The Riemann Zeta Function

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Introduction

Arithmetic, a fundamental branch of mathematics, plays a crucial role in our modern world. It is at the heart of many technologies, from cryptography that secures our digital communications to the optimization of algorithms used in artificial intelligence.

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Thanks to the fundamental theorem of arithmetic, prime numbers are crucial in arithmetic as they serve as the building blocks for the unique factorization of integers. They also play a pivotal role in areas like cryptography, where their indivisibility is leveraged to secure systems.

Introduction

In particular, understanding how these numbers are distributed would strengthen our knowledge in the field.

To study their distribution we introduce, for all $x \geq 2$, the prime counting function,

$$\pi(x) := \mathrm{card}\left(\{p \in \mathcal{P} \mid p \leq x\}\right).$$

We particularly have the prime number theorem, which gives,

$$\pi(x) \underset{x \to +\infty}{\sim} \frac{x}{\ln(x)}.$$

The approximation error is : $O\left(\frac{x}{\ln^2(x)}\right)$.

This approximation can be refined, for example, with the relation,

$$\pi(x) \underset{x \to +\infty}{\sim} \mathrm{li}(x),$$

where

$$\mathrm{li}(x) := \int_0^x \frac{\mathrm{d}t}{\ln(t)}.$$

Here the error is : $O\left(x \exp\left(-\sqrt{\frac{\ln(x)}{69}}\right)\right)$.

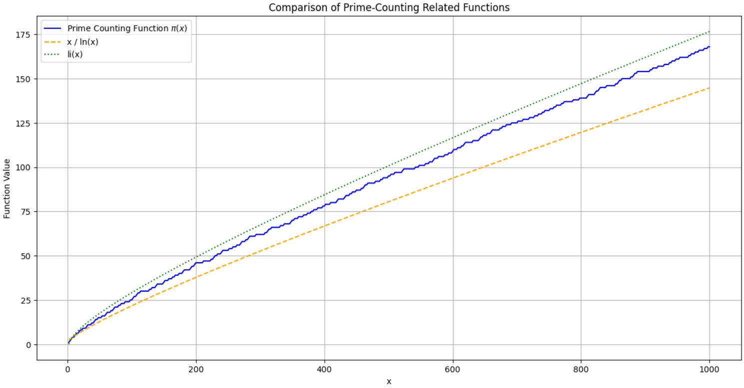

Figure: Approximations of π.

We currently have the possibility to **significantly improve these approximations** thanks to the **zeros** of a function called ζ, though the exact knowledge of these **zeros** still eludes us.

But what is the function ζ and how does it relate to prime numbers ?

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Domain of convergence of the Riemann series.

Let $s \in \mathbb{R}$. Thanks to series-integral comparisons, we know that the Riemann series, $\sum \frac{1}{n^s}$, converges if, and only if, $s > 1$.

Now let $s \in \mathbb{C}$. Then the Riemann series $\sum \frac{1}{n^s}$ converges absolutely if $\mathrm{Re}(s) > 1$, since $\left|\frac{1}{n^s}\right| = \frac{1}{n^{\mathrm{Re}(s)}}$, for all $n \geq 1$.

Definition 2.1

For all $s \in \{\mathrm{Re}(.) > 1\}$, we define :

$$\zeta(s) := \sum_{n=1}^{+\infty} \frac{1}{n^s}.$$

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A First Analytical Study of ζ on $\{\mathrm{Re}(.) > 1\}$.

Holomorphy is a very **significant property** in complex analysis. This condition is much stronger than real differentiability. It implies that the function is analytic: it is **infinitely differentiable** and, in the neighborhood of any point in the open set, is equal to the sum of its **Taylor series**.

It can be proven that ζ is holomorphic on $\{\mathrm{Re}(.) > 1\}$ through the **holomorphy theorem under the integral sign**, which is the complex analogue of the derivative theorem under the integral sign in the real case.

Theorem 2.2

Let $(X, \mathcal{B}, \mu)$ be a measured space. Let U be an open set in $\mathbb{C}$ and $f : U \times X \to \mathbb{C}$. Suppose that :

- for every $z \in U$, the mapping $x \mapsto f(z,x)$ is measurable;
- for almost every $x \in X$, the mapping $z \mapsto f(z,x)$ is holomorphic, with derivative denoted by $\frac{\partial f}{\partial z}$;
- there exists a measurable function $\varphi : X \to \mathbb{R}^+$ such that $\int\limits_X \varphi \, d\mu < +\infty$ and, for all $z \in U$ and almost every $x \in X$,

$$|f(z,x)| \leq \varphi(x).$$

Then the function $F : z \mapsto \int\limits_X f(z,x)\, d\mu(x)$ is holomorphic on U and, for all $z \in U$, the function $x \mapsto \frac{\partial f}{\partial z}(z,x)$ is integrable, and,

$$F'(z) = \int_X \frac{\partial f}{\partial z}(z,x)\, d\mu(x).$$

A sum being an integral over the natural integers with the counting measure μ on $\mathbb{N}$, we can thus apply Theorem 2.2 to ζ.

Theorem 2.3

ζ is holomorphic on $\{\mathrm{Re}(.) > 1\}$ and for all $s \in \{\mathrm{Re}(.) > 1\}$, for all $k \geq 1$:

$$\zeta^{(k)}(s) = (-1)^k \sum_{n=2}^{+\infty} \frac{(\ln(n))^k}{n^s}.$$

In particular ζ is of class $C^{+\infty}$.

Remark 2.3

The fact that ζ is holomorphic on $\{\mathrm{Re}(.) > 1\}$ is a fundamental property for our study. Indeed, we will later see that holomorphic functions are **analytically extendable**. It is thanks to this extension that we will be able to relate the zeros of ζ with π.

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Connection Between Riemann's ζ Function and Euler's Γ Function.

To be able to extend ζ, we will need a relation satisfied by Γ.

Let $s \in \mathbb{C}$. Since the function $t \mapsto t^{s-1}e^{-t}$ is integrable over $]0, +\infty[$ when $\mathrm{Re}(s) > 0$, we can extend the definition of Γ to the domain $\{\mathrm{Re}(.) > 0\}$:

$$\Gamma(s) := \int_0^{+\infty} t^{s-1}e^{-t}\,dt \quad \text{for all} \quad s \in \{\mathrm{Re}(.) > 0\}.$$

Connection Between Riemann's ζ Function and Euler's Γ Function.

We define also the function φ, which for all $x \in]0, +\infty[$ is given by :

$$\varphi(x) := \sum_{n=1}^{+\infty} e^{-n^2 \pi x}.$$

Remark 2.4

The function φ satisfies, for all $x > 0$:

$$\varphi(x) = \frac{1}{\sqrt{x}} \varphi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2}.$$

Connection Between Riemann's ζ Function and Euler's Γ Function

Theorem 2.5

For all $s \in \{\mathrm{Re}(.) > 1\}$,

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)(s-1)} - \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)s} + \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \int_1^{+\infty} f(x,s)\,dx.$$

where $f(x,s) := \frac{x^{\frac{1-s}{2}} + x^{\frac{s}{2}}}{x} \varphi(x)$.

Remark 2.6

Γ possesses a great analytical richness. The analytical properties of ζ largely inherit those of Γ.

Connection Between Riemann's ζ Function and Euler's Γ Function

Proof:

Let $s \in \{\mathrm{Re}(.) > 1\}$. Then $\Gamma\left(\frac{s}{2}\right) = \int\limits_0^{+\infty} t^{\frac{s}{2}-1} e^{-t}\, dt$.

Let $n \geq 1$. We perform the change of variable $t = n^2 \pi x$. Then:

$$\Gamma\left(\frac{s}{2}\right) \frac{1}{n^s} \pi^{-\frac{s}{2}} = \int_0^{+\infty} x^{\frac{s}{2}-1} e^{-n^2 \pi x}\, dx.$$

By summing these equalities, we obtain:

$$\Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \sum_{n=1}^{+\infty} \int_0^{+\infty} x^{\frac{s}{2}-1} e^{-n^2 \pi x}\, dx.$$

By Beppo-Levi's theorem the function φ appears in the relation, and we have:

Connection Between Riemann's ζ Function and Euler's Γ Function

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_0^{+\infty} x^{\frac{s}{2}-1}\varphi(x)\,dx.$$

We will now apply the Chasles relation to the integral. We have:

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_0^{1} x^{\frac{s}{2}-1}\varphi(x)\,dx + \int_1^{+\infty} x^{\frac{s}{2}-1}\varphi(x)\,dx. \tag{1}$$

Let us define $I := \int\limits_0^1 x^{\frac{s}{2}-1}\varphi(x)\,dx$. By the functional equation, we have:

$$I = \int_0^1 x^{\frac{s-3}{2}}\varphi\left(\frac{1}{x}\right)\,dx + \frac{1}{2}\int_0^1 x^{\frac{s-3}{2}}\,dx - \frac{1}{2}\int_0^1 x^{\frac{s}{2}-1}\,dx.$$

Connection Between Riemann's ζ Function and Euler's Γ Function

For the first integral, we apply the change of variables $y = \frac{1}{x}$, and we obtain:

$$I = \int_1^{+\infty} \frac{\varphi(y)}{y^{\frac{s+1}{2}}}\, dy + \frac{1}{s-1} - \frac{1}{s}$$

Substituting I back into (1), we get:

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_1^{+\infty} \frac{x^{\frac{1-s}{2}} + x^{\frac{s}{2}}}{x}\varphi(x)\, dx + \frac{1}{s-1} - \frac{1}{s}.$$

Using $f(x,s) = \frac{x^{\frac{1-s}{2}} + x^{\frac{s}{2}}}{x}\varphi(x)$, we finally have:

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)(s-1)} - \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)s} + \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \int_1^{+\infty} f(x,s)\, dx \quad .\blacksquare$$

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Theory of Analytic Continuation

In complex analysis, an analytic continuation (or complex analytic continuation) is a method for extending the domain of definition of a holomorphic function beyond the region where it was initially defined. We can summarize the main issues as follows :
Let f be a function defined and holomorphic on an open domain $D \subset \mathbb{C}$. Now suppose we wish to **extend** f outside of D. If we can find a function g defined on a larger domain $D' \supset D$, such that :

- g is **holomorphic** on D',
- g **coincides** with f on D, that is, $g(z) = f(z)$ for all $z \in D$,

then we call g an **analytic continuation** of f to D'.

Theory of Analytic Continuation

Theorem 3.1

Let U be an open subset of $\mathbb{C}$, a a point in U, and a holomorphic function $f : U \to \mathbb{C}$. Suppose that U is connected. Then the following statements are equivalent:

- f is identically zero on U;
- f is identically zero in a neighborhood of a;

This theorem guarantees the **uniqueness of analytic continuation** through the following corollary.

Corollary 3.2

If f and g are two holomorphic functions on a connected open set $U \subset \mathbb{C}$ and if f and g coincide on a neighborhood of a point in U, then $f - g = 0$ on this neighborhood. By Theorem 2.6, we thus have $f - g = 0$ on U and hence $f = g$ on U.

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Thanks to the theorem of holomorphy under the integral sign, it can be shown that Γ is a holomorphic function on $\{\mathrm{Re}(.) > 0\}$. By analytic continuation, we have the following result :

Theorem 3.3

$\frac{1}{\Gamma}$ is a holomorphic function on $\mathbb{C}$, satisfying, for all $s \in \mathbb{C}$:

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{+\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}.$$

where γ denotes Euler's constant.

Thanks to this analytical property and the relationship between Γ and ζ, we will be able to extend ζ.

Analytical continuation of ζ thanks to Γ

Recall that for all $s \in \{\mathrm{Re}(.) > 1\}$:

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)(s-1)} - \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)s} + \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \int_1^{+\infty} f(x,s)\,dx.$$

Since $\frac{1}{\Gamma}$ is holomorphic on $\mathbb{C}$ and the integral is holomorphic on $\mathbb{C}$ (by holomorphy under the integral sign), we have:

Theorem 3.4

The function ζ extends to $\mathbb{C} \setminus \{1\}$.

Analytical continuation of ζ thanks to Γ

Remark 3.5

Note that this does not mean that the sum $\sum \frac{1}{n^s}$ is defined for $s \in \mathbb{C} \setminus \{1\}$! Rather, there exists a unique holomorphic function on $\mathbb{C} \setminus \{1\}$ that coincides with $\sum \frac{1}{n^s}$ on $\{\mathrm{Re}(.) > 1\}$. This function is also called ζ.

Thanks to this extension, we obtain a very interesting functional relation that allows us to compute new values of ζ from those that were known before the extension.

Theorem 3.6

For all $s \in \mathbb{C} \setminus \{0, 1\}$:

$$\zeta(s) = 2^s \pi^{s-1} \zeta(1-s) \Gamma(1-s) \sin\left(\frac{\pi s}{2}\right).$$

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A surprising connection between ζ and prime numbers

In the following, $\mathcal{P}$ will denote the set of prime numbers.
Leonhard Euler established in 1737 a connection between Riemann series and infinite products over prime numbers.

Theorem 4.1

For all $s \in \{\mathrm{Re}(.) > 1\}$:

$$\zeta(s) = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Then we have a **strong relation** between ζ and the **prime numbers**.

Theorem 4.2

For all $s \in \{\mathrm{Re}(.) > 1\}$, we have :

$$\log(\zeta(s)) = s \int_2^{+\infty} \frac{\pi(x)}{x\,(x^s - 1)}\, dx.$$

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Thanks to the Euler product and the functional equation, we can deduce the **zeros** of the ζ function.

Theorem 4.3

ζ does not vanish on $\{\mathrm{Re}(.) > 1\}$. However, it vanishes on $\{\mathrm{Re}(.) < 0\}$ only at the points $s = -2k$, where $k \in \mathbb{N}^*$. They are called trivial zeros.

Regarding the plane $\{0 < \mathrm{Re}(.) < 1\}$, we know that ζ vanishes there. However we do not currently know where the zeros are located. We only know that many of them have a real part of $\frac{1}{2}$. These zeros are referred to as **non-trivial**.

In fact the knowledge of non-trivial zeros is fundamental for the study of π.

Theorem 4.4

For all $x > 2$, we have:

$$\pi(x) = R(x) - \sum_{\rho} R(x^{\rho}),$$

where ρ runs over the set of **non-trivial zeros** of ζ with

$$R(x) := \sum_{n=1}^{+\infty} \frac{\mu(n)}{n} \operatorname{li}(x^{1/n}) \quad \text{where } \mu \text{ is the Möbius function.}$$

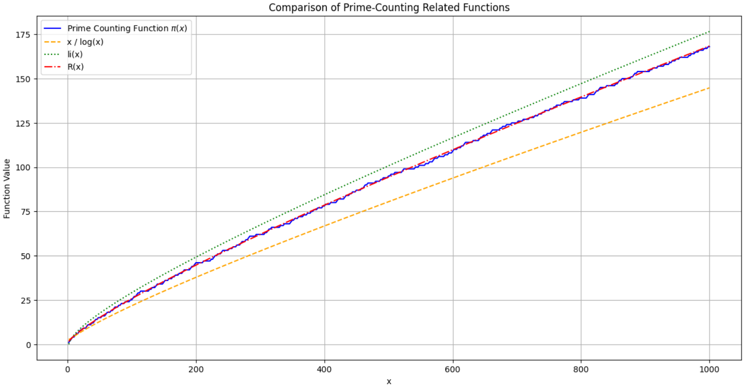

Figure: New approximations of π.

Theorem 4.4 actually provides one of the most powerful and precise expressions for approximating π. Indeed, the Riemann Hypothesis is equivalent to a much tighter bound on the error in the approximation of π, leading to a more regular distribution of prime numbers:

$$\pi(x) = \mathrm{li}(x) + O\left(\sqrt{x}\ln(x)\right).$$

Resolving the Riemann Hypothesis would not only merely refine what we already know; **it would pave the way for yet unforeseen questions**. By linking arithmetic problems to deep analytical structures, it would provide tools and perspectives to explore aspects of arithmetic in a more unified and coherent manner.

Thank you for your attention and for listening !