ENSAE - Computational Statistics

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Problem 10.15. If $K(x, x')$ is a Markov transition kernel with stationary distribution g, show that the Metropolis-Hastings algorithm where, at iteration t, the proposed value $y_t \sim K(x^{(t)}, y)$ is accepted with probability $\frac{f(y_t)g(x^{(t)})}{f(x^{(t)})g(y_t)} \wedge 1$ provides a valid MCMC algorithm for the stationary distribution f.

Metropolis-Hastings with Markov transition kernel and acceptance probability depending on the stationary distribution of the Markov transition kernel.
Let $K(,)$ a Markov transition kernel with stationary distribution g. We are considering the following modified Metropolis-Hastings algorithm :

Algorithm 1 Metropolis-Hastings

$\forall t$,

- Generate $Y_t \sim K(x^{(t)}, y)$
- $$x^{(t+1)} = \begin{cases} y_t, & \text{with probability } \rho(x^{(t)}, y_t) = min\left(\frac{(f(y_t)g(x^{(t)})}{f(x^{(t)})g(y_t)}, 1\right). \\ x^t, & \text{otherwise.} \end{cases}$$

Let's assume that $K(,)$ and g verify the detailed condition balance. Let $\tilde{K}(,)$ be the transition kernel of $(X^{(t)})$ We have,

$$\tilde{K}(x, y) = \rho(x, y)K(x, y) + (1 - r(x))\delta_x(y)$$

with

$$r(x) = \int \rho(x, y)K(x, y)dy$$

We notice

$$(1 - r(x))\delta_x(y)f(x) = \begin{cases} (1 - r(x))f(y) = (1 - r(y))f(x), & \text{if } x = y. \\ 0, & \text{otherwise.} \end{cases}$$

So,

$$(1-r(x))\delta_x(y)f(x) = (1-r(y))\delta_y(x)f(y)$$

And,

$$\begin{aligned}\rho(x,y)K(x,y)f(x) &= min\left(\frac{f(y)g(x)}{f(x)g(y)},1\right)K(x,y)f(x)\\ &= min\left(\frac{K(x,y)f(y)g(x)}{g(y)},K(x,y)f(x)\right)\end{aligned}$$

From the detailed condition balance, we know $K(x,y)g(x) = K(y,x)g(y)$ then

$$\begin{aligned}\rho(x,y)K(x,y)f(x) &= min\left(\frac{K(y,x)g(y)f(y)g(x)}{g(x)g(y)},\frac{K(y,x)g(y)f(x)}{g(x)}\right)\\ &= min\left(K(y,x)f(y),\frac{K(y,x)g(y)f(x)}{g(x)}\right)\\ &= min\left(1,\frac{f(x)g(y)}{f(y)g(x)}\right)K(y,x)f(y)\end{aligned}$$

$\rho(x,y)K(x,y)f(x) = \rho(y,x)K(y,x)f(y)$

Hence,

$$\tilde{K}(x,y)f(x) = \tilde{K}(y,x)f(y)$$

The detailed balance condition is verified and f is the stationary distribution of the Metropolis Hastings.