Lebanese University
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P1100 - Mechanics

Chapter 2 – Kinetics of a particle, Force and Acceleration

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Classical Dynamics

Central Force and Space Dynamics

Real and Fictitious Forces

Objectives

- **Newton's Second Law (NSL) of Motion**.
- Analysis of the accelerated motion of a particle using the equation of motion with **different coordinate systems**.
- Investigation of **central-force motion** and application to **space mechanics**.

Classical Dynamics

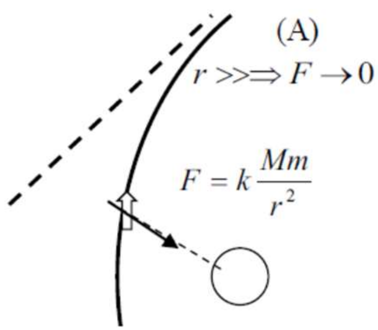
II-1- Newton's First Law, or the Principle of Inertia:

Suppose we send a spacecraft outside the solar system. After passing beyond the orbits of the farthest planets, the spacecraft follows a hyperbolic trajectory relative to the Sun. At a very great distance from the solar system, the spacecraft will have an almost straight-line trajectory, which is none other than the asymptote of the hyperbola.

Its velocity will be constant, and its acceleration will be zero.

Such motion is called uniform rectilinear motion or inertial motion.

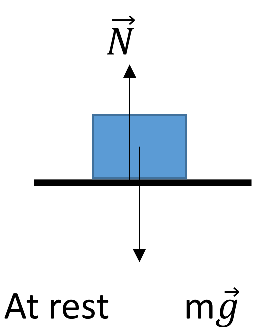
Note: A **straight line** and **constant velocity** are two essential concepts in the definition of **inertial motion**. Motion at constant speed along a circle is not inertial.



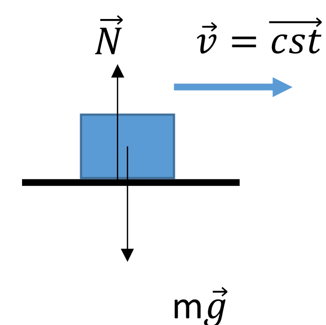
Thus, a body placed on a smooth horizontal table (no friction) is subjected to its weight $\vec{W}$ and the vertical reaction $\vec{N}$ that exactly counteracts the weight. It is experimentally verified that in this setup, when the body is launched from one edge of the table to the other, it exhibits inertial motion between its launch point A and its arrival point B. Everything happens as if the body were not subjected to any external influence.

An object that is not influenced by any other objects (isolated object):

- **Possesses uniform rectilinear motion (inertial motion) if it's moving OR**
- **Remains at rest if it were initially at rest**



$$\sum \vec{F}_{est} = \vec{0}$$



Newton's second law:

This principle establishes the relationship between the cause of motion and the variation of its state. Let a material point be subjected to a resultant force $\vec{F}$. Under the action of this force, the point experiences an acceleration $\vec{a}$. Newton's second law postulates that the relationship between $\vec{F}$ and $\vec{a}$ is given by $\vec{F} = \mathrm{m}\vec{a}$, where m is a characteristic property of the body called mass. $\vec{F}$ and $\vec{a}$ are both vector quantities. This relationship is known as the fundamental law of dynamics.

$$\boxed{\sum \vec{F}_{ext} = m\vec{a}}$$

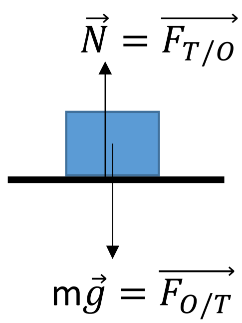
$$\vec{a} = \frac{\Sigma \vec{F}}{m}$$

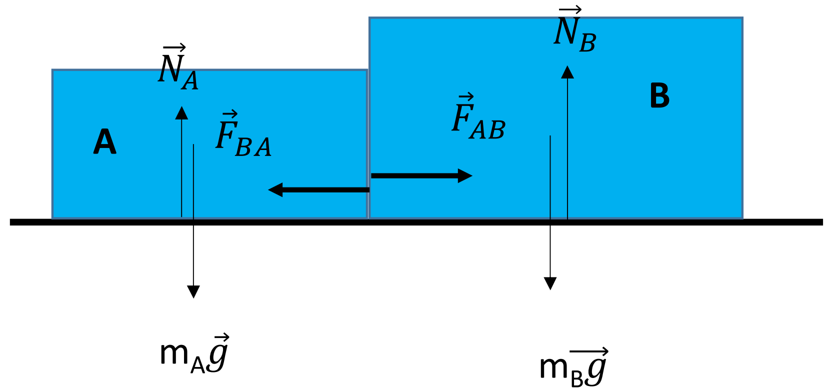
If we have $\vec{F}$, $\vec{v}$ will change

The mass is a measure of an object's resistance to acceleration when a force is applied.

Newton's Third Law or the Principle of Action and Reaction:

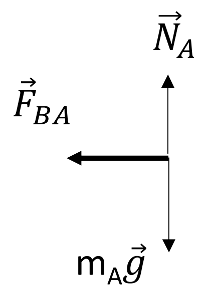
$$\vec{N} + m\vec{g} = 0 \rightarrow \quad \vec{N} = -m\vec{g}$$



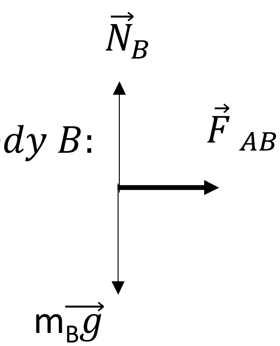


$\mathrm{m_A}\vec{g} = -\vec{N}_A \qquad \vec{F}_{AB} = -\vec{F}_{BA} \qquad \mathrm{m_B}\vec{g} = -\vec{N}_B$

$Free - body\ diagram\ for\ body\ A$:



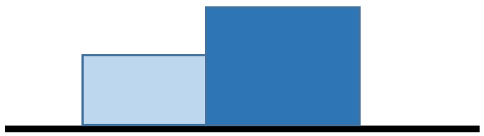
$Free - body\ diagram\ for\ body\ B$:



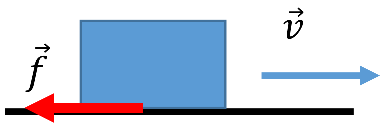
2. 2: Types of forces

1- Contact forces :

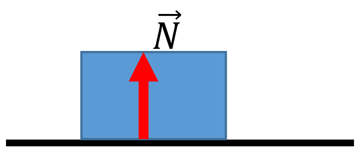
- Forces caused by other objects touching the object of interest.



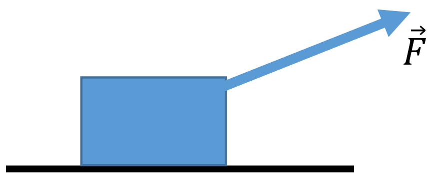
- The forces of friction



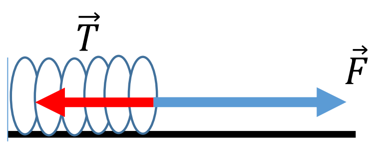
- The Normal contact force



- The forces exerted by ropes

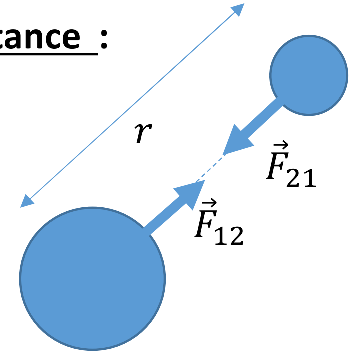


- The force exerted by spring



2- Forces acting at distance :

- The Gravitational force



$$\vec{F}_{12} = -\vec{F}_{21}$$

$$F_{12} = F_{21} = G\,\frac{M_1 M_2}{r^2}$$

- The weak nuclear force (hold the neutrons together)

- The strong nuclear force (hold neutrons and protons)

- The electromagnetic force

Applications: II-1- An artificial **satellite**, with **mass m**, orbits around the **Earth** with **mass M**, on an approximately flat orbit centered on that of the Earth, and located at an altitude **h = 800 km**. The Earth's radius is given as **R = 6400 km**, and the acceleration due to gravity at the surface is g = 10 m/s².
Determine the orbital speed v of the satellite and its period T around the Earth.

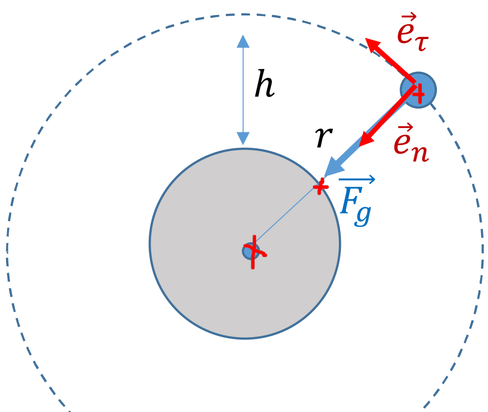
Solution $v \rightarrow a \rightarrow polar\ coord\ (circular\ motion)$

$$\boxed{F_g = G\frac{Mm}{r^2 = (R+h)^2}}$$

Newton's second law $\overrightarrow{F_g} = m\vec{a} = m(\vec{a}_n + \vec{a}_t)$

Projection $\vec{n}$: $F = ma_n \rightarrow$

$$G\frac{Mm}{(R+h)^2} = m\frac{v^2}{(R+h)}$$



$$v = \sqrt{\frac{GM}{R+h}} \qquad GM = ?$$

The only force acting on the satellite is its weight, mg.

At the height h=R, on the surface of the Earth: $F = mg$

$$G\frac{Mm}{(R+h)^2} = mg$$

$$GM = gR^2$$

Note: g changes with altitude!
$g = 10m/s^2$ at h=0 only

$$v = \sqrt{\frac{g_0R^2}{R+h}}$$

$$v = 6400.10^3\sqrt{\frac{10}{7200.10^3}} = 7{,}55\ km/s$$

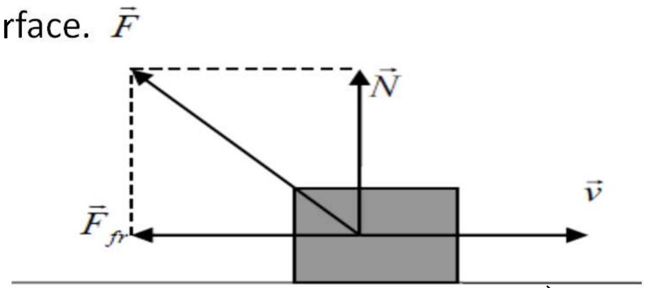
$$T = \frac{2\pi}{\omega} = \frac{2\pi}{v/(R+h)} = \frac{2\pi(R+h)}{v} = \frac{2\pi(R+h)}{\sqrt{\frac{g_0R^2}{R+h}}}$$

$\vec{v} = \vec{\omega} \wedge \vec{r}$

$$= \frac{2\pi(R+h)^{3/2}}{\sqrt{g_0R^2}} = 1^h\ 40^{min}$$

2.3 Friction forces

Friction represents the action of a rigid surface on a solid, an action that opposes the motion of the solid relative to the surface.



$\vec{\boldsymbol{F}}$ is the force exerted by the surface on the body. $\vec{\boldsymbol{F}}$ has two components: vertical (the normal force) and horizontal (the frictional force).

STATIC friction OR KINETIC friction??

- As long as the external force is less than or equal to f_s^{max} ($\boldsymbol{F_{ext}} \leq \boldsymbol{f_s^{max}}$), the object remains at **rest**: static friction takes place: $\boldsymbol{f_s} \leq \boldsymbol{f_s^{max}} = \mu_s N$.
- Once the object is in **motion**, static friction is no longer acting. Instead, **kinetic friction** takes over, which is generally lower than static friction: $f_k = \mu_k N$.

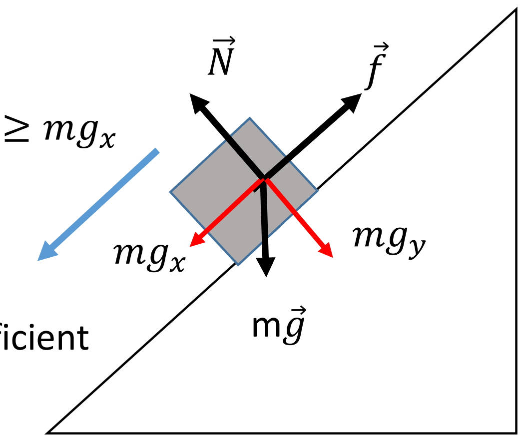
Example 1:

$$\sum \vec{F}_{ext} = m\vec{a}\,; Proj.\,on\ x'ox:$$

$$mg_x - f = ma_x$$

1. **System at rest**: $f = f_s \geq mg_x$

$$\boxed{\overrightarrow{f_s^{max}} = \mu_s \vec{N}}$$



μ_s is called the static coefficient of friction

2. Once the **object** starts **moving**, the frictional force f is slightly less than f_s but remains approximately proportional to N.

$$\boxed{\overrightarrow{f_k} = \mu_k \vec{N}}$$

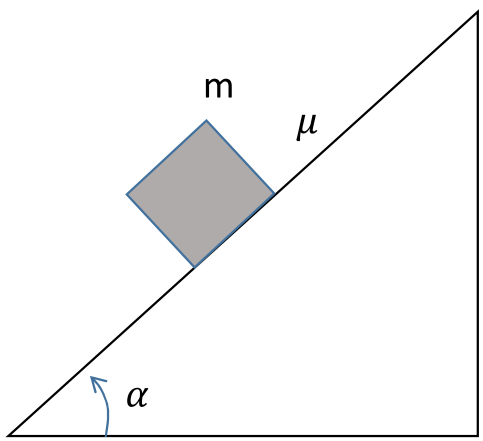
μ_k is is called the coefficient of kinetic friction ($\mu_k < \mu_s$)

Example 2:

An object of mass m is in equilibrium on an inclined plane with an angle α.
Determine the minimum value of the coefficient of static friction μ_s for the object to remain in equilibrium.

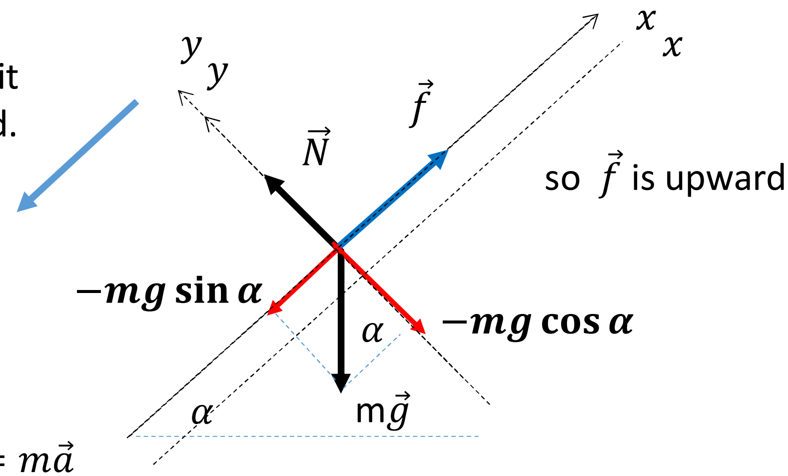
Solution

Step 1: identify the system: m



Free-body diagram

If there is movement, it is downward.



$$\sum \vec{F}_{ext} = m\vec{a}$$

$system\ at\ equilibrium,\qquad a = 0,\qquad m\vec{g} + \vec{N} + \vec{f} = \vec{0}$

Let's choose a reference frame:: (xOy)

Projection $x'Ox$: $-mg\sin\alpha \; +0 \; +f_s = 0 \rightarrow f_s = mg\sin\alpha$

$y'Oy$: $-mg\cos\alpha \; +N \; +0 = 0 \rightarrow N = mg\cos\alpha$

$$f_s \le f_s^{max} = \mu_s N$$

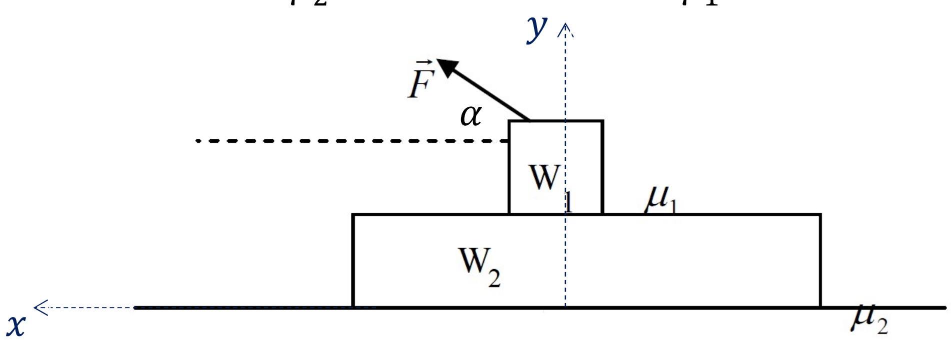
$$mg\sin\alpha \le \mu_s mg\cos\alpha$$

$$\mu_s \ge \frac{mg\sin\alpha}{mg\cos\alpha} = \tan\alpha \qquad \Rightarrow \mu_s^{min} = \tan\alpha$$

Application II-3- In the figure below, two blocks of weight W_1 and W_2 are superimposed on a horizontal plane. The coefficient of friction between W_1 and W_2 is μ_1 and that between W_2 and the plane is μ_2.

We exert a force F on W_1 at an angle α to the horizontal. If $\cot\alpha \geq \mu_1 \geq \mu_2$, demonstrate that the necessary and sufficient condition for **W_2 to move relative to the plane** *without W_1 moving relative to W_2* is as follows:

$$\frac{\mu_2(W_1+W_2)}{\cos\alpha+\mu_2\sin\alpha} < F < \frac{\mu_1 W_1}{\cos\alpha+\mu_1\sin\alpha}$$



We assume that the two blocks are at rest and form a single system.

Newton's second law on the system of the two blocs:

$$\underbrace{\vec{W}_1+\vec{W}_2+\vec{N}+\vec{F}}_{\sum \vec{F'}} + \vec{f}_2 = (m_1+m_2)\vec{a}$$

The system W_1+W_2 moves if : $\sum F'_x > f_2 = \mu_2 N$

? ?

Projection

$x'Ox$: $\sum F'_x = F\cos\alpha > \mu_2 N$

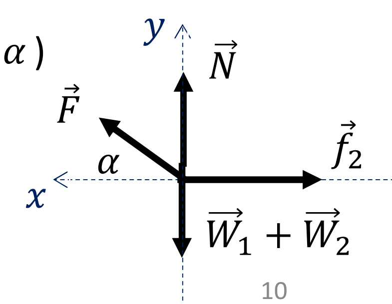
$y'Oy$: $-W_1 - W_2 + N + F\sin\alpha = 0$

$$N = W_1 + W_2 - F\sin\alpha$$

$\rightarrow f_2 = \mu_2(W_1+W_2-F\sin\alpha)$

$\rightarrow F\cos\alpha > \mu_2(W_1+W_2-F\sin\alpha)$

$$F > \frac{\mu_2(W_1+W_2)}{(\cos\alpha+\mu_2\sin\alpha)}$$



Let's take the system : weight W_1

$$\underbrace{\overrightarrow{W_1} + \vec{N}_1 + \vec{F} + \vec{f}_1}_{\sum \overrightarrow{F''}} = m_1\vec{a}$$

W_1 does not move relative to W_2 if: $\sum F''_x < f_1 = \mu_1 N_1$

$$x'Ox: \quad \sum F''_x = F\cos\alpha < f_1$$

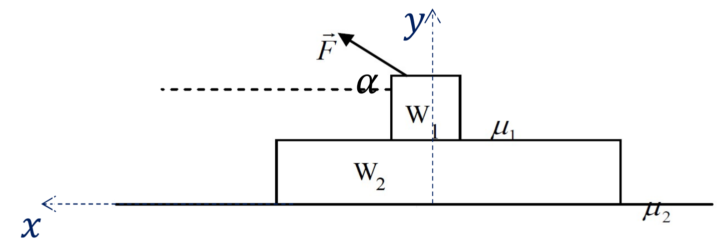
$$y'Oy: \quad F\sin\alpha - W_1 + N_1 = 0$$

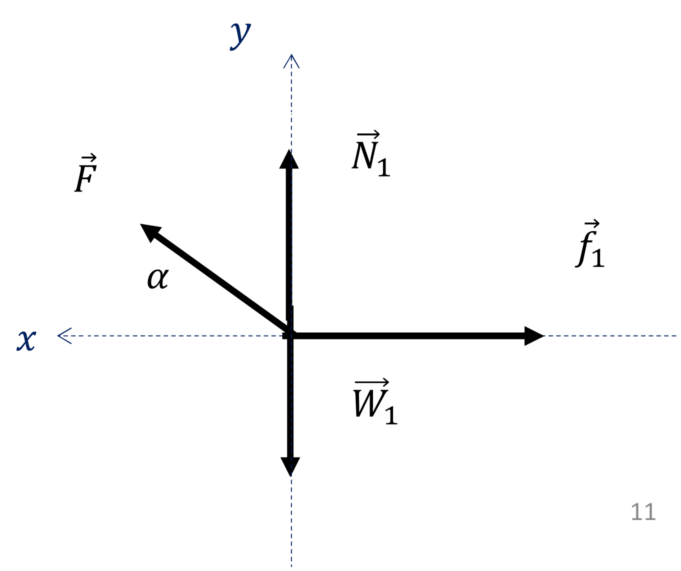
$$N_1 = W_1 - F\sin\alpha$$

$$\rightarrow F\cos\alpha < \mu_1(W_1 - F\sin\alpha)$$

$$\rightarrow F < \frac{\mu_1 W_1}{(\cos\alpha + \mu_1 \sin\alpha)}$$

$$\frac{\mu_2(W_1 + W_2)}{(\cos\alpha + \mu_2\sin\alpha)} < F < \frac{\mu_1 W_1}{(\cos\alpha + \mu_1\sin\alpha)}$$





2.4 Equation of motion

– Equation of motion in cartesian coordinates:

$$\sum \vec{F} = m\vec{a} = \sum F_x\hat{\imath} + \sum F_y\hat{\jmath} + \sum F_z\hat{k} = m(a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k})$$

By identification

$$\sum F_x = ma_x, \qquad \sum F_y = ma_y, \sum F_z = ma_z$$

- Equation of motion in normal and tangential coordinates:

$$\sum F_t\hat{u}_t + \sum F_n\hat{u}_n = m(a_t\hat{u}_t + a_n\hat{u}_n)$$

$$\sum F_t = ma_t \quad \text{et} \quad \sum F_n = ma_n$$

$$a_t = \frac{dv}{dt} \qquad a_n = \frac{v^2}{\rho}$$

$$\rho = \left| \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \right|$$

$$a_t ds = vdv$$

Application II-5- The equation of the trajectory of a ski slope is given by $y = \frac{1}{200}x^2$-200. The skier with a mass of m = 150 kg passes through the point A (0,−200 m) with a speed of v= 65 m/s.

Determine the **reaction of the slope** and the skier's **acceleration at point A**.

Solution

We know the eq of the traj → a_n and a_t

$$m\vec{g} + \vec{R} = m(\vec{a}_n + \vec{a}_t) \Rightarrow \begin{cases} \hat{n}: -mg + R = ma_n \\ \hat{\tau}: 0 + 0 = ma_t \end{cases}$$

$$\boldsymbol{R = m(g + a_n) = m(g + v^2/\rho)}$$

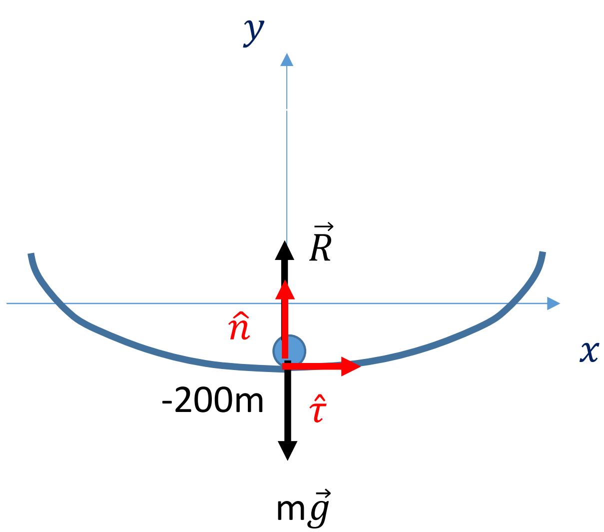
$$\rho = \left| \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{d^2y/dx^2} \right| \quad , \quad \frac{dy}{dx} = x/100 \quad , \quad d^2y/dx^2 = 1/100 \, ,$$

$$\begin{matrix} x = 0 \\ y = -200 \end{matrix} \Rightarrow dy/dx = 0 \Rightarrow \rho = \left| \frac{1+0}{1/100} \right| = 100m \, .$$

$$R = m(g + v^2/\rho) = 150(9{,}81 + 65^2/100) = 1569N \, .$$

$$\vec{a} = \vec{a}_n + \vec{a}_t \qquad \text{avec} \quad a_t = 0 \Rightarrow a = a_n = 42{,}2 m/s^2 \, .$$

$$R = m(g + v^2/\rho) = 150(9{,}81 + 65^2/100) = 7809N$$



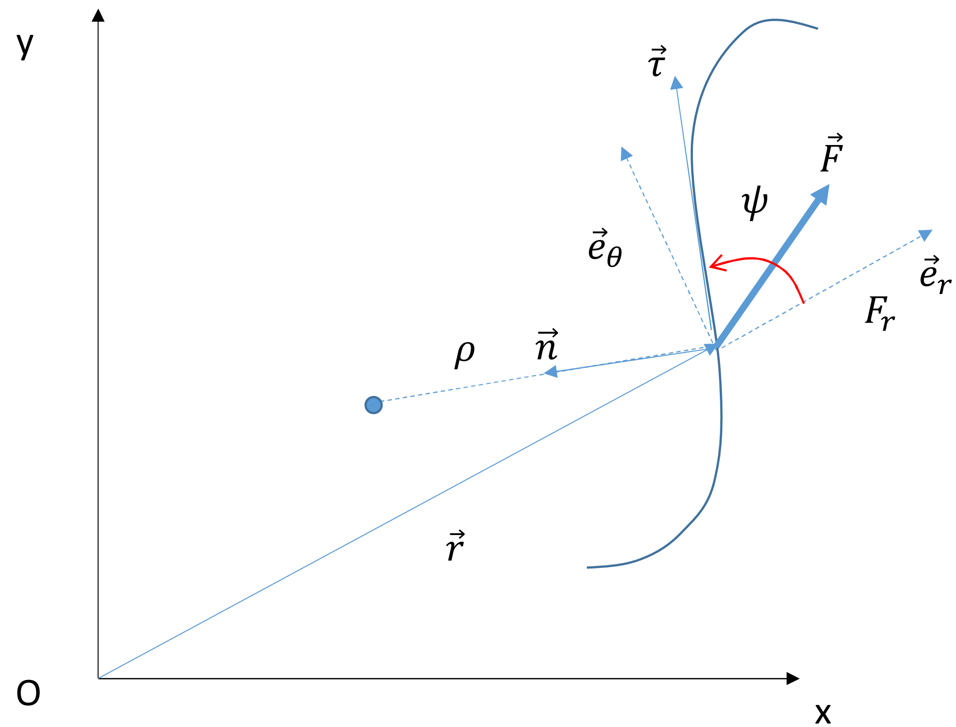
– Equation of motion in polar coordinates :

When the particle is required to move in a plane (ρ; θ), it may be more practical to express the FRD using the two unit vectors ($\vec{e}_\rho, \vec{e}_\theta$). The equations of motion will then be:

$$\sum F_r \vec{e}_r + \sum F_\theta \vec{e}_\theta = m(a_r \vec{e}_r + a_\theta \vec{e}_\theta)$$

$$\sum F_r = m(r'' - r\theta'^2) \qquad \sum F_\theta = m(r\theta'' + 2r'\theta')$$

We could determine F_n and F_τ relative to the polar components by the determination of the angle ψ:



$$\psi = (\vec{r}, \vec{\tau})$$

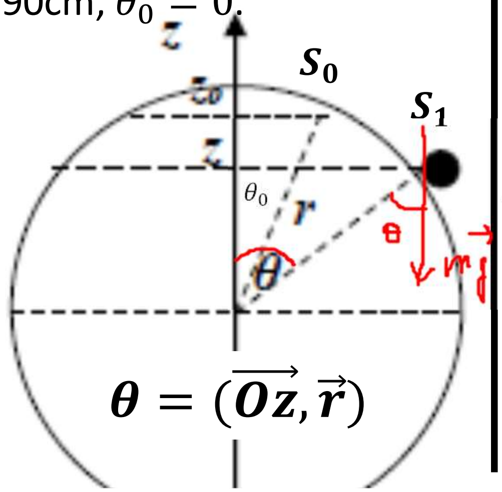
$$tan\,\psi = \frac{r}{dr/d\theta}$$

II-6- A particle of mass *m* is launched from the point S_0 (with an elevation ($z_0 = rcos\theta_0$) on a sphere with center *O* and radius *r* with an initial velocity v_0 (tangent to the sphere and in the vertical plane passing through *O*); it slides without friction on the sphere and then takes off, leaving the sphere at a point S_1 . Let *g* denote the acceleration due to gravity.

a- Express the **reaction *R* of the support** on the particle as a function of its elevation $z = rcos\theta$ at any given moment, and the parameters *m*, *r*, *g*, v_0, and Z_0.

b- Show that if $\boldsymbol{v_0 > V}$, the particle leaves the sphere right from the start at S_0. **Determine *V*.**
Note: $g = 10\, m/s^2$, *r*=90cm, $\theta_0 = 0$.

c- Calculate the **path traveled** by the particle on the sphere if it is released at S_0 with a velocity $\boldsymbol{v_0 = \frac{V}{2}}$.



$$\sum \vec{F}_{ext} = m\vec{a} \rightarrow \ m\vec{g} + \vec{R} = m(\vec{a}_t + \vec{a}_n)$$

$$\begin{cases} \vec{n} : mg\cos\theta - R = ma_n = m\dfrac{v^2}{r} & \rightarrow R = m\left(g\cos\theta - \dfrac{v^2}{r}\right) \\ \vec{\tau} : mg\sin\theta = ma_t = m\dfrac{dv}{dt} \end{cases}$$

$$vdv = a_t\, ds = g\sin\theta\, rd\theta$$

$$\int_{v_0}^{v} vdv = \int_{\theta_0}^{\theta} g\, r\sin\theta\, d\theta$$

$$\frac{1}{2}(v^2 - v_0^2) = -gr(\cos\theta - \cos\theta_0)$$

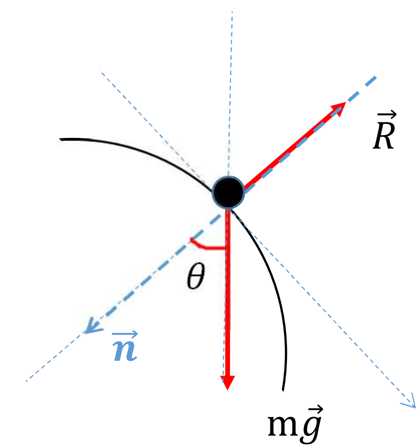
We have to eliminate θ

$$\cos\theta = z/r\,, \cos\theta_0 = z_0/r\,,$$

$$R = m\left[g\frac{z}{r} - \frac{1}{r}\left(-2gr\left(\frac{z}{r} - \frac{z_0}{r}\right) + v_0^2\right)\right]$$

$$R = \frac{3mgz}{r} - \frac{2mgz_0}{r} - \frac{mv_0^2}{r}$$

$$R = \frac{m}{r}(3gz - 2gz_0 - v_0^2)$$



$$R = \frac{m}{r}(3gz - 2gz_0 - v_0^2)$$

b) At the beginning $at\ s_0, z = z_0$ $R_0 = \frac{m}{r}(gz_0 - v_0^2)$,

The particle remains in contact with the surface of the sphere when $R > 0$;
The **particle escapes** from the surface if $R \leq 0$,

$$R_0 = \frac{m}{r}(gz_0 - v_0^2) \leq 0$$

$$gz_0 - v_0^2 \leq 0$$

$$gz_0 \leq v_0^2$$

$$v_0^2 \geq gz_0 = V^2$$

So for $v_0 \geq \sqrt{gz_0}$ the particle leaves the sphere at the beginning at S_0 . The lower limit of v_0 is

$$V = \sqrt{gz_0} = 3\, m/s$$

c) Path travelled before leaving = Arc S_0S_1

The traveled distance is: Arc $S_0S_1 = r(\theta_1 - \theta_0)$

At point S_1 ($z = z_1$), the particle leaves the sphere if $R = 0$

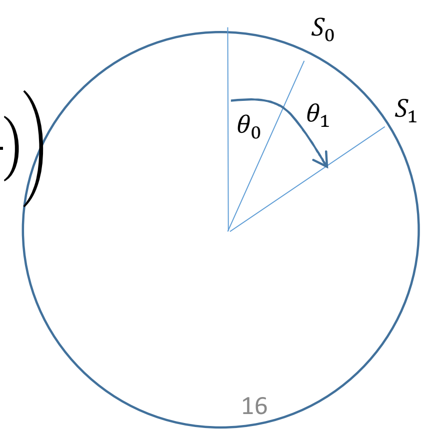
If $v_0 = \frac{V}{2} = \frac{\sqrt{gz_0}}{2}$, $\quad R = \frac{3mgz_1}{r} - \frac{2mgz_0}{r} - \frac{mgz_0}{4r}$

$$R = \frac{3mg}{r}\left(z_1 - \frac{3}{4}z_0\right) = 0$$

$$z_1 = \frac{3z_0}{4} = rcos\theta_1 \rightarrow \theta_1 = cos^{-1}\left(\frac{3z_0}{4r}\right)$$

The traveled distance is
Arc $S_0S_1 = r(\theta_1 - \theta_0)$

$$= r\left(cos^{-1}\left(\frac{3z_0}{4r}\right) - cos^{-1}\left(\frac{z_0}{r}\right)\right)$$



2.6 *Two–body problem and space dynamics*

In this paragraph, we study the most important applications of classical mechanics: the motion of an object subjected to a gravitational force proportional to $1/r^2$; the explanation of **planetary motion** and Kepler's laws, and the study of the motion of ballistic missiles, satellites, and interplanetary probes.

2-6-1- The law of universal gravitation

Two arbitrary particles with masses m_1 and m_2, separated by a distance r, exert on each other an attractive force acting along the line connecting them, with a magnitude given by: $F = G\frac{m_1 m_2}{r^2}$ where G represents a universal constant. In the MKSA system: G = 6,67.10^{-11} Nm^2Kg^{-2}

Properties :

1- The vector $\vec{h} = \overrightarrow{OP} \wedge \vec{v}$ (which is referred to as the areal velocity) is a constant of motion.

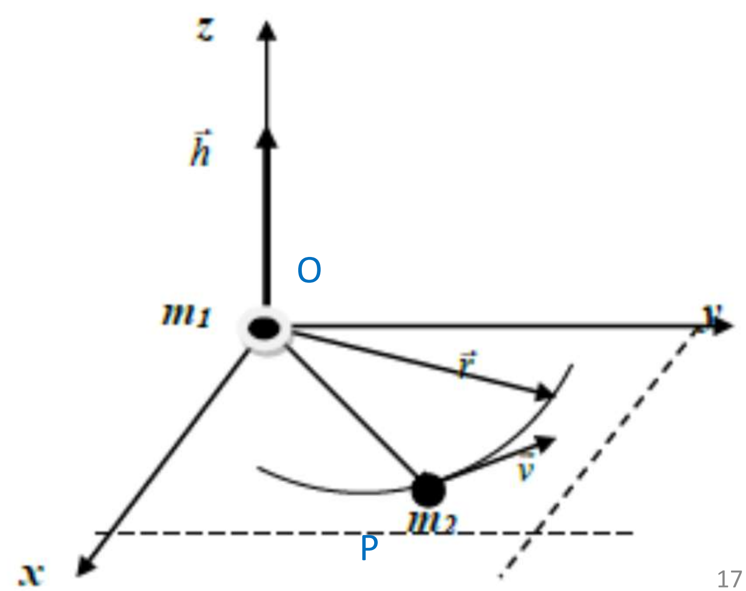
$$h = \frac{area}{time} \rightarrow speed!$$

$$\vec{h} = \overrightarrow{OP} \wedge \vec{v} = r\vec{e}_r \wedge (r'\vec{e}_r + r\theta'\vec{e}_\theta) = r^2\theta'\vec{k} = \overrightarrow{cst}$$

$$\frac{d\vec{h}}{dt} = \frac{d\overrightarrow{OP}}{dt} \wedge \vec{v} + \overrightarrow{OP} \wedge \frac{d\vec{v}}{dt} = \vec{v} \wedge \vec{v} + \overrightarrow{OP} \wedge \frac{d\vec{v}}{dt} = 0$$

$$\vec{h} = \overrightarrow{cst}$$

2- The motion occurs in a plane passing through O. Indeed, since $\vec{h}$ is constant, $\overrightarrow{OP}$ remains perpendicular to a fixed direction during the motion, and thus lies in a plane. To study this motion, it is most appropriate to switch to polar coordinates $(\vec{e}_r, \vec{e}_\theta)$.



3- The motion follows the law of areas (Kepler's second law), meaning that the radius vector sweeps out equal areas in equal intervals of time.

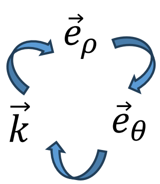
$$\vec{h} = \overrightarrow{OP} \wedge \vec{v} = r\vec{e}_r \wedge (r'\vec{e}_r + r\theta'\vec{e}_\theta)$$

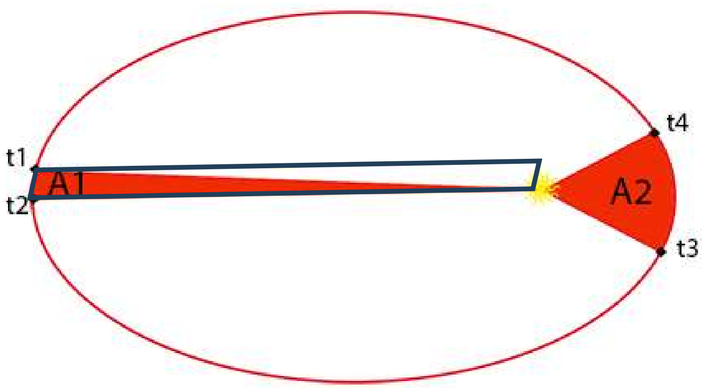
$$= r^2\theta'\vec{k}$$

On the other hand $\vec{h} = h\vec{k} = r^2\dfrac{d\theta}{dt}\vec{k}$

$$\boxed{hdt = r^2 d\theta}$$

Orthonormal base:





t2-t1=t4-t3 A2=A1

$$dA = \frac{1}{2}r.rd\theta = \frac{1}{2}r^2 d\theta$$

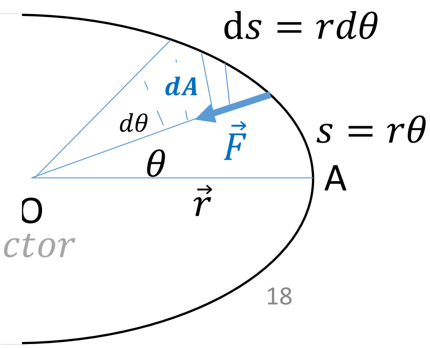
$$dA = \frac{1}{2}hdt \rightarrow \frac{dA}{dt} = \frac{1}{2}h = cste$$

4- **The total energy and angular momentum of the particle are conserved:**

We say that the central force (conservative) $\vec{f} = -\frac{C}{r^2}\vec{e}_r$ applied to the particle m derives from a potential $U(r)$, or in other words, the integral of this force gives a scalar function called potential energy, such that:

$$U(r) = -\int \vec{f}(r)d\vec{r} + k \rightarrow U(r) = -\int -\frac{C}{r^2}\vec{e}_r\, d\vec{r} + k$$

$$= C\int \frac{dr}{r^2} + k = -\frac{C}{r} + k$$



$d\vec{r}$: *elementary displacement vector*

$$U(r) = -\frac{C}{r} + k$$

We assume that $U(r) \to 0 \; when \; r \to \infty, so \; k = 0$

$C = Gm_1m_2 > 0$, and the potential is attractive

The **fundamental relation of dynamics** on the particule m :

$$m(r'' - r\theta'^2)\vec{e}_r + m(r\theta'' + 2r'\theta')\vec{e}_\theta = -\frac{C}{r^2}\vec{e}_r$$

$$\begin{cases} m(r'' - r\theta'^2) = -\frac{C}{r^2} & \times \boldsymbol{r'} \\ m(r\theta'' + 2r'\theta') = 0 & \times \boldsymbol{r\theta'} \end{cases}$$

2 differential eq. of motion

$$m(\boldsymbol{r'}r'' - r\boldsymbol{r'}\theta'^2 + \boldsymbol{r\theta'}\,r\theta'' + 2\boldsymbol{r\theta'}\,r'\theta') = -\frac{C}{r^2}\boldsymbol{r'}$$

$$m(r'r'' + rr'\theta'^2 + r^2\theta'\,\theta'') = -\frac{C}{r^2}r'$$

$$m(r'r'' + rr'\theta'^2 + r^2\theta'\,\theta'') + \frac{C}{r^2}r' = 0$$

The integration of this equation gives:

$$\frac{1}{2}m(r'^2 + r^2\theta'^2) - \frac{C}{r} = cte \to \; E_c + U(r) = E_m = cst$$

this represents the law of **conservation of total energy.**

<u>The angular momentum is conserved</u>

$$\vec{J} = \vec{r} \wedge m\vec{v} = m\,\vec{r} \,\wedge \vec{v} = m\,\vec{h} = \overrightarrow{cst}$$

2-6-2- Trajectories or Orbits

We will start with the 2 diff eq of motion obtained from the FRD

$$m(r'' - r\theta'^2) = -\frac{C}{r^2} \quad 1$$

$$m(r\theta'' + 2r'\theta') = 0 \rightarrow \frac{1}{r}\frac{d}{dt}(r^2\theta') = 0; \quad r^2\theta' = h = cst$$

Eq involving $r'', r', r, \theta'', \theta'$, all changing with time! We need to find another form of this eq → change of variables

$\boldsymbol{Let\ r = \frac{1}{u}}$ where r and u are functions of θ and t

$$\frac{\boldsymbol{dr}}{\boldsymbol{dt}} = \frac{dr}{du}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\theta'\frac{du}{d\theta}$$

$$= -r^2\theta'\frac{du}{d\theta} = -h\frac{du}{d\theta}$$

$$\frac{\boldsymbol{d^2r}}{\boldsymbol{dt^2}} = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\frac{d^2u}{dtd\theta} \times \frac{d\theta}{d\theta} = -h\theta'\frac{d^2u}{d\theta^2}$$

$$\boxed{r'' = -h^2u^2\frac{d^2u}{d\theta^2}}$$

$$1\ m(r'' - r\theta'^2) = -\frac{C}{r^2} \rightarrow m\left(-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3\right) = -Cu^2$$

$$-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = \frac{-Cu^2}{m} = \frac{-GMmu^2}{m} = -GMu^2$$

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}}$$

It is a second-order differential equation with a non-homogeneous term, whose **solution** is the sum of a particular solution ($\frac{GM}{h^2}$) with a solution of the equation without a non-homogeneous term ($A\cos\theta$).

$$u(t) = A\cos\theta(t) + \frac{GM}{h^2} \rightarrow \boxed{\frac{1}{r} = \frac{GM}{h^2}\left[1 + \frac{Ah^2}{GM}\cos\theta\right]}$$

General eq of a conic: $\frac{1}{r} = \frac{1}{ep} + \frac{\cos\theta}{p} = \frac{1}{ep}[1 + e\cos\theta]$

Let $e = \frac{Ah^2}{GM}$ is the eccentricity of the conic. Four cases are distinguished based on the value of the eccentricity e:

$0 < e < 1$	Ellipse	$e = 0$	Circle
$e = 1$	parabola	$e > 1$	Hyperbola

Calculation of h :

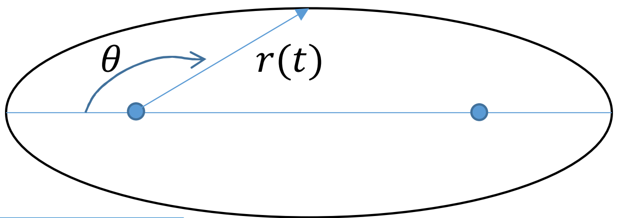
In polar coordinates, the velocity is given by $\vec{v} = r'\vec{e}_r + r\theta'\vec{e}_\theta$. At the initial moment, the radial velocity is zero; only the orthoradial component remains. . $\vec{v}_0 = r_0\theta'\vec{e}_\theta \rightarrow v_0 = r_0\theta'$

$$h = r_0^2\theta' = \frac{r_0^2 v}{r} = v_0 r_0$$

$\vec{v} = \vec{\omega} \wedge \vec{r}$

To determine the constant A, we use the equation at initial conditions

$$\frac{1}{r} = A\cos\theta(t) + \frac{GM}{h^2}$$



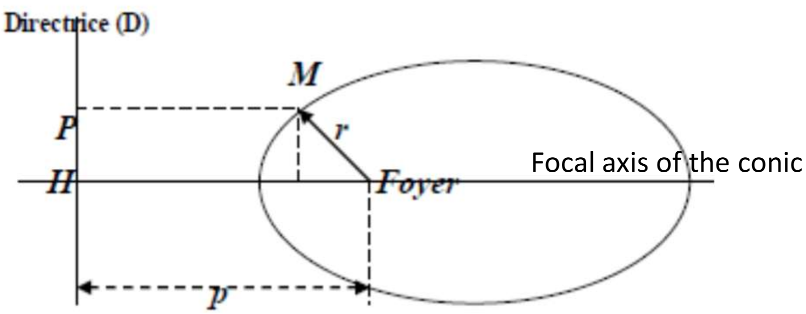
Let $r = r_0$ for $\theta = 0$

$$\frac{1}{r_0} = A + \frac{GM}{h^2} \rightarrow \boxed{A = \frac{1}{r_0} - \frac{GM}{h^2}}$$

The equation of the orbit becomes :

$$\boxed{\frac{1}{r} = \frac{1}{r_0}\left[1 - \frac{GM}{r_0 {v_0}^2}\right]\cos\theta + \frac{GM}{{v_0}^2{r_0}^2}}$$

<u>Qu'est ce qu'une conique ?</u>



Parameter p = cst

The number **e** is the excentricity of the conic

$$e = \frac{MF}{MP} = \frac{r}{p - r\cos\theta} \rightarrow r = e(p - r\cos\theta) \rightarrow r(1 + e\cos\theta) = ep$$

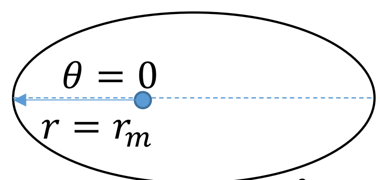
$$\frac{1}{r} = \frac{1 + e\cos\theta}{ep} = \frac{1}{ep} + \frac{\cos\theta}{p}$$

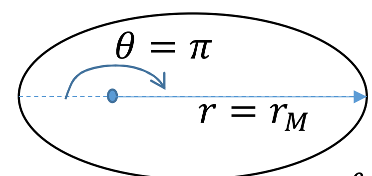
By comparison with $\frac{1}{r} = A\cos\theta(t) + \frac{GM}{h^2}$

We can deduce that $\boxed{p = \frac{1}{A}}$ and $\boxed{e = \frac{Ah^2}{GM}}$

<u>Case of an elliptic trajectory :</u>

$$r = \frac{e\,p}{1 + e\cos\theta} = \frac{\frac{Ah^2}{GM}\cdot\frac{1}{A}}{1 + e\cos\theta} = \frac{\ell_0}{1 + e\cos\theta}; \qquad \ell_0 = \frac{h^2}{GM} = cst$$





$\theta = 0;$ $r_m = \frac{\ell_0}{1 + e}$

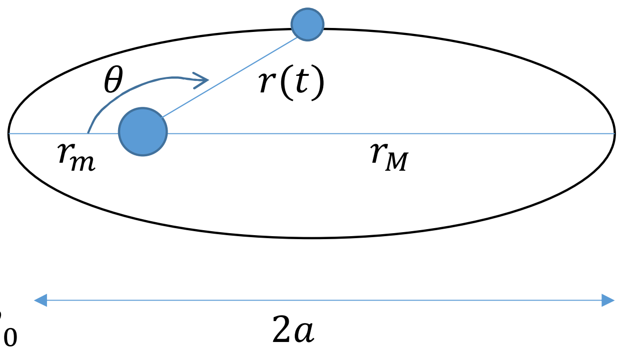
$\theta = \pi;$ $r_M = \frac{\ell_0}{1 - e}$

perigee

apogee

$$\frac{r_M}{r_m} = \frac{1+e}{1-e}$$

$$2a = r_m + r_M$$



$$a = \frac{r_m + r_M}{2} = \frac{\ell_0}{1-e^2}$$

Relation between excentricity and total energy of a particle :

We admit that the total energy is $\boldsymbol{E = -\frac{GMm}{2a}}$

$$\boldsymbol{E = \frac{(e^2-1)GMm}{2\ell_0}}$$

Escape velocity and velocity on a circular orbit:

To determine the escape velocity, we apply the conservation of total energy between two positions:

initial (r, v_0) and final $(\infty$,v)

$$\frac{1}{2}mv_0^2 - \frac{GMm}{r} \qquad\qquad \frac{1}{2}m\boldsymbol{v}^2 - \frac{GMm}{\infty}$$

$$\frac{1}{2}mv_0^2 - \frac{GMm}{r} = \frac{1}{2}m\boldsymbol{v}^2 - \frac{GMm}{\infty} \rightarrow \qquad v_0^2 = \boldsymbol{v}^2 + \frac{2GM}{r_0}$$

The minimum value of the initial velocity is called the escape velocity and corresponds to v=0 (when it reaches space).

$$\boxed{\boldsymbol{v_\ell = \sqrt{\frac{2GM}{R}}}}$$

$R: Radius\ of\ the\ planet$

To determine the velocity on a circular trajectory, we apply the fundamental relation of dynamics in polar coordinates (the orbital velocity).

$$\vec{F} = m\vec{a} \rightarrow \frac{GMm}{{r_0}^2} = ma_n = m\frac{v_0^2}{r_0} \rightarrow \boxed{\boldsymbol{v = \sqrt{\frac{GM}{r_0}}}}$$

$(r_0 = R + h)$

If the trajectory is closed, the period can be evaluated using the law of areas:

$$\boxed{\boldsymbol{T = 2\pi a^{\frac{3}{2}}\sqrt{\frac{1}{GM}}}}$$

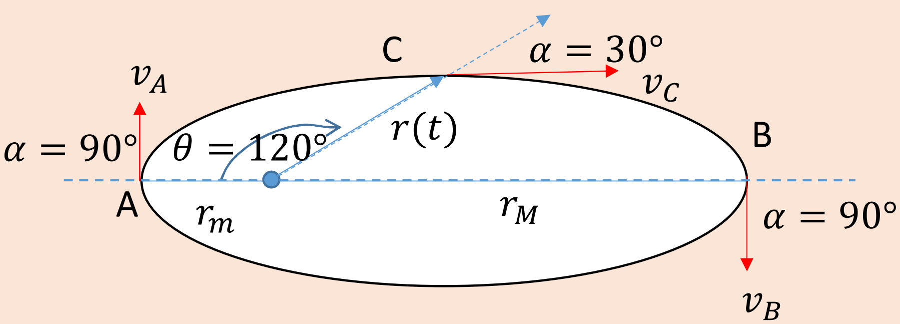
Summary:

$$\left.\begin{aligned} m(r'' - r\theta'^2) &= -\frac{C}{r^2} \\ m(r\theta'' + 2r'\theta') &= 0 \end{aligned}\right\} \rightarrow E_t = cst\ (conservation)$$

$$\downarrow$$

$$\frac{1}{r}\frac{d}{dt}(m\, r^2\theta') = 0 \rightarrow m\, r^2\theta' = cte \rightarrow mrv = cte \qquad \boldsymbol{\theta' = v/r}$$

$\vec{j} = m\,\vec{r} \wedge \vec{v} = mrv \sin\alpha\, \hat{k}$ (conservation of the angular momentum



$$\boldsymbol{E_t(A) = E_t(B) = E_t(C)}$$

$$\frac{1}{2}mv_A^2 - \frac{GMm}{r_m} = \frac{1}{2}mv_B^2 - \frac{GMm}{r_M} = \frac{1}{2}mv_C^2 - \frac{GMm}{r_C}$$

$$= E_t(ellipse) = -\frac{\boldsymbol{GMm}}{\boldsymbol{2a}} = -\frac{GMm}{r_m + r_M}$$

$$\boldsymbol{J(A) = J(B) = J(C)}$$

$$\boldsymbol{\vec{j} = m\,\vec{r} \wedge \vec{v} = mrv \sin\alpha\, \hat{k} = \overrightarrow{cst}}$$

$$\boldsymbol{mr_m v_A \sin 90° = mr_M v_B \sin 90° = mr_C v_C \sin 30°}$$

$$r = \frac{ep}{1 + e\cos\theta} = \frac{\ell_0}{1 + e\cos\theta}; \qquad \ell_0 = \frac{h^2}{GM}$$

$$\frac{r_M}{r_m} = \frac{1+e}{1-e} \qquad \boxed{h = v_0 r_0 = v_A r_m}$$

$$\boxed{r_M = \frac{r_0}{\left(\frac{2GM}{r_0 v_0^2}\right) - 1}}$$

<u>**Escape speed**</u>

$$\boxed{\boldsymbol{v_\ell} = \sqrt{\frac{\boldsymbol{2GM}}{R}}}$$

<u>**Circular orbital speed**</u>

$$\boxed{\boldsymbol{v} = \sqrt{\frac{\boldsymbol{GM}}{r}}} \qquad \boxed{r = R + h}$$

<u>**Period**</u>

$$\boxed{\boldsymbol{T = 2\pi a^{\frac{3}{2}}\sqrt{\frac{1}{GM}}}}$$

Application II-10-

An artificial satellite, with mass m=1 tonne, orbits the Earth, which has a mass M=6.1024×10^{24} kg, on a flat and circular orbit with the center of the Earth as its center and an altitude h=800 km. The Earth's radius is given as R=6400 km, the acceleration due to gravity at the surface is $g_0 = 10\ m/s^2$ and G= 0.66×10^{-10} SI. Determine:

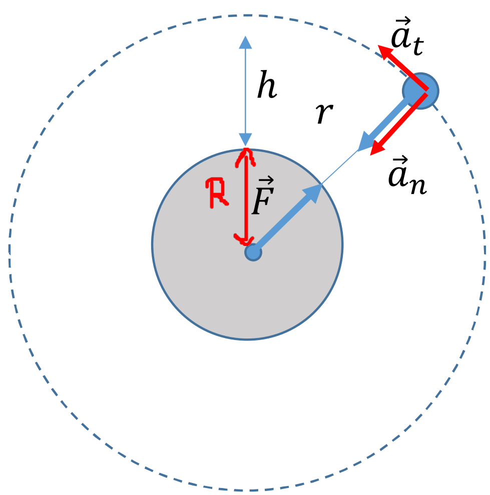
a- The potential, kinetic, and total energies of the satellite.

b- Its rotation period around the Earth.

c- The change in speed required to shift its orbit to an ellipse characterized by r_m =7200 km and r_M =8000 km.

Solution

a- Potential, kinetic, and total energies of the satellite?



$$U = -\frac{GMm}{r} = -\frac{GMm}{R+h}$$

$$U = -55.10^9 J$$

$$\boldsymbol{v} = \sqrt{\frac{\boldsymbol{GM}}{r_0}} = \sqrt{\frac{\boldsymbol{GM}}{R+h}}$$

$$KE = \frac{1}{2} m\boldsymbol{v}^2 = \mathbf{27.5}\ \times 10^9 J$$

$$E_{total} = U + KE = -\frac{GMm}{R+h} + \frac{1}{2}\frac{GMm}{R+h} = -\frac{\boldsymbol{GMm}}{\mathbf{2}(\boldsymbol{R}+\boldsymbol{h})} < \mathbf{0}$$

b- Its period of rotation around the Earth.

$$\boldsymbol{T} = \frac{\mathbf{2}\boldsymbol{\pi}(\boldsymbol{R}+\boldsymbol{h})}{\boldsymbol{v}} = \frac{\mathbf{2}\boldsymbol{\pi}(\boldsymbol{R}+\boldsymbol{h})}{\sqrt{\frac{\boldsymbol{GM}}{R+h}}} = \frac{\mathbf{2}\boldsymbol{\pi}(\boldsymbol{R}+\boldsymbol{h})^{\frac{3}{2}}}{\boldsymbol{R}\sqrt{\boldsymbol{g_0}}} = \mathbf{1h\ 40min}$$

c- The change in velocity required to alter its orbit.

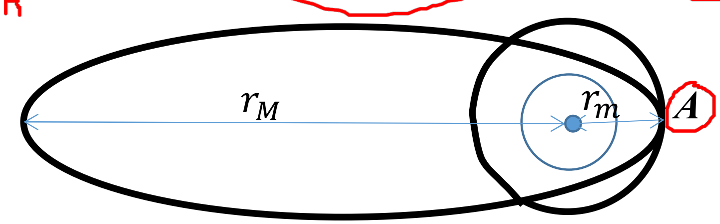
Calculation of the velocity on the ellipse at point A using total energy.

$$\frac{1}{2} m v_{2_A}^2 - \frac{GMm}{\underset{=r_A}{r_m}} = -\frac{GMm}{r_m + r_M} \rightarrow \quad v_2^2 = 2GM\left(\frac{1}{r_m} - \frac{1}{r_m + r_M}\right)$$

$v_2 = 7603.7\ m/s$

A^{ll}

$$\Delta v_A = v_{A,ellipse} - v_{A,circle} = 190\ m/s$$



2nd method

$$r_M = \frac{r_m}{\left(\frac{2GM}{r_m v_A^2}\right) - 1}$$ *where* v_A is the speed of the ellipse

ellip

3rd meth:

Variation of the total energy between the ellipse and the circle at point A.:

$E_{T,C}$ - $E_{T,E}$ =

$$\left[\frac{1}{2}mv_c^2 - \frac{GMm}{r_m}\right] - \left[\frac{1}{2}mv_e^2 - \frac{GMm}{r_m}\right] = -\frac{GMm}{2(R+h)} - \left(-\frac{GMm}{r_m + r_M}\right)$$

$$v_e^2 = v_c^2 + \frac{GM}{(R+h)} - \frac{2GM}{r_m + r_M} = 2GM\left(\frac{1}{r_m} - \frac{1}{r_m + r_M}\right)$$

$$\Delta v \sim 190\ m/s$$

$$r_M = \frac{r_m}{\left(\frac{2GM}{r_m v_A^2}\right) - 1}$$

where v_A is the speed of the ellipse

Variation of the total energy between the ellipse and the circle at point A.:

$$\left[\frac{1}{2}mv_c^2 - \frac{GMm}{r_m}\right] - \left[\frac{1}{2}mv_e^2 - \frac{GMm}{r_m}\right] = -\frac{GMm}{2(R+h)} - \left(-\frac{GMm}{r_m + r_M}\right)$$

$$v_e^2 = v_c^2 + \frac{GM}{(R+h)} - \frac{2GM}{r_m + r_M} = 2GM\left(\frac{1}{r_m} - \frac{1}{r_m + r_M}\right) \Delta v = 187{,}5\ m/s$$

2-7 - Real Forces and Fictitious forces

If a traveler is sitting in a stationary car or in a moving car at a constant speed in a straight line, he does not feel any unusual force and remain at rest in their seat. But when the car decelerates, they feel a force pushing them forward. This sensation is due to the acceleration of the reference frame linked to the car and is manifested by a fictitious force that combines with the real forces to maintain stability and motion.

$$\vec{a}_a = \vec{a}_d + \vec{a}_r + \vec{a}_c$$

$$\sum \vec{F}_{real} = m\vec{a}_a = m\vec{a}_d + m\vec{a}_r + m\vec{a}_c = \vec{f}_d + \vec{f}_r + \vec{f}_c$$

$$\boxed{\vec{f}_r = \sum \vec{F}_{real} - \vec{f}_e - \vec{f}_c}$$

This expression tells us: for an observer linked to the accelerated reference frame to apply the fundamental relation of dynamics, he must add to the real force $\vec{F}_{real}$ the termes $(-m\,\vec{a}_d)$ and $(-m\,\vec{a}_c)$.