ENSAE - Computational Statistics

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Problem 6.7. Given the transition matrix

$$P = \begin{pmatrix} 0 & 0.4 & 0.6 & 0 & 0 \\ 0.65 & 0 & 0.35 & 0 & 0 \\ 0.32 & 0.68 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.12 & 0.88 \\ 0 & 0 & 0 & 0.56 & 0.44 \end{pmatrix}$$

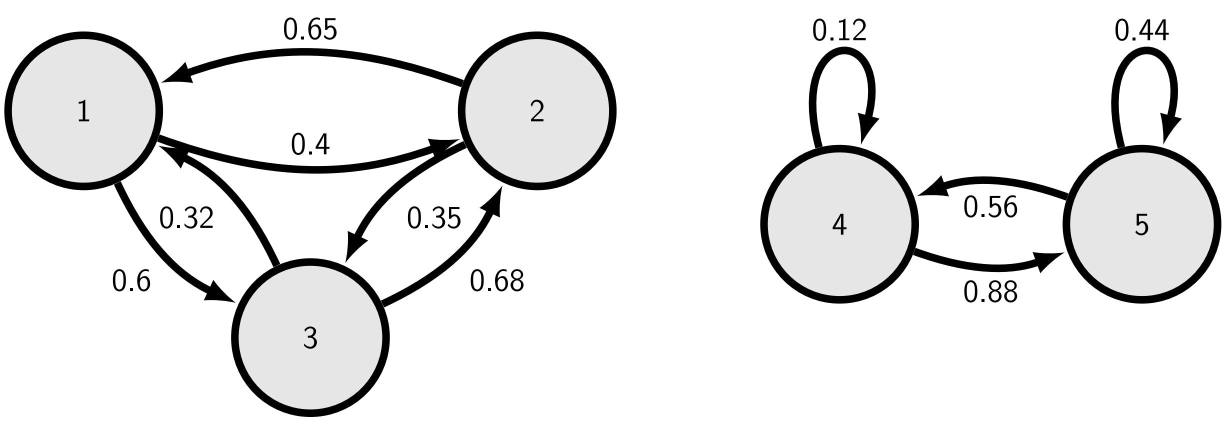
Examine whether the corresponding chain is irreducible and aperiodic.

A Markov Chain is irreducible if all states communicate. Meaning

$$\forall i, j, P(X_i = x_i | X_j = x_j) > 0$$

In the equation above, the transition matrix is not irreducible since we can reduce it in two matrices $P_{1,3}$ (sub-matrix composed by the states 1, 2 and 3) and $P_{4,5}$ (sub-matrix composed by the states 4 and 5). For example,

$$P(X_i = 5 | X_j = 1) = 0$$



For a state i the period is $d_i = hcf\{n \geq 1; P(X_{i+n} = x_i | X_i = x_i) > 0\}$, where hcf is the highest common factor. The state i is aperiodic if $d_i = 1$.

- State $i = 1$
  There are 2 paths : $1 \to 2 \to 1$ of length 2 and $1 \to 2 \to 3 \to 1$ of length 3.

  $$d_1 = hcf(2, 3) = 1$$

- State $i = 2$
  There are 2 paths : $2 \to 3 \to 2$ of length 2 and $2 \to 3 \to 1 \to 2$ of length 3.

  $$d_1 = hcf(2, 3) = 1$$

- State $i = 3$
  There are 2 paths : $3 \to 1 \to 3$ of length 2 and $3 \to 2 \to 1 \to 3$ of length 3.

  $$d_1 = hcf(2, 3) = 1$$

- State $i = 4$
  There are 2 paths : $4 \to 4$ of length 1 and $4 \to 5 \to 4$ of length 2.

  $$d_1 = hcf(1, 2) = 1$$

- State $i = 5$
  There are 2 paths : $5 \to 5$ of length 1 and $5 \to 4 \to 5$ of length 2.

  $$d_1 = hcf(1, 2) = 1$$

Hence, the Markov Chain of matrix P is aperiodic since all its states are.
Finally, the Markov Chain of matrix P is aperiodic but non irreducible.