Thesis on the Riemann Zeta Function

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Context and Acknowledgments

This thesis is part of the *Project of Initiation to Research* (PIR MA) in the Applied Mathematics track at INSA Rennes, a program designed to introduce students to both theoretical and applied mathematical methods. The subject of this work focuses on the analysis of the Riemann Zeta Function, a central element in the field of analytic number theory.

I would like to express my sincere gratitude to my supervisor, Professor Olivier Ley, for his invaluable support, guidance, and insightful feedback throughout the development of this thesis. His expertise and encouragement were key to the completion of this project.
I also wish to acknowledge all those who contributed in various ways to this work and provided helpful suggestions and advice.

Notations

- log corresponds to the principal logarithm determination on $\mathbb{C} \setminus \mathbb{R}_-$.

- $\{\mathrm{Re}(.) > a\}$: Represents the set of $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > a$ for $a \in \mathbb{R}$. The meaning is preserved with the symbols $\geq$, $<$, and $\leq$.

- $\mathcal{D}_n$ represents the set of positive divisors of $n \in \mathbb{N}$.

- $\mathcal{P}$: Denotes the set of prime numbers.

- $\mathscr{P}(E)$: Refers to the set of partitions of E.

- $C(a, r)$: Denotes the circle with center $a \in \mathbb{C}$ and radius $r > 0$, oriented in the counterclockwise direction.

- $D(a, r)$: Represents the open disk with center $a \in \mathbb{C}$ and radius $r > 0$.

- $\overline{D(a, r)}$: Refers the closed disk with center $a \in \mathbb{C}$ and radius $r > 0$.

Introduction

The **Riemann zeta function**, denoted by ζ, is one of the most studied objects in mathematics, particularly in number theory and complex analysis. Initially introduced by Euler in the 18th century, it was later extended to the complex domain by Riemann in 1859. The aim of this thesis is to explore the fundamental properties of the ζ function, its fascinating connections with prime numbers, as well as its crucial role in one of the most famous conjectures in mathematics: the **Riemann Hypothesis**.

We will investigate its convergence, its analytic continuation, and its connection with number theory. The study will also include an analysis of its functional equation and its role in the prime number theorem. Finally, numerical methods for approximating and visualizing the function will be discussed.

This thesis is structured as follows:

- **Chapter 1** introduces the definition and basic properties of the zeta function in the real case, including explicit calculations such as $\zeta(2)$.
- **Chapter 2** extends the function to the complex plane, establishing its domain of convergence and proving its holomorphy. Several key functional relationships are explored.
- **Chapter 3** focuses on the deep connections between ζ and number theory, particularly through its Euler product representation and implications for prime numbers.
- **Chapter 4** presents numerical methods for computing and graphically representing the zeta function.
- **Appendices** provide additional results from complex analysis and special functions that are used throughout the thesis.

In this thesis, certain statements will be justified in the appendix. These will be marked with the symbol "(*)".
The references at the end of the document provide further reading for those interested in more advanced aspects of the topic.

1 Study of the ζ Function in the Real Context

We will first introduce and analyze the properties of the ζ function for real variables.

1.1 Definition of the ζ Function in the Real Case

Let $s \in \mathbb{R}$. We consider the following series, called the Riemann series, $\sum \frac{1}{n^s}$.

Definition-proposition 1: The series $\sum \frac{1}{n^s}$ converges if $s > 1$ and diverges otherwise. Therefore, we introduce a function, denoted by ζ, as follows:

$$\zeta(s) := \sum_{n=1}^{+\infty} \frac{1}{n^s} \quad , \text{for all } s > 1.$$

This is the Riemann ζ function.

Proof: Since the general term is strictly positive, we know that the sequence of partial sums of the series admits a limit. We now examine the values of s for which the series converges.

Let us first assume $s \leq 0$: In this case, we have:

$$\lim_{n \to +\infty} \frac{1}{n^s} \neq 0,$$

and therefore the series is clearly divergent.

Now, let us consider the case where $s > 0$: We proceed with a comparison method between series and integrals. For the following, note that the function $x \mapsto \frac{1}{x^s}$ is strictly decreasing on $]0, +\infty[$.

Let $n \geq 2$, $k \geq 1$, and $t \in [k, k+1]$. Since $k \leq t$, we have:

$$\frac{1}{k^s} \geq \frac{1}{t^s}, \quad \text{then} \quad \frac{1}{k^s} \geq \int_k^{k+1} \frac{dt}{t^s}.$$

Thus by summing we have:

$$\sum_{k=1}^{n} \frac{1}{k^s} \geq \sum_{k=1}^{n} \int_k^{k+1} \frac{dt}{t^s}, \quad \text{and hence} \quad \sum_{k=1}^{n} \frac{1}{k^s} \geq \int_1^{n+1} \frac{dt}{t^s}.$$

Similarly, if $n \geq 2$, $k \geq 2$, and $t \in [k-1, k]$, we find that:

$$1 + \int_1^n \frac{1}{t^s} \geq \sum_{k=1}^{n} \frac{1}{k^s}.$$

Finally, we obtain the following bounds:

$$\int_1^{n+1} \frac{dt}{t^s} \leq \sum_{k=1}^{n} \frac{1}{k^s} \leq 1 + \int_1^n \frac{dt}{t^s}.$$

Now, let us take $s \neq 1$. This can be rewritten as:

$$\frac{(n+1)^{1-s} - 1}{1-s} \leq \sum_{k=1}^{n} \frac{1}{k^s} \leq 1 + \frac{n^{1-s} - 1}{1-s}. \tag{1}$$

Thus, if $s < 1$, the series diverges, and if $s > 1$, it converges.
Finally, for $s = 1$, we have:

$$\ln(n+1) \leq \sum_{k=1}^{n} \frac{1}{k} \leq 1 + \ln(n).$$

Therefore, the series diverges.

Thus, the real domain of convergence of the series is $]1, +\infty[$. $\square$

1.2 Some Properties of the ζ Function in the Real Context

Often, computing the sums of series explicitly proves challenging, posing a significant obstacle in the study of the ζ function. This difficulty stems from the fact that, while some series, such as geometric series, admit explicit formulas, most cannot be calculated analytically. As a result, advanced analytical techniques or approximations are typically required for a deeper investigation.

1.2.1 A Calculation of $\zeta(2)$

To illustrate the previous statements, let us focus on the calculation of $\zeta(2)$. Mathematicians knew for a long time that the series was convergent, but its exact value remained unknown for many years. It was only in 1734 that Euler succeeded in proving that $\zeta(2) = \frac{\pi^2}{6}$, marking a significant breakthrough by linking this infinite sum to the properties of trigonometry.

To compute that, we will develop the Fourier series of the function f, which is 2π-periodic and even, such that for all $x \in [0, \pi]$, $f(x) = x^2$. The Fourier coefficients are as follows:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi} \int_0^{\pi} x^2\,dx = \frac{\pi^2}{3}.$$

Then, for all $n \in \mathbb{N}^*$, we have:

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,dx \\
&= \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx)\,dx \\
&= -\frac{2}{\pi} \int_0^{\pi} \frac{2x \sin(nx)}{n}\,dx \\
&= -\frac{4}{n\pi} \left(-\frac{\pi(-1)^n}{n} + \frac{1}{n} \int_0^{\pi} \cos(nx)\,dx \right) \\
&= \frac{4(-1)^n}{n^2}.
\end{aligned}$$

Noting that by parity, for all $n \in \mathbb{N}^*$, we have $b_n = 0$.
Finally, f is continuous and piecewise C^1 on $[0, \pi]$. Thus, according to Dirichlet's theorem, we have for all $x \in \mathbb{R}$:

$$f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2} \cos(nx).$$

By evaluating this equality at π, we obtain:

$$f(\pi) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{+\infty} \frac{1}{n^2} = \pi^2 \quad \text{that is to say} \quad \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

We have just proven that $\zeta(2) = \frac{\pi^2}{6}$.

Since the calculations are not straightforward, it is clear that having explicit formulas to directly compute the values of the function ζ at many points would be highly desirable.

1.2.2 Calculation of ζ on strictly positive even integers

There exists a formula to calculate $\zeta(2k)$ in general for all $k \geq 1$.
We will elaborate on this formula using the theory of Bernoulli polynomials.

(*)Definition-proposition 2: There exists a unique sequence of polynomials $(B_n)_{n\in\mathbb{N}}$ such that:

- $B_0 = 1$.
- For all $n \in \mathbb{N}$, $\quad B'_{n+1} = (n+1)B_n$.
- For all $n \in \mathbb{N}$, $\quad \int\limits_0^1 B_{n+1}(x)\,dx = 0$.

The terms of this sequence are called Bernoulli polynomials. The numbers $B_n := B_n(0)$ are called Bernoulli numbers.

It will be useful to note that the Bernoulli polynomials satisfy the following property.

(*)Proposition 3: For all $n > 1$: $\quad B_n(0) = B_n(1)$.

Now, we will prove the following relation.

Proposition 4: For all $k \geq 1$:

$$\zeta(2k) = B_{2k}\frac{2^{2k-1}\pi^{2k}}{(-1)^{k-1}(2k)!}.$$

Unfortunately, no formula has been discovered so far to compute ζ at positive odd integers.
However, when using this formula, we have:

$$\zeta(2) = \frac{\pi^2}{6} \approx 1.64493, \quad \zeta(4) = \frac{\pi^4}{90} \approx 1.08232, \quad \zeta(6) = \frac{\pi^6}{945} \approx 1.01734.$$

Thus, we observe that $\zeta(2) < \zeta(4) < \zeta(6)$. This illustrates a potential decrease in the ζ function. We will confirm the monotonicity of this function in the next section.

The following proof is inspired by the work in [7, p. 2].

Proof: We introduce a 2π-periodic function, denoted by f_k, defined on $\mathbb{R}$, such that $f_k(x) = B_{2k}\left(\frac{x}{2\pi}\right)$, for all $x \in [0, 2\pi[$ and for all $k \geq 1$. In the following, we observe that f_k is piecewise C^1 with respect on $[0, 2\pi[$. Moreover, $\lim\limits_{x \to 2\pi^+} f_k(x) = B_{2k}(0) = B_{2k}(1)$ and $\lim\limits_{x \to 2\pi^-} f_k(x) = B_{2k}(1) = B_{2k}(0)$, according to Proposition 3. Therefore, f_k is continuous on $\mathbb{R}$. In the following we denote the Fourier coefficients of f_k by $a_n(k)$ and $b_n(k)$. Hence, Dirichlet's theorem applies, and the Fourier coefficients are:

$$a_0(k) = \frac{1}{2\pi}\int_0^{2\pi} B_{2k}\left(\frac{x}{2\pi}\right) dx = \frac{1}{2\pi}\int_0^{2\pi} \frac{B'_{2k+1}\left(\frac{x}{2\pi}\right)}{2k+1}\, dx = \frac{\left[B_{2k+1}\left(\frac{x}{2\pi}\right)\right]_0^{2\pi}}{2k+1} = 0.$$

Given the construction of f_k, it is easy to see that it is an even function. This implies that $b_n(k) = 0$ for all $n \geq 1$. We thus determine $a_n(k)$ for all $n \geq 1$.
By performing the change of variable $t = \frac{x}{2\pi}$ and through double integration by parts, we have for all $n \in \mathbb{N}^*$:

$$\begin{aligned}
a_n(k) &= \frac{1}{\pi}\int_0^{2\pi} f_k(x)\cos(nx)\, dx \\
&= 2\int_0^1 B_{2k}(t)\cos(2\pi n t)\, dt \\
&= -2\int_0^1 2k(2k-1)\frac{B_{2k-2}(t)\cos(2\pi n t)}{(2\pi n)^2}\, dt \\
&= -2\frac{2k(2k-1)}{(2\pi n)^2}\int_0^1 B_{2k-2}(t)\cos(2\pi n t)\, dt \\
&= -\frac{2k(2k-1)}{(2\pi n)^2} a_n(k-1) \\
&\vdots \\
&= (-1)^{k-1}\frac{(2k)!}{(2\pi n)^{2k-2}} a_n(1).
\end{aligned}$$

We compute $a_n(1)$ using double integration by parts and the fact that $B_1 = X - \frac{1}{2}$. We have, for all $n \in \mathbb{N}^*$:

$$\begin{aligned}
a_n(1) &= 2\int_0^1 B_2(t)\cos(2\pi n t)\, dt \\
&= -\int_0^1 2B_1(t)\frac{\sin(2\pi n t)}{2\pi n}\, dt \\
&= -\left[\frac{-2B_1(t)\cos(2\pi n t)}{(2\pi n)^2}\right]_0^1 + 2\int_0^1 \frac{\cos(2\pi n t)}{(2\pi n)^2}\, dt \\
&= \frac{2}{(2\pi n)^2}.
\end{aligned}$$

Finally, we have:

$$a_n(k) = (-1)^{k-1}\frac{(2k)!}{(2\pi n)^{2k-2}}\frac{2}{(2\pi n)^2} = (-1)^{k-1}\frac{(2k)!}{2^{2k-1}\pi^{2k}n^{2k}}.$$

From this we have for all $x \in \mathbb{R}$:

$$f_k(x) = \sum_{n=1}^{+\infty}(-1)^{k-1}\frac{(2k)!}{2^{2k-1}\pi^{2k}n^{2k}}\cos(nx).$$

Thus, by evaluating this formula at 0, we obtain:

$$f_k(0) = \sum_{n=1}^{+\infty}\frac{(-1)^{k-1}(2k)!}{2^{2k-1}\pi^{2k}n^{2k}}.$$

Then:

$$B_{2k}\frac{2^{2k-1}\pi^{2k}}{(-1)^{k-1}(2k)!} = \sum_{n=1}^{+\infty}\frac{1}{n^{2k}}.$$

Finally:

$$\zeta(2k) = B_{2k}\frac{2^{2k-1}\pi^{2k}}{(-1)^{k-1}(2k)!}.$$

The proof ends here. □

After having examined the properties of ζ over natural numbers, we will therefore, as stated before, focus on its regularity, which then relates to more general properties. Indeed, understanding the global behavior of this function, particularly regarding its continuity and differentiability, offers new perspectives on its analysis. This exploration will allow us to better comprehend the subtleties of its structure.

1.2.3 Study of the Regularity of ζ on $]1, +\infty[$

In this section, we will primarily focus on the differentiability of ζ on $]1, +\infty[$, particularly utilizing the dominated convergence theorems.
The study of differentiability will allow us to conclude about the decrease of ζ.

Proposition 5: The function ζ is differentiable and decreasing on $]1, +\infty[$ with for all $s \in]1, +\infty[$:

$$\zeta'(s) = -\sum_{n=2}^{+\infty}\frac{\ln(n)}{n^s}.$$

Moreover, $\lim\limits_{s\to+\infty}\zeta(s) = 1$, $\lim\limits_{s\to 1^+}\zeta(s) = +\infty$ and $\zeta(s) \underset{s\to 1^+}{\sim} \frac{1}{s-1}$.

In Section 2.2, we will see that ζ has much stronger analytic properties, particularly due to its holomorphy.

Proof: Let $a > 1$, $s \in [a, +\infty[$, and $n \geq 1$. The function $s \mapsto \frac{1}{n^s}$ is clearly differentiable on $[a, +\infty[$, and its derivative is given by:

$$\frac{d}{ds}\left(\frac{1}{n^s}\right) = \frac{d}{ds}\left(e^{-s\ln(n)}\right) = -\ln(n)e^{-s\ln(n)} = -\frac{\ln(n)}{n^s}, \quad \text{for all } s \in [a, +\infty[.$$

Next, for all $n \geq 1$, for all $s \in [a, +\infty[$, we have:

$$\left|\frac{d}{ds}\left(\frac{1}{n^s}\right)\right| = \frac{\ln(n)}{n^s} \leq \frac{\ln(n)}{n^a}, \quad \text{since } s \geq a.$$

Let $b := \frac{a+1}{2} > 1$. Then:

$$n^b \frac{\ln(n)}{n^a} = \frac{\ln(n)}{n^{a-b}} = \frac{\ln(n)}{n^{\frac{a-1}{2}}}.$$

Now, since $\frac{a-1}{2} > 0$, by comparative growth, we have:

$$\lim_{n \to +\infty} \frac{\ln(n)}{n^{\frac{a-1}{2}}} = 0.$$

Thus, $\frac{\ln(n)}{n^a} = o\left(\frac{1}{n^b}\right)$, and since $b > 1$, we conclude that $\sum \frac{1}{n^b}$ converges. Therefore, by the negligible terms property, $\sum \frac{\ln(n)}{n^a}$ also converges.
From this, we can apply the dominated convergence theorem. Thus, ζ is differentiable on $[a, +\infty[$, and we have for all $s \in [a, +\infty[$:

$$\zeta'(s) = \frac{d}{ds}\left(\sum_{n=1}^{+\infty} \frac{1}{n^s}\right) = \sum_{n=1}^{+\infty} \frac{d}{ds}\left(\frac{1}{n^s}\right) = -\sum_{n=2}^{+\infty} \frac{\ln(n)}{n^s}.$$

This being verified for all $a > 1$, it implies, by generalization of differentiation over the interval, that it is also valid over $]1, +\infty[$.

Given the preceding expression, we also deduce that $\zeta'(s) \leq 0$ for all $s \in]1, +\infty[$, and thus ζ is decreasing on $]1, +\infty[$.

To complete our study, we will focus on the limits of ζ as 1 and $+\infty$.
Again, we introduce $a > 1$ and $s \in [a, +\infty[$ to place ourselves in a neighborhood of $+\infty$. We have:

$$\lim_{s \to +\infty} \frac{1}{n^s} = \begin{cases} 0 & \text{if } n \geq 2, \\ 1 & \text{if } n = 1. \end{cases}$$

Also, since $s \geq a$, we have, $\frac{1}{n^s} \leq \frac{1}{n^a}$ for all $n \geq 1$. Since $\sum \frac{1}{n^a}$ converges, by applying the Dominated Convergence Theorem, we obtain:

$$\lim_{s \to +\infty} \zeta(s) = \lim_{s \to +\infty} \left(\sum_{n=1}^{+\infty} \frac{1}{n^s}\right) = \sum_{n=1}^{+\infty} \lim_{s \to +\infty} \left(\frac{1}{n^s}\right) = 1.$$

Finally, we recall (1) which states that for all $s > 1$ and $n \geq 2$, we have the following inequality:

$$\frac{(n+1)^{1-s} - 1}{1-s} \leq \sum_{k=1}^{n} \frac{1}{k^s} \leq 1 + \frac{n^{1-s} - 1}{1-s}.$$

Taking the limit as $n \to +\infty$, we obtain the following bounds for all $s > 1$:

$$\frac{1}{s-1} \leq \zeta(s) \leq 1 + \frac{1}{s-1}.$$

Thus, we conclude that:

$$\lim_{s \to 1^+} \zeta(s) = +\infty.$$

Note that we also obtain an asymptotic expansion of ζ as s approaches 1^+:

$$\zeta(s) \underset{s \to 1^+}{\sim} \frac{1}{s-1}.$$

The proof ends here. $\square$

Through numerical methods, specifically using the acceleration method based on the binomial transform (see Section 4.1.2), we can graphically represent our results.
Numerical methods for approximating the ζ function on $]1, +\infty[$ are discussed in Section 4.1.

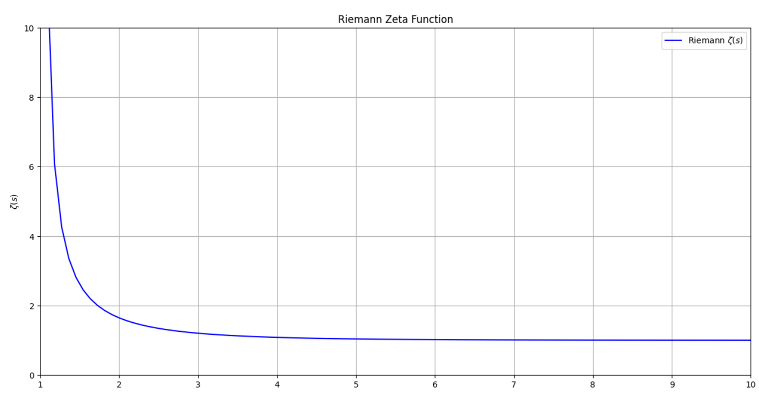


Figure 1: The function ζ on $]1, +\infty[$ from the binomial transform where $N = 500$.

N represents the number of terms used to approximate $\zeta(s)$ using the binomial transform method.

Although the study of the function ζ in the real case has revealed important properties, a deeper exploration requires approaching it within the complex framework, where new perspectives open up to us.
This will allow us to uncover an unexpected connection between the complex analysis of ζ and the arithmetic of prime numbers.

2 Extension of the ζ Function in the Complex Context

Let $x \in \mathbb{R}_+^*$ and $z \in \mathbb{C}$. We define, $x^z := e^{z\ln(x)}$. It is from this definition that we will be able to study the function ζ for variables that will now be complex numbers.

2.1 Domain of definition of the ζ Function in the Complex Context

In this section, we will explore whether it is possible to compute the sums of Riemannian series for complex variables. To do so, we have the following property.

Proposition 6: The domain of convergence of the Riemann series extends to and is limited to $\{\mathrm{Re}(.) > 1\}$.

Thus, similarly to the real case, ζ can be defined in the complex case.

We will use the following proposition to prove Proposition 6.

(*)Proposition 7: The domain of convergence of the Riemann integrals, $\int_1^{+\infty} \frac{dx}{x^s}$, extends to and is limited to the set $\{\mathrm{Re}(.) > 1\}$.

Now let's prove Proposition 6.

Proof: Let $s \in \mathbb{C}$.

<u>If $\mathrm{Re}(s) > 1$, then we have:</u>

$$\left|\frac{1}{n^s}\right| = \frac{1}{n^{\mathrm{Re}(s)}}.$$

Now, from the study conducted in the real case, we know that $\sum \frac{1}{n^{\mathrm{Re}(s)}}$ is convergent, since $\mathrm{Re}(s) > 1$. Thus, for $\mathrm{Re}(s) > 1$, $\sum \frac{1}{n^s}$ is absolutely convergent and therefore converges.

<u>If $\mathrm{Re}(s) \leq 1$, then we have two scenarios:</u>

First, if $\mathrm{Re}(s) \leq 0$, then:

$$\lim_{n \to +\infty} \left| \frac{1}{n^s} \right| = \lim_{n \to +\infty} \frac{1}{n^{\mathrm{Re}(s)}} \neq 0,$$

and therefore we have $\lim_{n \to +\infty} \frac{1}{n^s} \neq 0$, which implies that the series is grossly divergent.

Next, if $0 < \mathrm{Re}(s) \leq 1$, then:

We define $f_s(t) := \frac{1}{t^s}$ for $t \in [1, +\infty[$. The function f_s is differentiable on $[1, +\infty[$ and $f_s'(t) = -\frac{s}{t^{s+1}}$, for all $t \in [1, +\infty[$.
Let $k \geq 2$. By integration by parts, we have:

$$\begin{aligned} \int_{k-1}^{k} f_s(t)\, dt &= [f_s(t)(t-(k-1))]_{k-1}^{k} - \int_{k-1}^{k} f_s'(t)(t-(k-1))\, dt \\ &= f_s(k) - \int_{k-1}^{k} f_s'(t)(t-(k-1))\, dt. \end{aligned}$$

The expressions we integrate are continuous. Consequently, we can view the preceding integrals as integrals over $[k-1, k[$. Thus, for any $t \in [k-1, k[$, we have $\lfloor t \rfloor = k-1$. Therefore, we obtain:

$$f_s(k) = \int_{k-1}^{k} f_s(t)\, dt + \int_{k-1}^{k} f_s'(t)(t - \lfloor t \rfloor)\, dt$$

Let $n \geq 2$. By summing on both sides, we find that:

$$\sum_{k=2}^{n} f_s(k) = \int_1^n f_s(t)\, dt + \int_1^n f_s'(t)(t - \lfloor t \rfloor)\, dt = \sum_{k=2}^{n} \frac{1}{k^s} = \int_1^n \frac{dt}{t^s} - \int_1^n \frac{s(t - \lfloor t \rfloor)}{t^{s+1}}\, dt.$$

Then, thanks to Proposition 7, as $n \to +\infty$, the first integral diverges because it is a Riemann integral over $[1, +\infty[$ with $\mathrm{Re}(s) \leq 1$. In contrast, the second integral converges by domination. Indeed:

$$\left| \frac{s(t - \lfloor t \rfloor)}{t^{s+1}} \right| \leq \frac{|s|}{t^{\mathrm{Re}(s)+1}} \quad \text{and} \quad 0 < \mathrm{Re}(s) \leq 1 \quad \text{then} \quad 1 < \mathrm{Re}(s) + 1 \leq 2.$$

Consequently, $\sum \frac{1}{n^s}$ diverges. $\square$

Now that we have a broader domain of study, it would be interesting to examine the analytic properties of ζ over this domain.

2.2 Analytical Study of ζ on $\{\mathrm{Re}(.) > 1\}$

In complex analysis, the strongest property that a function can have is often associated with the property of holomorphicity. We will therefore investigate whether ζ possesses this remarkable characteristic on $\{\mathrm{Re}(.) > 1\}$. For this purpose, we will use the following theorem:

(*)Theorem 8: Let $(X, \mathcal{B}, \mu)$ be a measured space. Let U be an open set in $\mathbb{C}$ and $f : U \times X \to \mathbb{C}$. Suppose that:

- for every $z \in U$, the mapping $x \mapsto f(z, x)$ is measurable;
- for almost every $x \in X$, the mapping $z \mapsto f(z, x)$ is holomorphic, with derivative denoted by $\frac{\partial f}{\partial z}$;
- there exists a measurable function $\varphi : X \to \mathbb{R}^+$ such that $\int_X \varphi \, d\mu < +\infty$ and, for all $z \in U$ and almost every $x \in X$,

$$|f(z, x)| \leq \varphi(x).$$

Then the function $F : z \mapsto \int\limits_X f(z, x) \, d\mu(x)$ is holomorphic on U and, for all $z \in U$, the function $x \mapsto \frac{\partial f}{\partial z}(z, x)$ is integrable, and :

$$F'(z) = \int_X \frac{\partial f}{\partial z}(z, x) \, d\mu(x).$$

What is remarkable about this theorem is that the domination hypothesis applies to f and not to its partial derivative $\frac{\partial f}{\partial z}$, as is the case for the theorems of differentiability under the integral sign for functions of a real variable.

For ζ, since we are working with natural integers, we will refer to summability rather than integrability, and we will use the counting measure μ on $\mathbb{N}$.

Theorem 9: ζ is holomorphic on $\{\mathrm{Re}(.) > 1\}$ and for all $s \in \{\mathrm{Re}(.) > 1\}$, for all $k \geq 1$:

$$\zeta^{(k)}(s) = (-1)^k \sum_{n=2}^{+\infty} \frac{(\ln(n))^k}{n^s}.$$

Note that one could have used Dirichlet series theory to study the holomorphy of ζ, but the proof would have been more delicate. The theory of Dirichlet series is discussed in Section 2.3.4.

Proof: Let $a > 1$ and $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > a$.

The mapping $s \mapsto \frac{1}{n^s}$ is holomorphic on $\{\mathrm{Re}(.) > a\}$ for every $n \geq 1$, due to the fact that the function in question is a complex exponential. Furthermore, for every $s \in \{\mathrm{Re}(.) > a\}$ and for all $n \geq 1$, we have:

$$\left|\frac{1}{n^s}\right| = \frac{1}{n^{\mathrm{Re}(s)}} \leq \frac{1}{n^a},$$

and $\left(\frac{1}{n^a}\right)_{n\in\mathbb{N}^*}$ is summable, since $a > 1$. Therefore, by Theorem 8, ζ is holomorphic on $\{\mathrm{Re}(.) > a\}$. Since this holds for every $a > 1$, we can generalize that ζ is holomorphic on $\{\mathrm{Re}(.) > 1\}$. In particular, ζ is C^∞ on this domain. Theorem 8 also allows us to assert that for all $s \in \{\mathrm{Re}(.) > 1\}$:

$$\zeta'(s) = \sum_{n=1}^{+\infty} \frac{d}{ds}\left(\frac{1}{n^s}\right) = -\sum_{n=2}^{+\infty} \frac{\ln(n)}{n^s}.$$

This is the same formula as in the real case. However, we now know that we can differentiate ζ as many times as we want, and if we apply Theorem 8 several times, we derive the following relation for all $s \in \{\mathrm{Re}(.) > 1\}$, for all $k \geq 1$:

$$\zeta^{(k)}(s) = (-1)^k \sum_{n=2}^{+\infty} \frac{(\ln(n))^k}{n^s}.$$

We will prove this by induction.

The relation is obviously verified for $k = 1$.
Now, suppose it holds for some rank $k \geq 1$ and show that it is still valid at rank $k+1$.
Let $a > 1$ and $n \geq 2$. The function $s \mapsto \frac{(\ln(n))^k}{n^s}$ is holomorphic on $\{\mathrm{Re}(.) > a\}$, since it is still an exponential function. Next, for all $n \geq 2$ and for all $s \in \{\mathrm{Re}(.) > a\}$, we have:

$$\left|\frac{(\ln(n))^k}{n^s}\right| = \frac{(\ln(n))^k}{n^{\mathrm{Re}(s)}} \leq \frac{(\ln(n))^k}{n^a}.$$

Let $b := \frac{a+1}{2} > 1$. Then:

$$n^b \frac{\ln(n)^k}{n^a} = \frac{\ln(n)^k}{n^{a-b}} = \frac{\ln(n)^k}{n^{\frac{a-1}{2}}}.$$

Now, since $\frac{a-1}{2} > 0$, by comparison of growth:

$$\lim_{n\to+\infty} \frac{\ln(n)^k}{n^{\frac{a-1}{2}}} = 0.$$

Thus, $\frac{\ln(n)^k}{n^a} = o\left(\frac{1}{n^b}\right)$, and since $b > 1$, the series $\left(\frac{1}{n^b}\right)_{n\in\mathbb{N}^*}$ is summable. Consequently, $\left(\frac{\ln(n)^k}{n^a}\right)_{n\in\mathbb{N}^*}$ is also summable. Therefore, by Theorem 8, we can affirm that $\zeta^{(k)}$ is differentiable on $\{\text{Re}(.) > a\}$, and since this holds for any $a > 1$, by generalization, it is true on $\{\text{Re}(.) > 1\}$. Hence for all $s \in \{\text{Re}(.) > 1\}$:

$$\zeta^{(k+1)}(s) = (\zeta^{(k)})'(s) = (-1)^k \sum_{n=2}^{+\infty} \frac{d}{ds}\left(\frac{(\ln(n))^k}{n^s}\right) = (-1)^{k+1} \sum_{n=2}^{+\infty} \frac{(\ln(n))^{k+1}}{n^s}.$$

The proof is complete. □

After demonstrating differentiability in the real context, we graphically represented our results to visualize them. To broaden this study, we will now examine ζ using a representation in the complex number plane. This approach will allow us to highlight the regularity characteristics of the function in the complex domain by exploring its behaviors.

Numerical methods for approximating $|\zeta|$ on $\{\text{Re}(.) > 1\}$ will be discussed in Section 4.2, in particular the binomial transform method (see Section 4.1.2). By applying the numerical method based on the binomial transform, we obtain the following graph:

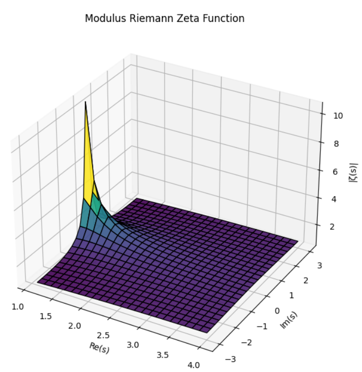


Figure 2: Function $|\zeta|$ using the accelerated method where $N = 25$.

We see that for values in the domain $\{\text{Re}(.) > 1\}$ that are close to 1, $|\zeta|$ seems to explode.

2.3 Some Relations Satisfied by ζ

The function ζ satisfies several important relations, which, combined with the holomorphy of ζ, will lead to the study of deep analytical properties. We present some of them here.

2.3.1 Link between the Riemann ζ function and the Dirichlet η function

Let $s \in \{\mathrm{Re}(.) > 1\}$. Thanks to Proposition 6, we immediately deduce that the sequence $\left(\frac{(-1)^{n-1}}{n^s}\right)_{n\in\mathbb{N}}$ is summable.

Definition 10: For all $s \in \{\mathrm{Re}(.) > 1\}$, we define:

$$\eta(s) := \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s}.$$

This is the Dirichlet function η.

This function will be useful for the numerical approximations of ζ. Indeed, the numerical method of the binomial transform is particularly suited for alternating series. The binomial transform method is discussed in Section 4.1.2.

In particular, we have the following relationship with the function ζ.

Proposition 11: For all $s \in \{\mathrm{Re}(.) > 1\}$, we have:

$$\zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}}.$$

Proof : We perform the following decomposition:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n \text{ even}} \frac{(-1)^{n-1}}{n^s} + \sum_{n \text{ odd}} \frac{(-1)^{n-1}}{n^s} = -\frac{1}{2^s}\sum_{k=1}^{+\infty} \frac{1}{k^s} + \sum_{k=0}^{+\infty} \frac{1}{(2k+1)^s}.$$

Now by the same procedure,

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = \frac{1}{2^s}\sum_{k=1}^{+\infty} \frac{1}{k^s} + \sum_{k=0}^{+\infty} \frac{1}{(2k+1)^s}.$$

Thus,

$$\sum_{k=0}^{+\infty} \frac{1}{(2k+1)^s} = \left(1 - \frac{1}{2^s}\right)\zeta(s).$$

Therefore,

$$\eta(s) = -\frac{1}{2^s}\zeta(s) + \left(1 - \frac{1}{2^s}\right)\zeta(s) = \left(1 - \frac{1}{2^{s-1}}\right)\zeta(s) = \left(1 - 2^{1-s}\right)\zeta(s). \quad \square$$

2.3.2 Connection Between Riemann's ζ Function and Euler's Γ Function

The Euler Γ function is a function defined from a parameterized integral. Originally defined on $\mathbb{R}_+^*$, its domain can also be extended to complex numbers. More precisely, its extended domain corresponds to the set $\{\mathrm{Re}(.) > 0\}$, and this will be formalized in the following definition.

(*)Definition-proposition 12: For every complex $s \in \{\mathrm{Re}(s) > 0\}$, we define the Gamma function as:

$$\Gamma(s) := \int_0^{+\infty} t^{s-1} e^{-t} \, dt.$$

The connection between the ζ function and the Γ function is as follows.

Proposition 13: For all $s \in \{\mathrm{Re}(.) > 1\}$,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{u^{s-1}}{e^u - 1} \, \mathrm{d}u.$$

The function Γ has important analytical properties that will be discussed in Section 2.4.2. This function will notably provide strong analytical results for ζ.

Proof: We take $s \in \{\mathrm{Re}(.) > 1\}$ and calculate $\zeta(s)\Gamma(s)$:

$$\zeta(s)\Gamma(s) = \left(\sum_{n=1}^{+\infty} \frac{1}{n^s} \right) \Gamma(s) = \sum_{n=1}^{+\infty} \frac{\Gamma(s)}{n^s} = \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{t^{s-1} e^{-t}}{n^s} \, dt.$$

Let $n \geq 1$. By making the change of variable $u = \frac{t}{n}$, we have:

$$\int_0^{+\infty} \frac{t^{s-1} e^{-t}}{n^s} \, dt = \int_0^{+\infty} u^{s-1} e^{-nu} \, du.$$

Thus, by summing:

$$\zeta(s)\Gamma(s) = \sum_{n=1}^{+\infty} \int_0^{+\infty} u^{s-1} e^{-nu} \, du.$$

Now:

$$\sum_{n=1}^{+\infty} \int_0^{+\infty} \left| u^{s-1} e^{-nu} \right| \, du = \sum_{n=1}^{+\infty} \int_0^{+\infty} u^{\mathrm{Re}(s)-1} e^{-nu} \, du.$$

By the Beppo Levi theorem, we can interchange the operators.

$$\begin{aligned}\sum_{n=1}^{+\infty}\int_0^{+\infty} u^{\mathrm{Re}(s)-1}e^{-nu}\,du &= \int_0^{+\infty}\sum_{n=1}^{+\infty} u^{\mathrm{Re}(s)-1}e^{-nu}\,du\\ &= \int_0^{+\infty}\frac{u^{\mathrm{Re}(s)-1}e^{-u}}{1-e^{-u}}\,du\\ &= \int_0^{+\infty}\frac{u^{\mathrm{Re}(s)-1}}{e^u-1}\,du.\end{aligned}$$

Finally:

$$\frac{u^{\mathrm{Re}(s)-1}}{e^u-1} = o\left(\frac{1}{u^2}\right) \quad \text{as} \quad u\to+\infty.$$

$$\frac{u^{\mathrm{Re}(s)-1}}{e^u-1} \sim \frac{1}{u^{2-\mathrm{Re}(s)}} \quad \text{as} \quad u\to 0^+.$$

Now, $u\mapsto\frac{1}{u^2}$ is integrable at $+\infty$ and since $2-\mathrm{Re}(s)<1$, $u\mapsto\frac{1}{u^{2-\mathrm{Re}(s)}}$ is integrable at 0^+. Therefore, $u\to\frac{u^{\mathrm{Re}(s)-1}}{e^u-1}$ is integrable at both $+\infty$ and 0^+, so:

$$\sum_{n=1}^{+\infty}\int_0^{+\infty}\left|u^{s-1}e^{-nu}\right|\,du = \int_0^{+\infty}\frac{u^{\mathrm{Re}(s)-1}}{e^u-1}\,du < +\infty.$$

Thus, we can interchange the sum and the integral in the initial calculation. We just need to redo our calculation, replacing $\mathrm{Re}(s)$ with s. Hence:

$$\zeta(s)\Gamma(s) = \sum_{n=1}^{+\infty}\int_0^{+\infty} u^{s-1}e^{-nu}\,du = \int_0^{+\infty}\frac{u^{s-1}}{e^u-1}\,du. \quad \square$$

2.3.3 Expression from Abel's Summation Formula

In mathematics, Abel's summation formula is a widely used tool in analytic number theory that allows us to determine numerical series. This formula can be seen as a form of integration by parts.
Specifically, this formula is the subject of the following proposition.

(*)Proposition 14: Let $(a_n)_{n\in\mathbb{N}^*}$ be a sequence of real or complex numbers and φ be a real or complex function of class C^1. We define, for any real x, the sum as:

$$A(x) := \sum_{n=1}^{\lfloor x\rfloor} a_n.$$

Then, for any real x, we have the following identity:

$$\sum_{n=1}^{\lfloor x\rfloor} a_n\varphi(n) = A(x)\varphi(x) - \int_1^x A(u)\varphi'(u)\,\mathrm{d}u.$$

If we apply this property to ζ, we get the following property.

Proposition 15: For all $s \in \{\mathrm{Re}(.) > 1\}$:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = s \int_1^{+\infty} \frac{\lfloor u \rfloor}{u^{1+s}}\, du.$$

Proof : Let $s \in \{\mathrm{Re}(.) > 1\}$. We set, for all $n \geq 1$, $a_n := 1$, and for all $y > 0$, we define $\varphi(y) := \frac{1}{y^s}$. Thus, φ is indeed a C^1 function on $]0, +\infty[$, and for all $y > 0$, we have $\varphi'(y) = -\frac{s}{y^{s+1}}$.
Let $x \geq 1$. Then, $A(x) = \lfloor x \rfloor$, and by the summation formula, we get:

$$\sum_{n=1}^{\lfloor x \rfloor} \frac{1}{n^s} = \frac{\lfloor x \rfloor}{x^s} + s \int_1^x \frac{\lfloor u \rfloor}{u^{s+1}}\, \mathrm{d}u.$$

Now, we estimate:

$$\left| \frac{\lfloor x \rfloor}{x^s} \right| \leq \frac{\lfloor x \rfloor}{x^{\mathrm{Re}(s)}} \leq \frac{x}{x^{\mathrm{Re}(s)}} = \frac{1}{x^{\mathrm{Re}(s)-1}} \to 0 \quad \text{as} \quad x \to +\infty, \quad \text{since } \mathrm{Re}(s) - 1 > 0.$$

Thus, as $x \to +\infty$, we conclude that:

$$\sum_{n=1}^{+\infty} \frac{1}{n^s} = s \int_1^{+\infty} \frac{\lfloor u \rfloor}{u^{s+1}}\, du.$$

This concludes the proof. $\square$

2.3.4 Dirichlet Series Expansions of ζ

The study of the ζ function is a particular case of the theory of Dirichlet series. The application of the theory to these series will allow us to find interesting relations satisfied by ζ.
We will see in particular that ζ has relations with arithmetic operators.

Definition 16: Let $s \in \mathbb{C}$ and $(a_n)_{n \in \mathbb{N}^*}$ be a sequence of complex numbers. Then, the series $\sum \frac{a_n}{n^s}$ is called a Dirichlet series.

Thus, by setting $a_n := 1$ for all $n \geq 1$, a Dirichlet series corresponds to a Riemann series. From here on, $\mathbf{1}$ will denote the sequence where all terms are equal to 1, Id will denote the identity sequence [1] and finally, δ_1 [2] is the indicator function of the singleton $\{1\}$.

Definition 17: An arithmetic function is any function of the form $f : \mathbb{N}^* \to \mathbb{C}$.

[1] $\mathrm{Id}(n) = n$, for all $n \in \mathbb{N}^*$.
[2] $\delta_1(1) = 1$, and for all $n > 1$, $\delta_1(n) = 0$.

$\mathbf{1}$ and Id are, of course, arithmetic functions.

Arithmetic functions play an important role in the theory of Dirichlet series. They give rise to the Dirichlet convolution theorem.

Definition 18: Let f and g be two arithmetic functions. The Dirichlet convolution of these two functions is the function $f * g$ defined by, for all $n \in \mathbb{N}^*$:

$$(f * g)(n) := \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

From this, we state the Dirichlet convolution theorem.

(*)Theorem 19: Let f and g be two arithmetic functions. Then, for all $s \in \mathbb{C}$ such that both series $\sum \frac{f(n)}{n^s}$ and $\sum \frac{g(n)}{n^s}$ converge absolutely, we have:

$$\sum_{n=1}^{+\infty} \frac{f(n)}{n^s} \sum_{n=1}^{+\infty} \frac{g(n)}{n^s} = \sum_{n=1}^{+\infty} \frac{(f * g)(n)}{n^s}.$$

This theorem will be particularly useful for determining new relations satisfied by ζ.

In particular, these relations will involve Euler's totient function φ and the Möbius function μ.

Definition 20: The Möbius function is defined as the function $\mu : \mathbb{N}^* \to \{-1, 0, 1\}$ such that:

- $\mu(n) = 0$, if n is divisible by a perfect square other than 1.
- $\mu(n) = 1$, if n is the product of an even number of distinct prime numbers.
- $\mu(n) = -1$, if n is the product of an odd number of distinct prime numbers.

Definition 21: Euler's totient function is defined as the function $\varphi : \mathbb{N}^* \to \mathbb{N}^*$ such that, for every $n \in \mathbb{N}^* : \varphi(n) := \operatorname{card}\left(\{k \in \{1, \ldots, n\} \mid k \wedge n = 1\}\right).$

In particular, μ and φ are arithmetic functions.

These functions satisfy the following convolution relations.

(*)Proposition 22: The functions μ and φ satisfy the following convolution relations on $\mathbb{N}^*$:

$$\delta_1 = \mathbf{1} * \mu \quad \text{and} \quad \varphi = \mathrm{Id} * \mu.$$

Thanks to this, we will be able to prove the following proposition.

Proposition 23: The ζ function satisfies the following four relations :

- For every $s \in \{\mathrm{Re}(.) > 1\}$,

$$\zeta^2(s) = \sum_{n=1}^{+\infty} \frac{\tau(n)}{n^s}.$$

- For every $s \in \{\mathrm{Re}(.) > 2\}$,

$$\zeta(s)\zeta(s-1) = \sum_{n=1}^{+\infty} \frac{\sigma(n)}{n^s}.$$

- For every $s \in \{\mathrm{Re}(.) > 1\}$,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}.$$

- For every $s \in \{\mathrm{Re}(.) > 2\}$,

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{+\infty} \frac{\varphi(n)}{n^s}.$$

Arithmetically, τ and σ represent the divisor count function and the sum of divisors function. In other words, they are defined as follows:

$$\tau(n) := \sum_{d|n} 1.$$

$$\sigma(n) := \sum_{d|n} d.$$

It is interesting to see that the previous developments involve elements specific to arithmetic. The arithmetic of integers and the function ζ are therefore linked. This connection will be explored in greater detail in Section 3.

There exist other relations involving Dirichlet series. These can be found in [8] and in Titchmarsh's work [6].

Proof: We consider $s \in \{\mathrm{Re}(.) > 1\}$. Then:

$$\zeta^2(s) = \left(\sum_{n=1}^{+\infty} \frac{1}{n^s}\right)\left(\sum_{n=1}^{+\infty} \frac{1}{n^s}\right) = \sum_{n=1}^{+\infty} \frac{(\mathbf{1} * \mathbf{1})(n)}{n^s}.$$

Now, for all $n \geq 1$:

$$(\mathbf{1} * \mathbf{1})(n) = \sum_{d|n} 1 = \tau(n).$$

Thus, we have:

$$\zeta^2(s) = \sum_{n=1}^{+\infty} \frac{\tau(n)}{n^s}.$$

Now consider $s \in \{\mathrm{Re}(.) > 2\}$. Then:

$$\zeta(s-1)\zeta(s) = \left(\sum_{n=1}^{+\infty} \frac{\mathrm{Id}(n)}{n^s}\right)\left(\sum_{n=1}^{+\infty} \frac{\mathbf{1}(n)}{n^s}\right) = \sum_{n=1}^{+\infty} \frac{(\mathrm{Id} * \mathbf{1})(n)}{n^s}.$$

Next, for all $n \geq 1$:

$$(\mathrm{Id} * \mathbf{1})(n) = \sum_{d|n} d = \sigma(n).$$

Thus, we have:

$$\zeta(s-1)\zeta(s) = \sum_{n=1}^{+\infty} \frac{\sigma(n)}{n^s}.$$

Let $s \in \{\mathrm{Re}(.) > 1\}$. Then, for all $n \geq 1$:

$$\left|\frac{\mu(n)}{n^s}\right| \leq \frac{1}{n^{\mathrm{Re}(s)}}.$$

By domination, the series $\sum \frac{\mu(n)}{n^s}$ is therefore absolutely convergent. Thus:

$$\zeta(s)\sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s} = \left(\sum_{n=1}^{+\infty} \frac{1}{n^s}\right)\left(\sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}\right) = \sum_{n=1}^{+\infty} \frac{(\mathbf{1} * \mu)(n)}{n^s} = \sum_{n=1}^{+\infty} \frac{\delta_1(n)}{n^s} = 1.$$

Hence:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}.$$

Finally, take $s \in \{\mathrm{Re}(.) > 2\}$. Then:

$$\frac{\zeta(s-1)}{\zeta(s)} = \left(\sum_{n=1}^{+\infty} \frac{n}{n^s}\right)\left(\sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}\right) = \sum_{n=1}^{+\infty} \frac{(\mathrm{Id} * \mu)(n)}{n^s} = \sum_{n=1}^{+\infty} \frac{\varphi(n)}{n^s}. \quad \square$$

2.4 Analytic continuation of ζ on $\mathbb{C} \setminus \{1\}$

Let us consider Proposition 4. We showed that for all $k \geq 1$, $\zeta(2k) = B_{2k} \frac{2^{2k-1}\pi^{2k}}{(-1)^{k-1}(2k)!}$. What happens when $k = 0$? The expression on the right-hand side becomes $-\frac{1}{2}$. It suggests extending ζ at 0 by setting $\zeta(0) := -\frac{1}{2}$.

Nothing prevents us from assigning values like 40, $3i$, or others for the continuation. However, $-\frac{1}{2}$ is the most natural, as this value arises from a relation that is truly satisfied by ζ.

In this section, we will explore whether it is possible to find a continuation of the ζ function that is as natural as possible while preserving most of its properties. In particular, we will aim for this extension to be holomorphic, which is possible through what is called complex analytic continuation.

2.4.1 Theory of Analytic Continuation

In complex analysis, an analytic continuation (or complex analytic continuation) is a method for extending the domain of definition of a holomorphic function beyond the region where it was initially defined. We can summarize the main issues as follows:

Let f be a function defined and holomorphic on an open domain $D \subset \mathbb{C}$. Now suppose we wish to extend f outside of D. If we can find a function g defined on a larger domain $D' \supset D$, such that:

- g is holomorphic on D',
- g coincides with f on D, that is, $g(z) = f(z)$ for all $z \in D$,

then we call g an analytic continuation of f to D'.

The theory of analytic continuation is built with the following elements.

Definition 24: Let A be a non-empty subset of $\mathbb{C}$, and $a \in \mathbb{C}$. We say that the point a is an accumulation point of A if $a \in \overline{A \setminus \{a\}}$.

(*)Theorem 25: Let U be an open subset of $\mathbb{C}$, a a point in U, and a holomorphic function $f : U \to \mathbb{C}$. Suppose that U is connected. Then the following four statements are equivalent :

- f is identically zero on U;
- f is identically zero in a neighborhood of a;
- For all $n \in \mathbb{N}$, $f^{(n)}(a) = 0$;
- f is identically zero on a set of points possessing an accumulation point in U.

This theorem guarantees the uniqueness of analytic continuation through the following corollary.

Corollary 26: If f and g are two holomorphic functions on a connected open set $U \subset \mathbb{C}$ and if f and g coincide on a neighborhood of a point in U, then $f = g$ on U.

Proof: Let f and g be two holomorphic functions on a connected open set $U \subset \mathbb{C}$. If f and g coincide on a neighborhood of a point in U, then $f - g = 0$ on this neighborhood. By Theorem 25, we then have $f - g = 0$ on U, and therefore $f = g$ on U. $\square$

2.4.2 Analytical continuation of ζ using a Jacobi theta function

The function ζ extends to a meromorphic function on $\mathbb{C} \setminus \{1\}$. The following method is inspired by Titchmarsh [6, p. 21], which closely resembles the original proof conducted by B. Riemann in 1859.

We define the function ϕ, which for all $x \in]0, +\infty[$ is given by:

$$\phi(x) := \sum_{n=1}^{+\infty} e^{-n^2 \pi x}.$$

(*)Proposition 27: The function ϕ defined in this way satisfies, for all $x > 0$, the following functional relation:

$$\phi(x) = \frac{1}{\sqrt{x}} \phi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2}.$$

To demonstrate the analytic continuation of ζ, we will need the analytic properties satisfied by Γ.

(*)Theorem 28: The function Γ is holomorphic on $\{\mathrm{Re}(.) > 0\}$. Moreover, $\frac{1}{\Gamma}$ is a holomorphic function on $\mathbb{C}$, satisfying, for all $z \in \mathbb{C}$:

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{+\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}.$$

where γ denotes Euler's constant[3].

We will now prove the following theorem.

Theorem 29: The function ζ extends to $\mathbb{C} \setminus \{1\}$ through the following relation, satisfied for all $s \in \{\mathrm{Re}(.) > 1\}$:

[3]Euler's constant is defined as $\gamma := \lim_{n \to +\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n) \right)$.

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)(s-1)} - \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)s} + \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \int_1^{+\infty} f(x,s)\,dx.$$

where $f(x,s) := \frac{x^{\frac{1-s}{2}} + x^{\frac{s}{2}}}{x} \phi(x)$.

It is notably through the complex analytic continuation that the connection between ζ and number theory will be significantly strengthened.

Proof: Let $s \in \{\mathrm{Re}(.) > 1\}$. Recall that $\Gamma\left(\frac{s}{2}\right) = \int\limits_0^{+\infty} t^{\frac{s}{2}-1} e^{-t}\,dt$.

Let $n \geq 1$. We perform the change of variable $t = n^2\pi x$. Then:

$$\Gamma\left(\frac{s}{2}\right) = n^2\pi \int_0^{+\infty} \left(n^2\pi x\right)^{\frac{s}{2}-1} e^{-n^2\pi x}\,dx = n^s \pi^{\frac{s}{2}} \int_0^{+\infty} x^{\frac{s}{2}-1} e^{-n^2\pi x}\,dx.$$

Thus, we deduce that:

$$\Gamma\left(\frac{s}{2}\right) \frac{1}{n^s} \pi^{-\frac{s}{2}} = \int_0^{+\infty} x^{\frac{s}{2}-1} e^{-n^2\pi x}\,dx.$$

By summing these equalities, we obtain:

$$\Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \sum_{n=1}^{+\infty} \int_0^{+\infty} x^{\frac{s}{2}-1} e^{-n^2\pi x}\,dx.$$

We will interchange the operators by applying Beppo-Levi's theorem.

$$\sum_{n=1}^{+\infty} \int_0^{+\infty} \left| x^{\frac{s}{2}-1} e^{-n^2\pi x} \right| dx = \sum_{n=1}^{+\infty} \int_0^{+\infty} x^{\frac{\mathrm{Re}(s)}{2}-1} e^{-n^2\pi x}\,dx.$$

By the change of variable $t = n^2\pi x$, this gives us:

$$\begin{aligned}
\sum_{n=1}^{+\infty} \int_0^{+\infty} \left| x^{\frac{s}{2}-1} e^{-n^2\pi x} \right| dx &= \sum_{n=1}^{+\infty} \frac{1}{n^{\mathrm{Re}(s)}} \pi^{-\frac{\mathrm{Re}(s)}{2}} \int_0^{+\infty} t^{\frac{\mathrm{Re}(s)}{2}-1} e^{-t}\,dt \\
&= \pi^{-\frac{\mathrm{Re}(s)}{2}} \zeta(\mathrm{Re}(s)) \Gamma\left(\frac{\mathrm{Re}(s)}{2}\right) < +\infty.
\end{aligned}$$

Through the interchange, the function ϕ appears in the relation, and we have:

$$\Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \int_0^{+\infty} x^{\frac{s}{2}-1} \phi(x)\,dx.$$

We will now apply the Chasles relation to the integral. We have:

$$\Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \int_0^1 x^{\frac{s}{2}-1} \phi(x)\,dx + \int_1^{+\infty} x^{\frac{s}{2}-1} \phi(x)\,dx. \tag{2}$$

Let us define $I := \int\limits_0^1 x^{\frac{s}{2}-1}\phi(x)\,dx$. By the functional equation in Proposition 27, we have:

$$\begin{aligned} I &= \int_0^1 x^{\frac{s}{2}-1}\left(\frac{1}{\sqrt{x}}\phi\left(\frac{1}{x}\right)+\frac{1}{2\sqrt{x}}-\frac{1}{2}\right)dx, \\ &= \int_0^1 x^{\frac{s-3}{2}}\phi\left(\frac{1}{x}\right)dx+\frac{1}{2}\int_0^1 x^{\frac{s-3}{2}}\,dx-\frac{1}{2}\int_0^1 x^{\frac{s}{2}-1}\,dx. \end{aligned}$$

For the first integral, we apply the change of variables $y=\frac{1}{x}$, and we obtain:

$$\begin{aligned} I &= \int_1^{+\infty}\frac{1}{y^{\frac{s-3}{2}}}\phi(y)\,\frac{dy}{y^2}+\frac{1}{2}\left[\frac{x^{\frac{s-1}{2}}}{\frac{s-1}{2}}\right]_0^1-\frac{1}{2}\left[\frac{x^{\frac{s}{2}}}{\frac{s}{2}}\right]_0^1, \\ &= \int_1^{+\infty}\frac{1}{y^{\frac{s+1}{2}}}\phi(y)\,dy+\frac{1}{s-1}-\frac{1}{s}. \end{aligned}$$

Substituting I back into (1), we get:

$$\begin{aligned} \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) &= \int_1^{+\infty}\frac{1}{x^{\frac{s+1}{2}}}\phi(x)\,dx+\frac{1}{s-1}-\frac{1}{s}+\int_1^{+\infty}x^{\frac{s}{2}-1}\phi(x)\,dx, \\ &= \int_1^{+\infty}\left(\frac{1}{x^{\frac{s+1}{2}}}+\frac{1}{x^{1-\frac{s}{2}}}\right)\phi(x)\,dx+\frac{1}{s-1}-\frac{1}{s}, \\ &= \int_1^{+\infty}\frac{x^{\frac{1-s}{2}}+x^{\frac{s}{2}}}{x}\phi(x)\,dx+\frac{1}{s-1}-\frac{1}{s}. \end{aligned}$$

Using $f(x,s)=\frac{x^{\frac{1-s}{2}}+x^{\frac{s}{2}}}{x}\phi(x)$, we finally have:

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)=\int_1^{+\infty}f(x,s)\,dx+\frac{1}{s-1}-\frac{1}{s}.$$

Thus, we deduce that:

$$\zeta(s)=\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)(s-1)}-\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)s}+\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\int_1^{+\infty}f(x,s)\,dx.$$

Now, let's examine the expressions in the equality.
First, $s\mapsto\frac{1}{\Gamma\left(\frac{s}{2}\right)}$ is holomorphic on $\mathbb{C}$ by Theorem 28. Therefore, $s\mapsto\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)(s-1)}$ is holomorphic on $\mathbb{C}\setminus\{1\}$.
Then, $s\mapsto\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)s}=\frac{\pi^{\frac{s}{2}}}{2\Gamma\left(\frac{s}{2}+1\right)}$, and here, by Theorem 28, it is a holomorphic function on $\mathbb{C}$.
Finally, for the last term in the expression, we will apply Theorem 8.
For all $x\geq 1$, $s\mapsto f(x,s)$ is holomorphic on $\mathbb{C}$. We then consider a compact

$K \subset \mathbb{C}$. Thus, there exist real numbers $a \leq b$ such that $\mathrm{Re}(s) \in [a,b]$, for all $s \in K$. We then perform the following domination for all $x \geq 1$ and $s \in K$:

$$\begin{aligned}
|f(x,s)| &= \left| \frac{x^{\frac{1-s}{2}} + x^{\frac{s}{2}}}{x} \phi(x) \right|, \\
&\leq \frac{x^{\frac{1-\mathrm{Re}(s)}{2}} + x^{\frac{\mathrm{Re}(s)}{2}}}{x} \sum_{n=1}^{+\infty} e^{-n\pi x}, \\
&\leq \frac{x^{\frac{1-a}{2}} + x^{\frac{b}{2}}}{x} \frac{1}{e^{\pi x} - 1}, \\
&= o\left(\frac{1}{x^2}\right) \quad \text{as} \quad x \to +\infty.
\end{aligned}$$

The upper bound corresponds to a function independent of s and integrable on $[1, +\infty[$. By multiplication, $s \mapsto \frac{\pi^{s/2}}{\Gamma(s/2)} \int_1^{+\infty} f(x,s)\, dx$ is therefore holomorphic on $\mathbb{C}$.

Hence, ζ extends analytically to a holomorphic function on $\mathbb{C} \setminus \{1\}$. $\square$

2.4.3 Functional Equation

In this section, we will primarily use Proposition 15.

Definition 30: Let $u \in \mathbb{R}$. The fractional part of u is defined by $\{u\} := u - \lfloor u \rfloor$.

The function ζ satisfies a functional equation, stated in the following theorem.

Theorem 31: For all $s \in \mathbb{C} \setminus \{0,1\}$:

$$\zeta(s) = 2^s \pi^{s-1} \zeta(1-s) \Gamma(1-s) \sin\left(\frac{\pi s}{2}\right).$$

This relation allows, in particular, to connect the images of ζ on the extended domain from the original domain where we already have analytical results (see Section 2.2).

The following proof follows the first approach presented in [5, p. 23], which itself was inspired by Luis Báez-Duarte's work from 2003.

Proof: Let $s \in \{\mathrm{Re}(.) > 1\}$. Then:

$$\zeta(s) = s \int_1^{+\infty} \frac{\lfloor u \rfloor}{u^{1+s}}\, du = s \int_1^{+\infty} \frac{u - \{u\}}{u^{1+s}}\, du = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{u\}}{u^{1+s}}\, du.$$

The result we just obtained can be rewritten when $0 < \mathrm{Re}(s) < 1$. Indeed:

For u near $+\infty$, we have:

$$\left|\frac{\{u\}}{u^{1+s}}\right| \leq \frac{1}{u^{1+\mathrm{Re}(s)}}.$$

and $u \mapsto \frac{1}{u^{1+\mathrm{Re}(s)}}$ is integrable at $+\infty$, since $\mathrm{Re}(s)+1>1$.
Similarly, for u near 0, we have:

$$\left|\frac{\{u\}}{u^{1+s}}\right| = \left|\frac{u}{u^{1+s}}\right| = \left|\frac{1}{u^{s}}\right| = \frac{1}{u^{\mathrm{Re}(s)}}.$$

and $u \mapsto \frac{1}{u^{\mathrm{Re}(s)}}$ is integrable at 0 because $\mathrm{Re}(s) < 1$.
Therefore, for $0 < \mathrm{Re}(s) < 1$, the function $u \mapsto \frac{\{u\}}{u^{1+s}}$ is integrable on $]0, +\infty[$.

The rewriting will always coincide with the function ζ for such s due to the uniqueness of its analytic continuation. Thus, if $0 < \mathrm{Re}(s) < 1$, we have:

$$\begin{aligned}
\zeta(s) &= \frac{s}{s-1} - s\left(\int_0^{+\infty} \frac{\{u\}}{u^{1+s}}\,du - \int_0^1 \frac{\{u\}}{u^{1+s}}\,du\right), \\
&= \frac{s}{s-1} - s\left(\int_0^{+\infty} \frac{\{u\}}{u^{1+s}}\,du - \int_0^1 \frac{du}{u^{s}}\right), \\
&= -s\int_0^{+\infty} \frac{\{u\}}{u^{1+s}}\,du.
\end{aligned}$$

Thus:

$$-\frac{\zeta(s)}{s} = \int_0^{+\infty} \frac{\{u\}}{u^{1+s}}\,du.$$

By making the change of variable $u = 2v$, we obtain:

$$-2^s\frac{\zeta(s)}{s} = \int_0^{+\infty} \frac{\{2v\}}{v^{1+s}}\,dv.$$

By subtracting the last two equalities, we obtain:

$$(2^s-1)\frac{\zeta(s)}{s} = \int_0^{+\infty} \frac{\{u\}-\{2u\}}{u^{1+s}}\,du.$$

To proceed with our calculations, we will use the following property.

(*)Proposition 32: For all $x \in \mathbb{R}\setminus\mathbb{Z}$, we have:

$$\{x\} = \frac{1}{2} - \sum_{n=1}^{+\infty} \frac{\sin(2n\pi x)}{n\pi}.$$

Since $\mathbb{Z}$ has Lebesgue measure zero, we can introduce this Fourier series expansion into the integral. Thus, we have:

$$(2^s - 1)\frac{\zeta(s)}{s} = \int_0^{+\infty} \sum_{n=1}^{+\infty} \frac{\sin(4n\pi u) - \sin(2n\pi u)}{n\pi u^{s+1}}\, du.$$

For $-1 < \mathrm{Re}(s) < 0$, the right-hand side of the equation can be rewritten by interchanging the sum and the integral, which is justified by the uniform boundedness of the partial sums. The uniqueness of the complex analytic continuation guarantees that the left-hand side will be equal to the rewritten form of the right-hand side for $-1 < \mathrm{Re}(s) < 0$. Therefore, for such s, we have:

$$(2^s - 1)\frac{\zeta(s)}{s} = \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{\sin(4n\pi u) - \sin(2n\pi u)}{n\pi u^{s+1}}\, du.$$

By the change of variable $t = 2n\pi u$, we obtain:

$$\begin{aligned}(2^s - 1)\frac{\zeta(s)}{s} &= \sum_{n=1}^{+\infty} \frac{(2n\pi)^s}{n\pi} \int_0^{+\infty} t^{-s-1}(\sin(2t) - \sin(t))\, dt \\ &= 2^s \pi^{s-1} \sum_{n=1}^{+\infty} \frac{1}{n^{1-s}}\,(2^s - 1) \int_0^{+\infty} t^{-s-1} \sin(t)\, dt.\end{aligned}$$

To proceed with our calculations, we will use the following result:

(*)Lemma 33: For all $s \in \mathbb{C}$ such that $0 < \mathrm{Re}(s) < 1$, we have:

$$\int_0^{+\infty} t^{s-1} \sin(t)\, dt = \Gamma(s) \sin\left(\frac{\pi s}{2}\right).$$

This is the Mellin transform of the sine function.

This leads to the following result:

$$(2^s - 1)\frac{\zeta(s)}{s} = 2^s \pi^{s-1} \zeta(1-s)(2^s - 1)\Gamma(-s) \sin\left(-\frac{\pi s}{2}\right).$$

Thus:

$$\zeta(s) = 2^s \pi^{s-1} \zeta(1-s)\Gamma(1-s) \sin\left(\frac{\pi s}{2}\right).$$

This functional equation being valid for $-1 < \mathrm{Re}(s) < 0$, it is extended to $\mathbb{C} \setminus \{0, 1\}$ by analytic continuation. $\square$

This functional equation demonstrates the consistency of the analytic continuation with the one performed by Proposition 4 in the introductory part of Section 2.4 where we have set $\zeta(0) = -\frac{1}{2}$.

To achieve this, we first show that:

$$\lim_{s\to 1}(s-1)\zeta(s) = 1.$$

By the analytic continuation performed using Theorem 29, we have, for all $s \in \mathbb{C} \setminus \{1\}$:

$$(s-1)\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} - \frac{\pi^{s/2}}{\Gamma(s/2)s}(s-1) + \frac{\pi^{s/2}}{\Gamma(s/2)s}(s-1)\int_1^{+\infty} f(x,s)\,dx.$$

Now, by the continuity of Γ on $\{\mathrm{Re}(s) > 0\}$, we obtain:

$$\lim_{s\to 1}\frac{\pi^{s/2}}{\Gamma(s/2)} = \frac{\sqrt{\pi}}{\Gamma(1/2)}.$$

Moreover,

$$\Gamma(1/2) = \int_0^{+\infty} \frac{e^{-t}}{\sqrt{t}}dt.$$

Using the change of variable $y = \sqrt{t}$, we get:

$$\Gamma(1/2) = 2\int_0^{+\infty} e^{-y^2}dy = \int_{-\infty}^{+\infty} e^{-y^2}dy = \sqrt{\pi}.$$

Thus,

$$\lim_{s\to 1}\frac{\pi^{s/2}}{\Gamma(s/2)} = 1.$$

Furthermore, we have

$$\lim_{s\to 1}\left(\frac{\pi^{s/2}}{\Gamma(s/2)s}(s-1)\right) = 0.$$

Finally, we apply the theorem of limit-integral interchange. Since we conduct our study in the setting where $s \to 1$, we may assume $\frac{1}{2} < \mathrm{Re}(s) < \frac{3}{2}$. Thus, for all $x \geq 1$ and $s \in \mathbb{C}$ such that $\mathrm{Re}(s) \in]\frac{1}{2}, \frac{3}{2}[$, we have:

$$|f(x,s)| \leq \frac{x^{1/4} + x^{3/4}}{x}\frac{1}{e^{\pi x} - 1} = o\left(\frac{1}{x^2}\right), \quad \text{as } x \to +\infty.$$

Thus,

$$x \mapsto \frac{x^{1/4} + x^{3/4}}{x}\frac{1}{e^{\pi x} - 1} \in L^1([1, +\infty[).$$

Moreover,

$$\lim_{s\to 1} f(x,s) = \frac{1-\sqrt{x}}{x}\phi(x), \quad \text{for all } x \geq 1.$$

By interchanging the limit and the integral, we obtain:

$$\lim_{s\to 1}\int_1^{+\infty} f(x,s)dx = \int_1^{+\infty}\frac{1-\sqrt{x}}{x}\phi(x)dx.$$

Thus,

$$\lim_{s\to 1}\left(\frac{\pi^{s/2}}{\Gamma(s/2)}(s-1)\int_1^{+\infty} f(x,s)dx\right) = 0.$$

Finally, we conclude that:

$$\lim_{s\to 1}(s-1)\zeta(s) = 1.^4 \tag{3}$$

We will use this result to prove that $\zeta(0) = -\frac{1}{2}$ by analytic continuation. By the functional equation, we have for all $0 < s < 1$,

$$(1-s)\zeta(s) = 2^s\pi^{s-1}\zeta(1-s)(1-s)\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right).$$

Since $1 - s > 0$, it follows that:

$$(1-s)\zeta(s) = 2^s\pi^{s-1}\zeta(1-s)\Gamma(2-s)\sin\left(\frac{\pi s}{2}\right).$$

Moreover, from (3), we have:

$$\lim_{s\to 1^-}(1-s)\zeta(s) = -1$$

and

$$\lim_{s\to 1^-} 2^s\pi^{s-1}\zeta(1-s)\Gamma(2-s)\sin\left(\frac{\pi s}{2}\right) = 2\zeta(0).$$

Thus, we get $2\zeta(0) = -1$, leading to $\zeta(0) = -\frac{1}{2}$.

Thanks to the functional equation, we can also attempt to represent ζ on $\mathbb{C}\backslash\{1\}$. Indeed, if we take, for example, $s \in \{\mathrm{Re}(s) < 0\}$, then $\mathrm{Re}(1-s) > 1$. Now, on this domain, ζ is nothing but a sum and Γ an integral involving only simple powers. Therefore, it is possible to approximate the values of ζ on $\{\mathrm{Re}(s) < 0\}$, and even to provide exact values given the properties we have for $\{\mathrm{Re}(s) > 1\}$. Let us compute $\zeta(-1)$ to illustrate this. By the functional equation, we have:

$$\zeta(-1) = 2^{-1}\pi^{-2}\zeta(2)\Gamma(2)\sin\left(-\frac{\pi}{2}\right).$$

Now, $\zeta(2) = \frac{\pi^2}{6}$ and $\Gamma(2) = 1$. Thus, we get:

$$\zeta(-1) = -\frac{1}{2}\cdot\frac{1}{\pi^2}\cdot\frac{\pi^2}{6} = -\frac{1}{12}.$$

Here we use the `zeta` function from the Python library `mpmath`. We obtain the following figure :

[4] This makes 1 a pole of order 1 for ζ.

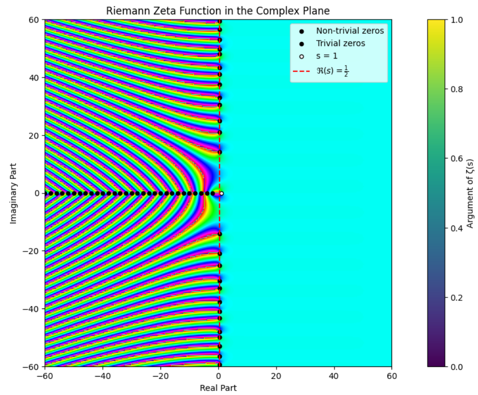


Figure 3: Graphical representation of the extended ζ function.

The figure displays the values of ζ in the complex plane. The color scale represents the argument of $\zeta(s)$, with the hue varying from $-\pi$ to π radians. The black dots represent the zeros of the function, which are plotted on the complex plane.
It is observed that some zeros lie along the vertical line defined by $\{\text{Re}(.) = \frac{1}{2}\}$. These zeros seem to cluster along this critical line. Additionally, there are other zeros located on the negative real axis, where $\{\text{Re}(.) < 0\}$.
The figure illustrates the approximate locations of these zeros in relation to the argument of the ζ function at each point in the complex plane.

3 The function ζ, a bridge between number theory and complex analysis

We have seen in Section 2.3.4 that the function ζ satisfies relations involving arithmetic operators, thereby establishing a connection between this function and number theory. In this section, we will explore much more explicit connections between ζ and arithmetic.

3.1 Eulerian Product

In this section, we will establish a relationship between the function ζ and prime numbers, stated in the following theorem.

Theorem 34: For all $s > 1$:

$$\zeta(s) = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

There is thus a strong link between ζ and the prime numbers, which are at the heart of arithmetic studies. This is an amazing explicit link.

Proof: In order to prove this, we will use probabilistic tools.
We consider the probability space $(\mathbb{N}^*, \mathscr{P}(\mathbb{N}^*))$ and we introduce the measure $\mathbb{P}_s$, with $s > 1$, defined by, for all $A \subset \mathbb{N}^*$:

$$\mathbb{P}_s(A) := \frac{1}{\zeta(s)} \sum_{n \in A} \frac{1}{n^s}.$$

This is indeed a probability [5] for $(\mathbb{N}^*, \mathscr{P}(\mathbb{N}^*))$. Indeed:

$$\mathbb{P}_s(\mathbb{N}^*) = \frac{1}{\zeta(s)} \sum_{n \in \mathbb{N}^*} \frac{1}{n^s} = 1.$$

If we consider $(A_n)_{n\in\mathbb{N}}$, a sequence of pairwise disjoint events, then:

$$\mathbb{P}_s\left(\bigsqcup_{k\in\mathbb{N}} A_k\right) = \frac{1}{\zeta(s)} \sum_{k\in\mathbb{N}} \sum_{n\in A_k} \frac{1}{n^s} = \sum_{k\in\mathbb{N}} \mathbb{P}_s(A_k).$$

Thus, $\mathbb{P}_s$ is indeed a probability, and therefore $(\mathbb{N}^*, \mathscr{P}(\mathbb{N}^*), \mathbb{P}_s)$ is a probability space.
We then define, for all $n \in \mathbb{N}^*$, $n\mathbb{N}^* := \{nk \mid k \in \mathbb{N}^*\}$. Hence:

$$\mathbb{P}_s(n\mathbb{N}^*) = \mathbb{P}_s\left(\bigsqcup_{k\in\mathbb{N}^*} \{nk\}\right) = \sum_{k\in\mathbb{N}^*} \mathbb{P}_s(\{nk\}) = \sum_{k\in\mathbb{N}^*} \frac{1}{\zeta(s)} \frac{1}{(nk)^s} = \frac{1}{n^s}.$$

Now, consider $(p_1, \ldots, p_n) \in \mathcal{P}^n$, an ordered tuple of pairwise distinct prime

[5] Note that the probabilistic measure given in this section allows us to define the zeta distribution, which is defined for discrete random variables.

numbers, then:

$$\begin{aligned}
\mathbb{P}_s\left(\bigcap_{i=1}^{n} p_i\mathbb{N}^*\right) &= \mathbb{P}_s\left(\{p_ik_i \mid i \in \{1,\dots,n\}, k_i \in \mathbb{N}^*\}\right) \\
&= \mathbb{P}_s\left(\{k\prod_{i=1}^{n} p_i \mid k \in \mathbb{N}^*\}\right) \\
&= \mathbb{P}_s\left(\prod_{i=1}^{n} p_i\mathbb{N}^*\right) \\
&= \frac{1}{\left(\prod\limits_{i=1}^{n} p_i\right)^s} \\
&= \frac{1}{\prod\limits_{i=1}^{n} p_i^s} \\
&= \prod_{i=1}^{n} \frac{1}{p_i^s} \\
&= \prod_{i=1}^{n} \mathbb{P}_s(p_i\mathbb{N}^*).
\end{aligned}$$

The second equality follows from the fundamental theorem of arithmetic [6]. Therefore, the events $p\mathbb{N}^*$, for $p \in \mathcal{P}$, are mutually independent. We thus deduce that the events $(p\mathbb{N}^*)^c$ are also mutually independent for $p \in \mathcal{P}$. We then define, for all $n \in \mathbb{N}^*$:

$$B_n := \bigcap_{k=1}^{n} (p_k\mathbb{N}^*)^c.$$

Concretely, B_n represents the set of integers that are not divisible by any of the first n prime numbers. The sequence $(B_n)_{n\in\mathbb{N}^*}$ is clearly decreasing with respect to inclusion. Thus:

$$\mathbb{P}_s\left(\bigcap_{n\in\mathbb{N}^*} B_n\right) = \lim_{n\to+\infty} \mathbb{P}_s(B_n).$$

But, for all $n \geq 1$:

$$\mathbb{P}_s(B_n) = \mathbb{P}_s\left(\bigcap_{k=1}^{n} (p_k\mathbb{N}^*)^c\right) = \prod_{k=1}^{n} \mathbb{P}_s\left((p_k\mathbb{N}^*)^c\right) = \prod_{k=1}^{n}\left(1 - \frac{1}{p_k^s}\right).$$

[6] Every strictly positive integer can be written as a product of prime numbers in a unique way, up to the order of the factors.

Then, by taking the limit:

$$\mathbb{P}_s\left(\bigcap_{n\in\mathbb{N}^*} B_n\right) = \prod_{n=1}^{+\infty}\left(1-\frac{1}{p_n^s}\right) = \prod_{p\in\mathcal{P}}\left(1-\frac{1}{p^s}\right).$$

Now, $\bigcap_{n\in\mathbb{N}^*} B_n = \{1\}$, since 1 is the unique positive integer divisible by no prime number. Thus:

$$\mathbb{P}_s\left(\bigcap_{n\in\mathbb{N}^*} B_n\right) = \frac{1}{\zeta(s)}.$$

We thus conclude that:

$$\zeta(s) = \prod_{p\in\mathcal{P}}\left(1-\frac{1}{p^s}\right)^{-1}.$$

The proof ends here. □

This relation is, in fact, verified for all $s \in \{\mathrm{Re}(.) > 1\}$, but the proof will not be provided.

3.2 An expression between the ζ function and the prime counting function π

In this section, we will delve into the connection between ζ and prime numbers.

Definition 35: Let $x \in \mathbb{R}$. Then, the prime counting function for primes less than or equal to x is defined by:

$$\pi(x) := \mathrm{card}\left(\{p \in \mathcal{P} \mid p \leq x\}\right).$$

What follows is partly based on the study by E.Royer [1].
The distribution function π is a cornerstone of arithmetic and number theory, playing a crucial role in understanding the distribution of prime numbers and their influence in many fields, particularly in modern cryptography.
Unfortunately, it is not possible to compute the exact values of $\pi(x)$ when x is very large. However, we have approximations, the most famous of which is the one given by the Prime Number Theorem, which states that:

$$\pi(x) \underset{x\to+\infty}{\sim} \frac{x}{\ln(x)}.$$

Here the approximation error is: $O\left(\frac{x}{\ln^2(x)}\right)$.

We can also mention a significantly better approximation than the previous one, using the function li:

$$\pi(x) \underset{x\to+\infty}{\sim} \mathrm{li}(x).$$

and,

$$\mathrm{li}(x) := \int_0^x \frac{\mathrm{d}t}{\ln(t)}.$$

The approximation error is better here, as it corresponds to: $O\left(x\exp\left(-\sqrt{\frac{\ln(x)}{69}}\right)\right)$.

It is possible to obtain even better approximations thanks to the connection between π and ζ through the zeros of the latter function, as will be explained in Section 3.3.

The link between ζ and π is made explicit by the following theorem.

Theorem 36: For all $s \in \{\mathrm{Re}(.) > 1\}$, we have:

$$\log(\zeta(s)) = s\int_2^{+\infty} \frac{\pi(x)}{x\,(x^s-1)}\,dx.$$

This relation is surprising because it presents an unexpected link between number theory and the theory of ζ.

Proof: Let $s \in \{\mathrm{Re}(.) > 1\}$. Then, from Theorem 34, we have:

$$\log(\zeta(s)) = \log\left(\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p^s}\right)^{-1}\right) = -\sum_{p\in\mathcal{P}}\log\left(1-\frac{1}{p^s}\right).$$

Let $n \geq 2$. Then two cases arise.

<u>If n is not a prime number, we have:</u> $\pi(n) = \pi(n-1)$, and therefore $\pi(n) - \pi(n-1) = 0$.

<u>If n is prime, then:</u> $\pi(n) = 1 + \pi(n-1)$, and therefore $\pi(n) - \pi(n-1) = 1$.

Furthermore, we know that the series $\sum\left((\pi(n)-\pi(n-1))\log\left(1-\frac{1}{n^s}\right)\right)$ is absolutely convergent because:

$$\left|(\pi(n)-\pi(n-1))\log\left(1-\frac{1}{n^s}\right)\right| \leq \left|\log\left(1-\frac{1}{n^s}\right)\right| \sim \frac{1}{n^{\mathrm{Re}(s)}}.$$

Thus, we perform the following summation:

$$\begin{aligned}\sum_{n=2}^{+\infty} (\pi(n) - \pi(n-1)) \log\left(1 - \frac{1}{n^s}\right) &= \sum_{n\geq 2, n\notin\mathcal{P}} (\pi(n) - \pi(n-1)) \log\left(1 - \frac{1}{n^s}\right) \\ &\quad + \sum_{n\geq 2, n\in\mathcal{P}} (\pi(n) - \pi(n-1)) \log\left(1 - \frac{1}{n^s}\right) \\ &= \sum_{p\in\mathcal{P}} \log\left(1 - \frac{1}{p^s}\right).\end{aligned}$$

Thus:

$$\begin{aligned}\log(\zeta(s)) &= -\sum_{n=2}^{+\infty} (\pi(n) - \pi(n-1)) \log\left(1 - \frac{1}{n^s}\right) \\ &= -\sum_{n=2}^{+\infty} \pi(n) \left(\log\left(1 - \frac{1}{n^s}\right) - \log\left(1 - \frac{1}{(n+1)^s}\right)\right) \\ &= \sum_{n=2}^{+\infty} \pi(n) \int_n^{n+1} \frac{s}{x(x^s - 1)}\, dx \\ &= \sum_{n=2}^{+\infty} \int_n^{n+1} \pi(x) \frac{s}{x(x^s - 1)}\, dx \\ &= s \int_2^{+\infty} \frac{\pi(x)}{x(x^s - 1)}\, dx.\end{aligned}$$

The proof ends here. $\square$

3.3 The zeros of the function ζ and the Riemann Hypothesis

In this section, we will see that there is a relation between the zeros of the function ζ and the function π.
It is precisely this relationship that is at the heart of the **Riemann Hypothesis**. Let us now study the zeros of ζ.

We will first prove the following theorem.

Theorem 37: ζ does not vanish on $\{\mathrm{Re}(.) > 1\}$. However, it vanishes on $\{\mathrm{Re}(.) < 0\}$ only at the points $s = -2k$, where $k \in \mathbb{N}^*$.

The zeros of the ζ function which lie in $\{\mathrm{Re}(s) < 0\}$ are referred to as trivial zeros.
Indeed, as we observed in Figure 3, ζ also vanishes on $\{0 \leq \mathrm{Re}(.) \leq 1\}$, and

it appears that all of these zeros have a real part of $\frac{1}{2}$. Such zeros are called non-trivial zeros and we still do not know their exact location.
In 1859, Bernhard Riemann conjectured that they should indeed all have a real part of $\frac{1}{2}$. This is the famous **Riemann Hypothesis**.

Proof: Let $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$:

Then, using the formula for the Euler product, we have:

$$\begin{aligned}
|\zeta(s)| &= \lim_{n\to+\infty} \left| \prod_{p\leq n} \frac{1}{1-\frac{1}{p^s}} \right| \\
&= \lim_{n\to+\infty} \exp\left(\ln\left(\left| \prod_{p\leq n} \frac{1}{1-\frac{1}{p^s}} \right| \right)\right) \\
&= \lim_{n\to+\infty} \exp\left(-\sum_{p\leq n} \ln\left(\left| 1-\frac{1}{p^s} \right| \right)\right).
\end{aligned}$$

We will now show that the sequence $\left(\sum_{p\leq n} \ln\left(\left| 1-\frac{1}{p^s} \right| \right)\right)_{n\geq 2}$ converges. It is known that, for every prime number p such that $p \leq n$, we have:

$$\ln\left(\left| 1-\frac{1}{p^s} \right| \right) = \frac{1}{2}\ln\left(\left| 1-\frac{1}{p^s} \right|^2 \right) = \frac{1}{2}\ln\left(1 - 2\mathrm{Re}\left(\frac{1}{p^s}\right) + \left|\frac{1}{p^s}\right|^2 \right).$$

Now,

$$\lim_{p\to+\infty} \left|\frac{1}{p^s}\right| = \lim_{p\to+\infty} \frac{1}{p^{\mathrm{Re}(s)}} = 0,$$

and thus:

$$\lim_{p\to+\infty} \left(-2\mathrm{Re}\left(\frac{1}{p^s}\right) + \left|\frac{1}{p^s}\right|^2 \right) = 0.$$

Hence, by applying a Taylor expansion, we obtain:

$$\frac{1}{2}\ln\left(1 - 2\mathrm{Re}\left(\frac{1}{p^s}\right) + \left|\frac{1}{p^s}\right|^2 \right) = \frac{1}{2}\left(-2\mathrm{Re}\left(\frac{1}{p^s}\right) + \left|\frac{1}{p^s}\right|^2 + o\left(\frac{1}{p^s}\right) \right) = O\left(\frac{1}{p^s}\right).$$

Thus,

$$\ln\left(\left| 1-\frac{1}{p^s} \right| \right) = O\left(\frac{1}{p^s}\right),$$

and since $\mathrm{Re}(s) > 1$, the sequence $\left(\frac{1}{p^s}\right)_{p\geq 2}$ is summable. Therefore, $\left(\ln\left(\left| 1-\frac{1}{p^s} \right| \right)\right)_{p\geq 2}$ is also summable. Hence, our series converges, and we deduce that:

$$|\zeta(s)| = \exp\left(-\sum_{p\in\mathcal{P}} \ln\left(\left| 1-\frac{1}{p^s} \right| \right)\right) > 0.$$

From this, we conclude that the function ζ does not vanish in the half-plane $\{\mathrm{Re}(.) > 1\}$.

Now suppose that $\mathrm{Re}(s) < 0$:

Thus, using the functional equation, We have that $\zeta(s) = 0$ if and only if:

$$\Gamma(1-s) = 0 \quad \text{or} \quad \zeta(1-s) = 0 \quad \text{or} \quad 2^s\pi^{s-1} = 0 \quad \text{or} \quad \sin\left(\frac{\pi s}{2}\right) = 0.$$

It is clear that $2^s\pi^{s-1} \neq 0$. Also, $\zeta(1-s) \neq 0$ because $\mathrm{Re}(1-s) > 1$. Finally, we cannot have $\Gamma(1-s) = 0$ by virtue of Theorem 28. Thus, this is equivalent to $\sin\left(\frac{\pi s}{2}\right) = 0$, which is equivalent to $\frac{\pi s}{2} = -k\pi$ with $k \in \mathbb{N}^*$, and finally equivalent to $s = -2k$, with $k \in \mathbb{N}^*$, because $\mathrm{Re}(s) < 0$.

Thus, ζ vanishes on $\{\mathrm{Re}(.) < 0\}$. $\square$

Let us now focus on the non-trivial zeros of ζ. The non-trivial zeros are of paramount importance because they establish a truly remarkable connection with the prime-counting function π. We will state the following theorem, which will not be proven in this paper.

Theorem 38: For all $x > 2$, we have:

$$\pi(x) = R(x) - \sum_{\rho} R(x^{\rho}),$$

where ρ runs over the set of non-trivial zeros of the ζ function, with

$$R(x) := \sum_{n=1}^{+\infty} \frac{\mu(n)}{n} \operatorname{li}(x^{1/n}) \quad \text{where } \mu \text{ is the Möbius function.}$$

Although this theorem provides an equality, it is impossible to compute the exact values of π using this relation. Indeed, the R function cannot be calculated explicitly. We can only approximate its values. However, to this day, Theorem 38 actually provides one of the most powerful and precise expressions for approximating π. Indeed, the Riemann Hypothesis is equivalent to a much tighter bound on the error in the approximation of π, leading to a more regular distribution of prime numbers:

$$\pi(x) = \operatorname{li}(x) + O\left(\sqrt{x}\ln(x)\right).$$

4 A Brief Algorithmic Analysis

In this section, we will mathematically justify the operation of the computer codes that allowed us to obtain our graphical representations.

4.1 Numerical method for the graphical representation of ζ on $]1, +\infty[$

Here, we aim to approximate the series sums that define ζ in order to plot the function for real values of $s > 1$.

4.1.1 Naive method

For numerical purposes, the code approximates $\zeta(s)$ using the following sum:

$$\zeta(s) \approx \sum_{n=1}^{N} \frac{1}{n^s} =: S_{N,1}(s),$$

where N represents the number of terms used for the approximation. Therefore, we need to choose sufficiently large values for N while respecting the machine's limits, and ensuring that the error is small enough. The approximation error is calculated as follows:

$$|\zeta(s) - S_{N,1}(s)| = \sum_{n=N+1}^{+\infty} \frac{1}{n^s} =: E_{N,1}(s).$$

By applying a series-integral comparison, we obtain:

$$\int_{N+1}^{+\infty} \frac{\mathrm{d}t}{t^s} \leq \sum_{n=N+1}^{+\infty} \frac{1}{n^s} \leq \int_{N}^{+\infty} \frac{\mathrm{d}t}{t^s}.$$

Thus:

$$\frac{(N+1)^{1-s}}{s-1} \leq E_{N,1}(s) \leq \frac{N^{1-s}}{s-1}.$$

From this, we deduce that:

$$E_{N,1}(s) \sim \frac{(N+1)^{1-s}}{s-1} \quad \text{as } N \to +\infty.$$

Therefore, for sufficiently large values of N, we have the approximation $E_{N,1}(s) \approx \frac{(N+1)^{1-s}}{s-1}$.

The following graph is plotted with a logarithmic scale on the y-axis.

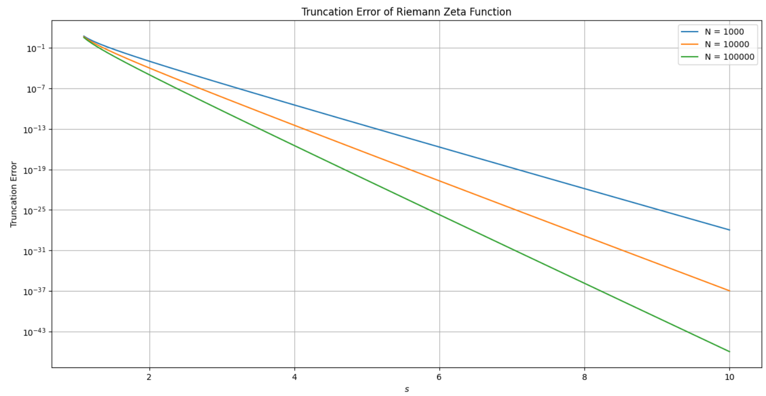


Figure 4: Truncation Error of ζ

The approximation is accurate for $N = 10^5$. A representation of ζ on $]1, +\infty[$ gives:

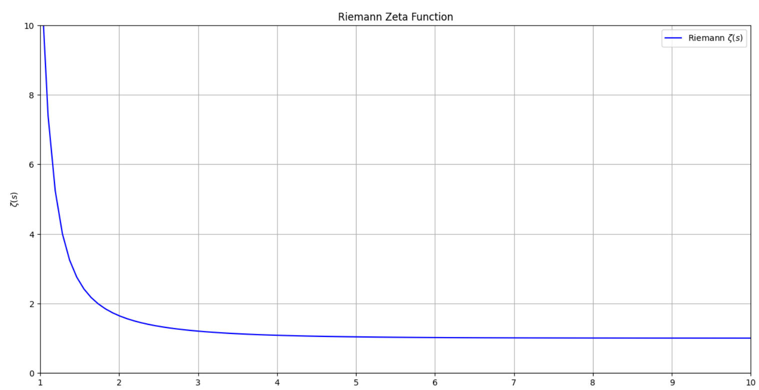


Figure 5: The function ζ on $]1, +\infty[$ from the naive method where $N = 10^5$.

The use of libraries such as `NumPy` and `Matplotlib` also ensures stable execution and professional-quality plots.

In conclusion, the method allows approximation and visualization of the function ζ for $s > 1$. Its simplicity and mathematical rigor ensure accurate results.

4.1.2 Accelerated Method

We consider $s > 1$. In this section, we will apply the method of binomial transformation. This is a method well-suited for alternating series that accelerates their convergence. The binomial transformation is formulated in the following proposition.

Proposition 39: Consider the alternating series $\sum (-1)^n a_n$. Then, defining:

$$\Delta^n a_0 := \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_{n-k},$$

we have:

$$\sum_{n=0}^{+\infty} (-1)^n a_n = \sum_{n=0}^{+\infty} \frac{(-1)^n \Delta^n a_0}{2^{n+1}}.$$

The acceleration of convergence comes from the fact that the terms on the right-hand side generally decrease much faster, allowing for a rapid numerical computation of the sum. Indeed, the factor 2^{n+1} in the denominator significantly reduces the contribution of higher-order terms, thereby decreasing the sum of the remaining terms. Moreover, the operator Δ^n reduces the variations of the terms in the series, which improves the convergence.

We will apply this to the Dirichlet function η, because it is an alternating series. This will allow us to obtain approximate values for η and hence ζ thanks to the link in Proposition 11.

Let us prove Proposition 39.

Proof: We start from the right-hand side:

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{2^{n+1}} \Delta^n a_0 = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^{n+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_{n-k} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^{n+1}} \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} a_m.$$

By Fubini's theorem, we get:

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{2^{n+1}} \Delta^n a_0 = \sum_{n=0}^{+\infty} \sum_{m=0}^{n} \frac{(-1)^m}{2^{n+1}} \binom{n}{n-m} a_m = \sum_{m=0}^{+\infty} \sum_{n=m}^{+\infty} \frac{(-1)^m}{2^{n+1}} \binom{n}{n-m} a_m.$$

Then:

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{2^{n+1}} \Delta^n a_0 = \sum_{m=0}^{+\infty} (-1)^m a_m \sum_{n=m}^{+\infty} \frac{1}{2^{n+1}} \binom{n}{n-m} = \sum_{m=0}^{+\infty} (-1)^m a_m \sum_{n=m}^{+\infty} \frac{1}{2^{n+1}} \binom{n}{m}.$$

Now, we have:

$$\sum_{n=m}^{+\infty} \frac{1}{2^{n+1}} \binom{n}{m} = \sum_{k=0}^{+\infty} \frac{1}{2^{k+m+1}} \binom{m+k}{m} = \frac{1}{2^{m+1}} \sum_{k=0}^{+\infty} \frac{1}{2^k} \binom{m+k}{m}.$$

We use the following relation:

$$\sum_{k=0}^{+\infty} (-1)^k \binom{m+k}{m} x^k = \frac{1}{(1+x)^{m+1}} \quad \text{valid when } |x| < 1.$$

Thus:

$$\sum_{n=m}^{+\infty} \frac{1}{2^{n+1}} \binom{m}{n} = \frac{1}{2^{m+1}} \frac{1}{\left(\frac{1}{2}\right)^{m+1}} = 1.$$

Therefore,

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{2^{n+1}} \Delta^n a_0 = \sum_{m=0}^{+\infty} (-1)^m a_m.$$

This concludes the proof. □

Since $\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(n+1)^s}$, we will apply the previous method by setting $a_n := \frac{1}{(n+1)^s}$. For sufficiently large $N \in \mathbb{N}$, we then have:

$$\eta(s) = \sum_{n=0}^{+\infty} (-1)^n a_n = \sum_{n=0}^{+\infty} \frac{(-1)^n \Delta^n a_0}{2^{n+1}} \approx \sum_{n=0}^{N} (-1)^n \frac{\Delta^n a_0}{2^{n+1}} =: S_{N,2}(s).$$

Hence the error is:

$$|\eta(s) - S_{N,2}(s)| = \left| \sum_{n=0}^{+\infty} (-1)^n \frac{\Delta^n a_0}{2^{n+1}} - \sum_{n=0}^{N} (-1)^n \frac{\Delta^n a_0}{2^{n+1}} \right| = \left| \sum_{n=N+1}^{+\infty} (-1)^n \frac{\Delta^n a_0}{2^{n+1}} \right| \leq \frac{|\Delta^{N+1} a_0|}{2^{N+2}}.$$

The last inequality follows from the fact that the series is alternating.

Thus the final error in the approximation of $\zeta(s)$ is dominated by:

$$E_{N,2}(s) := \frac{|\Delta^{N+1} a_0|}{2^{N+2} |1 - 2^{1-s}|}.$$

A representation of ζ on $]1, +\infty[$ gives:

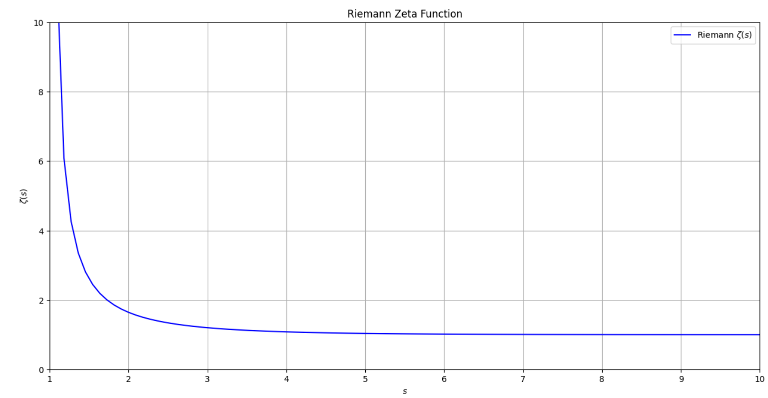


Figure 6: The function ζ on $]1, +\infty[$ from the accelerated method where $N = 500$.

We now need to determine which is the best method.

4.1.3 Comparison of Methods

We set $N = 10^3$ here and will compare the calculation errors for the two methods.

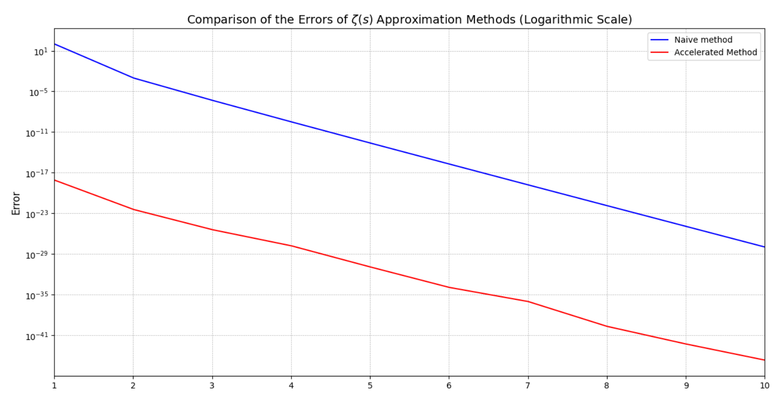


Figure 7: Comparison of the Errors of $\zeta(s)$ Approximation Methods (Logarithmic Scale)

The results are clear. The errors with method 2 are approximately 10^{19} times smaller. This method is much more efficient.

4.2 Numerical Method for Graphical Representation of $|\zeta|$ on $\{\mathrm{Re}(.) > 1\}$

In this part, we will approximate the modulus of ζ and represent this approximation on $\{\mathrm{Re}(.) > 1\}$.

4.2.1 Naive method

We will use a naive method similar to the previous one. We have:

$$|\zeta(s)| \approx \left| \sum_{n=1}^{N} \frac{1}{n^s} \right| = |S_{N,1}(s)|,$$

The approximation error is then calculated as follows:

$$||\zeta(s)| - |S_N(s)|| \leq |\zeta(s) - S_N(s)| = \left| \sum_{n=N+1}^{+\infty} \frac{1}{n^s} \right| \leq \sum_{n=N+1}^{+\infty} \frac{1}{n^{\mathrm{Re}(s)}} = E_{N,1}(\mathrm{Re}(s)).$$

We define $\sigma := \mathrm{Re}(s)$. Thus, $E_{N,1}(\sigma)$ acts as a control factor on the error. Since we are taking large values of N, we can again use that $E_{N,1}(\sigma) \approx \frac{(N+1)^{1-\sigma}}{\sigma-1}$.

The calculation methods follow the same approach as before, and by taking $N = 10^5$ as previously, we obtain the following graph.

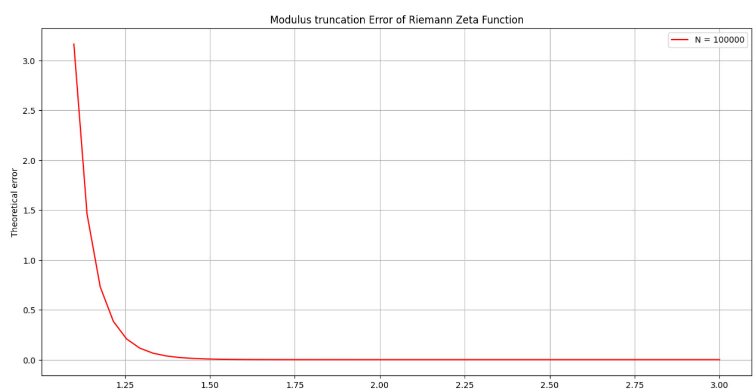


Figure 8: Modulus truncation Error of Riemann Zeta Function

The approximation is accurate for $N = 10^5$.
A representation of ζ on the given domain gives :

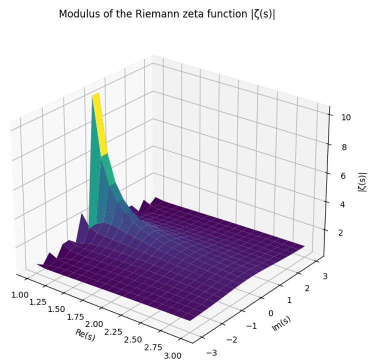


Figure 9: The function $|\zeta|$ on $\{\mathrm{Re}(.) > 1\}$ from the naive method where $N = 10^5$.

In conclusion, the method allows for the approximation and visualization of the modulus of the function ζ for $\mathrm{Re}(s) > 1$. Its simplicity and mathematical rigor ensure accurate results, unless $\mathrm{Re}(s)$ is too close to 1.

4.2.2 Accelerated Method

We apply the binomial transform method (see Section 4.1.2) to approximate the modulus of η. We have:

$$|\eta(s)| \approx \left| \sum_{n=0}^{N} (-1)^n \frac{\Delta^n a_0}{2^{n+1}} \right| = |S_{N,2}(s)|.$$

The approximation error is therefore calculated as follows:

$$||\eta(s)| - |S_{N,2}(s)|| \leq |\eta(s) - S_{N,2}(s)| = \frac{|\Delta^{N+1} a_0|}{2^{N+2}}.$$

Thus, the final error in the approximation of $\zeta(s)$ is dominated by:

$$\frac{|\Delta^{N+1} a_0|}{2^{N+2}|1 - 2^{1-s}|} \leq \frac{|\Delta^{N+1} a_0|}{2^{N+2}|1 - 2^{1-\mathrm{Re}(s)}|} = E_{N,2}(\mathrm{Re}(s)).$$

Let $\sigma := \mathrm{Re}(s)$, and we have the following error graph:

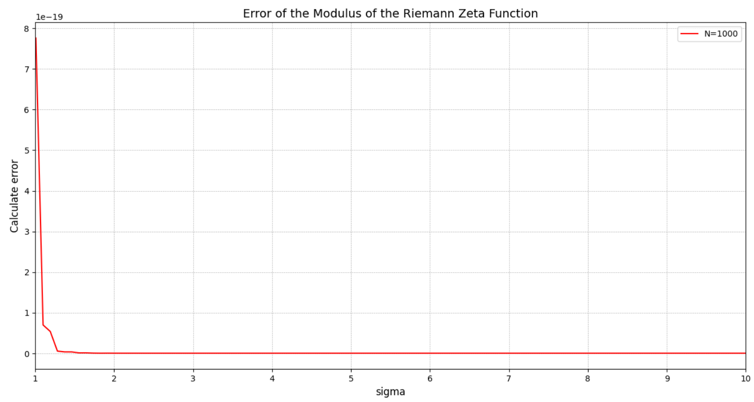


Figure 10: Error resulting from the approximation of $|\zeta|$

The approximation is very accurate for $N = 10^3$.
A representation of ζ on the given domain gives:

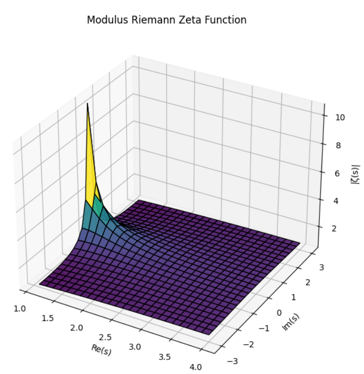


Figure 11: The function $|\zeta|$ on $\{\mathrm{Re}(.) > 1\}$ from the accelerated method where $N = 25$.

4.2.3 Comparison of Methods

Given that the errors are of the same type as in the real context, it is the accelerated method that, as expected, proves to be more efficient. This is particularly evident at points near 1, where the representations are of much better quality than in the naive case with only $N = 25$.

Conclusion

The study of ζ has allowed us to explore one of the most fascinating objects in **complex analysis** and **number theory**. Starting from its initial definition as a **Dirichlet series**, we progressively extended its domain to the complex plane, highlighting its fundamental analytical properties and its **analytic continuation** on $\mathbb{C}\setminus\{1\}$. We also demonstrated its famous **functional equation**, which establishes a deep connection between its values at s and $1-s$.

The significance of ζ is not limited to its analytical properties; it serves as a **true bridge** between complex analysis and the arithmetic of **prime numbers**. Its development as an Euler product reveals the intimate relationship between its **zeros** and the distribution of **prime numbers**. In particular, the **Riemann Hypothesis**, which posits that all non-trivial zeros have a real part equal to $\frac{1}{2}$, remains one of the most profound **unsolved problems in mathematics**, with considerable implications for understanding the distribution of primes.

Finally, we have discussed numerical methods for evaluating $\zeta(s)$, illustrating the importance of **algorithmic approaches** in its study. These tools not only allow us to explore the function at points that are difficult to analyze theoretically but also enable us to visualize its singular behaviors in the complex plane.

Far from being a mere theoretical curiosity, ζ appears in numerous mathematical domains and its study continues to **inspire researchers** and provides a gateway to an inexhaustible wealth of mathematical developments.

Appendices

What follows are the proofs of the theorems that were not proven in the thesis.

A Bernoulli Polynomial

This section contains the results concerning the Bernoulli polynomials.

We will prove, Definition-Proposition 2.

Proof: We show the existence and uniqueness of the sequence by constructing it term by term.
We start by setting $B_0 := 1$.
Then, we assume that the term B_n, with $n \in \mathbb{N}$, exists and is unique. We must now construct the term B_{n+1} and prove its uniqueness.
We consider P, a polynomial such that:

$$P' = (n+1)B_n.$$

Thus,
$$P = (n+1)\beta_n + k,$$
where β_n denotes the unique primitive of B_n vanishing at 0, and $k \in \mathbb{R}$ denotes the constant of integration.
We impose:
$$\int_0^1 P(x)\,dx = 0.$$
This gives:
$$k = -(n+1)\int_0^1 \beta_n(x)\,dx.$$
Thus,
$$P = (n+1)\left(\beta_n - \int_0^1 \beta_n(x)\,dx\right).$$
This completely determines P. By setting $B_{n+1} := P$, we have:
$$B'_{n+1} = (n+1)B_n \quad \text{and} \quad \int_0^1 B_{n+1}(x)\,dx = 0.$$
Moreover, the reasoning previously carried out guarantees the uniqueness of B_{n+1}.
Thus, the sequence $(B_n)_{n\in\mathbb{N}}$ is correctly constructed, and it is also unique. $\square$

We will now prove Proposition 3.

Proof: Let $n > 1$. Then:
$$\int_0^1 B_{n-1}(x)\,dx = 0, \quad \text{so} \quad \int_0^1 B'_n(x)\,dx = 0, \quad \text{which implies} \quad B_n(1) = B_n(0). \quad \square$$

Note that, $B'_1 = B_0 = 1$, and thus $B_1 = X + k$ where $k \in \mathbb{R}$. Now, $\int\limits_0^1 B_1(x)\,dx = 0$, so $k = -\frac{1}{2}$. Hence, $B_1 = X - \frac{1}{2}$.

The first Bernoulli polynomials are found in [3, p. 10].

B Important Results in Complex Analysis

Here, essential results in complex analysis are provided to justify statements left unproven in the thesis.
Also, some results of the thesis require some elements on sequences of holomorphic functions. This will therefore be addressed in this section.

Let's begin by proving Theorem 8.

Proof: For all $x \in X$ and $z \in U$, we define $g_x(z) := f(z,x)$. Let $(z_n)_{n\in\mathbb{N}} \in U^{\mathbb{N}}$ such that $\lim_{n\to+\infty} z_n = z$ and $z_n \neq z$ for all $n \in \mathbb{N}$. Then,

$$\frac{F(z)-F(z_n)}{z-z_n} = \int_X \frac{f(z,x)-f(z_n,x)}{z-z_n}\,d\mu(x) = \int_X \frac{g_x(z)-g_x(z_n)}{z-z_n}\,d\mu(x).$$

Since $z \mapsto f(z,x)$ is holomorphic on U for all $x \in X$, it follows that $z \mapsto g_x(z)$ is also holomorphic, and we have

$$g_x'(z) = \frac{\partial f}{\partial z}(z,x), \quad \text{for all } z \in U, x \in X.$$

Thus,

$$\lim_{n\to+\infty} \frac{g_x(z)-g_x(z_n)}{z-z_n} = g_x'(z), \quad \text{for all } x \in X.$$

We now need to dominate the variation rate. Since U is open, there exists $r > 0$ such that $D(z,r) \subset U$. For $r' > 0$ sufficiently small compared to r, we apply Cauchy's integral formula:

$$g_x'(z) = \frac{1}{2\pi i}\int_{C(z,r')} \frac{g_x(\omega)}{(\omega-z)^2}\,d\omega.$$

Hence,

$$|g_x'(z)| = \left|\frac{1}{2\pi i}\int_{C(z,r')} \frac{g_x(\omega)}{(\omega-z)^2}\,d\omega\right| \leq \frac{1}{2\pi}2\pi r' \sup_{\omega\in C(z,r')} \left|\frac{g_x(\omega)}{(\omega-z)^2}\right|.$$

Thus,

$$|g_x'(z)| \leq \frac{\varphi(x)}{r'}.$$

Since $z_n \to z$, for sufficiently large n, we have $z_n \in D(z,r')$. As $D(z,r')$ is convex, it follows that $[z_n,z] \subset D(z,r')$, so:

$$g_x(z) - g_x(z_n) = \int_{[z_n,z]} g_x'(\omega)\,d\omega.$$

This implies

$$|g_x(z)-g_x(z_n)| \leq |z-z_n| \sup_{\omega\in[z_n,z]} |g_x'(\omega)| \leq |z-z_n|\frac{\varphi(x)}{r'}, \quad \text{for all } x \in X.$$

Thus, for sufficiently large n, we have

$$\left|\frac{g_x(z)-g_x(z_n)}{z-z_n}\right| \leq \frac{\varphi(x)}{r'}, \quad \text{for all } x \in X.$$

Since φ is integrable, we apply the dominated convergence theorem:

$$F'(z) = \lim_{n\to+\infty} \frac{F(z)-F(z_n)}{z-z_n} = \int_X \lim_{n\to+\infty} \frac{g_x(z)-g_x(z_n)}{z-z_n}\,d\mu(x) = \int_X g_x'(z)\,d\mu(x).$$

Thus,

$$F'(z) = \int_X \frac{\partial f}{\partial z}(z, x)\, d\mu(x).$$

This concludes the proof. $\square$

Next, let's prove Theorem 25.
This proof will be carried out using two approaches, denoted (1) and (2). Approach (1) consists of proving the theorem by using the arcwise connectedness of U (since U is open), while approach (2) does not rely on paths.

Proof with (1): It is clear that 1 implies 2, which implies 3, which implies 4. In order to complete the chain of equivalence, it is enough to prove that 4 implies 1. We assume that f is identically zero on a set of points, which we will call A, that has an accumulation point, denoted a, in U. Thus, $f = 0$ on A.
Since f is holomorphic on U, it is analytic there. Therefore, we can develop it as a power series in the neighborhood of a. Hence, for $r > 0$, much smaller than R, the radius of convergence of the following power series, we have for all $z \in \mathbb{C}$ such that $|z| < r$,

$$f(z + a) = \sum_{n=0}^{+\infty} c_n z^n.$$

Now, a is an accumulation point of A. Therefore, it is a limit point of $A \setminus \{a\}$. Thus, there exists $(a_k)_{k\in\mathbb{N}}$ such that $a = \lim_{k\to+\infty} a_k$. Hence, there exists $N \in \mathbb{N}$, such that for $k \geq N$, $|a_k - a| \leq r$. Therefore, for all $k \geq N$,

$$f(a_k) = f(a_k - a + a) = \sum_{n=0}^{+\infty} c_n (a_k - a)^n.$$

Now, since $a_k \in A$, we have $f(a_k) = 0$. Thus,

$$\sum_{n=0}^{+\infty} c_n (a_k - a)^n = 0.$$

Therefore,

$$\lim_{k\to+\infty} c_n (a_k - a)^n = \begin{cases} c_0 & \text{if } n = 0, \\ 0 & \text{if } n \geq 1. \end{cases}$$

Moreover, for all $n \in \mathbb{N}$ and $k \geq N$,

$$|c_n (a_k - a)^n| \leq |c_n| r^n,$$

and

$$\sum_{n=0}^{+\infty} |c_n| r^n < +\infty,$$

since r is very small compared to R. Hence, by passing to the limit and interchanging sums, we obtain:

$$\lim_{k\to+\infty}\sum_{n=0}^{+\infty} c_n(a_k-a)^n = \sum_{n=0}^{+\infty}\lim_{k\to+\infty} c_n(a_k-a)^n = c_0 = 0.$$

Next, we show by induction that for all $p \in \mathbb{N}$, $c_p = 0$.
The base case has been verified above. Now, suppose $c_p = 0$ (and thus $c_0 = c_1 = \cdots = c_{p-1} = c_p = 0$). Thus, for all $k \geq N$, we have:

$$0 = \sum_{n=0}^{+\infty} c_n(a_k-a)^n,$$

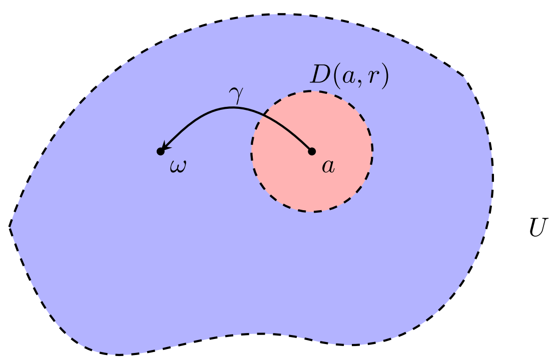
which implies :

$$0 = \sum_{n=p+1}^{+\infty} c_n(a_k-a)^n.$$

Since $a_k \in A \setminus \{a\}$, it follows that $a_k - a \neq 0$. Therefore,

$$\sum_{n=p+1}^{+\infty} c_n(a_k-a)^{n-(p+1)} = 0.$$

Thus, reasoning as before, by passing to the limit as $k \to +\infty$, we have $c_{p+1} = 0$. This concludes the induction.
Therefore, $c_p = 0$ for all $p \in \mathbb{N}$. Thus, f is zero on $D(a, r)$.



Let $\omega \in U$. Consider now the path γ, starting from a and ending at ω (thus $\gamma(0) = a$ and $\gamma(1) = \omega$), which does not loop back on itself. Such a path exists, as U is connected. We then define:

$$B := \{t \in [0, 1] \mid f \circ \gamma = 0 \text{ on } [0, t[\}.$$

Such a set is non-empty, as f is zero in the neighborhood of a. Since $\beta := \sup(B)$, for any $\epsilon > 0$ very small, $\beta - \epsilon$ is not an upper bound of B, so there exists $t \in B$ such that $\beta - \epsilon < t \leq \beta$. Thus, $f \circ \gamma = 0$ on $[0, \beta - \epsilon[$. Hence, by taking the limit as $\epsilon \to 0$, and by the continuity of $f \circ \gamma$, we have $f \circ \gamma = 0$ on $[0, \beta[$. Thus, $\beta \in B$.
Finally, by the continuity of $f \circ \gamma$, we have:

$$\lim_{t \to \beta^-} f(\gamma(t)) = 0 = f \circ \gamma(\beta).$$

We now proceed by contradiction and assume $\beta < 1$.
We have:

$$\overline{[0, \beta[} = [0, \beta].$$

Thus, there exists $(\beta_n)_{n\in\mathbb{N}} \in ([0, \beta[)^{\mathbb{N}}$ such that $\beta = \lim\limits_{n\to+\infty} \beta_n$. By the continuity of γ,

$$\lim_{n\to+\infty} \gamma(\beta_n) = \gamma(\beta).$$

Finally, $(\gamma(\beta_n))_{n\in\mathbb{N}} \in (\gamma([0, \beta[))^{\mathbb{N}}$. Therefore, $\gamma(\beta)$ is an accumulation point of $\gamma([0, \beta])$. Thus, as for the point a, f is zero in the neighborhood of $\gamma(\beta)$. Therefore, for sufficiently small $\epsilon > 0$, $f(\gamma(\beta + \epsilon)) = 0$. This leads to a contradiction because β would not be the upper bound.
Hence, $\beta = 1$ and $f \circ \gamma(1) = 0$, so $f(\omega) = 0$. Therefore, f is zero on U . □

Another alternative consists in exploiting the connectedness of U without involving paths. We will first show that 1, 2, and 3 are equivalent. Then, since it is clear that 3 implies 4, it will suffice to prove that 4 implies 1 using the established chain of equivalences. Thus, we will have proved that 1, 2, 3, and 4 are equivalent.

Proof with (2): As stated before, it is clear that 1 implies 2 and 2 implies 3. We now prove that 3 implies 1. Consider $a \in U$ such that $f^{(n)}(a) = 0$ for all $n \in \mathbb{N}$. Define

$$E := \{z \in U \mid f^{(n)}(z) = 0, \text{ for all } n \in \mathbb{N}\}.$$

Since $a \in E$, we have $E \neq \emptyset$. As f is holomorphic on U, it is C^∞ on U, and thus $f^{(n)}$ is continuous for all $n \in \mathbb{N}$. This ensures that E is closed.
We now show that E is open. Let $\xi \in E$. For $r > 0$ sufficiently small and $z \in D(\xi, r)$, we have the expansion

$$f(z) = \sum_{n=0}^{+\infty} c_n (z - \xi)^n.$$

For all $n \in \mathbb{N}$,

$$c_n = \frac{f^{(n)}(\xi)}{n!} = 0,$$

thus $f(z) = 0$. Since f is holomorphic, it follows that $f^{(n)} = 0$ in this domain. Hence, $D(\xi, r) \subset E$, proving that E is open.

Thus,

$$U = E \sqcup E^c.$$

Since E is both open and closed, E and E^c are open. Therefore, we necessarily have $E = U$, meaning f is identically zero on U. Thus, 1 is equivalent to 2, which is equivalent to 3.

We now show that 4 implies 1. Let $A \subset U$ be a set such that $a \in A$ is an accumulation point of A. Suppose f is zero on A. By reasoning as in the proof 1, we can justify the existence of $r > 0$, sufficiently small, such that f is identically zero on $D(a, r)$. Thus, for all $n \in \mathbb{N}$, $f^{(n)}(a) = 0$. By the implication $3 \Rightarrow 1$, we conclude that $f = 0$ on U. □

Sequences and Infinite Products of Holomorphic Functions.

Theorem 28 involves an infinite product of holomorphic functions. The legitimate question that arises is whether such a product remains holomorphic.

To approach the proof of this theorem, we therefore need results related to sequences and infinite products of holomorphic functions. The results of this section are inspired by the proofs from D.Hulin [4].

Infinite Products

This is a brief introduction to infinite products.

Definition 40: Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. For $n \in \mathbb{N}$, we define $P_n := \prod\limits_{k=0}^{n} u_k$. We say that the infinite product $\prod u_n$ converges if the sequence $(P_n)_{n\in\mathbb{N}}$ converges, and we define,

$$\prod_{n=0}^{+\infty} u_n := \lim_{n\to\infty} \prod_{k=0}^{n} u_k.$$

We now state a crucial result in the analysis of infinite products. It involves relating the study of infinite products to that of series.

Proposition 41: Let $(v_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$. If $\sum\limits_{n=0}^{+\infty} |v_n| < +\infty$, then $\prod(1 + v_n)$ converges.

Proof: The hypothesis ensures that $v_n \to 0$. Thus, $1+v_n \to 1$. Therefore, $1+v_n$ will be close to 1 from a certain index onward N. Here, we use the principal logarithmic determination, denoted log. We then have $\log(1 + v_n) \sim v_n$. Since $(v_n)_{n\in\mathbb{N}}$ is summable, we get $\sum \log(1 + v_n)$ converges, which implies that the product $\prod(1 + v_n)$ converges. □

Sequences of Holomorphic Functions.

Infinite products of holomorphic functions can be seen as sequences of finite products of holomorphic functions. To better understand such mathematical objects, it is essential to study sequences of holomorphic functions.

Definition 42: Let $f_n : U \to \mathbb{C}$. The sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges locally uniformly to $f : U \to \mathbb{C}$ if, for every point $z \in U$, there exists a neighborhood $V \subset U$ of this point such that the sequence $(f_n|_V)_{n\in\mathbb{N}}$ of restrictions converges uniformly to $f|_V$.

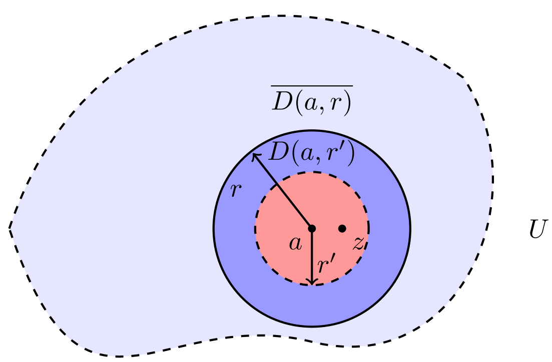
With such convergence, it is possible that the uniform limit is also holomorphic.

Theorem 43: Let U be an open subset of $\mathbb{C}$, and $(f_n)_{n\in\mathbb{N}}$ a sequence of holomorphic functions on U. Suppose that the sequence $(f_n)_{n\in\mathbb{N}}$ converges locally uniformly to $f : U \to \mathbb{C}$. Then f is holomorphic.

We will now justify this theorem.

Proof: Let U be an open subset of $\mathbb{C}$, and $(f_n)_{n\in\mathbb{N}}$ a sequence of holomorphic functions on U. Suppose that the sequence $(f_n)_{n\in\mathbb{N}}$ converges locally uniformly to $f : U \to \mathbb{C}$.
Let $n \in \mathbb{N}$. Since f_n is holomorphic on U, we can apply the Cauchy Integral Theorem. We consider $a \in U$ and $C(a, r)$, with $r > 0$, small enough. We then consider $z \in D(a, r')$, where $r' = \frac{r}{2}$. Hence, we have:

$$f_n(z) = \frac{1}{2\pi i} \int_{C(a,r)} \frac{f_n(\omega)}{\omega - z}\, d\omega.$$



By uniform local convergence, we know that $f_n \to f$ uniformly on $\overline{D(a,r)} \subset U$.

By uniform convergence, we get:

$$\int_{C(a,r)} \frac{f_n(\omega)}{\omega - z}\, d\omega \to \int_{C(a,r)} \frac{f(\omega)}{\omega - z}\, d\omega.$$

By the uniqueness of the limit, we then have:

$$f(z) = \frac{1}{2\pi i} \int_{C(a,r)} \frac{f(\omega)}{\omega - z}\, d\omega.$$

Now, $z \mapsto \frac{f(\omega)}{\omega - z}$ is holomorphic on $D(a, r')$. As the uniform limit of holomorphic functions and thus continuous, f is continuous on $\overline{D(a, r)}$. Therefore, it is bounded on this compact for $M > 0$. Thus, for every $\omega \in C(a, r)$, and for every $z \in D(a, r')$, we have:

$$\left| \frac{f(\omega)}{\omega - z} \right| \leq \frac{M}{r'}.$$

We can then apply Theorem 8 on $D(a, r')$. Thus, f is holomorphic on every open disk in U. Therefore, f is holomorphic on U. $\square$

From this result, we immediately deduce the following corollary.

Corollary 44: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of holomorphic functions. If the product $\prod f_n$ converges locally uniformly, then the limit of the product, given by $\prod_{n=0}^{+\infty} f_n$, is a holomorphic function.

This corollary is justified by applying the theorem that precedes it to the sequence $(\prod_{k=0}^{n} f_k)_{n\in\mathbb{N}}$.
We therefore have a context in which we are able to determine whether an infinite product of holomorphic functions is also holomorphic or not.

C On the Γ Function

This section provides the results on the behavior of the function Γ with respect to $\mathbb{C}$.

Let's prove Definition-Proposition 12.

Proof: Let $s \in \mathbb{C}$. It is sufficient to observe that $t \mapsto t^{s-1}e^{-t} \in L^1(]0, +\infty[)$ when $\mathrm{Re}(s) > 0$. $\square$

Let's continue by proving Theorem 28.

Proof: We will prove this Theorem. Therefore We will apply the results proven in Appendix B that is, Proposition 41 and Corollary 44, which provide results on infinite products.

Let us show that Γ is holomorphic on $\{\mathrm{Re}(.) > 0\}$. We use Theorem 8:
For all $t > 0$, $s \mapsto t^{s-1}e^{-t} = \exp((s-1)\ln(t) - t)$ is holomorphic. Then, consider $K \subset \{\mathrm{Re}(.) > 0\}$, a compact set : there exist $0 < a \leq b$ real numbers such that for all $s \in K$, $\mathrm{Re}(s) \in [a, b]$. If $t > 0$ and $s \in K$, then,

$$|t^{s-1}e^{-t}| = t^{\mathrm{Re}(s)-1}e^{-t} \leq \begin{cases} t^{a-1}e^{-t} & \text{if } t \leq 1, \\ t^{b-1}e^{-t} & \text{if } t \geq 1, \end{cases}$$

which shows that the function is integrable on $]0, +\infty[$, because $a > 0$.
Thus, Γ is holomorphic on $\{s \in \{\mathrm{Re}(.) > 0\}$.

We show the equality on $\mathbb{R}_+^*$ and will extend it to $\mathbb{C}$ by analytic continuation, since $\mathbb{R}_+^*$ has accumulation points.
For all $n \in \mathbb{N}$ and $x > 0$, define:

$$g_n(x) := \int_0^n t^{x-1}\left(1 - \frac{t}{n}\right)^n dt = \int_0^{+\infty} \mathbf{1}_{[0,n]}(t)t^{x-1}\left(1 - \frac{t}{n}\right)^n dt.$$

We know that for all $t > 0$,

$$\mathbf{1}_{[0,n]}(t)t^{x-1}\left(1 - \frac{t}{n}\right)^n \to t^{x-1}e^{-t}.$$

Moreover, for all $n \in \mathbb{N}$ and for all $t > 0$,

$$\left|\mathbf{1}_{[0,n]}(t)t^{x-1}\left(1 - \frac{t}{n}\right)^n\right| \leq t^{x-1}e^{-t}.$$

We can therefore apply the dominated convergence theorem and conclude that $g_n(x) \to \Gamma(x)$. Next, we explicitly calculate $g_n(x)$.
By making the substitution $u = \frac{t}{n}$, we get:

$$g_n(x) = n^x \int_0^1 u^{x-1}(1-u)^n \, du.$$

By performing several integrations by parts, we obtain:

$$\begin{aligned} g_n(x) &= n^x \frac{n}{x} \int_0^1 u^x (1-u)^{n-1} \, du \\ &= n^x \frac{n}{x} \frac{n-1}{x+1} \cdots \frac{1}{x+n-1} \int_0^1 u^{x+n-1} \, du \\ &= n^x \frac{n!}{x(x+1)\cdots(x+n-1)(x+n)}. \end{aligned}$$

Thus, we find:

$$g_n(x) = \frac{n^x}{x \prod_{k=1}^{n} \left(1 + \frac{x}{k}\right)}.$$

Then:

$$g_n(x) = \frac{n^x e^{-\sum\limits_{k=1}^{n} \frac{x}{k}}}{x \prod\limits_{k=1}^{n} \left(1 + \frac{x}{k}\right) e^{-\frac{x}{k}}}.$$

Thus:

$$g_n(x) = \frac{1}{x} e^{x\left(\ln(n) - \sum\limits_{k=1}^{n} \frac{1}{k}\right)} \prod_{k=1}^{n} \frac{e^{\frac{x}{k}}}{1 + \frac{x}{k}}.$$

Therefore:

$$\Gamma(x) = \lim_{n \to +\infty} g_n(x) = \lim_{n \to +\infty} \left(\frac{1}{x} e^{x\left(\ln(n) - \sum\limits_{k=1}^{n} \frac{1}{k}\right)} \prod_{k=1}^{n} \frac{e^{\frac{x}{k}}}{1 + \frac{x}{k}} \right).$$

Thus,

$$\frac{1}{\Gamma(x)} = \lim_{n \to +\infty} \left(x e^{x\left(\sum\limits_{k=1}^{n} \frac{1}{k} - \ln(n)\right)} \prod_{k=1}^{n} \left(1 + \frac{x}{k}\right) e^{-\frac{x}{k}} \right).$$

Now,

$$\sum_{k=1}^{n} \frac{1}{k} - \ln(n) \to \gamma \quad \text{as } n \to +\infty.$$

Moreover, for all $x > 0$,

$$\left(1 + \frac{x}{k}\right) e^{-\frac{x}{k}} = 1 + O\left(\frac{1}{k^2}\right) \quad \text{as } k \to +\infty.$$

In particular, the product $\prod \left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}}$ converges by Proposition 41. Therefore:

$$\frac{1}{\Gamma(x)} = x e^{\gamma x} \prod_{n=1}^{+\infty} \left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}}.$$

Similarly, it can be shown that the infinite product converges on $\mathbb{C}$. We will now prove that it converges uniformly locally on $\mathbb{C}$. For all $n \in \mathbb{N}^*$, $z \mapsto \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$ is holomorphic on $\mathbb{C}$. Consider $z \in \mathbb{C}$, and let $R > 0$ be sufficiently large such that $|z| < R$. Then,

$$\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} = \left(1 + \frac{z}{n}\right) \left(1 - \frac{z}{n} + O_R\left(\frac{1}{n^2}\right)\right) = 1 + O_R\left(\frac{1}{n^2}\right).$$

This is a domination independent of z. Therefore, the product $\prod \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$ converges uniformly locally on $\mathbb{C}$, and hence $z \mapsto \prod\limits_{n=1}^{+\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$ is holomorphic on $\mathbb{C}$ by Corollary 44. The equality being verified on $\mathbb{R}_+^*$, we extend it to $\mathbb{C}$ by complex analytic continuation. Thus, $\frac{1}{\Gamma}$ extends to a holomorphic function on $\mathbb{C}$. $\square$

D Integral Calculations and Summation Methods

Some results involving integrals and sums have been stated in the thesis.
We therefore prove them in this section.

We will prove Proposition 7.

Proof: We need to study the convergence of

$$\int_1^{+\infty} \frac{dx}{x^s}, \quad s \in \mathbb{C}.$$

<u>Suppose $\mathrm{Re}(s) > 1$:</u> Then $x \mapsto \frac{1}{x^s} \in L^1([1, +\infty[)$, because $\left|\frac{1}{x^s}\right| = \frac{1}{x^{\mathrm{Re}(s)}}$, for all $x \in [1, +\infty[$.

<u>Now suppose $\mathrm{Re}(s) \leq 1$:</u>

If $s = 1$, then the integral diverges.

Next, if $s \neq 1$, we proceed with partial integrals. Let $t \in [1, +\infty[$. Then :

$$\int_1^t \frac{dx}{x^s} = \frac{t^{1-s} - 1}{1 - s}.$$

If $\mathrm{Re}(s) < 1$, then since $|t^{1-s}| = t^{1-\mathrm{Re}(s)} \to +\infty$, the integral diverges as $t \to +\infty$.

Finally, if $\mathrm{Re}(s) = 1$, we have:

$$\int_1^t \frac{dx}{x^s} = \frac{i\left(t^{-i\,\mathrm{Im}(s)} - 1\right)}{\mathrm{Im}(s)},$$

which diverges as $t \to +\infty$.

Therefore, the integral converges if, and only if, $\mathrm{Re}(s) > 1$. $\square$

Here, we prove Proposition 14.

Proof: Let $x \in \mathbb{R}$. The equality holds immediately when $x < 1$, as the sums are empty, yielding $0 = 0$. Then, for $x \geq 1$, we have:

$$\begin{aligned}\sum_{1\leq n\leq x} a_n\varphi(n) &= A(x)\varphi(x) + \sum_{1\leq n\leq x} a_n\varphi(n) - A(x)\varphi(x)\\ &= A(x)\varphi(x) + \sum_{1\leq n\leq x} a_n\varphi(n) - \sum_{1\leq n\leq x} a_n\varphi(x)\\ &= A(x)\varphi(x) - \sum_{1\leq n\leq x} a_n(\varphi(x) - \varphi(n))\\ &= A(x)\varphi(x) - \sum_{1\leq n\leq x}\int_n^x a_n\varphi'(u)\,du.\end{aligned}$$

Next, since φ is $\mathcal{C}^1$ and the sum is indexed over a finite domain, by Fubini's theorem, we get:

$$\begin{aligned}\sum_{1\leq n\leq x} a_n\varphi(n) &= A(x)\varphi(x) - \int_1^x \sum_{1\leq n\leq u} a_n\varphi'(u)\,du\\ &= A(x)\varphi(x) - \int_1^x A(u)\varphi'(u)\,du.\end{aligned}$$

This concludes the proof. $\square$

Recall that for any x belonging to $]0, +\infty[$, we define:

$$\phi(x) := \sum_{n=1}^{+\infty} e^{-n^2\pi x}.$$

Let us now prove Proposition 27.

The arguments that follow are derived from Parreaux [12, p. 3].

Proof: To prove this result, we will use the function θ, defined for all $x > 0$ as:

$$\theta(x) := \sum_{n\in\mathbb{Z}} e^{-n^2 x}.$$

We begin by applying Poisson's summation formula to $f(t) := e^{-yt^2}$ for all $t \in \mathbb{R}$ and for all $y > 0$. The Fourier transform of f is:

$$\hat{f}(t) = \sqrt{\frac{\pi}{y}}e^{-\frac{\pi^2 t^2}{y}}.$$

Thus, Poisson's summation formula gives:

$$\sum_{n\in\mathbb{Z}} \hat{f}\left(\frac{n}{2\pi}\right) = 2\pi\sum_{k\in\mathbb{Z}} f(2\pi k).$$

Now,

$$\sum_{n\in\mathbb{Z}} \hat{f}\left(\frac{n}{2\pi}\right) = \sum_{n\in\mathbb{Z}} \sqrt{\frac{\pi}{y}} e^{-\frac{n^2}{4y}},$$

and,

$$\sum_{k\in\mathbb{Z}} f(2\pi k) = \sum_{k\in\mathbb{Z}} e^{-y4\pi^2k^2}.$$

Thus:

$$\sum_{n\in\mathbb{Z}} \sqrt{\frac{\pi}{y}} e^{-\frac{n^2}{4y}} = 2\pi \sum_{k\in\mathbb{Z}} e^{-y4\pi^2k^2}.$$

The equation then becomes:

$$\sqrt{\frac{\pi}{y}}\theta\left(\frac{1}{4y}\right) = 2\pi\theta(4\pi^2 y).$$

By the change of variable $x = 4\pi^2 y > 0$, we get:

$$\sqrt{\frac{\pi}{x}}\theta\left(\frac{\pi^2}{x}\right) = \theta(x).$$

Finally, for all $x > 0$, the sequence $(e^{-n^2x})_{n\in\mathbb{Z}}$ is summable. Therefore:

$$\theta(x) = 1 + 2\sum_{n=1}^{+\infty} e^{-n^2x}.$$

Thus, $\theta(\pi x) = 1 + 2\phi(x)$. Now, from the functional equation satisfied by θ, we have:

$$\sqrt{\frac{1}{x}}\theta\left(\frac{\pi}{x}\right) = \theta(\pi x).$$

This gives:

$$1 + 2\phi(x) = \sqrt{\frac{1}{x}}\left(1 + 2\phi\left(\frac{1}{x}\right)\right).$$

Thus:

$$\phi(x) = \frac{1}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\phi\left(\frac{1}{x}\right) - \frac{1}{2}.$$

This completes the proof. □

Let us proceed with the proof of Proposition 32.

Proof: We know that for every $x \in \mathbb{R}$,

$$\{x+1\} = x + 1 - \lfloor x+1 \rfloor = x - \lfloor x \rfloor = \{x\}.$$

Thus, $\{\cdot\}$ is a 1-periodic function. Moreover, for all $x \in]n, n+1[$, with $n \in \mathbb{Z}$, we have $\{x\} = x - n$. Therefore, the function is piecewise $\mathcal{C}^1$ on $\mathbb{R} \setminus \mathbb{Z}$. The Dirichlet theorem is then applicable.

The Fourier coefficients are given by:

$$a_0 = \int_0^1 \{x\}\, dx = \int_0^1 x\, dx = \frac{1}{2}.$$

Next, we compute a_n and b_n for all $n \in \mathbb{N}^*$ using integration by parts. Hence:

$$a_n = 2\int_0^1 x\cos(2\pi nx)\, dx = -\frac{1}{\pi n}\int_0^1 \sin(2\pi nx)\, dx = 0.$$

Then,

$$b_n = 2\int_0^1 x\sin(2\pi nx)\, dx = 2\left(-\frac{1}{2\pi n} + \int_0^1 \frac{\cos(2\pi nx)}{2\pi n}\, dx\right) = -\frac{1}{\pi n}.$$

Therefore, for all $x \in \mathbb{R} \setminus \mathbb{Z}$,

$$\{x\} = \frac{1}{2} - \sum_{n=1}^{+\infty} \frac{\sin(2\pi nx)}{\pi n}.$$

This completes the proof. $\square$

Finally, let us conclude with the proof of Lemma 33.

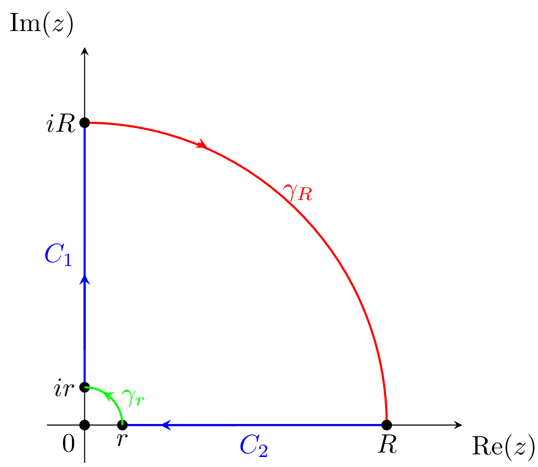
Proof: Let $s \in \mathbb{C}$ such that $0 < \mathrm{Re}(s) < 1$. To prove this, we first show that:

$$\int_{\mathbb{R}_+} z^{s-1}e^{-z}\, dz = \int_{i\mathbb{R}_+} z^{s-1}e^{-z}\, dz.$$

To do so, consider Γ, the contour defined by:

$$\Gamma := C_1 \sqcup \gamma_R \sqcup C_2 \sqcup \gamma_r,$$

with C_1 the segment $i[0, R]$, γ_R the quarter circle of radius R connecting iR to R, C_2, the segment $[0, R]$ traversed in the opposite direction, and finally γ_r, the quarter circle of radius r connecting r to ir.



Thus,

$$\int_{\Gamma} z^{s-1}e^{-z}\,dz = \int_{C_1} z^{s-1}e^{-z}\,dz + \int_{\gamma_R} z^{s-1}e^{-z}\,dz + \int_{C_2} z^{s-1}e^{-z}\,dz + \int_{\gamma_r} z^{s-1}e^{-z}\,dz.$$

In our calculations, we will use the principal logarithmic branch. Thus, the function $z \mapsto z^{s-1}e^{-z}$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}_-$ and has 0 as its singularity. Therefore, by the Green-Riemann theorem, we have:

$$\int_{\Gamma} z^{s-1}e^{-z}\,dz = 0.$$

Consequently,

$$\int_{C_1} z^{s-1}e^{-z}\,dz + \int_{\gamma_R} z^{s-1}e^{-z}\,dz + \int_{C_2} z^{s-1}e^{-z}\,dz + \int_{\gamma_r} z^{s-1}e^{-z}\,dz = 0.$$

Now,

$$\int_{C_1} z^{s-1}e^{-z}\,dz = \int_{i[r,R]} z^{s-1}e^{-z}\,dz,$$

and

$$\int_{\gamma_R} z^{s-1}e^{-z}\,dz = -iR^s \int_0^{\frac{\pi}{2}} e^{i(s-1)\theta - Re^{i\theta} + i\theta}\,d\theta.$$

with,

$$\int_{C_2} z^{s-1}e^{-z}\,dz = -\int_r^R z^{s-1}e^{-z}\,dz.$$

Finally,

$$\int_{\gamma_r} z^{s-1}e^{-z}\,dz = ir^s \int_0^{\frac{\pi}{2}} e^{i(s-1)\theta - re^{i\theta} + i\theta}\,d\theta.$$

Therefore,

$$\int_{i[r,R]} z^{s-1}e^{-z}\,dz - iR^s\int_0^{\frac{\pi}{2}} e^{i(s-1)\theta - Re^{i\theta}+i\theta}\,d\theta - \int_r^R z^{s-1}e^{-z}\,dz + ir^s\int_0^{\frac{\pi}{2}} e^{i(s-1)\theta - re^{i\theta}+i\theta}\,d\theta = 0.$$

Next,

$$\left| iR^s\int_0^{\frac{\pi}{2}} e^{i(s-1)\theta - Re^{i\theta}+i\theta}\,d\theta \right| \leq R^{\mathrm{Re}(s)}\int_0^{\frac{\pi}{2}} e^{-\theta \mathrm{Im}(s)}e^{-R\cos(\theta)}\,d\theta.$$

Now, by the concavity of the cosine function, we have $\cos(\theta) \geq -\frac{2}{\pi}\theta + 1$ for all $\theta \in [0, \frac{\pi}{2}]$.

From this, we deduce that:

$$e^{-\theta\,\mathrm{Im}(s) - R\cos(\theta)} \leq e^{\frac{2}{\pi}\theta R - R - \theta\,\mathrm{Im}(s)}, \quad \text{for all } \theta \in \left[0, \frac{\pi}{2}\right].$$

Hence,

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} e^{-\theta\,\mathrm{Im}(s) - R\cos(\theta)}\,d\theta &\leq \int_0^{\frac{\pi}{2}} e^{\frac{2}{\pi}\theta R - R - \theta\,\mathrm{Im}(s)}\,d\theta, \\
&= e^{-R}\left[\frac{e^{\left(\frac{2}{\pi}R - \mathrm{Im}(s)\right)\theta}}{\frac{2}{\pi}R - \mathrm{Im}(s)}\right]_0^{\frac{\pi}{2}}, \\
&= e^{-R}\left(\frac{e^{R - \frac{\pi}{2}\,\mathrm{Im}(s)} - 1}{\frac{2}{\pi}R - \mathrm{Im}(s)}\right), \\
&\sim e^{-R}\frac{e^{R - \frac{\pi}{2}\,\mathrm{Im}(s)}}{\frac{2}{\pi}R} \quad \text{as } R \to +\infty.
\end{aligned}$$

Thus,

$$\begin{aligned}
R^{\mathrm{Re}(s)}\int_0^{\frac{\pi}{2}} e^{-\theta\,\mathrm{Im}(s) - R\cos(\theta)}\,d\theta &\sim R^{\mathrm{Re}(s)}e^{-R}\frac{e^{R - \frac{\pi}{2}\,\mathrm{Im}(s)}}{\frac{2}{\pi}R} \quad \text{as } R \to +\infty. \\
&= \frac{\pi}{2}R^{\mathrm{Re}(s)-1}e^{-\frac{\pi}{2}\,\mathrm{Im}(s)}.
\end{aligned}$$

$$\frac{\pi}{2}R^{\mathrm{Re}(s)-1}e^{-\frac{\pi}{2}\,\mathrm{Im}(s)} \to 0, \quad \text{as } R \to +\infty, \text{ since } -1 < \mathrm{Re}(s) - 1 < 0.$$

Subsequently,

$$\left| ir^s\int_0^{\frac{\pi}{2}} e^{i(s-1)\theta - re^{i\theta}+i\theta}\,d\theta \right| \leq r^{\mathrm{Re}(s)}\int_0^{\frac{\pi}{2}} e^{-\theta \mathrm{Im}(s)}e^{-r\cos(\theta)}\,d\theta.$$

Moreover, for all $\theta \in [0, \frac{\pi}{2}]$,

$$e^{-\theta \mathrm{Im}(s)}e^{-r\cos(\theta)} \leq e^{-\theta \mathrm{Im}(s)},$$

and $\theta \mapsto e^{-\theta \mathrm{Im}(s)} \in L^1([0, \frac{\pi}{2}])$. By interchanging the limit and integral,

$$r^{\mathrm{Re}(s)} \int_0^{\frac{\pi}{2}} e^{-\theta \mathrm{Im}(s)} e^{-r\cos(\theta)} \, d\theta \to 0 \quad \text{as } r \to 0.$$

Thus,

$$\int_{i\mathbb{R}_+} z^{s-1} e^{-z} \, dz - \int_{\mathbb{R}_+} z^{s-1} e^{-z} \, dz = 0,$$

which implies that,

$$\int_{\mathbb{R}_+} z^{s-1} e^{-z} \, dz = \int_{i\mathbb{R}_+} z^{s-1} e^{-z} \, dz.$$

The computations are similar on $i\mathbb{R}_-$, and we can show that:

$$\int_{i\mathbb{R}_-} z^{s-1} e^{-z} \, dz = -\int_{\mathbb{R}_+} z^{s-1} e^{-z} \, dz.$$

Using this result, we can now compute the Mellin transform of the sine function:

$$\begin{aligned}\int_0^{+\infty} t^{s-1} \sin(t) \, dt &= \int_0^{+\infty} t^{s-1} \frac{e^{it} - e^{-it}}{2i} \, dt, \\ &= \frac{1}{2i} \left(\int_0^{+\infty} t^{s-1} e^{it} \, dt - \int_0^{+\infty} t^{s-1} e^{-it} \, dt \right).\end{aligned}$$

By performing the substitution $t = iz$ in the first integral and $z = it$ in the second, we have:

$$\begin{aligned}\int_0^{+\infty} t^{s-1} \sin(t) \, dt &= \frac{1}{2i} \left(-i^s \int_{i]-\infty,0]} z^{s-1} e^{-z} \, dz - \frac{1}{i^s} \int_{i[0,+\infty[} z^{s-1} e^{-z} \, dz \right) \\ &= \frac{1}{2i} \left(i^s \int_{[0,+\infty[} z^{s-1} e^{-z} \, dz - \frac{1}{i^s} \int_{[0,+\infty[} z^{s-1} e^{-z} \, dz \right) \\ &= \frac{1}{2i} \left(i^s - i^{-s} \right) \Gamma(s) \\ &= \frac{e^{i\frac{\pi s}{2}} - e^{-i\frac{\pi s}{2}}}{2i} \Gamma(s) \\ &= \sin\left(\frac{\pi s}{2} \right) \Gamma(s).\end{aligned}$$

This completes the proof. $\square$

E Multiplicative Functions and Dirichlet Convolutions

This section will present, in addition to Dirichlet's convolution theorem, some results on the notion of multiplicative functions, in order to prove Proposition

22.

Recall that for f and g, two arithmetic functions, the Dirichlet convolution of these two functions is the function $f * g$ defined by, for all $n \in \mathbb{N}^*$:

$$(f * g)(n) := \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

Let's prove Theorem 19.

Proof: Let $s \in \mathbb{C}$ such that the series $\sum \frac{f(n)}{n^s}$ and $\sum \frac{g(n)}{n^s}$ converge absolutely. Thus, we have:

$$\sum_{n=1}^{+\infty} \frac{f(n)}{n^s} \sum_{n=1}^{+\infty} \frac{g(n)}{n^s} = \sum_{d=1}^{+\infty} \sum_{k=1}^{+\infty} \frac{f(d)g(k)}{(dk)^s} = \sum_{(d,k) \in (\mathbb{N}^*)^2} \frac{f(d)g(k)}{(dk)^s}.$$

For $n \in \mathbb{N}^*$, we define:

$$I_n := \{(d,k) \in (\mathbb{N}^*)^2 \mid dk = n\}.$$

From this, we deduce that:

$$(\mathbb{N}^*)^2 = \bigsqcup_{n \in \mathbb{N}^*} I_n.$$

Thus, since both series converge absolutely, we get:

$$\sum_{n=1}^{+\infty} \frac{f(n)}{n^s} \sum_{n=1}^{+\infty} \frac{g(n)}{n^s} = \sum_{n=1}^{+\infty} \sum_{(d,k) \in I_n} \frac{f(d)g(k)}{(dk)^s} = \sum_{n=1}^{+\infty} \sum_{(d,k) \in (\mathbb{N}^*)^2 | dk = n} \frac{f(d)g(k)}{(dk)^s}.$$

Finally, we obtain:

$$\sum_{n=1}^{+\infty} \frac{f(n)}{n^s} \sum_{n=1}^{+\infty} \frac{g(n)}{n^s} = \sum_{n=1}^{+\infty} \sum_{d|n} \frac{f(d)g(\frac{n}{d})}{n^s} = \sum_{n=1}^{+\infty} \frac{(f * g)(n)}{n^s}. \quad \square$$

Multiplicative functions.

[11], as a particularly rich work on arithmetic functions, has been used as a basis for this study.

We begin by defining the notion of multiplicative functions.

Definition 45: An arithmetic function f is said to be multiplicative if:

- $f(1) \neq 0$,
- For all $(m,n) \in (\mathbb{N}^*)^2,$ if $m \wedge n = 1,$ then $f(mn) = f(m)f(n).$

In particular, if f is an multiplicative function, then $f(1) = f(1)^2$, and since $f(1) \neq 0$, we have $f(1) = 1$.

Multiplicative functions form a subgroup of the group of arithmetic functions under the Dirichlet convolution operation $*$.
This structure provides a powerful algebraic framework to study number-theoretic problems. In particular, the convolution operation allows for the decomposition of functions into simpler parts and the use of algebraic tools to study their properties.
One of the key applications of multiplicative functions is in the study of the distribution of prime numbers and the analysis of Dirichlet series, such as the ζ function. The Dirichlet series of a multiplicative function often admits an Euler product, which is a product over primes that reveals deep connections between the function and prime number theory.
Moreover, the inversion of the Dirichlet convolution, facilitated by the Möbius function, is a fundamental tool in analytic number theory. By using the algebraic structure of multiplicative functions, one can efficiently compute sums over divisors, study the behavior of arithmetic functions, and obtain results about prime number distribution and other central topics in the theory of numbers.

An important theorem about such functions is the uniqueness principle for multiplicative functions. This is stated as follows:

Theorem 46: Let f and g be two multiplicative functions. If, for all $p \in \mathcal{P}$ and all $k \in \mathbb{N}^*$, $f(p^k) = g(p^k)$, then $f = g$.

Proof: We prove this by induction. Specifically, we aim to show that, for all $n \in \mathbb{N}^*$, $f(n) = g(n)$.

Since f and g are multiplicative, we have $f(1) = 1 = g(1)$. This proves the base case.

Let $n \in \mathbb{N}^*$ such that $f(n) = g(n)$. By the fundamental theorem of arithmetic, there exists a unique $(p_1, \ldots, p_k) \in \mathcal{P}^k$ of distinct prime numbers, and a unique $(\alpha_1, \ldots, \alpha_k) \in (\mathbb{N}^*)^k$, such that:

$$n + 1 = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where $k \geq 1$.
For $i \neq j$, we have $p_i \neq p_j$, and hence, since they are prime numbers, we have:

$$p_i^{\alpha_i} \wedge p_j^{\alpha_j} = 1.$$

Since f is multiplicative, we get

$$f(p_i^{\alpha_i} p_j^{\alpha_j}) = f(p_i^{\alpha_i}) f(p_j^{\alpha_j}),$$

and the same holds for g. Thus, by multiplicativity, we have:

$$f(n+1) = f\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right) = \prod_{i=1}^{k} f(p_i^{\alpha_i}) = \prod_{i=1}^{k} g(p_i^{\alpha_i}) = g\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right) = g(n+1).$$

Thus, for all $n \in \mathbb{N}^*$, $f(n) = g(n)$.
Hence, $f = g$. $\square$

Moreover, the convolution operation defines an internal composition law on the set of multiplicative functions.

Theorem 47: If f and g are two multiplicative functions, then $f * g$ is also multiplicative.

Proof: Let f and g be two multiplicative functions. Let $(m, n) \in (\mathbb{N}^*)^2$ such that $m \wedge n = 1$. Consider $d \in \mathcal{D}_n$ and $d' \in \mathcal{D}_m$. Then, $d \wedge d' = 1$.
Since f is multiplicative, $f(dd') = f(d)f(d')$. Similarly, $\frac{n}{d} \in \mathcal{D}_n$ and $\frac{m}{d'} \in \mathcal{D}_m$. Thus, $\frac{n}{d} \wedge \frac{m}{d'} = 1$. By the multiplicativity of g, we have $g\left(\frac{n}{d}\frac{m}{d'}\right) = g\left(\frac{n}{d}\right) g\left(\frac{m}{d'}\right)$.
Finally, since $\mathcal{D}_n \times \mathcal{D}_m \to \mathcal{D}_{nm}$ is a bijection, we get:

$$\begin{aligned}
(f * g)(nm) &= \sum_{d \mid nm} f(d) g\left(\frac{nm}{d}\right) \\
&= \sum_{d \in \mathcal{D}_{nm}} f(d) g\left(\frac{nm}{d}\right) \\
&= \sum_{(d,d') \in \mathcal{D}_n \times \mathcal{D}_m} f(dd') g\left(\frac{n}{d}\frac{m}{d'}\right) \\
&= \sum_{d \in \mathcal{D}_n} \sum_{d' \in \mathcal{D}_m} f(d) f(d') g\left(\frac{n}{d}\right) g\left(\frac{m}{d'}\right) \\
&= \left(\sum_{d \mid n} f(d) g\left(\frac{n}{d}\right)\right) \left(\sum_{d' \mid m} f(d') g\left(\frac{m}{d'}\right)\right).
\end{aligned}$$

Thus, $(f * g)(nm) = (f * g)(n)(f * g)(m)$. Hence, $f * g$ is multiplicative. $\square$

With all these results, we can now prove Proposition 22.

To prove this, we will use the Chinese Remainder Theorem. This is an important theorem in arithmetic because it states that, for $(m, n) \in \mathbb{N}^2$ such that $m \wedge n = 1$, the map $\alpha : \mathbb{Z}/mn\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ defines a ring isomorphism. This allows for solving modular systems of the form:

$$\begin{cases} x \equiv a[m] \\ x \equiv b[n] \end{cases}$$

which, by this theorem, has a unique solution modulo mn.

Let us prove Proposition 22.

Proof: We prove that μ is multiplicative:

Let $(m,n) \in (\mathbb{N}^*)^2$ be two coprime integers. If $m = 1$, then $\mu(m) = 1$, and thus $\mu(mn) = \mu(n) = \mu(m)\mu(n)$. The same holds if $n = 1$.
Now assume $m \neq 1$ and $n \neq 1$. Decompose m and n into their prime factors. We have $m = \prod_{i=1}^{k} p_i^{\alpha_i}$ and $n = \prod_{i=1}^{l} q_i^{\beta_i}$. Since m and n are coprime, no p_i can correspond to a q_i.
Assume that either m or n contains a prime p with multiplicity greater than one in its factorization. In this case, $\mu(m) = 0$ or $\mu(n) = 0$, so $\mu(m)\mu(n) = 0$. Furthermore, mn contains p with multiplicity greater than one in its factorization, which implies $\mu(mn) = 0 = \mu(m)\mu(n)$.
Now assume that m and n each contain every prime factor with multiplicity one. Thus, $m = \prod_{i=1}^{k} p_i$ and $n = \prod_{i=1}^{l} q_i$, giving $mn = \prod_{i=1}^{k} p_i \prod_{i=1}^{l} q_i$.
Finally, $\mu(m) = (-1)^k$, $\mu(n) = (-1)^l$, and $\mu(mn) = (-1)^{k+l} = \mu(m)\mu(n)$. Therefore, μ is multiplicative.

Consider $p \in \mathcal{P}$ and $k \in \mathbb{N}^*$. Since μ and $\mathbf{1}$ are multiplicative, their convolution $(\mu * \mathbf{1})$ is also multiplicative. The divisors of p^k are p^i for $0 \leq i \leq k$. We have $\mu(1) = 1 \neq 0$, $\mu(p) = -1$, and $\mu(p^k) = 0$ for $k \geq 2$. Hence :

$$(\mu * \mathbf{1})(p^k) = \sum_{d \mid p^k} \mu(d)\mathbf{1}\left(\frac{p^k}{d}\right) = \sum_{i=0}^{k} \mu(p^i) = \mu(1) + \mu(p) = 1 - 1 = 0.$$

Since $\delta_1(p^k) = 0$, it follows that $(\mu * \mathbf{1})(p^k) = \delta_1(p^k)$. Thus, by Theorem 46, we conclude that $\mu * \mathbf{1} = \delta_1$.

We prove that φ is multiplicative:

Let $n \in \mathbb{N}^*$. Denote $U(\mathbb{Z}/n\mathbb{Z})$ the set of invertible elements of the commutative ring $(\mathbb{Z}/n\mathbb{Z}, +, \times)$. Hence, $\varphi(n) = \operatorname{card}(U(\mathbb{Z}/n\mathbb{Z}))$.
Let $(m,n) \in (\mathbb{N}^*)^2$ be two coprime integers. By the Chinese Remainder Theorem, the map $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/mn\mathbb{Z}$ is a ring isomorphism. It maps the invertible elements of the rings accordingly. Thus:

$$\begin{aligned}
\varphi(mn) &= \operatorname{card}(U(\mathbb{Z}/mn\mathbb{Z})) \\
&= \operatorname{card}(U(\mathbb{Z}/n\mathbb{Z}) \times U(\mathbb{Z}/m\mathbb{Z})) \\
&= \operatorname{card}(U(\mathbb{Z}/n\mathbb{Z})) \times \operatorname{card}(U(\mathbb{Z}/m\mathbb{Z})) \\
&= \varphi(n)\varphi(m).
\end{aligned}$$

Thus, φ is multiplicative.

Now consider $p \in \mathcal{P}$ and $k \in \mathbb{N}^*$. Then,

$$\varphi(p^k) = \mathrm{card}\left(\{i \in \{1, \ldots, p^k\} \mid i \wedge p^k = 1\}\right).$$

By complementing, this gives:

$$\varphi(p^k) = p^k - \mathrm{card}\left(\{i \in \{1, \ldots, p^k\} \mid i \wedge p^k \neq 1\}\right).$$

Let $i \in \mathbb{N}$ such that $1 \leq i \leq p^k$. Consider $d := i \wedge p^k$. Hence, $d \mid p^k$, so $d = 1, p, p^2, \ldots, p^k$. The case $d \neq 1$ is irrelevant. If $d = p, p^2, \ldots, p^k$, then $p \mid d$, and thus $p \mid i$. Therefore, the only reachable values of i are $p, 2p, 3p, \ldots, p^{k-1}p$. Thus :

$$\mathrm{card}\left(\{i \in \{1, \ldots, p^k\} \mid i \wedge p^k \neq 1\}\right) = \mathrm{card}(p, 2p, 3p, \ldots, p^{k-1}p) = p^{k-1}.$$

Finally, we obtain:

$$\varphi(p^k) = p^k - p^{k-1}.$$

Again consider $p \in \mathcal{P}$ and $k \in \mathbb{N}^*$. The divisors of p^k are p^i for $0 \leq i \leq k$. Recall that $\mu(1) = 1$, $\mu(p) = -1$, and $\mu(p^k) = 0$ for $k \geq 2$. Hence:

$$(\mathrm{Id} * \mu)(p^k) = \sum_{d \mid p^k} \mu(d)\mathrm{Id}(\frac{p^k}{d}) = \sum_{i=0}^{k} \mu(p^i)p^{k-i} = \mu(1)p^k + \mu(p)p^{k-1} = p^k - p^{k-1}.$$

Thus:

$$(\mathrm{Id} * \mu)(p^k) = \varphi(p^k).$$

Since the functions are multiplicative, we deduce by Theorem 46, that $(\mu * \mathbf{1})(n) = \varphi(n)$ for all $n \in \mathbb{N}^*$. $\square$

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