ENSAE - Computational Statistics

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December 15, 2017

Problem 7.22. Question (a)
Show that the problem can be described by (**i**) a permutation σ on $\mathbf{1}, \ldots, \mathbf{N}$ and (**ii**) a distance $\mathbf{d(i, j)}$ on $\mathbf{1}, \ldots, \mathbf{N}$.

The traveling salesman problem is a classic in combinatoric and operations research, where a salesman has to find the shortest route to visit each of his N customers.

The N customers are indexed $\{1, \ldots, N\}$.
The best way to optimize the salesman journey is to avoid visiting any customer twice except for the first customer who will be the first and last place to visit.
We will focus on this kind of journeys.
The index of the city visited at step i is defined as $\sigma(i)$ where $i \in \{1, \ldots, N\}$.
So $\{\sigma(1), \ldots, \sigma(N)\} = \{1, \ldots, N\}$. σ is a permutation of $\{1, \ldots, N\}$.
Let $d(i, j)$ be the distance between cities i and j. We can reformulate the problem as the minimization of the distance between two consecutives customers. Meaning :

$$\min_{\sigma} \sum_{i=1}^{N} d(\sigma(i), \sigma(i+1))$$

Question (b)
Deduce that the traveling salesman problem is equivalent to minimization of the function

$$\mathbf{H}(\sigma) = \sum_{\mathbf{i}} \mathbf{d}(\mathbf{i}, \sigma(\mathbf{i}))$$

From **(a)**, the total distance traveled is

$$\sum_{i=1}^{N} d(\sigma(i), \sigma(i+1))$$

. We can also interpret it as follows : Let σ be the permutation applied after the visit of the i^{th} customer. We then must minimize the following function :

$$H(\sigma) = \sum_{i=1}^{N} d(i, \sigma(i))$$

Question (c)
Propose a Metropolis-Hastings algorithm to solve the problem with a simulated annealing scheme (Section 5.2.3).

Using the distance function formulated in **(a)**, we suggest this Metropolis-Hastings with simulated annealing scheme algorithm :

Algorithm 1 Metropolis-Hastings with simulated annealing

Initialization :
Set $\sigma^{(0)}$, a random permutation of $\{1, \dots, N\}$.
Set T_0 (the initial temperature of the simulated annealing scheme) to 1.

At iteration t :

- Simulate σ' by swapping $\sigma(i)$ and $\sigma(j)$, with $(i, j) \in \{1, \dots, N\}^2$
- $$\sigma^{(t+1)} = \begin{cases} \sigma', & \text{with probability } \rho = min(exp^{-\frac{(H(\sigma')-H(\sigma^t))}{T_t}}, 1). \\ \sigma^t, & \text{otherwise.} \end{cases}$$
- $T_{(t+1)} = \beta T_t$ with $\beta = 0.99$ (arbitrarily)

Question (d)
Derive a simulation approach to the solution of $Ax = b$ and discuss its merits.

We can use simulated annealing with a Gaussian kernel to minimize $f(x) = \|Ax - b\|$

Algorithm 2 Metropolis-Hastings with simulated annealing

Initialization :
We take a random $x^0 \in \mathbb{R}^{\mathrm{N}}$

At iteration t :

- Simulate $x' \sim N(x^{(t)})$ and set $T_0 = 1$
- $$x^{(t+1)} = \begin{cases} x', & \text{with probability } \rho^{(t)} = min(exp^{-\frac{(f(x')-f(x^{(t)})}{T_t}}, 1). \\ x^t, & \text{otherwise.} \end{cases}$$
- Set $T_{(t+1)} = \beta T_{(t)}$