Chapter 3: Conservation of energy:

1) Introduction

2) Power and efficiency:

$$P_{(W)} = \vec{F}_{(N)} \cdot \vec{V}_{(m/s)}$$

P: power developped by $\vec{F}$; F: force; V: velocity

$$\text{Efficiency} = \varepsilon = \frac{\text{output power}}{\text{input power}}$$

$$P = \frac{\Delta E}{\Delta t}$$

3) work done by a force:

$$W_{\vec{F}} = \int_{t_1}^{t_2} P\,dt = \int \vec{F} \cdot \underbrace{\vec{V}}_{\frac{d\vec{r}}{dt}} dt = \int \vec{F} \cdot d\vec{r}$$

- In cartesian coordinate:

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\longrightarrow W_{\vec{F}} = \int \vec{F} \cdot d\vec{r} = \int (F_x\hat{i} + F_y\hat{j} + F_z\hat{k}) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$\Rightarrow W_{\vec{F}} = \int (F_x\,dx + F_y\,dy + F_z\,dz)$$

- In cylindrical coordinates:

$$\vec{V} = \frac{d\vec{r}}{dt} \Rightarrow \vec{V} = \frac{dr}{dt}\hat{e}_r + r\frac{d\theta}{dt}\hat{e}_\theta + \frac{dz}{dt}\hat{k}$$

$$\Rightarrow d\vec{r} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + dz\,\hat{k}$$

$$W_{\vec{F}} = \int (F_r\hat{e}_r + F_\theta\hat{e}_\theta + F_z\hat{k}) \cdot (dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + dz\,\hat{k})$$

$$\Rightarrow W_{\vec{F}} = \int (F_r\,dr + F_\theta\,r\,d\theta + F_z\,dz)$$

Application:

a) Find the work done by $\vec{mg}$ along the arc $\widehat{AB}$ of a circle with radius R.

$$W_{\widehat{AB}}^{\vec{mg}} = \int \vec{mg}\cdot\vec{dr} = \int mg\cos\theta\, dr$$

$$= mgR\int_0^{\frac{\pi}{2}} \cos\theta\, d\theta = mgR\left[\sin\theta\right]_0^{\frac{\pi}{2}}$$

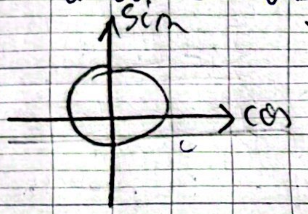
$$= mgR\underbrace{(\sin\tfrac{\pi}{2} - \sin 0)}_{1}$$

$$= mgR$$

or $\theta + \alpha = 90°$

or $dr = R\,d\theta$

and $\sin\theta = \cos\theta\, d\theta$



b) Calculate the work done by $\vec{mg}$ for a displacement AOB

$$W_{AOB}^{\vec{mg}} = W_{AO}^{\vec{mg}} + W_{OB}^{\vec{mg}} \cancel{= 0 + \int mg\cos\theta\, dr = mg\int dr}$$

$$= \int mg\cos(\tfrac{\pi}{2})\, dr + \int mg\cos(0°)\, dr$$

$$0 + mg\int_0^R dr = mg\left[r\right]_0^R = mgR$$

$\int \vec{mg}\cdot\vec{dr}$
$= \int mg\, dr \cos(\vec{mg}, \vec{dr})$
$= \int mg\cdot dr \cos(\tfrac{\pi}{2}) = 0$

c) Calculate the work done by $\vec{mg}$ for the closed curve ABOA

$$\cancel{W_{AOBA}^{\vec{mg}} = W_{AO}^{\vec{mg}} + W_{OB}^{\vec{mg}} + W_{BA}^{\vec{mg}}}$$

$$W_{ABOA}^{\vec{mg}} = W_{AB}^{\vec{mg}} + W_{BO}^{\vec{mg}} + W_{OA}^{\vec{mg}}$$

$$= mgR - mgR + 0$$

$$= 0$$

III: 4) Conservative forces:

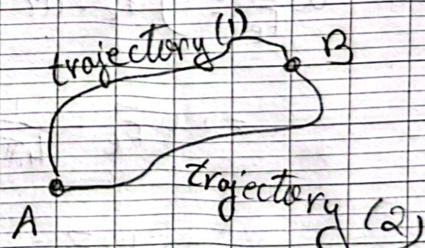
- **1st definition:**
Under the effect of a conservative force, the object returns to its starting point with the same kinetic energy.

- **2nd definition:**
A force is conservative if its work on a closed curve is zero.

- **3rd definition:**
The work of a conservative force does not depend on the path followed by the particle. It only depends on the starting point and end point.

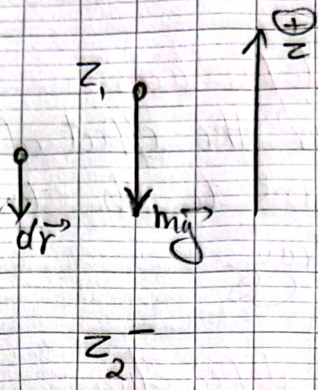
$$W_{A \underset{①}{\rightarrow} B} + W_{B \underset{②}{\rightarrow} A} = 0$$

$$\Rightarrow W_{A \underset{①}{\rightarrow} B} = - W_{B \underset{②}{\rightarrow} A} = W_{A \underset{②}{\rightarrow} B}$$



- **4th definition:**
For a conservative force, there exists a potential U such that F derives from this potential.
$\vec{F} = -\vec{\nabla} U$; in 1st Def.: $F_x = -\frac{dU}{dx}$; $U = -\int \vec{F} \cdot d\vec{r}$

Application III - 3:



$$W_{m\vec g} = \int m\vec g \cdot d\vec r$$

$$= \int_{z_1}^{z_2} -mg\,\hat k \cdot dz\,\hat k$$

$$= -mg\int_{z_1}^{z_2} dz$$

$$= -mg\,[z]_{z_1}^{z_2} = -mg(z_2 - z_1) = mg(z_1 - z_2)$$

or $\vec r = x\hat i + y\hat j + z\hat k \Rightarrow d\vec r = dx\,\hat i + dy\,\hat j + dz\,\hat k$

Application III - 5:

$$\vec F_e = \underbrace{\frac{1}{4\pi\varepsilon_0}}_{K_e} \cdot \frac{Qq}{r^2}\,\hat e_r$$

or $\vec v = \frac{d\vec r}{dt}$

$\cancel{\vec v} = \frac{dr}{dt}\,\hat e_r + r\frac{d\theta}{dt}\,\hat e_\theta$

$\Rightarrow d\vec r = dr\,\hat e_r + r\,d\theta\,\hat e_\theta$

$$W_{\vec F_e} = \int \vec F_e \cdot d\vec r = \frac{1}{4\pi\varepsilon_0}\cdot Qq\int \frac{1}{r^2}\,dr$$

$$\Rightarrow W_{\vec F_e} = \frac{1}{4\pi\varepsilon_0}\cdot Qq\int \frac{1}{r^2}\,\hat e_r\cdot(dr\,\hat e_r + r\,d\theta\,\hat e_\theta)$$

$$\Rightarrow W_{\vec F_e} = \frac{1}{4\pi\varepsilon_0}\cdot Qq\int_{r_A}^{r_B}\frac{1}{r^2}\,dr = \frac{1}{4\pi\varepsilon_0}\cdot Qq\left[\frac{-1}{r}\right]_{r_A}^{r_B}$$

$$\cancel{\frac{1}{4\pi\varepsilon_0}\cdot Qq\left(\frac{1}{r_B} - \frac{1}{r_A}\right)}$$

$$\Rightarrow W_{\vec F_e} = -\frac{1}{4\pi\varepsilon_0}\cdot Qq\left[\frac{1}{r}\right]_{r_A}^{r_B} = \frac{-1}{4\pi\varepsilon_0}\cdot Qq\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

Application III-6 :

displacement vector

$$W_{\vec{T}\,elongation} = \int \vec{T}\cdot d\vec{r}$$

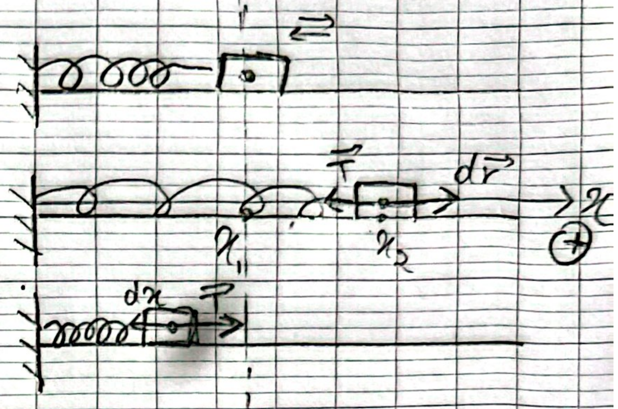
$$= \int_{x_1}^{x_2} -Kx\hat{i}\cdot dx\hat{i}$$

$$= -K\int_{x_1}^{x_2} x\,dx$$

$$= -K\left[\frac{x^2}{2}\right]_{x_1}^{x_2}$$

$$= -\frac{K}{2}\left[x^2\right]_{x_1}^{x_2}$$

$$= \underbrace{-\frac{K}{2}}_{\ominus}\underbrace{(x_2^2 - x_1^2)}_{\oplus} < 0$$



$$W_{\vec{T}\,compression} = \int \vec{T}\cdot d\vec{r} = \int_{x_2}^{x_1} -Kx\hat{i}\cdot dx\hat{i} = -K\int_{x_2}^{x_1} x\,dx$$

$$= -K\left[\frac{x^2}{2}\right]_{x_2}^{x_1} = -\frac{K}{2}\left[x^2\right]_{x_2}^{x_1}$$

$$= \underbrace{-\frac{K}{2}}_{\ominus}\underbrace{(x_1^2 - x_2^2)}_{\ominus} > 0$$

$W_{\vec{T}} = 0$; $\vec{T}$ is a conservative force its work on a closed curve is zero.

$x_1 \to x_2 \to x_1$