Holomorphic ODEs Around the Notion of Monodromy and Regular Singularities

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Research Project: Analysis of holomorphic ODEs and monodromy

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Context and Acknowledgments

This report is part of the *Research Initiation Project* (PIR MA) within the Applied Mathematics curriculum at INSA Rennes, a program aimed at familiarizing students with both abstract and practical aspects of mathematical sciences. The topic of this study centers around the investigation of holomorphic differential equations, a fundamental theme in the domain of complex analysis.
I am deeply thankful to my advisor, Professor Guy Casale, for his continuous support, expert guidance, and valuable insights throughout the course of this project. His knowledge and encouragement have been instrumental in bringing this work to completion.
I would also like to extend my appreciation to everyone who contributed to this endeavor, offering constructive suggestions and thoughtful advice along the way.

Notations

- $\mathrm{Sp}(A)$: the spectrum of $A \in \mathcal{M}_n(\mathbb{C})$.

- $E_\lambda(A)$: the set of eigenvectors of $A \in \mathcal{M}_n(\mathbb{C})$ associated with the eigenvalue λ.

- $D(a, r)$: Represents the open disk with center $a \in \mathbb{C}$ and radius $r > 0$.

- $C(a, r)$: Denotes the circle with center $a \in \mathbb{C}$ and radius $r > 0$, oriented in the counterclockwise direction.

Introduction

Linear differential equations with singularities play a central role in the theory of complex analysis and differential geometry. In particular, systems of holomorphic differential equations exhibiting regular singular points reveal rich structures related to **monodromy** representations, analytic continuation, and the fundamental group of the punctured complex plane. This document aims to provide an accessible yet rigorous introduction to the theory of holomorphic differential systems with singularities, focusing on the behavior of their solutions under analytic continuation around singularities.
The study of monodromy phenomena, which describe how solutions of differential equations transform as they are analytically continued along loops encircling singular points, is central to understanding global properties of differential systems. These concepts not only provide deep insight into the analytic structure of the solutions but also connect to diverse areas such as Riemann surfaces, representation theory, and algebraic geometry.
This exposition is intended for students with a background in complex analysis and linear algebra, and it progressively develops the necessary tools to study

the monodromy of differential equations. Through concrete examples and theoretical results, we aim to highlight both the geometric intuition and the analytic rigor underlying this beautiful theory.
This thesis is structured as follows:

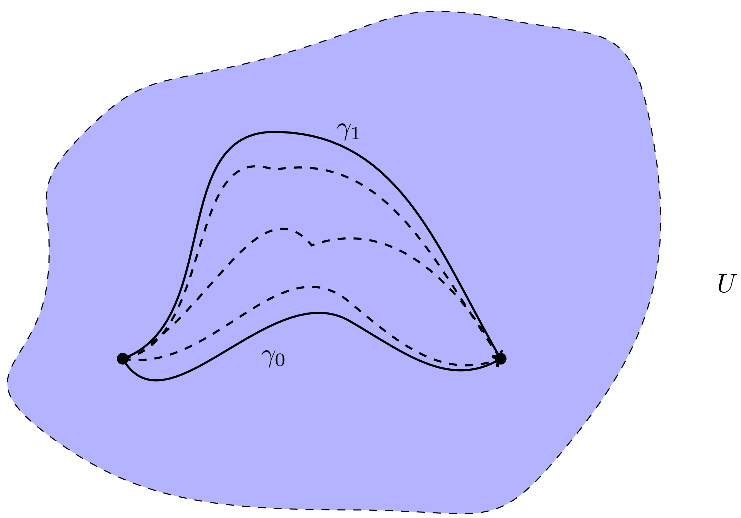
- **Chapter 1** introduces key concepts from algebraic topology necessary to understand analytic continuation and monodromy, including homotopy, fundamental groups, and the construction of monodromy representations.
- **Chapter 2** develops the notion of holomorphic primitives. It explores conditions for the existence of primitives, properties of contour integration, and consequences for holomorphic functions in simply connected domains.
- **Chapter 3** investigates holomorphic ordinary differential equations. It begins with linear differential equations of various orders, then studies their monodromy, the structure of their solution spaces, and systems with regular singularities. Applications include Frobenius-type equations and fundamental matrix solutions.

1 Algebraic Topology in Analytic Continuation

In this section, we will state some results from algebraic topology and examine their consequences in complex analysis. These results are thoroughly detailed in Chapter 8, Section 1, "Analytic Continuation" [1, p. 283].

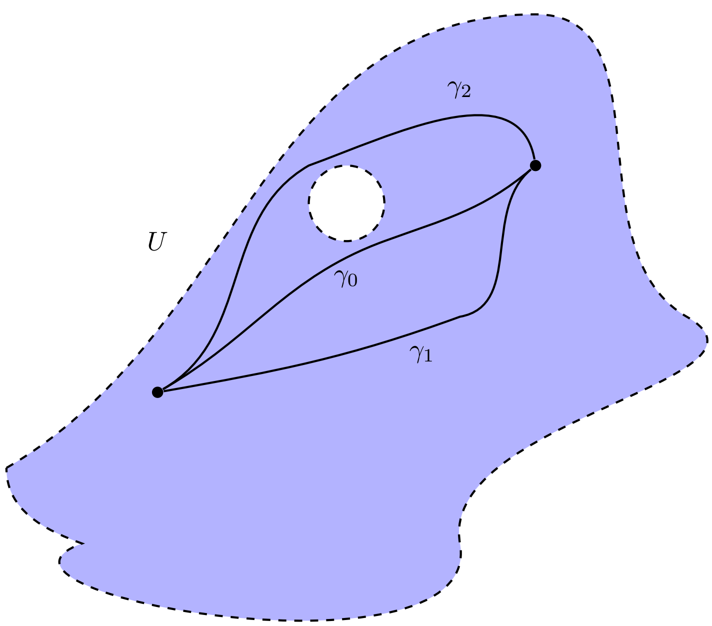
1.1 Homotopy

The topic of this subsection concerns the notion of continuous deformations of arcs. It is natural, when observing two paths on a plane, to ask whether one can be continuously deformed into the other.
To illustrate this, we consider two paths γ_0 and γ_1 with the same endpoints.



In the figure, we see in dotted lines some intermediate paths that may appear during the continuous deformation process from γ_0 to γ_1.

Now, suppose we have three paths γ_0, γ_1, and γ_2, all sharing the same endpoints.



We observe that γ_0 can be continuously deformed into γ_1, but γ_2 cannot be continuously deformed into γ_1 due to the presence of a hole separating the two paths.

This notion of continuous deformation between paths has so far been perceived from a purely visual perspective. We must now define it in a rigorous and mathematical way.

Definition 1: *Two paths, γ_0 and γ_1, are said to be homotopic in U if there exists a continuous function $H : [0,1] \times [0,1] \to U$ such that:*

$$\text{For all } t \in [0,1],\ H(t,0) = \gamma_0(t) \quad \text{and} \quad H(t,1) = \gamma_1(t).$$

In the case where the paths have the same endpoints, we speak of strict homotopy if:

$$\text{For all } s \in [0,1],\ H(0,s) = \gamma_0(0) = \gamma_1(0) \quad \text{and} \quad H(1,s) = \gamma_0(1) = \gamma_1(1).$$

From this definition, we will define the relation "to be homotopic to" for two given paths and we denote this relation by $\sim$. This relation then satisfies the following property.

Proposition 2: *The relation $\sim$ defines an equivalence relation between the paths.*

Thus, for a given path γ, we can introduce its equivalence class, denoted by $[\gamma]$, which will be called the homotopy class of γ. It is the set of paths homotopic to γ.

Proof: We must prove that the relation is reflexive, symmetric, and transitive.

For the first point, if we consider a given path γ, it is easily seen that the function defined by $H(t,s) = \gamma(t)$, for all $(t,s) \in [0,1] \times [0,1]$ defines a homotopy from γ to itself. Thus, $\gamma \sim \gamma$.

For the second point, we consider two paths γ_0 and γ_1 such that $\gamma_0 \sim \gamma_1$. Thus, there exists a homotopy H from γ_0 to γ_1. Therefore, we define the function $\tilde{H}$ by $\tilde{H}(t,s) = H(t, 1-s)$, for all $(t,s) \in [0,1] \times [0,1]$, which indeed corresponds to a homotopy from γ_1 to γ_0. Thus, we have $\gamma_1 \sim \gamma_0$.

Finally, for the third point, we consider three paths γ_0, γ_1, and γ_2 such that $\gamma_0 \sim \gamma_1$ and $\gamma_1 \sim \gamma_2$. Thus, there exists a homotopy H from γ_0 to γ_1 and a homotopy K from γ_1 to γ_2. Therefore, we define the function G by, for all $(t,s) \in [0,1] \times [0,1]$,

$$G(t,s) = \begin{cases} H(t,2s) & \text{if } s \leq \frac{1}{2}, \\ K(t,2s-1) & \text{if } s \geq \frac{1}{2}. \end{cases}$$

G indeed corresponds to a homotopy from γ_0 to γ_2, and thus $\gamma_0 \sim \gamma_2$. $\square$

In the rest of the work, we will focus on loops. We will call a loop based at $z_0 \in \mathbb{C}$, any loop that starts and ends at z_0.

Definition 3: *We denote by $\pi_1(U, z_0)$, the set of homotopy classes of loops in U based at z_0. Such a set will be called the fundamental group.*

$\pi_1(U, z_0)$ is referred to as the fundamental group because this set allows for the definition of a group structure with a given internal composition law.

Indeed, for two loops γ_1 and γ_2 based at z_0, we define the concatenation of γ_1 and γ_2, denoted by $\gamma_1.\gamma_2$, as

$$\gamma_1.\gamma_2(t) := \begin{cases} \gamma_1(2t), & \text{if } t \leq \frac{1}{2}, \\ \gamma_2(2t-1), & \text{if } t > \frac{1}{2}. \end{cases}$$

One can easily verify that this path is indeed based at z_0.
Before assigning an internal composition law to $\pi_1(U, z_0)$, we will prove the following proposition.

Proposition 4: *We consider four paths, γ_1, γ_2, Γ_1, and Γ_2 such that: $\gamma_1 \sim \Gamma_1$ and $\gamma_2 \sim \Gamma_2$. Then, $\gamma_1 \cdot \gamma_2 \sim \Gamma_1 \cdot \Gamma_2$.*

Proof : We consider H, a homotopy from γ_1 to Γ_1, and K, a homotopy from γ_2 to Γ_2. We define the function G by, for all $(t, s) \in [0, 1] \times [0, 1]$

$$G(t, s) = \begin{cases} H(2t, s) & \text{if } t \leq \frac{1}{2}, \\ K(2t-1, s) & \text{if } t \geq \frac{1}{2}. \end{cases}$$

This is a homotopy from $\gamma_1 \cdot \gamma_2$ to $\Gamma_1 \cdot \Gamma_2$. Thus, we have $[\gamma_1 \cdot \gamma_2] = [\Gamma_1 \cdot \Gamma_2]$. $\square$

Thanks to this property, we are now able to construct an internal composition law between the homotopy classes. This will be denoted by $*$ and we set, for two paths γ_1 and γ_2,

$$[\gamma_1] * [\gamma_2] := [\gamma_1.\gamma_2].$$

Proposition 5: *If we equip $\pi_1(U, z_0)$ with the law $*$, then we have a group structure.*

Proof : First, we define the loop based at z_0, $\mathrm{Id}_{z_0} := z_0$, then $[\mathrm{Id}_{z_0}]$ corresponds to the neutral element of the given operation. Indeed, for a given path γ, we prove that $[\gamma]*[\mathrm{Id}_{z_0}] = [\gamma] = [\mathrm{Id}_{z_0}]*[\gamma]$. To do this, we justify that $[\gamma \cdot \mathrm{Id}_{z_0}] = [\gamma]$, i.e., $\gamma \cdot \mathrm{Id}_{z_0} \sim \gamma$. We then define the function H by, for all $(t, s) \in [0, 1] \times [0, 1]$,

$$H(t, s) = \begin{cases} \gamma\left(\frac{2t}{s+1}\right) & \text{if } t \leq \frac{s+1}{2}, \\ z_0 & \text{if } t \geq \frac{s+1}{2}. \end{cases}$$

This is indeed a homotopy from $\gamma \cdot \mathrm{Id}_{z_0}$ to γ. Thus, we have $[\gamma] * [\mathrm{Id}_{z_0}] = [\gamma]$. In the same way, we show that $[\mathrm{Id}_{z_0}] * [\gamma] = [\gamma]$.

Next, for a loop γ based at z_0, if we consider the loop $\tilde{\gamma}$ based at z_0 defined by

$$\tilde{\gamma}(t) := \gamma(1-t), \quad \text{for all } t \in [0,1],$$

then $[\tilde{\gamma}]$ defines the inverse of $[\gamma]$ for $*$. Indeed, we have $[\tilde{\gamma}]*[\gamma] = [\mathrm{Id}_{z_0}] = [\gamma]*[\tilde{\gamma}]$. Thanks to the function H defined as follows, for all $(t,s) \in [0,1] \times [0,1]$,

$$H(t,s) = \begin{cases} \gamma(2t-1) & \text{if } t \geq \frac{s+1}{2}, \\ \gamma(s) & \text{if } \frac{1-s}{2} \leq t \leq \frac{s+1}{2}, \\ \tilde{\gamma}(2t) & \text{if } t \leq \frac{1-s}{2}. \end{cases}$$

We have a homotopy from $\tilde{\gamma} \cdot \gamma$ to Id_{z_0}. We verify in the same way that $\gamma \cdot \tilde{\gamma}$ is homotopic to Id_{z_0}. Thus, we can write $[\gamma]^{-1} := [\tilde{\gamma}]$.

Finally, the operation $*$ is associative because for three loops $\gamma_1, \gamma_2, \gamma_3$ based at z_0, we have

$$(\gamma_1.\gamma_2).\gamma_3 \sim \gamma_1.(\gamma_2.\gamma_3).$$

Indeed, this is manifested through the following homotopy, defined for all $(t,s) \in [0,1] \times [0,1]$,

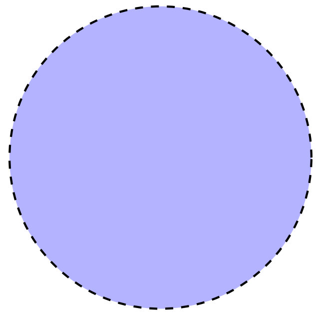
$$H(t,s) = \begin{cases} \gamma_1\left(\frac{4t}{s+1}\right) & \text{if } t \leq \frac{s+1}{4}, \\ \gamma_2\left(4t-(s+1)\right) & \text{if } \frac{s+1}{4} \leq t \leq \frac{s+2}{4}, \\ \gamma_3\left(\frac{4t-(2+s)}{2-s}\right) & \text{if } t \geq \frac{s+2}{4}. \end{cases}$$

This concludes the proof. $\square$

Thus, the algebraic properties of $\pi_1(U, z_0)$ allow us to rigorously define the notion of simple connectivity for U.
From a visual perspective, U is simply connected if it is connected and has no "holes". With the help of $\pi_1(U, z_0)$, we can construct a formal definition.
It turns out that for $z_0' \in U$, the fundamental group $\pi_1(U, z_0)$ is isomorphic to $\pi_1(U, z_0')$. Due to this bijective correspondence, the study at z_0 and z_0' is the same. Thus, we can refer to $\pi_1(U)$ without specifying the base point.
The formal definition is then as follows:

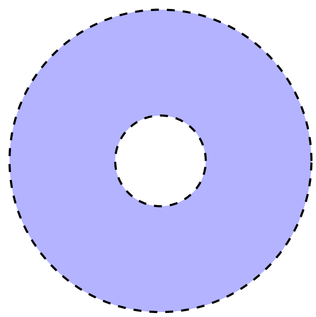
Definition 6: *Let U be a connected open set. Then U is said to be simply connected if its fundamental group $\pi_1(U)$ is trivial.*

We can illustrate this with the following figures.



D_1

Here we can visually see that the disk D_1 is simply connected because it has no holes. This is not the case for D_2, which is, on the other hand, punctured.



D_2

These sets will be particularly useful in the computation of holomorphic primitives presented in Section 2.3.

1.2 Analytic continuation along an arc

We consider $U \subset \mathbb{C}$, an open set, and a function f holomorphic on $D(z_0, r) \subset U$, where $r > 0$ and $z_0 \in U$. Naturally, one may ask whether it is possible to extend f to a larger domain in U, or even to all of U, in such a way that it remains holomorphic. We will construct a procedure to address this question using paths. Specifically, if there exists a path γ defined in U starting at z_0 and ending outside $D(z_0, r)$, then we will attempt to define f outside the disk by following γ. Rigorously, the process is defined as follows.

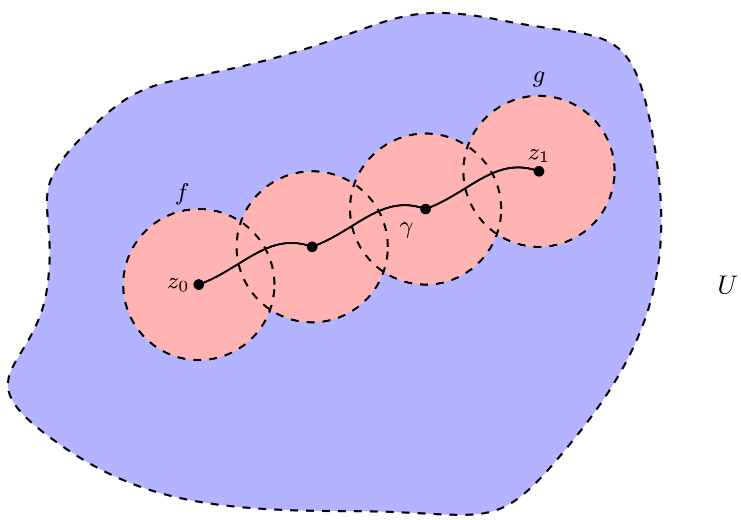
Definition 7: *Let $\gamma : [0, 1] \to U$ be a path with $\gamma(0) = z_0$ and $\gamma(1) = z_1$. Let $r > 0$ and let f be holomorphic on $D(z_0, r)$. We subdivide the interval $[0, 1]$ into n points*

$$0 =: t_0 < t_1 < t_2 < \cdots < t_n := 1.$$

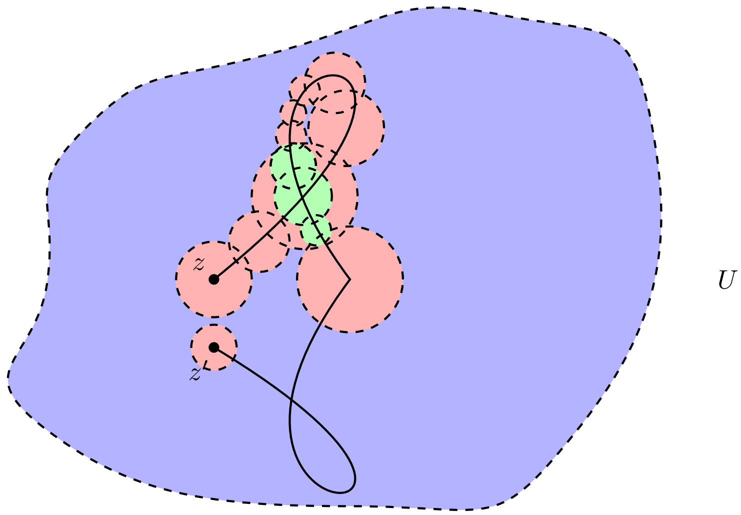
We then consider an open disk D_i and a holomorphic function f_i on it, for $0 \leq i \leq n$.
We call an analytic continuation of f along γ any sequence $(f_i, D_i)_{0 \leq i \leq n}$ such that:

- $f_0 = f$ *on* $D_0 = D(z_0, r)$,
- $\gamma(t_i) \in D_i$, *for all* $0 \leq i \leq n$,
- $f_i = f_{i+1}$ *on* $D_i \cap D_{i+1} \neq \emptyset$, *for all* $0 \leq i \leq n-1$,

The following figure provides an illustrative example of the previously defined process.



Here, the function g defines the analytic continuation of f at the point z_1 along the path γ.
It may also be interesting to visualize what happens when γ loops back on itself.



We can see in this figure that disks intersect without necessarily following one another immediately. These areas are shown in green. Thus, in these intersections, several values can be assigned to the points by the prolongation along γ. We will say that our function is multivalued. This will be at the heart of the concept known as **monodromy**.

We will show, using the following theorem, that the analytic continuation along an arc does not depend on the disks and functions that realize it.

Theorem 8: *Let $(f_i, D_i)_{0\leq i\leq n}$ and $(g_j, E_j)_{0\leq j\leq m}$ be two analytic continuations of a holomorphic function f along the same path γ. Then $f_n = g_m$ on a certain neighborhood of $z_1 := \gamma(1)$.*

Proof : Suppose first that the partition is fixed, so $n = m$. We know that $f_0 = g_0 = f$ on $D_0 = E_0$, corresponding to a neighborhood of z_0. We know that $z_0 \in D_0$. By hypothesis, $f_1 = f_0$ on $D_0 \cap D_1$ and $g_1 = g_0$ on $E_0 \cap E_1$. Hence, $f_1 = g_1$ on $D_0 \cap E_0 \cap D_1 \cap E_1$, which contains z_1. Since $D_1 \cap E_1$ is connected, it follows that $f_1 = g_1$ on $D_1 \cap E_1$. We can now proceed by induction to see that

$$g_n = f_n \text{ on } D_n \cap E_n,$$

thus concluding the proof in this case.

Next, let us consider a change in the partition. Any two partitions have a common refinement. To show the independence of the partition, it suffices to do so when we insert one point in the given partition, say we insert c in the interval $[a_k, a_{k+1}]$ for some k.

On one hand, we take the sequence of open disks $(D_0, \ldots, D_k, D_k, \ldots, D_n)$, where D_k is repeated twice, so that $\gamma([a_k, c]) \subset D_k$ and $\gamma([c, a_{k+1}]) \subset D_k$. Then $((D_0, f_0), \ldots, (D_k, f_k), (D_k, f_k), \ldots, (D_n, f_n))$ is an analytic continuation of (f_0, D_0) along this connected sequence.
On the other hand, suppose that $((E_0, g_0), \ldots, (E_k, g_k), (E_k^*, g_k^*), \ldots, (E_n, g_n))$ is an analytic continuation of (g_0, E_0) with respect to the new partition. Since $f_0 = g_0$ in a neighborhood of z_0, then by the first part of the proof, we know that $g_k = f_k$ on $D_k \cap E_k$. By hypothesis, $g_k^* = g_k$ on $E_k \cap E_k^*$ and $f_k = g_k$ on $D_k \cap E_k$, so $g_k^* = f_k$ on $D_k \cap E_k \cap E_k^*$, which contains $z_k^* := \gamma(c)$. Again, we can apply the first part of the proof to the second piece of the path, which is defined on the interval $[a_{k+1}, a_n]$ with respect to the partition $a_{k+1}, a_{k+2}, \ldots, a_n$, to conclude the proof of the theorem. $\square$

1.3 Concept of Monodromy

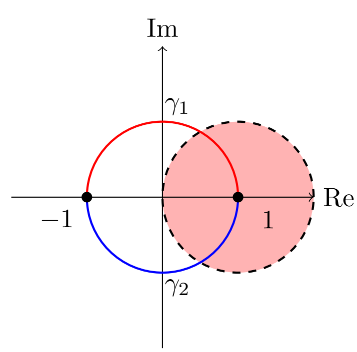
The concept of **monodromy** corresponds to the study of a mathematical object around a singularity. Specifically, in our context of study, we will attempt to analytically continue a given function along a path that bypasses a singularity of the function.

A striking example, also studied by [3], is that of complex logarithms. These functions are not defined at 0, which corresponds to a singularity. We can define a complex logarithmic function in the neighborhood of 1 by:

$$\ell(z) := \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n}(z-1)^n, \quad \text{for all } z \in D(1,1).$$

This function being holomorphic on $D(1,1)$, suppose we want to analytically extend it to -1. We will do this via a path. Let us consider two paths:

$$\gamma_1(t) := e^{i\pi t}, \quad \gamma_2(t) := e^{-i\pi t}, \quad \text{for all } t \in [0,1].$$



Then, the analytic extension of the function gives:

$$\ell(-1) = i\pi, \quad \text{with } \gamma_1, \quad \ell(-1) = -i\pi, \quad \text{with } \gamma_2.$$

We thus obtain two different values, which illustrates the multi-valued nature of the function.

The multivalued character is explained by the presence of 0 as a singularity. Indeed, as the logarithm is not defined at 0, our paths are not homotopic.
Homotopy indeed plays a major role with respect to the uniqueness of the analytic continuation along an arc. This is illustrated by the following theorem, known as the **Monodromy Theorem**.

Theorem 9: *Let f be holomorphic on $D(z,r)$. Suppose that along any loop γ starting from z, there exists an analytic continuation of f along γ. Then, if γ and $\tilde{\gamma}$ are two paths from z to z' that are strictly homotopic in U, the continuation g of f along γ at z' coincides with the continuation $\tilde{g}$ of f along $\tilde{\gamma}$ at z'.*

Before proving this theorem, we will need the following lemma:

Lemma 10: *Let $H : [0,1] \times [0,1] \to U$ be a homotopy between the paths γ and $\tilde{\gamma}$. For each $s \in [0,1]$, if $s' \in [0,1]$ is sufficiently close to s, then a continuation of f along γ_s is equal to a continuation of f along $\gamma_{s'}$ in some neighborhood of z'.*

Proof : We now prove the lemma. Given a continuation $((f_0, D_0), \dots, (f_n, D_n))$ along γ_s, it is immediately verified, by the uniform continuity of the homotopy H, that if s' is sufficiently close to s, then this is also a continuation along $\gamma_{s'}$. In Theorem 8, we have seen that the analytic continuation along a curve does not depend on the choice of partition and $D_0, \dots, D_n$. This proves the lemma. $\square$

Now we can prove the Theorem 9.

Proof : We denote by $H : [0,1] \times [0,1] \to U$ the homotopy between γ and $\tilde{\gamma}$. We say that two paths, Γ and $\tilde{\Gamma}$, connecting z to z' are f-equivalent if the continuations of f along Γ and $\tilde{\Gamma}$ are equal in a neighborhood of z'.
Let

$$S := \{s \in [0,1] \mid \gamma_0 \text{ and } \gamma_s \text{ are } f\text{-equivalent}\}.$$

We claim that $S = [0,1]$, which implies the theorem. Note that $0 \in S$, so $S \neq \emptyset$.
By Lemma 10, S is open in $[0,1]$.
We show that S is closed in $[0,1]$. Let $s \in \overline{S}$. Then there exist points $s' \in S$ close to s, so again by Lemma 10, γ_s is f-equivalent to $\gamma_{s'}$, and thus γ_s is f-equivalent to γ_0.
Therefore, $S = \overline{S}$, so S is closed. Since $[0,1]$ is connected and S is both open and closed, it follows that $S = [0,1]$, as was to be shown. $\square$

2 Notion of Holomorphic Primitives

Now we continue our study with the notion of holomorphic primitives. This is a **key point in solving holomorphic differential equations**, which will be examined in Section 3. In particular, we will examine whether it is possible to reduce the computation of holomorphic primitives to situations similar to the real case.

2.1 Definition and General Properties

We establish the notion of primitives in the same way as in the real case.
Let $f : U \to \mathbb{C}$ be a continuous function, where $U \subset \mathbb{C}$ denotes an open subset of $\mathbb{C}$.

Definition 11: *We say that $F : U \to \mathbb{C}$ is a primitive of f if F is holomorphic on U and $F' = f$ on U.*

Thus, if f admits a holomorphic primitive, it is itself holomorphic. Therefore, the definition only makes sense within the class of holomorphic functions.

We can provide a few simple examples for illustration:

- Let $n \in \mathbb{N}$, then $F : z \mapsto \frac{z^{n+1}}{n+1}$ is a primitive of $f : z \mapsto z^n$ on $\mathbb{C}$.
- $F : z \mapsto e^z + 3i$ is a primitive of $f : z \mapsto e^z$ on $\mathbb{C}$.
- $F : z \mapsto \log(z)$ is a primitive of $f : z \mapsto \frac{1}{z}$ on $\mathbb{C} \setminus \mathbb{R}_-$, with log denoting the principal logarithmic branch.

Since holomorphic functions are expandable in power series in the neighborhood of each of their points, we are therefore able to take their primitive locally. We express this through the following proposition:

Proposition 12: *Let $f : U \to \mathbb{C}$ be a holomorphic function and $a \in U$. Then, there exists $F_a : U \to \mathbb{C}$ such that for all $r > 0$ sufficiently small, $F_a' = f$ on $D(a, r)$.*

We will see in Section 2.2 under what conditions a holomorphic function admits a primitive over its entire domain of definition.

Proof : Let $f : U \to \mathbb{C}$ be holomorphic and $a \in U$. Then, for all $r > 0$ sufficiently small, we have for all $z \in D(a, r)$,

$$f(z) = \sum_{n=0}^{+\infty} c_n (z-a)^n.$$

We then define for all $z \in D(a, r)$,

$$F_a(z) := \sum_{n=0}^{+\infty} \frac{c_n}{n+1} (z-a)^{n+1}.$$

Thus, by the theory of power series, F_a is well-defined and corresponds to a primitive of f on $D(a, r)$. $\square$

To conclude this section, we state one last property relating contour integrals and primitives.

Proposition 13: *Let $f : U \to \mathbb{C}$ be a holomorphic function and $F : U \to \mathbb{C}$ a primitive. Let $(z_0, z_1) \in U^2$. Suppose there exists $\gamma : [0, 1] \to U$, a path connecting z_0 to z_1. Then, we have*

$$\int_\gamma f(\omega)\, d\omega = F(z_1) - F(z_0).$$

Thus, the integral does not depend on the path connecting the endpoints.

Proof : Let $f : U \to \mathbb{C}$ be holomorphic and $F : U \to \mathbb{C}$ a primitive. Consider $(z_0, z_1) \in U^2$ and suppose there exists a path γ connecting them. Then,

$$\int_\gamma f(\omega)\, d\omega = \int_0^1 f(\gamma(t))\gamma'(t)\, dt = \int_0^1 F'(\gamma(t))\gamma'(t)\, dt = [F(\gamma(t))]_0^1 = F(z_1) - F(z_0).$$

This completes the proof. $\square$

2.2 Condition of existence

In this section, we will see a necessary and sufficient condition for the existence of a primitive for a holomorphic function. This is described in the following theorem.

Theorem 14: *Let $f : U \to \mathbb{C}$ be a holomorphic function. The function f admits a primitive on U if and only if for every loop σ in U,*

$$\int_\sigma f(\omega)\, d\omega = 0.$$

Proof: Let $f : U \to \mathbb{C}$ be a holomorphic function. Suppose that f admits a primitive F on U. Then, by Proposition 13, we have

$$\int_\sigma f(\omega)\, d\omega = F(\sigma(1)) - F(\sigma(0)) = 0.$$

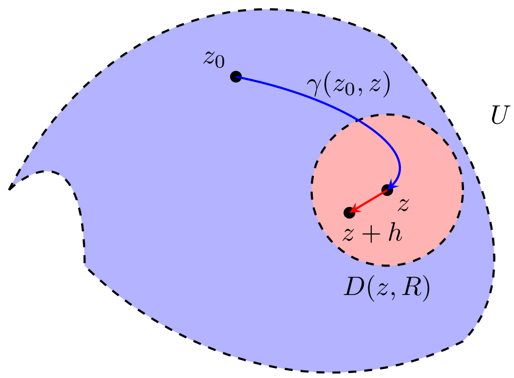
Now, suppose that for every loop σ in U, we have

$$\int_\sigma f(\omega)\, d\omega = 0.$$

Assume that U is connected (or work separately on each of its connected components otherwise). Since U is open, it is also arc-connected. Let $z_0 \in U$, and define for every $z \in U$ the function

$$F(z) := \int_{\gamma(z_0, z)} f(\omega)\, d\omega,$$

where $\gamma(z_0, z)$ denotes a path connecting z_0 to z. By the working assumption, F does not depend on γ. We now show that F is a primitive of f on U. Let $z \in U$. Since U is open, there exists $R > 0$ such that $D(z, R) \subset U$. Let $h \in \mathbb{C}^*$ such that is sufficiently small in modulus. Consider the path $\Gamma(z_0, z+h) := \gamma(z_0, z) \cup [z, z+h]$.



Due to the independence of F from the path taken, we have

$$F(z+h) = \int_{\Gamma(z_0,z+h)} f(\omega)\,d\omega = \int_{\gamma(z_0,z)} f(\omega)\,d\omega + \int_{[z,z+h]} f(\omega)\,d\omega = F(z) + \int_{[z,z+h]} f(\omega)\,d\omega.$$

Thus,

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\int_{[z,z+h]} f(\omega)\,d\omega = \int_0^1 f(z+th)\,dt.$$

Now, $\lim_{h\to 0} f(z+th) = f(z)$ by the continuity of f. Since h is sufficiently small in modulus, for all $t \in [0,1]$,

$$|f(z+th)| \leq 1 + |f(z)|.$$

Thus, by the interchange of limit and integral, we obtain

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = \int_0^1 f(z)\,dt = f(z).$$

Therefore, F is differentiable at z, and $F'(z) = f(z)$. Thus, F is a primitive of f on U. $\square$

In the following section, we will naturally seek open sets on which Theorem 14 is satisfied. Thus, we will determine the domains where holomorphic functions necessarily admit primitives.

2.3 Case of Connected Open Sets

Within a connected open set, the computation of primitives of a holomorphic function is carried out similarly to the real case. In particular, we have the following proposition.

Proposition 15: *Let $f : U \to \mathbb{C}$ be a holomorphic function and $U \subset \mathbb{C}$ a connected open set. Then f is constant on U if and only if $f' = 0$ on U.*

Proof: The first implication is immediate. For the second one, let us fix $z_0 \in U$. Then, for any $z \in U$, there exists a path γ connecting z_0 to z in U. Since f is holomorphic on U, it is a primitive of f' on U. Thus, by Proposition 13,

$$0 = \int_\gamma f'(\omega)d\omega = f(z) - f(z_0).$$

Hence, for all $z \in U$, we have $f(z) = f(z_0)$, which implies that f is constant on U. $\square$

This result is remarkable. Indeed, if f admits a primitive F on U, then the set of its primitives on U consists of functions of the form $G = F + c$, where $c \in \mathbb{C}$. This structure is identical to the real case. However, we do not yet have a guarantee that f possesses a primitive on U.
To illustrate this, consider $f(z) := \frac{1}{z}$ for all $z \in \mathbb{C}^*$. Although $\mathbb{C}^*$ is connected, f does not admit a primitive on $\mathbb{C}^*$. Indeed,

$$\int_{C(0,1)} \frac{dz}{z} = 2i\pi \neq 0,$$

and we conclude using Theorem 14. Therefore, additional conditions on U are necessary for f to admit a primitive.

Now we will assume that U is simple connected.
There exist several types of simply connected sets, such as star-shaped open sets.

Definition 16: *An open set $U \subset \mathbb{C}$ is star-shaped if there exists $z_0 \in U$ such that for all $z \in U$, the segment $[z_0, z]$ is contained in U.*

We have the following important result.

Proposition 17: *Let $f : U \to \mathbb{C}$ be a holomorphic function, where $U \subset \mathbb{C}$ is a simply connected open set. Then, for any loop σ in U, we have*

$$\int_\sigma f(\omega)\, d\omega = 0.$$

Thus, every holomorphic function admits a primitive on a simply connected open set by Theorem 14.

Proof : Since U is a simply connected open set, the path γ defines the boundary of a compact domain with boundary in U. Since f is holomorphic on U, we conclude by Green-Riemann's theorem[1]. $\square$

In a simply connected open set, we can state a complex domain version of the Fundamental Theorem of Calculus from real analysis. This result is particularly interesting for the study of holomorphic differential equations.

Theorem 18: *Let $f : U \to \mathbb{C}$ be a holomorphic function, where U is a simply connected open set. Let $z_0 \in U$. Then, for all $z \in U$, the function*

$$F(z) := \int_{\gamma(z_0,z)} f(\omega)\, d\omega,$$

is the unique primitive of f on U that vanishes at z_0 ($\gamma(z_0, z)$ denotes an arbitrary path from z_0 to z).

In particular, if U is a star-shaped open set with respect to z_0, then $F : z \mapsto \int_{[z_0,z]} f(\omega)\, d\omega$ is the unique primitive of f on U that vanishes at z_0.

Proof: A direct consequence of Theorem 14 and Proposition 15. $\square$

To conclude this section, we introduce a result linking path homotopy and contour integration in the complex plane.

Proposition 19: *Let $f : U \to \mathbb{C}$ be a holomorphic function, where $U \subset \mathbb{C}$ is an open set, and let γ_0 and γ_1 be two homotopic paths in U (either with common endpoints or as loops). Then,*

$$\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz.$$

Proof : First, we consider the homotopy between γ_0 and γ_1, which we write in the form

$$H(t, s) = \gamma_s(t), \quad \forall (t, s) \in [0, 1] \times [0, 1].$$

We set $\epsilon := \mathrm{d}(\mathrm{Im}(H), \mathbb{C} \setminus U) > 0$ if $U \neq \mathbb{C}$. Since $[0, 1] \times [0, 1]$ is compact and H is a continuous function, it follows from Heine's theorem that H is uniformly continuous on this domain. More precisely, there exists $\eta > 0$ such that for all $((v, u), (t, s)) \in ([0, 1]^2)^2$, if

$$\|(v, u) - (t, s)\|_2 < \eta,$$

[1]If f is holomorphic on U and $\Delta \subset U$ is a compact set with piecewise smooth boundary, then

$$\int_{\partial\Delta} f(z)\, dz = 0.$$

then

$$|\gamma_u(v) - \gamma_s(t)| < \epsilon.$$

We perform a subdivision of the set $[0,1] \times [0,1]$ with a step size of $\eta/2$, where

$$0 =: s_0 < \cdots < s_m := 1 \quad \text{and} \quad 0 =: t_0 < \cdots < t_n := 1.$$

For $1 \leq j \leq m-1$, we aim to deform the path γ_{s_j}. Consider the piecewise linear path $\tilde{\gamma}_j : [0,1] \to \mathbb{C}$ connecting the points

$$\gamma_{s_j}(t_0), \ldots, \gamma_{s_j}(t_n).$$

In particular, $\tilde{\gamma}_j$ is piecewise C^1.

Now, we show that:

$$\int_{\gamma_0} f(z)\, dz = \int_{\tilde{\gamma}_1} f(z)\, dz.$$

Let $0 \leq i \leq n$. By the choice of ϵ, the open disk $D_i := D(\gamma_0(t_i), 2\epsilon)$ is contained in the open set U. Since D_i is simply connected, f admits a holomorphic antiderivative F_i in D_i.
We define $a_i := \gamma_0(t_i)$ and $b_i := \tilde{\gamma}_1(t_i)$. Due to uniform continuity, the disks D_i and D_{i+1} contain the points a_i, b_i, a_{i+1} and b_{i+1}. Thus, the disks contain the segments $[a_i, a_{i+1}]$ and $[b_i, b_{i+1}]$.
Moreover, the functions F_i and F_{i+1} are antiderivatives of f on $D_i \cap D_{i+1}$, which is a connected set. Therefore, they differ by a constant, so we have:

$$F_{i+1}(a_{i+1}) - F_i(a_{i+1}) = F_{i+1}(b_{i+1}) - F_i(b_{i+1}).$$

Rearranging, we obtain:

$$F_{i+1}(a_{i+1}) - F_{i+1}(b_{i+1}) = F_i(a_{i+1}) - F_i(b_{i+1}).$$

Thus, we have:

$$\begin{aligned}
\int_{\gamma_0} f(z)dz - \int_{\tilde{\gamma}_1} f(z)dz &= \sum_{i=0}^{n-1}(F_i(a_{i+1}) - F_i(a_i)) - \sum_{i=0}^{n-1}(F_i(b_{i+1}) - F_i(b_i)) \\
&= \sum_{i=1}^{n-1}(F_{i+1}(a_{i+1}) - F_{i+1}(b_{i+1})) - \sum_{i=1}^{n-1}(F_i(a_i) - F_i(b_i)) \\
&= F_n(a_n) - F_n(b_n) - (F_0(a_0) - F_0(b_0)).
\end{aligned}$$

Since $a_0 = b_0$ and $a_n = b_n$ (or $a_0 = a_n$ and $b_0 = b_n$), we obtain the equality

$$\int_{\gamma_0} f(z)dz = \int_{\tilde{\gamma}_1} f(z)dz.$$

In the same way, we show that

$$\int_{\tilde{\gamma}_1} f(z)dz = \int_{\tilde{\gamma}_2} f(z)dz, \ldots, \int_{\tilde{\gamma}_{m-1}} f(z)dz = \int_{\gamma_1} f(z)dz.$$

Step by step, this proves the theorem. $\square$

Thanks to the results obtained in this section, we can now begin the study of holomorphic ODEs.

3 Study of Holomorphic ODEs

In the following, we will formally introduce the notion of holomorphic ODEs, examine the issues surrounding their resolution, and observe the behavior of the solutions when they can be explicitly computed.

3.1 Definition and Overview

Holomorphic ODEs are defined in the same way as in the real case.

Definition 20: *A holomorphic ODE is an equation that relates an unknown holomorphic function to its derivatives. More precisely, it is an equation of the form:*

$$F(z, y, y', \ldots, y^{(n)}) = 0,$$

where $n \in \mathbb{N}$, with:

- *$z \in U$, where $U \subset \mathbb{C}$ is an open set,*
- *$y : U \to \mathbb{C}$ is a holomorphic function,*
- *$F : U \times \mathbb{C}^{n+2} \to \mathbb{C}$ represents the equation field.*

We can mention several examples of holomorphic ODEs. The computations of primitives that we observed in Section 2 are particular cases of holomorphic ODEs. Indeed, they correspond to solving equations of the form:

$$y' = f,$$

where $f : U \to \mathbb{C}$ is a holomorphic function.
We can also mention the Painlevé II equation:

$$y'' = 2y^3 + zy,$$

as well as the Cauchy-Euler equation:

$$z^2 y'' + zy' - y = 0.$$

In the following, we will focus on so-called linear ODEs.

3.1.1 Linear Differential Equation of Order 1

We begin our study with first-order ODEs.

Definition 21: *A first-order holomorphic linear ODE is an ODE of the form:*

$$a_1(z)y' + a_0(z)y = b(z),$$

where $a_0, a_1, b : U \to \mathbb{C}$ are meromorphic functions.

In the following, we assume that the open set U is simply connected. Thus, by Theorem 18, we are in a situation analogous to the real case.
We first consider $a_1 = 1$ and a_0 and b holomorphic on U, and we define $a := -a_0$. Then we have the following theorem:

Theorem 22: *The linear holomorphic first-order ODE, $y' = a(z)y + b(z)$, has the solutions*

$$y(z) = \lambda \exp\left(\int_{\gamma(z_0,z)} a(\omega) d\omega \right) + y_p(z),$$

where $\lambda \in \mathbb{C}$, $\gamma(z_0, z)$ is any path from z_0 to z and y_p is a particular solution of the equation. Moreover, under the initial condition $y(\xi_0) = y_0$, where $\xi_0 \in U$, the equation admits a unique solution.

Proof: As the equation is linear, its solutions are composed of the homogeneous solutions and a particular solution. We thus begin by studying the homogeneous equation:

$$y' = a(z)y.$$

Let $\lambda \in \mathbb{C}$, $z_0 \in U$, $z \in U$, and $\gamma(z_0, z)$ be a path in U connecting z_0 to z. For all $z \in U$, we define

$$\alpha(z) := \int_{\gamma(z_0,z)} a(\omega)\, d\omega, \quad \text{and} \quad y_h(z) := \lambda \exp(\alpha(z)).$$

Since U is simply connected, Theorem 18 implies that α is a primitive of a on U. It can then be easily verified that y_h is a solution of the homogeneous differential equation.
Now, consider y_h, a solution of the homogeneous equation. For all $z \in U$, let us define

$$V(z) := y_h(z) \exp(-\alpha(z)).$$

Then V is differentiable on U, and for all $z \in U$, we have

$$V'(z) = y_h'(z) \exp(-\alpha(z)) - a(z)y_h(z) \exp(-\alpha(z)) = \big(y_h'(z) - a(z)y_h(z)\big) \exp(-\alpha(z)).$$

Since y_h satisfies the homogeneous equation, we obtain $V'(z) = 0$ for all $z \in U$. As U is connected, Proposition 15 implies that there exists $\lambda \in \mathbb{C}$ such that

$V(z) = \lambda$ for all $z \in U$. Hence,

$$y_h(z)\exp(-\alpha(z)) = \lambda, \quad \text{so that} \quad y_h(z) = \lambda \exp(\alpha(z)),$$

for all $z \in U$. We thus know the structure of the homogeneous solutions.
Let y_p be a particular solution on U. Then any solution y of the differential equation can be written as, for all $z \in U$,

$$y(z) = y_h(z) + y_p(z) = \lambda \exp(\alpha(z)) + y_p(z).$$

Finally, if we impose the initial condition $y(\xi_0) = y_0$, for some $\xi_0 \in U$, then

$$y_0 = \lambda \exp(\alpha(\xi_0)) + y_p(\xi_0),$$

which yields

$$\lambda = (y_0 - y_p(\xi_0))\exp(-\alpha(\xi_0)).$$

This determines the solution y uniquely.

□

To illustrate this Theorem, let's solve a simple ODE. It is of the form

$$\begin{cases} y' + y = z, & z \in \mathbb{C}, \\ y(0) = 0. \end{cases}$$

The right-hand side being polynomial, we easily find that $y_p(z) = z - 1$ provides a particular polynomial solution to the problem. By applying Theorem 22, we find that the solutions are $y(z) = \lambda e^{-z} + z - 1$. With the initial condition $y(0) = 0$, we necessarily have $\lambda = 1$. Therefore, the solution to the problem is:

$$y(z) = e^{-z} + z - 1.$$

At this stage, we have not seen any differences with the solution on $\mathbb{R}$. We will now consider a slightly more delicate case by assuming that a has an isolated singularity.

For example, let's study the ODE

$$y' = \frac{1}{z^2} y,$$

with $z \in \mathbb{C}^*$. It is easy to see that $y(z) = \exp\left(-\frac{1}{z}\right)$ provides a solution. It would be interesting to observe how the solution behaves around 0, which corresponds to a singularity of the solution. In the following, we will denote $z = x + iy$, with $(x, y) \in (\mathbb{R}^2)^*$, θ being an argument of z, and $r := |z|$. We will study $|y(z)|$ as $z \to 0$, i.e., when $r \to 0$. For all $z \in \mathbb{C}^*$, we have

$$|y(z)| = \exp\left(\mathrm{Re}\left(-\frac{1}{z}\right)\right) = \exp\left(-\frac{x}{x^2 + y^2}\right) = \exp\left(-\frac{\cos(\theta)}{r}\right).$$

We will distinguish three cases.

Suppose $x > 0$: In this case, $\cos(\theta) > 0$ and hence $|y(z)| \to 0$.

Now suppose $x < 0$: Then $\cos(\theta) < 0$ and thus $|y(z)| \to +\infty$.

Finally, suppose $x = 0$: In this case, $|y(z)| = 1$.

Here, we observe that depending on how we approach our singularity, the behavior of the solution is clearly not the same.

We can see this with the following figure, which was generated using a Python program. A logarithmic scale was applied in order to obtain a better visualization.

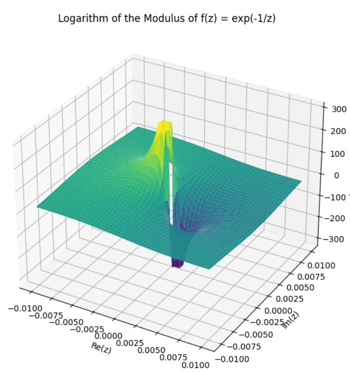


Figure 1: Logarithmic representation of the solution

In the following section, we will consider holomorphic ODEs of higher order than those studied previously.

3.1.2 Linear Differential Equation of Order n

We continue our study by examining holomorphic linear ODEs of order $n \in \mathbb{N}^*$. Formally, they are written as follows.

Definition 23: *An n-order holomorphic linear ODE is an ODE of the form:*

$$\sum_{k=0}^{n} a_k(z) \frac{d^k y}{dz^k} = b(z)$$

where $a_0, a_1, \ldots, a_n, b : U \to \mathbb{C}$ are meromorphic functions.

We always assume that the open set U is simply connected. We also assume that $a_n = 1$, $b = 0$ and that a_k are holomorphic functions on U for all $0 \leq k \leq n-1$. In everything that follows, we will study our scalar differential equation through an equivalent vector differential equation. This idea was taken from [2]. Specifically, we consider the vector function $Y : U \to \mathbb{C}^n$ fully defined by its components as follows:

$$y_k := \frac{d^{k-1}y}{dz^{k-1}}, \quad \text{for all } 1 \leq k \leq n,$$

where y denotes a solution of the scalar differential equation with the imposed conditions. Thus, Y satisfies:

$$\frac{dY}{dz} = AY \quad \text{on } U,$$

with

$$A := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{pmatrix}.$$

Thus, if we know a solution Y of this vector differential equation, then y_1 is a solution of the scalar differential equation, given the definition of A. The equivalence that can be established between the scalar and vector differential equations is formalized in the following lemma.

Lemma 24: *The function* $D : y \mapsto Y := \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}$ *is a linear isomorphism between the vector space of solutions of the scalar differential equation and the vector space of solutions of the vector differential equation.*

Proof : We consider the function $D : y \mapsto Y := \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}$, where y is a solution of the scalar ODE. As we have previously seen, the column vector Y is a solution of the vector ODE. Therefore, the function D is well-defined. We now show that it is a linear isomorphism.

First, D is clearly linear due to the linearity of differentiation.

We now study its injectivity. Let $y \in \ker(D)$. Then $D(y) = 0$, that is, $Y = 0$. In particular, the first component of Y is zero, hence $y = 0$. This proves the injectivity of D.

We now study the surjectivity of D. Let Z be a solution of the vector ODE $Z' = AZ$. We write $Z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$. Let $z := z_1$. Then $Z' = AZ$ gives

$$\begin{pmatrix} z_1' \\ \vdots \\ z_n' \end{pmatrix} = A \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{pmatrix}$$

Thus, we obtain

$$\begin{pmatrix} z_1' \\ z_2' \\ \vdots \\ z_n' \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ \vdots \\ -a_0 z_1 - a_1 z_2 - \dots - a_{n-1} z_n \end{pmatrix}$$

From this, we deduce that $z' = z_1' = z_2$, $z'' = z_1'' = z_2' = z_3$, and so on, until $z^{(n)} = z_1^{(n)} = z_n'$. But $z_n' = -a_0 z_1 - a_1 z_2 - \dots - a_{n-1} z_n$. Finally, using the previous equalities, we obtain

$$z^{(n)} = -a_0 z - a_1 z' - \dots - a_{n-1} z^{(n-1)}$$

Thus, z satisfies the scalar ODE. Moreover, we also have

$$Z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} z \\ z' \\ \vdots \\ z^{(n-1)} \end{pmatrix} = D(z)$$

Therefore, D is surjective. Hence, D is a linear isomorphism. $\square$

We now state the following theorem, whose result will be fundamental for our study.

Theorem 25: *Let U, an open set, be a simply connected region in $\mathbb{C}$, $z_0 \in U$, and $A : U \to M_n(\mathbb{C})$ a holomorphic map. For any $Y_0 \in \mathbb{C}^n$, there exists a unique holomorphic function $Y : U \to \mathbb{C}^n$ such that*

$$\frac{dY}{dz} = AY$$

in U, and $Y(z_0) = Y_0$.

Proof : Let $D := D(z_0, R)$, with $R > 0$ small enough, be an open disk centered at z_0 and contained in U. We will first carry out our study on D. Let us begin by proving the uniqueness of the solution. Since A is holomorphic, we can expand it into a power series on D. Thus, we have, for all $z \in D$:

$$A(z) = \sum_{p=0}^{+\infty} C_p (z - z_0)^p,$$

with $C_p \in \mathcal{M}_n(\mathbb{C})$, for all $p \in \mathbb{N}$. Let us consider Y, a solution of our vector ODE. Then Y is holomorphic on U and therefore Y can be expanded into a power series on D. Thus, we have, for all $z \in D$:

$$Y(z) = \sum_{p=0}^{+\infty} T_p (z - z_0)^p,$$

with $T_p \in \mathbb{C}^n$, for all $p \in \mathbb{N}$. Thus, as Y satisfies the equation $\frac{dY}{dz} = AY$ on D, we have consequently, for all $z \in D$:

$$\sum_{p=1}^{+\infty} p T_p (z - z_0)^{p-1} = \left(\sum_{p=0}^{+\infty} C_p (z - z_0)^p \right) \left(\sum_{p=0}^{+\infty} T_p (z - z_0)^p \right).$$

By a Cauchy product, we thus have, for all $z \in D$:

$$\sum_{p=1}^{+\infty} p T_p (z - z_0)^{p-1} = \sum_{m=0}^{+\infty} \left(\sum_{k=0}^{m} C_{m-k} T_k \right) (z - z_0)^m.$$

Finally, this rewrites, for all $z \in D$, as:

$$\sum_{m=0}^{+\infty} (m+1) T_{m+1} (z - z_0)^m = \sum_{m=0}^{+\infty} \left(\sum_{k=0}^{m} C_{m-k} T_k \right) (z - z_0)^m.$$

By uniqueness of the terms of a power series expansion, we have, for all $m \in \mathbb{N}$:

$$(m+1) T_{m+1} = \sum_{k=0}^{m} C_{m-k} T_k.$$

Hence, for all $m \in \mathbb{N}$:

$$T_{m+1} = \frac{1}{m+1} \sum_{k=0}^{m} C_{m-k} T_k.$$

Now $T_0 = Y(z_0) = Y_0$. From this recurrence relation and its initialization, the sequence $(T_m)_{m \in \mathbb{N}}$ is entirely and uniquely determined. Thus, if Y is a solution of the ODE on D, then Y is unique. This result easily generalizes to U due to the uniqueness of the analytic continuation of Y on U, since U is a simply connected open set. Thus, if our equation admits a solution on U, then it is unique.

We will now show the existence of a solution. To do this, we set, for all $z \in D$:

$$Y(z) := \sum_{p=0}^{+\infty} T_p (z - z_0)^p,$$

where

$$T_{p+1} = \frac{1}{p+1} \sum_{k=0}^{p} C_{p-k} T_k, \quad \text{for all } p \in \mathbb{N}, \text{ and } T_0 = Y_0.$$

Here, Y is indeed a solution of the equation on D, provided of course that the power series is convergent on D. We will thus show that it is convergent using Cauchy's majorant method. In what follows we will make a series of assumptions and verify their validity later. Let us first suppose that

$$\|C_p\|_\infty \le c_p, \quad \text{with } c_p \ge 0, \text{ for all } p \in \mathbb{N}.$$

We now consider the power series

$$a(z) := \sum_{p=0}^{+\infty} c_p (z - z_0)^p,$$

and we assume that it converges on $D' := D(z_0, r)$ where $0 < r \le R$. From this we study the scalar differential equation:

$$\frac{dy}{dz} = n a(z) y, \quad y(z_0) = \|Y_0\|_\infty, \text{ on } D'.$$

Since D' is a simply connected open set, then the function defined by:

$$y(z) := \|Y_0\|_\infty \exp\left(n \int_{[z_0, z]} a(\omega)\, d\omega \right), \quad \text{for all } z \in D',$$

is the unique solution of the ODE in question.
Let us now suppose that y is expandable as a power series on D':

$$y(z) = \sum_{p=0}^{+\infty} t_p (z - z_0)^p, \quad \text{for all } z \in D'.$$

Similarly to the first part of the proof, we prove that:

$$t_{m+1} = \frac{n}{m+1} \sum_{k=0}^{m} b_{m-k} t_k, \quad \text{for all } m \in \mathbb{N}.$$

Since $b_p \geq 0$, for all $p \in \mathbb{N}$, and $t_0 = y(z_0) = \|Y_0\|_\infty \geq 0$, we immediately deduce by induction that $t_m \geq 0$, for all $m \in \mathbb{N}$. We now prove by induction that for all $p \in \mathbb{N}$:

$$\|T_p\|_\infty \leq t_p.$$

The initialization $p = 0$ is trivial. Now, let $p \in \mathbb{N}$ such that $\|T_p\|_\infty \leq t_p$. Then:

$$\|T_{p+1}\|_\infty \leq \frac{1}{p+1} \sum_{k=0}^{p} \|C_{p-k} T_k\|_\infty.$$

Due to the structure of the vector $C_{p-k}T_k$, we deduce that:

$$\|T_{p+1}\|_\infty \leq \frac{n}{p+1} \sum_{k=0}^{p} \|C_{p-k}\|_\infty \|T_k\|_\infty \leq \frac{n}{p+1} \sum_{k=0}^{p} c_{p-k} t_k = t_{p+1}.$$

This proves the induction. If all the assumptions made so far are satisfied, then the radius of convergence of:

$$\sum_{p=0}^{+\infty} T_p (z - z_0)^p$$

is at least equal to r. Thus, the latter converges at least on D'. We now show that these assumptions are verified.
Let $r > 0$, such that $r < R$. Then, since A is holomorphic on D', from Cauchy estimates, there exists $M > 0$ such that, for all $p \in \mathbb{N}$:

$$\|C_p\|_\infty \leq \frac{M}{r^p}.$$

Here, one can choose $c_p = \frac{M}{r^p}$, for all $p \in \mathbb{N}$. Then, subject to convergence:

$$a(z) = \sum_{p=0}^{+\infty} c_p (z - z_0)^p = M \sum_{p=0}^{+\infty} \left(\frac{z - z_0}{r} \right)^p.$$

From this, we deduce that the convergence of this power series is verified on D', and moreover we have, for all $z \in D'$:

$$a(z) = \frac{Mr}{r - (z - z_0)}.$$

We thus deduce that y is indeed expandable into a power series on D'. Finally, the power series:

$$\sum_{p=0}^{+\infty} T_p (z - z_0)^p$$

converges on D'. Since this is verified for all $r > 0$ such that $r < R$, we deduce that it converges on D.

It remains to prove the existence on U. To do so, we will rely on the monodromy theorem.
Let $z \in U$ and let $\gamma : [0,1] \to U$ be a path connecting z_0 to z. Then, there exists $R > 0$ and a family of open disks of radius R, $D_0, D_1, \ldots, D_n$, such that the center $z_j \in D_j$ also lies in D_{j-1}, for all $1 \leq j \leq n$, and $z_n = z$.
Thus, we can find solutions $Y_0, Y_1, \ldots, Y_n$ to our equation on the disks $D_0, \ldots, D_n$ such that:
(i) $Y_0(z_0) = Y_0$
(ii) The function element (Y_j, D_j) is a direct analytic continuation of (Y_{j-1}, D_{j-1}), for all $1 \leq j \leq n$.
We can construct from (Y_0, D_0) an analytic continuation along γ. Since U is simply connected, by the monodromy theorem, Y_0 extends to a holomorphic function on U that is a solution to the differential equation.

This completes the proof of existence. The theorem is thus proven. □

From this, we have the following corollary:

Corollary 26: *The linear space of all solutions of the system*

$$\frac{dY}{dz} = AY$$

in the simply connected domain U is n-dimensional.

Proof: We consider $z_0 \in U$. Thus, by the result of Theorem 25, the linear map $Y \mapsto Y(z_0)$ defines an isomorphism between the vector space of solutions of the vectorial differential equation and $\mathbb{C}^n$. Hence, we deduce that the space of solutions of the vectorial differential equation has dimension n. □

These results transpose to the scalar differential equation. Thus, we have:

Theorem 27: *Let U, an open set, be a simply connected region in $\mathbb{C}$, $z_0 \in U$. For any complex numbers $y_0, y_1, \ldots, y_n$, there exists a unique holomorphic function y on U, such that*

$$\frac{d^n y}{dz^n} + a_{n-1}\frac{d^{n-1}y}{dz^{n-1}} + \cdots + a_1\frac{dy}{dz} + a_0 y = 0$$

in U, and

$$y(z_0) = y_0, \quad y'(z_0) = y_1, \quad \ldots, \quad y^{(n-1)}(z_0) = y_{n-1}.$$

Corollary 28: *The linear space of all solutions of the differential equation*

$$\frac{d^n y}{dz^n} + a_{n-1}\frac{d^{n-1}y}{dz^{n-1}} + \cdots + a_1\frac{dy}{dz} + a_0 y = 0$$

in the simply connected domain U is n-dimensional.

We prove Theorem 27 and Corollary 28 simultaneously.

Proof: Regarding Theorem 27, the scalar equation can be reformulated as follows:

$$\frac{dY}{dz} = AY, \quad Y(z_0) = Y_0.$$

Theorem 27 is thus verified as a consequence of Theorem 25.
Then, for Corollary 28, the isomorphism from Lemma 24 guarantees that the space of solutions of the scalar equation has dimension n. □

By Corollaries 26 and 28, we know that the solution space of our differential equations is of dimension n. Thus, we can attempt to find a vector basis for our solution set. This is done using the concept of the fundamental matrix.

Let U be a simply connected domain in $\mathbb{C}$, $A : U \to M_n(\mathbb{C})$ a holomorphic map, and

$$\frac{dY}{dz} = AY$$

a first order system in U. Fix a base point $z_0 \in U$. Let $e_1, e_2, \ldots, e_n$ be the canonical basis of $\mathbb{C}^n$. Then, by Theorem 25, we can find solutions $S_1, S_2, \ldots, S_{n-1}, S_n$ of our system in U satisfying the following initial conditions:

$$S_1(z_0) = e_1, \quad S_2(z_0) = e_2, \quad \ldots, \quad S_{n-1}(z_0) = e_{n-1}, \quad S_n(z_0) = e_n.$$

Let $S : U \to M_n(\mathbb{C})$ be the holomorphic function such that its columns are $S_1, S_2, \ldots, S_n$.
S satisfies the following matrix differential equation:

$$\frac{dS}{dz} = AS,$$

on U, with $S(z_0) = I_n$. As a consequence of Theorem 25, S is uniquely determined.

Definition 29: *S is called the fundamental matrix of the differential equation at the point z_0.*

In particular, we are able to explicitly express the solutions of our vector differential equations using S. For example, if we impose the initial condition $Y(z_0) = Y_0$, then the unique solution is given by:

$$Y(z) = S(z)Y_0,$$

for all $z \in U$.

In what follows, we will introduce a tool related to S that will be useful to us for constructing bases of solutions for vector ODEs.

Definition 30: *We call the Wronskian of S, the function Δ defined by:*

$$\Delta(z) := \det(S(z)),$$

for all $z \in U$.

In particular, since S is holomorphic on U, then Δ is also holomorphic. Note also that $\Delta(z_0) = 1$.

Moreover, the Wronskian satisfies the following property.

Proposition 31: *Δ satisfies the differential equation:*

$$\frac{d\Delta}{dz} = tr(A)\Delta,$$

on U.

Proof : Let $\mathfrak{S}_\mathfrak{n}$ be the group of permutations of order n. We define, by $\epsilon : \mathfrak{S}_\mathfrak{n} \to \{-1, 1\}$, the signature of our permutations. Thus, by definition of the determinant, we have, for all $z \in U$:

$$\Delta(z) = \sum_{\sigma \in \mathfrak{S}_\mathfrak{n}} \left(\epsilon(\sigma) \prod_{i=1}^{n} S_{i\sigma(i)}(z) \right).$$

Then we have, for all $z \in U$:

$$\frac{d}{dz}(\Delta(z)) = \sum_{\sigma \in \mathfrak{S}_\mathfrak{n}} \left(\epsilon(\sigma) \frac{d}{dz} \left(\prod_{i=1}^{n} S_{i\sigma(i)}(z) \right) \right)$$

$$= \sum_{\sigma \in \mathfrak{S}_\mathfrak{n}} \left(\sum_{i=1}^{n} \epsilon(\sigma) S_{1\sigma(1)}(z) \dots S_{(i-1)\sigma(i-1)}(z) \frac{d}{dz}(S_{i\sigma(i)}(z)) S_{(i+1)\sigma(i+1)}(z) \dots S_{n\sigma(n)}(z) \right)$$

$$= \sum_{i=1}^{n} \sum_{\sigma \in \mathfrak{S}_\mathfrak{n}} \left(\epsilon(\sigma) S_{1\sigma(1)}(z) \dots S_{(i-1)\sigma(i-1)}(z) (AS)_{i\sigma(i)}(z) S_{(i+1)\sigma(i+1)}(z) \dots S_{n\sigma(n)}(z) \right)$$

$$= \sum_{i=1}^{n} \sum_{\sigma \in \mathfrak{S}_\mathfrak{n}} \left(\epsilon(\sigma) S_{1\sigma(1)}(z) \dots S_{(i-1)\sigma(i-1)}(z) \left(\sum_{k=1}^{n} A_{ik}(z) S_{k\sigma(i)}(z) \right) S_{(i+1)\sigma(i+1)}(z) \dots S_{n\sigma(n)}(z) \right)$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik}(z) \sum_{\sigma \in \mathfrak{S}_\mathfrak{n}} \left(\epsilon(\sigma) S_{1\sigma(1)}(z) \dots S_{(i-1)\sigma(i-1)}(z) S_{k\sigma(i)}(z) S_{(i+1)\sigma(i+1)}(z) \dots S_{n\sigma(n)}(z) \right).$$

In the second sum, we fix $1 \leq k \leq n$. If $k \neq i$, then:

$$\sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) S_{1\sigma(1)}(z) \dots S_{(i-1)\sigma(i-1)}(z) S_{k\sigma(i)}(z) S_{(i+1)\sigma(i+1)}(z) \dots S_{n\sigma(n)}(z)$$

represents the determinant of a matrix where row i is the same as row k. Since the determinant is alternating, this determinant is zero. Thus we have:

$$\begin{aligned} \frac{d}{dz}(\Delta(z)) &= \sum_{i=1}^{n} A_{ii}(z) \left(\sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) S_{1\sigma(1)}(z) \dots S_{(i-1)\sigma(i-1)}(z) S_{i\sigma(i)}(z) S_{(i+1)\sigma(i+1)}(z) \dots S_{n\sigma(n)}(z) \right) \\ &= \sum_{i=1}^{n} A_{ii}(z) \det(S(z)) = \operatorname{tr}(A(z))\Delta(z). \end{aligned}$$

Thus, Δ indeed satisfies the equation:

$$\frac{d\Delta}{dz} = \operatorname{tr}(A)\Delta,$$

on U. $\square$

Thanks to this result, we will now be able to prove the following property.

Proposition 32: *The fundamental matrix S satisfies: $S(z) \in GL_n(\mathbb{C})$, for all $z \in U$.*

Thus, for all $z \in U$, $S_1(z), \dots, S_n(z)$ are vectors forming a linearly independent set in $\mathbb{C}^n$. Consequently, the solutions $S_1, \dots, S_n$ form a linearly independent set of the vector differential equation. They thus form a basis of solutions for the vector differential equation. We now prove Proposition 32.

Proof : By Theorem 22, since U is a simply connected open set, we have for all $z \in U$:

$$\Delta(z) = \exp\left(\int_{\gamma(z_0,z)} \operatorname{tr}(A(\omega))\, d\omega \right).$$

Therefore, $\Delta(z) \neq 0$ for all $z \in U$, and thus $S(z) \in GL_n(\mathbb{C})$ for all $z \in U$. $\square$

We will now use the notion of a fundamental matrix to study the monodromy of certain vectorial ODEs.
For example, let us study on $D(0,1) \setminus \{0\} =: D^*$ the equation:

$$\frac{dY}{dz} = \frac{A}{z} Y,$$

where $A \in M_n(\mathbb{C})$. Since the map $z \mapsto \frac{A}{z}$ presents, a priori, problems for being integrated, we will cut D^* using a branch (for example, $]0,1[$). On this domain, the solutions of our ODE are of the form:

$$Y(z) = z^A Y_0,$$

with

$$z^A := \exp(A\ell(z)),$$

where ℓ denotes a determination of the logarithm, and $Y_0 \in \mathbb{C}^n$.
Here, the fundamental matrix is $S = \exp(A\ell(z))$. To return to D^*, we will analytically continue the solutions we have found.
For example, we will do this along the loop γ. We denote by $S^{(\gamma)}(z)$ the result of the analytic continuation along the loop γ at the point z. Thus:

$$S^{(\gamma)}(z) = S(z)e^{2i\pi A}.$$

Therefore, in attempting to define our solution on D^*, we have clearly changed S. This is the reflection of the monodromy of our solutions on D^*.

We now study, on $D(0,2) \setminus \{0,1\}$, the equation:

$$\frac{dY}{dz} = \left(\frac{A}{z} + \frac{B}{z-1}\right) Y,$$

under the assumption that $AB = BA$. As before, the functions $z \mapsto \frac{A}{z}$ and $z \mapsto \frac{B}{z-1}$ present potential integration issues on $D(0,2) \setminus \{0,1\}$. Therefore, we will study our problem on $D(0,2) \setminus \{0,1\}$ by cutting it along two branches (for example, $]-1,0[$ and $]1,2[$).
In this context, since $AB = BA$, the solutions are given by:

$$Y(z) = z^A(z-1)^B Y_0,$$

with $z^A := \exp(A\,\ell_1(z))$, $(z-1)^B := \exp(B\,\ell_2(z-1))$, where ℓ_1 and ℓ_2 are two determinations of the logarithm, and $Y_0 \in \mathbb{C}^n$.
Here, the fundamental matrix is clearly $S = z^A(z-1)^B$. To return to $D(0,2) \setminus \{0,1\}$, we will analytically continue the solutions we have found. For example, we will do this along the loops γ_1 and γ_2. Since $AB = BA$, we have:

$$S^{(\gamma_1)}(z) = S(z)e^{2\pi iA}, \quad S^{(\gamma_2)}(z) = S(z)e^{2\pi iB}.$$

Thus, the effect of monodromy is manifested through two transformations of our fundamental matrices.

3.2 Monodromy group

We saw in the previous section that the effects of monodromy on the solutions of an ODE can be represented using matrices. The representation matrices are then called monodromy matrices. They are defined as follows.

Definition 33: *Let $U \subset \mathbb{C}$ be a connected open subset. We consider the ordinary differential equation*

$$\frac{dY}{dz} = A(z)Y$$

for all $z \in U$, where $A : U \to \mathcal{M}_n(\mathbb{C})$. Let $z_0 \in U$, and let γ be a loop based at z_0 in U. Denoting by S a fundamental matrix of solutions in a neighborhood of z_0, we define $M^{(\gamma)}$ as the monodromy matrix of the ODE associated with γ if the matrix

$$S^{(\gamma)} := SM^{(\gamma)}$$

is a fundamental matrix of the ODE obtained after analytic continuation along γ.

Since monodromy matrices are constructed from loops, there naturally exist properties of these matrices that depend on the paths traversed. More precisely, there is a connection between the monodromy matrices and the homotopy classes of loops, as expressed in the following property:

Proposition 34: *Let γ_1 and γ_2 be two loops in a connected open subset U, both based at a point $z_0 \in U$. If $\gamma_1 \sim \gamma_2$, then their associated monodromy matrices are equal:*

$$M^{\gamma_1} = M^{\gamma_2}.$$

Thus, in the following, for a given loop γ based at z_0, we denote by $M^{[\gamma]}$ the monodromy matrix associated with loops in the homotopy class of $[\gamma]$. We also denote by $\mathcal{M}$ the set of monodromy matrices of our vector-valued ODE.

Proof : This is an immediate consequence of Theorem 9. □

This set then has a special structure; it is, in fact, a matrix group. We will now prove the following theorem:

Theorem 35: The set $\mathcal{M}$ is a subgroup of $\mathrm{GL}_n(\mathbb{C})$.

Proof : The following proof relies on path traversal, and the specific paths chosen can be replaced by any homotopic ones without altering the result.

Let $z_0 \in U$, where $U \subset \mathbb{C}$ is a connected open subset. We solve our vector-valued ODE in a neighborhood of z_0, and denote by S the fundamental matrix of solutions associated with this local resolution.
Assume that γ_1 and γ_2 are two loops in U based at z_0. Then we have:

$$M^{[\gamma_1 \cdot \gamma_2]} = M^{[\gamma_2]} M^{[\gamma_1]}.$$

Indeed, when we follow the path $\gamma_1 \cdot \gamma_2$, we first traverse γ_1, so after this traversal, the fundamental matrix becomes

$$S^{[\gamma_1]} = SM^{[\gamma_1]}.$$

Then, continuing along γ_2 from the fundamental matrix $S^{[\gamma_1]}$, we obtain the new fundamental matrix

$$(S^{[\gamma_1]})^{[\gamma_2]} = S^{[\gamma_1]} M^{[\gamma_2]}.$$

Thus, $(S^{[\gamma_1]})^{[\gamma_2]}$ is the fundamental matrix of our system after traversing γ_2, assuming we had already followed γ_1. In other words, it is the fundamental matrix after having traversed the full path $\gamma_1 \cdot \gamma_2$. We conclude that:

$$S^{[\gamma_1 \cdot \gamma_2]} = (S^{[\gamma_1]})^{[\gamma_2]} = SM^{[\gamma_1]}M^{[\gamma_2]}.$$

Therefore,

$$M^{[\gamma_1 \cdot \gamma_2]} = M^{[\gamma_2]}M^{[\gamma_1]}.$$

Thus, the product of two monodromy matrices is itself a monodromy matrix. Next, we prove that $M^{[\mathrm{Id}_{z_0}]} = I_n$. Indeed, when following the identity path Id_{z_0}, no displacement is made along the path. Therefore, we have

$$S^{[\mathrm{Id}_{z_0}]} = S = S \cdot I_n,$$

and hence $M^{[\mathrm{Id}_{z_0}]} = I_n$.
Finally, from all this, we deduce that for any loop γ based at z_0,

$$I_n = M^{[\gamma \cdot \gamma^{-1}]} = M^{[\gamma^{-1}]}M^{[\gamma]}.$$

Thus, $M^{[\gamma]} \in \mathrm{GL}_n(\mathbb{C})$ and

$$M^{[\gamma^{-1}]} = \left(M^{[\gamma]}\right)^{-1}.$$

This concludes the proof. □

From the results obtained above, we immediately deduce the following theorem.

Theorem 36: *The map*

$$\begin{array}{rcl} \varphi : \pi_1(U, z_0) & \longrightarrow & \mathrm{GL}_n(\mathbb{C}) \\ [\gamma] & \longmapsto & M^{[\gamma]} \end{array}$$

defines a group homomorphism. It is called the representation morphism.

Thus, the study of our differential equations is partly reduced to the study of π_1. This means that there is a correspondence between the complex analysis of differential equations and the algebra of loops in the complex plane. We will explore this correspondence further by studying the notions of regular singularities and equivalent differential equations.

3.3 Systems with Regular Singularities and Equivalent Differential Equations

In this section, we aim to construct a method that allows us to equivalently transition from the solutions of one differential equation to another. We now define the notion of equivalent differential equations as follows.

Definition 37: *Let us consider the differential equations*

$$\frac{dU}{dz} = AU \quad \text{and} \quad \frac{dV}{dz} = BV,$$

where $A : D^* \to \mathcal{M}_n(\mathbb{C})$ and $B : D^* \to \mathcal{M}_n(\mathbb{C})$ are holomorphic. These systems are said to be equivalent if there exists a holomorphic function $\Phi : D^* \to \mathrm{GL}_n(\mathbb{C})$, with at most a pole at 0, such that

$$\frac{d\Phi}{dz} = B\Phi - \Phi A \quad \text{on } D^*.$$

From this definition, we will easily establish the following property.

Proposition 38: *The relation defined above is an equivalence relation among our differential equations.*

Proof : The relation is clearly reflexive. Indeed, it suffices to take $\Phi(z) := I_n$ for all $z \in D^*$.
We now show that the relation is symmetric. Let us consider the two differential equations

$$\frac{dU}{dz} = AU \quad \text{and} \quad \frac{dV}{dz} = BV$$

on D^*, and suppose there exists a holomorphic function $\Phi : D^* \to \mathrm{GL}_n(\mathbb{C})$, with at most a pole at 0, such that

$$\frac{d\Phi}{dz} = B\Phi - \Phi A \quad \text{on } D^*.$$

Then Φ^{-1} is also a holomorphic function from D^* to $\mathrm{GL}_n(\mathbb{C})$, with at most a pole at 0. Since $\Phi\Phi^{-1} = I_n$ on D^*, it follows that

$$\frac{d}{dz}(\Phi\Phi^{-1}) = 0 \quad \text{on } D^*.$$

From this, we deduce the identity

$$\frac{d\Phi^{-1}}{dz} = -\Phi^{-1}\left(\frac{d\Phi}{dz}\right)\Phi^{-1}.$$

Substituting the expression for $\frac{d\Phi}{dz}$, we obtain

$$\frac{d\Phi^{-1}}{dz} = -\Phi^{-1}(B\Phi - \Phi A)\Phi^{-1} = A\Phi^{-1} - \Phi^{-1}B.$$

This proves the symmetry of the relation.
We now prove the transitivity of the relation. To this end, we introduce a third differential equation

$$\frac{dW}{dz} = CW$$

on D^*, and assume that there exists a holomorphic function $\Psi : D^* \to \mathrm{GL}_n(\mathbb{C})$, with at most a pole at 0, such that

$$\frac{d\Psi}{dz} = C\Psi - \Psi B.$$

Then the product $\Psi\Phi$ is a holomorphic function from D^* to $\mathrm{GL}_n(\mathbb{C})$, with at most a pole at 0. Moreover, we have

$$\frac{d(\Psi\Phi)}{dz} = \left(\frac{d\Psi}{dz}\right)\Phi + \Psi\left(\frac{d\Phi}{dz}\right) = (C\Psi - \Psi B)\Phi + \Psi(B\Phi - \Phi A) = C\Psi\Phi - \Psi\Phi A.$$

This proves that $\Psi\Phi$ satisfies the required relation, and thus establishes the transitivity of the equivalence.

This concludes the justification of the proof. $\square$

We now explain how this relation is useful. Suppose that Y is a solution of the equation

$$\frac{dU}{dz} = AU$$

on some open subset $U \subset D^*$. Then we compute:

$$\frac{d(\Phi Y)}{dz} = \left(\frac{d\Phi}{dz}\right)Y + \Phi\left(\frac{dY}{dz}\right) = (B\Phi - \Phi A)Y + \Phi AY = B\Phi Y.$$

Thus, ΦY is a solution of

$$\frac{dV}{dz} = BV$$

on U.

Now let $\Phi : D^* \to \mathrm{GL}_n(\mathbb{C})$ be the equivalence matrix between our differential systems, and let S_A denote the fundamental matrix of the first system. Then the fundamental matrix of the second system is given by

$$S_B(z) = \Phi(z)S_A(z)\Phi(z_0)^{-1}.$$

Indeed, we have

$$S_B(z_0) = \Phi(z_0)S_A(z_0)\Phi(z_0)^{-1} = \Phi(z_0)\Phi(z_0)^{-1} = I_n.$$

Next, differentiating S_B, we find:

$$\frac{dS_B(z)}{dz} = \frac{d\Phi(z)}{dz}S_A(z)\Phi(z_0)^{-1} + \Phi(z)\frac{dS_A(z)}{dz}\Phi(z_0)^{-1}.$$

Using the relation $\frac{d\Phi}{dz} = B\Phi - \Phi A$ and the fact that $\frac{dS_A}{dz} = AS_A$, we get:

$$\frac{dS_B(z)}{dz} = (B\Phi(z) - \Phi(z)A)S_A(z)\Phi(z_0)^{-1} + \Phi(z)AS_A(z)\Phi(z_0)^{-1} = B\Phi(z)S_A(z)\Phi(z_0)^{-1} = BS_B(z).$$

This proves our assertion.

This result will be useful for proving the following proposition.

Proposition 39: *Two equivalent ODEs on D^* have conjugate monodromy matrices.*

Proof : Since Φ is holomorphic on D^*, the previous result implies that for any analytic continuation along a loop γ in D^* based at z_0, we have:

$$\begin{aligned}(S_B)^{[\gamma]}(z) &= \Phi(z)(S_A)^{[\gamma]}(z)\Phi^{-1}(z_0)\\ &= \Phi(z)S_A(z)(M_A)^{[\gamma]}\Phi^{-1}(z_0)\\ &= \Phi(z)S_A(z)\Phi^{-1}(z_0)\Phi(z_0)(M_A)^{[\gamma]}\Phi^{-1}(z_0)\\ &= S_B(z)\Phi(z_0)(M_A)^{[\gamma]}\Phi^{-1}(z_0),\end{aligned}$$

Hence,

$$(M_B)^{[\gamma]} = \Phi(z_0)(M_A)^{[\gamma]}\Phi^{-1}(z_0).$$

We have thus completed the proof. □

We will now prove the following theorem.

Theorem 40: *Consider the system*

$$z\frac{dY}{dz} = A(z)Y, \quad \textit{on } D^*, \quad \textit{where } A : D^* \to M_n(\mathbb{C}),$$

holomorphic and such that for all $z \in D^*$, we have

$$A(z) = \sum_{n=0}^{+\infty} A_n z^n.$$

The ODE then has a unique fundamental matrix of the form

$$S(z) = \phi(z)z^B \ \textit{on } D \quad \textit{with} \quad \phi(z) = \sum_{n=0}^{+\infty} \phi_n z^n, \quad \phi_0 = I_n \textit{ and } B \in M_n(\mathbb{C}).$$

The coefficients ϕ_n are uniquely determined by the relation

$$\phi_n\left(A_0 + nI_n\right) - A_0\phi_n = \sum_{k=0}^{n-1} A_{n-k}\phi_k, \quad \textit{for all } n \geq 1.$$

The following proof is based on arguments introduced in [5].

Proof : We now prove the following theorem in the special case where if $(\lambda, \mu) \in \mathrm{Sp}(A_0)^2$ with $\lambda \neq \mu$, then $\lambda - \mu \notin \mathbb{Z}$. Under this assumption, we show that there exists a holomorphic function $\phi : D \to \mathrm{GL}_n(\mathbb{C})$ such that

$$S(z) := \phi(z)z^{A_0}, \quad \text{for all } z \in D^*$$

is a fundamental matrix of the ODE. We write

$$\phi(z) = \sum_{n=0}^{+\infty} \phi_n z^n,$$

and determine the coefficients ϕ_n for all $n \in \mathbb{N}$.
Assuming such a matrix S is fundamental, we have for all $z \in D^*$:

$$z\frac{dS(z)}{dz} = A(z)S(z).$$

Therefore,

$$z\frac{d}{dz}(\phi(z)z^{A_0}) = A(z)\phi(z)z^{A_0}.$$

This yields:

$$\sum_{n=1}^{+\infty} n\phi_n z^n z^{A_0} + \sum_{n=0}^{+\infty} \phi_n z^n A_0 z^{A_0} = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} A_{n-k}\phi_k \right) z^n z^{A_0}.$$

Multiplying both sides by z^{-A_0}, we obtain:

$$\sum_{n=0}^{+\infty} \phi_n(nI + A_0)z^n = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} A_{n-k}\phi_k \right) z^n.$$

Thus, for all $n \in \mathbb{N}$,

$$\phi_n(A_0 + nI) = \sum_{k=0}^{n} A_{n-k}\phi_k.$$

In particular, for $n = 0$, we have:

$$\phi_0 A_0 = A_0\phi_0,$$

and so $\phi_0 = I_n$ satisfies the equation.
For $n \geq 1$, we get:

$$\phi_n(A_0 + nI) - A_0\phi_n = \sum_{k=0}^{n-1} A_{n-k}\phi_k.$$

Introducing the map $\kappa_{A_0} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ defined by $\kappa_{A_0}(\phi) := \phi A_0 - A_0\phi$, this becomes:

$$(n \cdot \mathrm{id} + \kappa_{A_0})(\phi_n) = \sum_{k=0}^{n-1} A_{n-k}\phi_k.$$

By the assumption on the spectrum of A_0, κ_{A_0} has no eigenvalues in $\mathbb{Z}^*$, so $n \cdot \mathrm{id} + \kappa_{A_0}$ is an isomorphism of $M_n(\mathbb{C})$. Thus, each ϕ_n is uniquely determined for $n \geq 1$ by:

$$\phi_n = (n \cdot \mathrm{id} + \kappa_{A_0})^{-1} \left(\sum_{k=0}^{n-1} A_{n-k}\phi_k \right).$$

To complete the proof, it remains to justify that the power series defining ϕ converges.

Since A is holomorphic on D^*, there exists $M \geq 0$ such that

$$\|A_n\|_\infty \leq \frac{M}{r^n}, \quad \text{for all } n \in \mathbb{N} \text{ and } r < 1.$$

We define, for all $n \in \mathbb{N}^*$,

$$B_n := L_n(\phi_n), \quad \text{where } L_n := n \cdot \mathrm{id} + \kappa_{A_0}, \text{ for all } n \in \mathbb{N}.$$

In particular, we have

$$B_n = \sum_{k=0}^{n-1} A_{n-k}\phi_k, \quad \text{for all } n \in \mathbb{N}^*.$$

By the triangle inequality,

$$\|B_n\|_\infty \leq \sum_{k=0}^{n-1} \|A_{n-k}\|_\infty \|\phi_k\|_\infty \leq M \sum_{k=0}^{n-1} \frac{\|\phi_k\|_\infty}{r^{n-k}}, \quad \text{for all } n \in \mathbb{N}^*.$$

Since L_n is invertible for all $n \in \mathbb{N}^*$, then so is $\frac{L_n}{n}$. Now,

$$\left(\frac{L_n}{n}\right)^{-1} = \left(\mathrm{id} + \frac{\kappa_{A_0}}{n}\right)^{-1}.$$

For n large enough, we have $\left\|\frac{\kappa_{A_0}}{n}\right\| < 1$, and thus

$$\left(\frac{L_n}{n}\right)^{-1} = \sum_{k=0}^{+\infty} (-1)^k \frac{\kappa_{A_0}^k}{n^k}.$$

Hence,

$$\left\|\left(\frac{L_n}{n}\right)^{-1}\right\| \leq \sum_{k=0}^{+\infty} \frac{\|\kappa_{A_0}\|^k}{n^k} = \frac{1}{1 - \|\kappa_{A_0}\|/n} \longrightarrow 1.$$

Therefore, $\left(\frac{L_n}{n}\right)^{-1}$ is bounded. So, there exists $\widetilde{M} \geq 0$ such that

$$\|L_n^{-1}\| \leq \frac{\widetilde{M}}{n}, \quad \text{for all } n \in \mathbb{N}^*.$$

Then,

$$\phi_n = L_n^{-1}(B_n), \quad \text{for all } n \in \mathbb{N}^*,$$

which gives

$$\|\phi_n\|_\infty \leq \|L_n^{-1}\| \|B_n\|_\infty \leq \frac{\widetilde{M}}{n} \|B_n\|_\infty, \quad \text{for all } n \in \mathbb{N}^*.$$

We now define $\varphi_0 := \|\phi_0\|$ and, for all $n \in \mathbb{N}^*$,

$$\varphi_n := \frac{\widehat{M}}{n} \sum_{k=0}^{n-1} \frac{\varphi_k}{r^{n-k}}, \quad \text{where } \widehat{M} := \widetilde{M} M.$$

We now prove by induction that

$$\|\phi_n\|_\infty \leq \varphi_n, \quad \text{for all } n \in \mathbb{N}.$$

The base case is trivial. Assume the inequality holds for all $n \in \mathbb{N}$. Then:

$$\|\phi_{n+1}\|_\infty \leq \frac{\widetilde{M}}{n+1} \|B_{n+1}\|_\infty \leq \frac{\widehat{M}}{n+1} \sum_{k=0}^{n} \frac{\|\phi_k\|_\infty}{r^{n+1-k}} \leq \frac{\widehat{M}}{n+1} \sum_{k=0}^{n} \frac{\varphi_k}{r^{n+1-k}} = \varphi_{n+1}.$$

Hence, the induction is complete.
Now, under the assumption of convergence, we define

$$f(z) := \sum_{n=0}^{+\infty} \varphi_n z^n.$$

Then,

$$f'(z) = \sum_{n=1}^{+\infty} n \varphi_n z^{n-1} = \widehat{M} \sum_{n=0}^{+\infty} \left(\sum_{m=0}^{n} \frac{\varphi_m}{r^{n+1-m}} \right) z^n.$$

This leads to

$$f'(z) = \frac{\widehat{M}}{r} \left(\sum_{n=0}^{+\infty} \varphi_n z^n \right) \left(\sum_{n=0}^{+\infty} \left(\frac{z}{r} \right)^n \right) = \frac{\widehat{M} f(z)}{r - z},$$

which implies in particular that $|z| < r$.
Thus, we find

$$f(z) = \frac{C}{(r-z)^{\widehat{M}}}, \quad C \in \mathbb{C}.$$

Therefore, the radius of convergence of the series defining f is r. Since this is valid for all $r > 1$, we deduce that the radius of convergence of ϕ is 1. Consequently, the series defining ϕ converges.
This completes the proof. $\square$

3.3.1 Application to Frobenius-type Equations

In this section, we study differential equations of the form

$$\frac{d^2 u}{dz^2} + \frac{p(z)}{z} \frac{du}{dz} + \frac{q(z)}{z^2} u = 0 \quad \text{on } D^*,$$

where p and q are holomorphic on D^*. Alternative solution methods, different from those that will be presented here, can be found in [4]. This equation is characterized by a singularity of order 2 at the origin.

To apply Theorem 40, we introduce the differential operator

$$\delta := z\frac{d}{dz}.$$

From this definition, we deduce in particular that

$$\delta^2 = \delta(\delta) = \delta + z^2\frac{d^2}{dz^2}.$$

Hence, the differential equation can be equivalently rewritten as

$$\delta^2 u + (p(z) - 1)\delta u + q(z)u = 0 \quad \text{on } D^*.$$

Setting $U := \begin{pmatrix} u \\ \delta u \end{pmatrix}$ and

$$A(z) := \begin{pmatrix} 0 & 1 \\ -q(z) & -(p(z) - 1) \end{pmatrix},$$

we obtain the equivalent first-order system

$$\delta U = A(z)U \quad \text{on } D^*.$$

Since $\delta U = z\frac{dU}{dz}$, we are exactly in the setting of Theorem 40. The fundamental matrix solution then takes the form

$$S(z) := \phi(z)z^{A_0},$$

where

$$A_0 = \begin{pmatrix} 0 & 1 \\ -q_0 & -(p_0 - 1) \end{pmatrix}, \quad \text{with } p_0 := p(0), \quad q_0 := q(0),$$

provided the eigenvalues of A_0, denoted λ_1 and λ_2, do not differ by an integer. Suppose A_0 is diagonalizable. Then there exists an invertible matrix P and a diagonal matrix D such that

$$A_0 = PDP^{-1}.$$

Thus, we may write

$$S(z) = \phi(z)Pz^D P^{-1}.$$

Let U be a solution of the equation. Then, we have

$$U(z) = S(z)C,$$

for some $C \in \mathbb{C}^2$. Since A_0 is diagonalizable, there exist $C_1 \in E_{\lambda_1}(A_0)$ and $C_2 \in E_{\lambda_2}(A_0)$ such that

$$C = C_1 + C_2.$$

Therefore,

$$U(z) = S(z)(C_1 + C_2) = \phi(z)z^{A_0}C_1 + \phi(z)z^{A_0}C_2 = \phi(z)z^{\lambda_1}C_1 + \phi(z)z^{\lambda_2}C_2.$$

This implies, in particular, that the general solution $u(z)$ can be written as

$$U(z) = \left(\sum_{n=0}^{+\infty} \phi_n z^{n+\lambda_1} \right) C_1 + \left(\sum_{n=0}^{+\infty} \phi_n z^{n+\lambda_2} \right) C_2,$$

Since the solution space is of dimension 2, the functions $\phi(z)z^{\lambda_1}$ and $\phi(z)z^{\lambda_2}$ form a basis of the solution space.

Note that in this context, the monodromy matrix is given by

$$M = \begin{pmatrix} e^{2i\pi\lambda_1} & 0 \\ 0 & e^{2i\pi\lambda_2} \end{pmatrix}.$$

If λ_1 and λ_2 differed by an integer, then

$$M = \begin{pmatrix} e^{2i\pi\lambda_1} & 2i\pi \\ 0 & e^{2i\pi\lambda_2} \end{pmatrix}.$$

It is therefore interesting to observe that the properties of A_0 directly influence the structure of the monodromy.

Conclusion

In this work, we have explored the theory of **linear differential equations** with holomorphic coefficients, from first-order scalar equations to higher-order systems and their matrix formulations. Starting with local solutions around regular points, we gradually introduced the notion of singularities and the powerful framework of **analytic continuation**.

A central theme of this study was the phenomenon of **monodromy**, which captures the multivalued nature of solutions upon analytic continuation along closed paths. The monodromy representation provides deep insight into the global structure of solution spaces and connects **differential equations** to **topological and algebraic structures**.

Throughout the chapters, we illustrated these concepts through classical methods, explicit computations, and examples that shed light on the rich interplay between complex analysis, linear algebra, and differential geometry.

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