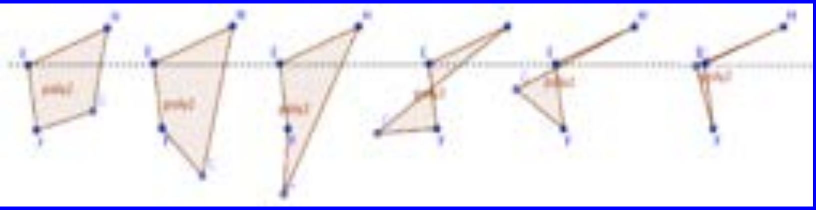
Gerry Stahl's assembled texts volume #13

# Essays in Collaborative Dynamic Geometry



Gerry Stahl

## Gerry Stahl's Assembled Texts

1. *Marx and Heidegger*
2. *Tacit and Explicit Understanding in Computer Support*
3. *Group Cognition: Computer Support for Building Collaborative Knowledge*
4. *Studying Virtual Math Teams*
5. *Translating Euclid: Designing a Human-Centered Mathematics.*
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# Essays in Collaborative Dynamic Geometry

**Gerry Stahl**

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Gerry Stahl

Gerry@GerryStahl.net

www.GerryStahl.net



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# Introduction

The essays in this volume are about the Virtual Math Teams project during its later years when it included a multi-user version of GeoGebra, supporting collaborative dynamic geometry. These essays supplement the more systematic presentations in *Translating Euclid* and *Constructing Dynamic Triangles Together*.

## References

The essays in this volume were originally published as: (Stahl et al. 2010; Stahl 2012a; 2012b; 2012c; Öner & Stahl 2015; 2016; Khoo & Stahl 2015; Çakir & Stahl 2015; Stahl, Weimar, Fetter & Mantoan 2014a; 2014b)

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# Contents

# 1. Multi-User GeoGebra for Virtual Math Teams

Gerry Stahl, Jimmy Xiantong Ou, Baba Kofi Weusijana, Murat Perit Çakir & Stephen Weimar

ABSTRACT. The Math Forum is an online resource center for pre-algebra, algebra, geometry and pre-calculus. Its Virtual Math Teams (VMT) service provides an integrated web-based environment for small teams to discuss mathematics. The VMT collaboration environment now includes the dynamic mathematics application, GeoGebra. It offers a multi-user version of GeoGebra, which can be used in concert with VMT's chat, web browsers, curricula and wiki repository.

## The Virtual Math Teams Project

The Virtual Math Teams (VMT) Project grew out of the Problem-of-the-Week (PoW) service at the Math Forum. The Math Forum is a well-established online resource for improving math learning, teaching and communication (Renninger & Shumar, 2002). Operating since 1992, the Math Forum is now visited by several million different visitors a month. Its PoW service provides challenging problems of K-12 students on a weekly basis. These problems are primarily oriented to individual student work, and exemplary student solutions are posted to the http://mathforum.org site. The original idea of the VMT Project was to provide similar stimulating problems for small groups of students to work on collaboratively over the Internet (Stahl, 2006; 2009).

In our design-based research at the VMT Project, we started by hosting student chats in a variety of commercially available environments, including AOL Instant Messenger, Babylon, WebCT and Blackboard. Based on these early investigations, we concluded that we needed to include a shared whiteboard for drawing geometric figures and for persistently displaying notes. We also found a need to minimize "chat confusion" by supporting explicit referencing of response threads. We decided to adopt and adapt ConcertChat, a research chat environment with special referencing tools (Mühlpfordt & Wessner, 2009). By collaborating with the software developers at Fraunhofer IPSI in Germany, our educational researchers have been able to successively try out versions of the environment with groups of students and to gradually modify the environment in response to what we find by analyzing the chat logs.

## Referencing support for collaboration

The ConcertChat environment integrates text chat with a shared whiteboard. A unique feature of ConcertChat is its support for graphical referencing (see Figure 1 below for an example). It allows for three forms of referencing from the text chat:

- A chat message can point to one or more earlier textual postings with a bold connecting line. When that message appears in the chat as the last posting or as a selected posting, a bold line appears connecting the text to the selected chat posting above.
- While someone types a new chat message, they can select and point to a rectangular area in the whiteboard. When that message appears in the chat as the last posting or as a selected posting, a bold line appears connecting the text to the area of the whiteboard.
- While someone types a new chat message, they can select and point to a graphical object in the whiteboard. When that message appears in the chat as the last posting or as a selected posting, a bold line appears connecting the text to the area of the whiteboard.

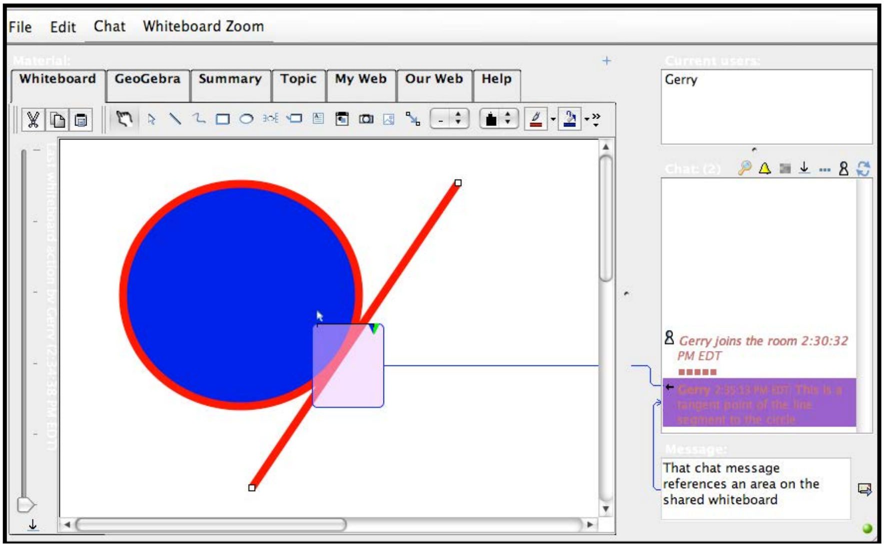


Figure 1. A VMT chat room. Note the multiple workspace tabs on the left and the chat area on the right. The selected chat message is referencing an area on the whiteboard and the new message being typed references that previous message. The small rectangles above the selected chat message provide awareness within the chat area that a series of actions have taken place within one of the workspaces. The rectangles are color coded to match the color of the chat messages of the user who made the actions.

Referencing is critical to supporting online collaboration. In face-to-face situations, like a group standing around a physical whiteboard, we tend to take for granted that people point to and gesture at items in the whiteboard. People also take turns drawing on the whiteboard by exchanging possession of the marker—and it is visible to everyone who is doing the drawing. In an online context, other forms of referencing and awareness are needed. The Concert-Chat referencing tool can be used to avoid or clarify confusions in text-chat discussions. The action indicators (shown in Figure 1) provide another form of awareness to someone focused on the chat that other participants are active in the whiteboard. In addition, notices are displayed announcing who is typing in the chat, editing text boxes in a whiteboard or creating new objects in a drawing. The box above the chat maintains a list of who is currently logged in the room.

# The VMT lobby, chat room and wiki

This referencing is just one form of integration of media in the VMT environment. The overall technological integration of the VMT Lobby (or portal), chat room/shared whiteboard, and wiki should be understood theoretically as a pedagogical integration of learning at the individual, small-group and community levels: (a) The VMT Lobby provides a portal for the *individual user* to browse the people and topics of the community and to select a room for group work. (b) The chat rooms are basically meeting and work places for the *small groups* as they engage in synchronous collaborative learning. (c) The wiki, on the other hand, primarily provides an asynchronous *community space* in which the work of all groups is coordinated, commented upon and perhaps summarized.

(a) The *VMT Lobby* provides a social networking portal for students to log into the system. It includes tools for defining and viewing personal profiles. In general, students in a VMT group have no knowledge about each other except for what is revealed in the chat interaction; with the functionality available in the VMT Lobby, they can define their own profiles and view profiles of each other, as well as send messages to individuals or groups in their communities (projects). Communities are defined for various VMT constituencies, such as participants in a given online contest or in a given course. There is also support for defining buddies, listing favorite chat rooms, etc. In addition, there is an interface for searching and browsing available chat rooms, usually listed for a given community. This provides access to chat rooms on different topics. Students may be told by their teachers to find certain rooms, may be invited by buddies, may search for rooms on interesting topics or may create new rooms and invite peers to join them.


Figure 2. The VMT Lobby. In this view, a list of chat rooms is displayed for Project eMath. The math subjects for which there are rooms in this project are listed; under each subject are math topics (here, a set of related combinatorics topics) that have rooms defined; under each topic are a list of rooms. Clicking on a room name opens the associated chat room.

(b) A typical *VMT chat room* consists of the text chat interface on the right and a shared whiteboard on the left (see Figure 1 above). The history of the whiteboard state can be scrolled through, much like that of the chat, but unlike the chat it usually retains inscriptions in the visible board as long as they are relevant. VMT chat rooms have a tabbed interface, with multiple workspaces—and users can add additional spaces as needed. One kind of workspace is the shared *Workspace*, supporting graphics and text boxes. Another is a similar shared whiteboard, intended for preparing a *Summary* of the group's work for posting to a special wiki page associated with this chat room. A third tab may display the *Topic* for the room, stored on a wiki page by an instructor. A *Wiki* tab displays a page of the VMT wiki; a special page is created for each room, linked to other pages on the Topic, math Subject or Community. A *Browser* tab provides a simple multi-user web browser that can support the graphical referencing tool from the chat and a history scrollbar. A final tab can display wiki pages containing the VMT *Help* manual and associated information. As described below, we have recently added an optional *GeoGebra* tab. This provides a complex, but integrated set of spaces for a group to work and communicate together. A group working on a math topic can bring in resources from the different tabs and everyone can see what the others are viewing and working on.

(c) The *VMT wiki* can act as a digital library repository for summaries of work posted by teams. If there is a course that involves multiple chats by several teams, a wiki home page can be constructed for the course. The home page would then point to pages describing the course and each assignment. Group assignments are all posted to linked wiki pages. The course wiki includes index pages that bring together the student assignments in various combinations and allow the instructor to post feedback that is visible to all. The student groups can also rate and provide feedback to each other's previous reports.

## Integration of tools in the environment

The VMT environment has come a long way from the simple AOL Instant Messaging system to the current lobby/chat/tabbed-spaces/wiki multiple-interaction space. In part, this increased complexity parallels the shift from simple math exercises to open-ended explorations of math worlds, from one-shot meetings to multiple-session Fests, from problem-solving tasks to knowledge-building efforts. Along with the considerable gain in functionality come substantial increases in complexity and the potential for confusion. This has been countered by trying to extend and supplement the integration approaches of ConcertChat. The graphical referencing and the history scrollbars have been extended to the multiple tabs. New social awareness notices have been added to track which tab each group member is viewing or referencing.

The VMT collaboration environment has been tuned to the needs of high-school math students. There are specifically math-oriented functions—like a partial implementation of MathML for displaying equations (see http://vmt.mathforum.org/VMTLobby/VMTHelp/ mathequations.html) and the whiteboard's stock of Euclidean shapes. In addition, there are tools for integrating the multiple workspaces—like the graphical referencing from chat, the creation of wiki pages corresponding to each chat room and the posting of summary text to the proper wiki page.

Integration across modules has been important. Logins and passwords have been unified across the Lobby, chat rooms and wiki, so that logging into one automatically logs into the others. People registered in one module show up in the profiles and messaging system, by their selected community. When a new chat room is created, it is categorized by a community (e.g., a school), subject (e.g., combinatorics), a topic (e.g., Week 3's assignment) and a group (e.g., Team D). A new wiki page is generated for posting the summary from this room. The MediaWiki functionality of categories automatically associates this new page with aggregation pages for the community, subject, topic and group.

## Porting GeoGebra to VMT

Our most recent enhancement to the VMT environment was to port the single-user GeoGebra application into VMT as a multi-user component of the tabbed chat room. This allows groups of users to co-develop and co-explore a GeoGebra geometric construction. They can chat about the drawing and reference parts of it from their chat postings. There is a history slider, so users can scroll back and forth, watching the changes take place in the drawing for convenient review and reflection.

The version of GeoGebra in VMT is fully multi-user. VMT integrates GeoGebra as a tab of the environment (see Figure 3). GeoGebra is a particularly appropriate dynamic math application for this project because its source code is freely available as open source, there is a development community to support on-going development, the lead developer and the founder are consulting with us, the application supports a wide range of math from Euclidean construction to calculus and 3-D, GeoGebra has won international prizes, and it has been translated into about 50 languages.

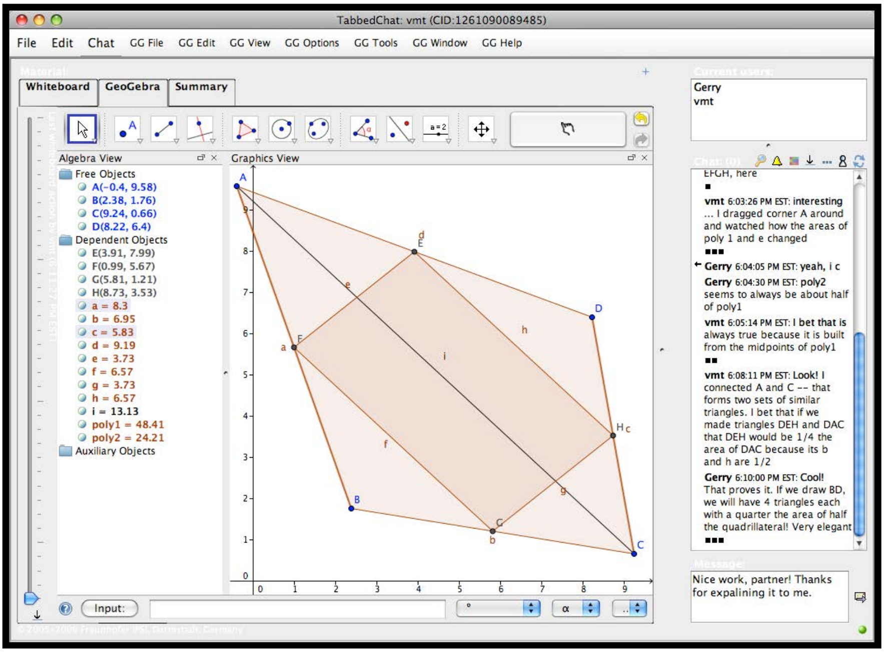

Figure 3. A GeoGebra construction created and discussed collaboratively in the VMT 2.0 learning environment.

Like all other dynamic math applications, GeoGebra previously existed only as a single-user application. While users could send their static constructions to each other, display screen images, or awkwardly include a view of the GeoGebra application within other environments (Blackboard, Moodle, Elluminate, etc.), only one person could dynamically manipulate the construction. Our port converted GeoGebra into a client-server architecture, allowing multiple distributed users to manipulate constructions simultaneously and to all observe everyone's actions in real time. Every action in the GeoGebra tab is immediately broadcast by the server to all collaborating clients.

In addition, incorporation of GeoGebra in the VMT environment framework allows users to engage in text chat while manipulating the construction. Importantly, users can graphically point from a chat posting to an area of the construction that they want to index—an important support for math discourse that is unique to VMT (or its now-defunct basis, ConcertChat). They can also scroll back and forth through the history of the GeoGebra construction, animating its evolution—a powerful way to explore many mathematical relationships. In addition, a complete record of the collaborative construction is available to the participants, their teachers and project researchers, allowing them to analyze and reflect upon the complete interaction, including the construction actions synchronized with the chat.

The VMT version of GeoGebra is compatible with the standard version. Thus, constructions can be imported and exported seamlessly between the two versions. This facilitates use of legacy GeoGebra curriculum within the collaborative VMT environment. Images of GeoGebra co-constructions can be created and pasted by users into the VMT wiki or into Word documents. Logs of the corresponding chats can also be saved as spreadsheet files and pasted into documents.

The integration of GeoGebra significantly enhances the mathematical domain-orientation of the VMT system. On the other hand, for the GeoGebra community, it makes available for the first time truly multi-user dynamic geometry support within a rich collaborative environment. With the flexible system of tabbed components, a curriculum designer, instructor or even a student can define topics for rooms with just GeoGebra and chat or with a more complicated mix of additional browsers and support components.

For researchers of math learning, the enhanced environment provides a flexible laboratory for hosting virtual math teams engaged in GeoGebra-based tasks. The entire interactions of these teams will be logged in detail. Not only can the logs be generated in a variety of convenient formats, but also the team interactions can actually be replayed from the logs like digital videos for careful study (see Figure 4). With these tools, researchers can explore the group cognition of small teams accomplishing creative problem solving involving geometric constructions that are shared, visible and dynamic.

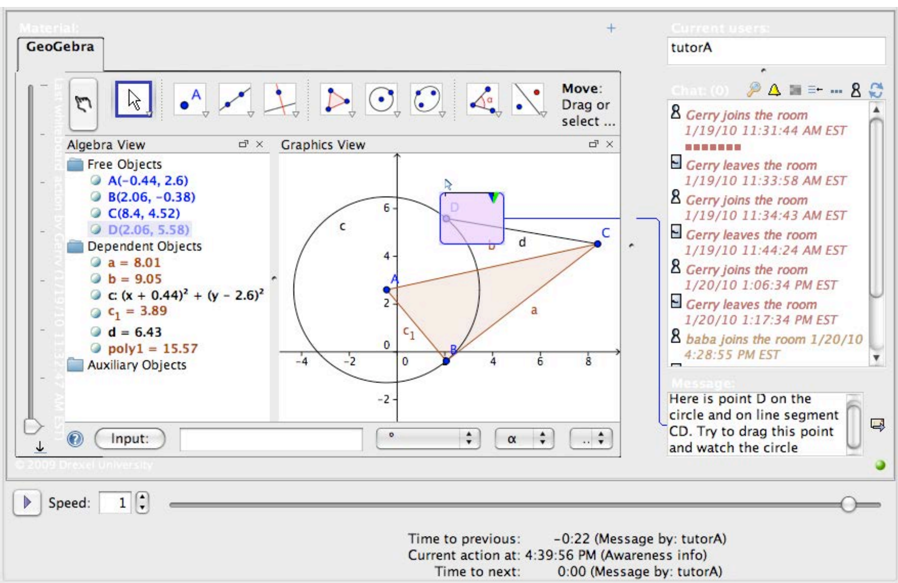

Figure 4. The VMT Replayer. The slider at the bottom can be set to scroll forward in real time or fast forward, as well as being dragged for browsing or stepped forward and backward action by action for detailed observation of coordinated construction and chat. Note that the new chat message being typed is graphically referencing a point in the GeoGebra construction for others to see.

# Designing for multi-user issues

Making GeoGebra multi-user has involved many technical, underlying changes to the software and has necessitated a number of trade-offs and design decisions. In terms of the software technology, we treated the GeoGebra application as a client and embedded it in a Concert-Chat tab. Every action performed in the tab is immediately broadcast across the Internet to the VMT server. The server logs the action in its database and then broadcasts the action to the client of every user who is logged into the same room, including the originating client. In this way, each action performed by someone in a given VMT room is displayed identically for everyone who is working together. Minimizing Internet traffic is a major concern, especially with potentially large GeoGebra interdependent objects, and we had to make changes to Concert-Chat and GeoGebra implementations to keep traffic under control.

A major issue with multi-user systems is what to do when two users try to do conflicting things at the same time. We have recently implemented a locking mechanism, so that when two clients are creating objects at the same time or are manipulating the same object simultaneously, the changes are not broadcast until the end of the operations. This causes some delay in sharing what people are doing; however, we believe it is necessary to avoid serious confusion. Imagine if several clients were moving point A in opposite directions at the same time. If the system broadcast changes every tenth of a second, point A would be jumping back and forth wildly, making it hard for either user to move it sensibly. Where would point A end? We have decided to have point A end where the last user to release it leaves it. If two clients are simultaneously creating an initial triangle ABC, then without locking we would get multiple points with the same names. Our locking mechanism avoids these problems by noting the conflict and assigning different names to the points, but at some cost to mutual awareness.

In the near future, we plan to try to implement two mechanisms to counteract the problem of delayed mutual awareness: (1) labeling actions and (2) simulating dragging. (1) We would like to display awareness notices in the drawing area stating who is creating, editing or moving a graphical object. This would indicate when multiple users are simultaneously at work, and perhaps some of the users would then wait to see what the others have done. (2) If point A is dragged to a new position, ending up, say, 5 units to the right, rather than having point A suddenly jump to the new position in everyone's client, we would simulate the dragging motion by interpolating 10 steps at tenth-of-a-second intervals. Then point A would appear to move to its new position through a smooth and straight motion. This would not be true to the actual dragging motion, but would give more of a feel for a dragging manipulation, which we believe to be important to the GeoGebra manipulation experience.

Of course, other trade-offs are possible, depending upon the technical architecture. We are trying certain approaches and testing them out. We hope to soon have students trying our system. Gradually, we will learn of additional problems and evolve some solutions. The experience will never be the same as having a group of geometers standing around a physical whiteboard—although in some ways it will be better because there will be a permanent record of all interactions, which can be replayed for reflection and analysis. We hope that the integration of GeoGebra with text chat will help to overcome problems that arise from imperfect mutual awareness by allowing people to discuss in text what they are doing in constructions.

Mathematics is often thought of as a solitary experience. However, our findings in the Virtual Math Teams Project show that it can be an exciting, engaging, motivating and rewarding experience when conducted collaboratively. To promote this effectively online, one must provide a carefully crafted set of tools. We believe that GeoGebra can play an important role as a central tool in the VMT environment and we look forward to working with the GeoGebra development and user community to tune our environment to meet the needs of math education globally.

# Acknowledgements

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We would like to thank Markus Hohenwarter and Michael Borcherds for their support and assistance in porting GeoGebra to VMT.

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# Biographies

**Gerry Stahl** teaches, publishes and conducts research in human-computer interaction (HCI) and computer-supported collaborative learning (CSCL). His books are *Group Cognition: Computer Support for Building Collaborative Knowledge* (2006, MIT Press) and *Studying Virtual Math Teams* (2009, Springer). He is founding Executive Editor of the *International Journal of Computer-Supported Collaborative Learning* (*ijCSCL*). He is the Principal Investigator of the *Virtual Math Teams Project*, a major research effort in collaboration with the Math Forum@Drexel. He served as Program Chair for the international CSCL 2002 and 2011 conferences. He teaches undergraduate, masters and PhD courses in HCI, CSCW and CSCL at the College of Information Science and Technology at Drexel University in Philadelphia, USA. See http://GerryStahl.net.

**Jimmy Xiantong Ou** is the Java programmer who ported GeoGebra to the Virtual Math Teams system, converting it to a multi-user system integrated with chat, wiki and social networking. Born in China, he is now a US citizen. He graduated in computer science from Drexel University in Philadelphia, USA.

**Baba Kofi Weusijana** was the software engineer and learning scientist who organized the source code for the Virtual Math Teams system, including GeoGebra, and deployed VMT 2.0. He earned a PhD in learning sciences from Northwestern

University and was a post-doc at the LIFE NSF Science of Learning Center before joining the Math Forum team at Drexel University in Philadelphia, USA.

**Murat Perit Çakir** is now a faculty member in the cognitive science program at the Informatics Institute at the Middle East Technical University in Ankara, Turkey. He was a research assistant in the Virtual Math Teams (VMT) Project from its beginning in 2003 through 2009. He earned a PhD from the College of Information Science and Technology at Drexel University in Philadelphia, USA; his dissertation analyzed the coordination of visual representations with chat discussion in online math problem solving using VMT.

**Stephen Weimar** is the Director of The Math Forum, http://mathforum.org/, a leading online mathematics education community run out of the Goodwin College of Professional Studies at Drexel University. A former middle and high school math teacher, he has been with the Math Forum since it began as an NSF-funded research project at Swarthmore College in 1991. He focuses on research, teacher education and school-based programs concerning the use of technology in K-12 math education, particularly those that support the development of mathematical problem solving and communication.

# 2. Designing a Learning Environment to Promote Math Discourse

Designing a software environment for online learning of mathematics in small collaborative groups requires innovation in multiple dimensions. There has to be generic support for collaborative learning at a distance and also special functionality for mathematical work and communication. We combine the Virtual Math Teams environment with a multi-user version of GeoGebra. We also develop curricular activities through an iterative process of evaluating the discourse that is stimulated by drafts of the activities in prototypes of the technology.

## Introduction

Mathematics education in the future faces enormous opportunities from the availability of ubiquitous digital networks, from innovative educational approaches based on theories of collaborative learning and from rich resources for interactive, online, dynamic math exploration.

The fact that more and more teachers and students are learning online—with distance education, online masters programs, home schooling, online high schools, etc.—makes the incorporation of virtual collaborative learning environments a growing need.

This paper reports on the design of a virtual learning environment that integrates synchronous and asynchronous media with an innovative multi-user version of a dynamic math visualization and exploration toolbox. This VMT-with-GeoGebra environment is designed to support the production of significant math discourse.

# An Online Math Collaborative-Learning Environment

The VMT-with-GeoGebra learning environment integrates two forms of technology to support math learning with collaborative and interactive tools:

a) Computer-supported collaborative learning (CSCL) software and

b) Dynamic mathematics (software that allows users to manipulate geometric diagrams, equations, etc.).

(a) CSCL provides virtual-learning environments in which teams of students can interact synchronously and asynchronously to build knowledge together. This student-centered approach has many advantages, including increased motivation, sharing of skills, engaging in significant discourse and practicing teamwork. The system reported here extends the Virtual Math Teams (VMT) environment, which has already been prototyped and tested (Stahl, 2009b).

(b) Dynamic math (such as Geometer's Sketchpad, Mathematica, Cabri or GeoGebra) has profoundly impacted math education (Goldenberg, 1995; Hoyles & Noss, 1994; King & Schattschneider, 1997; Laborde, 1998; Myers, 2009; Scher, 2002), with Geometer's Sketchpad and GeoGebra used in many US classrooms and globally. Yet, research on math education has not analyzed how students use dynamic math tools in sufficient detail (compare Cakir, Zemel & Stahl, 2009; Stahl, 2009b). GeoGebra (http://www.geogebra.org) is an open-source system for dynamic geometry, algebra and beginning calculus—including trigonometry, conics, matrices, graphing and Euclidean constructions. It offers multiple representations of objects in its graphics, algebra and spreadsheet views—which are all dynamically linked—making GeoGebra a particularly flexible tool for exploration. The VMT-with-GeoGebra system provides the first multi-user version of dynamic math, so that student teams can explore math collaboratively; it integrates the GeoGebra dynamic math tools into the larger VMT virtual collaborative-learning environment with text chat and wiki to support persistent discourses about math—that can be shared, reflected on and researched.[1]

[1] For a demo of the prototype system, go to http://vmt.mathforum.org/VMTLobby. Log in as "guest" with password "guest". The Lobby should open showing the List of All Rooms. Select Project "VMT Research". Click on "Apply filters". Open "Geometry". Open "Polygons". Click on "GeoGebra Demo Room" Eventually a JavaWebStart chat room should open. Explore its different tabs and functions.

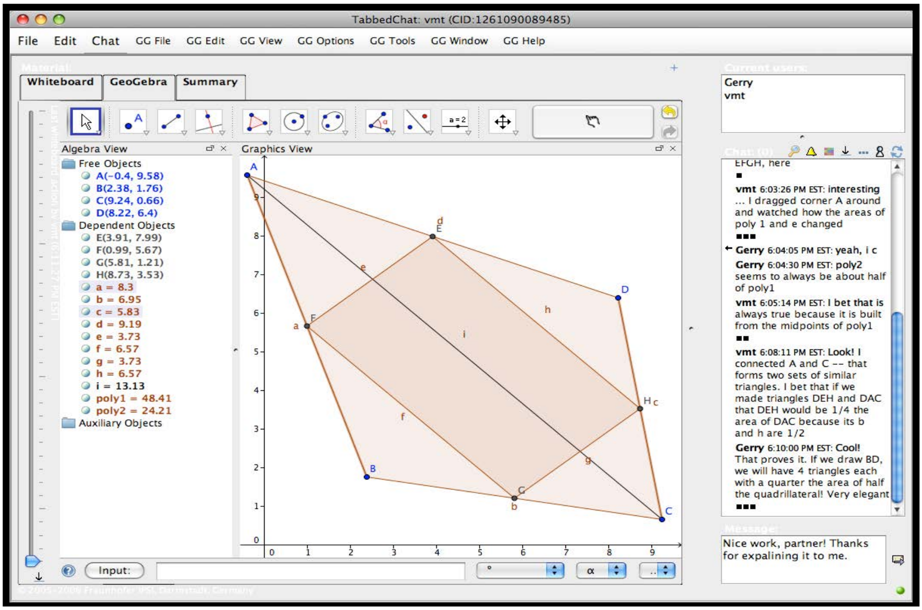


Figure 1. A demo (not real student interaction data) GeoGebra construction created and discussed collaboratively in a multi-user prototype of the learning environment, based on the VMT system. The VMT system includes (not shown here): a Lobby with social networking and tools for teachers, integration with a wiki, and Web browsers.

The VMT-with-GeoGebra system grew out of the successful Virtual Math Teams (VMT) Project. The VMT Project developed an open-source virtual learning environment for math students between 2003 and 2010. The system integrated a social-networking portal, synchronous text chat, a shared whiteboard, an asynchronous wiki, a referencing tool, mathML expressions and a web browser. Student actions and chat postings are automatically logged; they can be replayed for reflection, assessment and analysis by students, teachers and researchers. Over a thousand student-hours of piloted usage were logged. A qualitative micro-analytic approach to interaction analysis was developed based on ethnomethodologically inspired conversation analysis (Garfinkel, 1967; Sacks, 1962/1995; Stahl, 2009a; 2009c; Zemel, Çakir & Stahl, 2009). A large number of publications have appeared from the project (see http://GerryStahl.net/vmt/pubs.html), including 2 books (Stahl, 2006; 2009b) and 8 doctoral dissertations (Çakir, 2009; Litz, 2007; Merges, 2010; Mühlpfordt, 2008; O'Hara, 2010; Sarmiento-Klapper, 2009; Wee, 2009; Zhou, 2010).

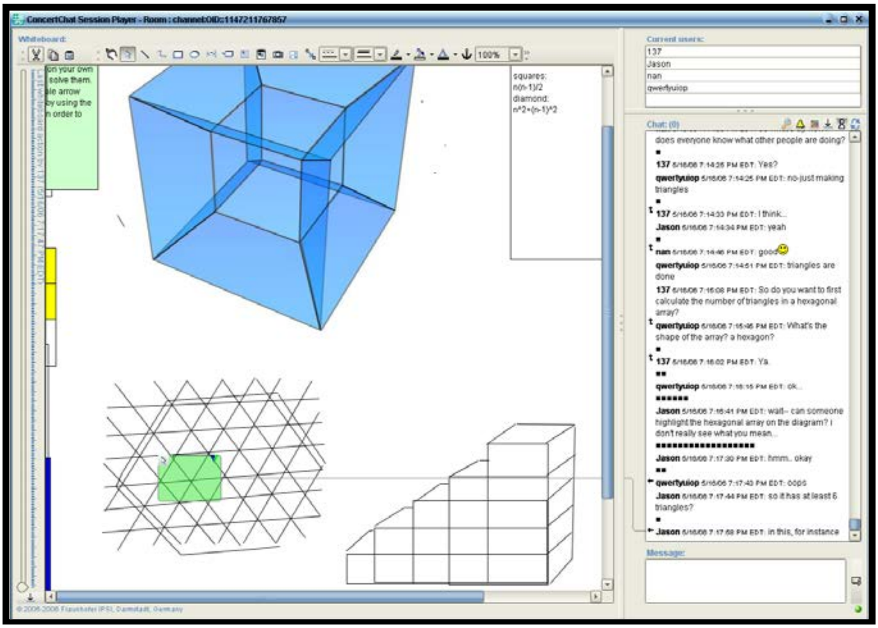

*Figure 2*. Image of actual student online collaborative work on patterns; a student points from a chat message to a smallest hexagon pattern composed of 6 triangles illustrating VMT's unique integration of chat and whiteboard with its deictic reference tool.

# A Design-Based Research Approach

The VMT Project pioneered the study of online collaborative math discourse—both its nature and modes of computer support for it. The 28 studies in (Stahl, 2009b) present some of the most important of the 169 publications related to the project. They include a number of case studies of interactions in the VMT environment by middle-school, high-school and junior-college students, which analyze: how math problem solving can be effectively conducted collaboratively among students who have never met face-to-face; how the structure of text chat interaction differs from spoken conversation; how the media of graphical diagrams, textual narratives and symbolic representations can be intimately interwoven to build deep math understanding; how deictic referencing is important to establishing shared understanding; how students co-construct a joint problem space; how collaborative meaning making and knowledge building are accomplished in detail; how online math discourse can be supported by a software environment that integrates synchronous and asynchronous media with specialized math tools; and how a methodology based on interaction analysis can be used for a science of group cognition. (See Figures 2 and 3.)

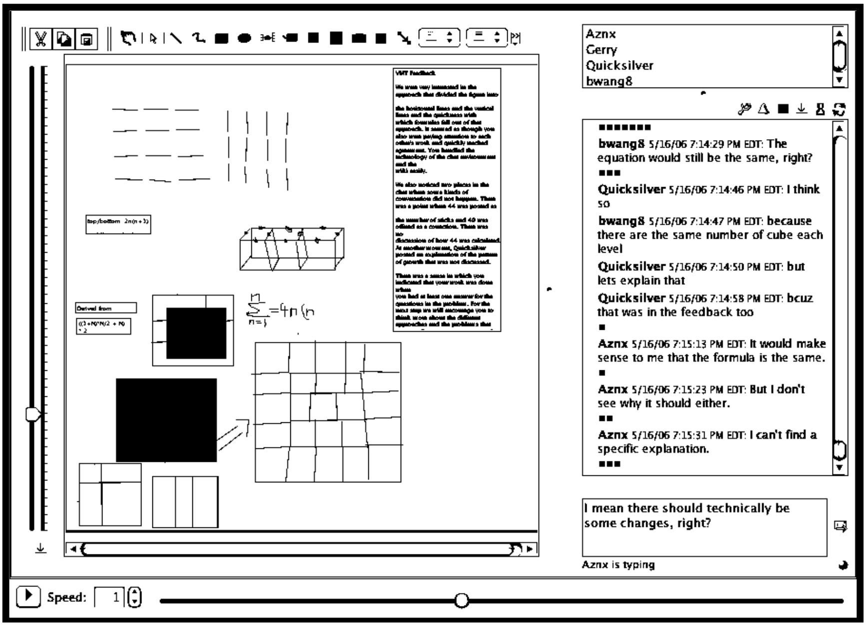


Figure 3 shows the Replayer tool interface across the bottom.

The VMT Project was structured as design-based research, with the technology, research and theory co-evolving through dozens of iterations. The VMT Project demonstrated both the practicality of the VMT-with-GeoGebra system and the need for it. While the VMT Project prototyped a rich cyber-learning environment and studied student interaction, it did not develop the range of supports that we know are needed for classroom use: robust software, problem sets, guidelines, etc. Furthermore, it did not include a dynamic-math component. The VMT-with-GeoGebra system extends the environment to cover these needs.

The VMT Project was widely recognized as an important example of synchronous support for online collaboration and was studied by several international researchers. The VMT Replayer allows complete replay of a user session, including all actions and system notices, as though the session was digitally video-recorded. The researcher's view is guaranteed to be identical to the user's view since it is generated from the same data as sent to a client computer. The log information is also made available in convenient textual or spreadsheet formats for student reflection and reporting as well as for researcher analysis.

# Technology Development

In the VMT-with-GeoGebra system, GeoGebra version 4 has been ported into the VMT system, making the dynamic math tools fully multi-user. GeoGebra is integrated as a tab in VMT (see Figure 1 above). GeoGebra is a particularly appropriate dynamic-math application for this project because its source code is freely available as open source, there is an active international development community to support on-going development, the application supports a wide range of math from algebra and geometry construction to calculus and 3-D, GeoGebra has won international prizes, and it has been translated into about 50 languages. Like all other dynamic-math applications, GeoGebra has until now only existed as a single-user application. While users can send their static constructions to each other, display screen images, or awkwardly include a view of the GeoGebra application within other environments through screen sharing (e.g., in Blackboard, Moodle, Elluminate, etc.), only one person can dynamically manipulate the construction. The port into VMT converted GeoGebra to a client-server architecture, allowing multiple distributed users to manipulate constructions and to all observe everyone's actions in real time. Every action in the GeoGebra tab is immediately broadcast by the server to all collaborating clients (and logged in detail for replay and research).

We have been exploring turn-taking mechanisms (see Figure 4) to avoid conflicts in the construction and modification of GeoGebra drawings; although it is important in synchronous chat to allow multiple users to type simultaneously, we have found that it is natural for a group to allow one member at a time to change a graphical construction and for group members to take turns editing and rearranging.

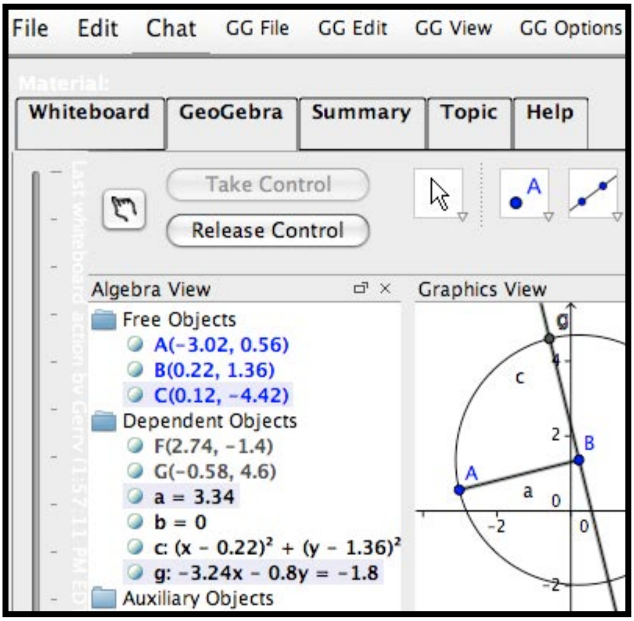


Figure 4. The GeoGebra tab with turn-taking button to avoid conflicts.

# Designing Activities

The VMT-with-GeoGebra system is not a walk-up-and-use simple app. It requires orientation of students to its purposes and introduction to its functionality. The system therefore includes sets of Activities, which step students through interactions with each other, with the technology and with the mathematics. Each Activity stresses the use of the chat medium to support coordination and collaboration as well as to reflect on the mathematical actions engaged in and to investigate the relationships among the dynamic math objects. These Activities are correlated with math content presented in the U.S. *Common Core State Standards for Mathematics* and in selected math textbooks.

Math teachers are trained in the use of the VMT-with-GeoGebra environment by having them work in it on Activities in small groups of teachers, and reflect on their experiences and on how they might use the Activities in their classrooms.

These Activities have been designed to promote collaborative learning, particularly as exhibited in significant mathematical discourse about geometry. Collaborative learning involves a subtle interplay of processes at the individual, small group and class levels of engagement, cognition and reflection. Accordingly, the Activities are structured with sections for individual work, small-group collaboration and whole-class discussion. It is hoped that this mixture will enhance motivation, extend attention and spread understanding.

# Curricular Goals

The goal of the set of Activities is to improve the following skills in math teachers and students:

- To engage in significant mathematical *discourse*; to collaborate on and discuss mathematical activities in supportive small online groups
- To collaboratively *explore* mathematical phenomena and dependencies; to make mathematical phenomena visual in multiple representations; and to vary their parameters
- To *construct* mathematical diagrams – understanding and exploring their structural dependencies
- To notice, wonder about and form conjectures about mathematical relationships; to justify, explain and *prove* mathematical findings
- To understand core concepts, relationships, theorems and constructions of basic high-school *geometry*

The working hypothesis of the activities is that these goals can be furthered through an effective combination of:

- Collaborative experiences in mathematical activities with guidance in collaborative, mathematical and accountable geometric *discourse*
- *Exploring* dynamic-mathematical diagrams and multiple representations
- Designing dependencies in dynamic-mathematical *constructions*
- Explaining conjectures, justifications and *proofs*
- Engagement in well-designed activities around basic high-school *geometry* content

In other words, the Activities seek a productive synthesis of *collaboration*, *discourse, visualization, construction, and argumentation* skills applied in the domain of beginning geometry. They operationalize "deep conceptual learning" of mathematics in terms of these measurable outcomes:

- The quality and quantity of significant mathematical *discourse* in collaborative interactions

- Group *explorations* of mathematical objects and representations, including noticing and wondering
- *Constructions* of mathematical objects with dependencies
- Explanations, justifications and *proofs* of conjectures
- Engagement in significant mathematical discourse involving *geometric* notions of congruence, symmetry, dependencies, relationships, transformations and deduction

# Geometric Dependencies

Our focus has centered increasingly on facilitating and supporting lessons in geometric dependency. GeoGebra allows one to construct systems of interdependent geometric objects. Students have to learn how to think in terms of these dependencies. They can learn through visualizations, manipulations, constructions and verbal articulations. These can all be modeled and these skills can be developed gradually.

Our concerns are incorporated in a focus on dependency as follows:

- Increase the ability of math teachers and students to engage in significant mathematical *discourse* about geometric dependencies.
- Provide math teachers and students with a coherent sequence of activities *exploring* mathematical dependencies.
- Empower math teachers and students to *construct* their own mathematical dependencies among objects in a dynamic-mathematics environment, which they can use in the future as well
- Increase the understanding of math teachers and students in why mathematical objects behave in the ways they are constrained to by their dependencies, possibly *proving* why the dependencies have specific consequences
- Increase the understanding of math teachers and students in the content of basic high-school *geometry* dependencies, including how to discuss them, explore them, visualize them, prove them and extend them

We are now drafting and piloting versions of curricular activities designed to develop significant mathematical discourse focused on dependencies among geometric objects. Concomitantly, we are implementing software support for teachers and students to explore the dependencies and assembling materials for professional development to prepare teachers to enact this curriculum with their students.

# Conclusion

Incorporation of GeoGebra in the VMT environment framework allows users to engage in text chat while manipulating geometric constructions. Importantly, users can graphically point from a chat posting to an area of the construction that they want to index (see Figure 2)—a handy support for math discourse that is unique to VMT. They can also scroll back and forth through the history of the GeoGebra construction, animating its evolution—a powerful way to explore many mathematical relationships. In addition, a complete record of the collaborative construction is available to the participants, their teachers and project researchers, allowing them all to analyze and reflect upon the complete interaction, including the construction actions synchronized with the chat. GeoGebra in VMT provides an exciting collaborative experience and a rich dataset for research on collaborative learning of mathematics. A set of carefully designed Activities introduces students to the VMT environment, the construction of objects in GeoGebra and the approach of dynamic geometry in a collaborative setting.

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# 3. Supporting Group Cognition, Individual Learning and Community Practices in Dynamic Geometry

*Abstract*—Group cognition is analyzed at the small-group unit of analysis. It involves the semantics, syntactics and pragmatics of natural language, gestures, inscriptions, etc. The meaning-making processes involve inputs from individuals, based on their interpretation of the on-going context. They are also responses to the on-going social/historical/cultural/linguistic context, which they can reproduce and modify. Technologies play a central role in mediating the multi-level, intertwined processes. Emergent technologies should be designed to support this mediation. Collaboration environments should be designed to prepare groups, individuals and communities to take advantage of the technical functionality and to promote learning at all levels. This paper reports on the design of a curriculum in dynamic geometry to support group cognition, individual learning and community practices in a coordinated way.

## Introduction

Group cognition is analyzed at the small-group unit of analysis. It involves the semantics, syntactics and pragmatics of natural language, gestures, inscriptions, etc. The meaning-making processes involve inputs from individuals, based on their interpretation of the on-going context (Stahl, 2006, esp. Ch. 16). They also take into account the larger social/historical/cultural/linguistic context, which they can reproduce and modify (Stahl, 2013). Applying this perspective to the learning of mathematics, we adopt a discourse-centered view of mathematical understanding as the ability to engage in significant mathematical discussion (Sfard, 2008; Stahl, 2008). Here, "discourse" includes gesture, inscription, representation and symbol, as well as speech and text; these are often closely interwoven in effective interactions (**Error! Hyperlink reference not valid.**; Çakir, Zemel & Stahl, 2009).

Technologies play a central role in mediating the multi-level, intertwined problem-solving, learning and knowledge-building processes. Emergent technologies should be designed to support this mediation. This involves considering within the design process of collaboration environments how to prepare groups, individuals and communities to take advantage of the designed functionality and to promote mathematical thinking at all levels. This paper reports on the design of a curriculum in dynamic geometry to support group cognition, individual learning and community practices in a coordinated way.

We have been developing a collaboration environment for small groups of students to explore mathematics – especially dynamic geometry – together online (Stahl, 2009). Our Virtual Math Teams (VMT) environment now includes a multi-user version of GeoGebra, an open-source dynamic-geometry tool (Stahl et al., 2010). Shared chat rooms in this VMT environment can include:

- Personal GeoGebra tabs for an individual to experiment with dynamic-geometry explorations and constructions.
- Group GeoGebra tabs for a team of students to experiment together with dynamic-geometry explorations and constructions.
- A text-chat window for a team to discuss its collaborative explorations, while it is working together or to ask questions when team members have problems in their individual work.
- A shared whiteboard and a group wiki page for the group to summarize its findings.
- The wiki can be used by a whole class or a community of teams to view and comment on what each team has accomplished.
- Logs of the text chat and a replayer, which allows anyone to replay a collaboration session in complete detail for purposes of reflection and/or analysis.

We have conducted pilot trials of the VMT-with-GeoGebra environment and have found that this relatively complex system requires some preparation and training for students, student groups and classes to use effectively without encountering frustration. In response to issues identified in the analysis of the multi-user GeoGebra use sessions, we have drafted a set of dynamic-geometry curricular activities, interspersed with tutorial tours of the technology features (Stahl, 2012a). These materials are designed for use both by teachers in professional-development contexts and by students in online-classroom or after-school settings.

The curriculum activities have been designed to promote collaborative learning, particularly as it occurs in significant mathematical discourse about geometry. Collaborative learning involves a subtle interplay of processes at the individual, small-group and classroom levels of engagement, cognition and reflection. Accordingly, the activities are structured with sections for individual work, small-group collaboration and whole-class discussion. It is hoped that this mixture will enhance motivation, extend attention and spread understanding.

# Increasing Skill Levels

The set of activities should gradually increase student skill levels in each of the identified dimensions. The design starts out assuming relatively low skill levels and gradually increases the level of skill expected. Concomitant with this is a progressive shift from scaffolded instruction to open-ended inquiry.

1. The *discourse* begins with having students greet each other online and then negotiate about who will do what, when in the online environment. Students are next asked to comment on their noticings and wonderings. Later, they are to make conjectures. Finally, they are expected to explain things to each other, make sure that everyone understands, and produce presentations of group findings. Linguistic, conceptual and procedural skills developed in collaborative work eventually contribute to individual skills.

2. The *exploration* begins with being introduced to software widgets and tools. It goes on to increasingly complicated geometric drawings. Then, students are expected to construct geometric objects themselves and in small groups. Finally, they are given open-ended scenarios and encouraged to figure out how to explore unknown mathematical territory.

3. *Construction* skills gradually grow from dragging existing dynamic objects, to constructing with step-by-step instructions, to figuring out how to construct objects with specific dependencies, to defining their own custom construction tools, to constructing objects of their own design in open-ended micro-worlds. The skill level progresses from novice to a reasonable command of GeoGebra's geometry tools. A transition to GeoGebra's algebra connection (analytic geometry) is provided at the end, opening up GeoGebra's multiple representations of geometric diagrams, analytic-geometry graphs, spreadsheet data, 3-D transformations and a computer-algebra system.

4. *Proof* in geometry is introduced slowly, with a focus on noticing and wondering. This is followed by formulation of text-chat-based explanations and multi-media documentation of findings. The explanations gradually entail increased levels of justification, finally approaching formal proofs, without ever reaching the completely formalized version of routinized two-column proof.

5. The *geometry* content starts by covering many of the activities in Book I of Euclid's *Elements* (300 BCE/2002), but implemented in the computer-supported

collaborative-learning medium of multi-user dynamic geometry. It incorporates the beginning standards for high school geometry in the new *Common Core Standards* (CCSSI, 2011), including congruence, symmetry and rigid transformations. The fundamental features of triangles are examined first, and then students are encouraged to explore similar features for quadrilaterals. For instance, students are involved in designing hierarchies of kinds of triangles or quadrilaterals based on alternative representations and dependencies of congruence, symmetry and rigid transformations. Finally, a sampling of creative objects, micro-worlds and challenge problems are offered for student-centered exploration.

There is a theoretical basis for gradually increasing skill levels in terms of both understanding and proof in geometry. Here "understanding" and "proof" are taken in rather broad senses. The van Hiele theory (see deVilliers, 2003, p. 11) specifies several levels in the development of students' understanding of geometry, including:

1. *Recognition*: visual recognition of general appearance (something looks like a triangle).
2. *Analysis*: initial analysis of properties of figures and terminology for describing them.
3. *Ordering*: logical ordering of figures (a square is a kind of rectangle in the quadrilateral hierarchy).
4. *Deduction*: longer sequences of deduction; understanding of the role of axioms, theorems, proof.

The implication of van Hiele's theory is that students who are at a given level cannot properly grasp ideas presented at a higher level until they reach that higher level. Thus, a developmental series of activities pegged to the increasing sequence of levels is necessary to effectively present the content and concepts of geometry, such as, eventually, formal proof. Failure to lead students through this developmental process is likely to reinforce student feelings of inadequacy and consequent negative attitudes toward geometry.

Citing various mathematicians, deVilliers (2003) lists several roles and functions of proof, particularly when using dynamic-geometry environments:

1. *Communication*: proof as the transmission of mathematical knowledge.
2. *Explanation*: proof as providing insight into why something is true.
3. *Discovery*: proof as the discovery or invention of new results.
4. *Verification*: proof as concerned with the truth of a statement.
5. *Intellectual challenge*: proof as the self-realization/fulfillment derived from constructing a proof.
6. *Systematization*: proof as the organization of various results into a deductive system of axioms, major concepts and theorems.

In his book, deVilliers suggests that students be introduced to proof by gradually going through this sequence of levels of successively more advanced roles of proof through a series of well-designed activities. In particular, the use of a dynamic-geometry environment can aid in moving students from the early stages of these sequences (recognition and communication) to the advanced levels (deduction and systematization). The use of dragging geometric objects to explore, analyze and support explanation can begin the developmental process. The design and construction of geometric objects with dependencies to help discover, order and verify relationships can further the process. The construction can initially be highly scaffolded by instructions and collaboration; then students can be guided to reflect upon and discuss the constructed dependencies; finally they can practice constructing objects with gradually reduced scaffolding.

This can bring students to a stage where they are ready for deduction and systematization that builds on their exploratory experiences. Furthermore, by working through the different roles of proof, math teachers and students are exposed to a richer conception of proof, in line with contemporary theories of proof, such as those by Lakatos (1976) and Livingston (1999).

# Discourse and Technology about Dependencies

The curricular activities center particularly on facilitating and supporting lessons in geometric dependency. GeoGebra allows one to construct systems of inter-dependent geometric objects. The dependencies built into dynamic-geometry constructions are intimately related to proofs illustrated by those constructions. Often, to understand a dependency and to be able to implement it in a construction is tantamount to being able to articulate a proof and to explore its validity dynamically (Stahl, 2012b). Students have to learn how to think in terms of these dependencies. They can learn through visualizations, manipulations, constructions and verbal articulations. These can all be modeled by examples, and these skills can be developed gradually.

The view of mathematical understanding as a communications skill suggests the central role of *mathematically significant discourse* and collaborative group practices in the growth of the abilities of students as they move from level to level in geometric understanding and proof. The activities for VMT-with-GeoGebra should support increasing fluidity with mathematically significant discourse.

The set of activities is designed to provide an educational experience in basic geometry to math teachers and students, taking them from a possibly novice level to a more skilled level, from which they can proceed more effectively without such designed activities. It is hoped that by providing activities on different levels for each of the dimensions, it can help most math teachers and students to increase their relevant skills – probably in quite different ways for different people.

Our design work is guided by socio-technical implications of continuing pilot studies as the technology and pedagogy of our project co-evolve. We are countering the problems that caused technical and cognitive distractions in our pilot studies by improving the software and testing the curriculum. The curriculum integrates tutorials about using the VMT and GeoGebra interfaces with carefully structured dynamic-geometry activities for virtual math teams. The activities systematically build up the background knowledge, group practices and problem-solving orientation needed for engaging in significant mathematical discourse.

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# 4. Evaluating Significant Math Discourse in a Learning Environment

Integral to our development of an environment for online collaborative learning of mathematics is the evaluation of the discourse of teachers and students using the system. We use multiple approaches to analyzing the interaction of small groups of people engaged in exploring geometry together, guided by a set of curricular activities.

## Math Cognition as Math Discourse

To mathematicians since Euclid, *geometry represents the paradigm of creative intellectual activity. Its methods set the standard throughout Western civilization for rigorous thought, problem solving and argumentation.* Many schools teach geometry in part to instill in students a sense of deductive reasoning. Yet, too many students—and even some math teachers—end up saying that they "hate math" and that "math is boring" or that they are "not good at math" (Boaler, 2008; Lockhart, 2009). They have somehow missed the intellectual math experience—and this may limit their lifelong interest in science, engineering and technology.

According to a recent "cognitive history" of the origin of deduction in Greek mathematics (Netz, 1999), the primordial math experience in 5th and 4th Century BCE was based on the confluence of labeled geometric diagrams (*shared visualizations*) and a language of written mathematics (*asynchronous collaborative discourse*), which supported the rapid evolution of math cognition in a small community of math discourse around the Mediterranean, profoundly extending mathematics and Western thinking.

The vision behind the research described in this paper is to foster *communities of math discourse* in networks of math teachers, in classrooms of K-12 math students and in online communities associated with the Math Forum. We want to leverage the potential of networked computers and dynamic math applications to catalyze groups of people exploring math and experiencing the intellectual excitement that Euclid's colleagues felt—refining and testing emerging 21st Century media of *collaborative math discourse* and *shared math visualization* to support math discourse in both formal and informal settings and groupings.

The learning sciences have transformed our vision of education in the future (Sawyer, 2006; Stahl, Koschmann & Suthers, 2006). New theories of mathematical cognition (Bransford, Brown & Cocking, 1999; Brown & Campione, 1994; Greeno & Goldman, 1998; Hall & Stevens, 1995; Lakatos, 1976; Lemke, 1993; Livingston, 1999) and math education (Boaler, 2008; Cobb, Yackel & McClain, 2000; Lockhart, 2009; Moss & Beatty, 2006), in particular, stress collaborative knowledge building (Bereiter, 2002; Scardamalia & Bereiter, 1996; Schwarz, 1997), problem-based learning (Barrows, 1994; Koschmann, Glenn & Conlee, 1997), dialogicality (Wegerif, 2007), argumentation (Andriessen, Baker & Suthers, 2003), accountable talk (Michaels, O'Connor & Resnick, 2008), group cognition (Stahl, 2006) and engagement in math discourse (Sfard, 2008; Stahl, 2008a). These approaches place the focus on problem solving, problem posing, exploration of alternative strategies, inter-animation of perspectives, verbal articulation, argumentation, deductive reasoning and heuristics as features of *significant math discourse* (Maher, Powell & Uptegrove, 2010; Powell, Francisco & Maher, 2003; Powell & López, 1989).

To learn math is to participate in a mathematical discourse community (Lave & Wenger, 1991; Sfard, 2008; Vygotsky, 1930/1978) that includes people literate in and conversant with topics in mathematics beyond basic arithmetic. Learning to "speak math" is best done by sharing and discussing rich math experiences within a supportive math discourse community (Papert, 1980; van Aalst, 2009). By articulating thinking and learning in text, students make their cognition public and visible. This calls for a reorientation of the teaching profession to facilitate dialogical student practices as well as requiring content and resources to guide and support the student discourses. Teachers and students must learn to adopt, appreciate and take advantage of the visible nature of collaborative learning. The emphasis on text-based collaborative learning can be well supported by computers with appropriate computer-supported collaborative learning (CSCL) software, such as that prototyped in the Virtual Math Teams (VMT) Project (Stahl, 2009).

# A Learning Environment for Math Discourse

In order to support our vision of significant mathematical discourse, we have integrated an online environment for synchronous and asynchronous communication (VMT) with a system for exploring dynamic mathematics (GeoGebra). We have described this dual system elsewhere (Stahl, 2012; Stahl et al., 2010). We attempt to support the combination of *collaborative math discourse* and *shared math visualization* by allowing small groups of students to engage in text chat while they are exploring a dynamic math workspace together. We have created a multi-user version of GeoGebra and integrated it with chat (as well as wiki and shared whiteboard) communication with the VMT functionality. This is designed to pool the advantages of dynamic math visualization with collaborative learning and math discourse.

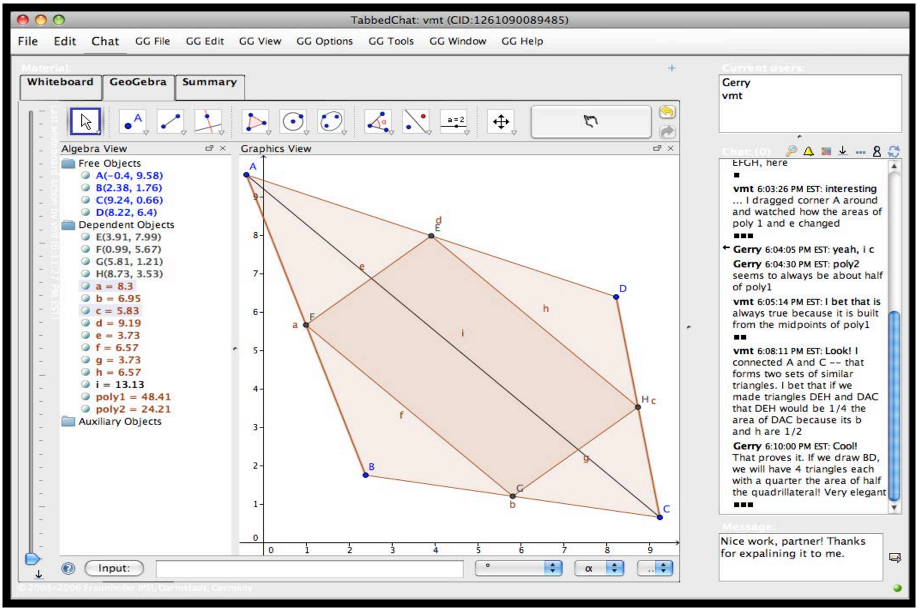


Figure 1. The VMT environment with multi-user GeoGebra and chat.

# Researching Discourse Practices

Our research centers on measurements of *group* math discourse rather than on assessment of *individual* learning of math content—in accordance with the socio-cultural view that effective individual math learning can be an indirect product of *participation* in group math discourse (Lave & Wenger, 1991; Sfard, 1998; 2008; Stahl, 2006; Vygotsky, 1930/1978). Vygotsky's notion of the *zone of proximal development* suggests that students may be able to engage in mathematical work within groups at a level that they will not be able to engage in for a couple years as individuals—and that such group work can be essential for their individual development in the long run (Vygotsky, 1930/1978, pp. 84-91). As a result, there is a need to assess the educational effectiveness of group interactions as such, beyond pre/post tests of the individuals.

In addition, the striking finding within CSCL research of *productive failure* (Barron, 2003; Kapur & Kinzer, 2009; Patak et al., 2011; Schwartz, 1995) shows that there can be a paradoxical inverse relationship between measures of successful learning by small groups versus by the individual members of those groups because of group processes that reveal deep mathematical relationships but that do not lead immediately to high test scores of the individuals. For these reasons, we evaluate engagement in mathematics *in terms of the quantity and quality of the math discourse* that takes place during the small-group problem-solving interactions, looking for increases for groups as they participate and in successive project years as our teaching model, collaboration technology and curricular resources are iteratively developed.

The analysis of significant math discourse is a task and goal for students using the system, for their teachers assessing their learning as well as for researchers studying collaborative math education. Reflection on interaction logs by teachers and students primarily involves trying to follow the problem-solving path of participants and to notice critical collaboration moves. They will be encouraged to look for examples of accountability to the group, to standards of math reasoning and to the characteristics of their math objects. They will look for instances where someone poses a productive inquiry that initiates effective group exploration—or where the group fails to come up with a useful proposal or fails to take up a proffered proposal. Examples will be culled and shared on the community wiki.

# A Design-Based Research Approach

Formative evaluation is a constant process built into the design of our work. As a design-based research effort, our project involves designing and exploring an iteratively refined solution—and by documenting its impact on the quantity and quality of math discourse by teachers and students. The interlocking components of the project will be reviewed at weekly project team meetings. Team meetings include interaction-analysis data sessions (Jordan & Henderson, 1995; Stahl, 2010), in which the research group collaboratively discusses new data from logs of teachers or students—and makes design decisions for refining the co-evolving components of our research. The project team discusses what seems to be working and what does not. It decides what to modify for the next iteration. Our ultimate goal is to *increase the quality and quantity of both teacher and student mathematical discourse*. Therefore, teacher professional development is oriented to improving the math discourse of their students.

Other research has documented the efficacy of dynamic-math visualization tools for *individual learning*; for instance, a study of geometry students in eleven Florida schools revealed a significant difference in the FCAT mathematics scores of students who were taught geometry using Geometer's Sketchpad compared to those who used the traditional method—regardless of differences based on SES or gender (Myers, 2009). Our project has a different focus. We have developed coding schemes and analysis approaches oriented to the *group unit of analysis* based on conversation analysis of adjacency pairs and longer sequences (Sacks, 1962/1995; Schegloff, 2007; Stahl, 2009, Chs. 20, 22, 23, 26; 2011b; Stahl et al., 2011). This approach serves both quantitative and qualitative analysis, by simultaneously specifying the structure of meaningful discourse moves and providing countable categories of group interaction units, in order to document changes over time—comparing discourse characteristics in selected time slices within teams or across cohorts.

The project will automatically produce raw data in the form of log files of participant online interactions. The log files are anonymous, but allow tracking of individual users through consistent login handles. The VMT environment is instrumented to capture all user actions in the chat and whiteboard—this has been extended to multi-user GeoGebra. A database of all sessions is automatically maintained and provides spreadsheet logs in handy formats and Replayer files. Software tools will be used for automated and manual log analysis of discourse measures and their evolution during training. While low-level group processes (e.g., number, length and rate of chat postings and drawing actions in different time slices) can be tracked automatically and analyzed statistically, higher-level math-discourse processes have to be interpreted manually. Raw and coded logs are maintained in a database to facilitate analysis of changes over time for groups across sessions and across successive cohorts of participants.

Quantitative analysis—based largely on the coding of discourse moves in teacher and student VMT logs—will track changes in key measures of significant math discourse. Discourse will be coded and measured along the following dimensions: (1) volume of discourse and level of participation, (2) percentage of on-task math discourse, (3) use of representations, (4) integration of chat and drawing, (5) use of accountable talk moves, (6) adoption of socio-mathematical norms and practices, (7) speaking meaningfully with explanation and argumentation, (8) involvement in posing, exploring and solving problems and (9) additional dimensions to be developed based on project experience.

The theory of math learning through participation in math discourse (Sfard, 2008; Stahl, 2008b) specifies important mathematical discourse moves, such as encapsulation, reification, saming, routines, deeds, explorations and rituals. The theory of accountable talk (Michaels, O'Connor & Resnick, 2008; Resnick, 1988) specifies discourse moves that promote accountability to the group, to standards of math reasoning and to the characteristics of the math objects. Speaking meaningfully in math discourse "implies that responses are conceptually based, conclusions are supported by a mathematical argument and explanations include reference to the quantities in the problem context [as opposed to a focus on merely] describing the procedures and calculations used to determine the answer" (Clark, Moore & Carlson, 2008, p.298).

Socio-mathematical norms include what counts as an acceptable, a justifiable, an easy, a clear, a different, an efficient, an elegant and a sophisticated explanation (Yackel, 1995; Yackel & Cobb, 1996). Mathematical practices emerge from interaction, are taken up by participants and are applied repeatedly (Medina, Suthers & Vatrapu, 2009; Stahl, 2011a). These dimensions of significant math discourse are associated with typical sentences and discourse moves that can be identified by coders. A coding scheme will be validated with acceptable inter-rater reliability, as in (Stahl, 2009, Chs. 22, 23; 2011b).

# Discourse-Building Activities

These theories suggest the central role of *mathematically significant discourse* and collaborative group practices in the growth of the skills of students as they move from level to level in geometric understanding and proof. Based on our analyses of teacher and student discourse within teams using our VMT-with-GeoGebra system, we are designing curricular activities to build mathematical discourse practices. The activities for VMT-with-GeoGebra should support increasing fluidity with mathematically significant discourse.

The set of activities is therefore designed to:

- Increase the ability of math teachers and students to engage in significant mathematical *discourse*
- Provide math teachers and students with a coherent sequence of activities *exploring* mathematical relationships and representations
- Empower math teachers and students to *construct* their own mathematical objects in a dynamic-mathematics environment, which they can use in the future as well
- Increase the understanding of math teachers and students in why mathematical objects behave in the ways they do, possibly *proving* why they do
- Increase the understanding of math teachers and students in the content of basic high-school *geometry* content, including how to discuss it, explore it, visualize it, prove it and extend it

The set of activities is designed to provide an educational experience in basic geometry to math teachers and students, taking them from a possibly novice level to a more skilled level, from which they can proceed more effectively without such designed activities. It is hoped that by providing activities on different levels for each of the dimensions, it can help most math teachers and students to increase their relevant skills – probably in quite different ways for different people.

Detailed interaction analysis of selected cases will show *how* the math discourse actually evolves. Quantitative analysis can establish the statistical significance of changes in learning outcomes, but it generally does not provide much insight into the mechanisms of change; these mechanisms will become visible in detailed case studies in which the specifics of the interactions can be studied. By combining quantitative and qualitative analysis of discourse transformations, the project evaluation will determine how the online interaction involves engagement in significant mathematical discourse.

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# 5. Tracing the Change in Discourse in a Collaborative Dynamic Geometry Environment: From Visual to More Mathematical

Diler Oner and Gerry Stahl

**Abstract.** This case study investigates the development of group cognition by tracing the change in mathematical discourse of a team of three middle-school students as they worked on a construction problem within a virtual collaborative dynamic-geometry environment. Sfard's commognitive framework was employed to examine how the student team's word choice, use of visual mediators, and adoption of geometric construction routines changed character during an hour-long collaborative problem-solving session. The findings indicate that the team gradually moved from a visual discourse toward a more formal discourse—one that is primarily characterized by a routine of constructing geometric dependencies. This significant shift in mathematical discourse was accomplished in a CSCL setting where tools to support peer collaboration and pedagogy are developed through cycles of design-based research. The analysis of how this discourse development took place at the group level has implications for the theory and practice of computer-supported collaborative mathematical learning. Discussion of which features of the specific setting proved effective and which were problematic suggests revisions in the design of the setting.

# Introduction

Documenting processes by which learning takes place in collaborative settings has been one of the most important research agendas for CSCL researchers. This endeavor is even more challenging in the context of learning geometry, which has been considered a classic example of individual intellectual development (Stahl, 2016). Shifting the focus from individual cognition to group cognition, this study examines the development of a group of students' geometrical thinking in the Virtual Math Teams (VMT) environment (Stahl, 2009). VMT is an open-source, virtual, collaborative learning setting that affords synchronous text-based interaction (chat) with an embedded multi-user dynamic-geometry application, GeoGebra (www.GeoGebra.org). VMT is regarded as the first sustained effort supporting a collaborative form of dynamic geometry (Stahl, 2013a).

Learning within a dynamic geometry environment (DGE) is indicated by the ability to construct figures, which marks the transition toward formal mathematics. There is a crucial distinction between *drawing* and *construction* within a DGE. Drawing refers to the juxtaposition of geometrical objects that *look* like some intended figure (Hoyles & Jones, 1998). Construction, however, depends on creating theory-based relationships, in other words *dependencies* (Stahl, 2013a), among the elements of a figure. Once relationships are constructed accordingly, the dynamic figure maintains these theoretical relationships even under dragging.

The transition from visual to formal mathematics is, however, neither straightforward nor easy for students working with dynamic geometry (Jones, 2000; Marrades & Guttierez, 2000). Students often think that it is possible to construct a geometric figure based on visual cues (Laborde, 2004), although constructing dynamic-geometry figures requires defining dependencies. Corresponding to this contrast, one can distinguish between two different *mathematical discourses* (Sfard, 2008) in which students may engage when working within DGEs. Within one of these, students may talk about geometrical figures as if they are merely visually perceptible entities without making any connections between them and the theoretical relationships they signify. When presented with a geometry construction problem, students might adopt a solution *routine* (Sfard, 2008) that is based on visual placement and verification, which produces a *drawing* (Hoyles & Jones, 1998). Taking a more sophisticated mathematical discourse, however, they would frame the problem as *construction*, that is, one that involves establishing dependencies.

Sfard (2008) argues that such a discursive jump to more sophisticated discourses takes place "while participating in the discourse with more experienced interlocutors" (p. 191). However, this study will show that participation within a well designed collaborative learning setting, such as VMT, can also help students move forward from visual toward more formal ways of dealing with construction problems. That is, interacting with expert interlocutors (e.g., teachers) may not be the only path toward advancing one's mathematical discourse. This developmental process may also take place within a virtual collaborative setting where feedback from dynamic-geometry software, collaboration with peers and guidance from task instructions collectively fulfill a role similar to that of the discourse of experts.

## Constructing Dependencies with Dynamic Geometry

In geometry, entering the theoretical domain is challenging given that students need to deal with the double role that diagrams play. On the one hand, diagrams refer to *theoretical* properties of geometrical objects and their relations. On the other hand, they are *spatio-graphical* figures that are immediately accessible through *visual* perception (Laborde, 2004). These two worlds come in close contact in DGEs. When one uses theory to *construct* a geometrical object, theoretical relationships are preserved even when the elements of the construction are visually altered through dragging. That is, spatio-graphical aspects of the construction keep reflecting dynamically invariant theoretical properties dynamically. For instance, when one properly constructs two line segments to be perpendicular bisectors of each other, not only will the segments look and measure as though they bisect each other at 90°, but they will remain so even if the points of the construction are dragged into other positions. Within a DGE, in order to construct a perpendicular bisector, one needs to create *dependencies* by defining the theoretical relationships that determine perpendicularity. The counterpart of the classical Euclidean compass-and-straightedge construction within a DGE makes uses of circle and line software tools, which can, for instance, create a rhombus whose diagonals bisect at right angles (see Figure 2b in the Methods section). In that way, dynamic-geometry constructions provide a computer-based context in which the connections between spatio-graphical and theoretical worlds are maintained.

Although dynamic geometry affords unique possibilities for learning geometry, there have been concerns regarding the nature of mathematical truth that students may be deriving when working in DGEs (Chazan, 1993a; Hadas et al., 2000; Hoyles & Jones, 1998). Some researchers and teachers worry that when students can easily generate empirical evidence, the need and motivation for formal explanations may vanish. More fundamentally, students may not make the transition toward the theoretical aspects of geometry (Marrades & Guttierez, 2000) and build the connection between spatio-graphical and theoretical worlds that is an essential aspect of meaning in geometry (Laborde, 2004). Learners may become stuck in the transition area between a visually produced solution and the underlying theoretical relationships (Hölzl, 1995).

On the other hand, it can be argued that focusing on constructing dependencies may help students move toward noticing relevant mathematical relationships (Jones, 2000). Dynamic geometry constructions are associated with formal geometry because created dependencies can correspond to elements of a mathematical proof (Stahl, 2013a). One starts with creating dependencies as if listing the givens in a mathematical-proof task. These built-in relationships in turn constrain the elements of a figure in certain ways that lead to further relationships, which reflect the ideas underlying a corresponding explanatory proof.

Some researchers stress the differences between Euclidean geometry and dynamic geometry. For instance, Hölzl (1996) argues that dynamic-geometry software imposes a hierarchy of dependencies that alters the relational character of geometric objects. He states that a distinction arises between free points (that can be dragged) and restricted points (such as intersections), which may not be geometrical or necessary in a paper-and-pencil environment. This is not surprising given that Euclidean geometry and dynamic geometry rely on "qualitatively different technologies" (Shaffer & Kaput, 1999). Despite the lack of complete congruence between the two, many researchers believe that explicitly stating the steps of a dynamic-geometry construction can break down the separation between deduction and construction (Chazan & Yerulshalmy, 1998; Healy & Jones, 1998; Stahl, 2013a). That is, well-designed DGEs may be able to help students to transition toward formal mathematics.

Constructions are also taken as a form of *mathematization* (Gattegno, 1988; Treffers, 1987; Wheeler, 1982) by Jones (2000), who defined the term for elementary-school geometry using dynamic-geometry software. When mathematizing, students can be said to be involved in modeling the geometrical situation using the tools available in the software. This involves setting up a construction and seeing if it is appropriate, and quite probably having to adjust the construction to fit the specification of the problem (p. 62). Thus, when students move forward from a visual solution toward one that is based on constructing dependencies in a DGE, this is taken as an indication of the development of students' geometric thinking.

## Theoretical Framework

In this study, Sfard's (2008) commognitive framework is used to examine students' mathematical discourse. Defining learning as the development of discourses, Sfard frames (mathematical) thinking as an individualized form of communication. Thus, she suggests a developmental unity between the processes of thinking and communicating, which leads to naming her approach "commognitive." Commognitive researchers are interested in mathematical discourses, as this is where one can trace the processes of learning. Sfard distinguishes mathematical discourses in terms of their tools—*words* and *visual means*—and the form and outcomes of their processes—*routines* and *narratives* (Figure 1). Each of these constructs is explained below, but the focus will be on the notion of routines, which is the most relevant construct for the analysis in this study.

Different mathematical discourses employ different mathematical *words*, which might signify different things in different discourses, and *visual objects*, such as figures or symbolic artifacts. In addition to using these discourse tools, participants functioning in different discourses produce what Sfard calls *narratives*, that is, sequences of utterances about mathematical objects and relations among them. Narratives are subject to endorsement or rejection under certain substantiation procedures by the community. Endorsed narratives usually take the form of definitions, axioms, theorems and proofs. In order to produce mathematical narratives, participants engage in mathematical tasks in certain ways. They follow what are called *metarules*, which are different than object-level rules. Rules that express patterns about mathematical objects, say about triangles, are defined as object-level rules (e.g., the sum of interior angles of a triangle is 180°). Metarules, on the other hand, are about actions of participants, and they relate to the production and substantiation of object-level rules. The set of metarules that describe a patterned discursive action are named *routines*, since they are repeated in specific types of situations.

Routines take two forms: the how and the when of a routine. The how of a routine, which may be called *course of action* or *procedure*, refers to a set of metarules describing the course of the patterned discursive action. The when of a routine, on the other hand, is a collection of metarules used by participants to determine the appropriateness of the performance. The researcher might observe the how of a routine more easily when a specific task is assigned. Examining the when of a routine, however, requires extended periods of observation, when participants are asked to solve problems that are more complex. In this study, given that students were provided with a well-defined task, the how of a routine is analyzed.

| Tools of math discourses | | Form and outcomes of math discourses | |
|---|---|---|---|
| Words | Visual means | Routines | Narratives |
| Use of certain keywords that signify different things in different discourses. | Visible objects that are operated upon within communication. | Set of metarules that describe a patterned discursive action and that relate to the production and substantiation of object-level rules. | Sequences of utterances about mathematical objects and relations among them. |

Figure 1. The four distinguishing aspects of mathematical discourses

Sfard (2008) states that metarules and routines are the researcher's construct based on observations of participants' discursive actions. Therefore, they are about the observed past. They are useful constructs for the researcher because "constructed metarules allow us to map the trajectory of one's discursive development" (p. 209).

# Method

This is a case study of a team of three eighth-grade students (about 14 years old) who worked on a geometry construction problem collaboratively within the Virtual Math Teams (VMT) environment. These three students were participants in the VMT Project, the larger design-based research (DBR) project that incorporates cycles of data collection and analysis to refine technology, curriculum and theory for collaborative learning. As part of the VMT project, the participants worked on the tasks of a geometry curriculum for the VMT environment written by Stahl (2013b) for about a semester. Although the participants had very little formal background in geometry, this particular team was able to solve a challenging task (Oner, 2013) in session 5. That brought this team to the attention of the project research team leading to this study to understand the team's mathematical development (see Stahl [2015] for an analysis of all eight of their sessions).

The study focuses on one of the team's problem-solving sessions, namely, session 3. This session was chosen for presentation here as it represented an "extreme case" (Patton, 1990) given that it displayed characteristics from which one could learn the most for the purposes of the larger DBR project. Detailed analyses of such cases could suggest ways of refining the VMT technology, pedagogy and curriculum to provide better support for future online groups.

## *The Context and Participants*

The team was named the "Cereal Team," because the members selected their online handles to be Cheerios, Cornflakes and Fruitloops. None of the team members had previously studied geometry; they were taking first-year algebra at the time of data collection. They are all females. Before the session analyzed in this study, they had met within the VMT online environment for two hour-long sessions, trying basic GeoGebra tools, such as the software tools for creating points, lines and line segments, or working on the task of equilateral-triangle construction (in sessions 1 & 2).

In session 3, students worked on Topic 3 of the VMT dynamic-geometry curriculum (Stahl, 2013b) that involved two tasks:

*Task 1*: Construct two lines that are perpendicular bisectors of each other. A list of steps is provided so that students can construct the diagonals (AB and CD) of a rhombus (ACBD). A completed construction is provided as an illustration for students (Figure 2a).

*Task 2*: Construct a perpendicular line to a given line through a given point. The expected solution for this task is provided in Figure 2b. Here, one first needs to define the given point H as a midpoint between two points using the circle tool (i.e., drawing the circle at center H with radius AH). Since H is the center of this circle AH and HB are congruent, which are the radii of this smaller circle. Now one can use points A and B (the intersections of line FG and the small circle) as centers and line segment AB as the radius to construct the two larger circles. As line segments DB, BC, CA and AD are all radii for these circles (r), they are equally long. Connecting these line segments would create four congruent triangles (by the SSS congruency theorem involving triangles CHB, CHA, DHA, and DHB). This implies that angle CHB is a right angle and line CD is perpendicular to the line FG at H.

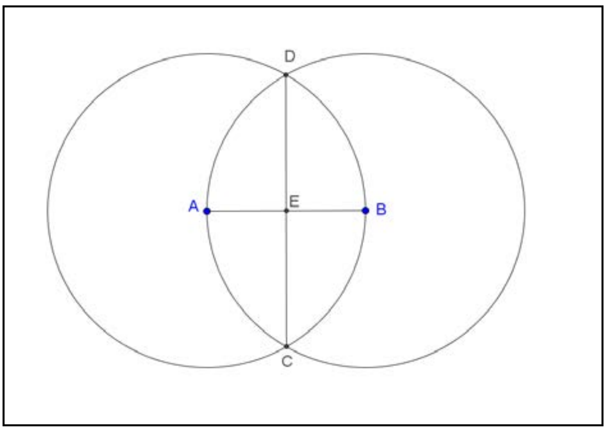


Figure 2a. Construction of two line segments that are perpendicular bisectors of each other (Task 1).

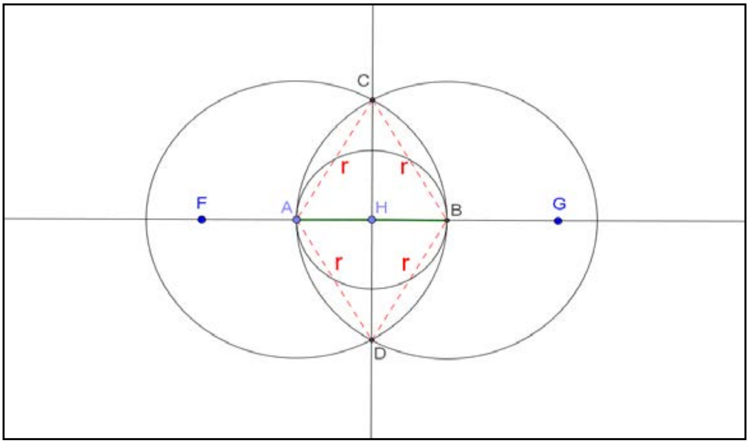


Figure 2b. Construction of the perpendicular to the line FG through a given point H (solution for Task 2).

Participants work on geometry problems in the VMT software environment within chat rooms created for each session. Figure 2c shows the VMT room created for session 3. The screenshot was taken at the very beginning of the session. Note that a completed perpendicular bisector construction is provided for students. In VMT rooms, there is a chat panel on the right hand side and a whiteboard area for multi-user GeoGebra. One can post a chat anytime during the session. However, in order to manipulate objects in the GeoGebra area one has to click on the "Take Control" button (at the bottom). Thus, only one person at a time can interact with the dynamic-geometry section of the room. The GeoGebra view is, however, shared by everyone in the team so they can all observe changes to the figures as they are made.

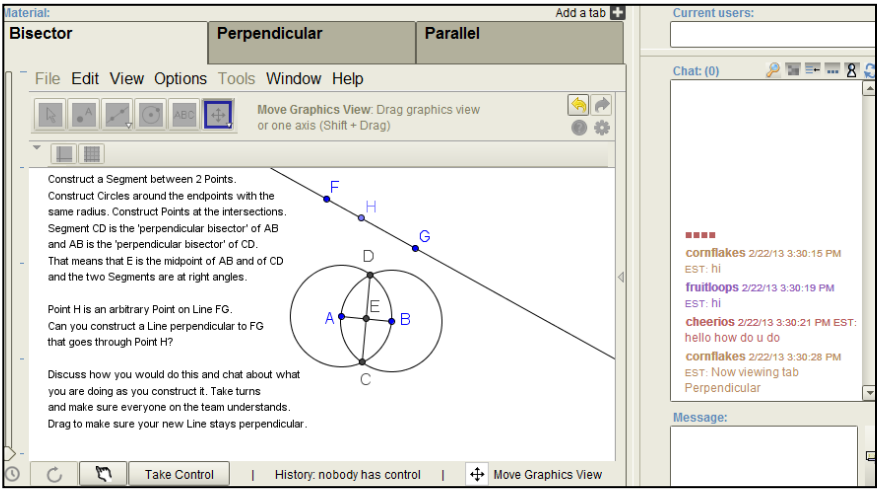


Figure 2c. The VMT window at the start of work on session 3. Note the task instructions and example figures. The chat section is in the panel on the right.

## *Data Collection and Analysis*

The team's meeting in the VMT environment was part of an after-school club organized by their math teacher in an American public school. The Cereal Team worked on Topic 3 for about an hour. The problem-solving session was recorded as a VMT log file to be replayed later allowing subsequent observation of the team's problem-solving process in micro-detail. All chat postings and GeoGebra actions produced by the team members are automatically logged and digitally recorded.

In order to investigate the changes in participants' discourse, both the chat postings and the actions of the participants recorded in their VMT session were examined through Sfard's (2008) discursive lens. As summarized in Figure 3, the particular focus was on the changes in: (a) the team's use of the word "perpendicular," (b) the visual mediators they acted upon (i.e., the perpendicular bisector construction), and (c) their mathematical routines, since the changes in these features were the most salient aspects of their changing discourse.

Given the nature of the assigned geometry tasks, this study investigated two routines:

- *The production of the perpendicular*: This routine involved the use of a set of procedures referring to the repetitive actions in producing a perpendicular line, such as construction (by creating dependencies) or visual placement (drawing)
- *The verification of perpendicularity*: This routine is a set of procedures describing the repetitive actions in substantiating whether a solution (a line produced) is in fact perpendicular to a given line. These procedures could include visual judgment,

numerical measurements, or use of theoretical geometry knowledge to justify proposed solutions.

| words | visual means | routines |
|---|---|---|
| The use of the word "perpendicular" | The perpendicular bisector construction | • The production of the perpendicular<br>• The verification of perpendicularity |

Figure 3. Sfard's (2008) three discourse aspects used in the present analysis

Two discourses are considered different when they are *incommensurable*, that is, when they have different rules for the same type of task (Sinclair & Moss, 2012). One can therefore distinguish between two mathematical discourses when they entail two different ways of solving the tasks in Topic 3 as summarized in Figure 4. In one discourse, students' production of the perpendicular and verification of perpendicularity are exclusively based on spatio-graphical cues without any concern for theoretical relationships. More specifically, the solution and verification routine is based on visual placement of a perpendicular-looking line (spatio-graphical solution), which produces a drawing (Hoyles & Jones, 1998). Along the same lines, the use of the word "perpendicular" reflects a visual image in which two lines perceptually look perpendicular. Thus, this discourse is categorized as *visual*. In another discourse, which is called *formal*, the production of the perpendicular line involves constructing dependencies—that is, defining relationships using the software tools. The verification routine within this discourse is theoretical, deriving from geometrical relationships. The word "perpendicular" within this discourse signifies a theoretical relationship between geometrical objects.

| Visual discourse | Formal discourse |
|---|---|
| • Production of the perpendicular is based on visual placement of a perpendicular-looking line (spatio-graphical)<br>• Verification of perpendicularity involves visual check (spatio-graphical) | • The production of the perpendicular is based on constructing dependencies<br>• Verification of perpendicularity derives from theoretical relationships<br>• The use of the word "perpendicular" signifies a |

| | |
|---|---|
| • The use of the word "perpendicular" reflects a visual image of which two lines look perpendicular | theoretical relationship between geometrical objects |

Figure 4. Characteristics of v*isual* vs. *formal* mathematical discourses in session 3.

As the first step in the analysis, the chat postings and GeoGebra actions of the Cereal Team were divided into episodes, mainly based on the detected changes in participants' routines of solving the task (i.e., routines of production and verification). In each episode, what is said and done was examined focusing on the three aspects of their mathematical discourse when relevant: their use of the word "perpendicular," the visual means acted upon, and routines of the production of the perpendicular or verification of perpendicularity in each episode. In what follows, an analysis of the most notable moments of these episodes will be presented by providing excerpts from the chat postings and VMT room screenshots.[2]

# Analysis

Based on the team's routines of production and verification, the interaction is divided into the following episodes: (1) constructing the perpendicular bisector; (2) drawing a perpendicular-looking line; (3) drawing the perpendicular using the perpendicular bisector construction (PBC) as straightedge; (4) use of circles with no dependencies defined; (5) constructing dependencies; and (6) discussing why the construction worked.

## *Episode 1: Constructing the perpendicular bisector (3:32:15-3:40:20)*

As the first task, the team was asked to construct two line segments that are perpendicular bisectors of each other. They were provided the steps to construct a line segment first and then to construct two circles around its endpoints, with the line segment as their radii (see Figure 2a for the expected answer, above). By constructing the two intersections of the two circles and connecting them, the participants would obtain two line segments perpendicular to each other at their midpoints.

[2] The full log for Session 3 is available at: http://gerrystahl.net/vmt/icls2014/Topic3.xlsx. The VMT Player is available at: http://gerrystahl.net/vmt/icls2014/vmtPlayer.jnlp. The replayer file for Session 3 is available at: http://gerrystahl.net/vmt/icls2014/Topic3.jno.

At the start of the first episode, Fruitloops and Cheerios were active with the construction of the two line segments as perpendicular bisectors of each other. The team decided that Fruitloops should take control and tackle the task (Excerpt 1, Lines 14-16). However, Fruitloops asked how she could make a line segment after creating two points (I and J). At that moment, the segment tool was not visible; it needed to be pulled down in the toolbar. Cornflakes provided some direction by saying that the segment tool is next to the circle tool (Excerpt 1, Line 19). This information was sufficient for Fruitloops, as she was then able to construct a line segment (IJ).

Excerpt 1.

| Line | Post time | User | Message |
|---|---|---|---|
| 11 | 3:31:02.6 | fruitloops | who wants to take control |
| 12 | 3:31:16.1 | fruitloops | do you was to delete the instruction |
| 13 | 3:31:21.5 | fruitloops | want* |
| 14 | 3:32:11.4 | fruitloops | want me to start? |
| 15 | 3:32:13.4 | cheerios | take control |
| 16 | 3:32:16.0 | cornflakes | Yes |
| 17 | 3:33:03.9 | fruitloops | how do i make the line segment? |
| 18 | 3:33:08.0 | cheerios | do u need help |
| 19 | 3:33:26.1 | cornflakes | its by the circle thingy |
| 20 | 3:33:38.1 | fruitloops | got it thanks |
| 21 | 3:34:06.5 | cornflakes | no problem |
| 22 | 3:35:54.1 | fruitloops | i did it |
| 23 | 3:36:02.0 | cheerios | good job my peer |
| 24 | 3:36:14.4 | cornflakes | Nice |
| 25 | 3:36:15.6 | fruitloops | someone else want to continue? |
| 26 | 3:36:23.6 | fruitloops | thankyou thankyou |
| 27 | 3:36:32.5 | cheerios | release control |
| 28 | 3:37:40.4 | fruitloops | so now you need to construck points at the intersection |
| 29 | 3:38:12.1 | fruitloops | no you dont make a line you make a line segment |

| | | | |
|---|---|---|---|
| 30 | 3:38:35.1 | fruitloops | good!! |
| 31 | 3:39:20.4 | fruitloops | so continue |
| 32 | 3:39:29.9 | cheerios | i just made the intersecting line and point in the middle |
| 33 | 3:39:40.0 | cheerios | it made a perpindicular line |

Another problem Fruitloops had difficulty with was constructing circles at the endpoints of the line segment with the same radius, which establishes the dependency crucial for the construction. She created two circles centered at points I and J with radius IK and JL respectively, which were not congruent but *looked* the same (Figure 5a). To define the radii of the circles centered at points I and J, she used arbitrary points (K & L), not the line segment IJ. That is, her circles *looked* to have the same radius, but they were not *constructed* based on an equal-radius relationship. Later, however, after playing with the circle tool for a while, Fruitloops did the construction again and managed to construct two circles around the endpoints (points I and J) with the same radii (IJ) (Figure 5b).

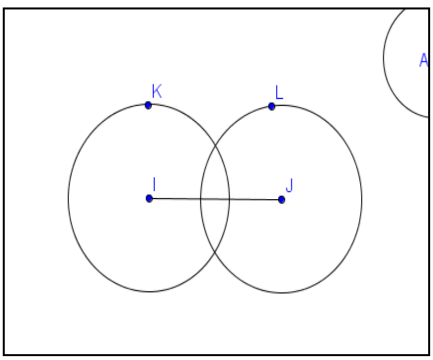


Figure 5a.

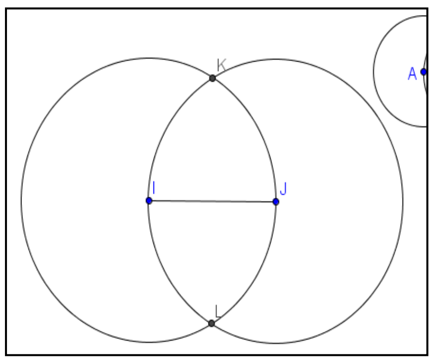


Figure 5b.

Next, Cheerios took control and continued the work by constructing the intersection points of the two circles (new points K and L) and the line that passed through them. Yet, as the following move, Cheerios removed the line she just constructed. Next, she reconstructed it, and then again deleted it and the intersection points. Finally, she reconstructed the intersections. At this point, Fruitloops drew attention to the instructions, saying they needed to construct a line segment not a line (Excerpt 1, Line 29). This time, Cheerios constructed the line segment KL through the intersections, and created point M, the intersection of the two segments (KL and IJ). Cheerios explained her actions by saying "i just made the intersecting line and point in the middle," calling M "the point in the middle." She continued, "it made a perpindicular line" (Excerpt 1, Lines 32-33).

In this episode, the routine for solving the first task simply involved following the instructions. Yet, Fruitloops had two difficulties. While one had to do with finding the needed menu item in the software, the other was related to constructing the key dependency, that is, same-radius circles at the endpoints of the line segment. Cheerios also had to pay attention to the wording in the instructions (i.e., the difference between "line" and "line segment"). She used the word "perpendicular" once (Excerpt 1, Line 33). At this point, it seems reasonable to argue that the word "perpendicular" was just a revoicing of the task instructions.

### *Episode 2: Drawing a perpendicular-looking line (3:40:27-3:55:30)*

Moving to the second part of the given task, the team now had to work on a more challenging problem, which was constructing a perpendicular to a line through a given point. In this episode, the team's problem-solving discourse took a visual character, which was evidenced by (a) producing a perpendicular-looking line (a drawing), (b) verifying perpendicularity by visual perception, and (c) using the word "perpendicular" to refer to a visual image. One other important aspect of this episode was Cornflakes' bringing the illustrative perpendicular-bisector construction to the team's attention.

On their screen, a line FG and the point H was provided to them (Figure 2c). Initially, however, how to use these givens was not clear to any of the team members. For Cornflakes and Cheerios, the production of the perpendicular first required creating another reference line that was somehow related to the line FG, as they both tried to construct lines that either *looked* parallel to or intersected the line FG. Fruitloops elegantly suggested using the line that was already there (Excerpt 2, Line 37). Furthermore, she next uttered the word "perpendicular." She said "perpindicular no intersecting" (Excerpt 2, Line 39). This use was different than that of Cheerios in the first episode. Fruitloops used the word to evaluate Cheerios' line, which intersected the line FG. At this stage, this use of "perpendicular" may have just implied a visual image rather than a construct with mathematical properties.

Excerpt 2.

| Line | Post time | User | Message |
|---|---|---|---|
| 34 | 3:40:27.5 | fruitloops | okay cornflakes go next |
| 35 | 3:41:11.5 | cornflakes | what are you supposed to do? |

| | | | |
|---|---|---|---|
| 36 | 3:41:42.6 | fruitloops | just follow the instructions |
| 37 | 3:43:48.5 | fruitloops | were we supposed to just use the line that was already there? |
| 38 | 3:44:10.2 | cornflakes | i think so |
| 39 | 3:44:44.2 | fruitloops | perpindicular no intersecting |
| 40 | 3:44:46.1 | fruitloops | not* |

After this initial stage, Cornflakes took control. She constructed a point N and a line through N and H that looked perpendicular to line FG at H (Figure 6a). Then she removed this line but later reconstructed it in the same manner, and deleted it once more. She was just picking a location for point N such that a line NH would visually appear to look perpendicular to line FG.

Next, however, she did something rather unexpected: she started moving the perpendicular-bisector constructions (PBCs) around. She dragged both the one that was given with the topic and the one they had just constructed in episode 1 changing their shape and location. Not seeing any of the use of the PBC immediately, she repeated her production of a line that seemed (visually) perpendicular to line FG through H, after creating points N and O. While the line looked as if it passed through O, N and H, it was only passing through O and H (Figure 6b).

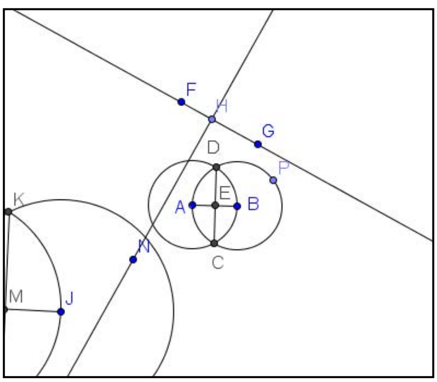


Figure 6a.

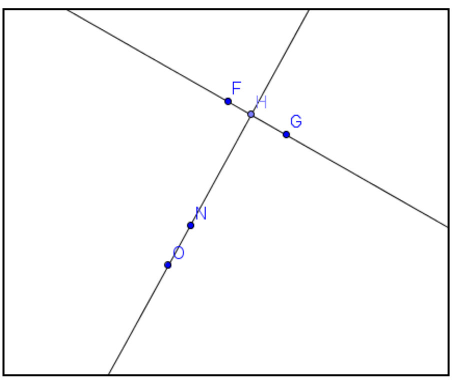


Figure 6b.

After Cornflakes' attempt to provide a solution, Fruitloops took control. She first deleted the line Cornflakes constructed (line OH), the one that appeared to be perpendicular to FG at H (Figure 6b). She played with constructing some other points and line segments, which did not seem relevant. It is reasonable to argue that she was not happy with Cornflakes' seemingly perpendicular line. She then released control and asked in the chat: "can you remake it?" (Excerpt 3, Line 43). In response, Cheerios took control and added points O and Q and a line through them that passed through H (Figure 7a). This line again was a visual solution that looked perpendicular to FG through H.

Cheerios then added another point (R) on the line placing it in the upper plane. Fruitloops, however, questioned defining extra points (O and Q) (Excerpt 3, Line 44) while Cornflakes was fine with them (Excerpt 3, Line 45). In response, Cheerios removed point R, and then her line OQ. She reconstructed point R and constructed another line through R, which this time did not even look perpendicular to line FG at H (Figure 7b). She then asked if the line was ok (Excerpt 3, Line 46). Fruitloops once again evaluated the line Cheerios constructed saying "its not perpinicuklar" (Excerpt 3, Line 48). Then Cornflakes deleted this line and constructed a more perpendicular-looking one first through H and S (a new point) and then, deleting line HS, through H and N (Figure 7c). Even though Fruitloops seemed satisfied this time saying, "I think that's good," (Excerpt 3, Line 49) Cornflakes erased the perpendicular-looking line (line HN) once more.

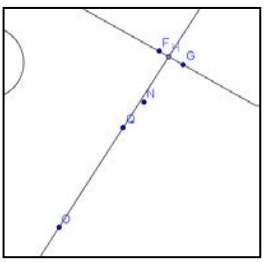

Figure 7a.

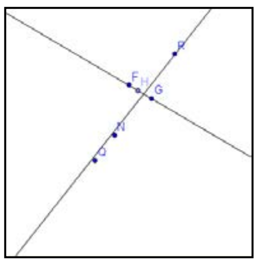

Figure 7b.

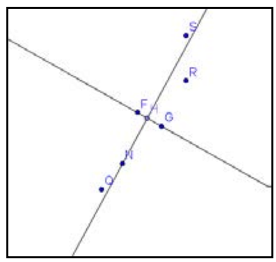

Figure 7c.

The solution offered by Cornflakes included placing a perpendicular-looking line visually (a spatio-graphical solution), which did not depend on creating dependencies. Cheerios also worked toward producing a line that would look perpendicular to the line FG at point H. However, there was also some level of discomfort with this solution, which was evidenced by deletion actions immediately followed by creating such lines. Fruitloops did not explicitly undertake the same production routine. She used the word "perpendicular," judging Cheerios' line as not fitting her notion of perpendicular. However, she eventually agreed on the line produced by Cornflakes in response (Excerpt 3, Line 49). Therefore, at this stage, one can say that all team members' production of the perpendicular routine involved creating a line that was a *drawing*. An important aspect of this episode was Cornflakes' little play with the available PBC. Even though the PBC had not been used as a mediator of the production of the perpendicular routine just yet, Cornflakes made its presence known and highlighted it as a potential tool.

Excerpt 3.

| Line | Post time | User | Message |
|---|---|---|---|
| 41 | 3:48:09.7 | fruitloops | sorry i did it by accident |
| 42 | 3:48:23.5 | cheerios | its fine :) my dear peer |
| 43 | 3:48:38.3 | fruitloops | can you remake it |
| 44 | 3:48:52.7 | fruitloops | why did you make point o and q |
| 45 | 3:48:55.0 | cornflake s | its alright |
| 46 | 3:49:09.5 | cheerios | is the line ok |
| 47 | 3:49:16.0 | cornflake s | i didnt make point o and q |
| 48 | 3:49:23.0 | fruitloops | its not perpinicuklar |
| 49 | 3:50:57.7 | fruitloops | i think thats good |

As the team did not seem completely satisfied with their (visual) solution, some of their efforts next focused on finding ways to judge perpendicularity. This stage was marked and initiated by Cheerios when she suggested rotating the line FG (she referred to it as FHG) "so it is easier to make it horizontal" (Excerpt 4, Line 50). With this statement, she meant dragging the given line FG into a horizontal-looking position so that one can test when a line was perpendicular to it more easily. Presumably, the prototypical visual image of perpendicularity for her involves a horizontal base line and a vertical perpendicular to it. This statement added a new routine to the problem: verification of perpendicularity along with a production routine.

However, neither Cornflakes nor Fruitloops took up this suggestion. Cornflakes was busy reconstructing another perpendicular-looking line passing through H. Fruitloops also adjusted this line so that it would look more perpendicular. Cheerios first helped Fruitloops by removing some of the extra points on or around that line and adjusting the line. Next, she implemented what she suggested by making the line FG horizontal looking, so that the team could better test the perpendicularity of the line it was to construct (Figure 8a). This would of course be a visual test, not a mathematical one. Seeing the line FG in a horizontal position, Cornflakes asked Cheerios to construct the perpendicular line (Excerpt 4, Line 53). Cheerios then constructed another two points (R and O) and a line through them that looked perpendicular to FG, but this did not go through point H. Cheerios deleted her first construction and then cleared the area deleting some extra points. Then she constructed line NH, which looked nearly perpendicular to FG through H (Figure 8b). Cornflakes seemed satisfied with the new line, saying, "that's good" (Excerpt 4, Line 54). Fruitloops said, "I think its perpendicular cause they are all 90 degree angles" (Excerpt 4, Line 55).

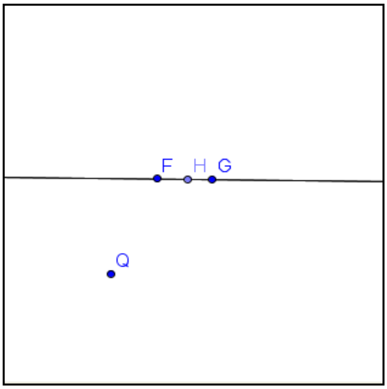


Figure 8a.

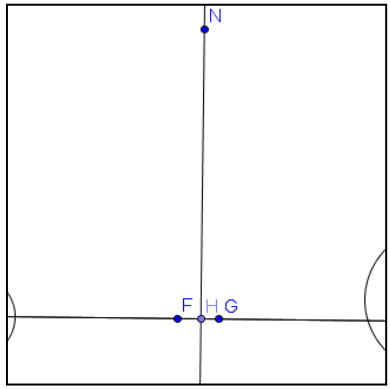


Figure 8b.

| Line | Post time | User | Message |
|---|---|---|---|
| 50 | 3:50:59.8 | cheerios | turn line fhg so its easier make it horizontal |
| 51 | 3:52:54.4 | fruitloops | Hey |
| 52 | 3:54:06.9 | fruitloops | which point did you move to get the line like that |
| 53 | 3:54:07.5 | cornflakes | now construct the line |
| 54 | 3:55:10.7 | cornflakes | thats good |
| 55 | 3:55:30.5 | fruitloops | i think its perpendicular cause they are all 90 degree angles |

Excerpt 4.

To summarize, Cheerios produced yet another drawing (Line NH, Figure 8b) at this point and Cornflakes and Fruitloops agreed on that solution (Excerpt 4, Lines 54-55). Furthermore, Fruitloops' approval involved the use of the word "perpendicular." She said: "i think its perpendicular cause they are all 90 degree angles" (Excerpt 4, Line 55). With this sentence, it became clearer that she used the word as representing a visual image of perpendicularity as she referred to the measure of the angles without measuring. Thus, all group members were still realizing the perpendicular line as a figure that could be produced perceptually. Moreover, Cheerios felt the need to verify their solution. She suggested producing the perpendicular line in a horizontal-vertical arrangement of two lines (the prototypical visual image for perpendicularity), which allowed a visual verification. Therefore, at this stage, a new routine for verifying perpendicularity emerged, although it was also spatio-graphical.

Table 1 provides a summary of the analysis presented for Episode 2.

Table 1. Summary of Episode 2 in terms of discourse characteristics

| Production of the perpendicular routine | Verification of perpendicularity routine | Use of the word *perpendicular* | Use of visual mediators |
|---|---|---|---|
| **Creating another reference line in relation to line FG (Cornflakes and Cheerios)** | | Signifying a visual image of perpendicular to disagree with a spatio-graphical solution (Fruitloops) | |
| **Spatio-graphical solution / drawing a perpendicular-looking line (Cornflakes)** | | | PBC-random dragging (Cornflakes) |
| **Spatio-graphical solution / drawing a perpendicular-looking line (Cheerios & Cornflakes)** | | Signifying a visual image of perpendicular to disagree and then agree with a spatio-graphical solution (Fruitloops) | |
| **Spatio-graphical solution (Cornflakes, Fruitloops, Cheerios)** | Spatio-graphical verification / vertical-horizontal alignment of the lines (Cheerios) | Signifying a visual image of perpendicular to agree with a spatio-graphical solution (Fruitloops) | |

### *Episode 3: Drawing the perpendicular using the PBC as straightedge (3:55:55-3:58:26)*

Something interesting happened next. Cornflakes started moving the PBC around as if she wanted to use it as a protractor—to verify the right angles. She was not able to get the orientation correct. Getting the idea, Fruitloops took control and dragged the PBC (the one they constructed) placing the middle point M on top of H and aligning with the line FG (Figure 9a). Cornflakes was satisfied, as she responded with a "yes" (Excerpt 5, Line 56). These moves signaled a new and different verification routine of perpendicularity, one that is based on measurement rather than based on a visual judgment.

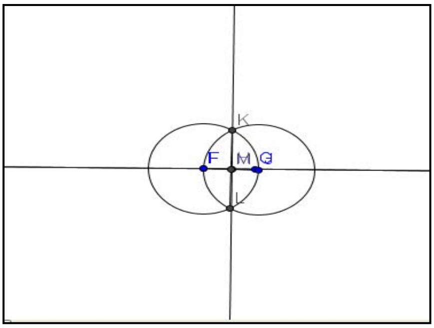


Figure 9a.

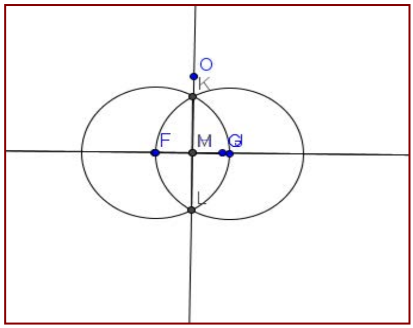


Figure 9b.

Meanwhile, Fruitloops realized another procedure for producing the perpendicular. Even though she was able to superimpose the two figures well, she deleted the perpendicular-looking line (Line NH). This move suggested that rather than using the PBC as a tool for measuring the angles, she could use it as a straightedge to draw the perpendicular. This still represented a visual production of the perpendicular (a spatio-graphical solution); meanwhile it perhaps marked the point of new possibilities for approaching the problem. Cornflakes was following Fruitloops one step behind saying "so after construting the line we put the circle on top" (Excerpt 5, Line 57). She was still seeing the PBC as a tool for checking perpendicularity rather than as a tool for drawing. Fruitloops, on the other hand, constructed another line (line OH) that looked like it concurred with the line segment KL (the segment perpendicular to segment IJ in the PBC construction, Figure 9b). Cornflakes then realized what Fruitloops was trying to do as she typed "so put the line thru the line on the circle" (Excerpt 5, Line 58). Fruitloops, however, was not sure how to proceed. She deleted her line (line OH), and even constructed an intersecting line (not a perpendicular). She next deleted that too, and finally said "I don't know what I am doing help" (Excerpt 5, Line 59).

In this episode, two new routines emerged. First, initiated by Cornflakes, the routine of verification shifted from one that is based on perception to one that is based on measurement by making use of a new visual mediator, the PBC. She wanted to use the PBC, which is known to be perpendicular, to check perpendicularity. She got help from Fruitloops to do that. Secondly, the production of the perpendicular also changed character involving the same visual mediator. While helping Cornflakes, Fruitloops wanted to imitate a paper-pencil routine of drawing the perpendicular using the PBC as a straightedge, yet she left the work unfinished. Cornflakes adopted this new routine as well.

Excerpt 5.

| Line | Post time | User | Message |
|---|---|---|---|
| 56 | 3:56:28.6 | cornflakes | Yes |
| 57 | 3:57:05.2 | cornflakes | so after construting the line we put the circle on top |
| 58 | 3:57:56.8 | cornflakes | so put the line thru the line on the circle |
| 59 | 3:58:18.5 | fruitloops | i dont know what i am doing help |
| 60 | 3:58:24.8 | fruitloops | sonmeone else take control |

Table 2 provides a summary of the analysis presented for Episode 3.

Table 2. Summary of Episode 3 in terms of discourse characteristics.

| Production of the perpendicular routine | Verification of perpendiculari ty routine | Use of the word *perpen dicular* | Use of visual mediators |
|---|---|---|---|
| **Spatio-graphical solution / imitation of paper-pencil routine of drawing the perpendicular using PBC as straightedge (Fruitloops & Cornflakes)** | Measurement -based verification using PBC (Cornflakes & Fruitloops) | | -PBC as protractor (Cornflakes) -PBC as straightedge (Fruitloops) |

## *Episode 4: Use of circles with no dependencies (3:58:27- 3:59:52)*

Taking control after Fruitloops, Cornflakes first dragged the PBC away. For a while, she seemed to play with the PBC: randomly constructing points on it, dragging them, and moving the labels of the points. Then, Cheerios jumped in, suggesting to "make the line first" (Excerpt 6, Line 61). One can infer that Cheerios was still trying to produce the perpendicular line visually. In response, Fruitloops clarified her approach: "i think you need to make the circles first" (Excerpt 6, Line 62). This statement signaled a new routine regarding the production of the perpendicular. That is, Fruitloops proposed using the construction of circles to produce the perpendicular just as the team had done with the PBC and the equilateral triangle in the previous topic (Topic 2).

Following her statement, Fruitloops took control and embarked on constructing. At this moment, Cornflakes said "put point m on tp of h" (Excerpt 6, Line 63). That is, she proposed moving the PBC back on top of point H. This statement suggested that she was not yet following Fruitloops. She either wanted to use the PBC to check perpendicularity, or more plausibly, to use it as a guide to draw the perpendicular. Fruitloops, on the other hand, started the construction by creating two circles with centers at F and G and with radii GQ and FR, respectively (Figure 10). However, although GQ and FR looked the same, they were not constructed as equal. This was, in fact, the same procedure she had initially followed with the PBC construction at the very beginning of their session (Figure 5a). She later constructed another and larger circle with center H and radius HS around these two circles, but immediately deleted it. Thus, although she realized that there had to be a construction involving circles, she failed to create the dependency for equal-radius circles. She then released control.

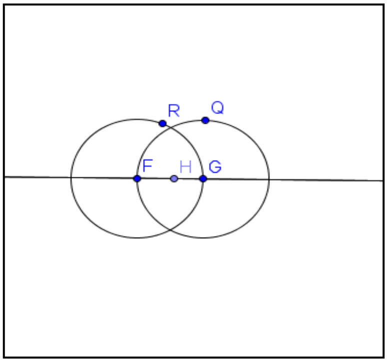


Figure 10.

At this stage, Fruitloops suggested a new routine for the production of the perpendicularity, the one that included creating circles. It is quite plausible that this newly emerged routine had been triggered by the presence of the PBC in the problem-solving environment. Although she wanted to follow a procedure that involved constructing circles, she was not able to build the necessary dependencies. Neither Cornflakes nor Cheerios was at this level yet.

Excerpt 6.

| Line | Post time | User | Message |
|---|---|---|---|
| 61 | 3:58:35.8 | cheerios | make the line first |
| 62 | 3:58:51.2 | fruitloops | i think you need to make the circles first |
| 63 | 3:59:19.0 | cornflakes | put point m on tp of h |

Table 3 provides a summary of the analysis presented for Episode 4.

Table 3. Summary of Episode 4 in terms of discourse characteristics

| Production of the perpendicular routine | Verification of perpendicularity routine | Use of the word *perpendicular* | Use of visual mediators |
|---|---|---|---|
| **Use of circles with no dependencies defined (Fruitloops)** | | | PBC as image of construction (Fruitloops) |

## *Episode 5: Constructing dependencies (3:59:53-4:14:15)*

Although Fruitloops was not able to complete what she had started immediately, Cheerios eventually took up her new reframing of the problem. After Fruitloops, Cheerios took control. She constructed a line through points T and S (new points) and adjusted it so that line TS would look like it passed through not only H, but also the intersections of the circles that Fruitloops constructed (Figure 11a). Cheerios tried several strategies to make the line TS to go through the intersections of the circles and point H, such as constructing a point very close to H (point U) and a line through that. However, as Fruitloops observed, the line was not going through H (Excerpt 7, Line 64). Thus, although Cheerios was now building on what Fruitloops had started, there were two problems with their attempts to construct the perpendicular. First, H was not defined as the midpoint of a line segment. Secondly, the circles around the endpoints did not have the same radius. In other words, although their production of the perpendicular routine now included the use of circles, no dependencies were constructed.

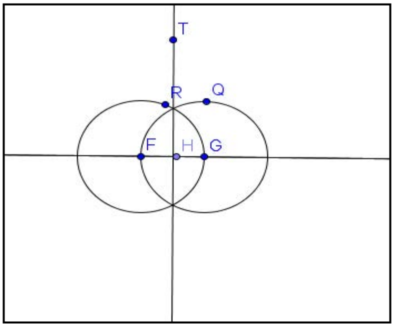


Figure 11a.

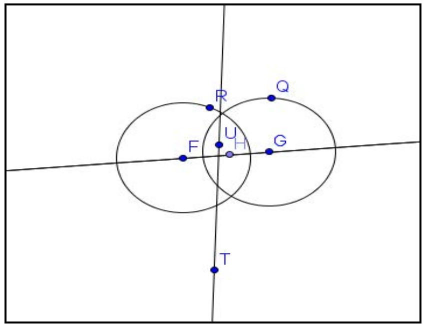


Figure 11b.

At this point, Cornflakes provided a definition for bisection, saying, "bisection is a division of something into two equal parts" (Excerpt 7, Line 65), which was not given to them with this task. Cheerios then took control and moved point H to the line, however it did not attach to the line. Next, Cornflakes played with the line as well moving it around point H and seeing that it was not set to pass through H. Then Fruitloops realized the problem saying, "we didn't put a point between the circles so the libne isnt perpendicular" (Excerpt 7, Line 66) and later adding "the part where the circles intersect" (Excerpt 7, Line 69).

Although Fruitloops was not using a formal mathematical language to explain her reasoning, this statement provided a new perspective on the production of the perpendicular as creating certain dependencies (which she demonstrated by actually performing the construction later). In response, Cornflakes dragged line FG and saw that dragging messed up their solution (Figure 11b). Cheerios agreed with Fruitloops immediately saying "oh I see now" (Excerpt 7, Line 68). Cornflakes, however, kept on moving other parts of the figure (such as points H, F) to make intersections and their perpendicular-looking line (TH) concur. Observing that Cornflakes was not convinced, Fruitloops suggested that she look at the examples. Finally, Cornflakes said, "ok I see" (Excerpt 7, Line 71).

Now that the team members seemed to be all on the same page, they spent some time discussing who would do the construction. Finally, Fruitloops took control and cleared up the space first by removing some points and their perpendicular-looking line. Then she created two circles at centers F and G with the same radius FG correctly (Figure 12). She also constructed the intersections (points Q and R) and explained what she did: "so i madfe two circles that intersect and the radius is the same in both circles right?" (Excerpt 7, Line 79). Cheerios agreed, "yea they are the same" (Excerpt 7, Line 80). Fruitloops highlighted once more that their radii were FG: "and segment fg is the radius" (Excerpt 7, Line 81). These statements confirmed that Fruitloops wanted to focus the group's attention on constructing certain relationships.

Cornflakes followed with a "yes" (Excerpt 7, Line 82). Cheerios said, "now we have to make another line" (Excerpt 7, Line 83). However, Fruitloops did not want to continue, saying: "yeah someone else can do that" (Excerpt 7, Line 84).

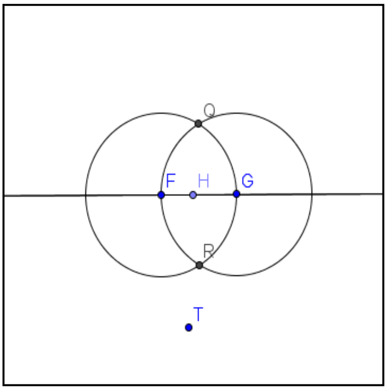


Figure 12.

Excerpt 7.

| Line | Post time | User | Message |
|---|---|---|---|
| 64 | 4:02:26.9 | fruitloops | the line isnt going through part h |
| 65 | 4:02:39.5 | cornflakes | bisection is a division of something into two equal parts |
| 66 | 4:04:58.2 | fruitloops | we didnt put a point between the circles so the libne isnt perpendicular |
| 67 | 4:05:03.8 | fruitloops | line* |
| 68 | 4:05:19.4 | cheerios | oh i see now |
| 69 | 4:05:20.6 | fruitloops | the part where the circles intersect |
| 70 | 4:05:34.8 | fruitloops | look at the examples and youll see |
| 71 | 4:05:46.9 | cornflakes | ok i see |
| 72 | 4:05:51.8 | cheerios | r u fixing it |
| 73 | 4:05:54.7 | fruitloops | do you want to do it? |
| 74 | 4:06:02.0 | cornflakes | so we have to put a poijt bewtween the circles |
| 75 | 4:06:19.4 | fruitloops | yeah you can do it if you want |
| 76 | 4:06:43.5 | fruitloops | or should i do it? |

| | | | |
|---|---|---|---|
| 77 | 4:06:49.4 | cornflakes | you can |
| 78 | 4:06:49.6 | cheerios | yea u should |
| 79 | 4:08:23.3 | fruitloops | so i madfe two circles that intersect and the radius is the same in both circles right? |
| 80 | 4:08:41.9 | cheerios | yea they are the same |
| 81 | 4:08:55.1 | fruitloops | and segment fg is the radius |
| 82 | 4:08:58.4 | cornflakes | yes |
| 83 | 4:09:04.1 | cheerios | now we have to make another line |
| 84 | 4:09:14.8 | fruitloops | yeah someone else can do that |

In this episode, Fruitloops identified one of the problems with the construction in line 66 (Excerpt 7): the need to create equal-radius circles. Although one can argue that she was not fully aware of the mathematical meaning of this dependency, she must have come to a realization that the way circles are constructed matters. She furthermore carried out the construction and drew attention to the defined relationships (circles with the same radius). The team members agreed upon this procedure. Thus, Fruitloops turned the routine of production of the perpendicular into a construction, one that is based on defining dependencies. Her use of the word "perpendicular" in line 66 (Excerpt 7) also reflected this change in the production routine. Here "perpendicular" was not used to represent a visual image or to evaluate a figure based on that image, as in her previous uses of the word. Rather, the word referred to a mathematical relationship that results from the way the circles were constructed.

There was still one other dependency the team needed to consider. This issue came up when Cornflakes responded to Fruitloops' invitation and constructed a line passing through Q and R (the circle intersections) and U (Figure 13a). Seeing that it did not pass through H, Cornflakes deleted almost half of Fruitloops' construction hoping to solve it, even going back to making the same mistake Fruitloops made (not noticing the role of equal-radius circles at the endpoints of a line segment). However, she eventually repeated the same construction steps and went back to the point where she started. Since H was not defined as the midpoint of the radius, the line through the circles' intersection points was not going through it. At this point, Fruitloops suggested a solution with the problem of H: "you make the points go through qr and then you move h ontop of the line" (Excerpt 8, Line 85). Q and R were the intersection points of the circles Cornflakes deleted. Next, Fruitloops took control and she performed what she said; she constructed the intersection points Q and R back again and the line through them, and attached H to that line by simply dragging it (Figure 13b). Then she announced that she finally did it (Excerpt 8, Line 86).

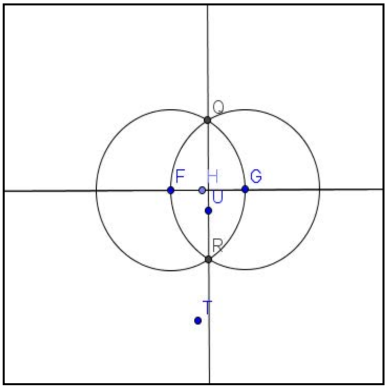


Figure 13a.

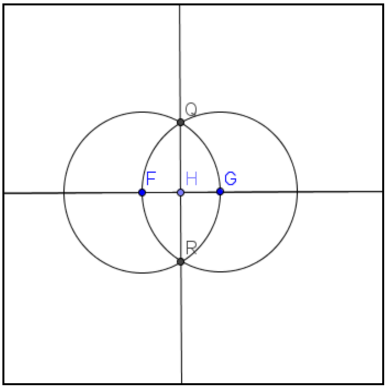


Figure 13b.

Although the team seemed to be on the same page regarding one of the dependencies (constructing equal-radius circles), the dependency regarding the point H was overlooked. Fruitloops simply attached the arbitrary point to their perpendicular line and this procedure seemed to work. Therefore, her routine of constructing a perpendicular through an arbitrary point did not involve taking that arbitrary point as the reference point as the task author intended. Rather, she took advantage of the dynamic geometry by simply dragging the point to the perpendicular.

Excerpt 8.

| Line | Post time | User | Message |
|---|---|---|---|
| 85 | 4:11:09.8 | fruitloops | you make the points go through qr and then you move h ontop of the line |
| 86 | 4:13:08.4 | fruitloops | i think i did it finallyu |
| 87 | 4:13:49.1 | cornflakes | the klines bisec the circle |
| 88 | 4:14:15.3 | cornflakes | *the lines bisect the circle |

Table 4 provides a summary of the analysis presented for Episode 5.

Table 4. Summary of Episode 5 in terms of discourse characteristics

| Production of the perpendicular routine | Verification of perpendicularity routine | Use of the word *perpendicular* | Use of visual mediators |
|---|---|---|---|
| **Constructing dependencies / use of equal-radius circles (Fruitloops)** | | Signifying a mathematical relationship (Fruitloops) | |
| **Dynamic solution / attaching the arbitrary point H to the line (Fruitloops)** | | | |

### *Episode 6: Discussing why the construction worked (4:14:29-4:16:17)*

Immediately after producing a solution, Fruitloops raised the question, "but how do we know for sure that the line is perpinmdicular" (Excerpt 9, Line 89). Cheerios said she was not sure (Excerpt 9, Line 90). Cornflakes first mentioned the spatio-graphical aspect of the figure by saying: "there 90 degree angles" (Excerpt 9, Line 91). However, Fruitloops was looking for another explanation. She said, "but you cant really prove that by looking at it" (Excerpt 9, Line 93). In response, Cornflakes participated within this new discourse sensing that the explanation had to do with the circles. She said, "they intersect throught the points that go through the circle" (Excerpt 9, Line 94). Fruitloops built on that and said, "it has to do with the perpendicular bisector" (Excerpt 9, Line 95). The two continued the discussion with Cornflakes saying "they 'bisect' it" (Excerpt 9, Line 96). Fruitloops must have thought Cornflakes referred to the line segment by "it" and added, "and the circles" (Excerpt 9, Line 97). Cheerios was relatively quiet when Fruitloops and Cornflakes were looking for a deeper understanding. She simply said, "oh I see" (Excerpt 9, Line 98) as a response. However, before they moved to the next tab, she was the one who dragged their perpendicular construction extensively, confirming the integrity of the construction as suggested by the final step in the topic instructions.

In this episode, it became clear that Fruitloops was not content with a spatio-graphical verification routine. She completed the task, yet also wondered why it worked. This may indicate that she was ready for a formal mathematical explanation. While Cheerios remained silent, Cornflakes participated within this conversation.

Fruitloops' use of the word "perpendicular" in line 89 (Excerpt 9) sounded more mathematical as she asked, "how do we know for sure the line is perpendicular?" She further mentioned the PBC as if highlighting its significant role within this problem solving session.

Excerpt 9.

| Line | Post time | User | Message |
|---|---|---|---|
| 89 | 4:14:29.8 | fruitloops | but how do we know for sure that the line is perpinmdicular |
| 90 | 4:14:39.6 | cheerios | im not sure |
| 91 | 4:14:42.1 | cornflakes | there 90 degree angles |
| 92 | 4:14:45.4 | cheerios | do u cornflakes |
| 93 | 4:14:59.4 | fruitloops | but you cant really prove that by looking at it |
| 94 | 4:15:06.8 | cornflakes | they intersect throught the points that go through the circle |
| 95 | 4:15:17.7 | fruitloops | it has to do with the perpendicular bisector |
| 96 | 4:15:19.8 | cornflakes | they"bisect" it |
| 97 | 4:15:31.2 | fruitloops | and the circles |
| 98 | 4:15:37.2 | cheerios | oh i see |

Table 5 provides a summary of the analysis presented for Episode 6.

Table 5. Summary of Episode 6 in terms of discourse characteristics

| Production of the perpendicular routine | Verification of perpendicularity routine | Use of the word *perpendicular* | Use of visual mediators |
|---|---|---|---|
| | -Spatio-graphical (Cornflakes)<br>-Looking for a verification | Signifying a mathematical | |

| | | | |
|---|---|---|---|
| | routine beyond spatio-graphical evidence (Fruitloops & Cornflakes) | relationship (Fruitloops) | |

# Discussion

Mathematical experiences at the middle-school level are considered critical for students to develop deductive and formal thinking (Ellis et al., 2012). Harel and Sowder (1998) note that it would be unreasonable to expect that students will instantly appreciate sophisticated forms of mathematics in high school, where expectations regarding mathematical rigor are higher. Therefore, it is important to provide learning opportunities for middle-school students to advance their geometric thinking. The VMT environment is designed to serve this purpose by affording virtual collaborative problem solving with a multiuser GeoGebra component. It is important to study the ways in which teams of students using the VMT software and its curriculum are learning geometry and what problems they encounter. Toward this end, Sfard's (2008) discursive lens was employed to investigate the change in mathematical discourse of a team of three middle-school students as they worked on a geometry construction problem in the VMT environment. The analysis focused on how the team's use of the word "perpendicular," its use of the PBC as a visual mediator, and its use of routines (for production of a perpendicular and for verification of perpendicularity) shifted during an hour-long collaborative problem-solving session. The findings indicated that the Cereal team, whose members had very limited formal geometry background, moved forward from a visual discourse toward a more sophisticated formal mathematical discourse.

To be specific, the team started constructing two line segments as perpendicular bisectors of each other following the instructions of Topic 3 (Episode 1). In this part, Cheerios' use of the word "perpendicular" was copied from the task instructions as if using a foreign language word in a sentence. The team next moved to the second task, which was built on the first one. This presented a challenge as the team needed to figure out how to construct a perpendicular to a line through a given point, which they had not done before.

Table 6 summarizes the team's use of the word "perpendicular," their use of visual mediators, their routines of production of a perpendicular, and their verification of perpendicularity in episodes 2 to 6, where the team worked on the second task in Topic 3.

Table 6. The change in discourse in episodes 2-6 (summary).

| Episode | Production of the perpendicular routine | Verification of perpendicularity routine | Use of the word *perpendicular* | Use of visual mediators |
|---|---|---|---|---|
| 2 | Creating another reference line in relation to line FG (Cornflakes and Cheerios) | | Signifying a visual image of perpendicular to disagree with a spatio-graphical solution (Fruitloops) | |
| | Spatio-graphical solution / drawing a perpendicular-looking line (Cornflakes) | | | PBC - random dragging (Cornflakes) |
| | Spatio-graphical solution / drawing a perpendicular-looking line (Cheerios & Cornflakes) | | Signifying a visual image of perpendicular to disagree and then agree with a spatio-graphical solution (Fruitloops) | |
| | Spatio-graphical solution (Cornflakes, Fruitloops, Cheerios) | Spatio-graphical verification / vertical-horizontal alignment of the lines (Cheerios) | Signifying a visual image of perpendicular to agree with a spatio-graphical solution (Fruitloops) | |
| 3 | Spatio-graphical solution / imitation of paper-pencil | Measurement-based verification using PBC (Cornflakes | | - PBC as protractor |

| | | | | |
|---|---|---|---|---|
| | routine of drawing the perpendicular using PBC as straightedge (Fruitloops & Cornflakes) | & Fruitloops) | | (Cornflakes) - PBC as straightedge (Fruitloops) |
| 4 | Use of circles with no dependencies defined (Fruitloops) | | | PBC as image of construction (Fruitloops) |
| 5 | Constructing dependencies / use of equal-radius circles (Fruitloops) | | Signifying a mathematical relationship (Fruitloops) | |
| | Dynamic solution / attaching the arbitrary point H to the line (Fruitloops) | | | |
| 6 | | -Spatio-graphical (Cornflakes) -Looking for a verification routine beyond spatio-graphical evidence (Fruitloops & Cornflakes) | Signifying a mathematical relationship (Fruitloops) | |

In the *production of the perpendicular routine column* in the summary Table 6, one can see that the team started by producing spatio-graphical solutions including placing the perpendicular line visually and imitating the paper-and-pencil procedure of drawing the perpendicular by using the PBC as a straightedge guide (in Episodes 2 & 3). These routines, however, evolved into first using circles (in Episode 4) and then defining certain relationships with the circles, such as the use of equal-radius circles with the construction allowing the group to successfully complete the task (in Episode 5). The second dependency, however, was bypassed by simply attaching the arbitrary point H to the perpendicular line. Although no dependencies were created here, as Sfard (personal communication, June, 2014) observed, this could be considered a legitimate move in GeoGebra. In a dynamic-geometry world where everything moves, the point of reference may be redefined as well, as long as the software supports this use.[3]

A parallel progression can also be observed in the *verification of the perpendicularity routine column*. The team first felt the need to verify their solution, which was not explicitly asked in the instructions. Initially, this took a spatio-graphical form, with Cheerios wanting to arrange the lines into a vertical-horizontal position, which represents the prototypical visual image for perpendicularity (in Episode 2). Then Cornflakes, who received help from Fruitloops, wanted to use the PBC as a protractor turning the verification routine into one that is based on measurement (in Episode 3). Eventually, Fruitloops, upon completing the construction, asked how they could be sure if the line was perpendicular (in Episode 6). In this episode, Cornflakes pointed at the visual appearance of the figure to convince Fruitloops. However, Fruitloops seemed to be looking for a verification routine that would go beyond the spatio-graphical. She even used the word "proof"—though not necessarily in a deductive mathematical sense. This situation is quite contrary to the findings in the literature, as students' validation of a mathematical statement often takes the form of testing it against a few examples, even at the more advanced levels (Chazan, 1993b; Coe and Ruthven, 1994). In the case of dynamic geometry, students often think that they can justify a claim by empirically checking the diagram (Laborde, 2004)—that is, by dragging.

This situation and the difficulty the team had with defining point H as the middle point suggest revisions in Topic 3. The group constructed the PBC at the beginning of their session following scripted steps. Completing the task with Fruitloops, Cheerios said, "i just made the intersecting line and point in the middle," continuing, "it

[3] The instructions specified that, "point H is an arbitrary point on line FG." In Euclidean geometry, that would mean that even though H can be any point on line FG, it is not something that moves. Thus, although one looks for a solution that would work for any point H, any treatment of H would be static. In dynamic geometry, however, an arbitrary point H is a free point that can be dragged along line FG. Thus, there is some legitimacy to the students' solution. Ultimately, however, the solution fails the drag test of dynamic geometry. If one properly constructs the perpendicular through point H, then one should be able to drag point H along line FG and have the perpendicular to FG move with it so that it always passes through H and remains perpendicular to FG. Cheerios, however, had only dragged their final construction by moving point G.

made a perpindicular line" (Excerpt 1, Lines 32-33). However, there was not much discussion of its mathematical aspects. The group immediately moved to the next task of constructing a perpendicular to a line through a given point. It may be necessary to lead students explicitly to discuss their constructions mathematically when scaffolding the development of higher-level discourses. If participants are genuinely wondering about the relationships and asking questions, as in the present case, additional task instructions could even provide the geometrical theory behind such constructions. Encouraging students to make explicit connections between their deduction and construction knowledge is important since otherwise, as Schoenfeld (1988) cautioned, students may be learning about dynamic constructions merely as a set of procedures to follow.

The word *perpendicular* was first used by Cheerios in the first part of the task (Episode 1). She uttered the word only once, as if to revoice the instructions. Fruitloops, on the other hand, used the word throughout the problem-solving session. Her use of the word also represented a parallel advancement along with the production and verification routines. Initially the word signified a visual image of perpendicularity and was used to evaluate produced visual solutions (in Episodes 2&3). Later, however, her use of the word came to refer to a certain relationship between figures (in Episodes 5&6).

Finally, it is reasonable to argue that the *PBC*, the already completed construction, functioned as *the key visual mediator* of the session. The PBC figure is derived from Euclid and was presented as a resource in the Topic 3 instructions. The group was also asked to construct the PBC at the beginning of their session following very specific steps. In the second task, Cornflakes brought it to the team's attention when the team seemed to be out of ideas (in Episode 2). Although at first she only played with it randomly, she later figured out a way to use it as a protractor, thus as a tool for verifying perpendicularity (in Episode 3). This use may have led Fruitloops to view it as a straightedge that could be used to draw the perpendicular (in Episode 3). More importantly, however, the PBC became a crucial resource that probably triggered Fruitloops' use of circles, which led to the framing of the problem as a construction task (in Episode 4).

These observations about the PBC are important for at least three reasons: First, when students appear to be stuck with the problem or run out of ideas, they seem to make use of every resource within their problem-solving space. Ryve, Nilsson, and Pettersson (2013) underline the crucial role that visual mediators play in effective communication. However, along with visual mediators, they have also observed that technical terms (i.e., technical mathematical words) were equally important for communication that is effective. In Episode 5, just before Fruitloops framed the task as construction, Cornflakes provided a definition for the term "bisection" (Excerpt 7, Line 65). This definition was not given to the team with the task, thus Cornflakes must have found it somewhere else. A little later, when Fruitloops realized the problem with their circles, she was lacking the mathematical terms to express the situation. She said "we didn't put a point between the circles so the libne isnt perpendicular" and then "the part where the circles intersect" (Excerpt 7, Lines 66, 69). Hence, CSCL task designers should pay considerable attention to the type of resources to be provided to students with the problems. These resources should encompass not only visual mediators but also the technical mathematical words.

Secondly, Cornflakes initially was not able to place the PBC on top of line FG correctly, but Fruitloops completed what Cornflakes had in mind, and Cornflakes responded with a "yes." Afterwards, Fruitloops realized another procedure for producing the perpendicular (i.e., use the PBC as a straightedge to draw the perpendicular). All these suggest that in a setting like VMT, "transactive dialogue" (Berkowitz & Gibbs, 1985, as cited in Barron 2000) can take place through participants' actions using visual mediators on the shared computer screen. This seems more likely when students lack the technical terms to express themselves, as in this case. The "take control" button opens up a "joint problem space" for dynamic manipulations and affords action-based dialogue, in addition to the conversational turns supported by the chat platform. In that way, as Roschelle & Teasley (1995) observed, participants can still interact productively even when they lack the technical vocabulary to talk about the problem.

Third, and most importantly, one could observe that the PBC accompanied the moments of change in mathematical routines: first from the vertical-horizontal alignment of the lines to the use of PBC as a straight-edge guide in Episode 3, and then to the use of circles in producing the perpendicular in Episode 4. Thus, it is reasonable to assume that it played a significant role in the change in mathematical discourse in this problem-solving session.

Along with the PBC, other aspects of the VMT environment also seemed to play a role in the moments of discourse shifts. In Episode 4, Fruitloops introduced a new production routine when she suggested making the circles first (Excerpt 6, Line 62) and started constructing the circles. The team constructed circles in the first part of Topic 3 (to construct PBC) and the equilateral triangle in the previous topic in the VMT curriculum (Topic 2), which also required using circles in defining dependencies. Thus, the VMT curriculum, particularly the sequence of the topics in that curriculum, might have also played an important role in supporting students' discourse development.

Initially Fruitloops' circles were not created using the necessary dependencies such as the equal-radius relationship. As no dependencies were defined, the team had problems creating the line that would go through the intersections of the circles and the point H. That is, the dynamic-geometry software provided the essential feedback until Fruitloops realized that they needed to construct the circles with certain relationships (in Episode 5). Both Cheerios and Cornflakes played with their construction to see that there was something missing with their solution at that stage.

This situation also confirms Roschelle & Teasley (1995), who observed that when students had differing ideas, they were able to experiment with the computer representation. In a dynamic-geometry environment, the drag function enables testing the construction if dependencies are correctly defined. Eventually, this experimentation leads the participants to generate new ideas, when they see that their solution is not supported by the software.

This analysis was conducted at the group unit of analysis, involving the team discourse rather than the individual cognition of the students.[4] This analysis is not necessarily meant to suggest that the individual team members, including Fruitloops, decisively moved beyond the visual discourse. Nor is the observed discursive jump by the team necessarily an indication of "individualization" (Sfard, 2008): that the team members will henceforth follow more formal mathematical procedures and employ more formal word uses irrespective of the context. One can observe that Cornflakes and Cheerios were mostly attending to the spatio-graphical aspects of their figure, even toward the end of the session. Even Fruitloops was not able to clearly articulate why and how circles worked.

This team of novices succeeded in participating within a collective discourse that gradually took a more mathematical character. Yet, this more formal discourse was, as Baruch Schwarz (personal communication, June 2014) suggested, *rooted* in the spatio-graphical solutions—i.e., solutions that rely on reasoning and recognition of geometric figures with their appearances without any regard to their mathematical properties (Laborde, 2004). Thus, similar to what Sinclair and Moss (2012) noted, the process of discourse change may be better described as oscillating—rather than simply shifting—between the visual and more formal discourse levels.

Sfard's commognitive framework provided an account for the development of geometrical thinking observed within this episode. Rather than talking about fixed-ordered geometrical cognitive levels, as in van Hiele levels (1986), Sfard (2008) talks about incommensurable mathematical discourses. Saying that two discourses are incommensurable does not mean that one cannot participate in both of them at the same time. It simply means that "they do not share criteria for deciding whether a given narrative should be endorsed" (Sfard, 2008, p. 257). However, moving towards higher discourse levels requires "student's acceptance and rationalization (individualization) of the discursive ways of the expert interlocutor" (p. 258). Thus, students need to interact with expert others in order to develop sophisticated

[4] In a similar analysis of all eight sessions of the Cereal Team, Stahl (2016) conceptualizes the development of the group's mathematical cognition in terms of the successive adoption of *group practices*, rather than *routines*, in order to emphasize that they are being theorized as group-level rather than individual phenomena. As illustrated in the six episodes here, the Cereal Team questions, negotiates and adopts new practices through their discourse (including shared GeoGebra actions). This meaning-making process creates a shared understanding within the team. Once the team agrees to use a routine, it may become a group practice, which can be used in the future without further discussion.

mathematical discourses. The findings in this study indicate that an environment such as VMT may provide a context in which students can engage in higher-level mathematical discourses with their peers.

Thus, along with instruction by expert mathematicians, well-designed virtual collaborative learning environments can provide a form of interaction that supports significant mathematical discourse development. In that regard, the findings support Sinclair and Moss (2012), who suggested that dynamic-geometry software could function as a stand-in or alternative for the discourse of experts. In the present case, multi-user dynamic geometry was a component of the VMT software, which was built to support collaborative learning with a specific geometry curriculum (Stahl, 2013b). Therefore, in addition to the dynamic geometry component, the curriculum and the collaborative interaction aspects of the VMT environment also played crucial roles in supporting students' mathematical discourse development.

There is a tendency in educational research to reduce cases of group cognition to psychological phenomena of individual cognition. Considering the Cereal Team's problem-solving session, one may be inclined to think that Fruitloops was the higher thinker in this session. Not only did she appear to be the one solving the second task, she also wondered why it worked. However, that was not where she started. Initially, her notion of perpendicular referred to a visual image. It evolved into one that represented a mathematical relationship. Similarly, at the beginning, her routine of the production of the perpendicular involved a spatio-graphical solution, the same as for everyone else in the team, which only later became one that was based on defining dependencies. These transformations took place within the context of interacting with her team members, enacting task instructions, and interacting with the VMT software. Furthermore, most of the time, her lead was negotiated with the other team members, as part of the team's coordination of social resources (Oner, 2013). These took the form of the others building on her actions (as in Episode 5) as well as engaging in *transactive dialogue* (Berkowitz & Gibbs, 1985, as cited in Barron, 2000) with Cornflakes (as in Episode 6). She received help from other team members (as in Episode 1). The PBC was brought to her attention by other team members (Cornflakes) as well. Thus, the team's success was the product of group cognition, not simply attributable to one team member (Stahl, 2006).

Would the findings be applicable for other online groups? Qualitative case studies, such as this one, are not usually designed to make grand generalizations concerning the population. They, however, allow making what Stake (1978) calls "naturalistic generalizations." That is, the findings from a case with a unique team in a particular situation can provide insight and understanding of potential computer-supported collaborative learning and development. Session 3 was just one hour of the team's eight-hour experience with dynamic geometry; the whole longitudinal development is analyzed in a similar way by Stahl (2016), providing more insight into the team's development of geometric commognition. Furthermore, this case study should not be viewed as a summative assessment of the VMT environment, but as part of one cycle in an iterative DBR investigation. Accordingly, it was more concerned with documenting learning and *how* a team of novice students accomplished significant advance in mathematical discourse within the VMT environment in order to guide modifications in technology, pedagogy and curriculum—so that more student groups might undergo similar mathematical development in future versions of VMT. We can now conduct such analyses of other teams during other DBR cycles or using different CSCL supports.

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# 6. Constructing Knowledge: A Community of Practice Framework for Evaluation in the VMT Project

Michael Khoo and Gerry Stahl

**Abstract:** This paper describes a formative evaluation of the Virtual Math Teams (VMT) Project as it adopts the GeoGebra dynamic-mathematics application. The project team is considered as a Community of Practice, which communicates with student users through boundary objects. An ethnographic action-research approach is used to analyze three sources of data: VMT user manuals; screenshots of tool interfaces and assignments; and logs of student chat sessions addressing those assignments. The analysis focuses on how the topic of 'construction' is articulated in each data set. The results show that the team understands 'construction' in terms of a complex web of knowledge of dynamic geometry, while the students develop their own emergent notions of 'constructing' from the boundary objects produced by the team. Recommendations for boundary object design are considered.

## Introduction

A basic distinction in project evaluation is that between summative and formative evaluation (Frechtling, 2002). Summative evaluation assesses the final outcomes of a project, while formative evaluation assesses ongoing processes, focusing on how that project is achieving its goals. Evaluations reported in the literature are often summative in nature, and explicit cases of formative evaluation are infrequent. One issue here is that the ongoing nature of formative evaluation, and the iterative generation of interim data, present a less tidy unit of analysis than summative evaluation. However, formative evaluations are useful, for instance in design research, where projects follow iterative design cycles of development, implementation, and assessment. Where a summative evaluation may conclude that one intervention has better outcomes than another, a formative evaluation may suggest how to adjust the intervention to be more effective.

This paper introduces a formative evaluation of work with the Virtual Math Teams (VMT) Project as it adopts the GeoGebra dynamic-mathematics application. One aim of the evaluation is to evaluate the ongoing ways in which the VMT project team is codifying its knowledge of dynamic geometry into the project's online tools and documentation, and how well these tools and documents support students to engage in dynamic geometry. This contributes to the ongoing incremental improvement of the project's artifacts, and is a typical use of formative evaluation (it would be of little use to wait until the project has been completed in order to evaluate these materials).

The evaluation approach described in this paper draws on theories of Communities of Practice (Lave & Wenger, 1991; Wenger, 1998) to consider VMT not just as a series of tools, assignments and users, but as an ensemble of people, practices, processes, and other phenomena, which all aim at education in dynamic geometry. The analysis focuses on recent pedagogical artifacts produced by the project, the ways in which these artifacts represent the educational intentions of the project team, and how these artifacts are used by the project's student users. The analysis focuses on the team's understanding of dynamic geometry, the reifications of these understandings in informational artifacts, and the understanding gained by students after interacting with these reifications. An important issue here is the extent to which the design of VMT tools and curricula help students to understand dynamic-geometric construction in the same ways as the project team.

# The VMT Project with GeoGebra

VMT is a CSCL project spanning over a decade (Stahl, 2013). The project's technological, pedagogical and analytic components provide an integrated online platform for middle- and high-school students to engage in online mathematical discourse as they explore dynamic geometry. Geometrical construction and explanation are emphasized. In classical geometry, these practices involve the use of a straight edge and a compass. In VMT, these concrete affordances are 'translated' (Stahl, 2013a) into a virtual environment and dynamic screen tools. VMT incorporated GeoGebra for the past several years. The system now includes an online environment with a chat window, a virtual whiteboard, and a range of interactive dynamic-geometry tools, which students use to learn about dynamic geometric construction (Stahl, 2011).

The project follows a design-based research approach, with iterative cycles of user-centered design, implementation and evaluation. The dynamic-geometry tools in

VMT have been developed over a number of years by a team of pedagogists, learning scientists, coders, discourse analysts, social scientists, HCI experts, evaluators, and others. Key components of the design process are the team's weekly meetings, in which members discuss topics such as the student chat logs, curriculum design, technological issues, and paper writing. Outputs from the meetings include revised assignments, analyses of project data, drafts of papers, and technical bug reports and fixes. These and other outputs feed into the iterative development of the educational artifacts that mediate the project to its users: the online tools, the tool documentation, the curricula, specific assignments, and so on. It is a form of mutual bootstrapping, with implementations of the tools and curricula generating data for the research team, the analysis of which supports improvements in the tools and curricula, which in turn leads to the generation of new data for analysis. The project team's hope is that by following a design-research process, the educational artifacts they produce will be more useful than a 'one-shot' design approach.

Communities of practice are often glossed somewhat simply as 'groups of people working together on a common task.' The concept has however considerable theoretical depth, including attempting to account for how knowledge is constituted and shared within and between groups. Lave and Wenger (1991) draw on theories of practice by Giddens (1979), Bourdieu (1977), and others, to theorize how newcomers become members of a CoP and gain knowledge of its practices not just by learning what a community knows, but what community members do. This includes knowing *what* to talk about in that community, and also *how* to talk and to support community discourse and memory. Experienced community members guide less experienced members through the community's practices, a process known as legitimate peripheral participation, "an engagement in social practice that entails learning as an integral constituent." Wenger (1998) further elaborates membership in a CoP in terms of a duality of participation and reification (Figure 1). Participation involves "the social experience of living in the world in terms of membership in social communities and active involvement in social enterprises," while reification includes a range of activities (making, designing, representing, naming, encoding, and describing, as well as perceiving, interpreting, using, reusing, decoding and recasting") which generate traces of that membership. The processes are distinct yet mutually constitutive: "[Reification] always rests on participation: what is said, represented, or otherwise brought into focus always assumes a history of participation as a context for its interpretation. In turn, participation always organizes itself around reification because it always involves artifacts, words and concepts that allow it to proceed."

# Communities of Practice

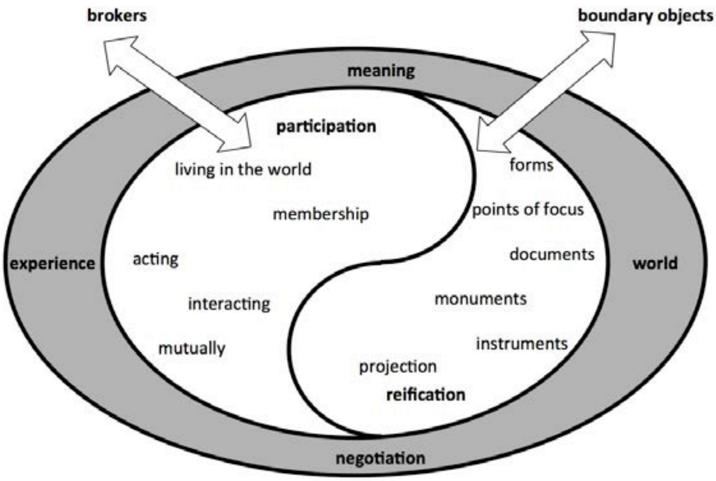


Figure 1. Participation and reification in Communities of Practice. (c.f. Wenger, 1998, Figs 1.1 and 4.1.)

Given that meaning is constituted internally in CoPs in terms of the participation-reification duality, what might individuals external to a CoP make of the reifications of that CoP? Here, says Wenger, boundary objects play an important role. According to Star and Griesemer (1989), boundary objects are:

> objects which are both plastic enough to adapt to local needs and constraints of the several parties employing them, yet robust enough to maintain a common identity across sites … They may be abstract or concrete. They have different meanings in different social worlds but their structure is common enough to more than one world to make them recognizable, a means of translation. The creation and management of boundary objects is key in developing and maintaining coherence across intersecting social worlds.

While CoPs regularly appear in the CSCL literature, boundary objects appear less frequently. The rest of this paper therefore focuses on VMT as a CoP and its use of boundary objects.

# Boundary objects in VMT

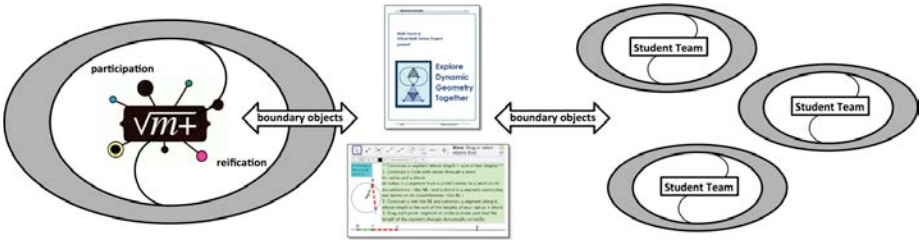


Figure 2. Boundary objects in the VMT project

From the point of view of CoP theory, participation in the VMT project team is built partly on community practices that support project members' knowledge of math, dynamic geometry, the project tools, and the team's educational and pedagogical philosophy. Reification in VMT takes this knowledge and produces artifacts such as the tool interfaces and documentation that represent this knowledge to users. Ongoing cycles of participation and reification have led to the creation of formalized webs of meaning that in turn support the team's practices and identity. This process has been going on for a number of years, with the team's knowledge of dynamic geometry reified in the form of a range of informational artifacts. Many of these artifacts are internal to the project, such as document drafts, meeting agendas and minutes, email, and so on. Other artifacts – boundary objects – have been designed to communicate the project team's understandings of dynamic geometry to wider audiences. Distillations of the team's thinking, these boundary objects are tailored for particular audiences, and include documents such as conference papers, journal articles, funding proposals, and reports. The boundary objects that we are interested in here are designed for students, and consist of sociotechnical ensembles of tools, interfaces, instructions, curricula, and other artifacts. As explicit reifications of the team's knowledge, they are designed to lead students, through a series of activities and exercises, to an understanding of dynamic geometry.

Figure 2 shows, on the left, the CoP of the project team; on the right, a series of small and emerging CoPs that represent various student teams; and in the middle, the boundary objects created by the project team and intended to serve as sense-making artifacts for the students. An important evaluation question here is: *Do these boundary objects – which make sense to the team – also make sense to the students and support them to learn about dynamic geometry?*

To address this question, the students' discourse and actions in VMT are analyzed, using as a starting point Laborde's (2004) distinction between spatio-graphical and theoretical reasoning. According to Laborde, when students construct and use geometrical figures, they can understand these figures at 'face value,' in terms of what they see, and also in terms of wider theoretical reasoning. In learning, students oscillate between these two modes of reasoning, with theoretical hypotheses explored partly through the construction of diagrams, leading to new hypotheses being formed, and an increase in their overall understanding. From this perspective, when examining the VMT student logs, do we see students working their way through the assignments based on 'surface' comprehension of the visual forms that they see on their screens, or do they build theoretical arguments and reasoning to account for the underlying dynamic geometrical forms? The analysis that follows evaluates this question by examining the ways in which the project team and the students use the word 'construction' and its synonyms. Analyzing the use of 'construction' is useful, as such usage can be a marker of both spatio-graphical and theoretical reasoning, depending on the context in which it is used (c.f. Wittgenstein, 2001).

# Methodological approach

As noted in the introduction, formative evaluation focuses on descriptions of ongoing dynamic processes, rather than static one-off measurements of project outcomes. It calls for different methods than may traditionally be used in summative evaluation. The research in this paper follows an ethnographic action-research approach (Tacchi, Slater, & Hearn, 2003), combining ethnographic methods with ongoing analyses of and contributions to the field site. The aim is to develop iterative improvements in theoretical and practical understanding, useful for both the ethnographer and the research subjects. It is a method suitable for complex field sites exhibiting organizational and technological development, such as VMT (Baskerville & Pries-Heje, 1999).

In this analysis, it is assumed that the VMT team produces reifications both for internal use and also for communicating with external stakeholders. The focus here is on boundary objects produced for students, and the students' responses to these boundary objects. The data examined were as follows. First, curricula and manuals related to the VMT tools were analyzed, including introductory assignments, and overall reviews of the project work (Stahl 2012, 2013b, 2014a, and 2014b). Second, the VMT interface, and instructions and assignments were analyzed (screenshots of many of these are included in the documents just cited). Third, the chat of a student group in the VMT Fall Fest 2013 was analyzed (see http://gerrystahl.net/vmt/icls2014/Topic3.xlsx; http://gerrystahl.net/vmt/icls2014/). The analysis followed a general grounded-discourse-analysis approach, in which 'construction' had previously been identified as a main category, and axial codes related to 'construction' were identified. The analysis was carried out using NVIVO coding software.

# Results

As might be expected, the curricula, manuals, and the VMT interface, evidenced richer uses of 'construction' than the discourse of the student teams. At the same time, the students used what knowledge they had acquired from the reifications to engage, at times, in creative hypothesis generation. For reasons of space, the analysis in this section is abbreviated.

## *Curricula and manuals*

Across the four documents, 'construction,' as well as 'constructing,' 'constructed,' and a range of synonyms, were used in a wide variety of practical and technical senses. Approximately 180 examples were identified. A common sense usage that emerged from the coding was that of an activity that can be carried out by users:

> GeoGebra lets you *construct* dynamic-mathematics figures

A related usage was that of a thing that was being made or had been made in GeoGebra by students or tutors:

> Take turns being in control of the *construction*. Say what you are doing in the chat.
> Use the chat to let people know when you want to 'take control' of the GeoGebra construction. Use the chat to tell people what you notice and what you are wondering about the *construction*.
> Sliding the history slider shows you previous versions of *constructions* in the GeoGebra tab, so you can review how your group did its work.

Construction was seen as an activity carried out with tools. For instance, the VMT tool has a 'construction area' (i.e. the whiteboard area and associated tool buttons):

> Here is how to use these tool buttons. Try each one out in the construction area of your own GeoGebra tab. First click on the button for the tool in the tool bar, then click in the *construction* area to use the tool.

Different uses of 'construction' were often combined in the same chunk of discourse, for example:

> You can even let someone else take control in your tab to help you construct something or to explore your *construction*. After your group constructs something in the group GeoGebra tab, you should make sure that you can do it yourself by doing the *construction* in your own tab.

A wide range of synonyms was also used to describe actions involved in construction, for example:

> Use the Compass to *draw* a circle whose radius is equal to the distance between two points and whose center is at a third point. First *click* on two points to *define* the length of the radius.
> Then without releasing the cursor, *drag* the circle to the point where you want its center to be.

A 'construction' was often seen as the goal or outcome of an assignment. This usage included subsidiary actions, such as creating, dragging, moving, and placing, and subsidiary components such as points, lines, segments, rays, and so on. While this whole/subsidiary distinction was often observed, there were also places where these usages overlapped, for instance where students were expected to construct a figure with underlying dependencies. In these cases, a dependency, although part of the overall figure, could be referred to in itself as a construction:

> Note: You must *construct* the dependencies among the objects, (lines and circle), not just *draw* something that looks like this.
> Can you think of any ways you could use the dependency created with the compass tool or circle tool to construct other geometric figures or relationships?

Finally, where one task of the assignment was to come up with a component, this could also be referred to as a construction. For instance, a segment – if it was the outcome of an assignment – could be constructed:

> Challenge: *Construct* a segment DG along ray DE, whose length is equal to the sum of the length of a radius AB of a circle plus the length of a segment BC connecting two points on the circumference of the circle.

## *Interface and assignments*

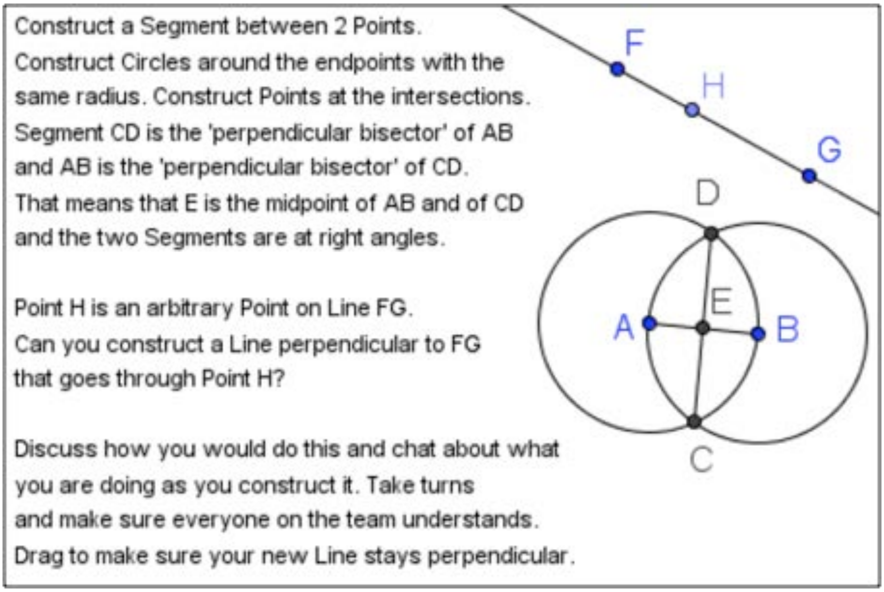


Figure 3. An example of an assignment

The project manuals (previous section) are useful documents, but students may not always read them, and their first encounter with VMT is often through the tool interface. The interface includes several interrelated functional elements, including a whiteboard area in which students can construct dynamic geometry figures with a range of tools, a chat window where they can communicate with other students in their group, and a list of users currently online. Often included in the whiteboard area are informational graphics related to the assignments. The content and format of the graphics varies, but they can be seen as summaries of the description of the assignment in the manual, consisting of a few sentences, step-by-step instructions, and figures (see Figure 3). Graphics are sometimes posted alongside pre-built GeoGebra constructions that students can interact with. The assignments summarized in the graphics run from simple tasks that introduce the students to the functions and affordances of VMT, to more complex tasks that test students' understanding of dynamic geometry. As a boundary object, the assignment figure provides a good reflection and synopsis of the associated course materials, although it should be noted that as the description and accompanying figures are succinct summaries, they might require further background understanding in order to be usefully understood (such understanding as is, for instance, provided in the extended explanations in the accompanying manuals). The graphic illustrated in Figure 3 was presented to students in Session 3 of the Fall Fest of 2013, in which the students had to construct the perpendicular of a line they have previously constructed, and after that to construct a perpendicular bisector through a given point. The word 'construction' is used in several places in this assignment description, and refers both to the construction of various parts of assignment, and the overall goal of the assignment itself (the construction of perpendiculars and bisectors). The assignment also includes elements of dynamic geometry, where the students are instructed to 'drag to make sure your new Line stays perpendicular.'

### *Student chat logs and constructions*

Students use VMT tools to discuss how to approach each assignment, to coordinate their use of the whiteboard and tools, and to discuss what they have learned and discovered in each session. The system is instrumented to capture the traces of student discussion and tool use in GeoGebra. These logs are regularly analyzed by project members, both individually, and in collective data sessions. The discussion in this section refers to a student session associated with the GeoGebra assignment outlined in the previous section. An excerpt from this log is shown in Figure 4, which shows (left to right) the event number; the time a student started to write a comment; the time the student actually posted the comment; and the comment itself. Because of their login names, this student team, who are middle school students, are referred to by the VMT project as the 'Cereal Team' (sessions by the Cereal team are also reported in Çakir & Stahl, 2015, and Öner & Stahl, 2015).

| Line | Start Time | Post Time | cornflakes | cheerios | fruitloops |
|---|---|---|---|---|---|
| 28 | 37:27.9 | 37:40.4 | | | so now you need to construck points at the intersection |
| 29 | 38:02.8 | 38:12.1 | | | no you dont make a line you make a line segment |
| 30 | 38:31.3 | 38:35.1 | | | good!! |
| 31 | 39:17.4 | 39:20.4 | | | so continue |
| 32 | 38:26.4 | 39:29.9 | | i just made the intersecting line and point in the middle | |
| 33 | 39:30.8 | 39:40.0 | | it made a perpindicular line | |

Figure 4. An example of a VMT chat log

Here we can see the three team members, Cornflakes, Cheerios, and Fruitloops. An interesting feature of this session is that while the students often referred to using the tools, they used the term 'construction' relatively infrequently: they used 'make' fourteen times; 'put,' 'move' and other related terms seven times; and 'construct' only twice. For example:

(17) fruitloops: how do i make the line segment?
(29) fruitloops: no you dont make a line you make a line segment
(30) cheerios: i just made the intersecting line and point in the middle
(50) cheerios: turn line fhg so its easier make it horizontal
(57) cornflakes: so after construting the line we put the circle on top

(63) cornflakes: put point m on tp of h
(85) fruitloops: you make the points go through qr and then you move h ontop of the line
(28) fruitloops: so now you need to construck points at the intersection (Assignment: Construct Points at the intersections)
(53) cornflakes: now construct the line (Assignment: Can you construct a Line perpendicular to FG that goes through Point H?)

Where construction is part of the students' chat, it seems to be in terms of quoting from the assignment. For instance:

| Line | Start Time | Post Time | cornflakes | cheerios | fruitloops |
|---|---|---|---|---|---|
| 89 | 14:10.8 | 14:29.8 | | | but how do we know for sure that the line is perpinmdicular |
| 90 | 14:37.1 | 14:39.6 | | im not sure | |
| 91 | 14:36.9 | 14:42.1 | there 90 degree angles | | |
| 92 | 14:41.0 | 14:45.4 | | do u cornflakes | |
| 93 | 14:46.1 | 14:59.4 | | | but you cant really prove that by looking at it |
| 94 | 14:55.4 | 15:06.8 | they intersect throught the points that go through the circle | | |
| 95 | 15:02.1 | 15:17.7 | | | it has to do with the perpendicular bisector |
| 96 | 15:15.2 | 15:19.8 | they"bisect" it | | |
| 97 | 15:20.0 | 15:31.2 | | | and the circles |
| 98 | 15:34.9 | 15:37.2 | | oh i see | |

Figure 5. "You can't really prove that by looking at it."

One question that could be asked of the students' actions here, following Laborde's model of spatio-graphical and theoretical thinking, is: Are they generally making, moving, putting, etc., or are they 'constructing' in the dynamic-geometrical sense intended by the term? The discourse of the students describes various acts of manipulation, but there is little explicitly technical reference to any higher order dynamic-geometry principles that could be informing the construction. At the same time, however, the students also engage in some incipient attempts at proof, even if they lack the technical language to describe this in formal terms. This occurs towards the end of their session, where Fruitloops makes some suggestions for summing up the knowledge gained from the assignment (Figure 5). Here, Fruitloops seems to be working towards a distinction between what the students see, what they know, and what they should be able to prove, noting that "you can't really prove that [the line is perpendicular] by looking at it," but rather that the students should aim towards an understanding guided by deeper underlying geometrical thinking.

# Discussion

One finding from the analysis is that the VMT project team used 'construction' and related terms in complex ways, while the students used the same terms imprecisely. On one level this should not be surprising. At this stage in their learning of dynamic geometry, the students do not necessarily have at their disposal the range of language and concepts that would enable them to make the rich connections between dynamic-geometry concepts and practices that the project team does. Further, the students seemed to respond literally to the assignment prompts, with the inference that they are demonstrating spatio-graphical responses; that is, they are talking about constructing without having full knowledge of what this might mean in dynamic geometry.

From the point of view of the evaluation framework proposed in this paper, one counter-argument is that the students did in fact develop an understanding of dynamic geometry, but that they did not (yet) possess a sophisticated enough vocabulary to express this in the same terms as the project team. Towards the end of the session described in this paper, and in response to the assignment instruction "Point H is an arbitrary Point on Line FG. Can you construct a Line perpendicular to FG that goes through H?", they test hunches by moving the construction around to see if it aligned. They appear to be working towards a preliminary hypothesis regarding dynamic-geometrical proof, even if they do not formally test this, or use the same terminology that a member of the VMT team might have used. While from one perspective, this may be seen as spatio-graphical behavior, and lining up different figures with no regard for why they align, at the same time, these actions can also be seen as nascent forms of hypothesis testing based on early understandings of the possibilities for proof in dynamic-geometry environments. This has been suggested in other analyses of the Cereal Team. Çakir and Stahl (2015) studied another VMT session in which the team was instructed to drag previously constructed quadrilaterals, in order to make inferences regarding the underlying dependencies and constructions of these quadrilaterals. They note: "through an interactive process of calibrating and recalibrating their indexical references (Zemel & Koschmann, 2013) to the evolving visual configurations witnessed during different dragging performances, the team members were able to collectively notice several dependencies among constituent elements, describe them in colloquial and semi-formal terms, and produce conjectures for the underlying causes of those dependencies." Similarly, Öner and Stahl (2015), who have analyzed the same session described in this paper, but using Sfard's commognitive framework, suggest that by the end of the session the students are engaged in working towards a preliminary discussion of proof, even if this is not framed in rigorous terms.

From the perspective of Laborde, the Cereal Team iterated rapidly between spatio-graphical responses and theoretical reasoning. This rapid iteration and reasoning was supported by the dynamic nature of the VMT tools and the ability to drag, as well as by the collaborative affordances of the interface, such as the ability to watch other students manipulating constructions in real time, and the chat window for discussion of these manipulations. This iteration was productive; while (as the students observed) the perpendicular bisector figure may look the same from either a spatio-graphical or theoretical perspective, as they interacted with it over the course of the session, the students started to specify differences between these perspectives. They began to posit proto-rules for evaluating whatever it is that they have been instructed to notice in the assignment (constructing a perpendicular bisector), and move from spatio-graphical reasoning and towards theoretical reasoning. Thus, while their discourse consciously reflects at least partly the steps presented in the assignment images and texts – and they recognize that following these steps correctly should result in achieving the assignment goal – they are also aware that the assignment calls on them to provide theoretical accounts beyond those same steps. They recognize that their dynamic constructions can be seen both as spatio-graphical assemblages of components (points, lines, circles), and also as objects that can be explained in terms of more abstract underlying dynamic-geometrical principles. At this stage, however, this reasoning is emergent, not least because of the lack of practice with and reflection upon dynamic geometry, at least beyond the immediate goals of the assignment.

Overall, the VMT tools and documentation functioned well as boundary objects that reified and externalized the project team's knowledge for students. This is not surprising, given the design-research approach that the project team uses, which produces regular ongoing revisions to these artifacts. At the same time, the observations, framed within CoP theory, regarding the differences in meaning of 'construction' in both the vocabularies and practices of the project team and the students, suggest further topics for investigation in this process. What are the ontological grounds that the team and the students bring to their understanding of geometry and dynamic geometry? Do the students understand their actions in terms of classical geometry, in which they have yet to be formally trained, or in terms of a translation of dynamic geometry, or in terms of something else again? Further, how do they interpret the affordances of the VMT tools? Do they assume that objects move, unless otherwise specified, and if so, what sort of understanding of dynamic geometry then emerges? These questions lead to a consideration of how VMT can be developed further to support students in theory building and proof. One strategy suggested by the evaluation is to continue to refine the boundary objects produced by the team (in the form, for instance, of assignment images and texts), and to gain further traction with the students' understanding of what is meant by 'construction.' A second strategy is to understand further the contribution the connections between the collaborative nature of the tools, and the rapid iterations between spatio-graphical and theoretical thinking displayed by the students. (Note: this section benefited from discussion at the 2014 ICLS workshop *Interaction Analysis of Student Teams Enacting the Practices of Collaborative Dynamic Geometry*; http://gerrystahl.net/vmt/icls2014/.)

# Conclusion

A formative evaluation of the VMT project, based on Communities of Practice and boundary objects, identified various uses of 'constructing' by project members and students. For the VMT team, the idea of constructing was constituted within a web of dynamic-geometry knowledge, and reified in boundary objects such as instructional manuals, assignments, and interfaces. The students drew on these boundary objects, developing notions of 'constructing' which were more emergent. The evaluation recommendations are that the project's boundary objects should continue to be refined, and also that further understanding be gained of what team members and students understand by 'constructing' and related terms. Overall, the CoP approach usefully pulled back the evaluation lens from the tools, and brought into view the project as a whole, covering not just technology use, but organizational levels of design and implementation. The evaluation design allowed insights to be fed back to the project team on an iterative basis, complementing the design-research approach of the team.

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# 7. Dragging as a Referential Resource for Mathematical Meaning Making in a Collaborative Dynamic-Geometry Environment

Murat P. Çakır and Gerry Stahl

**Abstract:** This paper focuses on the referential roles played by dragging moves on dynamic-geometry representations in a collaborative-geometry problem-solving context. Through an interaction analysis of chat excerpts where dragging is used by a team of students to explore the geometric properties of a given polygon, the paper investigates the role of dragging on the facilitation of joint mathematical meaning making online. Our qualitative findings suggest that the indexical properties of the dynamic constructions are specified and recalibrated through the coordination of dragging actions with textual chat, where the two types of actions mutually elaborate each other.

## Introduction

Dynamic Geometry Systems (DGS) such as GeoGebra, Geometer's Sketchpad and Cabri offer unique affordances for exploring and making sense of geometry (Arzarello et al., 2002; Hölzl, 1996). The visual interface provided by such micro-worlds allows students to construct geometric objects by using elements of Euclidean Geometry such as points, lines and circles through a digital analog of compass and straight-edge constructions. More importantly, the object-oriented design of these micro-worlds allows students to dynamically act on these constructions by dragging their constitutive elements, which helps them to interactively explore the implications of the dependencies within those constructions (Stahl, 2013). By developing increasingly more purposeful dragging strategies, students may notice how a family of Euclidean constructions relate to each other and whether specific invariants are present in that family of figures (Arzarello et al., 2002). Therefore, such dynamic representations can be instrumental in helping students develop a deeper understanding of geometry by making otherwise obscure ideas/theorems in geometry more accessible. The possibility of testing an invariant across a continuum of cases can also help students to develop intuitions for generalizations that go beyond the particular construction view at hand (Leung, 2008).

The nature of the dragging actions through which geometric constructions are manipulated and explored, and their role in facilitating students' understanding of geometry concepts have been investigated by several studies in the math-education literature (Arzarello et al., 2002; Baccaglini-Frank & Mariotti, 2010; Leung, 2008; Lopez-Real & Leung, 2006; Hölzl, 1996). In particular, Arzarello et al. (2002, p.67) proposed a hierarchy of dragging modalities that distinguish *wandering dragging* (randomly moving basic points to fish for interesting configurations or regularities in the dynamic diagram), *bounded dragging* (moving a restricted point), *guided dragging* (moves aimed to give the dynamic drawing a particular shape), *dummy locus dragging* (moves that reveal that a point is restricted to move on a specific path), *line dragging* (drawing new points along a line in order to keep the regularity of the figure), *linked dragging* (linking a point to an object and moving it onto that object) and *the dragging test* (moves aimed to test if a particular property of the current shape is preserved). These dragging actions are employed by students at different stages of their problem-solving activity, which provide insights into their reasoning with dynamic representations. In particular, wandering and guided dragging are employed during exploration/discovery phases, dummy locus dragging often hints at the construction of a conjecture, and the dragging test is often used to validate/justify conjectures. Therefore, dragging is treated as a key process facilitating the development of cognitive structures that bridge perceptual observations with formal accounts of deductive reasoning in geometry (Arzarello et al., 2002).

The focus of the dragging studies reviewed thus far has been on the individual learner developing a sense of understanding through his/her interaction with dynamic representations. However, acting on these dynamic resources also has a social significance, which changes the problem context not only for the actor himself but also for collaborators witnessing those actions. At the individual unit of analysis, the meaning-making role of the dragging actions may be difficult to investigate from the actions themselves or think-aloud protocols. In a collaborative problem-solving situation, such actions become resources for joint meaning making, which are acted upon, referred to, reasoned with, and questioned in collaborative discourse (Stahl, 2009). Therefore, such collaborative activities present a perspicuous setting for researchers to explore how actions with and around dynamic-geometry objects facilitate the development of shared mathematical understanding.

This paper focuses on the referential roles played by dragging moves on dynamic-geometry representations in a collaborative geometry problem-solving context. Our interest is motivated by recent CSCL studies that treat collaborative problem solving as "discovery work," in which collaborators work out the indexical details of their joint situation by calibrating and recalibrating references to relevant constituent elements of their shared task and its evolving solution (Zemel & Koschmann, 2013; Koschmann & Zemel, 2011). Indexical expressions refer to those linguistic resources whose sense depends on the context of the utterance. Through a process of calibrating and recalibrating references to an evolving space of persistently available diagrams, participants increasingly specify what those representations mean for them as part of an evolving solution account. Referring expressions initially function as a place holder for what is currently not known, which gets specified further (i.e., thingified) as subsequent actions and references modify their sense in interaction (Koschmann & Zemel, 2011). We argue that dragging actions have a similar referential role, which facilitates the discovery process with dynamic representations in geometry. Through an interaction analysis (Jordan & Henderson, 1995) of excerpts where dragging is used to explore the geometric properties of a given polygon, we identify the role of dragging on the facilitation of joint mathematical meaning making by studying how the dragging is used to increasingly specify the indexical properties of the dynamic construction.

# Methods and data

The excerpts analyzed in this paper are obtained from the Virtual Math Teams (VMT) Spring Fest organized by the Math Forum in 2013. The analyzed chat session is part of a broader curriculum-development activity including the use of dynamic geometry in math classes supported through online collaborative-learning activities in the VMT system. The team consisted of Fruitloops, Cornflakes and Cheerios who are female students about 14 years old, who have not yet studied geometry. This team completed seven hour-long chat sessions of dynamic-geometry tasks in the VMT environment before they met for the session from which the excerpts were obtained. In this session, the team was given a set of 21 different quadrilaterals and was asked to (a) identify their dependencies, and (b) tell how each of them was constructed. The task description suggests participants drag the vertices of each quadrilateral to see what is special about each one. The GeoGebra application also hints at the presence of dependencies/constraints by shading vertices that are dependent on other points. Excerpts from this chat session were subjected to interaction analysis to investigate the referential roles fulfilled by dragging actions.

During the session, participants interacted through the VMT environment (Stahl, 2009), which provides a chat interface with an integrated electronic drawing area with collaborative dynamic-geometry drawing capabilities. The dynamic drawing area is based on GeoGebra, a popular dynamic-geometry application. The VMT environment allows a group of users to co-construct and discuss shared dynamic-geometry objects online. Access to the drawing area is managed through a turn-taking mechanism, which allows only one user at a time to construct or manipulate dynamic objects. The VMT system also supports researchers by providing re-playable logs of these sessions for analysis, allowing step-by-step walkthroughs of drawing and typing actions that took place during the online student sessions. The excerpts discussed in this paper involve manipulation of dynamic objects through dragging moves—which is challenging to present in a text document. For that reason, screenshots that capture intermediary states of the dragging actions are provided to complement the chat logs.

# Analysis

### *Excerpt 1*

The excerpt starts when the team decides to move on to polygon #2 (i.e., EFGH). Before exploring polygon #2, the team took turns to explore polygon #1 (i.e., ABCD) by dragging each of its four vertices; they quickly concluded that none of the vertices had any dependencies (i.e., they are free points). In line 1, Cornflakes announces that she will explore polygon EFGH, and then she takes control of the drawing area. Cornflakes first drags vertex F. The series of screenshots displayed in Figure 1 shows how polygon #2 changes while Cornflakes is dragging point F. Cornflakes drags point F up, left, down and then right, tracing almost a complete circle around point E in a counter-clockwise direction. Note that points E and H are unaffected by this drag, but point G is apparently moving as F is dragged.

1. cornflakes (3:20:26): ill do polygon efgh
2. cornflakes (3:20:33): takes control of the drawing area
3. cheerios (3:20:37): just say the number its easier
4. cornflakes (3:20:40-3:20:50): drags points F (Figure 1), H, D, E (Figure 2) and G (drags on H, D, and G were not visualized in the figures. D is part of another polygon the team worked on prior to EFGH, which is not displayed).
5. cornflakes (3:20:52): releases control of the drawing area
6. cornflakes (3:21:17): okasy polygon 2 has all points moving except point g

7. cornflakes (3:21:28): and point g is also a different color
8. cheerios (3:21:40): do u think it is restricted
9. cheerios (3:21:44): or constrained
10. fruitloops (3:21:49):i feel like poly 1 and poly 2 are almost exactly the same except that poly 2 had one point that is a lighter shade
11. fruitloops (3:22:04): can i try moving it?
12. cornflakes (3:22:17): sure
13. fruitloops (3:22:25): and @ cheerios , i dont know for sure

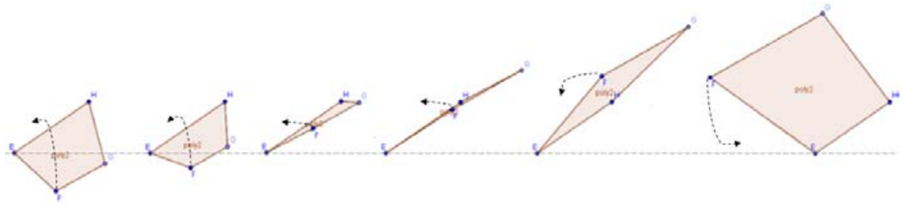

Figure 1. Cornflakes drags vertex F counter-clockwise around point E. Each screen shot corresponds to different stages of the polygon as F is dragged. The dashed lines are provided to aid the interpretation of the dynamic changes enacted by the dragging action. The dashed line over E, which was not affected by the drag, is provided as an anchor to aid the visual comparison between stages. Arrows show the direction of the drag.

Next, Cornflakes drags vertex H up and down. Then she begins to drag point E. The steps of this dragging action are displayed in Figure 2 below. Point E is slightly moved to up-right and then to bottom-left, which did not seem to affect any other vertex, except very minor shifts on G's position. Finally, Cornflakes drags point G. Point G is moved up-left and then down-right slightly as a consequence of this dragging action. None of the other vertices seem to be affected.

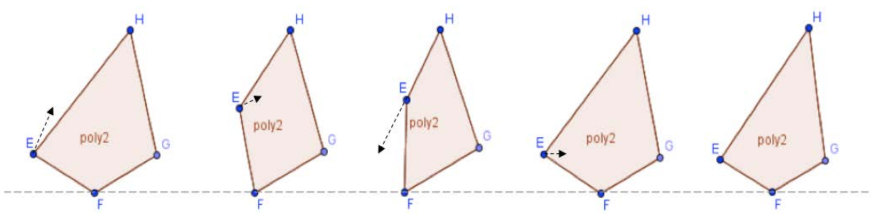


Figure 2. Cornflakes drags point E. Black arrows indicate the direction of the next drag.

After dragging point G, Cornflakes posts two chat messages (lines 6 & 7), announcing that "polygon 2 has all points moving except point g", and notes that point G is also marked with a different color. Cornflakes' account marks G as different from other points based on the claim that all points can move except G, even though she has explicitly moved point G during her last drag. The way she formulates her observations from her drags suggests that she is oriented towards whether a point can be freely moved or not. Cheerios responds to Cornflakes in line 8 by asking if she thinks "it" is restricted or constrained. The indexical "it" can be read as a reference to point G, since this point was the most salient object mentioned in Cornflake's chat messages besides polygon #2. By explicitly mentioning the terms "restricted" and "constrained," Cheerios invokes relevant terminology that encodes specific distinctions among the kinds of points one can make in GeoGebra. Hence, Cheerios' message can be read as an implicit assessment of Cornflake's account in terms of its descriptive adequacy, as well as an attempt to orient Cornflake's proposal towards a more formal account. So, Cheerios' statements can be seen as a *recalibration* move. Both terms seem to index a kind of limitation in the movement of a point based on their dependencies on other points, but the specific distinction encoded in this terminology has not been explicitly specified yet.

Next, Fruitloops posts the message she has been typing while Cheerios' messages appeared in chat, which states that polygon #1 and polygon #2 are almost the same except for the point with the lighter shade (line 10). Fruitloops' account seems to be informed by her observation of Cornflakes' prior dragging moves, and makes a visual reference to the color-coding of each vertex. She then requests the team's permission to try moving the polygon on her own in line 11. In line 12, Fruitloops responds to the question raised by Cheerios, that she is not sure about the restricted/constrained distinction.

Overall, in excerpt 1, the team seems to be oriented towards how the points can be moved around based on the witnessed dragging actions enacted by Cornflakes. The team does not mention more specific dependencies among the vertices, as the drags are rather minimal and hence have not yet hinted at the more complicated structure underlying polygon #2's construction. The team members seem to be oriented towards visually salient features of polygon #2, such as having a point G that is lighter in shade. Based on what is revealed by the drags performed by Cornflakes, the team seems to endorse the interpretation that polygon #2 is very similar to polygon #1 except for the vertex with the light blue shade. Cheerios contributes to the discussion by making the concepts "constrained" and "restricted" relevant to ongoing interaction as a means to categorize vertices. Yet, it is still not clear how these terms should be applied to the problem at hand.

### *Excerpt 2*

Fruitloops takes control of the GeoGebra area and begins to manipulate the shared dynamic drawing (line 15). Figure 3 shows a chronologically ordered series of screen shots from her dragging of point G. The dashed reference lines crossing over point E, which remained stationary while Fruitloops was dragging point G, are provided to aid the comparison of different stages. Fruitloops' drag gradually moves point G in a circular motion, first in a clockwise and then in a counter-clockwise direction, which is followed by several full circles in both directions. As Fruitloops is performing the drag on G, she seems to gradually notice the path that point G is constrained to, which is evidenced in the way she moves the vertex back and forth repeatedly in clockwise and counter-clockwise directions in this episode (Figure 3). Meanwhile, Cheerios requests permission to access the drawing area, which is acknowledged by Cornflakes. However, Fruitloops holds onto her turn in the drawing area, while she is typing what will appear in line 18, which states, "so point g only moves in like a circular motion around point f." Fruitloops' account is a reflection on or noticing of what has been discovered in the dragging. The message specifies the relationship she notices between points G and F without making any reference to more technical terms such as restrictions or constraints, but using only colloquial terms or descriptions. In the next line, Cornflakes agrees. This is followed by further drags of point G around F by Fruitloops, which seem to complement her exposition in line 18 with an enactment of the verbally described movement pattern. These drags also simultaneously verify the proposed relationship, which recalibrates the status of vertex G for the group in this context.

14. fruitloops (3:22:42): takes control of the drawing area
15. fruitloops (3:23:04): drags Point G (Figure 3)
16. cheerios (3:23:18): ok can i try
17. cornflakes (3:23:22): sure
18. fruitloops (3:23:23): so point g only moves in like a circular motion around point f
19. cornflakes (3:23:35): @fruitloops yea
20. fruitloops (3:23:50): drags Point G

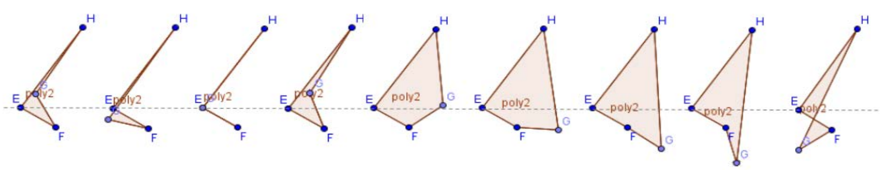

Figure 3. Chronologically ordered screenshots taken while Fruitloops is dragging vertex G.

In this episode, through her rigorous drags of vertex G, Fruitloops uncovers an important property of polygon #2: that vertex G always follows a specific circular path around point F. As soon as she gains control of the drawing area, Fruitloops starts dragging the point with the lighter shade. Fruitloops' initial drags on G seem rather exploratory, which gradually becomes more orderly and purposeful as she notices the constraint imposed on G. While Fruitloops is communicating this noticing to her teammates, she coordinates her actions across both chat and drawing areas in such a way that her verbal description in chat can be read in relation to her ongoing enactment on the shared drawing. In short, the sequential organization of Fruitloops' actions across both interaction spaces made her point witnessable by her teammates. This instance highlights another important aspect of dragging in dynamic geometry. The progression of drags from exploratory trials to purposeful demonstrations/tests serves both as a public display of an evolving understanding and as a resource for communicating abstract visuo-spatial relationships that may be difficult to articulate in text. Thus, drags also have a social role in this context, as demonstrable actions embodying specific conjectures about dependencies among geometric objects.

### *Excerpt 3:*

Following Fruitloops' demonstration, Cheerios asks about the difference between the terms "constrained" and "restricted" in line 23. Cornflakes states that "constrained is limited function," which provides some specificity for one of the terms. In the meantime, Fruitloops continues to drag vertices of polygon #2. She first drags point H slightly to the bottom. No other point seems to be affected by this move. Next, she begins to drag point E (Figure 4). Point E is gradually moved away and towards point F, which simultaneously moves point G out and towards point F. Fruitloops carefully and slowly drags E around F, which suggests that she is oriented towards the relationship among E, G and F triggered by the dragging action on E.

21. fruitloops (3:24:09): drags Point H (Figure 4)
22. fruitloops (3:24:15): drags Point E (Figure 4)
23. cheerios (3:24:16): what si the difference between constrained and restricted
24. fruitloops (3:24:17): drags Point G (Figure 4)
25. fruitloops (3:24:21): drags Point E (Figure 4)
26. cheerios (3:24:24): is*
27. cornflakes (3:24:41): constrained is limited function
28. fruitloops (3:24:46): also when you move e, g moves away or closer to f
29. fruitloops (3:25:08): so i think g it definitly constrained
30. fruitloops (3:25:12): drags Point H
31. cornflakes (3:25:13): yes
32. cornflakes (3:25:19): i think that too

33. cheerios (3:25:25): why though
34. fruitloops (3:25:31): drags Point,F
35. fruitloops (3:25:59): and g moves whenever you move point e and f but it doesnt move when you move h
36. cheerios (3:26:20): okay
37. fruitloops (3:26:31): releases control of the drawing area

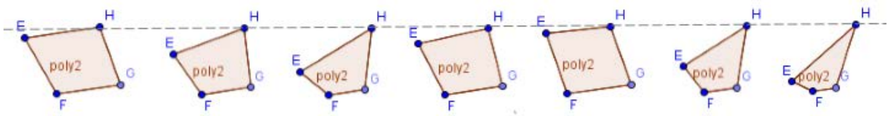


Figure 4. Fruitloops drags vertex E, first towards F, then away from F and then towards it again.

Fruitloops then posts in line 28 that "also when you move e, g moves away or closer to f." This can be read as a verbalization of the recent drag on E, which demonstrates the relationship among points E, G and F, where dragging E affects the position of G with respect to F. Similar to the previous instances, the verbal announcement of the noticed properties immediately follow the dragging actions. In line 29, Fruitloops elaborates on the prior message by proposing that G must be constrained. This statement also responds to the ongoing discussion of the distinction between constraints and restrictions, by proposing point G as an instance of a constrained object. In other words, the posting indexes G as a particular instance of a constrained point. In lines 31 and 32, Cornflakes concurs with Fruitloops' observation. In line 33 Cheerios posts a message wondering why the proposed relationship holds, which problematizes for the whole team the underlying cause of the relationship proposed by Fruitloops. In line 35, Fruitloops summarizes her observations after performing another drag on F. She states that point G moves whenever points E and F are moved, but G does not move when H is moved.

In this episode, Fruitloops identifies additional key relationships among points E, F, G and H through her systematic dragging of these points. She is oriented toward observing how each point is influenced by her drags on other points. Through initially explorative and progressively deliberate drags, Fruitloops notices that G's position is influenced by moving E or F, but not H. However, her statements do not specify the nature of those relationships in terms of concepts such as lengths or angles yet. The verbal accounts are primarily characterizations of visual effects triggered by drags of different points. Based on the relationships identified between E, F and G, Fruitloops proposes that G must be constrained, which provides further specificity to (i.e., a recalibration of) what is referred to by the term "constraint" by proposing G as an exemplar (lines 28, 29).

## Excerpt 4

In line 38 Fruitloops takes up the prior discussion of the distinction between constraint and restriction. Fruitloops suggests that G is constrained because it can be moved, but the function is limited (i.e., limited to move on a circle around F). Then Fruitloops posts a question asking for the definition of "dependant" (sic) in line 40. This concept is mentioned in the task description, which asks the team to identify the dependencies in each polygon. About a minute later, a chat message from Cheerios appears, stating the need for the other line or point: otherwise "it wont work." In line 43, Cornflakes agrees and states that some points depend on each other. The definitions are rather implicit and ambiguous at this point, but the concept of dependence gradually attains its meaning as a kind of connection between two or more objects in this exchange.

38. fruitloops (3:26:42): @ cheerios. i think its constrained because it moves but the function is limited
39. cheerios (3:27:36): oh i see
40. fruitloops (3:27:37): what is the definition of dependant
41. cheerios (3:28:52): u need the other line or point otherwise it wont work
42. fruitloops (3:28:54): do you guys have any idea of how this was made?
43. cornflakes (3:29:15): yeah some points are dependent on others
44. cornflakes (3:29:43): maybe some invisible circles and the shapes could be dependent on thos circles
45. cheerios (3:30:02): yea maybe like the triangles
46. fruitloops (3:30:20): maybe because point g only moves in a circular motion around point f
47. cornflakes (3:30:35): but why?
48. fruitloops (3:30:55): i think it has to do with how it was constructed
49. cheerios (3:31:03): i agree
50. cornflakes (3:31:29): YES

51. fruitloops (3:31:44): cause erememember how before in the other topic we would sometimes use circles to construct stuff and then hide the circles? well maybe thiis quad was made using a circle
52. cornflakes (3:31:58): yeah and one of the points was on the circle
53. cheerios (3:32:38): yeah that makes sense remember when we made the triangle the same thing happened
54. cornflakes (3:32:43): yes
55. fruitloops (3:33:10): but i dont really know how it could have been made?
56. fruitloops (3:33:48): releases control of the drawing area
57. cheerios (3:34:14): maybe they used another shape instead of circles
58. fruitloops (3:34:17): do you thinkk point e is the same distance away from f as g?
59. fruitloops (3:34:25): takes control of the drawing area
60. fruitloops (3:34:26-3:35:02): drags Point G (Figure 5)

In line 42, Fruitloops asks the team if they have any idea how the polygon was made. In line 44, Cornflakes proposes that there may be invisible circles accounting for the dependencies they have uncovered. In line 45, Cheerios agrees and states that this situation is "like triangles." Cheerios seems to be referring to the team's past constructions during previous sessions, where they used circles to make equilateral and inscribed triangles. In line 46, Fruitloops endorses the possibility of a hidden circle, based on the observation that G is only moving in circles around point F. In line 51, Fruitloops elaborates further by reminding other members about a past exercise where they used a circle as part of a larger construction and then hid it from view by making it invisible in GeoGebra. In line 57, Cheerios proposes the possibility that the point may even be constrained to an object other than a circle.

In line 58, Fruitloops solicits other members' assessment about the observations that points E and G are equally distant from point F. Next, she drags point G on the GeoGebra board, making circles around point F. Snapshots from Fruitloops' dragging actions are given in Figure 5. Fruitloops slows down when point G gets near point E, and drags it back and forth as the two points coincide with each other. This drag seems to explore the possibility that EF and FG have the same length. This is the first instance where a group member mentioned distance as a way to characterize a dependency among a set of points.

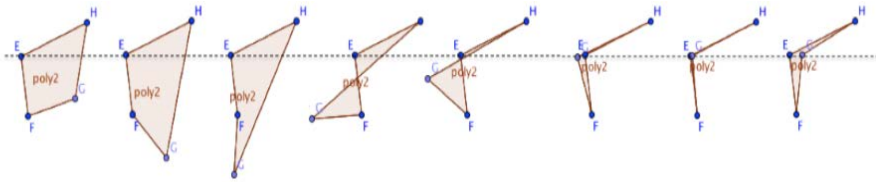


Figure 5. Fruitloops' drag on vertex G. G is seen as moving in a circle around F. Fruitloops slows down when G is about to move near vertex E.

In this episode, the team starts to reflect on the relationships uncovered between the points, and the terminology that should be used to characterize those relationships. The team begins to develop conjectures about possible ways polygon #2 might have been constructed. At this time, they relate the observed behavior to their prior experiences using circles in earlier sessions. The team seems to agree on the idea/conjecture that at least G would be constrained on such a circle. The notion of *dependence* makes its first appearance in the chat, gradually becoming a resource for describing how the polygon might have been constructed. In this episode, the team makes another key observation regarding the underlying structure of polygon #2: that the edges EF and FG have the same length. Fruitloops' drags of point E around F led her to realize that her drags influence the length of EF and FG in similar ways, where E and G can even be collapsed onto the same point.

Towards the end of their discussion of polygon #2, the team discusses how the dependencies they had discovered could be implemented in GeoGebra. The possibility of using an invisible circle for constraining G and the use of the compass tool to define two line segments of the same length are mentioned as possible steps in the construction. The team also discusses the order of the construction steps that might have been used to produce polygon #2. The proposed steps of the construction can be considered as an informal proof account, explaining why the polygon has the discovered properties. However, the team disagrees about which point would be plotted first, and cannot account for the joint relationship between points E, F and G, which precludes them from proceeding further in their joint inquiry.

# Discussion

Previous literature characterizing dragging moves has primarily focused on how an *individual* learner's cognitive processes are shaped through interaction with dynamic representations, without emphasizing the social and practical significance of such actions. In this paper, we underlined the social-interactional implications of dragging actions in a *collaborative* CSCL problem-solving context. The excerpts analyzed in this paper present a detailed view of the lived work of joint reasoning performed by a team of students while they were working together to discover the geometric properties of a given dynamic polygon. The team went through a sequence of sense-making steps, including dragging, noticing, stating in chat, bridging to past meaningful experiences of intersubjective shared understanding, and using technical terms like dependency. Our analysis of the excerpts suggest that through an interactive process of *calibrating and recalibrating their indexical references* (Zemel & Koschmann, 2013) to the evolving visual configurations witnessed during different dragging performances, the team members were able to collectively notice several key dependencies among constituent elements, describe them in colloquial/semi-formal terms and produce conjectures for the underlying causes of those dependencies.

The progression of dragging performances from exploratory trials to purposeful demonstrations serves both as a public display of an evolving understanding, and as a resource for noticing and communicating abstract visuo-spatial relationships that may be difficult to describe and follow in textual communication. In the excerpts analyzed above, the availability of the intermediary stages of dragging actions made the reasoning that goes with the unfolding dragging activity witnessable by the group. The witnessed unfolding of visual changes served as an indexical ground which (a) gave sense to subsequent utterances that refer to the noticed regularities, and (b) provided further specificity to technical terms that distinguish relevant geometric relationships such as constraints and dependencies by enacting them in the dynamic figure. Moreover, the emerging purposefulness of the drags was made evident with verbal glosses following an episode of dragging, which accounted for what was there to be noticed. Hence, actions in both interaction spaces mutually elaborate each other, where (a) drags highlight key relationships and eliminate the need to verbalize every complex detail, while (b) verbal accounts direct others' attention to relevant parts of the figure where the regularities can be located.

The analyzed excerpts also suggest that not all drags are equally effective for noticing key geometric properties. This point is supported by a comparison of the dragging performances of Cornflakes and Fruitloops, and the subsequent proposals the team members had made in the discussion following those drags. Initial drags by Cornflakes led to the conjecture that polygon #2 is very similar to #1 (whose vertices had no dependencies), except for the vertex that was marked with a different color. Only after Fruitloops took over and performed more strategic drags, did the team realize that there was more to the underlying geometric structure of polygon #2. In particular, the team noticed the following regularities: (a) G moves around F in a circle and when G is moved no other vertex moves, (b) when H is moved, no other vertex moves, (c) G moves when F is moved, (d) G moves when E is moved, and (f) E and G are always equidistant from F.

The analysis of the team's work in dragging its figures shows how collaborative learning about the nature of geometric dependencies develops gradually through hands-on exploration guided by challenging tasks in a computer-supported environment. In the remaining part of their chat session, which is not covered in the above excerpts, the team members continued to explore similar polygons by taking turns dragging. The dragging strategies developed in the excerpts above were appropriated by other members during those explorations. This exemplifies the gradual transformation of one member's public display of dragging-mediated reasoning into a shared practice of geometric reasoning for the team. Through calibration and recalibration of indexical references that refer to the discovered properties of the shared dynamic drawing, team members gradually made sense of key geometry concepts as they were enacted by dragging actions on shared figures. The affordances of the VMT environment for making the results of intermediary stages of drags available for all participants and the way participants coordinated such actions with their chat messages were consequential for collaborative meaning making online. For this reason, a key design requirement to support collaborative learning in CSCL settings should be the inclusion of mechanisms that help participants effectively coordinate representational affordances, especially in contexts like geometry, where diagrams and concepts need to be closely aligned with each other. Likewise, an important part of CSCL methodology should include the analysis of discourse and actions as referential components of intersubjective meaning making.

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# 8. Collaborative Exploration of Geometric Dependencies in Dynamic Geometry

Gerry Stahl, Stephen Weimar, Annie Fetter, Anthony Mantoan

**Abstract.** The Virtual Math Teams Project (2002-2014) at the Math Forum developed a collaborative-learning environment for mathematics, combining text chat and a multi-user version of GeoGebra. It created curricular activities aligned with Common Core and provides teacher professional development. It has deployed the technology and curriculum with groups of students year after year and analyzed some of the student interactions in micro-detail. Study of how collaborative learning takes place in this GeoGebra-based environment has been used to refine the environment and curriculum. Student teams learn how to collaborate, work online, use GeoGebra, analyze and construct dynamic-geometry figures, think about dependencies among geometric objects and talk about mathematics. This presentation demos the approach and shows how learning about dynamic-geometric dependencies is displayed in a data excerpt.

## Research Context and Problem: Guiding Students to Math Cognition

The contemporary fields of science, technology, engineering and mathematics (STEM), in particular, require a mindset that emerged historically within the community of ancient Greek geometers (Heath, 1921). For many people, learning basic geometry still represents a watershed event that determines if an individual will or will not be comfortable with the cultures of mathematical cognition. GeoGebra provides a promising tool for supporting transformational mathematical thinking.

At the Math Forum (www.MathForum.org), we have embedded GeoGebra in an online collaboration environment (Stahl, 2009) and converted it to a multi-user version (Stahl, 2013), so that groups of students can construct and drag figures together, while chatting about what they are doing. To guide student exploration, we have developed a cohesive curriculum focused on the construction of figures with geometric dependencies (Stahl, 2015b)—for use by student teams as well as in teacher professional development. Collaborative GeoGebra is now available for iPad, tablets and laptops (vmtdev.mathforum.org). The curriculum is in a GeoGebraBook (http://ggbtu.be/b154045), which is not yet multi-user.

In this paper, we demo collaborative GeoGebra and illustrate how we analyze case studies of students engaged over multiple sessions with our online collaboration-learning environment: multi-user GeoGebra, challenging topics and inquiry pedagogy. For the past decade, such analysis of student usage has been driving the iterative design of our approach. We want to indicate how student teams under these conditions display that they are learning fundamental insights about dynamic geometry.

## Theory of Collaborative Dynamic Geometry: Group Cognition and Dependencies

Learning is often conceived as a change in propositional knowledge possessed by an individual student. Opening up an alternative to this view, Vygotsky argued that students could accomplish knowledge-building or learning tasks in small groups before they could accomplish the same tasks individually—and that much individual learning actually resulted from the earlier group interactions (Vygotsky, 1930/1978), rather than the group being reducible to its members as already formed individual minds. Vygotsky conceived the group interactions as mediated by artifacts, such as representational images and communication media. More recently, educational theorists have argued that student processes of becoming mathematicians or scientists, for instance, are largely a matter of mastering the linguistic practices of the field (Lemke, 1993; Sfard, 2008).

Our pedagogical approach emphasizes collaborative learning through discourse in small groups (Stahl, 2015c). Carefully designed topics guide student exploration and bring in historically developed concepts from the mathematics community. Teachers prepare students before sessions and discuss findings and conjectures in whole-class discussions after the collaborative sessions. The group cognition (Stahl, 2006) that takes place in the group work can lead to learning by individual students in their zones of proximal learning, based on their joint meaning making and task accomplishments.

Our curriculum focuses on learning to construct *geometric dependencies* (Stahl, 2013) in GeoGebra, a challenging but important skill. While much classroom use of dynamic geometry today merely uses it as a visualization tool, to allow students to drag existing diagrams around, the technology has a greater potential: to empower students to construct their own diagrams, to build their own dependencies into the objects and even to fashion their own custom construction tools. Then they can view Euclid's propositions as guides to designing and constructing their own interesting mathematical objects, rather than as impersonal eternal truths to be memorized.

# Research Method: Sequential Interaction Analysis

The data we collect from hundreds of students using our system each year includes a complete record of their interactions, which we can replay just as it appeared to the students. We also have detailed logs generated automatically.

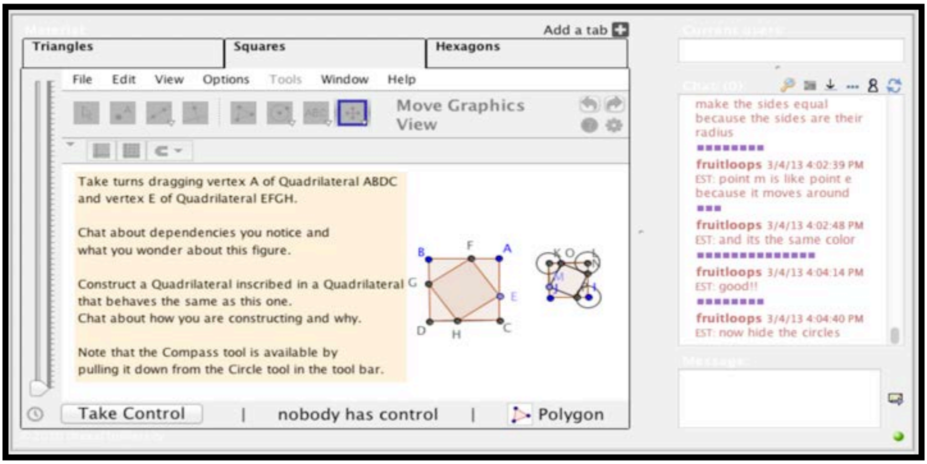


Figure 6: The interface of the collaboration environment, showing multi-user GeoGebra and text chat.

We use methods of interaction analysis or conversation analysis (Jordan & Henderson, 1995; Schegloff, 2007), adapted to our online math-education setting. This looks at how student groups engage in shared attention, joint representation and intersubjective meaning making. Although we recognize that processes at different levels are inextricably intertwined in reality, we focus methodologically on the group unit of analysis, which is where individual learning, group becoming and community practices are often most visibly displayed (Stahl, 2015c).

# Findings: Collaborative Learning of Dynamic Geometry Core Principles

In our case study for this presentation, three 14-year-old girls engaged in our environment for eight hour-long sessions (Stahl, 2015a). In their sixth session, they worked on the problem shown in Figure 1, constructing inscribed squares. They had previously solved the challenge of constructing inscribed triangles, but had never constructed a square. We follow their explorations, which led to an elegant construction of a square. They were then able quickly and collaboratively to construct the inscribed squares, based on their previous experience with inscribed triangles. They displayed their group and individual learning through their GeoGebra actions, text chat and building on each other. They explicitly discussed the need to construct various geometric dependencies to accomplish this task.

# Conclusions: Designing an Integration of Software, Curriculum and Practices

The Virtual Math Teams (VMT) Project (http://gerrystahl.net/vmt) at the Math Forum (www.MathForum.org) has been researching the integration of: an online collaboration environment, multi-user versions of GeoGebra, sequences of curricular units, data analysis methods and pedagogical approaches for over a decade. We now believe that a collaborative approach to dynamic geometry can support the learning of core components of mathematical cognition. Our approach integrates online software (for all browsers on computers, tablets, iPad), student-centered collaboration (with text chat), teacher orchestration of student teams and a carefully scripted sequence of curricular units (emphasizing exploration, reflection and group mathematical discourse). The curriculum is aligned with Common Core standards and focuses on the mathematical notion of dependency and techniques for constructing dependency in GeoGebra. Dependency is central to dynamic geometry, to deductive thinking and to student understanding of explanatory proofs. The dependencies constructed with GeoGebra tools—often following Euclid's procedures—result in figures with desired invariants. We have shown that even young students in groups can begin to understand, analyze, design and construct dynamic-geometric dependencies with GeoGebra.

The tablet version of multi-user GeoGebra with chat has just become available (vmtdev.mathforum.org). Teachers and groups of students can use it for free. A set of 50 GeoGebra activities (in a GeoGebraBook: http://ggbtu.be/b140867) introduces student teams to the role of geometric dependencies in exploring, articulating, creating and explaining dynamic-geometry figures and relationships—within a gaming-like context of sequenced challenges.

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# 9. Working Group: Developing Comprehensive Open-Source Geometry Curricula using GeoGebra

Gerry Stahl, Stephen Weimar, Annie Fetter, Anthony Mantoan

**Abstract**. Imagine combining the best characteristics of your favorite new geometry textbook with GeoGebraTube. It would cover all the material required for your ideal version of a full course on geometry, easily accessible and usable by teachers and students. However, it would also be free, flexible, up-to-date, easily revised and downloadable as needed by teachers and students. It would include activities tested in diverse classrooms, reviewed by teachers and flexibly adaptable to different languages, cultures or pedagogical preferences. Perhaps most importantly, it would take full advantage of GeoGebra for a dynamic, hands-on, visual, drag-able, constructible, personalizable exploratory-learning approach to geometry. The currently missing piece for moving GeoGebra into the center of contemporary mathematics education is the availability of comprehensive curriculum aligned with standards. GeoGebraTube provides a medium for shared resources, but requires coordinated efforts to develop model curricula and an interface for flexible, collaborative usage.

## Working Group Goal

The goal of this working group is to stimulate development of comprehensive geometry curriculum centered on student use of GeoGebra. This will support the use of GeoGebra by geometry teachers around the world by helping them to integrate student use of GeoGebra into their classroom activities, enhancing the pedagogy. This working group is only intended to start the process. Perhaps it will stimulate people thinking seriously and strategically about possible approaches and put them in contact to pursue next steps. Success with basic geometry could provide a model for other areas of mathematics.

# Problem Statement

While some new textbooks and the US Common Core standards recommend use of dynamic-geometry environments to "provide students with experimental and modeling tools that allow them to investigate geometric phenomena," they put the burden on the teacher of realizing this in the classroom. However, curriculum development and the construction of the corresponding well-designed GeoGebra files is a sophisticated and time-consuming task. Teachers have neither the time nor the resources to do this on their own for a whole course on geometry. They need well worked out curricula that they can choose from and adapt to their local needs.

Curricular items are currently made available through GeoGebraTube. However, that software does not support the assemblage of comprehensive, well-organized and easily adapted curricula. Nor does it support collaborative usage by student teams. GeoGebraBook can be a first step, but more is needed.

Curriculum in GeoGebraTube is currently unorganized; it is not systematic or comprehensive; it is not tied to progressive pedagogies. The consequence of this is a serious under-utilization of the potential of GeoGebra in typical classrooms. Without well-tested tutorials and curricula for important topics like construction, proof or custom-tool programming, teachers tend to fall back on using GeoGebra for fancy visualizations, and students use it to create pretty pictures. The power of dynamic geometry to stimulate mathematical thinking and cognitive development of students is barely touched.

# Working Group Focus

This working group will focus on enumerating the major issues and the main tasks that need to be addressed initially. The central question is how to support the integration of GeoGebra into geometry courses around the world. This includes approaches to both collaborative learning in small groups and individual learning. A particular opportunity of the Internet-based single-user and multi-user versions of GeoGebra is their use by online schools and for networking home-schooled students or students in countries with dispersed populations. Although intended to be useful for students world-wide, the curriculum might be aligned with the US Common Core standards as a framework. Although it is not necessary that GeoGebra be used for every aspect of school geometry or other math courses, the target curriculum should support a strategic, systematic approach to the aspects that it does address.

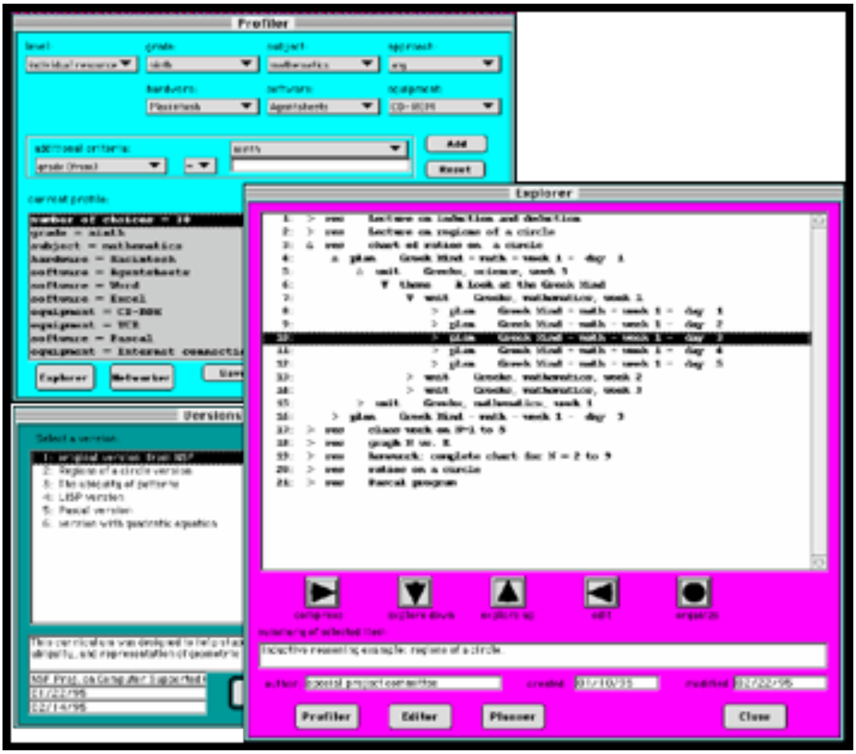
# Background Information

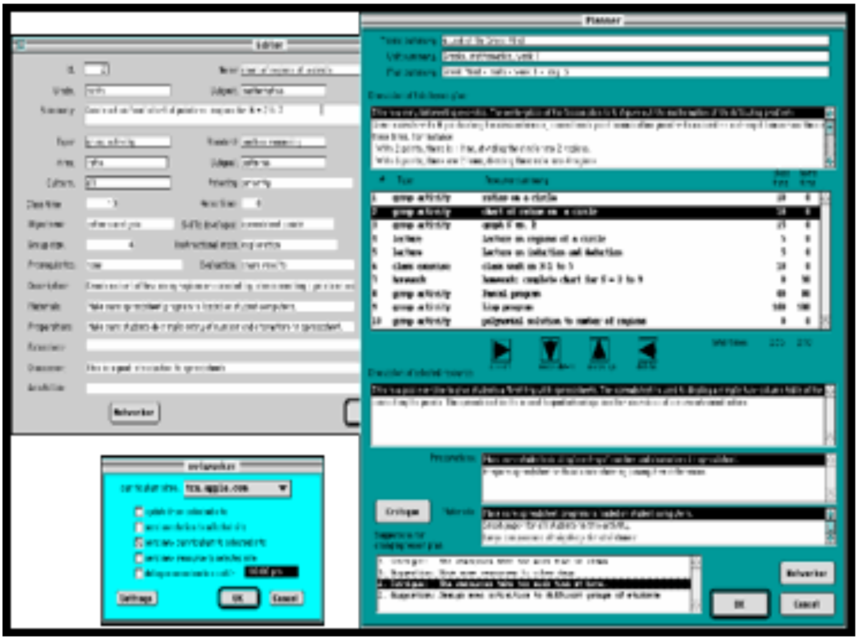
The Virtual Math Teams (VMT) Project (2003-2014) developed an online environment for collaborative dynamic geometry using a multi-user version of GeoGebra and text chat. (Stahl, 2006; 2009; 2013). It developed an associated mini-curriculum focused on collaborative learning of construction of geometric dependencies. This curriculum has been tested and revised each year. A version is now available as a GeoGebraBook (Stahl, 2015); it focuses on developing an understanding of how to construct geometric dependencies based on the beginning of Euclid's *Elements* and explores many notions recommended by the Common Core for middle school. This active book lets students work on 50 individual challenges in GeoGebra. Unfortunately, it is not multi-user, it is not persistent, there is no chat and it is not instrumented for researcher analysis, student learning analytics or teacher supervision.

An earlier project, the Teachers Curriculum Assistant (TCA) designed in 1994, explored the possibility of searching and browsing a database of curricular materials even before the Web existed (Stahl, 2006, Ch.1; Stahl, Sumner & Owen, 1995). It focused on five principles for a shared repository of constructivist educational resources, which could be applied to GeoGebraTube as follows:

1. Carefully structured summaries (meta-data) of the resources must be defined (when they are uploaded) and maintained, to support search. (GeoGebraTube begins to do this.)
2. The search process should be supported through a combination of query and browsing tools that help teachers explore what is available. (GeoGebraTube provides a simple search.)
3. Adaptation of tools and resources to teachers and students is critical for developing and benefiting from constructivist curriculum. (GeoGebraTube allows editing, but not versioning.)
4. Resources must be organized into carefully designed curricular units to provide developmental learning sequences. (GeoGebraTube has tags and Books, which are a start for this.)
5. GeoGebraTube should be a medium for sharing and combining curriculum ideas, not just accessing them. (In GeoGebraTube, "sharing" is just sending a link through social media.)

The following components of TCA were designed: a Profiler, Explorer and Versions (see first figure below) as well as a Planner, Editor and Networker (other figure). They are suggestive of useful functionality. These allow curriculum developers and teachers around the world to search for resources, try them, edit, annotate and store new versions. They also facilitate the aggregation and structuring of coherent sequences of curricular resources into course modules adapted to local needs of specific countries, schools, teachers or students.





# A MOOC Model for Collaborative GeoGebra

Massive Open Online Courses (MOOCs) and sites like Khan Academy provide useful educational resources, but they generally involve passive watching of video lectures, rather than engaged social learning. The VMT approach suggests a collaborative model, integrated with local classrooms and teachers. GeoGebra Institutes can provide teacher professional development in the proposed curricula. Then teachers can adapt the curriculum to integrate with their courses. Teachers organize small groups of their students to work collaboratively on GeoGebra curriculum, motivating each session in advance, then sharing group findings in whole-class discussions. The teachers guide the exploratory-learning trajectory and manage the grading (with automated support from the software). This overcomes the problems of MOOCS, takes advantage of large-scale resources and supports local mathematics education.

# Discussion Structure

The author team will begin by (a) motivating and illustrating the topic with the example of the VMT Project, its pedagogical approach to exploratory collaborative learning, and its sample GeoGebraBook curriculum. It will then (b) facilitate open discussion, starting with the questions and topics listed below. Finally, there will be (c) a wrap-up enumerating priorities, next steps and potential participants.

- How can comprehensive curriculum advance teaching and learning with GeoGebra?
- What new features should be designed into GeoGebraTube and GeoGebraBook to support meta-data, searching, browsing, adapting, annotating, reviewing, linking etc.?
- How should GeoGebra Institutes be involved? Should there be a form of MOOCs?
- Can curriculum be designed to support and assess both collaborative learning and individual learning?
- How can teachers be supported to adapt curricular units to their classrooms and how can they be involved in evolution of the materials?
- How can examples of teacher approaches, student work, assessment instruments, etc. be integrated into the materials?
- What resources are currently available and what further resources—such as research funding—should be sought?
- Who is interested in collaborating in further work on this?

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# Notes