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I-8-Change of reference frames

The choice of reference frame is often dictated by the nature of the motion. However, it is important to understand the changes that occur in the kinematics of a point's motion when adopting a different reference frame. For example, the motion of a body in free fall inside a train is different for a passenger in the train and an observer on the train station.

Absolute motion, relative motion, driving motion.

Motion observed by an observer at rest

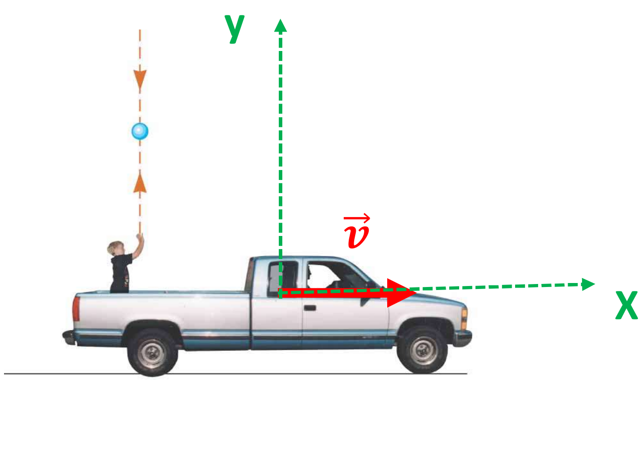
Motion observed by a moving observer

Motion of the moving frame

The truck moves with a constant velocity with respect to the ground.

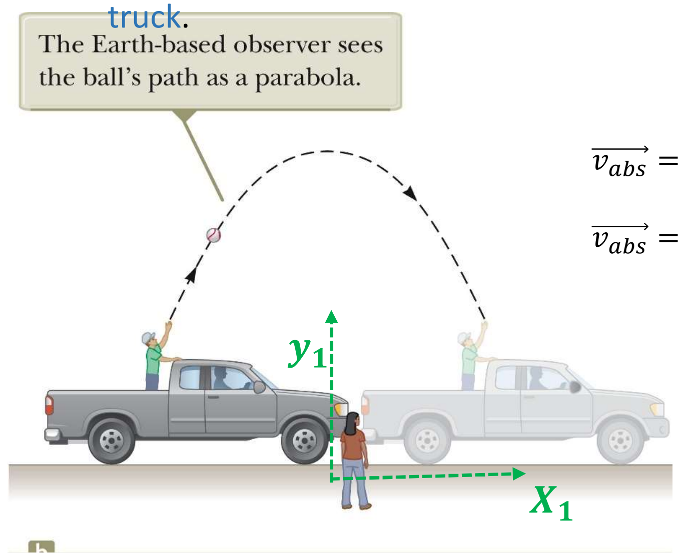
The **observer in the truck** throws a ball straight up.

It appears to move in **a vertical path**.



There is a stationary observer on the ground. He views the path of the ball thrown to be a **parabola**.

The ball has a velocity to the right equal to the velocity of the truck.

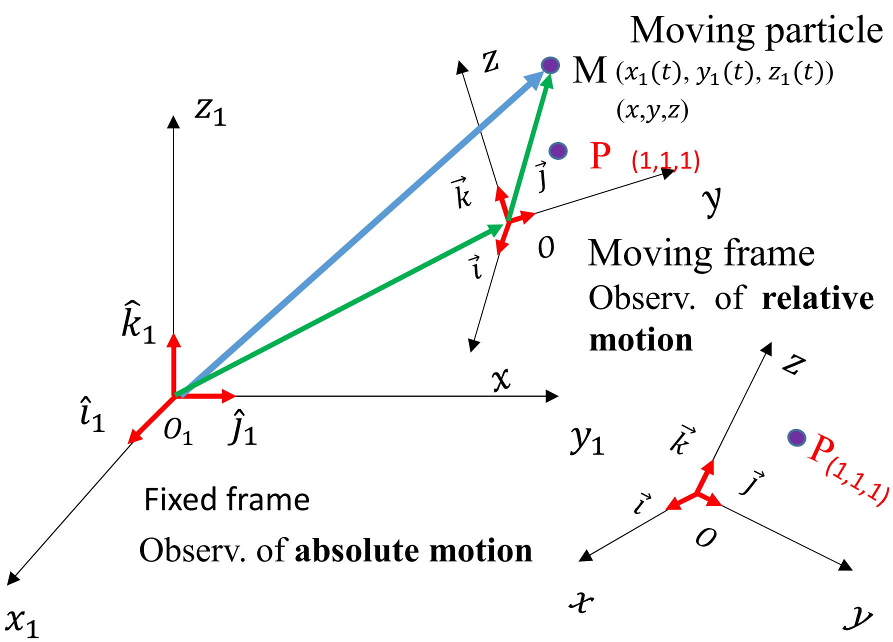


$$\overrightarrow{v_{abs}} = \overrightarrow{v_{truck}} + \overrightarrow{v_{free\ fall}}$$

$$\overrightarrow{v_{abs}} = \overrightarrow{v_d} + \overrightarrow{v_r}$$

Drift velocity= Velocity of the moving frame

Absolute motion, relative motion, driving motion:



$O_1(x_1, y_1, z_1)$ is a **fixed cartesian frame** with unit vectors $(\hat{\imath}_1, \hat{\jmath}_1, \hat{k}_1)$. This frame allows the observation of the **absolute motion** of the moving object M if its position is given by the vector:

$$\overrightarrow{O_1M} = x_1\hat{\imath}_1 + y_1\hat{\jmath}_1 + z_1\hat{k}_1$$

Let us now consider another reference frame to which the moving frame $(Oxyz)$ is attached, relative to the first one (with velocity $\vec{v}_O$).

The motion of the same point M, observed from O, is called relative motion and is defined by the time-dependent components of the Cartesian vector $\overrightarrow{OM}$

$$\overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k}$$

A fixed point in the second frame, with constant coordinates (x,y,z), appears to be moving in the first frame and has coordinates $(x_1(t), y_1(t), z_1(t))$ that depend on time. The motion thus defined relative to the space of the frame $O_1(x_1, y_1, z_1)$ is called the driving motion associated with the point considered fixed in the space defined from the frame $Oxyz$.

Composition of velocities

The equation of motion of the point MM with respect to the fixed frame is given by the vector.

$$\overrightarrow{O_1M} = \overrightarrow{O_1O} + \overrightarrow{OM}$$

The absolute velocity of the moving object M, i.e., the velocity of the absolute motion observed from the reference frame $O_1(x_1,\ y_1,\ z_1)$ is obtained by differentiating with respect to time: $\overrightarrow{O_1M} = \overrightarrow{O_1O} + \overrightarrow{OM}$

$$\vec{v}_a = \frac{d\overrightarrow{O_1M}}{dt} = \frac{d\overrightarrow{O_1O}}{dt} + \frac{d\overrightarrow{OM}}{dt} = \vec{v}_O + \frac{d\left(x\vec{\imath} + y\vec{\jmath} + z\vec{k}\right)}{dt}$$

$Attention: vectors\ \vec{\imath}, \vec{\jmath}, \vec{k}$ are moving

$$\vec{v}_a = \vec{v}_O + \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k} + x\frac{d\vec{\imath}}{dt} + y\frac{d\vec{\jmath}}{dt} + z\frac{d\vec{k}}{dt}$$

$$we\ had \quad \frac{d\vec{e}_r}{dt} = \vec{\omega} \wedge \vec{e}_r \quad so \quad \frac{d\vec{\imath}}{dt} = \vec{\omega} \wedge \vec{\imath}$$

$$likewise \quad \frac{d\vec{\jmath}}{dt} = \vec{\omega} \wedge \vec{\jmath} \qquad \frac{d\vec{k}}{dt} = \vec{\omega} \wedge \vec{k}$$

$$\vec{v}_a = \vec{v}_O + \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k} + x\vec{\omega} \wedge \vec{\imath} + y\vec{\omega} \wedge \vec{\jmath} + z\vec{\omega} \wedge \vec{k}$$

$$\vec{v}_a = \vec{v}_O + \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k} + \vec{\omega} \wedge (x\vec{\imath} + y\vec{\jmath} + z\vec{k})$$

$$\vec{v}_a = \vec{v}_O + \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k} + \vec{\omega} \wedge \overrightarrow{OM}$$

$$\vec{v}_a = \vec{v}_d + \vec{v}_r \quad \begin{cases} \vec{v}_d = \vec{v}_O + \vec{\omega} \wedge \overrightarrow{OM} & \text{Driving velocity} \\ \vec{v}_r = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k} & \text{Relative velocity} \end{cases}$$

Compositions of accelerations:

$$\vec{a}_a = \frac{d\vec{v}_a}{dt} = \underbrace{\frac{d\vec{v}_O}{dt}}_{\boxed{1}} + \underbrace{\frac{d(\dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k})}{dt}}_{\boxed{2}} + \underbrace{\frac{d(\vec{\omega} \wedge (x\vec{\imath} + y\vec{\jmath} + z\vec{k}))}{dt}}_{\boxed{3}}$$

$$\boxed{1} \quad \frac{d\vec{v}_O}{dt} = \vec{a}_O$$

$$\boxed{2} \quad \frac{d(\dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k})}{dt} = \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath} + \ddot{z}\vec{k} + \dot{x}\frac{d\vec{\imath}}{dt} + \dot{y}\frac{d\vec{\jmath}}{dt} + \dot{z}\frac{d\vec{k}}{dt}$$

$$= \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath} + \ddot{z}\vec{k} + \dot{x}\vec{\omega} \wedge \vec{\imath} + \dot{y}\vec{\omega} \wedge \vec{\jmath} + \dot{z}\vec{\omega} \wedge \vec{k}$$

$$= \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath} + \ddot{z}\vec{k} + \vec{\omega} \wedge (\dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k})$$

$$= \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath} + \ddot{z}\vec{k} + \vec{\omega} \wedge \vec{v}_r$$

$$\boxed{3} \quad \frac{d(\vec{\omega} \wedge (x\vec{\imath} + y\vec{\jmath} + z\vec{k}))}{dt} = \vec{\alpha} \wedge \left(x\vec{\imath} + y\vec{\jmath} + z\vec{k}\right) +$$

$$\vec{\omega} \wedge (\dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k}) + \vec{\omega} \wedge \left(x\frac{d\vec{\imath}}{dt} + y\frac{d\vec{\jmath}}{dt} + z\frac{d\vec{k}}{dt}\right) =$$

$$\vec{\alpha} \wedge \overrightarrow{OM} + \vec{\omega} \wedge \vec{v}_r + \vec{\omega} \wedge \left(x\vec{\omega} \wedge \vec{\imath} + y\vec{\omega} \wedge \vec{\jmath} + z\vec{\omega} \wedge \vec{k}\right) =$$

$$\vec{\alpha} \wedge \overrightarrow{OM} + \vec{\omega} \wedge \vec{v}_r + \vec{\omega} \wedge \vec{\omega} \wedge \overrightarrow{OM}$$

$$\vec{a}_a = \vec{a}_O \ + \ddot{x}\,\vec{i} + \ddot{y}\,\vec{j} + \ddot{z}\,\vec{k} + \ \vec{\omega} \wedge \vec{v}_r + \vec{\alpha} \wedge \overrightarrow{OM} + \vec{\omega} \wedge \vec{v}_r + \vec{\omega} \wedge \vec{\omega} \wedge \overrightarrow{OM}$$

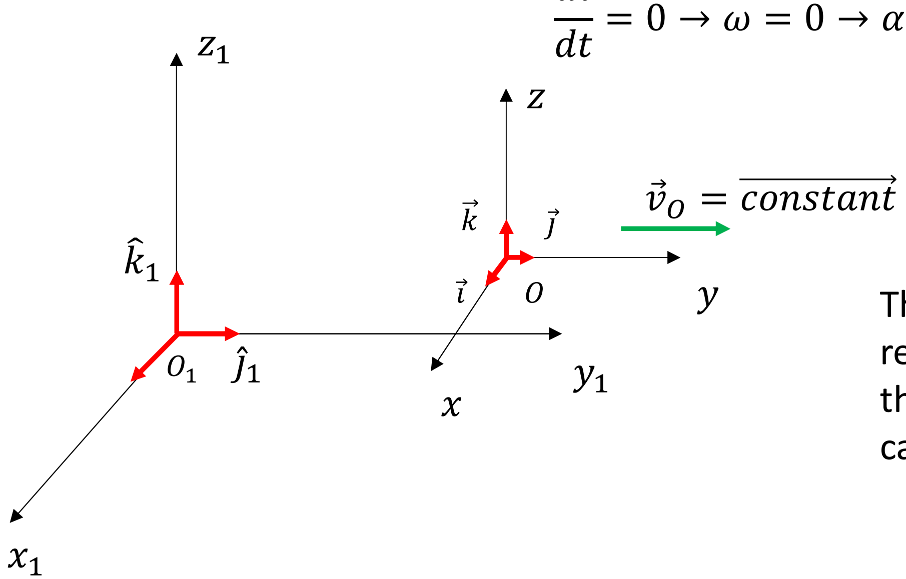
$$\vec{a}_a = \underbrace{\vec{a}_O + \vec{\alpha} \wedge \overrightarrow{OM} + \vec{\omega} \wedge \vec{\omega} \wedge \overrightarrow{OM}}_{d} + \underbrace{\ddot{x}\,\vec{i} + \ddot{y}\,\vec{j} + \ddot{z}\,\vec{k}}_{\vec{a}_r} + \underbrace{2\vec{\omega} \wedge \vec{v}_r}_{\vec{a}_c} = \vec{a}_d + \vec{a}_r + \vec{a}_c$$

Driving acceleration *Relative acceleration* *Coriolice* acceleration

Particular case A particular and important case is when the motion of the moving frame is a uniform rectilinear translation. $\vec{v}_O = \overrightarrow{constant} \rightarrow \vec{a}_O = 0$

$$\frac{d\vec{i}}{dt} = 0 \rightarrow \omega = 0 \rightarrow \alpha = 0, \vec{a}_d = \vec{a}_c = 0$$

Therefore $\vec{a}_a = \vec{a}_r$



Then the acceleration of the moving object is identical in both reference frames. Such a change of reference frame ensures the invariance of the kinematic quantity, acceleration, and is called a ***Galilean transformation***.

I-24- A swimmer starting from point A moves at a constant speed v_s relative to the water in a river of width d, where the water has a constant current of speed v_w ($v_w < v_s$).

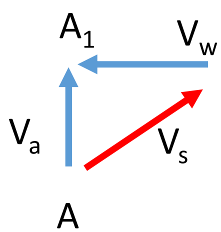
The swimmer completes the round-trip journeys AA_1A in a time t_1 and AA_2A in a time t_2.

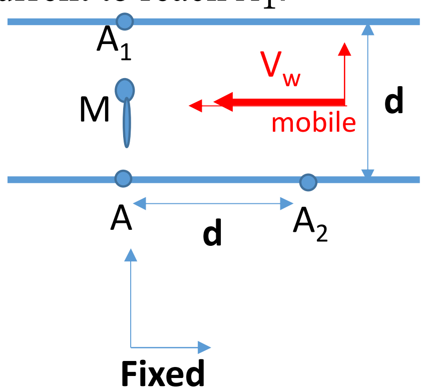
1- Express the ratio t_2/t_1 in terms of the ratio of speeds $v_w/\ v_s$.

2- Given that $t_2 = 2t_1$, determine the direction of the swimmer's speed v_s as he moves against the current to reach A_1.

Solution

1-





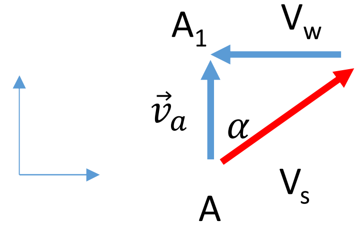
$$t = \frac{d}{v_a}\ ;\quad \vec{v}_a = \vec{v}_d + \vec{v}_r$$

$$\vec{v}_{swimmer/Fixed} = \vec{v}_{moving/Fixed} + \vec{v}_{swimmer/moving}$$

$$\vec{V}_a = \vec{V}_w + \vec{V}_s$$

$$V_s^2 = V_w^2 + v_a^2$$

$$v_a = \sqrt{V_s^2 - V_w^2}$$



The round-trip time AA_1A: $\quad t_1 = 2t = \dfrac{2d}{v_a} = \dfrac{2d}{\sqrt{V_s^2 - V_w^2}}$

The round-trip time AA_2A:

$$t_2 = t(\mathrm{AA_2}) + t(\mathrm{A_2}A) = \frac{d}{V_s - V_w} + \frac{d}{V_s + V_w} = \frac{2dV_s}{V_s^2 - V_w^2}$$

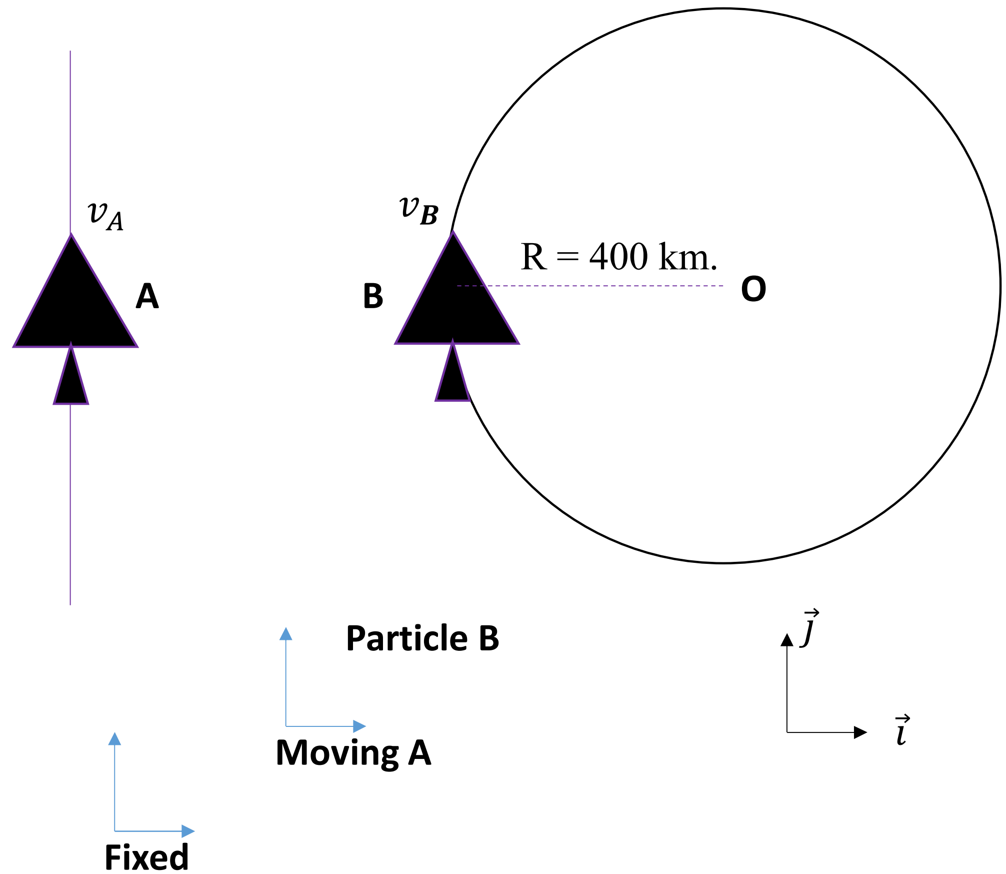
$$\frac{t_2}{t_1} = \frac{\dfrac{2dV_s}{V_s^2 - V_w^2}}{\dfrac{2d}{\sqrt{V_s^2 - V_w^2}}} = \frac{V_s\sqrt{V_s^2 - V_w^2}}{V_s^2 - V_w^2} = \frac{V_s\sqrt{V_s^2\left(1 - \dfrac{V_w^2}{V_s^2}\right)}}{V_s^2\left(1 - \dfrac{V_w^2}{V_s^2}\right)} = \frac{1}{\sqrt{\left(1 - \dfrac{V_w^2}{V_s^2}\right)}}$$

2- $t_2 = 2t_1$ $\quad \left(1 - \dfrac{V_w^2}{V_s^2}\right) = \dfrac{1}{4} \rightarrow \dfrac{V_w^2}{V_s^2} = \dfrac{3}{4} \rightarrow \dfrac{V_w}{V_s} = \dfrac{\sqrt{3}}{2}$

$$\sin\alpha = \frac{V_w}{V_s} = \frac{\sqrt{3}}{2}; \alpha = 60°$$

I-25- Two planes, A and B, are flying at the same altitude. The first follows a straight trajectory, while the other moves along a circle with center O and radius R = 400 km. Calculate, at the instant when the three points OAB are aligned, the speed and acceleration of B relative to the pilot of A.

Given: $v_A = 700\ km/h$, $v_B = 600\ km/h$, $a_{B,t} = -100\ km/h^2$, and $a_A = 50\ km/h^2$.



Solution Velocity and acceleration of B relative to pilote of A

$$\vec{v}_a = \vec{v}_d + \vec{v}_r$$

$\vec{v}_{B/A}$? $\vec{v}_{B/fixed} = \vec{v}_{A/fix} + \vec{v}_{B/A}$

$$\vec{v}_{B/A} = \vec{v}_{B/fix\ d} - \vec{v}_{A/fixe} = \vec{v}_B - \vec{v}_A$$

$$\vec{v}_{B/A} = (0-0)\vec{\imath} + (600-700)\vec{\jmath} = -100\vec{\jmath}\ (Km/h)$$

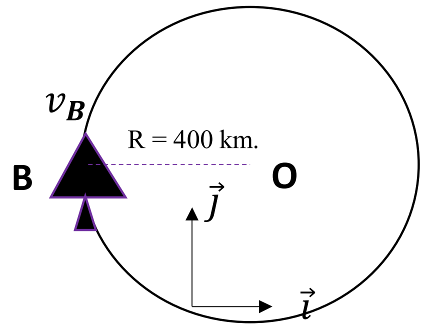
Magnitude: $v_{B/A} = 100\ km/h$

Direction : $-\vec{\jmath}$

$\vec{a}_{B/A}$? $\vec{a}_{B/fix\ \ d} = \vec{a}_{A/fixe} + \ \ \vec{a}_{B/A}$

$$\vec{a}_{B/A} = \quad \vec{a}_{B/fixe} \ \ - \vec{a}_{A/fixe} = \vec{a}_{B} - \vec{a}_{A}$$

$$\vec{a}_{B/A} = (\vec{a}_{t\ B} + \vec{a}_{n\ B}) - \vec{a}_{A}$$

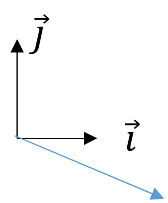


$$\vec{a}_{B/A} = (a_{t\ B}\vec{j} + \frac{{v_B}^2}{R}\vec{i}\,) - a_{tA}\vec{j}$$

$$\vec{a}_{B/A} = \frac{{v_B}^2}{R}\vec{i} + (a_{t\ B} - a_{tA})\,\vec{j}$$

$$\vec{a}_{B/A} = \frac{600^2}{400}\vec{i} + (-100 - 50)\,\vec{j}$$

$$\vec{a}_{B/A} = 900\,\vec{i} - 150\,\vec{j}$$



Magnitude: $a_{B/A} = 912{,}4\,Km/h^2$

Direction : $\tan\alpha = \dfrac{-150}{900} \rightarrow \alpha = -9{,}4°$