}$$

$\rightarrow$According to the figure: $x_0 = y_0 = 0$.

$$\rightarrow y_B(t) = -\frac{1}{2}g(t)^2 + v_{y0}t \Longrightarrow -45 = -4.9(t)^2 + 10t$$

$$\Longrightarrow -4.9(t)^2 + 10t + 45 = 0 \Longrightarrow t_1 < 0\text{(rejected) and } t_B = 4.22\ sec\text{(accepted)}.$$

b) Find the ball's speed at impact.

b) $v_B = ?$

$$\longrightarrow \vec{v} = v_x\vec{i} + v_y\vec{j} \Longrightarrow v = \sqrt{(v_x)^2 + (v_y)^2}$$

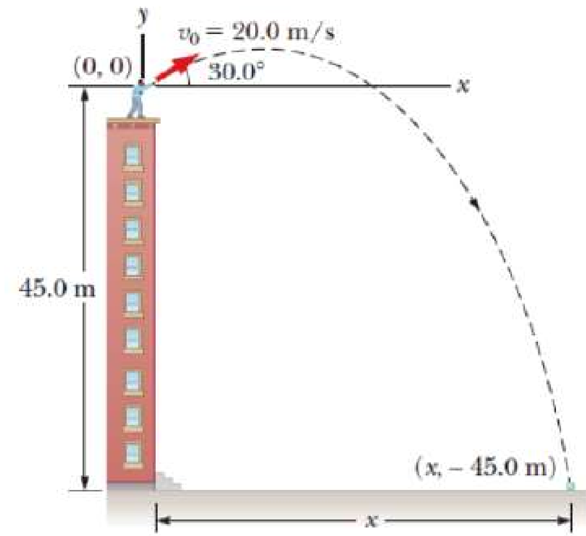
$$\rightarrow \overrightarrow{v_B} = v_{xB}\vec{i} + v_{yB}\vec{j} \Longrightarrow v_B = \sqrt{(v_{xB})^2 + (v_{yB})^2}\ .$$

$$\longrightarrow \text{For a projectile: } \begin{cases} v_x = v_{x0} \\ v_y = v_{y0} - gt \end{cases}$$

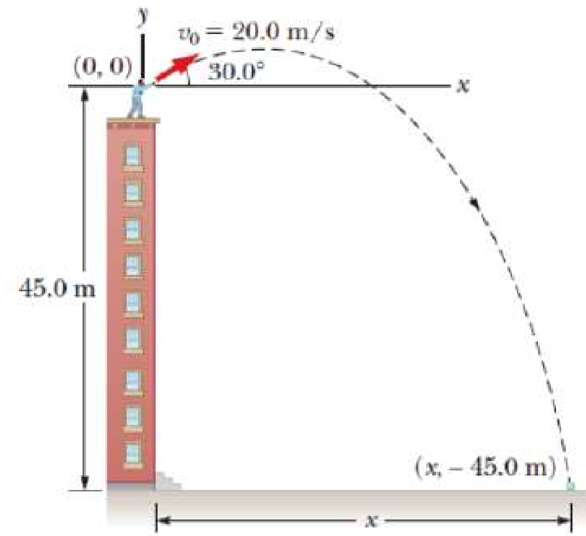
$$\Longrightarrow \begin{cases} v_{xB} = v_{x0} = +17.3\ m/sec \\ v_{yB} = v_{y0} - gt_B = +10 - 9.8 \times 4.22 \end{cases}$$

$$\Longrightarrow \begin{cases} v_{xB} = +17.3\ m/sec \\ v_{yB} = -31.4\ m/\sec \end{cases}$$

$$\longrightarrow v_B = \sqrt{(v_{xB})^2 + (v_{yB})^2} = \sqrt{(+17.3)^2 + (-31.4)^2} = 35.9\frac{m}{sec}\ .$$



c) Find the horizontal distance (x) of the stone from the building to the point on the ground where the stone lands.

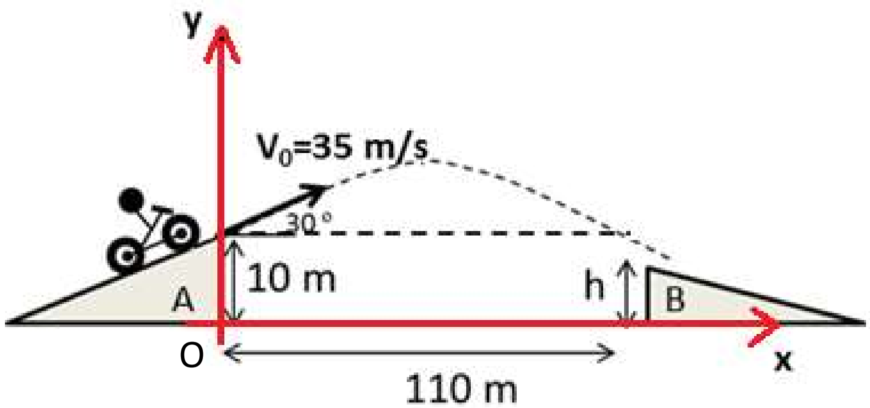


c) $x = ?$

$\longrightarrow x = x_B = v_{x0} t_B = +17.3 \times 4.22 = +73\ m$ with respect to the base.

Exercise 7:

A motorbike rider had a speed of 35 m/s when he jumped off a racing track of 30°-slope (see figure).



a) Give vectors of velocity and acceleration of the motorbike at time t = 0.

b) Give vectors of velocity and acceleration of the motorbike at time t.

c) Deduce the height h of ramp B necessary for the motorcycle to land safely.

d) At what time is the speed minimum?

Solution

a)

The initial position vector is given from the initial position as (y is measured relative to the ground level, not as shown the figure): $\vec{r}_0 = \ x_0\vec{\imath} + y_0\vec{\jmath}$, with $x_0 = 0\ et\ y_0 = 10$ so $\vec{r}_0 = 10\vec{\jmath}$

Similarly for the velocity vector : $\vec{v}_0 = v_{0x}\vec{\imath} + v_{0y}\vec{\jmath} = v_0\, cos\, \theta_0\, \vec{\imath} + v_0\, sin\, \theta_0\, \vec{\jmath}$ with $v_0 =$ $35\ m/s$ and $\theta_0 = 30°$ so $\vec{v}_0 = 35\, cos\, 30\, \vec{\imath} + 35\, sin\, 30\, \vec{\jmath} = 30.3\vec{\imath} + 17.5\vec{\jmath}$

$$\boldsymbol{\vec{r}_0 = 10\vec{\jmath}\ ;\ \vec{v}_0 = 30.3\vec{\imath} + 17.5\vec{\jmath}} \qquad \vec{a} = -10\hat{\jmath}$$

b)

We have a projectile so the equations are simply given by :

$$\begin{cases} a_x = 0; v_x = v_{0x} = v_0 cos\theta_0;\ x = v_{0x}t = v_0 cos\theta_0 t \\ a_y = -g; v_y = -gt + v_{0y} = -gt + v_0 \sin\theta_0\ ; y = -\frac{1}{2}gt^2 + v_{0y} = -\frac{1}{2}gt^2 + v_0 \sin\theta_0\ \ t + y_0 \end{cases}$$

c) Deduce the height h of ramp B necessary for the motorcycle to land safely.

$$y_{motorcycle} = y_B$$

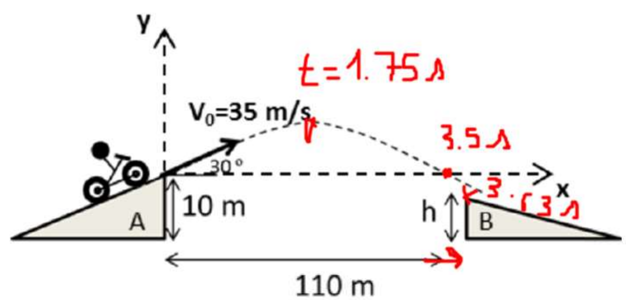
c)

To land safely, the value of h should be so that the point (x=110 m, h) belongs to the path (trajectory) of the motorbike.

So we calculate y that corresponds to x=110 m then take h=y:

$$\begin{cases} x = 30.3t \Rightarrow t = \frac{x}{30.3} = 3.63 \Rightarrow y = -5\left(\frac{x}{30.3}\right)^2 + 17.5\frac{x}{30.3} + 10 = 7.63m = h. \\ y = -5t^2 + 17.5t + 10 \end{cases}$$

Therefore, the height is $\boldsymbol{h = 7.63m}$



d) At what time is the speed minimum?

d)

As $v_x = cst$, and as v= $\sqrt{v_x^2 + v_y^2}$, the speed is minimum when $v_y = 0 \Rightarrow -10t + 17.5 = 0 \Rightarrow$

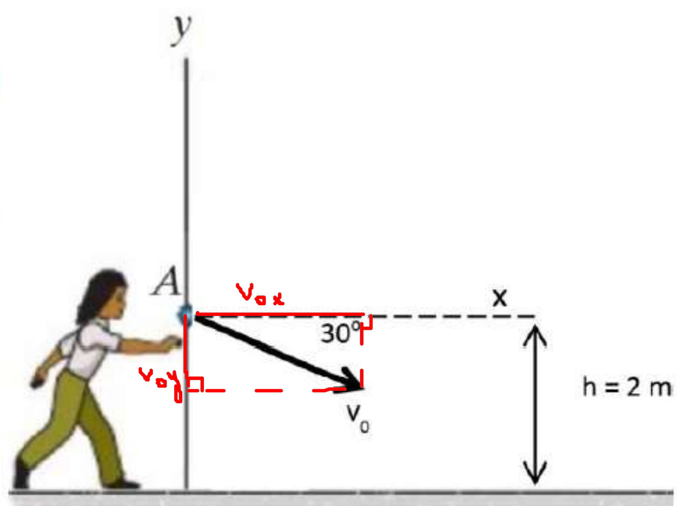
$$\boldsymbol{t = \frac{17.5}{10} = 1,75s.}$$

Exercise 8:

A ball is thrown dowward at an angle of 30° to the horizontal at speed $v_0 = 8m/s$.

a) Calculate its speed and tangential acceleration at $t = 0.25s$.

b) If the ball is thrown from point A of height h = 2 m. Calculate its normal acceleration just when the ball hits the ground. (g =10 m/s²)



Solution

a)

The ball is through from A (0,2m) with an angle of 30°.

Its initial velocity along x and y is: $\begin{cases} v_{0x} = v_0 \cos\theta = 8.\frac{\sqrt{3}}{2} = 4\sqrt{3}\ \mathrm{m/s} \\ v_{0y} = -v_0 \sin\theta = -8\frac{1}{2} = -4\ \mathrm{m/s} \end{cases}$

The tangential acceleration is the derivative with respect to time of the speed:

$$a_t = \frac{dv}{dt}.$$

And to find the speed we need the velocity. So let's find first the velocity.

As $a_x = 0$ and $a_y = -g$, the velocity is given by: $\begin{cases} v_x = v_{0x} = 4\sqrt{3}\text{ m/s} \\ v_y = -gt + v_{0y} = -(10t + 4) \end{cases}$ and

the speed by:

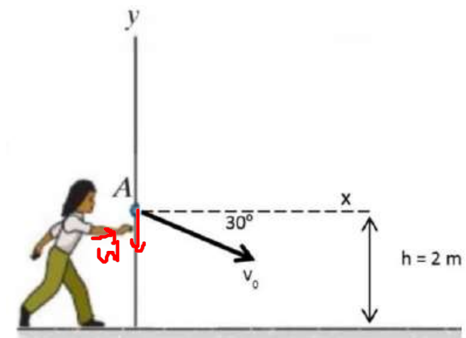
$$v = \sqrt{{v_x}^2 + {v_y}^2} = \sqrt{48 + (10t + 4)^2}$$

$$\boxed{\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}}$$

So the tangential acceleration becomes:

$$a_t = \frac{dv}{dt} = \frac{200t + 80}{2\sqrt{48 + (10t + 4)^2}}$$

At t=0.25 s, we obtain: $\begin{cases} v = \sqrt{48 + (2.5 + 4)^2} = 9.5\text{ m/s} \\ a_t = \frac{50+80}{2\sqrt{48+(2.5+4)^2}} = 6.84\text{ m/s}^2 \end{cases}$.



b) If the ball is thrown from point A of height h = 2 m. Calculate its normal acceleration just when the ball hits the ground. (g =10 m/s^2)

b)

The acceleration vector can always be written as the sum of two vectors perpendicular to each other, the tangential acceleration vector $a_t\mathbf{u_t}$ ($\mathbf{u_t}$: unit vector along the tangent to the path) and the normal acceleration vector $a_n \mathrm{u_n}$ ($\mathbf{u_n}$: unit vector along the normal to the path and towards the center of curvature):

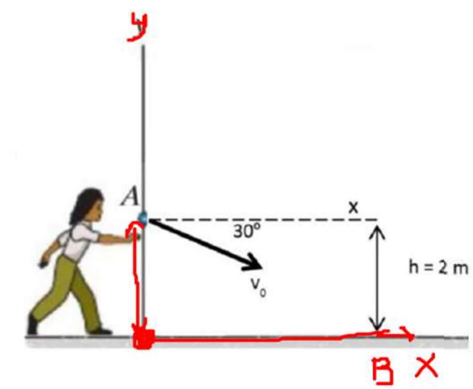
$$a_n \begin{cases} \dfrac{v^2}{\rho} \\ \sqrt{ - a_t^2} \end{cases}$$

$$\mathbf{a} = a_t\mathbf{u_t} + a_n\mathbf{u_n}$$

with $a_t = \frac{dv}{dt}$, $a_n = \frac{v^2}{\rho}$, $a = \sqrt{a_t^2 + a_n^2}$, ρ radius of curvature given by:

$$\rho = \left| \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{d^2y/dx^2} \right|$$

We need t & y=f(x)!



Method 1: we know a, so we find a_t as the speed derivative, and deduce a_n

The balls touches the ground when y=0.

$v_y = -10t - 4 \Rightarrow y = -5t^2 - 4t + y_0 = -5t^2 - 4t + 2$.

$y_B = -5t^2 - 4t + 2 = 0 \Rightarrow$ t=0.35 s.

$a_t = \frac{200t+80}{2\sqrt{48+(10t+4)^2}} = 7.35\ m/s^2$.

$a_n = \sqrt{a^2 - {a_t}^2} = \sqrt{g^2 - {a_t}^2} = 6.7 m/s^2$.

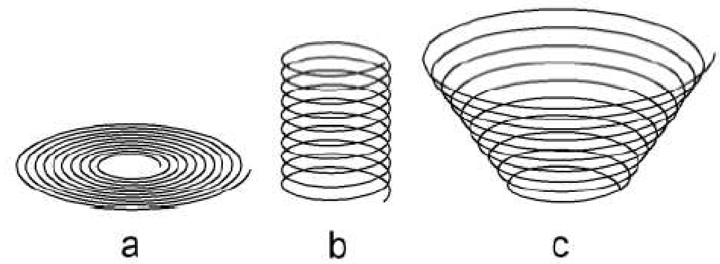
Method 2: we find the equation of the trajectory in terms of y and x, calculate ρ and the speed v and deduce a_n: do it by yourselves and verify!!!

Exercise 9:

A particle moves along a curve whose parametric equations are:

$$x = tsin\omega t, y = tcos\omega t, z = t.$$

Choose from the shown curves the trajectory of this particle.



Solution

$x^2 + y^2 = t^2(\sin^2 \omega t + \cos^2 \omega t) = t^2 \Rightarrow$ the projection of the path to the plane (x,y) is a cercle of variable radius $\Rightarrow$ Figure (b) is to exclude (because its radius is constant).

On the other side, $z = t$, so z is variable, and Figure (a) is to exclude (because its z is constant).

So finally, the path is that of Figure (c).

Exercise 10:

A moving object has a constant speed v_0 = 2 m/s at t = 0; it then undergoes an acceleration $a = 3v$ (rectilinear movement). We give: at t = 0, x_0 = 0.

- Calculate the speed after 1 second
- Determine the instantaneous position.

Solution

In this exercise, we are asked to calculate the velocity and the position in terms of the time while we are given the acceleration in terms of the velocity.

So we replace a in $a = 3v$ by $\frac{dv}{dt}$ and integrate in order to find v in terms of t.

Then we integrate v to find x.

$$a = \frac{dv}{dt} = 3v \Rightarrow \frac{dv}{v} = 3dt \Rightarrow \int_2^v \frac{dv}{v} = 3\int_0^t dt \Rightarrow ln(v/2) = 3t \Rightarrow v = 2e^{3t}$$

After 1 s: $v = 40.17\ m/s$.

$$v = \frac{dx}{dt} = 2e^{3t} \Rightarrow dx = 2e^{3t}dt \Rightarrow x = \frac{2}{3}e^{3t} + C.$$

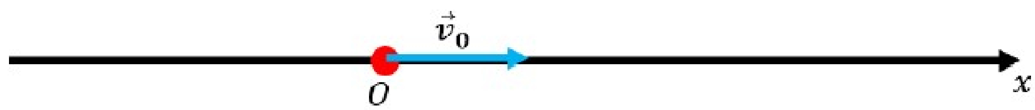
At $t = 0,\ x = 0 \Rightarrow C = -\frac{2}{3} \Rightarrow x = \frac{2}{3}(e^{3t} - 1)$.

Exercise 11:

The acceleration of a point in motion along the axis Ox is $a = 6x + 2$. For $x = 0$, we have $v_0 = 10\ \mathrm{cm/s}$.

Find v as a function of x.

Solution



We have the acceleration in terms of the position and we need the velocity in terms of the position. So we make use of the differential equation: $adx = vdv$.

$$adx = vdv \Rightarrow (6x + 2)dx = vdv$$

$$\Rightarrow \int_0^x (6x + 2)dx = \int_{v_0}^{v} vdv \Rightarrow \left(\frac{6x^2}{2} + 2x\right)\Big|_0^x = \frac{v^2}{2}\Big|_{v_0}^{v}$$

$$\Rightarrow 3x^2 + 2x = \frac{v^2}{2} - \frac{v_0^2}{2} \Rightarrow v^2 = 6x^2 + 4x + v_0^2$$

$\Rightarrow v = \pm\sqrt{6x^2 + 4x + 100}$. But for $x = 0$, we have $v = 10\mathrm{cm/s} > 0$ so the solution becomes:

$$v = \sqrt{6x^2 + 4x + 100}.$$

Exercise 12:

It is assumed that the speed of a particle along a straight and horizontal road is defined by $v = \frac{5}{4+x}$.

a) What is the nature of the motion?

b) Calculate the rate of change of its **velocity at time t = 4s.**

Solution

a) When an object's velocity and acceleration are in the same direction, the object is speeding up, otherwise, the object is slowing down.

$$\mathrm{v} = \frac{5}{4+x} \Rightarrow \textit{when } x \nearrow \Rightarrow v \searrow \Rightarrow \qquad \textit{decelerated}$$

v > 0
a < 0

b)

The rate of change of velocity is the acceleration.

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$$

$$\mathrm{v} = \frac{5}{4+x}$$

The acceleration can be obtained from the differential equation: $adx = vdv$ as:

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{f(x)^2}$$

$$\mathrm{a} = \mathrm{v}\frac{d\mathrm{v}}{dx} = \frac{5}{4+x}\cdot\frac{-5}{(4+x)^2} = \frac{-25}{(4+x)^3}.$$

At t=4s , x=???

Relation between x and t??

To find the acceleration at a given time, we need to know the position at this time. So we need to use an equation that involves the time, that is $v = \frac{dx}{dt}$.

$$v = \frac{dx}{dt} = \frac{5}{4+x}$$

$$\Rightarrow \int_{x_0}^{x}(4+x)dx = \int_0^t 5\mathrm{d}t \Rightarrow \left(\frac{x^2}{2} + 4x\right)\Big|_0^x = 5\mathrm{t}|_0^t$$ (we have assumed that at t=0, x=0)

$$\Rightarrow \frac{x^2}{2} + 4x = 5t.$$

$$t = 4\ \mathrm{s} \Rightarrow \frac{x^2}{2} + 4x = 20 \Rightarrow x^2 + 8x - 40 = 0$$

$\Rightarrow$ Two possible solutions: $x = -11.5$ m (unacceptable, because at t=0, x=0, then v=5/4 m/s>0, so x will increase and continue being positive) and x = 3.5 m (acceptable)

$$\Rightarrow a = \frac{-25}{(4+3.5)^2} = -0.059\ \mathrm{m/s^2}.$$

Exercise 13:

A ball is thrown from the ground with an initial speed $v_{01} = 25\ m/sec$; at the same instant, another ball is dropped from a building 15 m high. After how long will the balls be at the same height?

Solution

At the initial instant $t_0 = 0\ s$, the ball 1 is launched from ground vertically upwards with an initial speed $v_{01} = 25\ m/s$ while the ball 2 is released without initial speed from the top of a building of height h = 15m.

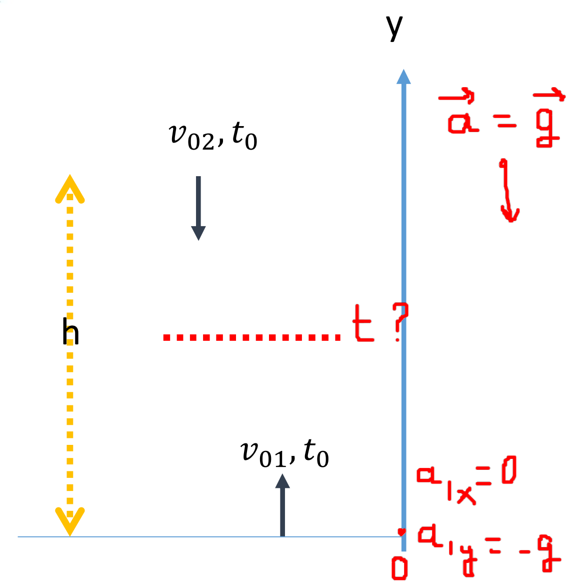
We must find t when the two balls are at the same height $(y_1 = y_2)$.

Ball 1: $a_1 = -g; v_1 = -gt + v_{01}; y_1 = -\frac{1}{2}gt^2 + v_{01}t + \cancel{y_{01}}^{0} = -\frac{1}{2}gt^2 + v_{01}t.$

Ball 2: $a_2 = -g; v_2 = -gt + \cancel{v_{02}}^{0} = -gt; y_2 = -\frac{1}{2}gt^2 + v_{02}t + y_{02} = -\frac{1}{2}gt^2 + h.$

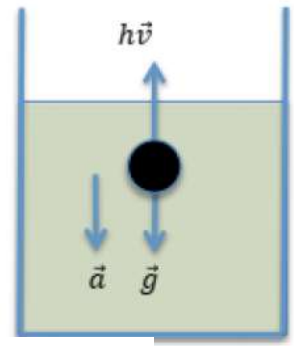
The balls will be at the same height when $y_1 = y_2$ So: $-\frac{1}{2}gt^2 + v_{01}t = -\frac{1}{2}gt^2 + \text{h}.$

$$\Rightarrow t = \frac{h}{v_{01}} = \frac{15}{25} = 0.6\ s.$$



Exercise 14:

A small object is released from rest in a tank of oil. The downward acceleration of the object is $g - hv$, where g is the constant acceleration due to gravity, h is a constant which depends on the viscosity of the oil and shape of the object. Derive expressions for the velocity v and vertical drop y as function of the time t after release.



Solution

First, let us find, $v(t)$.

$\longrightarrow a = g - hv$ (1) and $a = \frac{dv}{dt}$ (2).

$$\rightarrow (1) = (2) \Rightarrow g - hv = \frac{dv}{dt} \Rightarrow dt = \frac{dv}{g - hv} \Rightarrow \int_{t_i}^{t} dt = \int_{v_i}^{v} \frac{dv}{g - hv}$$

$\Rightarrow t - t_i = \int_{v_i}^{v} \frac{dv}{g-hv}$ (3)

$\rightarrow \int_{v_i}^{v} \frac{dv}{g-hv} = ?$

Let $u = g - hv \Rightarrow du = -hdv \Rightarrow dv = -\frac{du}{h}$.

So, $\int \frac{dv}{g-hv} = -\frac{1}{h}\int \frac{du}{u} = -\frac{1}{h}\ln|u|$

and $\int_{v_i}^{v} \frac{dv}{g-hv} = -\frac{1}{h}[\ln(g-hv)]_{v_i}^{v} = -\frac{1}{h}[\ln(g-hv) - \ln(g-hv_i)]$.

$\longrightarrow$ Initial conditions: $t_i = 0, v_i = 0$.

$\rightarrow (3) \Rightarrow t = -\frac{1}{h}[\ln(g-hv) - \ln(g)] = -\frac{1}{h}\ln\left(\frac{g-hv}{g}\right)$

$$\Rightarrow -ht = \ln\left(\frac{g-hv}{g}\right) \Rightarrow e^{-ht} = \frac{g-hv}{g} \Rightarrow ge^{-ht} - g = -hv$$

$\Rightarrow v = \frac{g}{h} - \frac{g}{h}e^{-ht} \Rightarrow v(t) = \frac{g}{h}(1 - e^{-ht})$ (4) .

Second, let us find $y(t)$.

$\longrightarrow v = \frac{dy}{dt}$ (5)

$\rightarrow (5) = (4) \Rightarrow \frac{dy}{dt} = \frac{g}{h}(1 - e^{-ht}) \Rightarrow \int_{y_i}^{y} dy = \frac{g}{h}\int_{t_i}^{t}(1 - e^{-ht})dt.$

$\longrightarrow$ Initial conditions: $t_i = 0, v_i = 0$.

$\therefore \boxed{\int e^{ax}dx = \frac{1}{a}e^{ax}}$

$\rightarrow \int_0^y dy = \frac{g}{h}\int_0^t (1 - e^{-ht})dt \Rightarrow y(t) = \frac{g}{h}\int_0^t dt - \frac{g}{h}\int_0^t e^{-ht}dt$

$\Rightarrow y(t) = \frac{g}{h}[t]_0^t - \frac{g}{h} \times \frac{1}{-h}[e^{-ht}]_0^t$

$\Rightarrow y(t) = \frac{g}{h}t + \frac{g}{h} \times \frac{1}{h}(e^{-ht} - 1)$

$\Rightarrow y(t) = \frac{g}{h}\left[t + \frac{1}{h}(e^{-ht} - 1)\right].$

Exercise 15:

The polar coordinates of a particle are given by: $r = 2e^{\omega t}$ et $\theta = \omega t$.

a) Find its speed.

b) Deduce the path length S after 2 s. We give $\omega = 2\ rd/s$.

Solution

a)

To find the speed we need the velocity vector.

The velocity vector in polar coordinates is given by: $\mathbf{v} = \dot{r} \cdot \mathbf{u}_r + r \cdot \dot{\theta} \cdot \mathbf{u}_\theta$ with:

$\dot{r} = 2\omega e^{\omega t}$ et $\dot{\theta} = \omega$.

The speed is the magnitude of $\mathbf{v}$:

$$v = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2} = \sqrt{4\omega^2 e^{2\omega t} + 4e^{2\omega t}\omega^2} = 2\omega e^{\omega t}\sqrt{2} = 2\sqrt{2}\omega e^{\omega t}.$$

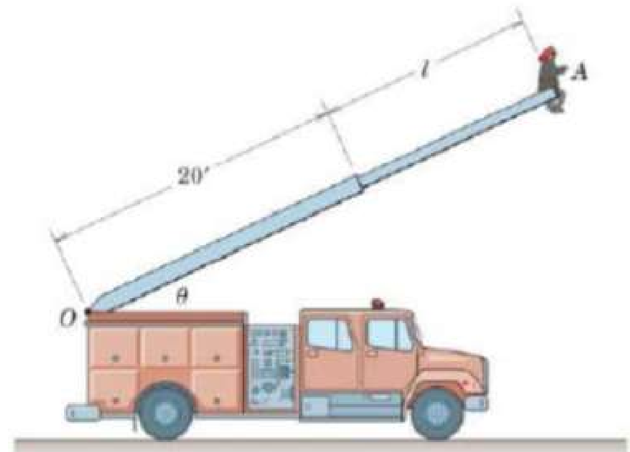
b)

The path length is the integral over time (here from time: 0 to time: 2 s) of the speed:

$$S = \int_0^2 v dt = \int_0^2 2\sqrt{2}\omega e^{\omega t} dt = 2\sqrt{2}e^{\omega t}\Big|_0^2 = 2\sqrt{2}(e^{2\omega} - 1) = 2\sqrt{2}(e^4 - 1) = 151.5\ m.$$

Exercise 16:

The ladder of a fire truck is designed to be extended at the constant rate $\dot{L} = 0.5\ m/s$ and to be elevated at the constant rate $\dot{\theta} = 2\ deg/s$. As the position $\theta = 50\ ^\circ$ and $L = 15\ m$ is reached, determine the magnitudes of the velocity v and the acceleration a of the fireman at A.



Solution

We need the magnitudes of the velocity and acceleration vectors at $\theta = 50°$ and L=15 m.

So let's first find these vectors.

In polar coordinates we can write the velocity and acceleration vectors as:

$$\vec{v} = v_r\,\overrightarrow{e_r} + v_\theta\,\overrightarrow{e_\theta} = \dot{r}\,\overrightarrow{e_r} + r\dot{\theta}\,\overrightarrow{e_\theta}$$

$$\vec{a} = a_r\,\overrightarrow{e_r} + a_\theta\,\overrightarrow{e_\theta} = \left(\cancel{\ddot{r}} - r\dot{\theta}^2\right)\overrightarrow{e_r} + \left(r\cancel{\ddot{\theta}} + 2\dot{r}\dot{\theta}\right)\overrightarrow{e_\theta}$$

We can see that $r = 20 + L \Rightarrow \dot{r} = \dot{L} = 0.5\ m/s$.

Velocity

$v_r = \dot{r} = \dot{L} = 0{,}5 m/s$

$v_\theta = r\dot{\theta} = (20 + 15)\left(2° \times \frac{\pi}{180}\right) = 1{,}22 m/s$ $\qquad 1°/s \rightarrow \frac{\pi}{180)} rad/s$

$$\boxed{v = \sqrt{v_r^2 + v_\theta^2} = 1{,}32 m/s.}$$

Acceleration

Note that $\ddot{r} = 0$ because $\dot{r}$ is constant and $\ddot{\theta} = 0$ because $\dot{\theta}$ is constant.

$a_r = \cancel{\ddot{r}} - r\dot{\theta}^2 = -35\left(2° \times \frac{\pi}{180}\right)^2 = -0{,}0426\ m/s^2$

$a_\theta = r\cancel{\ddot{\theta}} + 2\dot{r}\dot{\theta} = 2\dot{r}\dot{\theta} = 0{,}0349 m/s^2$

$$\boxed{a = \sqrt{a_r^2 + a_\theta^2} = 0{,}0551 m/s^2.}$$

Exercise 17:

A point M moves on a spiral polar equation:

$$r = r_0 e^{\theta}$$

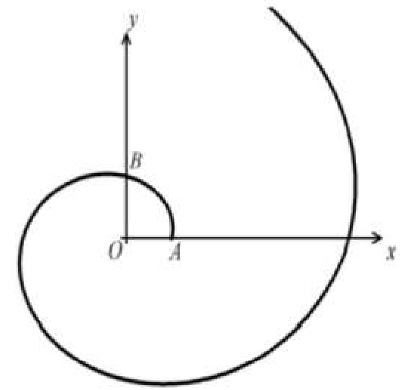
with $\theta = \omega t$ is the direction with the x-axis; r_0 and ω are constants.

a) What are the cartesian coordinates of A and B?

b) Knowing that the polar unit vectors are $\overrightarrow{e_r}$ and $\overrightarrow{e_\theta}$;
$(\overrightarrow{e_r}, \overrightarrow{e_\theta}) = \frac{\pi}{2}$ and $\overrightarrow{OM} = r\overrightarrow{e_r}$.
Calculate, in terms of ω and r, the radial component V_r and transverse component V_θ , and the magnitude of the velocity vector $\vec{V}$.

c) Determine the angle between $\vec{V}$ and $\overrightarrow{e_r}$.

d) Find the radial component a_r and transverse component a_θ, and the magnitude of the acceleration vector $\vec{a}$.



Solution

We can see that

At $t = 0, \theta = 0, r = r_0 e^0 = r_0$; when t increases, θ increases and r increases; so at t=0, we was at point A.

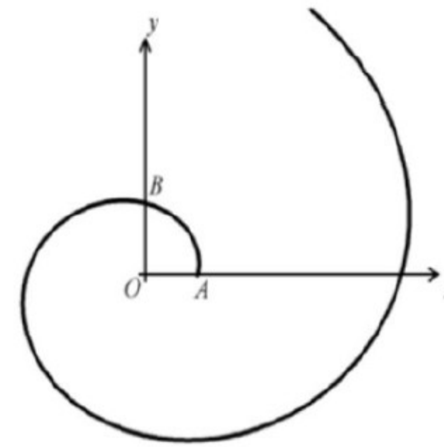
A: $\theta = 0$; $r = r_0 e^0 = r_0 \Rightarrow$ x= $r cos\theta = r_0$

B: $\theta = \frac{\pi}{2}$; $r = r_0 e^{\frac{\pi}{2}} \Rightarrow$ y= $r sin\theta = r_0 e^{\frac{\pi}{2}}$

b) Knowing that the polar unit vectors are $\overrightarrow{e_r}$ and $\overrightarrow{e_\theta}$;
$(\overrightarrow{e_r}, \overrightarrow{e_\theta}) = \frac{\pi}{2}$ and $\overrightarrow{OM} = r\overrightarrow{e_r}$.
Calculate, in terms of ω and r, the radial component V_r and transverse component V_θ , and the magnitude of the velocity vector $\vec{V}$.

c) Determine the angle between $\vec{V}$ and $\overrightarrow{e_r}$.

d) Find the radial component a_r and transverse component a_θ, and the magnitude of the acceleration vector $\vec{a}$.



a) $r = r_0 e^{\theta} = r_0 e^{\omega t}$; r_0 and ω are constants

$x = r\cos\theta = r_0 e^{\theta}\cos\theta$; $A(\theta = 0)$

$y = r\sin\theta = r_0 e^{\theta}\sin\theta$; B $(\theta = \frac{\pi}{2})$

$$\begin{cases} x_A = r_0 \\ y_A = 0 \end{cases} \text{ and } \begin{cases} x_B = r_0 e^{\frac{\pi}{2}}.\cos\frac{\pi}{2} = 0 \\ y_B = r_0 e^{\frac{\pi}{2}}.\sin\frac{\pi}{2} = r_0 e^{\frac{\pi}{2}} \end{cases} \Rightarrow A(r_0, 0) \text{ and B } (0; r_0 e^{\frac{\pi}{2}})$$

b) $\overrightarrow{OM} = r\vec{e}_r \Rightarrow \vec{V} = \frac{d\overrightarrow{OM}}{dt} = \dot{r}\,\vec{e}_r + r\dot{\theta}\vec{e}_\theta$

$$\begin{cases} v_r = \dot{r} = r_0\omega e^{\omega t} \\ v_\theta = r\dot{\theta} = r_0\omega e^{\omega t} \end{cases} \Rightarrow V = \sqrt{v_r^2 + v_\theta^2} = \sqrt{2}r_0\omega e^{\omega t}$$

c) $\vec{\boldsymbol{V}}.\vec{e}_r = (\dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta).\vec{e}_r = \dot{r} = \|V\|\cos\alpha \Rightarrow \cos\alpha = \frac{\dot{r}}{\|V\|} = \frac{r_0\omega e^{\omega t}}{\sqrt{2}r_0\omega e^{\omega t}} = \frac{\sqrt{2}}{2} \Rightarrow \alpha = \frac{\pi}{4}$

$$\boxed{\alpha = \frac{\pi}{4}}$$

d) $$\begin{cases} a_r = \ddot{r} - r\dot{\theta}^2 = r_0\omega^2 e^{\omega t} - r_0\omega^2 e^{\omega t} = 0 \\ a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 + r_0\omega^2 e^{\omega t} = 2r_0\omega^2 e^{\omega t} \end{cases}$$

$$\boxed{a = \sqrt{a_r^2 + a_\theta^2} = 2r_0\omega^2 e^{\omega t}}$$

Exercise 18:

A car travels along the level curved road with a speed which is decreasing at the constant rate of 0.6 m/s each second. The speed of the car as it passes point A is 16 m/s. $\frac{dv}{dt} = -0.6\ m/s^2$

Calculate the magnitude of the total acceleration of the car as it passes point B which is 120 m along the road from A. The radius of curvature of the road at B is 60 m.

Solution

A decreasing speed at a constant rate of 0.6 m/s every second means that the tangential acceleration is equal to -0.6 m/s^2.

$$v_B^2 - v_A^2 = +2.a.S \Rightarrow v_B = \sqrt{v_A^2 + 2.a.S} = \sqrt{16^2 - 2 \times 0,6 \times 120} = 10,85 m/s.$$

At point B:

$$a_n = \frac{v_B^2}{R} = \frac{(10,58)^2}{60} = 1,867\ m/s^2$$

$a^2 = a_t^2 + a_n^2$

Hence the total acceleration at point B will be:

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{0,6^2 + 1,867^2} = 1,961\ m/s^2.$$

$$\boxed{a = 1,961\ m/s^2}$$

$$\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}.$$

Exercise 19:

The motion of a particle M is described by its position vector: $\overrightarrow{OM} = t\overrightarrow{e_x} + \frac{t^2}{2}\overrightarrow{e_y} + t\overrightarrow{e_z}$.

a) Determine the speed and acceleration at point M.

b) Give the expressions of the tangential and normal components of the acceleration.

c) Deduce the expression of the radius of curvature of the trajectory.

Solution

a) $\vec{v} = v_x\vec{e}_x + v_y\vec{e}_y + v_z\vec{e}_z = \vec{e}_x + t\vec{e}_y + \vec{e}_z$

$\Rightarrow v = \sqrt{1 + t^2 + 1} = \sqrt{2 + t^2} \quad m/s$

$\vec{a} = a_x\vec{e}_x + a_y\vec{e}_y + a_z\vec{e}_z = \vec{e}_y \Rightarrow a = 1 \quad m/s^2$

b) $\vec{a} = \vec{a}_n + \vec{a}_t \Rightarrow \vec{a}_t = \dot{v}\vec{e}_t = \frac{dv}{dt}\vec{e}_t$

With $v = \sqrt{2 + t^2} \Rightarrow \vec{a}_t = \frac{t}{\sqrt{2+t^2}}\vec{e}_t$

$$a_n = \sqrt{a^2 - a_t^2} = \sqrt{1 - \frac{t^2}{2 + t^2}} = \sqrt{\frac{2}{2 + t^2}}$$

c) $a_n = \frac{v^2}{\rho} \Rightarrow \rho = \frac{v^2}{a_n} = \frac{2+t^2}{\sqrt{\frac{2}{2+t^2}}}$

$$\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}.$$

Exercise 20:

A particle moving on a parabolic curve of equation $y = 4t^2$ where $x = 2t$.

a) Determine the velocity and the acceleration of the particle at an instant t in the (Ox, Oy) system.

b) Determine the normal $\vec{a}_n$ and tangential $\vec{a}_t$ component of the acceleration

c) Deduce the radius of curvature of the trajectory at any instant t.

Solution

a) The parametric equations of the motion are:

$$\vec{r}\begin{cases} x = 2t \\ y = 4t^2 \end{cases} \Rightarrow \vec{v}\begin{cases} v_x = 2 \\ v_y = 8t \end{cases} \Rightarrow \vec{a}\begin{cases} a_x = 0 \\ a_y = 8 \end{cases}$$

b) $\vec{a}_t = a_t\vec{t}$

$$a_t = \frac{d||\vec{v}||}{dt} = \frac{d}{dt}\sqrt{2^2 + 64t^2} = 2\frac{d}{dt}\sqrt{1 + 16t^2}$$

$$a_t = 2\frac{32t}{2\sqrt{1 + 16t^2}} = \frac{32t}{\sqrt{1 + 16t^2}}$$

$$\vec{t} = \frac{\vec{v}}{\|v\|};$$

$$\vec{t} = \begin{cases} \frac{1}{\sqrt{1 + 16t^2}} \\ \frac{4t}{\sqrt{1 + 16t^2}} \end{cases}$$

$$\vec{a}_t = (\vec{a}.\vec{t})\vec{t} = \frac{32t}{\sqrt{1 + 16t^2}}\left(\frac{1}{\sqrt{1 + 16t^2}}\vec{\imath} + \frac{4t}{\sqrt{1 + 16t^2}}\vec{\jmath}\right) = \frac{32t}{1 + 16t^2}\vec{\imath} + \frac{128t^2}{1 + 16t^2}\vec{\jmath}$$

$$\vec{a}_n = \vec{a} - \vec{a}_t = -\frac{32t}{1+16t^2}\vec{\imath} + \frac{8}{1+16t^2}\vec{\jmath};\ a_n = \sqrt{\left(-\frac{32t}{1+16t^2}\right)^2 + \left(\frac{8}{1+16t^2}\right)^2} = \frac{8}{\sqrt{1+16t^2}}$$

c) $\rho(t) = \frac{v^2}{a_n} = \frac{4(1+16t^2)}{\frac{8}{\sqrt{1+16t^2}}}$

Exercise 21:

The position of a particle M is specified in polar coordinate by:

$$\begin{cases} r(t) = 2\sin t \\ \theta(t) = t \end{cases}$$

a) Find the components radial (v_r) and orthoradial (v_θ) of the speed of M.

b) Deduce the magnitude of the instantaneous speed.

c) Determine the components radial (a_r) and the orthoradial (a_θ) of the acceleration of M.

d) Deduce the magnitude of the vector acceleration $\vec{a}$.

e) Calculate the tangential acceleration a_t.

f) Deduce the normal acceleration a_n.

g) Calculate the radius of curvature ρ.

Solution

a) $\overrightarrow{OM} = r\vec{e}_r$, the instantaneous velocity: $\vec{V} = \dot{r}\,\overrightarrow{e_r} + r\dot{\theta}\,\overrightarrow{e_\theta}$

$$\vec{V} = (2cost)\vec{e}_r + (2sint)\frac{d\theta}{dt}\vec{e}_\theta = (2cost)\vec{e}_r + (2sint)\vec{e}_\theta$$

$v_r = 2cost$ radial	$v_\theta = 2sint$ orthoradial

b) $\|V\| = \sqrt{v_r^2 + v_\theta^2} = 2 \quad m/s$

c) $\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}[(2cost)\vec{e}_r + (2sint)\vec{e}_\theta] = -(2sint)\vec{e}_r + (2cost)\frac{d\vec{e}_r}{dt} + 2cost\vec{e}_\theta + 2sint\frac{d\vec{e}_\theta}{dt}$

OR

$\vec{a} = (\ddot{\rho} - \rho\dot{\theta}^2)\hat{e}_\rho + (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta})\hat{e}_\theta$

$$\vec{a} = -2sint\vec{e}_r + 2cost\frac{d\theta}{dt}\vec{e}_\theta + 2cost\vec{e}_\theta - 2sint\vec{e}_r = (-4sint)\vec{e}_r + (4cost)\vec{e}_\theta$$

$a_r = -4sint$ radial	$a_\theta = 4cost$ orthoradial

d) Deduce the magnitude of the vector acceleration $\vec{a}$.

e) Calculate the tangential acceleration a_t.

f) Deduce the normal acceleration a_n.

g) Calculate the radius of curvature ρ.

Solution

d) $a = \sqrt{a_r^2 + a_\theta^2} = 4 \quad m/s^2$

e) $a_t = \frac{d\|V\|}{dt} = 0$ because $\|V\| = 2 \quad m/s$

f) $\vec{a} = \vec{a}_n + \vec{a}_t = \vec{a}_n \ (a_t = 0)$ and $a_n = 4 \quad m/s^2$

g) The radius of curvature:

$$\rho = \frac{v^2}{a_n} = \frac{4}{4} = 1 \quad m$$

Exercise 22:

The position of a particle is given by the vector

$$\vec{r} = R(\cos\omega t)\hat{\imath} + R(\sin\omega t)\hat{\jmath} + (ct)\hat{k}$$

where R, ω *and* c are constant.

a) Find a_n *and* a_t.

b) Find the radius of curvature of its trajectory

Solution

a) In general, the acceleration of a particle is given by:

$\vec{a} = \vec{a}_n + \vec{a}_t = a_n\vec{n} + a_t\,\vec{t}$

where $\vec{n}$ and $\vec{t}$ are the unit vectors of the normal and tangential directions.

The tangential acceleration is nothing other than the derivative of the speed (magnitude of velocity) with respect to time:

$$\vec{v} = \frac{d\vec{r}}{dt} = \begin{cases} -R\omega(\sin\omega t) \\ R\omega(\cos\omega t) \\ c \end{cases}$$

$$a_t = \frac{dv}{dt} = \frac{d}{dt}\left(\sqrt{(-R\omega(\sin\omega t))^2 + (R\omega(\cos\omega t))^2 + (c)^2}\right) = \frac{d}{dt}\sqrt{(R\omega)^2 + (c)^2} = 0$$

The normal acceleration can be obtained as:

$\vec{a} = a_n\vec{n} + a_t\,\vec{t} \Rightarrow a^2 = {a_n}^2 + {a_t}^2 \Rightarrow {a_n}^2 = a^2 - {a_t}^2 = a^2$ (because $a_t = 0$).

So let's first find the acceleration vector: $\vec{a} = \frac{d\vec{v}}{dt} = \begin{cases} -R\omega^2(\cos\omega t) \\ -R\omega^2(\sin\omega t) \\ 0 \end{cases}$

$\Rightarrow a = \sqrt{(-R\omega^2(\cos\omega t))^2 + (-R\omega^2(\sin\omega t))^2} = \sqrt{R^2\omega^4} = R\omega^2 \Rightarrow a = a_n = R\omega^2$

b) We know that $a_n = \frac{v^2}{\rho}$ so $\rho = \frac{v^2}{a_n} = \frac{(R\omega)^2+(c)^2}{R\omega^2}$

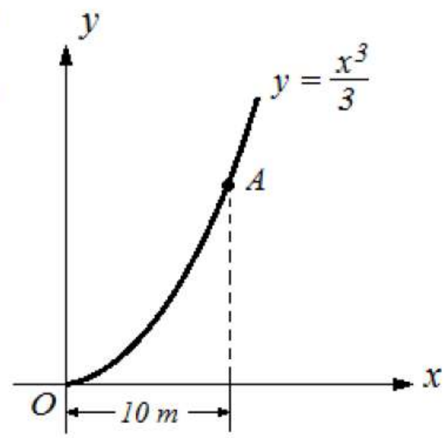
Exercise 23:

A car moves in a vertical plane on a path of equation $y = \frac{x^3}{60}$. It passes passes by a point A of abscissa $x_A = 10\,m$ at a constant speed of *10 m/s*.

a- Find the radius of curvature of its path at point A.

b- Determine the acceleration vector (modulus and direction).

(We give $\rho = \left| \frac{\left[1+(dy/dx)^2\right]^{3/2}}{d^2y/dx^2} \right|$)



Solution

a-

The radius of curvature is given by: $\rho = \left| \frac{\left[1+(dy/dx)^2\right]^{3/2}}{d^2y/dx^2} \right|$

with $y = \frac{x^3}{60}$ so $\frac{dy}{dx} = \frac{x^2}{20}$ and $\frac{d^2y}{dx^2} = \frac{x}{10}$.

At $= 10m$: $\frac{dy}{dx} = \frac{x^2}{20} = \frac{10^2}{20} = 5$ and $\frac{d^2y}{dx^2} = \frac{x}{10} = 1$ so

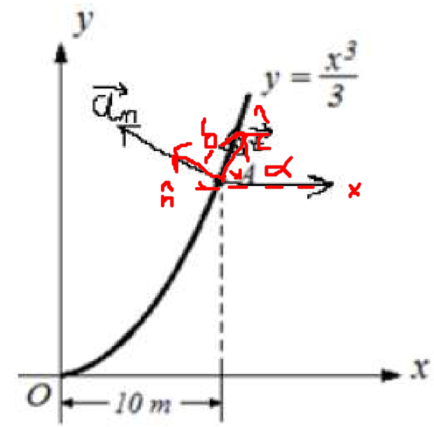
$$\boxed{\rho = \left| \frac{\left[1+(dy/dx)^2\right]^{3/2}}{d^2y/dx^2} \right| = \frac{(1+5^2)^{3/2}}{10/10} = 132.6 \text{ m.}}$$

b-

Let's first find the acceleration vector.

Tangential acceleration: $a_t = \frac{dv}{dt} = 0$ because $v = 10 \text{ m/s} = \text{cst}$.

Normal acceleration: $a_n = \frac{v^2}{\rho} = \frac{100}{132.6} = 0.75 m/s^2$.



So $\vec{a} = \vec{a}_n + \vec{a}_t = a_n \vec{n} + a_t \vec{t}$ ($\vec{n}$ and $\vec{t}$ are the normal and tangential unit vectors).

Magnitude of acceleration: $a^2 = {a_n}^2 + {a_t}^2 \Rightarrow a = a_n = 0.75 \; m/s^2$ (because $a_t = 0$).

To find the direction of $\vec{a}$ with respect to the x axis, we find it with respect to $\vec{t}$, then find the direction of $\vec{t}$ with respect to the x axis.

Angle between $\vec{a}$ and $\vec{t}$: $\cos\delta = \frac{\vec{a}.\vec{t}}{a} = \frac{(a_n \vec{n} + a_t \vec{t}).\vec{t}}{a} = \frac{a_t}{a} = 0 \Rightarrow \delta = 90°$.

Angle between $\vec{t}$ and the x axis: slope: $\tan\alpha = \frac{dy}{dx} = \frac{x^2}{20} = 5 \Rightarrow \alpha = \text{atan}\, 5 = 78.7°$;

Angle between $\vec{a}$ and the x axis: $\beta = \alpha + \delta = 168.7°$.

Exercise 24:

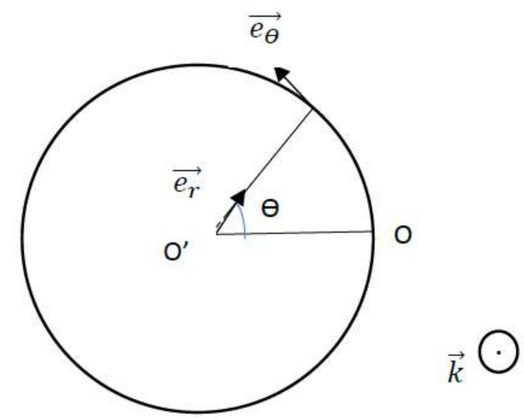
A point M moves along a circle of radius R = 2m whose angular motion equation with respect to a given origin, can be written as: $\theta = 2t^2 + t$.

a) Determine the angular velocity of M as a function of time. Deduce its velocity $\vec{v}(t)$.

b) Determine the angular acceleration of M. What is then the nature of the motion of M?

c) Find the value of the tangential acceleration a_t of M.

d) Determine the acceleration vector $\vec{a}$ of M at time t = 1 s.



Solution

a- $\theta = 2t^2 + t$

Angular velocity: $\dot{\theta} = \frac{d\theta}{dt} = 4t + 1 = \omega$

From the course: $\vec{v} = \vec{\omega} \wedge \vec{r} = \omega\vec{k} \wedge R\overrightarrow{e_r} = \omega R\overrightarrow{e_\theta}$

b- $\ddot{\theta} = 4\ rd/s^2 = \text{cst} > 0 \Rightarrow$ uniformly accelerated circular motion

c- $\overrightarrow{a_t} = \vec{\alpha} \wedge \vec{r} = \ddot{\theta}\vec{k} \wedge R\overrightarrow{e_r} = R\ddot{\theta}\overrightarrow{e_\theta} = 4R\overrightarrow{e_\theta} = 8\overrightarrow{e_\theta}$

d- $\vec{a} = \overrightarrow{a_t} + \overrightarrow{a_n}$

We have: $\overrightarrow{a_n} = \vec{\omega} \wedge \vec{\omega} \wedge \vec{r} = -\omega^2 R\overrightarrow{e_r} = \omega^2 R\vec{n} = (4t+1)^2 \times 2\vec{n}$ $\quad (\vec{n} = -\overrightarrow{e_r})$

Therefore: $\vec{a} = 8\overrightarrow{e_\theta} + (4t+1)^2 \times 2\vec{n}$

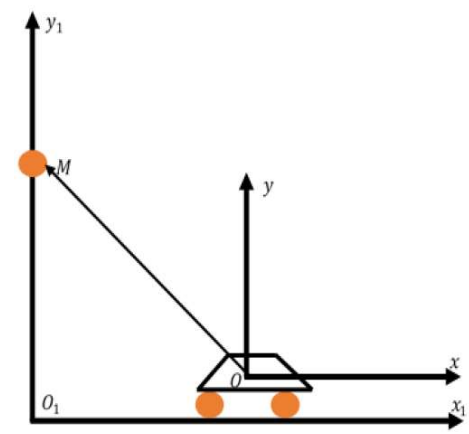
At t = 1 s : $\boldsymbol{\vec{a} = 8\overrightarrow{e_\theta} + 50\vec{n}}$

Exercise 25:

A particle M released without initial speed from a height h falls by performing a uniformly accelerated rectilinear motion of acceleration g along the vertical axis o_1y_1.

a) Determine the position vector of this particle with respect to a car that is moving at constant speed $\vec{v} = V\widehat{\iota_1}$ along the axis o_1x_1. We recall: $\vec{r}_a = \vec{r}_e + \vec{r}_r$.

b) Deduce its relative trajectory.



Solution

a)

We want to determine the position vector of M with respect to the car.

So there are two reference systems, the absolute reference system: $O_1x_1y_1$ and the relative system linked to the car: Oxy.

As $\vec{r}_a = \vec{r}_e + \vec{r}_r$, we can write : $\vec{r}_{aM} = \vec{r}_C + \vec{r}_{M/C}$ with $\vec{r}_{aM} = \overrightarrow{O_1M}$ the absolute position vector of M, $\vec{r}_C = \overrightarrow{O_1O}$ the absolute position vector of the car, and $\vec{r}_{M/C} = \overrightarrow{OM}$ the relative position vector M with respect to the car.

Moreover, $\overrightarrow{O_1M} = y_{1M}\vec{J_1}$

$= \left(-\frac{1}{2}gt^2 + y_{10M}\right)\vec{J_1} = \left(-\frac{1}{2}\boldsymbol{g}\boldsymbol{t}^2 + \boldsymbol{h}\right)\vec{\boldsymbol{j}}_1$ because the motion of M is rectilinear uniformly accelerated with acceleration –g.

$\overrightarrow{O_1O} = x_C\vec{\imath}_1 = \boldsymbol{Vt\vec{\imath}_1}$ because the car is moving at a constant speed.

Finally, the relative position vector is:

$\vec{r}_{M/C} = \overrightarrow{OM} = \overrightarrow{O_1M} - \overrightarrow{O_1O} = \left(-\frac{1}{2}\boldsymbol{gt^2} + \boldsymbol{h}\right)\boldsymbol{\vec{j}_1} - \boldsymbol{Vt\vec{\imath}_1} = \left(-\frac{1}{2}\boldsymbol{gt^2} + \boldsymbol{h}\right)\boldsymbol{\vec{j}} - \boldsymbol{Vt\vec{\imath}}$ (because $\vec{\imath}_1 = \vec{\imath}$ and $\vec{j}_1 = \vec{j}$).

b)

To determine the equation of the trajectory we must look for $y = f(x)$ with: $\begin{cases} x = -Vt \\ y = -\frac{1}{2}\boldsymbol{gt^2} + \boldsymbol{h}. \end{cases}$

$x = -Vt \Rightarrow t = -\frac{x}{V}$;

$y = -\frac{1}{2}\boldsymbol{gt^2} + \boldsymbol{h} = -\frac{1}{2}\boldsymbol{g}\left(\frac{-x}{V}\right)^{2} + \boldsymbol{h} = -\frac{1}{2}\boldsymbol{g}\frac{x^2}{V^2} + \boldsymbol{h}.$

So the trajectory of M in the system linked to the car Oxy is a parabola.

Exercise 26:

Two cars A and B have a two-dimensional uniform motion, such as $v_A = v_B = 10\ m/s$.

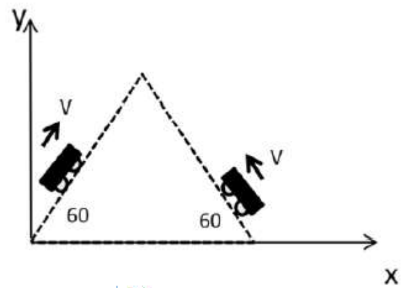
We give at t = 0: $x_A = y_A = 0$; $x_B = 25m$, $y_B = 0$.

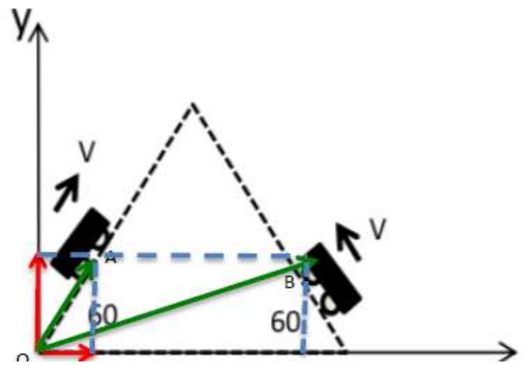
Determine at t = 2s:

a) The position vectors $\vec{r}_A$ and $\vec{r}_B$.

b) The position vector $(\vec{r}_{A/B})$ of car A with respect to car B.

c) The velocity $(\vec{v}_{A/B})$ of car A with respect to car B.





Solution

a) Notice that both cars A and B experience rectilinear uniform motion.

We have the speeds of A and B. To find the position vectors of A and B, we have to find the velocity vectors then integrate with respect to time.

Velocity vectors

$$\begin{cases} \overrightarrow{v_A} = v_A\,\overrightarrow{u_A} = v_{Ax}\,\vec{\imath} + v_{Ay}\,\vec{\jmath} = v_A\ \cos 60\,\vec{\imath} + v_A\ \sin 60\,\vec{\jmath} = 5\vec{\imath} + 8.66\vec{\jmath} \\ \overrightarrow{v_B} = v_B\,\overrightarrow{u_B} = v_{Bx}\,\vec{\imath} + v_{By}\,\vec{\jmath} = v_B \cos 120\,\vec{\imath} + v_B\ \sin 120\,\vec{\jmath} = -5\vec{\imath} + 8.66\vec{\jmath} \end{cases}$$

Position vectors

At t = 0; car A at (0,0) and car B at (25,0).

$$\begin{cases} \overrightarrow{r_A} = \int \overrightarrow{v_A}\,dt = \int(5\vec{\imath} + 8.66\vec{\jmath})\,dt = (5t + x_{A0})\vec{\imath} + (8.66t + y_{A0})\vec{\jmath} = 5t\vec{\imath} + 8.66t\vec{\jmath} \\ \overrightarrow{r_B} = \int \overrightarrow{v_B}\,dt = \int(-5\vec{\imath} + 8.66\vec{\jmath})\,dt = (-5t + x_{B0})\vec{\imath} + (8.66t + y_{B0})\vec{\jmath} = (-5t + 25)\vec{\imath} + 8.66t\vec{\jmath} \end{cases}$$

At t=2 s: $\begin{cases} \overrightarrow{r_A} = 5t\vec{\imath} + 8.66t\vec{\jmath} = 10\vec{\imath} + 17.32\vec{\jmath} \\ \overrightarrow{r_B} = (-5t + 25)\vec{\imath} + 8.66t\vec{\jmath} = 15\vec{\imath} + 17.32\vec{\jmath}. \end{cases}$

b)

Position vector of A with respect to B:

$$\overrightarrow{r_{A/B}} = \overrightarrow{r_A} - \overrightarrow{r_B} = (5t\vec{\imath} + 8.66t\vec{\jmath}) - \left((-5t + 25)\vec{\imath} + 8.66t\vec{\jmath}\right) = (10t - 25)\vec{\imath}.$$

At t=2 s: $\overrightarrow{r_{A/B}} = -5\vec{\imath}.$

c)

Velocity vector of A with respect to B:

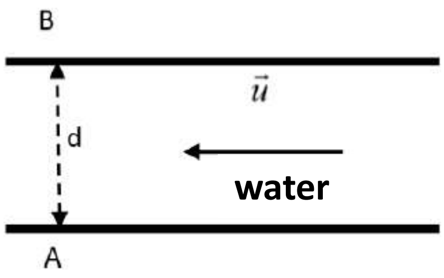
$\overrightarrow{v_{A/B}} = \frac{d\overrightarrow{r_{A/B}}}{dt} = 10\vec{\imath}$: constant all the time.

Or also (the reference frames are in translation with respect to each other there is no rotation):

$$\overrightarrow{v_{A/B}} = \overrightarrow{v_A} - \overrightarrow{v_B} = 10\vec{\imath}.$$

Exercise 27:

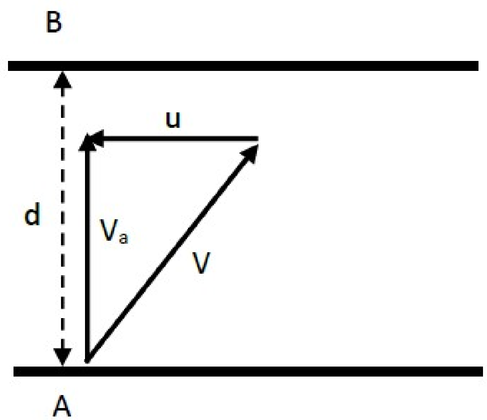
A swimmer departed from point A and moves at a constant speed V = 4m/s with respect to the water of a river of width d = 50m and whose waters are animated by a current of constant speed u = 2m/s . Find the time needed to get to point B directly opposite to point A.



Solution

Speed relative to river's water: V = 4 m / s.

Let $\vec{V}$ be the relative velocity with respect to water, $\vec{u}$ the velocity of water, and $\vec{V_a}$ the absolute velocity. We have: $\vec{V} = \vec{V_a} - \vec{u} \Rightarrow \vec{V_a} = \vec{V} + \vec{u}$.



Let us suppose a direct path (thus linear) between A and B. The absolute speed then has this direction.

The vectors $\vec{V}$, $\vec{u}$ and $\vec{V_a}$ then form a right triangle as illustrated below.

We deduce: $V^2 = u^2 + V_a^2 \Rightarrow V_a = \sqrt{V^2 - u^2} = \sqrt{16 - 4} = \sqrt{12}\, m/s = cst \Rightarrow$ uniform rectilinear motion.

Time required: $t = \frac{d}{V_a} = \frac{50\, m}{\sqrt{12}\, m/s} = 14.4\, s$.

Note that other trips at uniform relative speed but variable direction are possible (too complicated).