Holomorphic ODEs Around the Notion of Monodromy and Regular Singularities

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Differential equations play a central role in mathematics and applied sciences. They indeed model the dynamics of evolving phenomena and allow expressing how one quantity varies as a function of another, often time or a spatial variable, through relationships involving derivatives.

When defined within the framework of complex analysis, they reveal a richness of singular and profound phenomena that have no direct counterparts in the real context.

Let us focus, for instance, on the equation:

$$y' = \frac{1}{z^2} y, \quad z \in \mathbb{C}^*.$$

It is easy to see that $y(z) := \exp\left(-\frac{1}{z}\right)$, for all $z \in \mathbb{C}^*$, defines a solution to our equation.

Introduction

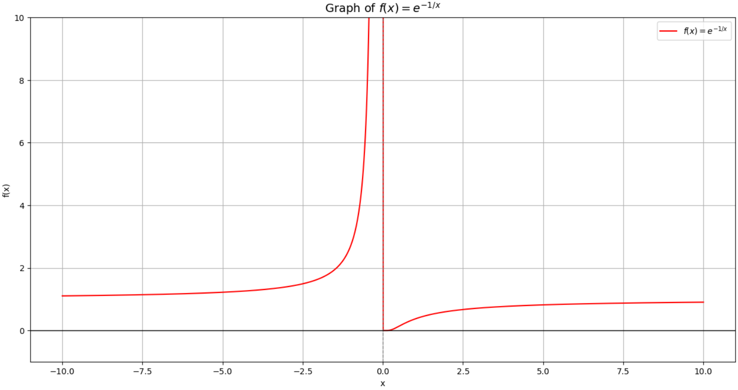
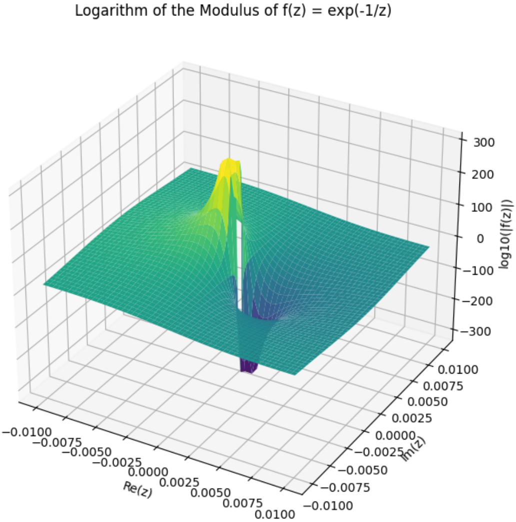


Figure: The solution on $\mathbb{R}^*$.

Introduction



This shows that the geometry of the complex plane profoundly influences the study of solutions to a holomorphic differential equation, in contrast to the real setting.
Thus, in this presentation, we aim to gain a deeper understanding of the structure and properties of solutions to holomorphic differential equations, highlighting in particular the distinctive features offered by the holomorphic setting compared to the real case.

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Homotopic paths

Let $U \subset \mathbb{C}$ be an open set, and let γ_0 and γ_1 be two paths contained in U. We say that γ_0 is **homotopic** to γ_1 if there exists a continuous deformation between the two paths within U. The rigorous mathematical definition is as follows:

Definition 2.1

Two paths, γ_0 and γ_1, are said to be homotopic in U if there exists a continuous function $H : [0,1] \times [0,1] \to U$ such that:

$$\text{For all } t \in [0,1], \; H(t,0) = \gamma_0(t) \quad \text{and} \quad H(t,1) = \gamma_1(t).$$

In the case where the paths have the same endpoints, we speak of strict homotopy if:

$$\text{For all } s \in [0,1], \; H(0,s) = \gamma_0(0) = \gamma_1(0) \quad \text{and} \quad H(1,s) = \gamma_0(1) = \gamma_1(1).$$

Homotopic paths

Homotopy defines an equivalence relation. The homotopy class of a path γ is denoted by $[\gamma]$. Let $z_0 \in U$; we denote by $\pi_1(U, z_0)$ the set of homotopy classes of loops in U based at z_0. For two loops γ_0 and γ_1 in U, we define the operation $*$ by

$$[\gamma_0] * [\gamma_1] := [\gamma_0 \cdot \gamma_1],$$

where $\gamma_0 \cdot \gamma_1$ is the concatenation of the two loops. We thus have the following theorem.

Theorem 2.2

If we equip $\pi_1(U, z_0)$ with the law $*$, then we have a group structure.

We say that U is simply connected when this group is trivial.

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Definition 2.3

Let $\gamma : [0,1] \to U$ be a path with $\gamma(0) = z_0$ and $\gamma(1) = z_1$. Let $r > 0$ and let f be holomorphic on $D(z_0, r)$. We subdivide the interval $[0,1]$ into n points

$$0 =: t_0 < t_1 < t_2 < \cdots < t_n := 1.$$

We then consider an open disk D_i and a holomorphic function f_i on it, for $0 \leq i \leq n$. We call **an analytic continuation of f along γ** any sequence $(f_i, D_i)_{0 \leq i \leq n}$ such that:

- $f_0 = f$ on $D_0 = D(z_0, r)$,
- $\gamma(t_i) \in D_i$, for all $0 \leq i \leq n$,
- $f_i = f_{i+1}$ on $D_i \cap D_{i+1} \neq \emptyset$, for all $0 \leq i \leq n-1$,

Remark 2.4

This does not guarantee the uniqueness of the continuation, and there are even cases where it fails completely. Sometimes, counterintuitive phenomena may also occur.

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Monodromy

The concept of **monodromy** refers to the behavior of a mathematical object when analytically continued around a singularity. In our setting, we focus on continuing a given function analytically along a path that encircles one of its singularities. These phenomena can lead to the existence of multiple possible analytic continuations for a given function.

A striking example is that of complex logarithms. We can define a complex logarithmic function in the neighborhood of 1 by:

$$\ell(z) := \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n}(z-1)^n, \quad \text{for all } z \in D(1,1).$$

Since this function is holomorphic on $D(1,1)$, suppose we want to analytically continue it outside this domain along the loop $\gamma(t) := e^{2i\pi t}$, with $t \in [0,1]$.

Then, the analytic extension of the function gives:

$$\ell(1) = 2i\pi, \quad \text{with } \gamma,$$

But originally we had $\ell(1) = 0$!

Monodromy

We will see that this kind of phenomenon will have a significant impact on the study of holomorphic differential equations.

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Definition

The concept of primitives is introduced in a manner analogous to the real case. Let $f : U \to \mathbb{C}$ be a continuous function, where $U \subset \mathbb{C}$ denotes an open subset of $\mathbb{C}$.

Definition 3.1

We say that $F : U \to \mathbb{C}$ is a primitive of f if F is holomorphic on U and $F' = f$ on U.

Example 3.2

The complex logarithmic function defined by

$$\ell(z) := \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n}(z-1)^n, \quad \text{for all } z \in D(1,1),$$

is a primitive of the function $z \mapsto \frac{1}{z}$ on $D(1,1)$.

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Condition of existence

Let $f : U \to \mathbb{C}$ be a holomorphic function and let $z_0 \in U$. We define

$$\begin{aligned} \mu_{z_0} : \pi_1(U, z_0) &\longrightarrow \mathbb{C} \\ [\sigma] &\longmapsto \int_\sigma f(\omega)\, d\omega \end{aligned}$$

A necessary and sufficient condition determines the existence of a primitive. We have, in fact, the following theorem:

Theorem 3.3

The function f admits a primitive on U if and only if, for every point $z_0 \in U$ and every loop σ based at z_0, we have

$$\mu_{z_0}([\sigma]) = 0.$$

Condition of existence

This theorem always holds on simply connected open sets, from which we deduce the following theorem:

Theorem 3.4

Let $f : U \to \mathbb{C}$ be a holomorphic function, where U is a simply connected open set. Let $z_0 \in U$. Then, for all $z \in U$, the function

$$F(z) := \int_{\gamma(z_{\mathbf{0}},z)} f(\omega)\, d\omega,$$

is the unique primitive of f on U that vanishes at z_0 ($\gamma(z_0, z)$ denotes an arbitrary path from z_0 to z).

Remark 3.5

In this context, we have a complex version of the Fundamental Theorem of Calculus from real analysis. This result is particularly relevant for the study of holomorphic differential equations.

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Definitions and Example

We will focus exclusively on the linear case.

Definition 4.1

An n-order holomorphic linear ODE is an ODE of the form:

$$\sum_{k=0}^{n} a_k(z)\frac{d^k y}{dz^k} = b(z)$$

where $a_0, a_1, \ldots, a_n, b : U \to \mathbb{C}$ are meromorphic functions.

We also assume that $a_n = 1$ and $b = 0$ on U. In everything that follows, we will study our scalar differential equation through an equivalent vectorial formulation. Specifically, we consider the vector-valued function $Y : U \to \mathbb{C}^n$, defined component-wise by:

$$y_k := \frac{d^{k-1}y}{dz^{k-1}}, \quad \text{for all } 1 \leq k \leq n,$$

where y denotes a solution to the scalar differential equation under the imposed assumptions. Then, the function Y satisfies the following first-order system:

$$\frac{dY}{dz} = AY \quad \text{on } U,$$

where the matrix $A \in \mathcal{M}_n(\mathbb{C})$ is given by:

$$A := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{pmatrix}.$$

By Theorem 3.4, we obtain the following result:

Theorem 4.2

Let U, an open set, be a simply connected region in $\mathbb{C}$, $z_0 \in U$, and $A : U \to M_n(\mathbb{C})$ a holomorphic map. For any $Y_0 \in \mathbb{C}^n$, there exists a unique holomorphic function $Y : U \to \mathbb{C}^n$ such that

$$\frac{dY}{dz} = AY$$

in U, and $Y(z_0) = Y_0$.

Definitions and Example

Let $e_1, e_2, \ldots, e_n$ denote the canonical basis of $\mathbb{C}^n$. According to Theorem 25, there exist n solutions $S_1, S_2, \ldots, S_n$ to our system on U, each satisfying the initial conditions:

$$S_1(z_0) = e_1, \quad S_2(z_0) = e_2, \quad \ldots, \quad S_n(z_0) = e_n.$$

Define $S : U \to M_n(\mathbb{C})$ as the holomorphic matrix-valued function whose columns are the vectors $S_1, S_2, \ldots, S_n$. Then S satisfies the matrix differential equation

$$\frac{dS}{dz} = AS$$

on U, with initial condition $S(z_0) = I_n$. By Theorem 25, such a solution S is uniquely determined.

Definition 4.3

S is called the fundamental matrix of the differential equation at the point z_0.

Let us consider the differential equation on $D(0,1) \setminus \{0\} =: D^*$:

$$\frac{dY}{dz} = \frac{A}{z} Y,$$

where $A \in M_n(\mathbb{C})$. We begin by restricting D^* to a simply connected domain via a branch cut. On this restricted domain, a fundamental matrix of solutions is given by

$$S(z) = \exp(A\ell(z)) =: z^A,$$

where ℓ denotes a chosen branch of the logarithm.
To extend the solution back to the whole of D^*, we analytically continue S along a loop γ around the origin, based at some point $z \in D^*$. We denote the continuation by $S^{(\gamma)}(z)$. Then we have:

$$S^{(\gamma)}(z) = S(z)e^{2i\pi A}.$$

Thus, attempting to define a single-valued solution on all of D^* results in a change in the value of S after continuation.

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Monodromy group

The phenomena observed in the previous example naturally arise in a wide class of holomorphic differential equations. We denote by $\mathcal{M}$ the set of monodromy matrices associated with a given holomorphic differential equation.

Theorem 4.4

The set $\mathcal{M}$ is a subgroup of $(\mathrm{GL}_n(\mathbb{C}), \times)$.

Monodromy group

If our differential equation is solved in a neighborhood of a point $z_0 \in U$, then we have the following theorem:

Theorem 4.5

The map

$$\begin{array}{rcl} \varphi : \pi_1(U, z_0) & \longrightarrow & \mathrm{GL}_n(\mathbb{C}) \\ [\gamma] & \longmapsto & M^{[\gamma]} \end{array}$$

defines a group homomorphism. It is called the representation morphism.

This means that there exists a real correspondence between the complex analytic behavior of differential equations and the algebraic structure of loops in the complex plane.

Monodromy group

We now consider the differential equation on $D(0,2) \setminus \{0,1\}$:

$$\frac{dY}{dz} = \left(\frac{A}{z} + \frac{B}{z-1}\right) Y,$$

under the assumption that the matrices A and B commute, i.e., $AB = BA$. To study this system, we restrict our attention to a simply connected domain by introducing two branch cuts.

On this domain, a fundamental matrix of solutions is given by:

$$S(z) = z^A (z-1)^B,$$

where $z^A := \exp(A\,\ell_1(z))$ and $(z-1)^B := \exp(B\,\ell_2(z-1))$, with ℓ_1 and ℓ_2 denoting chosen branches of the complex logarithm.

To extend the solution to the entire domain $D(0,2) \setminus \{0,1\}$, we analytically continue S along loops encircling the singularities. Denote by γ_1 a loop around 0, and by γ_2 a loop around 1. Then, the monodromy of the system is expressed as:

$$S^{(\gamma_1)}(z) = S(z)e^{2\pi i A}, \quad S^{(\gamma_2)}(z) = S(z)e^{2\pi i B}.$$

Hence, the effect of monodromy manifests itself through the action of two commuting matrices $e^{2\pi i A}$ and $e^{2\pi i B}$ on the fundamental solution S.

Conclusion

To conclude, we have seen that differential equations defined within the framework of complex analysis reveal a far deeper structural richness than in real analysis. **Singularities, analytic continuations, multivalued solutions, and monodromy** are all phenomena illustrating the geometric influence of the complex plane on the dynamics of solutions.

Thank you for your attention and for listening !