Can Modal Structuralism be Adequately Reinterpreted in Extensional Leśniewskian Mereology?

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AQA Extended Project Qualification

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- Problems concerning (α):
  - Any definition of 'existence' excludes the traditional conception of the *'mathematical object'*:
    1. They do not consist in space;
    2. Nor consist in time;
    3. Nor belong to a causal chain.

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- *Conclusion:* Numbers are *not objects*.
- Arithmetical claims reference an *objectless* domain; they are thus reduced to *vacuity*, and arguably to *falsity*.

What is *Structuralism*?

- *"That a system of objects exhibits the structure of the integers implies that the elements of that system have some properties not dependent on structure. It must be possible to individuate those objects independently of the role they play in that structure. But this is precisely what cannot be done with the numbers." (Benacerraf, 1998)*

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- Mathematical objects are *places* or *positions* in relative *structures;* mathematics is thus the *"science of structures".*
- Accordingly, it is nonsensical to consider numbers individually, *i.e.,* without considering the *structure* wherein they consist
  - The real number 3, for example, is such by virtue of its *succeeding* 2 and *preceeding* 4...
  - ...it is substantial only as part of a *continuum.*

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- Truth in an *absolute* mathematico-logical sense is preserved

WHY INVOKE MODALITY?

- Modal Structuralism (MS) supplements the general framework of Structuralism with the primitive modality operator*:*
  - $\Box x$ *(it is necessary that x)*
  - $\Diamond x$ *(it is possible that x); i.e., it is not necessary that not x* $(\neg\Box\neg x)$

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- *'Ante Rem'* (*'universals before the thing'*) *Structuralism* postulates *extant* structures....
- ...we wish to avoid *existential assumptions* about the ontological status of such structures

LEŚNIEWSKI'S *Ontology*

- Axiomatises the most intuitive meaning of the English *'is'* (or the Latin *'est'*) – *symbollically, 'ε':*

$$\forall A \forall B \; [A \varepsilon B \iff \exists C \; (A \varepsilon C \wedge C \varepsilon B)]$$

for all objects *A* and *B*, *A is B* *if and only if* *for some C*, *A* is *C and C* is *B*

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-

Socrates ε Socrates

Socrates ε animal

Canis ε animal

Leśniewski's System: *Ontology*

- Existentially neutral; Ontology lacks the capacity to *derive a theorem* of the form

$$\exists B(B\varepsilon A).$$

- The quantifier '$\exists$' is not understood as *'there exists'* in Ontology.

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- The most fundamental example of a *structure* is that of the natural numbers: $0, 1, 2, 3, \ldots$
- ...but how is this structure determined? Which rules (*axioms*) govern its structure?

The *Hypothetical* Component

$$0 \varepsilon \mathbb{N}$$

'Zero is a natural number'

$$\forall A[A \varepsilon \mathbb{N} \Longrightarrow \mathbf{s}(A) \varepsilon \mathbb{N}]$$

'If A is a natural number, so is its successor'

$\vdots$

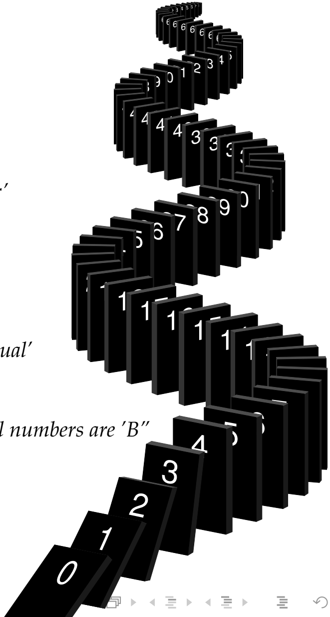
'No number is succeeded by zero'

$\vdots$

'Any numbers with identical successors are equal'

$\vdots$

'If zero is 'B' and the successor of any number is 'B', then all numbers are 'B''



THE *Hypothetical* COMPONENT

- The sequence of all natural numbers ordered by the successor relation (an *omega-sequence*) is thus the most basic structure:

All natural numbers $\{0, 1, 2, 3, \dots\}$

$$\Omega \; := \; \langle \mathbb{N}, \; \mathbf{s} \rangle$$

The *successor* relation (*i.e.* $1 = \mathbf{s}(0)$)

THE *Hypothetical* COMPONENT

- Translation scheme:

$$\boxed{\Box}\ \forall A\ [\ \boxed{A \varepsilon \Omega} \implies \boxed{(A \vdash S)}\ ]$$

Necessarily, If A were any *omega-sequence,* S would *hold* in A.

THE *Hypothetical* COMPONENT

$$'2+3=5'$$

$$\Box\forall A[A\varepsilon\Omega \Longrightarrow (A \vdash S)]$$

'*Necessarily*, in any *possible* natural number system *(omega-sequence)* 'A',
any object in the 2-place of A that is A-added to the object in the 3-place of A
is the object in the 5-place of A'.

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- *Possible existence claims* form the 'categorical component' of MS
- It is reasonable to assert:

$$\Diamond\exists A(A\varepsilon\Omega)$$ ✓

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- Not:

$$\exists A\Diamond(A\varepsilon\Omega)$$ ✗

'There exists A such that A is possibly an omega-sequence'

The *Categorical* Component

$$\begin{gathered}\Diamond\infty \\ \Big\Downarrow \textit{Modality, Assumption } (\Diamond\infty) \\ \Diamond\exists A(A\varepsilon\Omega)\end{gathered}$$

- Affirms the logical possibility of a progression (this is, of course, compatible with the absence thereof)

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PROBLEMS WITH MODAL STRUCTURALISM

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  - The notion of *successor* is ambiguous (1)
  - The Peano axioms violate the *'free logicism'* of Leśniewski's Ontology (2)

Extensional Mereology: RESOLUTION OF (1)

- Based upon the principles of Ontology

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- Based upon the principles of Ontology
- Characterises the relation of *'part'* to *'whole'*
- Axiom of Atomicity:

$$\underbrace{\forall A}_{\textit{for all } A,} \; [\; \underbrace{A \varepsilon atom}_{A \text{ is an } \textit{atom}} \; \underbrace{\Longleftrightarrow}_{\textit{if and only if}} \; (\; \underbrace{A \varepsilon A}_{\textit{A is A}} \; \underbrace{\wedge \forall B}_{\textit{and for all A}} \; \underbrace{\{\neg[B \varepsilon part(A)]\}}_{\textit{B is not a part of A}} \;)]$$

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- Thus, for brevity, we can introduce as primitive a relation $O(x, y, z)$, meaning *'z is atom correlated with the ordered pair $\langle x, y\rangle$'.*

Extensional Mereology: RESOLUTION OF (2)

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- Instead of asserting as axioms, we can derive as theses the rules for arithmetic.

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- ...Modal Structuralism is adequately reinterpretable with respect to philosophical considerations

ANY QUESTIONS?