CAN MODAL STRUCTURALISM BE ADEQUATELY REINTERPRETED IN EXTENSIONAL LEŚNIEWSKIAN MEREOLOGY?

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Part I

1 INTRODUCTION

[. . .] in a particular substance there is nothing substantial except the particular form, the particular matter, or the composite of the two. [. . .] And, therefore, every essence and quiddity and whatever belongs to substance, if it is really outside the soul, is just matter, form, or the composite of these or, following the doctrine of the Peripatetics, a seperated and immaterial substance.

– William of Ockham, *Summa Logicae* [58], *p. 97*.

A curious thing about the ontological problem is its simplicity. It can be put in three Anglo-Saxon monosyllables: 'What is there?'

– Willard van Orman Quine, *On What There Is* [62], *p. 1*.

1.1 THE THEORY OF *structure*

Structuralism of the mathematical concern forms a type of foundational interruption of the metaphysical theses birthing modern mathematical practice. Its thesis is one of holism; a radical countenace to the traditional platonist conception of ontological independence, in principle a rejection of 'individuation as primary subject-matter'. Structuralism's roots are found in the intuitions of those who extolled the virtues of the axiomatic method – that is, amongst the works of Hilbert, the schematic spirit of the *Bourbaki*, and, most overtly, in Dedekind. One finds, however, confliction and hostility between the various ways in which structuralism may be explicated: thus, whilst no precise, immutable set of positions maintained by *'the'* structuralist can be offered, one revolutionary perspective is mutual; focus is shifted from the *one* to the *many.*

Mathematicalia, on this view, serve thus to mark *positions* within a greater relational system; we might imagine, for instance, the absurdity of positing a single natural number, isolated from the continuum on which its very identity ostensibly depends. This shift of perspective, whilst absolving of some fundamental ontological issues, clearly does not offer a full dissolution of the pernicious dilemma we seek to repair; rather, we merely extend the problem of abstraction from objects themselves to multiplicities thereof.

1.2 LEŚNIEWSKI'S *protothetic* AND *ontology*

We speak, irrespective of *actual* spatiotemporal unity, of the *spatial* as a seperate *conceptual* component (if one is a presentist, a Putnamian relativist or a McTaggartian irrealist[1], *space* as represented through a formal axiomatic system remains the same conceptually).[2] The spatial 'component' demands the nominalist to be also a substantivalist[3], rather than a relationist (reductive[4] or eliminative[5]); as argued by Field, the nominalist-substantivalist is granted justification in *quantification* over *regions of space,* acting as *eo ipso* entities:

1 The irrealist in this McTaggartian sense may go as far as to deny the 'reality' of time, a position noteworthily defended by Dummett [20] and, more recently, by Oaklander [57]. See McTaggart [55], *p. 456:* 'It doubtless seems highly paradoxical to assert that Time is unreal, and that all statements which involve its reality are erroneous. Such an assertion involves a far greater departure from the natural position of mankind than is involved in the assertion of the unreality of Space or of the unreality of Matter. So decisive a breach with that natural position is not to be lightly accepted. And yet in all ages the belief in the unreality of time has proved singularly attractive.'

2 An 'isolation' of the constituents of space-time is perhaps indefensible. Thus, it must be understood that *conceptual* isolation is not a technical 'splitting' of the fabric of Einsteinian space-time; the semantic use of 'space' as 'apart' from time is 'conceptual' in the sense that no one is purporting that space does not *exist* (in any sense of the word) *in the present moment* (at least certainly not if one is to assert that space is, indeed, real).

3 In its most unembellished form, '[space-time substantivalism is the] claim that spacetime enjoys an existence analogous to that of material bodies' (Belot [3], *p. 3*).

4 '[. . .] the doctrine that space or spacetime has no independent existence independent of matter' (Saunders [71], *p. 152*).

5 'Relationism is usually taken to be a reductionist account of space, that space is *nothing more* than the system of spatial relations between actual or possible distributions of matter' (*Ibid., p. 152*).

'It is clear that reductive relationism is unavailable to the nominalist: for according to that form of relationism, points and regions of space-time are mathematical entities, and hence entities that the nominalist has to reject. So a nominalist must either be a substantivalist or be an eliminative relationist, and only in the first case can he find Hilbert's theory acceptable' (Field [23], *p. 34*).[6]

This consideration of 'quantification' is a crucial one, for it is therefrom that the very problem of nominalism in a more 'sophisticated' sense arises. The fundamental (and rather platitudinous) presupposition of the nominalist is that, if anything at all exists, it is certainly *concrete*, and that which is 'quantifiable' is similarly certainly concrete. It is this concreta, suitably, that is therefore 'quantified' over – *e.g.*, for some concretum x, the first-order statement '$\exists x$' is a relation of *ontological commitment* postulating x to be, indeed, extant.

Within this context, the existential quantifier is bestowed with greater ontological substance than perhaps traditionally conceived. If we consider, for example, the sentence $\exists x$(x does not exist) – considered by Quine to be 'existential' inasmuch as $\exists x$ – the instrinsic ontological vacuousness is apparent, although the form is wholly logical. Thus, it is not only in the nature of $\exists$ but in the nature of x, or any such bound variable to be *quantified over,* that the very foundations of a nominalist ontology come forth.

A similar sentiment is evident in the writings of Leśniewski, who, in demarcating a shift from speculative metaphysical ambiguity to the rigorous constructivity of formal axiomatic systems, considered ontology as solely explanatory of *individual being* in its most fundamental of senses:

'[...] Ontology forms a certain kind of modernized 'traditional logic', and which in content and 'power' most nearly approaches the Schröder 'Klassenkalkül', considered as including the theory of 'individuals" (Leśniewski [43], *p. 176*).

The *reistic* – or, more explanatorily, the *concretistic* – thesis that arises thus is that the totality of the *extant* is the totality of *things*; all that is *existent* is composed in entirety of *realia* or *concreta.* Interpreted in an appropriate first-order functional calculus, this view is easily misinterpreted as the claim that existence is composed of 'the totality of the values of the bound variables of the existential quantifier', or some similar iteration of that sentiment; in the words of Quine, to *be is to be the value of a bound variable,* or more explicitly, *the value of a variable bound to the existential quantifier.* However, Leśniewskian nominalism, and the manner in which his *Ontology*[7] is constructed, must not be understood strictly in this sense.

The system upon which Ontology itself is constructed is incontrovertibly important, for it is the logical rules demarcated by such a system – namely, *Protothetic* – that governs the manner in which Ontology (and any subsequent logical extension) addresses existential inquiries. Protothetic is often characterised as a Leśniewskian counterpart to the propositional calculus, an indefinitely extensible logic of propositions based on the primitive coimplicative constant wherein quantifiers bind variables of *all semantic categories*.

A number of these defining characteristics are worthy of examination. On the inclination for Protothetic to be built upon a sole primitive constant (that is, the functor of coimplication, symbolically represented henceforth as $\iff$[8]), we observe the following from Leśniewski [48], an excerpt especially illustrative of Leśniewski's endeavour for succinctness in axiomatisation:

'In 1921 I realised that a system of the theory of deduction containing definitions [...] would actually be constructed from a single term only if the definitions were written [...] down with just that primitive term and without recourse to a special equal-sign for [...] definitions' (Leśniewski [48], *p. 143*).

6 *Ibid., p. 26:* '[...] the claim that physical space is Euclidean is translated into the claim that each of the spatial slices of space-time is Euclidean. It is trivial to rewrite Hilbert's axiomatization of the geometry of space so that that is explicitly what it says; if we do so, then the objects in the domain of the quantifier are really space-time points rather than points of space, and there can be no danger of viewing the theory as being committed to the idea that absolute rest is a physically significant notion.'

7 I write the three systems composing Leśsniewski's holistic system with capital letters (*viz. Protothetic, Ontology* and *Mereology*), due partly to historical convention but primarily due to the polysemous manner in which the term *ontology* may be interpreted.

8 Leśniewski's preferred notation for *coimplication* is '$\equiv$'.

The doctoral work of Tarski [87] provided the developments necessary to satisfy precisely this *desideratum;* for a propositional calculus wherein all extensional propositional connectives are definable by virtue of its primitive symbols, the coimplicative sign functions capaciously as the sole sentential connective, provided the admission of the *universal quantifier* binding the variables of the calculus. To further examine the capacity of this *universal quantifier,* it is useful to examine Leśniewski's original axioms for the system of Protothetic; one must become acquainted with the structure of Leśniewski's propositions in order to appreciate the manner in which Protothetical quantifiers operate. Fundamentally a type theory whose lowest type is that of *propositions* – p, q, $r, \ldots$ – Protothetic, on a syntactical level, is characterised in particular by the following metarules, which we observe from Leśniewski [44]:[9]

$$\forall p \forall q \forall r\{[(p \iff r) \iff (q \iff p)] \iff (r \iff p)\}$$
$$\forall p \forall q \forall r\{[p \iff (q \iff r)] \iff [(p \iff q) \iff r]\}$$

It is illustrated, therefore, that the coimplication (i.e., *material equivalence*) of universally bound propositions is in turn coimplicative of the bivalence between similarly formed expressions. Using *unary, binary, tertiary, ...* functors, we combine expressions to form further, new propositions:

$$f(p),\ g(p,q),\ h(p,q,r),\ \ldots$$

And develop thus the metarule

$$\forall g \forall p\{\forall f[g(p,p) \iff (\forall r[f(r,r) \iff g(p,p)] \\ \iff \forall r\{f(r,r) \iff g[p \iff \forall q(q,p)]\}) \iff \forall q[g(p,p)]]\}.$$

We reduce the logic of coimplication, as detailed by the previous three metarules, to a single axiom for brevity. Whilst a number of single axiom forms of the Protothetical system were utilised by Leśniewski, the following is the shortest such form (as observed in Leśniewski [44]), reconstructed in the local logistic and including coimplication as the sole undefined term[10]:

AX. 1:1: $\forall p \forall q \Big((p \iff q)$ *Axiom of Protothetic*

9 The local notation, which we use throughout, is rather dissimilar (and considerably more intuitive) to that employed by Leśniewski. We observe the following correlates, in the *genuine logistic,* for the above given metarules:

$$\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi(\phi(pr)\phi(qp))\phi(rq)\Big)\urcorner$$
$$\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi(p\phi(pr))\phi(\phi(pq)r)\Big)\urcorner$$
$$\llcorner gp \lrcorner \ulcorner \phi\left(\llcorner f \lrcorner \ulcorner \phi\left(g(pp)\phi\left(\llcorner r \lrcorner \ulcorner \phi(f(rr)g(pp))\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)g(\phi(p\llcorner q \lrcorner \ulcorner q \urcorner)p)\Big)\urcorner\right)\right)\urcorner \llcorner q \lrcorner \ulcorner g(qp)\urcorner\right)\urcorner.$$

Syntactical similarity is more immediately observable when we consider the (respective) reconstruction of the above in the (historically) later Peano-Russellian notation:

$$[pqr]. \cdot .p \equiv r. \equiv .q \equiv p :\equiv .r \equiv q$$
$$[pqr]. \cdot .p \equiv .q \equiv r :\equiv: p \equiv q. \equiv r$$
$$[gp] : \cdot : [f] :: g(pp). \equiv . \cdot .[r] : f(rr). \equiv .g(pp) :\equiv: [r] : f(rr). \equiv .g(p \equiv .[q].qp) ::\equiv .[q].g(q,p).$$

10 It is known, however, that the Protothetic system $\mathfrak{S}_5$ was developed earlier in 1923; *cf.* Srzednicki and Stachniak [85], the Editor's Foreword. The original explication of the system $\mathfrak{S}_5$ takes the form:
$[fp] :: f([pq] : p \equiv q. \equiv .q \equiv p,p). \equiv . \cdot .f([hs] : \cdot : h([pqr]. \cdot .p \equiv r. \equiv .q \equiv p :\equiv .r \equiv q,s). \equiv:: h([kt] : \cdot : k([pqr]. \cdot .p \equiv .q \equiv r :\equiv: p \equiv q. \equiv r,t). \equiv:: k([gp] : \cdot : [f] :: g(pp). \equiv . \cdot .[r] : f(rr). \equiv .g(pp) :\equiv: [r] : f(rr). \equiv .g(p \equiv .[q].q,p) ::\equiv .[g].g(qp),t). \equiv . \cdot .[pqr]. \cdot .p \equiv .q \equiv r :\equiv: p \equiv q. \equiv r,s). \equiv . \cdot .[pqr]. \cdot .p \equiv r. \equiv .q \equiv p :\equiv .r \equiv q,p). \equiv: [pq] : p \equiv q. \equiv .q \equiv p.$
Given in the *genuine logistic*:

$$\iff \{\forall f[f(pf\{p[\forall u(u)]\})$$
$$\iff \forall r[f(qr) \iff (q \iff p)]]\}\Big)$$

Fundamentally, the propositional type theory that springs forth from the logical conditions parametrised in AX. 1:1 – the system we may refer to as $\mathfrak{S}5$, for historical considerations – is a system contingent on the necessary criteria for coimplication between universally bound propositions.

The earlier established importance of nominalism is especially consonant with Leśniewski's system thus far, even at such a foundational, primitive level. Notably, we omit an extensive and highly elaborate set of *rule statements* – which Leśniewski termed *Terminological Explanations* (*Terminologische Erklarungen*) – formed for the express intent of defining terms that *can* appear in logical formulae and subsequently *describing* formulae that may be added to the system as theses at any developmental stage. The forty-nine Protothetical Terminological Explanations are distinctly nominalistic insofar as they do not lay ground for an *ideal system*, but rather govern a conceivable infinitude of systems. Such an 'ideal system' is perhaps abstractly analagous to the haecceitous (uniquely individual) mathematical object instantiated in physical quantities; the nominalist certainly does not posit there to be *the* 'number one', but as many as are 'produced' in some ontological manner[11]. It is in precisely this sense that Protothetical systems, though convenient to refer to solitarily (*i.e.*, as '*the* Protothetic'), are not one but many – Leśniewski's Terminological Explanations concern concrete *tokens* as opposed to abstract *types*. The *sameness* of systems, therefore, arises as a consequence of formal similarity – ideals serve as convenient fictions.

According to Sobociński, the system $\mathfrak{S}5$ '[. . .] has only one rule of procedure divided into five points' (*cf.* Sobociński [81], *p. 57*). For brevity, we consider only those with direct or appreciable influence on the subsequent systems hierarchically dependent on Protothetic.

Foremostly, the *semantic category* – an implicit suggestion of *grammatical distinction* – of each non-quantificational or non-parenthetical term in a well-formed expression is determinate; that is, names take the place of M or N in the expression 'M is N':

Canis *est* animal
Socrates *est* coniunx Xanthippe
Chimera *est* Chimera

We denote semantic *sentences* by the symbol s, and similarly, semantic *names* by the symbol n. Thus, propositional variables and constants take the form s, and name variables and constants the form n. We observe the following exemplification, explicating the semantic categories of the multitude of terms in classical propositional logic:[12]

$$\llcorner fpqrst\lrcorner\ulcorner\phi\left(\phi(pq)\llcorner g\lrcorner\ulcorner\phi\left(f(pf(p\llcorner u\lrcorner\ulcorner u\urcorner))\phi\left(\llcorner u\lrcorner\ulcorner f(qu)\urcorner\phi\left(g\left(\phi(\phi(rs)t)q\right)g\left(\phi(\phi(st)r)p\right)\right)\right)\right)\urcorner\right)\urcorner$$

More concisely expressible due to Sobociński 1945:

$$\llcorner pq\lrcorner\ulcorner\phi(\phi(pq)\llcorner f\lrcorner\ulcorner\phi(f(pf(p\llcorner u\lrcorner\ulcorner u\urcorner))\llcorner r\lrcorner\ulcorner\phi(f(qr)\phi(qp))\urcorner)\urcorner)\urcorner$$

And further simplified in the Peano-Russellian notation (*cf.* Sobociński [78]):

$$[pq] :: p \equiv q. \equiv .\cdot.[f].\cdot.f(pf(p[u].u)). \equiv: [r] : f(qr). \equiv .q \equiv p.$$

11 Modal Structuralism in particular – as shall come to be shown – is certainly exemplary of this. One may infer from the principles of Structuralism, as thus vaguely outlined, the manner in which a multiplicity of ostensibly haecceitous mathematicalia may consist in reality.

12 It is inadmissible to have a functor belonging to the same semantic category as one of its arguments. We can imagine, for instance, the situation $\overbrace{q}^{s}\overbrace{\vee}^{s}\overbrace{r}^{s}$ – the index of the disjunction (which is definable in primitive Protothetical terms; see 4.1) is by necessity equal to the indices of its arguments. However, the entirety of the indexical fraction would have to be equal to a proper part of the fraction.

$$\underbrace{\overbrace{\neg}^{\frac{s}{s}}\underbrace{[\overbrace{p}^{s}\overbrace{\Longrightarrow}^{\frac{s}{ss}}\underbrace{(\overbrace{q}^{s}\overbrace{\vee}^{\frac{s}{ss}}\overbrace{r}^{s})}_{s}]}_{s}}_{s}$$

Universal and existential quantifiers – ubiquities in formal logic – are understood fairly heterogenously. It would not be disingenous to allow the following excerpt to summarise precisely this commonplace manner of understanding:

> '[We] can dispense with amost the entire category of indefinite singular terms. To begin with, the need for a distinction between 'any' and 'each' or 'every' is already removed by our recourse to 'such that' [...]. The essential forms of indefinite singular terms reduce thus to 'every *F*' and 'some *F*' (in the sense of 'a certain *F*'), where '*F*' stands for any general term in substantival form. But now for the striking economy; these two classes of indefinite singular terms are in turn dispensible in favor of just the two indefinite singular terms 'everything' and 'something'. [...] Usual notations for these respective purposes are '(x)' and '$(\exists x)$', conveniently read 'everything is x such that' and 'something is x such that'. These prefixes are know, for unobvious buth traceable reasons, as *quantifiers,* universal and existential' (Quine [64], *pp. 161–162*).

Perhaps unsurprisingly, Leśniewski's utilisation of the quantifier deviates somewhat considerably from the traditional Quinean conception. In Protothetic, quantifiers are solely universal, or of the form $\llcorner pq\ldots\lrcorner$; as observed, for example, in the expression $\llcorner pqr\lrcorner\ulcorner\phi\Big(\phi(\phi(pr)\phi(qp))\phi(rq)\Big)$. As an immediate consequence, and as clarification of an earlier discussed point, Protothetic has no 'genuine' existential quantifier; $[\exists p].f(p)$ abbreviates $\sim[p].\sim.f(p)$[13]. Thus, Protothetical quantifiers must not be understood in the convential sense; somewhat counter-intuitively, $\forall x$ is not read as *'every x is such that',* nor ∃x as *'some x is such that'*.

There remains thus a degree of vacuity to the existential quantifier that must be eschewed should one take seriously nominalistic claims of *existence*; *what precisely is the significance the quantifier ascribes to the variable it binds?* The methodological approach we maintain to be most ontologically tenable – that is, a generalised system of what there *is,* and, to some extent, what there *is not* – may perhaps be philosophically strengthened with, in finality, the introduction of a functor demarcating existence in the highest order. Such an introduction seems no less than a necessity if one is to resolve ostensibly paradoxical expressions, *viz.,* the Quinean '$\exists x$(x does not exist)' – as Leśniewski's Protothetic suggests, the universal and existential quantifiers as traditionally conceived do not provide an adequate basis for translating logical reasoning about *existence* as a metaphysically serious investigation of *being.*

The introduction of such a functor is by no means novel, for it was Leśniewski who developed a general theory wherein precisely this concern is addressed – and subsequently resolved

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(This is made explicit by Le Blanc [37].) Thus, by enforcing the restrictions of semantic categories, one avoids contradictions reminiscient of the *vicious circle* paradoxes described by Russell and Whitehead in the *Principia:*

> 'The principle which enables us to avoid illegitimate totalities may be stated as follows: "Whatever involves *all* of a collection must not be one of the collection"; or, conversely; "If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total." We call this the "vicious-circle principle," because it enables us to avoid the vicious circles involved in the assumption of illegitimate totalities' (Russell and Whitehead [68], *p. 37*).
>
> 'In the cases of names and definitions, the paradoxes result from considering non-nameability and indefinibility as elements in names and definitions. [...] In each contradiction something is said about *all* cases of some kind, and from what is said a new case seems to be generated, which both is and is not of the same kind as the cases of which *all* were concerned in what was said' (Russell and Whitehead [68], *p. 62*).

13 Quine makes note of this 'economical' abbreviation – *cf.* Quine [64], *p. 162:* 'A small further economy is still possible: of our two surviving indefinite singular terms, 'everything and 'something', only one is needed. In other words, existential quantifiers can be paraphrased with help of universal ones and vice versa, as is well known: '$(\exists x)(\ldots x\ldots)$' becomes 'not$(x)not(\ldots x\ldots)$' and conversely.'

– concomitantly. In order to absolve ontology – as conceived as a science of what *is,* an inquiry into the nature of *being,* the Aristotelian 'first philosophy' – of existential ambiguity, Leśniewski supplemented his Protothetic with the system aptly entitled *Ontology.* It is not without the introduction of an additional primitive, akin to the coimplicative sentential function as previously surveyed, that one can begin to interrogate the nature of *existence* with the austere predilection for rigour and formality at once observed in Leśniewski's Protothetic.

The *epsilon connective* decidely assumes this role, functioning as an adjudicator of *what 'is'* in the most literal of senses. Symbolically, we denote the epsilon connective as ε; phonetically, the copula is read simply as *'is'*. Most elementarily, we accordingly construct expressions of the form $A\varepsilon B$; however, an inexhaustive list of conditions governing the manner in which one can validly employ the epsilon connective must be respected.

Foremostly, one must respect the intended individuality of the argument A in $A\varepsilon B$; the expression $A\varepsilon B$ is taken to be well-formed *if and only if* A is a singular term naming an object which is amongst the objects B. Therefore, *'Socrates ε a man'* is well-formed on the basis that Socrates is amongst those who are men; likewise, the expression *'Socrates ε Socrates'* is well-formed, but satisfied by only one unshared name, namely that of Socrates. Thus, well-formed sentences of the form $A\varepsilon B$ where A is non-individual are rendered invalid, regardless of their syntactical correctness. To return to the sentences exemplary of the Protothetical semantic categories, we observe that *'Socrates ε coniunx Xanthippe'* and *'Chimera ε chimera'* serve as intended interpretations of the Ontological primitive; *'Canis ε animal'*, whilst well-formed, violates the necessity for individuality (requiring that a singular term take the position of A in the expression $A\varepsilon B$) by fault of its generality. Accordingly, ε is of semantic category $\frac{s}{nn}$, taking two *name* arguments and forming a sentential expression.

The sentence *'Chimera ε chimera'*, realised as well-formed, at once illustrates an odd capacity of Leśniewski's Ontology. The singular term A in the well-formed expression *B does not require a referent;* as concisely summarised by Sagal [70], '*Names* need not name anything'[14]. Prior [60] provides exposition on this peculiarity:

> 'Principia Mathematica contains a theorem, namely ∗24.52, which asserts that the universal class is not empty, that is, that there is at least one individual. And this is a theorem which Russell found an embarassment – [. . .] he describes it as "a defect in logical purity". In Leśniewski's ontology this defect, if it is one, doesn't exist – ontology is compatible with an empty universe. What is puzzling is the explanation which is commonly given of this achievement. The lowest-type variables of ontology are described, like Russell's lowest-type variables, as standing for *names;* but it is said that whereas Russell's variables stand for singular names only, Leśniewski's stand equally for empty names, singular names and plural names' (Prior [60], *p. 149*).

The *Principia* indeed assumes that its lowest-type variables have singular names – *names with only one referent* – as substituends; it is thus a natural consequence that elementary quantification in the *Principia* entails *assumptions of existence.* Conversely, Ontology – a pure *calculus of names* – is essentially reflective of Leśniewski's *Weltanschauung* in its existentially non-discriminatory formal procedure; no provision is made for the terminological designation of abstracta, and consequently, one may adequately reason about such abstracta in full concordance with Leśniewski's system. In formulating a general science of *being* – a "theory of what there is, or general principles of being" – one must resist temptation to regard the *quantified variable* as explicitly representative of concreta, precisely in order to circumnavigate the prejudice illustrated by the Heideggerian dictum *'definitio fit per genus proximum et differentiam specificam'*. It would be little more than a bias to assert axiomatically that only *concrete objects* or *things* are *existent* in any sense, and thus, whilst Leśniewski's philosophy is stringently nominalistic, the formalisation of his Ontology reflects a respect for the eschewal of preconceived notions of *existence* as a *discriminatorily prescriptive* predicational property. Devoid of existential assumptions, Ontology retains the spirit of Lambertian *free logic,* in a manner not dissimilar to Meinong's *Gegendstandstheorie* or Twardowski's *intentional object theory.*[15]

Therefore, the introduction of the existential quantifier in Ontology is one of pure informality, a mere symbolic convenience; quantifiers in Leśniewski's system do not carry ontological commitment. Whilst equitable with concern to existents, Ontology is not, of course, free from

14 *cf.* Sagal [70], *p. 260.*

15 Łukasiewicz describes Leśniewski's Ontology as a *'daseinsfreie Wissenschaft':* 'If we assume that there is no object, then both sides of Leśniewski's axiom are false, and, hence, the axiom by virtue of the meaning of the equivalence connector is true. Hence, the theorems of Ontology are true in the empty domain [. . .]. So Leśniewskian Ontology is a *'daseinsfreie Wissenschaft'* [. . .]' (Łukasiewicz [53], *p. 22*).

existential import – one need only consider the expression $A\varepsilon B$ to observe that its very truth value rests on the existence of A. One cannot, by the lucidity of intuition, claim that A is *something* if there is not an A to exemplify such a quality.

Appreciating the existential quantifier as a device of expression and the primitive epsilon copula as the sole Ontologically existential term, we proceed to define *existence* in a fairly intuitive manner:

$$\forall A[ex(A) \iff \exists B(B\varepsilon A)].$$

Evidently, the functor '*ex*' is indeed predicational, and intuitively significant of the property *canis* has, *Socrates* had and *chimera* lacks; but such a property is *ontological* rather than *descriptive*, and thus Ontology lacks the capacity to derive as *theorem* a statement asserting or refuting the existence of any such A regardless of the ostensible ontological status of its referent. A theorem of existence may indeed take the rudimentary form of the latter conjunct of the above coimplicate – $\exists B(B\varepsilon A)$ – but any perceived innocuity fails to affect the non-provability of existential formulae in Ontology. We see this 'peculiar characteristic', nevertheless, as virtuous.[16]

In precisely the manner exemplified by Protothetic, Ontology is axiomatisable by means of a single postulate, invoking the epsilon connective as its sole undefined term. Leśniewski provides an informal exposition of his Ontology as follows:

> 'It so happened that as a result of semantic analysis to which I subjected various categories of propositions, and in connection with my considerations on the possibility of 'reducing' [...] some of the types of propositions into others [...], 'singular' propositions of the type '*Aεb*', and the mutual relations between such propositions, temporarily became the central point of my interest in 1920. [...] I began to use the 'symbolic' equivalent of the thesis
>
> *A* is *a* if and only if ((for some *B*–(*B* is *A*)), (for all *B* and *C*–, if *B* is *A* and *C* is *A* then *B* is *C*) and (for all *B*–, if *B* is *A* then *B* is *a*)) [...]
>
> As the sole axiom of the deductive theory I have invented. This axiom has proved sufficient for the achievement of all theoretical results that I hoped to obtain through the axiomatic basis of the theory in question. Since I needed a name for the theory under construction, I have decided to use the word 'ontology'.' (Leśniewski [47], *pp. 156–159*).

Thus – as explicitly formalised by Leśniewski [45] – the *verbatim* symbolisation of the above thesis takes the following Peano-Russellian form:[17]

$$[Aa] :: A\varepsilon a. \equiv .\cdot.[\exists B].B\varepsilon A.\cdot.[BC] : B\varepsilon A.C\varepsilon A. \supset .B\varepsilon C.\cdot.[B] : B\varepsilon A. \supset .B\varepsilon a.$$

16 A sentiment shared by Kearns [33], *p. 77:* 'In systems containing both free and bound variables, such a formula [in reference to the formula $(\exists A)(A\varepsilon\phi)$] is usually provable, because of an axiom of substitution for bound variables. But this cannot be done in Ontology, which is surely a virtue of Ontology.'

17 As Leśniewski notes in *O podstawach ontologji: Über die Grundlagen der Ontologie,* the expression '$\sim ((B). \sim (\varepsilon\{BA\}))$' serves as a formal expansion of the informal abbreviation '$(\exists B).\varepsilon\{BA\}$' (*cf.* Leśniewski [45], *p. 114*). Thus, Leśniewski gives the following:

$$(A,a) :: \varepsilon\{Aa\}. \equiv .\cdot. \sim \big((B). \sim (\varepsilon\{BA\})\big).\cdot. \\ (B,C) :\varepsilon\{BA\}.\varepsilon\{CA\}. \supset .\varepsilon\{BC\}.\cdot.(B) : \varepsilon\{BA\}. \supset .\varepsilon\{Ba\}.$$

Whilst Leśniewski [41] provides the Peano-Russellian form of the Ontological axiom, the axiom in Leśniewski's authentic logistic would take the form

$$\llcorner Aa \lrcorner \ulcorner \text{ϕ} \left(\varepsilon\{Aa\} \text{ϙ} \left(\vdash (\llcorner B \lrcorner \ulcorner \vdash (\varepsilon\{BA\})\urcorner) \llcorner BC \lrcorner \ulcorner \text{ϕ-}(\text{ϙ}(\varepsilon\{BA\}\varepsilon\{CA\})\varepsilon\{BC\})\urcorner \llcorner B \lrcorner \ulcorner \text{ϕ-}(\varepsilon\{BA\}\varepsilon\{Ba\})\urcorner \right) \right) \urcorner.$$

Whilst adequate in its present form, Leśniewski's perpetual eagerness for brevity produced the following shorter axiom a year later in 1921[18]:

$$[Aa] :: A\varepsilon a. \equiv .\cdot.[\exists B].B\varepsilon A.B\varepsilon a.\cdot.[BC] : B\varepsilon A.C\varepsilon A. \supset .B\varepsilon C.$$

Historically, considerable reduction has been done to the sole Ontological axiom. The most radical of such reductions was accomplished by Sobociński, who, by the following intermediary theses –

$$\begin{aligned} [Aa] &: A\varepsilon a. \supset .[\exists B]B\varepsilon A \\ [AaB] &: A\varepsilon a.B\varepsilon A. \supset .B\varepsilon a \\ [AaBC] &: A\varepsilon a.B\varepsilon A.C\varepsilon A \supset .B\varepsilon C \\ [AaB] &:: B\varepsilon A.\cdot.[C] : C\varepsilon A. \supset .C\varepsilon a.\cdot.[CD] : C\varepsilon A.D\varepsilon A. \supset .C\varepsilon D.\cdot. \supset .A\varepsilon a \end{aligned}$$

– has shown the earlier single form of the axiom to be inferentially equivalent *(the full disposition is schematically detailed in 1)* to the intuitive and felicitously succinct expression:

$$[Aa] : A\varepsilon a. \equiv .[\exists B].A\varepsilon B.B\varepsilon a.$$

We introduce the above axiom, again in the local logistic, into the formal system:

AX. 1:2: $\forall A \forall a[A\varepsilon a \iff \exists B(A\varepsilon B \wedge B\varepsilon a)]$ *Axiom of Ontology*

The sole Ontological axiom is read intuitively as *'for all A and a, A is a if and only if for some B, A is B and B is a'*.

Leśniewski's analysis of Twardowski's theory of *general objects* is arguably the catalyst of his nominalism[19]. Equipped with the rather concise single axiom of Ontology as explicated above, one can formalise an argument with substantial (and, naturally, philosophically favourable) consequences for nominalism, due largely to Waragai [90], Luschei [54][20], Gryganiec [27] and Urbaniak [89][21]. The *general object,* as understood by Leśniewski, is synonymous with the *universal,* as understood conventionally; '[...] Leśniewski defined a universal as a general object that possesses only those properties, which are common to all the individual objects, corresponding to it' (Rojek [67], *p. 36*). For obvious reasons, one must ensure that one does not misrepresent the status of universals as described by the proponents thereof. Consider Leśniewski's original characterisation:[22]

> '[...] with reference to all those who, by reason of the meaning they give to expressions of the type 'general object' with respect to objects *a*, are inclined to state the proposition 'if *X* is a general object with respect to objects *a*, *X* is *b* and *Y* is *a*, then *Y* is *b*', I wish to state here that this proposition entails the proposition 'if there exist at least two different *a*, then a general object with respect to objects *a* does not exist' (Leśniewski [46], *p. 184*).

18 *cf.* Leśniewski [45], *p. 131:*

$$\llcorner Aa \lrcorner \ulcorner \phi \left(\varepsilon\{Aa\} \varphi \left(\vdash \left(\llcorner B \lrcorner \ulcorner \vdash (\varphi(\varepsilon\{BA\}\varepsilon\{Ba\}))\urcorner \right) \llcorner BC \lrcorner \ulcorner \phi\!\!-\!(\varphi(\varepsilon\{BA\}\varepsilon\{CA\})\varepsilon)\urcorner \right) \right) \urcorner.$$

19 *cf.* Łukasiewicz [53], *p. 20:* 'Leśniewski's Ontology is rooted in Twardowski's theory of objects, which finds its roots in Brentano's. Leśniewski's aim is to establish nominalism and to express it in a precise logical language. Ontology is to be such a language. Leśniewski's nominalism can be regarded as a result of his criticism of Twardowski's doctrine of "general" objects.' Woleński considers the inspiration of Anton Marty as a catalyst for Leśniewski's initial analysis of *general objects;* 'Leśniewski was strongly influenced by Marty. [...] In his polemics against various theories of abstract objects, Leśniewski [*cf.* Leśniewski [39], *p. 318*] refers explicitly to Marty's criticism of Husserl's theory' (Woleński [92], *p. 220*).

20 *cf. pp. 308–310.*

21 *cf. p. 109.*

22 Translation courtesy of Urbaniak [89], *pp. 96–97*. Subsequent formalisations of Leśniewski's argument remain true to the authentic symbolism, and thus deviate from characterisations offered by both Urbaniak [89] and Luschei [54].

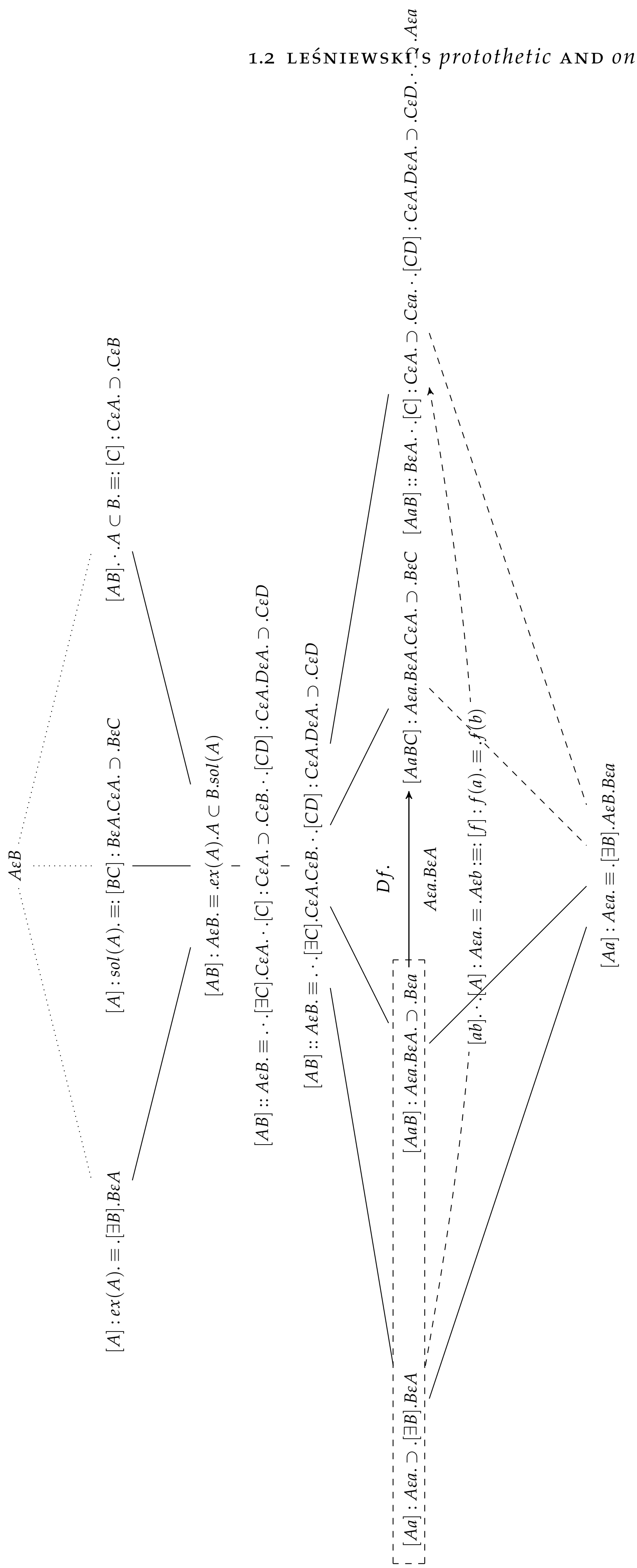

Figure 1: Inferential disposition of Leśniewski's 1929 Ontological axiom

One can immediately draw a parallel between the former of the above detailed propositions and an overtly obvious example: *'if 'red' is a general object with respect to red apples, 'red' is b and Y is a red apple, then Y is b'.* According to Leśniewski, such a proposition would by necessity entail the proposition *'if there exists at least two different red apples, then a general object ('redness') with respect to red apples does not exist'.* Immediately, it is clear that the latter proposition is in direct opposition to the most widely-understood notion of the Platonic universal – the abstract 'object' which is *instantiated in concrete objects* and thus *brought into existence.*

Designating general objects as *name arguments* of the argumentally unary functor '*repr*', the initial characterisation can be formalised as follows, where the Ontological epsilon connective functions as previously detailed:

CL. 1:3.0: $\forall X \forall a\{X\varepsilon repr(a) \iff [X\varepsilon X \wedge \forall b(X\varepsilon b \iff a \subset b)]\}$

That is, *'X is a universal with respect to a if and only if X is X and, for all b, X is c if and only if for all φ, if φ is a then φ is b'*[23]. Informally, the characterisation states in effect that any general object representing certain individual objects has *only* any property common amongst precisely those objects; as evidenced by the expression $X\varepsilon X$, the characterisation captures the existential import distinctive of the platonistic position, *viz.*, that universals are indeed *objects* in the most ontologically significant sense.

Leśniewski proceeds to detail schema which he claims to correspond with the platonist's[24] proposition ('if X is a general object with respect to objects a, X is b and Y is a, then Y is b'), beginning with an initial assumption:

'(1) If X is a general object with respect to objects a, X is b, [and] also Y is a, then Y is b.'

Provided that the above characteristic is indeed adequately representative of the platonist's thesis, we show that the following thesis, directly analagous to Leśniewski's initial proposition, is a naturally deductive consequence:[25]

TH. 1:3.1: $\forall X \forall Y \forall a \forall b\{[X\varepsilon repr(a) \wedge X\varepsilon b \wedge Y\varepsilon a] \implies Y\varepsilon b\}$

PF. $\forall X \forall Y \forall a \forall b$

1.	$X\varepsilon repr(a)$	*Supp.*
2.	$X\varepsilon b$	*Supp.*
3.	$Y\varepsilon a$	*Supp.*
4.	$X\varepsilon b \iff a \subset b$	$\forall b$ CL. 1:3.0, 1
5.	$a \subset b$	*Subs.*, *Det.*, 2, 4
6.	$(a \subset b \wedge Y\varepsilon a) \implies Y\varepsilon b$	$\forall Y \forall a \forall b$ AX. 1:2
7.	$Y\varepsilon b$	3, 5, 6

23
$$\forall X \forall a\big(X\varepsilon repr(a) \iff \{X\varepsilon X \wedge \forall b[X\varepsilon b \iff \forall\varphi(\varphi\varepsilon a \implies \varphi\varepsilon b]\}\big).$$

The idiosyncratic 'φ' name is denoted as such in order to avoid confusion with the names of type a, b, c and X, Y, Z.

24 As it is emphasised that – in Leśniewski's parlance –the term *'general object'* is understood as synonymous with *'universal'*, I refer to those who admit such universals into their ontology as *platonists.*

25 An alternative characterisation contains no instance of the Protothetical primitive, that is, the coimplicative function; rather, one employs material implication as the sole sentential connective of the expression. We observe the following from Urbaniak:

> 'If we represent *is* using the epsilon, and treat '*gen*' as a one-place functor which with one name argument, say a, constructs a name $gen(a)$ ('The general objects with respect to a's'), this assumption can be formalized [...]' (Urbaniak [89], *p. 97*).

The functor '*gen*' is explicitly equivalent to the local functor '*repr*', and thus all instances of the former in Urbaniak's proof are substitutable with the latter. Appreciating this, the alternative characterisation as formalised by Urbaniak takes the form:

$$\forall X \forall Y \forall a \forall b\{X\varepsilon repr(a) \implies [X\varepsilon b \implies (Y\varepsilon a \implies Y\varepsilon b)]\}.$$

The below proof includes two elementary sub-proofs of Ontology, left unproved by Urbaniak, corresponding to theses 4 and 13.

By virtue of the innocuous initial supposition – namely, that X is a universal with respect to a – and by appreciation of the claim 1:3.0, we can infer, under the scope of the universal quantifier binding b, that X is b *if and only if* all a are b. Substitution and detachment – operations guaranteed by Protothetic – allow the determination of the objects a being precisely those objects b; thus, by an inferential Ontological thesis, it follows that Y is b.

> '(2) If X is a general object with respect to objects a, X is different from Z, and Z is a, then Z is different from Z.
>
> (3) If X is a general object with respect to objects a, X is identical with Z and Y is a, then Y is identical with Z. [...]
>
> (4) If X is a general object with respect to objects a, and Z is a, then X is identical with Z.'

Thus, we can formalise as follows:[26]

PF.	$\forall X \forall Y \forall a \forall b$	
1.	$X \varepsilon repr(a)$	*Supp.*
2.	$Y \varepsilon a$	*Supp.*
3.	$Z \varepsilon a$	*Supp.*
4.	$X \varepsilon X$	$\forall X$ *Th.*, 1
4.1.	$X \varepsilon repr(a) \implies \exists A[A \varepsilon X \wedge A \varepsilon repr(a)]$	AX. 1:2
4.2.	$X \varepsilon repr(a) \implies X \varepsilon X$	*Th.*
4.3.	$\forall B \forall C$	
4.4.	$[X \varepsilon repr(a) \wedge B \varepsilon X \wedge C \varepsilon X] \implies B \varepsilon C$	AX. 1:2
4.5.	$\{A \varepsilon X \wedge [(B \varepsilon X \wedge C \varepsilon X) \implies B \varepsilon C]\} \implies X \varepsilon X$	AX. 1:2, *Th.*
4.5.1.	$A \varepsilon X \wedge A \varepsilon repr(a) \wedge (B \varepsilon X \wedge C \varepsilon X \implies B \varepsilon C) \implies X \varepsilon repr(a)$	AX. 1:2
5.	$X \varepsilon repr(a) \implies [X \varepsilon X \implies (Y \varepsilon a \implies Y \varepsilon X)]$	*Subs.*
6.	$X \varepsilon repr(a) \implies [X \varepsilon X \implies (Z \varepsilon a \implies Z \varepsilon X)]$	*Subs.*
7.	$X \varepsilon X \implies (Y \varepsilon a \implies Y \varepsilon X)$	1, 5
8.	$X \varepsilon X \implies (Z \varepsilon a \implies Z \varepsilon X)$	1, 6
9.	$Y \varepsilon a \implies Y \varepsilon X$	4, 7
10.	$Z \varepsilon a \implies Z \varepsilon X$	4, 8
11.	$Y \varepsilon X$	2, 9
12.	$Z \varepsilon X$	3, 10
13.	$(Y \varepsilon X \wedge Z \varepsilon X \wedge X \varepsilon X) \implies (Y \varepsilon Z \wedge Z \varepsilon Y)$	*Subs.*, *Th.*
14.	$Y \varepsilon Z \wedge Z \varepsilon Y$	4, 11, 12, 13

Thus, the above proof entails the intended conclusion:

$$\forall X \forall Y \forall Z \forall a \{[X \varepsilon repr(a) \wedge Y \varepsilon a \wedge Z \varepsilon a] \implies (Y \varepsilon Z \wedge Z \varepsilon Y)\}.$$

That is, *'if X is a universals with respect to objects a, Y is a and Z is a, then Y is identical with Z'*; there are thus no universals unique to more than one object.

26 Luschei details a number of defining characteristics of the functor '*Id*':

> '[...] the (sole individual which is) B; (individual) identical with B; the same individual as B; what is identical with (resp. is the same individual as) B' (Luschei [54], *p. 10*).

In direct comparison with conventional *equality*, a certain amiguity becomes apparent:

> '[...] *Identification (singular equality, individual equality, or mutual being):* (The sole) A is B and (the sole) B is A; A is the same individual as B; A is identical with (individual) B; (the sole) A is the sole B; individual A is B, and conversely; only the sole A is B' (Luschei [54], *p. 11*).

It appears that, whilst the function of '*Id*' is exact to that of equality, a certain logical elegance is achieved by its implementation. It is appreciated, for example, that the expression

$$\forall X \forall b [X \varepsilon repr(b) \implies X \varepsilon Id(b)]$$

TH. 1:3.2.1: $\forall X \forall Y \forall Z \forall a \{[X \varepsilon repr(a) \wedge Z \varepsilon a \wedge Y \varepsilon a] \implies Y \varepsilon Id(Z)\}$

PF.	$\forall X \forall Y \forall Z \forall a$	
1.	$X \varepsilon repr(a)$	*Supp.*
2.	$Z \varepsilon a$	*Supp.*
3.	$Y \varepsilon a$	*Supp.*
4.	$X \varepsilon \neg[Id(Z)] \implies Z \varepsilon \neg[Id(Z)]$	TH. 1:3.1, 1, 2
5.	$\forall b\big[\big(X \varepsilon c \wedge \{X \varepsilon \neg(b) \implies Z \varepsilon \neg[Id(Z)]\}\big) \implies X \varepsilon b\big]$	$\forall X \forall Z \forall b$ AX. 1:2
6.	$X \varepsilon Id(Z)$	1, 4, 5
7.	$Y \varepsilon Id(Z)$	TH. 1:3.1, 1, 3, 6

'(5) If X is a general object with respect to objects a, Z is a and Y is a, then X is a general object with respect to objects a, X is identical with Z, and Y is a. [. . .]

(6) If X is a general object with respect to objects a, Z is a and Y is a, then Y is identical with Z.'

TH. 1:3.2.2: $\forall Y \forall Z \forall a \big(\{Y \varepsilon a \wedge Z \varepsilon a \wedge \neg[Y \varepsilon Id(Z)]\} \implies \forall X \{\neg[X \varepsilon repr(a)]\}\big)$

PF.	$\forall Y \forall Z \forall a$	
1.	$Y \varepsilon a$	*Supp.*
2.	$Z \varepsilon a$	*Supp.*
3.	$\neg[Y \varepsilon Id(Z)]$	*Supp.*
	$\forall X$	
4.	$X \varepsilon repr(a) \implies Y \varepsilon Id(Z)$	TH. 1:3.2.1, 1, 2
5.	$\neg[X \varepsilon repr(a)]$	3, 4

TH. 1:3.3: $\forall X \forall a [X \varepsilon repr(a) \implies X \varepsilon Id(a)]$

PF.	$\forall X \forall a$	
1.	$X \varepsilon repr(a)$	*Supp.*
2.	$\forall b (X \varepsilon b \iff a \subset b)$	CL. 1:3.0, 1
3.	$\forall c \{a \subset a \wedge [X \varepsilon c \implies X \varepsilon Id(X)]\}$	$\forall X \forall b$ AX. 1:2
4.	$a \subset a \wedge X \varepsilon Id(X)$	1, 3
5.	$X \varepsilon a \wedge a \subset Id(X)$	2, 4
6.	$[X \varepsilon a \wedge a \subset Id(X)] \implies X \varepsilon Id(a)$	$\forall X \forall b$ AX. 1:2
7.	$X \varepsilon Id(a)$	5, 6

A number of intermediary steps lead Leśniewski to the following conclusion –

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is analagous to

$$\forall b[repr(b) \subset Id(b)].$$

Were one to represent identity solely be means of the epsilon connective – *viz.,* the expression $A \varepsilon B \wedge B \varepsilon A$ – one should require *two* name arguments, thus resulting in a loss of preferable generality. We observe, for instance, the fifth proposition of TH. 1:3.3: it is expressible that *all* objects a are precisely those objects identical with X, without necessary recourse to additional terms. Thus, Id is useful as a one-place functor.

Typical expressions of *identity* in Leśniewski's system are given below, juxtaposed with their negative counterparts:

$$\begin{array}{r|l} A \varepsilon Id(B) & A \varepsilon \neg Id(B) \\ A = B & \neg(A = B) \\ A \varepsilon B \wedge B \varepsilon A & \neg(A \varepsilon B \wedge B \varepsilon A) \end{array}$$

Noteworthily, $A \neq B$ is not the same expression as $\neg(A = B)$, for the expansion of the former is $A \varepsilon A \wedge B \varepsilon B \wedge \neg(A \varepsilon B \wedge B \varepsilon A)$ whilst the latter takes the form $\neg(A \varepsilon B \wedge B \varepsilon A)$.

'(6) If X is a general object with respect to objects a, z is a and Y is a, then Y is identical with Z.'

From (6) however it follows that if there exist at least two different a, then a general object with respect to objects a does not exist.'

The above proof competently illustrates the concretistic rigor with which Leśniewski's Ontology deals with objectual metaphysics; one is readily equipped with the underlying notion of *individuation* whilst working in Leśniewski's system, and only the least 'ontologically burdensome' objects are admitted. It must be noted, however, that the universal bears significance to *mathematicalia* beyond what is ostensibly evident; it is claimed by a number of authors than sets are, in and of themselves, *universals.* Perhaps the most marked of whom in this regard is Quine:

'The connection between quantification and entities outside language, be they universals or particulars, consists in the fact that the truth or falsity of a quantified statement ordinarily depends in pat on what we reckon into the range of entities appealed to by the phrases 'some entity x' and 'each entity x' – the so-called range of values of the variable. That classical mathematics treats of universals, or affirms that there are universals, means simply that classical mathematics requires universals as values of its bound variables. When we say, for example,

$$(\exists x)(x \text{ is prime}.x > 1{,}000{,}000),$$

We are saying that *there is* something which is prime and exceeds a million; and any such entity is a number, hence a universal' (Quine [63], *p. 103*).

Quine asserts nonchalantly that the ontological status of the number is not merely *comparable* to, but *identical* to that of the universal; Leśniewski's argument, taken to be adequately representative of universals, thus illustrates disastrous consequences for the nominalist who wishes to ontologically commit themselves to mathematicalia. If one is to concur that the ontological status of the mathematical object is indeed akin to that of the universal, then any doubt regarding the ostensible irrelevance of Leśniewski's argument ought be removed.

Leśniewski's assault on the concept of the universal thus serves not as an assisting factor in the ultimate desired outcome of our endeavour, *viz.* a nominalistic reparation of mathematicalia; on the contrary, it illustrates the *necessity* for such a reparation to occur, and the nuanced manner by which one must approach such a task.

1.3 EXTENDING LEŚNIEWSKI'S *mereology*

The *'third layer'* of Leśniewski's holistic system is precisely the system that shall allow for a nominalistic reparation of mathematics. *Mereology,* as originally titled by Leśniewski (from the Greek *μὲροζ*; literally, *'part'*) and identified as such in modern literature, is, in essence, a theory of *parts* and *wholes* and their relations; on the Mereological view, an aggregate of material objects – *realia,* or *concreta* - is itself nothing more than a concrete *whole* constituted by those *parts* taken as a plurality. Although without explicit reference to nominalism, Cantor succintly characterises the core tenet of the system in *Mitteilungen zur Lehre vom Transfiniten*:

'Any set of distinct things can be regarded as a single thing in which those objects are constituents or constitutive elements' (Cantor [7], *p. 83*).

As evidenced by Cantor's further writings, particularly the revolutionary Cantorian reconstruction of set theory, it is probable that Cantor did not understand sets in a mereological sense. Conversely, it is the ontological dubiousness of set theory that urged Leśniewski to formally construct his extralogical Mereology:

'The arrangement of definitions and truths, which I established in the present work dedicated to the most general problems of the theory of sets, has for me, in comparison to other previously known arrangements of definitions and truths, this advantage that it eliminates the 'antinomies' of the general theory of sets without narrowing the original domain of Cantor's term 'set' [...]' (Leśniewski [40], *pp. 129–130*).

As implied by the characterisation of Mereology as constuting a *third layer* of Leśniewski's system of logic, it is emphasised that, analagous to the manner in which Ontology presupposes Protothetic, Mereology presupposes both the former and latter of these systems. Thus, we understand Mereology precisely as detailed by Clay:

'Mereology, it may be recalled, is Leśniewski's system consisting of:

(1) A system of propositional logic, upon which is based

(2) A system for characterizing the meaning of 'is', upon which is based

(3) A system for characterizing the relation of 'part' to the 'whole" (Clay [12], *p. 467*).

Whilst we shall deviate slightly from Leśniewski in our most fundamental motivations for the use of a mereological system, it shall be shown that Leśniewski's original definitions correspond closely to the modern characterisation of mereology that we utilise.

Of the canonical texts on Mereology, Leonard and Goodman's seminal *The Calculus of Individuals and its Uses* [26] is perhaps the most well known and widely appreciated. Leonard and Goodman adopt the symbolism and logistic of Russell and Whitehead's *Principia,* introducing a novel dyadic propositional relation of discretion; symbolically, $x \urcorner y$, read intuitively as *'x and y have no parts in common'*.[27] Herefrom, classical mereology as it is commonly understood is constructed, and notions of atomism with respect to *parts* and *wholes* can be inferred thusly.[28]

Whilst conceptually identical, the system of Mereology outlined henceforward is technically dissimilar to that of Leonard and Goodman, insofar as we favour the notion of *proper parthood* as primitive over the notion of discretion. We define proper parthood intuitively, by the following axiom system:[29]

$$\forall A \forall B [A \varepsilon prpt(B) \implies B \varepsilon \neg(A)]$$
$$\forall A \forall B \forall C \{[A \varepsilon prpt(B) \wedge B \varepsilon prpt(C)] \implies A \varepsilon prpt(C)\}$$

General parthood – that is, non-strict parthood – can be defined *by virtue* of proper parthood:[30]

DF. 2:1.1: $\forall A \forall B \{A \varepsilon pt(B) \iff A \varepsilon A \wedge [A = B \vee A \varepsilon prpt(B)\}$ *Parthood*

Parthood in this sense is precisely equivalent to the '*el*' functor employed by Leśniewski; considering Mereology a nominalistic and non-paradoxical analogue of the theory of sets *(klasa)*, Leśniewski regarded *general parthood* as conceptually homologous to the more common set-theoretic *inclusion* (*i.e., $A \in B$*).

Leśniewski introduces the functor '*extr*' in *O Podstawach Matematyki,* to be intuitively read as *'exterior to'*; thus, the expression $A \varepsilon extr(B)$ is read as *'A is exterior to B'*. In the manner of Sobociński [80], one may wish to produce Leonard-Goodmanian discretion with proper parthood as primitive:[31]

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27 The symbolism given here is that of *Principia Mathematica* [68]; we opt, rather, to remain true to Leśniewski's notation. The logistic of the *Principia* produces, for example, the definition of *Parthood* (DF. 2:2.1.1) as:

$$x < y =_{df} .z \urcorner y \supset_z z \urcorner x.$$

As noted also by Leonard and Goodman, if one wishes to isolate the symbolism of the calculus from other treatments of logistic, identity can be defined as *mutual part-whole*, *i.e.:*

$$(x = y) \iff [A \varepsilon prpt(B) \wedge B \varepsilon prpt(A)].$$

We refrain from this; to define identity in this manner is bestow it with a greater 'philosophical weight' than observed in the definition offered by pure Ontology.

28 It is from these fundamental axioms that the *inseparability* of epistemology from ontology, as we wish to express preliminarily, becomes especially apparent.

29 As partially adopted from Lejewski [38], who terms the system containing these axioms 'System $\mathfrak{A}$'.

30 Thus, conversely taking *pt* as primitive, we define *prpt* accordingly:

DF. 2:1.3: $\forall A \forall B [A \varepsilon prpt(B) \iff A \varepsilon pt(B) \wedge \neg(A = B)]$ *Proper Parthood*

31 Conversely, one can define *pt* by taking *extr* as primitive:

$$\forall A \forall B \big(A \varepsilon extr(B) \iff A \varepsilon A \wedge \exists C [C \varepsilon pt(B)] \wedge \forall C \{ C \varepsilon pt(B) \implies C \varepsilon \neg [pt(A)] \} \big).$$

Leśniewski's original system of Mereology [40], dating from 1916, consists of four axioms – notably, we observe one of these axioms to function as an illustrative theorem of asymmetry:[32]

$$\forall A \forall B \{ A \varepsilon prpt(B) \implies B \varepsilon \neg [prpt(A)] \}$$

Leśniewski also established the transitivity of *proper parthood* as follows:

$$\forall A \forall B \forall C \{ [A \varepsilon prpt(B) \wedge B \varepsilon prpt(C)] \implies A \varepsilon prpt(C) \}.$$

Betti [4] notes that *proper parthood* can be defined on the basis of the primitive '*ingr*', an abbreviation for *'ingredjens' ('ingredient')* as used in the following manner: *each object is an ingredient of itself and each proper part of this object is an ingredient of this object, and there are no other ingredients of this object than itself or its proper parts.* Note that this primitive is introduced in Leśniewski's 1920 axiomatisation [41], and discarded with the introduction of the 1921 axiomatisation [42], with '*extr*' in favour of '*ingr*' as the sole primitive term. The definition takes the following form:

$$\forall A \forall B \{ A \varepsilon prpt(B) \iff [A \varepsilon ingr(B) \wedge \neg (A \varepsilon B \wedge B \varepsilon A)] \}.$$

Thus, it is inferrably a necessary condition for Leśniewskian parthood – as defined by the functor *pt* – to exclude instances of equality between *parts* and *wholes* (that is, instances of the form $A \varepsilon B \wedge B \varepsilon A$, for both the parthood relation $A \varepsilon pt(B)$ and the converse $B \varepsilon pt(A)$); the strict condition is therefore captured alternatively by the functor *ingr*.

Of what advantage is taking Leonard-Goodmanian *discretion* as the sole mereological primitive in place of Leśniewskian *proper parthood* (or the later Leśniewskian *ingredjens*)?[33] Whilst

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$$A \forall B \{ A \varepsilon pt(B) \iff \forall C [C \varepsilon extr(B) \implies C \varepsilon extr(A)] \}$$

Producing, by substitution, a tentative single-axiom for the mereological notion of *proper part:*

$$\begin{aligned} \forall \forall A \forall B \big\{ A \varepsilon pt(B) \iff \forall C \big[\big(C \varepsilon C \wedge \exists D [D \varepsilon pt(B)] \wedge \forall D \{ D \varepsilon pt(B) \implies D \varepsilon \neg [pt(C)] \} \big) \\ \implies \big(C \varepsilon C \wedge \exists E [E \varepsilon pt(A)] \wedge \forall E \{ E \varepsilon pt(A) \implies E \varepsilon \neg [pt(C)] \} \big) \big] \big\}. \end{aligned}$$

32 Leśniewski details relational asymmetry and transitivity as follows, respectively:

$$[AB] : A \varepsilon pt(B). \supset .B \varepsilon \sim (pt(A))$$

$$[ABC] : A \varepsilon pt(B).B \varepsilon pt(C). \supset .A \varepsilon pt(C).$$

We note that Leśniewski's '*pt*' does not function in the same sense as the local '*pt*'.

33 It is appreciated, however, that Goodman later adopted the two-place predicate *overlap* (see DF. 2:2.1.3) as the sole mereological primitive, in place of *discretion.* See Goodman [25], *pp. 47–48:*

> 'The sole primitive needed is the two-place predicate "overlaps", which I abbreviate in symbolic contexts by "∘". Two individuals overlap if they have some common content, whether or not either is wholly contained in the other. The predicate "∘" is symmetric and reflexive but not transitive. Moreover, it is ubiquitous among individuals in that all and only those things that overlap something are individuals. Thus "individual" may be easily defined in terms of overlapping; but in a nominalistic system, "(is an) individual" is a universal predicate and so of little use.'

Therefore, discretion can be defined by virtue of the *overlap* predicate as follows (and Goodman does precisely this; *cf. ibid., p. 48*):

$$x \mathbin{\urcorner} y =_{df} \neg (x \circ y).$$

It ought be noted that Goodman favours the above primitive over the local *discretion* due not to any logical amelioration (it is by no means a recantation of *The Calculus of Individuals and its Uses*); the choice of primitive functions only as an expository convenience.

the choice of primitive is largely one of preference, it is submitted by Sobociński that *discretion* is, *in principle,* the *weakest* primitive term of Mereology[34]:

> 'The expression '$dscr(a)$' may be read 'a's are discrete', which means that the a's are outside one another. The functor '*dscr*' is the weakest mereological functor known to me which is strong enough to be a single primitive term of the theory.' (Sobociński [80], *p. 221*).

Concerning the manner in which the term is employed by Sobociński, and as evidenced explicitly by the above passage, *discretion* does not function in the familiar sense; that is, discretion function not as a two-place predicate for *(presently)* singular terms, but as a functor demarcating the pairwise discretion of some arbitrary *group* of individuals, which we may term a *plurality.* We superficially refrain from using the term *collection* – lest the term *class* or *set* – precisely due to the manner in which Sobociński wishes to express the notion of *collection* in its own right.[35]

One notes distinct parallels to Leśniewski's later axiomatisation, composed in 1921. We observe that the following axiom, in Leśniewski's logistic[36], is closely analogous to our notion of *common parthood,* as given in DF. 2:1.3:

$$[AB] : A\varepsilon extr(B). \equiv .[C]. \cdot .C\varepsilon C. \supset .[\exists D] : \cdot : D\varepsilon extr(A). \vee .D\varepsilon extr(B) : .D\varepsilon \sim extr(C).$$

That is, for any two objects A and B, A is exterior to B if and only if for all objects C, there is an object D which *overlaps* with C but does not overlaps with A nor B. In the local logistic:

$$\forall A \forall B \big[A\varepsilon extr(B) \iff \forall C \big(C\varepsilon C \implies \exists D \{ [D\varepsilon extr(A) \vee D\varepsilon extr(B)] \wedge D\varepsilon \neg extr(C) \} \big) \big].$$

Whilst discretion (as Sobociński employs the term) is not primitive in Leśniewski's system, one can easily define a functor of discretion based *only* on exteriority:

$$\forall a \big(dscr(a) \iff \forall A \forall B \{ (A\varepsilon a \wedge B\varepsilon a) \implies [A = B \vee A\varepsilon extr(B)] \} \big)$$

Which is inferrable from the following thesis, provable in Mereology:[37]

$$\forall A \forall B \{ A\varepsilon extr(B) \iff [A\varepsilon A \wedge B\varepsilon B \wedge \neg(A = B) \wedge dscr(A \cup B)] \}.$$

The functor *dscr* is immediately identifiable as taking a plurality as its sole argument; therefore, we observe a dissimilarity between Leśniewskian discretion and the (conceptually) 'Leonard-Goodmanian' two-place functor *extr* (*via* which statements of the form $A\varepsilon extr(B) \wedge B\varepsilon extr(A)$ are expressible). Does the weakness of the primitive thus continue to hold? It seems that, whilst the functor *dscr* may be on a highly technical level the weakest of primitives, the functor *extr* proves to be certainly adequate.[38] Therefore, one is certainly at no palpable disadvantage in taking discretion as primitive, and, *mutatis mutandis,* at no disadvantage taking proper parthood as primitive.[39]

We must also define the notion of *overlap,* or *common parthood:*

---

34 Concerning the notion of *weakness* with respect to *primitive terms*, *cf.* Lindenbaum [52].

35 See **??** for further discussion.

36 The early writings of Leśniewski utilise the above notation, the well-known Peano-Russell logistic with slight modifications; *cf.* Sobociński [79]. The logical form of Leśniewski's exteriority axiom is courtesy of Urbaniak [89], *p. 123:*

$$\forall a, b[a\varepsilon ext(b) \equiv \forall c(c\varepsilon c \to \exists d((d\varepsilon ext(a) \vee d\varepsilon ext(b)) \wedge d\varepsilon \sim ext(c)))].$$

Whilst Urbaniak expresses Leśniewski's axioms of exteriority *via* the functor *ext*, the original form *pace* Leśniewski is known to be of the form *extr* (*cf.* Sobociński [79], *p. 220*).

37 We understand logical union in the usual sense.

38 So much so that, as noted, Leśniewski's later Mereology adopted *extr* in favour of both *ingr* and *pt*.

39

$$[ab] : a \subset b.dscr(b). \supset .dscr(a)$$

$$[a] :: dscr(cl(a)). \equiv . \cdot .[AB] : A\varepsilon a.B\varepsilon a. \supset .A\varepsilon B.B\varepsilon A$$

$$[ab] : dscr(a).dscr(b) : cl(a) \subset cl(b).cl(b) \subset cl(a) :\supset .a \subset b.b \subset a$$

DF. 2:1.2: $\forall A \forall B\{A\varepsilon ov(B) \iff A\varepsilon A \wedge \exists C[C\varepsilon pt(A) \wedge C\varepsilon pt(B)]\}$ *Common Parthood*

Obviously, $A\varepsilon ov(B) \implies B\varepsilon ov(A)$ and conversely.

It is well known amongst those familiar with Mereology that the system remains formally equivocal with regards to the ontological status of the *atom*, indifferent towards its existence or lack thereof[40]; thus, should one by philosophical motivation arbitrate *atomicity* or *atomlessness* by way of axiom, it is there that one makes a minor departure from classical mereology *pace* Leśniewski. Appreciating this, we indeed make a departure, with the bias of *atomicity*. Of course, atomicity cannot be proved without supplementing Mereology with additional axioms, as Leśniewski's original system was *not neccessarily* atomistic.

Atom, in the mereological sense, is understood as that which has *no proper part*. In *Contributions to Mereology*, Clay defines the mereological atom in Leśniewski's language as follows:

$$[A] : A\varepsilon A.[B]. \sim (B\varepsilon pr(A)). \equiv .A\varepsilon atm.$$

We want to express precisely this in the local logistic:

DF. 2:2.1: $\forall A\{A\varepsilon atm \iff A\varepsilon A \wedge \forall B[B\varepsilon\neg prpt(A)]\}$ *General Atomicity*

DF. 2:2.1*: $\forall A\{A\varepsilon atm \iff A\varepsilon A \wedge \forall B[B\varepsilon pt(A) \implies B = A]\}$ *General Atomicity**

DF. 2:2.2: $\forall A \forall B[A\varepsilon at(B) \iff A\varepsilon atm \wedge A\varepsilon pt(B)]$ *Specific Atomicity*

The latter of this set of definitions introduces the functor '*at*'; expressions of the form $A\varepsilon at(B)$ have the intuitive reading 'A is an atom of B'.[41]

In 4.2, we show that the system consisting of the following axiom **A**

$$\forall A \forall B\Big[A\varepsilon pt(B) \iff \Big(B\varepsilon B \wedge \forall C \forall a\big\{\forall D\big[D\varepsilon C \iff \big(\forall E[E\varepsilon a \implies E\varepsilon pt(D)] \wedge \forall E\{E\varepsilon pt(D) \implies \exists F \exists G[F\varepsilon a \wedge G\varepsilon pt(F) \wedge G\varepsilon pt(E)]\}\big)\big] \wedge [B\varepsilon pt(B) \wedge B\varepsilon a \implies A\varepsilon pt(C)]\big\}\Big)\Big]$$

and **V** (following the conventions of Sobociński [82, 83]

$$\forall A\big(A\varepsilon A \implies \exists B\{B\varepsilon pt(A) \wedge \forall C[C\varepsilon pt(B) \implies C = B]\}\big)$$

is inferrentially equivalent to the system $\mathfrak{B}$:

$$\begin{aligned}
\forall A[A\varepsilon at(B) &\implies B\varepsilon B]\\
\forall A \forall B \forall C\{[A\varepsilon at(B) \wedge C\varepsilon at(A)] &\implies C = A\}\\
\forall A \forall B\big(\{A\varepsilon A \wedge B\varepsilon B \wedge \forall C[C\varepsilon at(A) \iff C\varepsilon at(B)]\} &\implies A = B\big)\\
\forall A \forall a\big[A\varepsilon a \implies \exists B\big(\exists E([E\varepsilon at(B)] \wedge \forall C\{C\varepsilon at(B) &\iff \exists D[C\varepsilon at(D) \wedge D\varepsilon a]\})\big)\big].
\end{aligned}$$

Thus, as the system $\mathfrak{B}$ adequately expresses the desiderata of atomistic Mereology, we obtain the following single axiom as the conjugate of **A** and **V**:

AX. 2:2.3: *Axiom of Atomistic Mereology*

$$\forall A \forall B\Big(A\varepsilon at(B) \iff B\varepsilon B \wedge \forall C \forall D \forall a\big\{\big[\forall E\big(E\varepsilon C \iff \forall F\{F\varepsilon at(E) \iff \exists G[F\varepsilon at(G) \wedge G\varepsilon a]\}\big) \wedge D\varepsilon at(B) \wedge B\varepsilon a\big] \implies at(A)\varepsilon at(C)\big\}\Big)$$

Compositional notions can also be introduced. In the expression $A\varepsilon\sum(a,b)$, the mereological *fusion* A defines the 'summation' of a and b, such that the objects overlapping of A are just those objects overlapping *either a or b*; dually, the *product* $A\varepsilon\prod(a,b)$ defines the objects B, as being precisely those objects that are parts of *both a and b*:

40 Clay claims Leśniewski's Mereology to be 'neutral with respect to the existence of atoms' (Clay [14], *p. 345*).

41 This functor was introduced by Rickey; *cf.* Sobociński [82, 83].

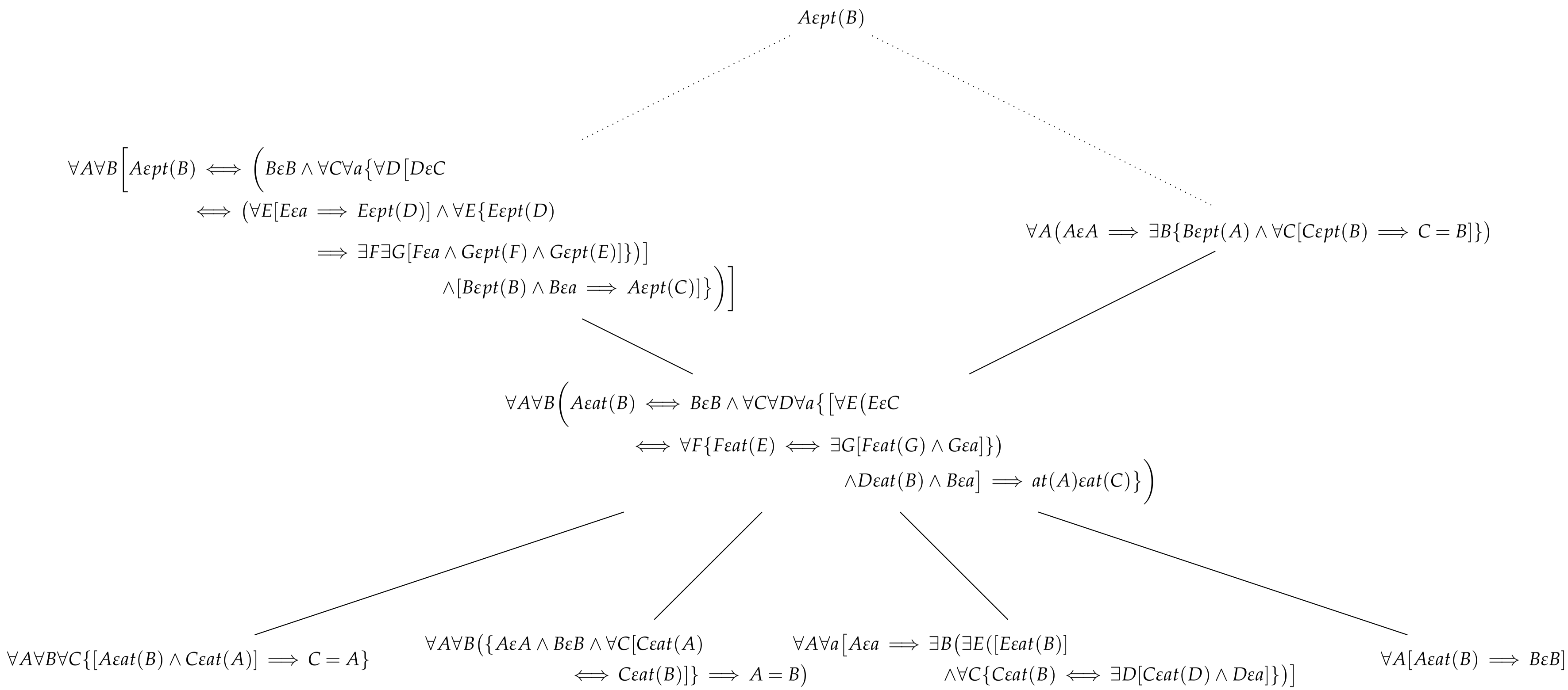


Figure 2: Inferential disposition of Lejewski-Clay's single axiom for Atomistic Mereology

DF. 2:2.4.1: $\forall A \forall a \forall b \big(A \varepsilon \sum(a,b) \iff \forall B \{ B \varepsilon ov(A) \implies [B \varepsilon ov(a) \vee B \varepsilon ov(b)] \} \big)$ *Fusion*

DF. 2:2.4.2: $\forall A \forall a \forall b \big(A \varepsilon \prod(a,b) =_{df} \forall B \{ B \varepsilon pt(A) \implies [B \varepsilon pt(a) \wedge B \varepsilon pt(b)] \} \big)$ *Product*

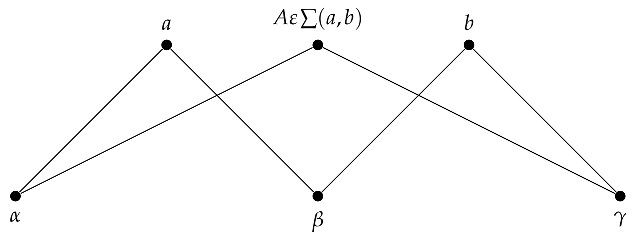
Mereological fusion, as previously detailed, is by no means unanimously defined. Tarski (and, similarly, Lewis[42]), offering an early characterisation of primitive mereological notions in *Foundations of the Geometry of Solids,* defines *sum (fusion)* as follows:

> DEFINITION III. An individual X is called a *sum* of all elements of a class α of individuals if every element of α is a part of X and if no part of X is disjoint from all elements of α.

The Tarskian terms *disjoint* and *sum* directly correlate to the Leśniewskian *exterior* and *set*, in turn (ostensibly) correlative of the local *discrete* and *fusion*[43]. Symbolically interpreting the above statement in familiar variable notation, we have the following definition:

$$\forall A \forall a \forall b \big(A \varepsilon \sum(a,b) \iff a \varepsilon pt(A) \wedge b \varepsilon pt(A) \wedge \forall B \{ B \varepsilon pt(A) \implies [B \varepsilon ov(a) \vee B \varepsilon ov(b)] \} \big).$$

Tarski's intended characterisation of *fusion* is thus captured by virtue of the *common parthood* functor; we observe that A is the sum of a and b if and only if both a and b are parts of A, and, for all B, the parthood of B relative to A implies that B *overlaps* a *or* B *overlaps* b. The above proposition bears close resemblance to DF. 2:2.5.1, insofar as the latter argument of the conditional remains identical and materially contingent on the universal relation of B to A; the difference lies explicitly in the functor by which this relation is governed. Consider the following diagrammatic sum:



$$\alpha \varepsilon pt(A) \wedge \neg[(\alpha \varepsilon A) \wedge (A \varepsilon \alpha)] \quad \gamma \varepsilon pt(A) \wedge \neg[(\gamma \varepsilon A) \wedge (A \varepsilon \gamma)]$$
$$\wedge \alpha \varepsilon pt(a) \wedge \neg[(\alpha \varepsilon a) \wedge (a \varepsilon \alpha)] \quad \wedge \gamma \varepsilon pt(b) \wedge \neg[(\gamma \varepsilon b) \wedge (b \varepsilon \gamma)]$$
$$\wedge \beta \varepsilon pt(a) \wedge \neg[(\beta \varepsilon a) \wedge (a \varepsilon \beta)] \quad \wedge \beta \varepsilon pt(b) \wedge \neg[(\beta \varepsilon b) \wedge (b \varepsilon \beta)]$$

$$\big(\forall B \{ B \varepsilon ov(A) \implies [B \varepsilon ov(a) \vee B \varepsilon ov(b)] \} \big)$$
$$:\neq \big(a \varepsilon pt(A) \wedge b \varepsilon pt(A) \wedge \forall B \{ B \varepsilon pt(A) \Longrightarrow [B \varepsilon ov(A) \vee B \varepsilon ov(b)] \} \big)$$

Figure 3: A *proper extension* network of the mereological fusion z in accordance to DF. 2:2.3.1, ostensibly inequivalent to Tarskian fusion

Here, z can be considered – in the most ontologically profound sense – a composition of *precisely those atoms* that constitute the arbitrary individuals a and b in their entirety; the Tarskian characterisation, however, demands that, with the appropriate auxiliary axiom[44], an arbitrary mereological binary condition ξ between two entities implies by *material conditionality* the existence of an upper bound wherein all parts are common with respect to a or b. The Tarskian

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42 We observe in Lewis' *Parts of Classes [51], p. 73:*

> *Definition:* Something is a *fusion* if some things iff it has all of them as parts and has no part that is distinct from each of them.

43 Such correlations are superficial to the extent that *(relatively)* minor discrepancies exist between *fusion* as presently conceived and the notion of *set* in the original Leśniewskian spirit. Leśniewski himself offered a notion of *sum* which may be considered the *de facto* correlate of mereological fusion; unsurprisingly, given Leśniewski's motivation for the initial conception of Mereology, the notion relies on set-theoretic (and *class-theoretic*) developments. Such exigencies are discussed in 3.2.

44 See AX. 2:1.2, where the binary functor ξ takes the form of the *common whole* relation *ov*. Briefly considering the general form of the aforementioned axiom –

offering thus violates 3, and DF. 2:2.4.1 serves as compromise; fusion defined in this sense also excludes the situation where A serves as a minimal upper bound of a and b, wherein a disjoint entity B consists as a part.

The following non-trivial axioms can be put forth, asserting fundamental mereological concepts of universal partial ordering, and guaranteeing the existence of atomic concreta and the compositions thereof, as previously defined. One can easily compare the axiom asserting the *existence of mereological fusions* with the above Leśniewskian axiom for the existence of classes (for, in the Leśniewskian system, the two are indiscernible):[45]

TH. 2:2.5.1: $\forall A[A\varepsilon V \implies A\varepsilon pt(A)]$

TH. 2:2.5.2: $\forall A\forall B\{[A\varepsilon pt(B) \wedge B\varepsilon pt(A)] \implies A = B\}$

PF. $\forall A\forall B$

1.	$A\varepsilon pt(B)$	*Supp.*
2.	$B\varepsilon pt(A)$	*Supp.*
3.	$A = B \vee A\varepsilon prpt(B)$	DF. 2:1.1, 1
4.	$B = A \vee B\varepsilon prpt(A)$	DF. 2:1.1, 2
5.	$\neg[A\varepsilon prpt(B) \wedge B\varepsilon prpt(A)]$	DF. 2:1.1
6.	$A = B$	3–5

TH. 2:2.5.3: $\forall A\forall B\forall C\{[A\varepsilon pt(B) \wedge B\varepsilon pt(C)] \implies A\varepsilon pt(C)\}$

PF. $\forall A\forall B$

1.	$A\varepsilon pt(B)$	*Supp.*
2.	$B\varepsilon pt(C)$	*Supp.*
3.	$A = B \vee A\varepsilon prpt(B)$	DF. 2:1.1, 1
4.	$B = C \vee B\varepsilon prpt(C)$	DF. 2:1.1, 2
5.	$A = C \vee A\varepsilon prpt(C)$	AX. 2:2.3, DF. 2:1.1, 1, 2
6.	$A = C \implies A\varepsilon pt(C)$	DF. 2:1.1
7.	$A\varepsilon prpt(C) \implies A\varepsilon pt(C)$	DF. 2:1.1
8.	$A\varepsilon pt(C)$	5–7

TH. 2:2.5.4: $\forall A\forall B\forall C\big(B\varepsilon\neg pt(A) \implies \exists C\{C\varepsilon C \wedge C\varepsilon pt(B) \wedge \neg\exists D[D\varepsilon pt(C) \wedge D\varepsilon pt(A)]\}\big)$

$$a\xi b \implies \exists C[C\varepsilon \textstyle\sum(a,b)]$$

– the explicit form, where the functor $\sum$ is defined in the manner of Tarski, takes the following form (in the sense of the general schema $[\varphi(B) \implies B\varepsilon prpt(A)] \wedge \{C\varepsilon prpt(A) \implies \exists B[\varphi(B) \wedge B\varepsilon ov(C)]\}$):

$$a\xi b \implies \exists A\big(a\varepsilon prpt(A) \wedge b\varepsilon prpt(A) \wedge \forall B\{B\varepsilon prpt(A) \implies [B\varepsilon ov(A) \vee B\varepsilon ov(b)]\}\big).$$

45 Some spatial mereological frameworks make the dimensional restriction of variables explicit. The na theory of dimension for qualitative spatial regions given in Hahmann and Gruninger [28] exhibits the notion of dimensionality in its axiom schemata: transitivity, for example, is given as $x =_{dim} y \wedge y =_{dim} z \rightarrow x =_{dim} z$. The Brentanian extensional mereology presented in Baumann [2] defines equidimensionality as the quality of occupying spatial regions ($eqdim(x,y) \iff (SReg(x) \wedge SReg(y))$), and, therefrom, axiomatises *'range restriction'* as $\forall xy(spart(x,y) \rightarrow eqdim(x,y))$. This can be re-formulated in the local logistic, and taken as a presupposed axiom:

$$\forall A\forall B[A\varepsilon pt(B) \implies (A\varepsilon\mathbb{D}^3 \wedge B\varepsilon\mathbb{D}^3)]$$

With $A\varepsilon\mathbb{D}^n \wedge B\varepsilon\mathbb{D}^n$ meaning – rather informally – *'A and B are n-dimensional'*. Classical mereology, as presented, is, most basally, atemporal – thus, as currently developed, variables in the framework consist in $\mathbb{D}^3$.

PF. $\forall A \forall B$

1.	$B\varepsilon\neg pt(A)$	*Supp.*
	$\exists C$	
2.	$C\varepsilon pt(B)$	AX. 2:2.3
3.	$C\varepsilon C$	TH. 1:3.4, 2
4.	$\exists D[D\varepsilon pt(C)]$	*Supp.*
	$\forall D$	
5.	$D\varepsilon pt(B)$	TH. 2:2.5.3, 2, 4
6.	$D\varepsilon\neg pt(A)$	1, 5
7.	$\neg[D\varepsilon pt(A)]$	*Th.*, 6
8.	$\neg\exists D[D\varepsilon pt(C) \wedge D\varepsilon pt(A)]$	4, 7

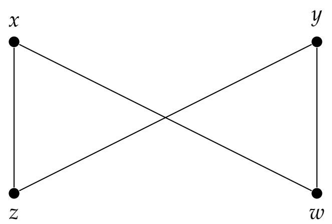
Whilst intuitive, *strong supplementation* – as a mereological principle – causes bivalent inconsonance in the *elementary* system (establishing *reflexivity*, *antisymmetry* and *transitivity*). Consider the weaker supplementation axiom:

$$\forall A \forall B\{A\varepsilon prpt(B) \implies \exists C[C\varepsilon pt(B) \wedge C\varepsilon\neg ov(A)]\}$$

The above axiom expresses the uncontroversial principle that if A is a *proper part* of B, then there exists some part C of B that does not overlap A. We observe, however, that whilst implication holds from the *strong* to the the *weak* supplementation principles (by virtue of the ordering axioms), the converse does not. The valid implication is written symbolically:

$$\begin{aligned}\forall A \forall B \forall C\big\{\big(&A\varepsilon pt(A) \wedge \{[A\varepsilon pt(B) \wedge B\varepsilon pt(A)] \implies A = B\}\\ &\wedge\{[A\varepsilon pt(B) \wedge B\varepsilon pt(C)] \implies A\varepsilon pt(C)\}\big) \implies \big[\big(\neg[B\varepsilon pt(A)]\\ &\implies \exists C\{C\varepsilon pt(B) \wedge \neg[C\varepsilon ov(A)]\}\big) \implies \big(A\varepsilon prpt(B) \implies \exists C\{C\varepsilon pt(B) \wedge \neg[C\varepsilon ov(A)]\}\big)\big]\big\}.\end{aligned}$$

4 illustrates a supplemented model in which the weaker axiom holds, whilst the stronger axiom is violated.



$$\begin{aligned}&z <_p x \wedge \neg[(z\varepsilon x) \wedge (x\varepsilon z)]\\ &\quad\wedge z <_p y \wedge \neg[(z\varepsilon y) \wedge (y\varepsilon z)]\\ &\qquad\wedge w <_p y \wedge \neg[(w\varepsilon y) \wedge (y\varepsilon w)]\\ &\qquad\quad\wedge w <_p x \wedge \neg[(w\varepsilon x) \wedge (x\varepsilon w)]\end{aligned}$$

Figure 4: A *supplemented model* consisting in the elementary system AX. 2:2.4.1–AX. 2:2.4.3 and in violation of AX. 2:2.5

Again, we draw comparison to Leśniewski's axiomatisations, wherein the underlying set-theoretic motivations again become evident:[46]

$$\forall A \forall \varphi\big\{\forall B\big[B\varepsilon\varphi(A) \iff B\varepsilon B \wedge \exists D\big(D\varepsilon A \wedge \forall C\{B\varepsilon extr(C) \iff \forall D[D\varepsilon A \implies D\varepsilon extr(C)]\}\big)\big] \implies \varphi(A)\varepsilon\varphi(A)\big\}.$$

46 This particular expression in a very comparable form is given in Urbaniak [89]:

$$a, \phi[\forall b[b\varepsilon\phi(a) \equiv b\varepsilon b \wedge \exists d\; d\varepsilon a \wedge \forall c(b\varepsilon ext(c) \equiv \forall d(d\varepsilon a \to d\varepsilon ext(c)))] \to \phi(a)\varepsilon\phi(a)].$$

The above axiom states that, if something is *a*, there is a *class* of *a*'s. More explicitly, if *a* is itself *non-empty*, then there is exactly one fusion of *a*'s, namely *b*. Leśniewski is therefore equating the *mereological fusion* of some objects with the *class* – ostensibly the *set* – of those very objects.[47]

47 We adopt a similar mereological characterisation of the *set*, although such a characterisation differs *slightly* from merely the *fusion* of those elements that would compose it. See 3.2.

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\begin{array}{l} \forall p\forall q((p \iff q) \iff \{\forall f[f(pf\{p[\forall r(r)]\} \\ \qquad \iff [\forall s(fqs) \iff (q \iff p)])]\}) \end{array} \arrow[r, dash, "\textit{Protothetic}"] \arrow[r, dashed, dash, bend right, "s,\ \frac{s}{ss},\ \frac{s}{sss}\cdots"'] \arrow[d, "\varepsilon"'] & \begin{array}{r} [fpqrst] :\cdot: p \equiv q. \equiv:: [g] :: f(pf(p[u].u)). \\ \equiv .\cdot.[u].f(qu). \equiv: g(r \equiv s. \equiv tq). \equiv .g(s \equiv t. \equiv rp) \end{array} \arrow[d, "\varepsilon"] \\
\forall A\forall a[A\varepsilon a \iff \exists B(A\varepsilon B \wedge B\varepsilon a)] \arrow[r, dash, "\textit{Ontology}"] \arrow[r, dashed, dash, bend right, "n,\ \frac{n}{n},\ \frac{s}{n},\ \frac{n}{\frac{s}{n}},\ \frac{\frac{s}{n}}{n}\cdots"'] \arrow[d, "prpt/at"'] & \begin{array}{r} [Aa] :: A\varepsilon a. \equiv .\cdot.[\exists B].B\varepsilon A.\cdot.[C,D] : C\varepsilon A. \\ D\varepsilon A. \supset .C\varepsilon D.\cdot.[E] : E\varepsilon A. \supset .E\varepsilon a \end{array} \arrow[d, "el"] \\
\begin{array}{l} \forall A\forall B\Big(A\varepsilon at(B) \iff B\varepsilon B \wedge \forall C\forall D\forall a\{[\forall E(E\varepsilon C \\ \qquad \iff \forall F\{F\varepsilon at(E) \iff \exists G[F\varepsilon at(G) \wedge G\varepsilon a]\}) \\ \qquad\quad \wedge D\varepsilon at(B) \wedge B\varepsilon a] \implies at(A)\varepsilon at(C)\}\Big) \end{array} \arrow[r, dash, "\textit{Mereology}"] & \begin{array}{l} [AB] :.: A\varepsilon el(B). \equiv:: B\varepsilon B :: [fa] :: [c] : .C\varepsilon f(a). \\ \quad \equiv: [D] : D\varepsilon a. \supset .D\varepsilon el(c) : [D] : D\varepsilon el(c). \\ \qquad \supset .[\exists EF].E\varepsilon a.F\varepsilon el(E).F\varepsilon el(D) : .B\varepsilon el(B).B\varepsilon a :. \supset .A\varepsilon el(f(a)) \end{array}
\end{tikzcd}
$$

Figure 5: Single-axiom disposition of Leśniewski's holistic system in the local and original logistics

Part II

2 NOMINALISTICALLY REINTERPRETING MODAL STRUCTURALISM

It is only through the purely logical process of building up the science of numbers and by thus acquiring the continuous number-domain that we are prepared accurately to investigate our notions of space and time by bringing them into relation with this number-domain created in our mind.
– Richard Dedekind, Preface to *Was sind und was sollen die Zahlen?* [19], *pp. 31–32.*

It is uncontroversially true that things might be otherwise than they are.
– David Lewis, *Counterfactuals* [49], *p. 84.*

2.1 STRUCTURALIST ARITHMETIC

The foundational construction of the natural numbers occupies a kind of metaphysical juncture, formed by both Mereology and Modal Structuralism. With concern to Mereology, the natural numbers possess the ability to be both the most pernicious and immediately reparable of abstract concepts; of the Modal Structuralist concern, their illustrative capacity and unparalled fundamentality earn them the prestigious status as the natural starting-point of structuralist study. As appreciated by Hellman, *'[the natural numbers] lend themselves most readily to a structuralist interpretation according to which they [...] are dispensed with entirely.'*

Dedekind, an influential figure in foundational mathematics, can be rightly considered the first 'true' structuralist – and it is through the manner in which he carried out his construction of the natural numbers, a construction which remains largely unchanged to the present day, that one at once observes the aforementioned *illustrative capacity* of the natural numbers. It is through numbers themselves that one observes the most fundamental of linear, progressive structures; a structure of pure relationship, amongst arbitrary objects whose ontological status is of no concern to the higher abstraction that constitutes mathematical practice. Structuralism, as outlined in the first chapter, absolves the individuality of places in structures, focusing rather on the larger relational system – and it is rather uncontroversial to declare the realm of arithmetic to behave in correspondence with precisely this principle.

Whilst distinctly non-nominalistic – perhaps even platonistic – Dedekind's structuralist perspective lays ground for even the most staunchly nominalistic of philosophies. We can imagine, for instance, the reduction of natural numbers to mereological atoms as a particularly parsimonious treatment of nominalistic arithmetic; it would, however, be a great disservice to the concept of *number* to perform such a bold reduction without thoroughly critical analysis. For this reason, we progress to detail a Leśniewskian (and thus, naturally, nominalistic) explication of *number,* which shall prove to be an apt introduction to the novel *modal* variant of classical Dedekindian structuralism.

The fundamentality of number extends past the restrictions of Mereology and rests firmly in Ontology – so much so, in fact, that the explanatory capacity of Ontology is sufficient for the full derivation of modern Peano arithmetic. Preliminarily, we follow in suit of the *Principia,* defining the *cardinal number* by virtue of the Ontological concept of *equinumerosity:*

$$\begin{aligned}\forall a \forall b \big[a \infty b \implies \exists \varphi \big(\forall A \forall B \forall C \{ [(\varphi\{AC\} \wedge \varphi\{BC\}) \implies A = B] \\ \wedge [(\varphi\{CA\} \wedge \varphi\{CB\}) \implies A = B] \} \wedge \forall A [A \varepsilon a \iff \exists B(\varphi\{BA\})] \\ \wedge \forall B [B \varepsilon b \iff \exists A(\varphi\{BA\})] \big) \big].\end{aligned}$$

Intuitively, we establish reflexivity, symmetry and transitivity for equinumerosity:

TH. 3:1.1: $\forall a (a \varepsilon V \implies a \infty a)$

TH. 3:1.2: $\forall a \forall b (a \infty b \implies b \infty a)$

TH. 3:1.3: $\forall a \forall b \forall c [(a \infty b \wedge b \infty c) \implies a \infty c]$

The distinction between *numericality* and *quantity* is drawn by definition – we consider an argumentally unary proposition-forming functor *numerical* if and only if, for all a and b, if $\varphi\{a\}$ and a is equinumerous with b then $\varphi\{b\}$. We immediately obtain the thesis that if a is equinumerous, a is numerical:

DF. 3:2.1: $\forall\varphi\{Num(\varphi) \iff \forall a\forall b[(\varphi\{a\} \wedge a\infty b) \implies \varphi\{b\}]\}$

TH. 3:2.2: $\forall a[Num(\infty\langle a\rangle)]$

In a largely analagous fashion, we define a functor to be *quantitative* if and only if $\varphi\{a\}$ and $\varphi\{b\}$ imply in conjunction the equinumerosity of a and b:

DF. 3:2.3: $\forall\varphi\{Qua(\varphi) \iff \forall a\forall b[(\varphi\{a\} \wedge \varphi\{b\}) \implies a\infty b]\}$

TH. 3:2.4: $\forall a[Qua(\infty\langle a\rangle)]$

WIth quantity and numericality established, the cardinal number is straightforwardly definable; if φ is numerical, quantitative, and non-empty (*i.e.,* it satisfies the Ontological expression $\forall a[!(\varphi) \iff \exists a(\varphi\{a\})]$), we consider φ *cardinal* ($Crd(\varphi)$):

DF. 3:2.5: $\forall\varphi\{Crd(\varphi) \iff [!\{\varphi\} \wedge Num(\varphi) \wedge Qua(\varphi)]\}$

TH. 3:2.6: $\forall a[Crd(\infty\langle a\rangle)]$

We consider the situation wherein the conjunction of an Ontologically existential name A – *i.e.,* in satisfaction of the expression $A\varepsilon a$ – and the proposition-forming functor φ binding the name b altogether implies that the logical union of A and b is also bound by φ. Should this condition in conjunction with the functor φ binding a sole objectual argument ($\varphi\{\bigwedge\}$) imply the condition $\varphi\{a\}$, for all such functors φ, we regard a to be *finite name:*[1]

DF. 3:3.1: $$\forall a\big[Finite\{a\} \iff \forall\varphi\big(\{\varphi\{\bigwedge\} \wedge \forall A\forall b[(A\varepsilon a \wedge \varphi\{b\}) \implies \varphi\{b \cup A\}]\} \implies \varphi\{a\}\big)\big]$$

TH. 3:3.2: $Finite\{\bigwedge\}$

Finitude, expressed as a functor, is trivially numerical; that is, we have $\forall a\forall b[(Finite\{a\} \wedge a\infty b) \implies Finite\{b\}]$.[2] We define zero to be arbitrarily objectual – thus, we introduce the first instance of a complete absolution of the individuality of a platonically haecceitious Quinean universal. *Zero* is arbitrary in the sense that we claim any entity designated by the Ontological name a – singularly bound by a certain functor and satisfying the condition of coimplicative non-emptiness – to harbour the capacity for zero-identity:[3]

DF. 3:4.1: $\forall a[\forall\varphi(\varphi\{a\} \iff \varphi\{\bigwedge\}) \iff 0\{a\})]$

1 We appreciate the given definition of finiteness to be essentially an abbreviation of that offered by Clay [13]. As observed in Clay's *Note on Inductive Finiteness in Mereology,* one may express the concept of a finite name by considering *inductive finiteness:*

$$\forall\varphi\forall a\big(InR\langle\varphi\rangle\{a\} \iff \{\varphi\{\bigwedge\} \wedge \forall A\forall b[(A\varepsilon a \wedge \varphi\{b\}) \implies \varphi\{b \cup a\}]\}\big)$$
$$\forall\varphi\forall a\{Finite\{a\} \iff [InR\langle\varphi\rangle\{a\} \implies \varphi\{a\}]\}.$$

2 We have, in addition, the following theses:

$$\forall A\forall a[(Finite\{a\} \wedge A\varepsilon A) \implies Finite\{a \cup A\}]$$
$$Num(Finite)$$
$$\forall a\forall b\{[Finite\{a\} \wedge b \subset a \wedge \neg(a \subset b)] \implies \neg(a\infty b).$$

3

$$0\{\bigwedge\}$$
$$\forall\varphi\forall a[(0\{a\} \wedge \varphi\{a\}) \implies \varphi\{\bigwedge\}].$$

As immediate, trivial theses, we establish the numericality and quantity of *zero* ($Num(0)$ and $Qua(0)$, respectively). We define *successor* as follows – φ is the successor of a if and only if there exists some A such that A is a and φ binds the *complement* of a relative to A (*i.e.,* $\forall A \forall a \forall b[A\varepsilon a - b \iff A\varepsilon a \wedge \neg(A\varepsilon b)]$; A is the complement of a relative to b if and only if A is a and it is not the case that A is b):

DF. 3:4.2: $\forall\varphi\forall a[\mathbf{s}\langle\varphi\rangle\{a\} \iff \exists A(A\varepsilon a \wedge \varphi\{a - A\})]$

We obtain the theses:

TH. 3:4.2.1: $\forall\varphi[\neg\exists a(0\{a\} \wedge \mathbf{s}\langle\varphi\rangle\{a\})]$

TH. 3:4.2.2: $\forall\varphi\forall\psi\forall a\{[Num(\varphi) \wedge \mathbf{s}\langle\varphi\rangle\{a\} \wedge \mathbf{s}\langle\psi\rangle\{a\}] \implies \exists b(\varphi\{b\} \wedge \psi\{b\})\}$

TH. 3:4.2.3: $\forall\varphi[Num(\varphi) \implies Num(\mathbf{s}\langle\varphi\rangle)]$

TH. 3:4.2.4: $\forall\varphi[Qua(\varphi) \implies Qua(\mathbf{s}\langle\varphi\rangle)]$

Thus, we take the following proposition as the *axiom of infinity,* declaring a to be finite if there exists any such A that is not amongst a (assuming a *multiplicity* thereof):

AX. 3:5: $\forall a\{Finite\{a\} \implies \exists A[A\varepsilon A \wedge \neg(A\varepsilon a)]\}$

We define the *natural number* in the traditional fashion – if a succesive progression of names are common to a functor μ, we consider all such instances of succession *natural:*

DF. 3:6: $\forall\varphi\big[Nat(\varphi) \iff \forall\mu(\mu(0) \wedge \forall\psi\{[\mu(\psi) \implies \mu(\mathbf{s}\langle\psi\rangle)] \implies \mu(\varphi)\})\big]$

A *higher epsilon* denotes various functorial characteristics:4

DF. 4:1: $\forall\Phi\forall\varphi\forall\psi\big[\varepsilon\langle\psi\rangle(\Phi\varphi) \iff \big(\exists a(\Phi\{a\} \wedge \varphi\{a\}) \wedge \forall a\forall b\{[(\Phi\{a\} \wedge \Phi\{b\})$
$\implies \psi\{ab\}] \wedge [(\Phi\{a\} \wedge \psi\{ab\}) \implies \Phi\{b\}] \wedge [(\varphi\{a\} \wedge \psi\{ab\})$
$\implies \varphi\{b\}]\}\big)\big]$

We shall be concerned with the following specific case, wherein we introduce the concept of the *numerical epsilon:*5

$$\begin{aligned}\forall\Phi\forall\varphi\forall\psi\big[\varepsilon\langle\infty\rangle(\Phi\varphi) \iff \big(&\exists a(\Phi\{a\} \wedge \varphi\{a\}) \wedge \forall a\forall b\{[(\Phi\{a\} \wedge \Phi\{b\}) \implies a\infty b]\\ &\wedge[(\Phi\{a\} \wedge a\infty b) \implies \Phi\{b\}] \wedge [(\varphi\{a\} \wedge a\infty b) \implies \varphi\{b\}]\}\big) \iff \Phi \underset{\infty}{\varepsilon} \varphi\big]\\ &\iff \exists a[\Phi\{a\} \wedge \varphi\{a\} \wedge Qua(\Phi) \wedge Num(\Phi) \wedge Num(\varphi)]\big].\end{aligned}$$

We prove the *numerical epsilon* analogue of Leśniewski's 1921 axiom of Ontology. (See 4.3 for proof.)

$$\forall\Phi\forall\psi([\Phi \underset{\infty}{\varepsilon} \varphi \iff \underset{\infty}{\varepsilon}\langle\Phi\rangle(\varphi)]$$

$$\forall\Phi\forall\varphi\big(\Phi \underset{\infty}{\varepsilon} \varphi \iff \{\exists\Psi(\Psi \underset{\infty}{\varepsilon} \Phi \wedge \Psi \underset{\infty}{\varepsilon} \varphi) \wedge \forall\Psi\forall A[(\Psi \underset{\infty}{\varepsilon} \Phi \wedge A \underset{\infty}{\varepsilon} \Phi) \implies \Psi \underset{\infty}{\varepsilon} A]\}\big)$$

$$\forall\Phi\forall\varphi\forall\alpha\big(\{\varphi \underset{\infty}{\varepsilon} \Phi \wedge \forall\Psi(\Psi \underset{\infty}{\varepsilon} \Phi \implies \Psi \underset{\infty}{\varepsilon} \alpha) \wedge \forall\Psi\forall\psi[(\Psi \underset{\infty}{\varepsilon} \Phi \wedge \psi \underset{\infty}{\varepsilon} \Phi) \implies \Psi \underset{\infty}{\varepsilon} \psi]\} \implies \Phi \underset{\infty}{\varepsilon} \alpha\big)$$

$$\forall A\forall a\forall B\big(\{B\varepsilon A \wedge \forall C(C\varepsilon A \implies C\varepsilon a) \wedge \forall C\forall D[(C\varepsilon A \wedge D\varepsilon A) \implies C\varepsilon D]\} \implies A\varepsilon a\big).$$

4 Brief exposition and utilisation of such 'higher' epsilon connectives are offered in Canty [8] and Davis [18].

5 Due in entirety to Canty [8].

Equipped with a logically strengthened apparatus for interrogating the nature of numerical concepts, it is useful at this stage to consider the system of arithmetic – as used in practice – from a Dedekindian perspective. We invoke a Dedekindian notion of the of the *chain* as some set K relative to ϕ such that the ϕ-image of K, K', is a (possibly non-strict) subset of K. Thus, we proceed to define A_0 as the intersection of all the chains in U for which A is a subset:

$$A_0 = \bigcap \{U_1, U_2, U_3, \dots \mid (A \subseteq U_n) \wedge [\phi(U_n) \subseteq U_n]\}$$

$$u_1 \longrightarrow u_1' \longrightarrow u_1'' \longrightarrow u_1''' \longrightarrow u_1'''' \longrightarrow \cdots$$

Figure 6: The Dedekind chain U_1 with distinct elements $u_1', u_1'', u_1''', \dots$

Introducing further set-theoretic definitions, two sets U and V are said to be *similar* $(U \sim V)$ if there exists a *one-to-one correspondence* by virtue of the function ϕ mapping V to U (and thus, trivially, $\phi^{-1}(U) = V$). The set U is defined to be be *infinite* ($Inf(U)$) if it is similar (*i.e.* can be mapped *one-to-one-and-onto*) to a *proper subset* of itself:

$$\begin{aligned} U \sim V &\iff \exists \phi : V \mapsto U \\ Inf(U) &\iff U \sim A(A \subset U). \end{aligned}$$

Axiomatising these principles, we can say that there indeed exists a one-to-one function $\phi : \mathbb{N} \mapsto \mathbb{N}$, such that the ϕ-image $\mathbb{N}'$ is a subset of $\mathbb{N}$ and the chain of the *base element* 1 not contained in $\mathbb{N}'$ is $\mathbb{N}$. From the above definition, it follows that $\mathbb{N}$ is infinite, and therefrom the fundamental Peano-Dedekind axioms are concretised:

$$\exists \phi : \mathbb{N} \mapsto \mathbb{N} \qquad \mathbb{N}' \subseteq \mathbb{N} \qquad 1_0 = \mathbb{N} \qquad 1 \notin \mathbb{N}'$$

$$Inf(\mathbb{N})$$

The Peano-Dedekind axioms, of course, take the following form (which we express in the language of Ontology, as modified[6] from Srzednicki and Stachniak [84], *p. 129*; Leśniewski's '*nat*' is taken as primitive, and is thus in some regard dissimilar to the local functor '*Nat*'):

$$\begin{aligned} &0\varepsilon nat \\ &\forall A[A\varepsilon nat \implies \mathbf{s}(A)\varepsilon nat] \\ &\forall A\{A\varepsilon nat \implies \neg[\mathbf{s}(A) = 0]\} \\ &\forall A \forall B\big(\{A\varepsilon nat \wedge B\varepsilon nat \wedge [\mathbf{s}(A) = \mathbf{s}(B)]\} \implies A = B\big) \\ &\forall A \forall a\big(\{0\varepsilon a \wedge \forall B[(B\varepsilon nat \wedge B\varepsilon a) \implies \mathbf{s}(B)\varepsilon a] \wedge A\varepsilon nat\} \implies A\varepsilon a\big). \end{aligned}$$

It is in the latter of these axioms, that of *numerical induction*, that the immediacy of a conceptual infinity proves itself to be a necessity. We wish to express this axiom system by virtue of the numerical epsilon, recovering arithmetic nominalistically and applying our reparation to the thesis of Modal Structuralism. Expressing the first of these axioms is a straightforward task:

TH. 4:3: $0 \overset{\varepsilon}{\underset{\infty}{}} Finite$

It is noted that the above thesis is indeed a consequence of earlier principles, specifically of those detailed in 4.5. The respective second axiom – establishing the nature of *successor* – is reinterpreted in the form of two theses:

6 Srzednicki and Stachniak [84] do not follow the convention of defining *zero* to be a natural number; we therefore modify the axioms to include this condition.

TH. 4:4.1: $\forall\Phi\big[\big(\forall a\{\mathit{Finite}\{a\} \implies \exists A[A\varepsilon a \wedge \neg(A\varepsilon a)]\} \wedge \Phi \underset{\infty}{\overset{\varepsilon}{}} \mathit{Finite}\big) \implies \mathbf{s}\langle\Phi\rangle \underset{\infty}{\overset{\varepsilon}{}} \mathit{Finite}\big]$

PF. $\forall\Phi$

1.	$\forall a\{\mathit{Finite}\{a\} \implies \exists A[A\varepsilon a \wedge \neg(A\varepsilon a)]\}$	*Supp.*
2.	$\Phi \underset{\infty}{\overset{\varepsilon}{}} \mathit{Finite}$	*Supp.*
3.	$\mathit{Qua}(\Phi) \wedge \mathit{Num}(\Phi)$	DF. 4:1, 2
	$\exists a$	
4.	$\Phi\{a\} \wedge \mathit{Finite}\{\Phi\}$	DF. 4:1, 2
	$\exists A$	
5.	$A\varepsilon A \wedge \neg(A\varepsilon a)$	1, 6
6.	$(a \cup A) - A \circ a$	*Th.*, 5
7.	$A\varepsilon a \cup A$	*Th.*, 5
8.	$\Phi\{(a \cup A) - A\}$	*Th.*, 5, 6
9.	$\mathbf{s}\langle\Phi\rangle\{a \cup A\}$	*Th.*, 7, 8
10.	$\mathit{Finite}\{a \cup A\}$	*Th.*, 4, 5
11.	$\mathbf{s}\langle\Phi\rangle \underset{\infty}{\overset{\varepsilon}{}} \mathit{Finite}$	*Th.*, 3, 9, 10

TH. 4:4.2: $\forall\Phi(\Phi \underset{\infty}{\overset{\varepsilon}{}} \mathit{Finite} \implies \mathbf{s}\langle\Phi\rangle \underset{\infty}{\overset{\varepsilon}{}} \mathit{Finite})$

By consequence of the axiom of infinity, we observe TH. 4:4.2 to be a *numerical epsilon* correlate of the second Peano-Dedekind axiom. We progress to derive the third corrrelate of Peano-Dedekind's axioms, which establishes that if A is any natural number, the successor of A cannot be identical to zero:

TH. 4:5.1: $\forall\varphi[\neg(0 \underset{\infty}{\overset{=}{}} \mathbf{s}\langle\varphi\rangle)]$

DF. 4:5.2: $\forall\Phi\forall\Psi\{[\Phi \underset{\infty}{\overset{\varepsilon}{}} \Phi \wedge \Psi \underset{\infty}{\overset{\varepsilon}{}} \Psi \wedge \neg(\Phi \underset{\infty}{\overset{\varepsilon}{}} \Psi)] \iff \Phi \underset{\infty}{\overset{\neq}{}} \Psi\}$

TH. 4:5.3: $\forall\Phi(\Phi \underset{\infty}{\overset{\varepsilon}{}} \mathit{Finite} \implies 0 \underset{\infty}{\overset{\neq}{}} \mathbf{s}\langle\Phi\rangle)$

Penultimately, we declare that natural numbers with identical successors are, in turn, identical:

TH. 4:6: $\forall\Phi\forall\Psi\{[\Phi \underset{\infty}{\overset{\varepsilon}{}} \mathit{Finite} \wedge \Psi \underset{\infty}{\overset{\varepsilon}{}} \mathit{Finite} \wedge (\mathbf{s}\langle\Phi\rangle \underset{\infty}{\overset{=}{}} \mathbf{s}\langle\Psi\rangle)] \implies \Phi \underset{\infty}{\overset{=}{}} \Psi\}$

PF. $\forall\Phi\forall\Psi$

1.	$\Phi \underset{\infty}{\overset{\varepsilon}{}} \mathit{Finite}$	*Supp.*
2.	$\Psi \underset{\infty}{\overset{\varepsilon}{}} \mathit{Finite}$	*Supp.*
3.	$\mathbf{s}\langle\Phi\rangle \underset{\infty}{\overset{=}{}} \mathbf{s}\langle\Psi\rangle$	*Supp.*
4.	$\mathit{Qua}(\Phi) \wedge \mathit{Num}(\Phi)$	*Th.*, 1
5.	$\mathit{Qua}(\Psi) \wedge \mathit{Num}(\Psi)$	*Th.*, 2
6.	$\mathbf{s}\langle\Phi\rangle \underset{\infty}{\overset{\varepsilon}{}} \mathbf{s}\langle\Psi\rangle$	*Th.*, 3
7.	$\exists a(\mathbf{s}\langle\Phi\rangle\{a\} \wedge \mathbf{s}\langle\Psi\rangle\{a\})$	*Th.*, 6
8.	$\exists b(\Phi\{b\} \wedge \Psi\{b\})$	*Th.*, 5, 7
9.	$\Phi \underset{\infty}{\overset{\varepsilon}{}} \Psi$	*Th.*, 4, 5, 8
10.	$\Psi \underset{\infty}{\overset{\varepsilon}{}} \Phi$	*Th.*, 4, 5, 8
11.	$\Phi \underset{\infty}{\overset{=}{}} \Psi$	*Th.*, 9, 10

As a thesis, we obtain the principle of mathematical induction (the extensive proof is committed to the appendix; see 4.7):

TH. 4:7.8: $\forall\Phi\forall\varphi\big(\{0 \underset{\infty}{\overset{\varepsilon}{}} \varphi \wedge \forall\Psi[(\Psi \underset{\infty}{\overset{\varepsilon}{}} \mathit{Finite} \wedge \Psi \underset{\infty}{\overset{\varepsilon}{}} \varphi) \implies \mathbf{s}\langle\Psi\rangle \underset{\infty}{\overset{\varepsilon}{}} \varphi] \wedge \Phi \underset{\infty}{\overset{\varepsilon}{}} \mathit{Finite}\} \implies \Phi \underset{\infty}{\overset{\varepsilon}{}} \varphi\big)$

An inspection of the Peano-Dedekind axioms, as constructed in this manner, may lead one to expect the following equivalence:

$$\forall\Phi[\Phi \underset{\infty}{\overset{\varepsilon}{}} \Phi \iff \mathit{Crd}(\Phi)].$$

That is, cardinal numbers are, *on a fundamentally Ontological level,* mere numerical individuals – similarly, it is rightly expected that natural numbers are just finite numerical individuals.

2.2 THE INVOCATION OF MODALITY

An outline for general *desiderata* of the nominalist school of thought were earlier touched upon. However, what was left unconsidered – and what remains at present unappreciated – is the *modal* context within which these desiderata were originally formulated; *'I say that worlds are individuals, not sets. I say that worlds are particulars, not universals. [...] I say that worlds are concrete'* (Lewis [50], *p. 83*).

Lewis talks of a plurality of concrete *possible worlds*, the *'logical space'* wherein concrete *possibilia* reside; all that *could have been* is extant in some *other* world, inasmuch as that which is uncontroversially extant is indeed extant in ours. Should one take the possible world to be *abstract*, the concept indeed becomes inferrably counter-intuitive to nominalism (even in its most liberal of formulations).

One need not be as confident and resolute as Lewis with regards to the ontology of modality in order to gather a firm conceptual appreciation thereof. The elegance of modality as applied to a structuralist context lies precisely in its intuitionistic nature – Lewisian possible worlds theory, whilst highly explanatory and philosophically rich, need not be fatedly pervasive to the structuralist. Expectedly, it is this structuralist context within which modality shall be incorporated.

The practise of modal logic, to the extent of our concern, is the interrogation of the *possible* and *necessary* (denoted by the symbolic operators $\Diamond$ and $\Box$ respectively; *i.e.*, '$\Diamond\varphi$' is the proposition that φ is possible), an investigation into the contingent *modes* of actuality. In concerning ourselves with the *'counterfactual aspect'* of mathematical statements and the mathematico-logical notion of *possibilia*, the underlying framework is – naturally – the modal logic *S5*, characterised by the axiom system:

AX. 5:1.1: $\Box(\varphi \implies \psi) \implies (\Box\varphi \implies \Box\psi)$

AX. 5:1.2: $\Box\varphi \implies \varphi$

AX. 5:1.3: $\Diamond\varphi \implies \Box\Diamond\varphi$

The precise manner in which these modal operators function will naturally become apparent through utilisation.

It is of course desirable, provided the earlier analysis of the natural numbers, to take a pure number-theoretical statement as *elliptical* for a statement pertaining to a natural-number structure, *i.e.*, a *progression* as determined by the derived Peano-Dedekind thesis (conventionally termed an ω-sequence). The modal variant of structuralism is appealing to the nominalist for a multitude of reasons, not least due to the nature of the ω-sequence. Consider, for instance, the abstract *ante rem* structure of the ω-sequence, henceforth $\omega_{Seq.}\langle\mathbb{N},\phi,1\rangle$, where 1 is the 'initial place'[7]. Appreciating the *abstract order type* $<_{\mathbb{N}}$ as determined by the process of iteration, the second-order logical proposition expressed by the infinitary formula (defining the *isomorphism type* of $\omega_{Seq.}$) can be formalised as follows:

$$\exists X[[\bigwedge_{i\neq j}(x_i \neq x_j)] \wedge (\bigwedge_{i\in\mathbb{N}} Yx_i) \wedge \forall x[Yx \implies \bigvee_{i\in\mathbb{N}}(x = x_i)] \wedge (\bigwedge_{i,j\in\mathbb{N}}\{[(i <_{\mathbb{N}} j) \implies Zx_ix_j]$$

$$\wedge[\neg(i <_{\mathbb{N}} j) \implies \neg Zx_ix_j]\})]$$

Wherein Y, Z are free second-order variables and $X = \{x_0, x_1, x_2, \ldots\}$. The *ante rem* (ostensibly, the *'non-modal'*) structuralist is faced with an issue reminiscent of Einsteinian *'hole arguments'* in relativistic space-time theory – allowing $\pi : \mathbb{N} \mapsto \mathbb{N}$ to be a non-trivial permutation, the resultant arbitrary order type can be denoted as $<^{\pi}_{\mathbb{N}}$ and the corresponding ω-sequence is inferrably $\omega^{\pi}_{Seq.}\langle\mathbb{N},<^{\pi}_{\mathbb{N}}\rangle$. It follows that $\omega_{Seq.} \neq \omega^{\pi}_{Seq.}$ *and* $\omega_{Seq.} \cong \omega^{\pi}_{Seq.}$, thus producing insurmountable ambiguity; which ω-sequence can be said to be the *'real' ante rem* abstraction? This,

7 Hellman defines the ω-sequence in this fashion:

> 'Suppose we had the ante rem structure for the natural numbers, call it $\langle N, \varphi, 1\rangle$, where φ is the privileged successor function, and 1 the initial place. Obviously, there are indefinitely many other progressions, explicitly definable in terms of this one, which qualify equally well as referents for our numerals and are just as "free from irrelevant features"; simply permute any (for simplicity, say finite) number of places, obtaining a system $\langle N', \varphi', 1'\rangle$, made up of the same items but "set in order" by an adjusted transformation, φ'' (Hellman [32], *p. 520*).

we observe, is elevated beyond the status of a trivial *pseudo*-problem, manifesting itself as fundamentally problematic;[8] as noted by Hellman, '[...] with purported *ante rem* structures, we can see again why not, since multiple, equally valid identifications compete with one another as "uniquely correct" [...] Hyperplatonist astraction, far from transcending the problem, leads straight back to it' (Hellman [32], *p. 521*).

Nominalistic modal structuralism provides precisely this necessary transcendence. Generalising the notion of the previously given ω-sequence to all *possible* ω-sequential forms – that is, to all sequences of the form $\Omega := \langle \omega, <, I \rangle$. where ω is a (possibly infinitary) arithmetic set, *e.g.* the set of finite von Neumann ordinals, $<$ is the ordering relation for the elements of the aformentioned set, and I is the relative *initial place* (*i.e.,* the zero-entity described by the thesis $0 \underset{\infty}{\overset{\varepsilon}{}} Finite$) – we detail a translative pattern that sends an arithmetical sentence S to a modal conditional:

$$\Box \forall A[A \varepsilon \Omega \implies (A \vdash S)].$$

The manner in which one may avoid existential import whilst constructing the natural numbers has been detail at length, and it is therefore appreciable that the sole axiom of infinity possesses the capacity for on a megetheological basis. However, we assert categorically – and perhaps tentatively – the proposition

$$\Diamond \exists X(X \varepsilon \Omega)$$

Where the background logic $S5$ excludes the Barcan formula. Thus, we have instances of the following:

$$\Diamond \exists x[\varphi(x)] \implies \Box \Diamond \exists x[\varphi(x)]$$

But not instances of the following, which would require the pernicious quantification over Lewisian *possibilia* – *i.e.,* the inference of actually possible existence from possibly actual existence:

$$\Diamond \exists x[\varphi(x)] \implies \exists \Diamond x[\varphi(x)].$$

By a Leśniewskian standard, the expression '$A\varepsilon\Omega$' as earlier employed in the assertion of categoricity is unforgivably imprecise. We may wish, for instance, to form a model-theoretic statement based on the *satisfaction* of the conjunction of the second-order Dedekind-Peano axioms:

$$\Box \forall A[(A \vdash \bigwedge \mathbf{PA}^2) \implies (A \vdash S)].$$

We consider model-theoretic satisfaction in this instance to ensure the second-order quantifier in the (higher-order) numerical induction principle[9] –

$$\forall \psi \big[\big(\forall a \{ \forall b [a \varepsilon \neg \mathbf{s}(b)] \implies \psi(a) \} \wedge \forall n \{ \psi(n) \implies \psi[\mathbf{s}(n)] \} \big) \implies \forall n [P(n)] \big]$$

– ranges over the subsets of the domain of A, where A is understood as a pair consisting of a domain and a valuation interpreting the predicate constants of the language of second-order Peano arithmetic. This is precisely Dedekind's characterisation of a *simply infinite system:* any two full models of the second-order axioms are isomorphic, where a second-order model is *full* in case the range of the k-ary relation quantifiers is the set of *all* subsets of D^k, with D the domain of the model. Dedekind's ambiguous (and philosophy suspicious) 'proof' of such infinite structures exemplifies the distinction between *actual* and *possible* infinity, a distinction that shall soon be drawn.

The second-order induction scheme is evidently an instance of set-theory, which, for respect of parsimony, we wish to avoid. As noted by Hellman [31], if we were to take the infinite conjunction of the first-order inductive Peano-Dedekind axioms as the antecedent of a modal conditional, we would not adequately express the restriction to ω-sequences, as illustrated in particular by the *Henkin compactness argument*. If one accepts an incomplete restriction in this fashion, however, the full truth-determinateness of arithmetical translates are dealt a great disservice; the modal scheme would allow sentences undecided by the first-order schemata to be rendered *untrue,* insofar as the would fail to hold in all models of the precisely these

8 As a direct result of the assault cast on abstract universals by Leśniewski's Ontology.

9 Intuitively, any class containing zero and closed under the successor relation contains every natural number.

schemata. Structuralism, therefore, would lapse into what Kessler [34] regards as a *semantic equivalent of deductivism.*

Constructing number theory in modal structuralism carries over *mutatis mutandis* to real analysis, etc.

Second-order axiom schemata are uniquely capable of expressing what is is to *be an ω-sequence;* thus, our second-order background theory takes the form of a functorial (extensional) Leśniewskian Mereology as outlined in 4.6. In *Mathematics without Numbers,* Hellman expresses a distinctly mereological sentiment whilst hypothesising an adequate background logic (parallelled also by Lewis in *Parts of Classes* [51]): *'If we abstract from the concerns of a strict nominalism, this extensional interpretation of second-order logical notation is a natural candidate for articulating structuralism [] we allow classes as occurring exclusively on the right of $\in$: we collect whatever individuals of a domain we ae given (actual, or hypothetical), but we go no further in "collecting these collections"'* (Hellman [31], *p. 20*). Evidently, the background logic by which we articulate structuralism is indeed of an appreciable degree of strictness.

2.2.1 *The Successor Function and Infinitude*

Of the few structuralist principles that rupture the fortified edifice of nominalism, the ω-sequence's *successor function* is certainly worthy of note. Lewis briefly appeals to this notion of the ω-sequence in his characterisation of structuralism: '[structuralism] says that there's no one sequence that is *the* number sequence; rather arithmetic is the general theory of omega-sequences'. It is immediately intuitive upon explication of the nature of the ω-sequence that any particular number is without 'haecceity'; each number, structuralistically construed, is a result of repeated iteration of some successor function to some entity 'acting' as *zero* (naturally, we present this nominalistically as concreta). Following Kreisel and Dedekind, any appropriate functor φ can act precisely as this successor function, with the intuitive property that $\varphi(x)$ is distinct from x and from all y iterated prior to x.

It seems deceptively apropos, therefore, to simply take an arithmetical sentence S as elliptical for the following modal conditional:

$$\Box(\bigwedge \mathbf{PA}^2 \implies S).$$

The above conditional, however, does not respect the modal-logical situation wherein an *object were the successor of no object; i.e.,* the successor requires requires a more sophisticated treatment, both Ontologically and Mereologically. Provided that a successor function circumvents the aforementioned issue, we may proceed to form the sentence:

$$\Box\forall A\forall f(\bigwedge \mathbf{PA}^2 \implies F)^{A}\langle{}^{\mathbf{s}}_{f}\rangle$$

Where A is a class variable, f is a unary function variable, F is a non-modal formula, all logical quantifiers are *relativised* to the domain of A, and the far right expression abbreviates the systematic subsitution of all instances of the successor function $\mathbf{s}$ with any such appropriate 'f' throughout. We achieve a *modal-mathematical* way of expressing the statement that an arithmetical sentences holds in *any possible ω-sequence*. First-order quantifiers are relativised as follows –

$$\forall a\varphi \text{ becomes } \forall a[A(a) \implies \varphi], \quad \exists x\varphi \text{ becomes } \exists x[A(a) \wedge \varphi]$$

– and higher-order quantifiers take the more grandiloquent form:

$$\forall R\varphi \text{ becomes } \forall R\{\forall a_1, \forall a_2, \ldots, \forall a_k[R(a_1, a_2, \ldots, a_k) \implies \bigwedge_{i=1}^{k} A(a_i)] \implies \varphi\},$$

$$\exists R\varphi \text{ becomes } \exists R\{\forall a_1, \forall a_2, \ldots, \forall a_k[R(a_1, a_2, \ldots, a_k) \implies \bigwedge_{i=1}^{k} A(a_i)] = \wedge\varphi\}.$$

In order to detail the specific nature of the successor function, we introduce the following definitions and theses:[10]

DF. 5:2.1 $\quad \forall\varphi\forall\psi[\circ\not\{\varphi\psi\not\} \iff \forall a\forall b(\varphi\{ab\} \iff \psi\{ab\})]$

10 As adopted from Davis [18].

DF. 5:2.2: $\forall\varphi\forall\theta[\mathfrak{R}\langle\theta\rangle\llparenthesis\varphi\rrparenthesis \iff \forall a(\theta\{a\} \implies \varphi\{aa\})]$

DF. 5:2.3: $\forall\varphi\forall\theta\{\mathfrak{T}\langle\theta\rangle\llparenthesis\varphi\rrparenthesis \iff \forall a\forall b\forall c[(\theta\{a\}\wedge\theta\{b\}\wedge\theta\{c\}\wedge\varphi\{ab\}\wedge\varphi\{bc\}) \implies \varphi\{ac\}]\}$

DF. 5:2.4: $\forall\varphi\forall\theta\{\mathfrak{A}\langle\theta\rangle\llparenthesis\varphi\rrparenthesis \iff \forall a\forall b[(\theta\{a\}\wedge\theta\{b\}\wedge\varphi\{ba\}\wedge\varphi\{bc\}) \implies a\circ b]\}$

DF. 5:2.5: $\forall\varphi\forall\theta\{\mathfrak{O}^{pt.}\langle\theta\rangle\llparenthesis\varphi\rrparenthesis \iff [\mathfrak{R}\langle\theta\rangle\llparenthesis\varphi\rrparenthesis\wedge\mathfrak{T}\langle\theta\rangle\llparenthesis\varphi\rrparenthesis\wedge\mathfrak{A}\langle\theta\rangle\llparenthesis\varphi\rrparenthesis]\}$

DF. 5:2.6: $\forall\varphi\forall\theta\forall a\{\mathfrak{P}\llparenthesis\theta\varphi\rrparenthesis\{a\} \iff [\mathfrak{O}^{pt.}\langle\theta\rangle\llparenthesis\varphi\rrparenthesis\wedge\theta\{a\}\wedge\forall b(\theta\{b\} \implies \varphi\{ab\})]\}$

TH. 5:2.7: $\forall\varphi\forall\theta\forall a\forall b[(\mathfrak{P}\llparenthesis\theta\varphi\rrparenthesis\{a\}\wedge\mathfrak{P}\llparenthesis\theta\varphi\rrparenthesis\{b\}) \implies a\circ b]$

DF. 5:2.8: $\forall\varphi\forall\theta\big(\mathfrak{O}^{wl.}\langle\theta\rangle\llparenthesis\varphi\rrparenthesis \iff \{\mathfrak{O}^{pt.}\langle\theta\rangle\llparenthesis\varphi\rrparenthesis\wedge\forall\rho[(\rho\subset\theta\wedge!\llparenthesis\rho\rrparenthesis) \implies \exists a(\mathfrak{P}\llparenthesis\rho\varphi\rrparenthesis\{a\})]\}\big)$

Ultimately, we obtain a statement of the *well-ordering principle:*

$$\forall\theta[\exists\varphi(\mathfrak{O}^{wl.}\langle\theta\rangle\llparenthesis\varphi\rrparenthesis)].$$

Following Lewis et al. [51], we translate a quantifier over relations of mereological atoms, utilising a mereological translate of the above principle. For the dyadic case, we translate the expressions of the form 'For any relation $R : aRb$' into the uneconomic 'For any θ that orders the atoms φ, for some diatoms R_1 and R_2, and for some atoms R_3: either $\Sigma(a,b)\varepsilon R_1$ and a precedes b with respect to θ, or $\Sigma(a,b)\varepsilon R_2$ and b precedes a with respect to θ, or $a = b$ and $a\varepsilon R_3$'. Thus we have a nominalistic concept of relation, and thus of *successor;* the principle of well-ordering ensures that the successor relation functions only within adequate mereological conditions. All arithmetic operations can therefore be defined from successor and second-order quantification.

Appreciating this, we wish to outline a framework for the Modal Structuralist reinterpretation of a given theorem of Peano Arithmetic. Initially, we introduce

$$\Box\exists R\forall a_1,\forall a_2,\ldots,\forall a_k[R(a_1,a_2,\ldots,a_k) \iff F]$$

Where R is not free in the formula F, F is restricted from modal operators, and the names $a_i\ldots$ are individual variables. The latter condition ensures a restriction built into axiomatic second-order logic, to block Russell's paradox, for instance. If a theorem of PA is correctly deduced from finitely many first-order axioms, Alongside rules for the necessity operator, analagoues for Peano Arithmetic theorems will be deduced in second-order logic.

The modal existence principle –

$$\Diamond\exists A\exists f(\bigwedge\mathbf{PA}^2)^A\langle{}^s_f\rangle$$

– ensures the coherency of the very concept of a *ω-sequence*. In its absence, Modal Structuralism would collapse to what Hellman considers a form of *"if-thenism"*[11], treating all authentic mathematical sentences as *vacuously true* even if no relevant mathematical structures are *possible*. Given the above categorical assumption, the use of numbers as *apparently* referring *constants* receives a natural but deceptive interpretation (*i.e.,* sets as *'universals'* or *'haecceities'*).

In avoiding the ontological weight of postulating an *actual* infinity, the Modal Structuralist is, we submit, justified in accepting that, logically, it *may be true* that infinitude is possible:

$$\Diamond\big(\forall a\{Finite\{a\} \implies \exists A[A\varepsilon A\wedge\neg(A\varepsilon a)]\}\big).$$

Appealing to *comprehension*, we can prove:

$$\Box\big[\infty \implies \exists A\forall z\big(\{A(z) \iff \exists x[F(x,z)]\}\wedge\Omega\varepsilon A\big)\big]$$

Where, for a constructive rule R (a rule that establishes *successor*), $F(x,y)$ is the predicate *'y is generated after x in accordance with rule R'*. The field of the relation F is infinite:

$$\exists x\exists y\big(F(x,y)\wedge\{F(x,y) \implies \neg[F(y,z)]\}\wedge\forall z\{[F(x,y)\wedge F(y,z)] \implies F(x,z)\}\big)\wedge\forall x\exists!y\{[F(x,y)\wedge\neg\exists z[F(x,z)\wedge F(z,y)]\}.$$

And thus, appealling to the *axiom of infinity*:

$$\Diamond\exists A\forall z\{[A(z) \iff \exists xF(x,z)]\wedge\Omega\varepsilon X\}.$$

In addition, we can reconstruct the second-order comprehension scheme by virtue of extensional Leśniewskian Mereology. Specifically, by use of the *common part* functor *'ov'*:

11 *cf.* Hellman [31], *p. 26.*

$$\exists x \varphi(x) \implies \exists x \forall y \{ y \varepsilon ov(x) \iff \exists z [\varphi(z) \wedge z \varepsilon ov(y)] \}.$$

It was earlier noted that the natural numbers possessed the elusive capacity for both immediate reparability and great perniciousness – and thus, upon the execution of a successful reparation, it is only appropriate to investigate the latter of these defining characteristics. One need only briefly contemplate the concept of *infinity* to sympathise with the ontological difficulty at once beset on the nominalist. We observe the following passage from Quine:

> 'The nominalist has repudiated the infinite universe of universals as a dream world; he is not going to impute infinitude to his universe of particulars unless it happens to be infinite as a matter of objective fact – attested to, say, by the physicist' (Quine [63], *pp. 128–129*).

Whilst it may be appreciate Quine's sentiment in its entirety, there is a manner in which a certain concept of the infinity can greatly assist nominalistic Modal Structuralism, by way of extensional Mereology. Due to the recent developments of Carrara and Martino [9], it is a logical truth that, provided one accepts a number of Lewisian assumptions, an *uncountable* infinity of objects can be established. Carrara and Martino note that, whilst Lewis' plurally quantificatory mereology is solitarily unable to guarantee an infinity of concrete objects, once the existence of any infinite object is admitted, Lewis' mereology assures the uncountable infinity of objects. Generalising this result to extensional Leśniewskian Mereology – *i.e.,* demarcating the uncountable infinity of objects to be mereological atoms – one is able to establish a richer conception of mathematical infinity, and thus *'repair'* a substantial quantity of modern mathematics in a Modal Structuralist fashion.

Part III

3 CONCLUSION

3.1 ON THE TENABILITY OF PRIMITIVE MODALITY

Of the main causes for nominalistic concern pertitent to Modal Structuralism, *primitive modality* proves often to be of high regard. Leśniewski did not write extensively on temporality and the nominalistic interpretation thereof; on the contrary, the philosophical school of Reism, with which Leśniewski is closely associated, often reject temporal concepts in their entirety.[1] The consonance between nominalism – and thus Leśniewski's Mereology – and modality is often contentiously discussed.

'Can time go on apart from events?'[2] This is a very crucial question; not solely due to what is *governed* by the event, but what, precisely, *constitutes* the event. Ultimately, and perhaps rather counter-intuitively, we shall follow in Davidsonian suit and characterise the *event* as indeed *particular;* that is, *concrete* inasmuch as anything is incontestably considered as such.

A Davidsonian analysis of Chisholm's presentation of the event as belonging to the ever-present *particular-universal* dichotomy allows one to conclusively arrive at this characterisation. Chisholm speaks of events as *repeatable entities* or *'states of affairs'*, which can certainly be construed as a rendering of the event as 'universal':

> 'Are the states of affairs we have been discussing "abstract objects"? Only if we take "abstract object" negatively, and incorrectly, to mean any object that is not a concrete individual thing, but not if we take it positively to refer to objects that may be exemplified or instantiated in other objects. Given the third view singled out in paragraph one above, the present view might be called "Platonistic," but otherwise not' (Chisholm [10], *p. 22*).

The notion of the 'state of affairs', congruent with the intended meaning in the above passage, is explicated, also by Chisholm, as follows:

> 'According to the view that I defended, there are such things as states of affairs, some of which occur and some which do not occur. [. . .] some occur exactly once [. . .], some occur exactly twice [. . .], some occur exactly three times [. . .], and so on for many other numbers of times' (Chisholm [11], *p. 179*).

It precisely in this notion of 'recurrence' of the event that entails its 'universal' characteristic; there is some *property,* to use the term broadly (and perhaps inaccurately), that distinguishes a multitude of events, by definition temporally disjoint, as of *similar* (or, even, exact) nature. Chisholm provides the example of *John F. Kennedy being elected President twice;* the *event* here remains the same, though both occurences of the particular circumstance in question occur at different instantiations, and thus with different 'concrete compositions'. An allusion to the most basic illustration of the particular-universal dichotomy is immediately apparent; whilst two red apples are materially (and, trivially, temporally) distinct, they share an exact characteristic, that of *redness.* Acknowledging the universality of the Chisholmian *event,* one can apply Leśniewskian criticisms, precisely in the manner previously done so – thus, rejecting the event as *universal,* especialy in a Platonic sense, is certainly defensible.

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1 See Smith [77], *p. 117:* 'Stage 2 consists in the rejection of events, processes, states of affairs, other putative individuals falling outside the category *thing.*'
Stages 1 and 3, respectively, are precisely the rejection of 'universals, properties, or general objects' and the rejection of 'sets or classes'. One may imagine that Leśniewski would follow in this trend and reject temporal concepts, as it is clear that Leśniewski was deeply concerned with (and staunchly in opposition to) the *universals* and *sets* as previously mentioned. Indeed, Smith notes Leśniewski's influence of Reism: 'The set-theoretical antinomies had resulted, in Leśniewski's view, not from any inherent contradiction in the notion of set as originally conceived by Cantor, but from a departure from this notion in the direction of a conception of sets as abstract entities'. How the distinction between *abstract* and *concrete* was drawn in Leśniewski's writings remains ambiguous.

2 See Wittgenstein [91], *pp. 14–15:* 'Can time go on apart from events? What is the criterion for time involved in "Events began 100 years ago and time began 200 years ago"? Has time been created, or was the world created in time? [. . .] "Time" as a substantive is terribly misleading. We have got to make the rules of the game before we play it.'

To construe the event in this fashion is far from uncontroversial, but should not be viewed as an assumption of utmost radicality (if considered assumptive at all); how is the event dissimilar to the particular if not only by the criterion of *endurance?* This reading of the event is, much like many ontological considerations, inextricable from Quinean theory. In a passage from which Pianesi and Varzi note as exemplary of the materialistic 'extreme' of event philosophy[3], Quine unambiguously renders the event as the totality of what it occupies at a certain spatiotemporal region:

> 'Physical objects [...] are not to be distinguished from events. [...] Each comprises simply the content, however heterogeneous, of some portion of space-time, however disconnected or gerrymandered' (Quine [65], *p. 171*).

Of course, one observes a similar sentiment in the writings of Davidson:

> 'Our language encourages us in the thought that there are, by supplying not only appropriate singular terms, but the full apparatus of definite and indefinite articles, sortal predicates, counting , quantification, and identity-statements; all the machinery, it seems, of reference. If we take this grammar literally, if we accept these expressions and sentences of having the logical form they appear to have, then we are committed to an ontology of events as unrepeatable particulars ('concrete individuals')' [...]. An adequate theory must give an account of adverbial modification; for example, the conditions under which (1) 'Sebastian strolled through the streets of Bologna at 2 a.m.' is said to be true must make clear why it entails (2) 'Sebastian strolled through the streets of Bologna.' If we analyse (1) as 'There exists an x such that Sebastian strolled x, x took place in the streets of Bologna, and x was going on at 2 a.m.' then the entailment is explained as logically parallel with (many cases of) adjectival modification; but this requires events as particulars' (Davidson [16], *p. 181*).

It is unclear, on the Davidsonian view, whether the event is precisely the *endurance* of the three-dimensional *through* time (or at some temporal *instance*), or the *eo ipso* four-dimensional[4]. But this 'distinction', if one to ascribe such a generous term, is, it seems, a semantic discrepancy rather than a divergence of metaphysical perspectives. If the object, conventionally (or, rather, intuitively) three-dimensional, endures for some length of time, the *totality* of the endurance constitutes the event; at any irreducible instance, even, the object is ensnared in four-dimensionality. However one may perceive the object, to concede even a weakly-Quinean ontology of the event is to make uncontroversial one basal claim; the event, much like the particular, is concrete. So much so, in fact, that to draw a distinction between the event and the particular is superfluous.[5]

A close scrutinisation of the event carries even further importance when one examines precisely how the *de facto* Lewisian concrete-abstract distinction is drawn. To recall, in *On the Plurality of Worlds,* Lewis calls this distinction the 'Negative Way'; *'abstract entities have no spatiotemporal location; they do not enter into causal interaction; they are never indiscernible one from another.'* Clearly, Lewis approaches the distinction by detailing precisely what the abstract is *not;* and in *at least* two of these criteria – those concerning *spatiotemporality* and *causality* – the notion of time bears great significance.

Elements of *branching-time temporal logic* ([35], [21], [22] [6]) can be utilised in order to enrichen the spatio-temporal relations as currently developed. The concept of the *time-frame,* often called a *flow of time* ([24], [66], [69]), provides adequate expressivity for precisely the aforementioned notions. In *Łukasiewiczian Logic of Tenses and the Problem of Determinism,* Trzęsicki employs

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3 See Pianesi and Varzi [59], *p. 9:* 'Quine's view occupies one extreme position on the thick/thin continuum, corresponding to the thickest possible theory. At the other extremity, the continuum is open-ended: there is no thinnest possible account of events as particulars.'

4 Quine's succint physicalistic characterisation of the *event* is distinctly *four-dimensionalist.* For an extensive defense of the philosophy of four-dimensionalism, see Sider [74].

5 Irrespective of this, it ought to be noted that some elucidation is offered by Davidson. See Davidson [17], *p. 310–215*: 'The undulations of the ocean cannot be identified with the wave or the sum of waves that cross the sweep of ocean, nor can the complex event composed of condensations and evaporations of endless water molecules be identified with the lenticular cloud. Occupying the same portion of space-time, event and object differ. One is an object which remains the same object through changes, the other a change in an object or objects. Spatiotemporal areas do not distinguish them, but our predicates, our basic grammar, our ways of sorting do.' Thus, it is appreciated that Davidson's view was, at least at the time the previous passage was written, distinctly non four-dimensionalist.

the time-frame in his conception of Priorean and Łukasiewiczian tense operators. The definition for the time-frame and auxiliary temporal concepts are presented in the original paper as follows:

> Let $\alpha, \beta, \gamma, \ldots$ be formulas of propositional language (*Form*), *i.e.*, formulas constructed as usual from a denumerable set of propositional variables (*At*) by means of binary connective: $\rightarrow$; and the unary connectives: $\sim, p, f$. [...] A time-frame (a time) is a structure $\boldsymbol{T}(= \langle T, R \rangle)$ comprising a non-empty set T of times (moments, instants, events) on which R is a binary relation of precedence (earlier-later). [...] $\langle T, R \rangle$ is a Kripke frame. A valuation V on a frame $\langle T, R \rangle$ is a mapping from $At \times T$ to Lv, where Lv is a set of logical values. Each valuation V can be uniquely extended to a mapping from $Form \times T$ to Lv (Trzęsicki [88], *p. 296*).[6]

Following the fundamental concepts of this system, we introduce a Kripke frame $\mathfrak{F} = \langle \boldsymbol{T}, \preceq_t \rangle$, containing a *root* $\mathfrak{r}$ in $\boldsymbol{T}$. Intuitively, the root functions as the interval which is preceded by none; thus, $\forall i(\mathfrak{r} \preceq_t i)$. This temporal logic is extensible to express *realistic* situations such as the following:

$$(\Diamond \Diamond_t x) \wedge (\Diamond \neg \Box_t x).$$

As keenly noted by Putnam in *Time and Physical Geometry,* the underlying convictions of the layman – or the *'"man on the street's" view of the nature of time'* – could perhaps be summarised effectively by the sentiment *'All (and only) things that exist* ***now*** *are real'* (Putnam [61], *p. 240*). The notion of *reality* here is explicated not by virtue of defining the term itself, but by employing the term in three assumptions:

firstly, that *I-now* am real;

secondly, that *at least one* other observer is real, where it is possible for said observer to be in relative motion to oneself (I-now);

and thirdly, if it is the case that all and only things standing in the relation R to Me-now are real, it is also the case that all and only things in standing also in R to *You-now* are real, granted the existence of You-now[7].

The ultimate remarkable discovery is that, after some difficulty, assuming Special Relativity and taking R to be tenseless, *coordinate-system-independent* and transitive – that is, $[(xRy) \wedge (yRz)] \vdash xRz$ – then some event E in one's relative future is *real* by means of R-relation via some observer. Naturally, it is not as simple as this, and the argument does not remain unassailed[8].

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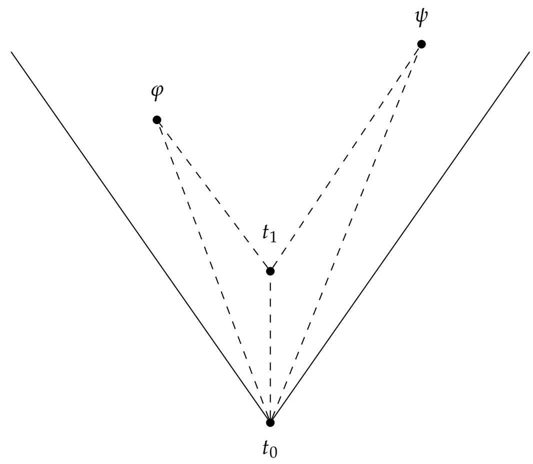
6 The original logistic as used by Trzęsicki is retained, with $\sim$ corresponding to $\neg$ in the local logistic and $\rightarrow$ corresponding to $\implies$. Trzęsicki uses the symbols p, f to denote *past* and *future* tense, respectively. It is noted that the Priorian and Peircean conceptions of *past-tense* are equivalent: 'For some philosophers, e.g. Peirce, the sentences '$p\alpha$' and '$f\alpha$' are true if and only if it is necessary that, respectively, it was the case that α and it will be the case that α. The rather strong 'will be' and 'was' are simply the Priorean 'necessarily will be' and 'necessarily was', respectively, while the Priorean 'will be' and 'was' are untranslatable. [...] Thus Priorean 'p' and Peirce's 'p' are materially equivalent' (Trzęsicki [88], *p. 295*). Trzęsicki notes also, however, the *non*-equivalence between the Priorian and Peircean conception of *future-tense:* 'The equivalence does not hold for Priorean and Peircean 'f'. [...] Peircean 'will be' [...] means: on every route into the future there is somewhere a point at which α is the case. His attitude to the future can be described as limited prederminism.' The tense operators employed locally differ markedly; most notably, the *past-tense* is not considered.

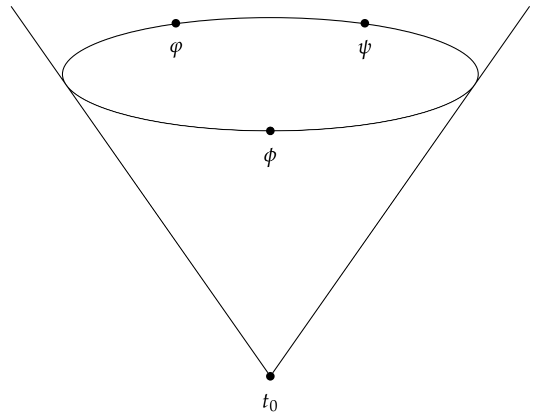
7 The principle that *There Are No Privileged Observers.*

8 For example, Saunders [72] raises an objection concerning the nature of R, subsequent to inferring that R must also be symmetric in Putnam's argument. The objection is precisely that, taking E_1 to be a fiduciary event (*real* by virtue of the first assumption) and defining events that are *real* as those which stand in relation R to E_1, then the set $\{E_2 : E_1 R E_2\}$ is specified by a *particular* element of it and not merely *any* element. Thus, when the 'element' in question is observer's event (and since R is 'surely reflexive', forcing R to be an equivalence relation), the principle that *There Are No Privileged Observers* is violated.
Stein also notes an error in Putnam's talk of the 'truth-value of statements', rather than of the *reality of events:* '[...] "having or not having a truth value" [...] must be understood classically to mean "at a given time" [...] but "at a given time" is not a relativistically invariant notion, and the question of definiteness of truth value, to make sense *at all* for Einstein-Minkowski space-time, has to be interpreted as meaning "definiteness at a given space-time point (or event) – to be vivid, "definiteness for me now". The "Privileged Observer" (or, rather, privileged event) is – in effect – named in the question, and therefore has every right to be considered germane to the answer. Putnam's objection has an exact analogue, whose appropriateness is plain, in the pre-relativistic case; namely, the question "why should a statement's having or not having a truth value depend upon the relation of the events referred to in the statement of just one special time, *now?*"' (Stein [86], *p. 15*).



$$\Diamond\varphi \wedge \Diamond\psi \Longrightarrow \Diamond(\Diamond\varphi \wedge \Diamond\psi)$$

Figure 7: Possible events at t_0 under the causal relation $\prec$?



$$((\bigwedge_i \Diamond\varphi_i) \wedge \{\bigwedge_{i\neq j} \Box[\varphi_i \Longrightarrow (\neg\varphi_i \wedge \neg\Diamond\varphi_j)]\}) \Longrightarrow [\bigvee_{i\neq j} \Diamond(\Diamond\varphi_i \wedge \Diamond\varphi_j)]$$

Figure 8: Future light-cone of t_0 in Minkowski space-time $\mathbb{T}^3$

But, what is of current scientific consensus[9] is the fundamental contrariety of Special Relativity and *presentism* (the *'"man on the street's" view'*): *'The notion of the present time that is so crucial to presentism is meaningless within Minkowski spacetime, in which there is no distinguished partition of spacetime into space and time, and no observer-independent notion of simultaneity'* (Sider [74], *p. 42*). We can thus, insofar as Relativity theory remains scientifically consensual, take this notion of 'persistent realness' of E to be a valid one; and thus, on a certain level, provide a phyicalistic account of primitive modality.

Further solidifying this point, due to proofs by Shehtman and Goldblatt, the logic of Minkowski spacetime is shown to be precisely the modal logic $S4$, extended with the confluence axiom

$$\Diamond\Box\varphi \implies \Box\Diamond\varphi$$

to form the modal logic $S4.2$. (See Figures 7, 8 and 9.)

9 Recent literature that argues for the inconsistency of presentism and Special Relativity (with the latter position taken as the more tenable of the two) includes: Zimmerman [93]; Savitt [73]; Balashov and Janssen [1]; Meyer [56].

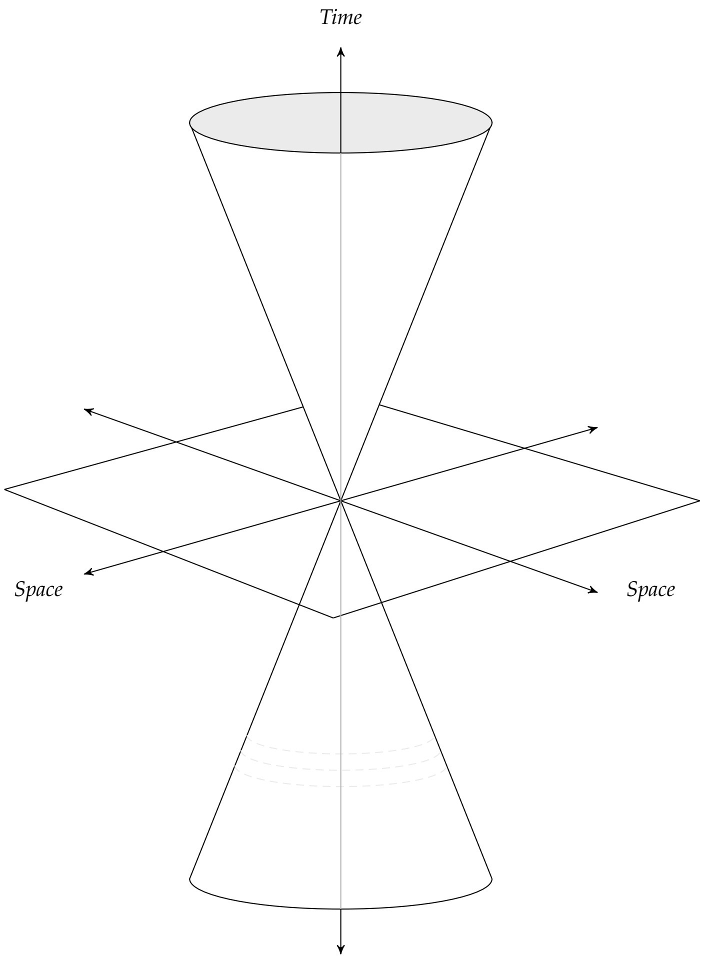


Figure 9: Minkowski space-time event, characterised by *hyperplanarity* and *achronality*

3.2 RECOVERING SET THEORY

Any philosophically–inclined individual acquainted with modern set theory is perhaps aware of a fundamental dubiety amidst the foundations thereof. The following excerpt is especially unsettling to those in firm repudiation of abstract universals:

> '[...] can a realist believe in sets, without apostasy from physicalism? If mathematics deals with universals (physical properties and relations), then what business do sets have in modern mathematics? [...] The answer is, that sets too are universals' (Bigelow [5], *p. 294*).

Whilst it is certainly commonplace for mathematicians to accept the philosophically tenebrous foundations of set theory on account of demonstrable efficacy, this was predictably not the attitude reflected by Leśniewski, for whom the ontological enigma of set theory was sufficient cause for the construction of Mereology – and, subsequently, the prior two sub-systems. With concern to the most fundamental of set-theoretic concepts, Leśniewski's Mereology indeed appears constitutionally consonant with its 'authentic' counterpart; more abstruse concepts, however, beset an exigent demand on the nominalist, with no immediately straightforward reparation. Irrespective of such difficulties, however, we maintain that Mereology is indeed *capable* of such a reparation.

In a manner closely echoed by the *manifold theory*[10] as developed Simons [75], mereological quantification can define inclusion predicates with a striking degree of identicality to the primitive set-theoretic concepts with which we are presently concerned. The principal idea in such a theory of manifolds is, as given by Simons, that *'sets are to plural terms as individuals are to singular terms'*. The composition of the individual is presented as the sum of its mereological atoms, and the sum of any multitude of individual – or, rather, the *fusion* – forms also a 'new' individual, compositionally identical to its parts. In what sense, then, does a *manifold,* or even a *set,* differ from simply a *fusion* or *sum?* The expressivity of mathematical set theory allows for not only the desired quantification over *pluralities* – that is, *'many'* individuals – but also profoundly unintuitive notions; the *countably infinite* set, the *uncountably infinite* set, and, more perplexingly still, the *null* set. Of what sense is the nominalist to make of the *null set,* the *set with no elements?* Surely, if one is to adopt the thesis of the *set* being simply the fusion of its members, the null set is rendered non-existent in the most concrete of senses; a dramatic disservice to the set-theoretic hierarchy, understood ultimately as an 'inverted pyramid of sets balanced on the empty one'[11]. Thus, it is this expressivity that ought be nominalistically reconstructable, and one ought concern oneself with the discrepancy between a *set-element* and an *individual amongst a plurality.*

The *manifold-element,* however, can be considered precisely this; *an individual amongst a plurality.* We understand quantification over pluralities to be 'ontologically innocent' – as articulated by Lewis, *'we have many things, we speak of them as many, [and] in no way do we mention one thing that is the many taken together'* (*Lewis et al. [51], p. 87).* To preserve this innocence, the manifold, or any similar basic construction, presents itself as a necessity.

Briefly returning to an earlier mentioned notion of *collection*, we observe in the following excerpt certain similarities to our current understanding of the *plurality:*

> 'A collection of *a*'s is an *object* which consists of a number of *a*'s. Unlike the class of *a*'s it need not comprise all *a*'s. The functor '*cl*' is one of the fundamental mereological constants' (Sobociński [80], *p. 220*; my emphasis).

I wish to emphasise the ontological status bestowed by Sobociński upon the *collection* in the above excerpt – namely, that the *collection* of *a*'s is, itself, an object. To illustrate precisely why such an ontological discrepancy is worthy of consideration, it is appropriate to recall a thought experiment courtesy of Putnam[12]. Consider a hypothetical *world* wherein three *distinct* objects – say, a, b and c – remain; then, consider the possible mereological fusions of precisely those objects, $\sum(a,b)$, $\sum(a,c)$, $\sum(b,c)$ and $\sum(a,b,c)$. Putnam submits that whether one considers the world to thus consists of *three objects,* solely the initial distinct a, b and c, or of *seven objects*, the fusions thereof inclusive, to be *framework-relative.* Whilst such a situation involving *fusions* remains undoubtedly problematic, the situation involving the *collection,* as detailed

10 The similarities between nominalistic manifold theory and Leśniewskian class theory are appreciated.
11 To borrow a phrase from Hand [29], *p. 429*.
12 *cf.* Putnam 1987.

above, proves to be considerably moreso. Foremostly, we omit the formal definition of *collection* and draw attention to the following elementary theses[13]:

$$[Aa] : A\varepsilon a. \supset .A\varepsilon cl(a)$$
$$[Aa] : A\varepsilon cl(a). \equiv .A\varepsilon cl(cl(a))$$

The former thesis states that, for any object a, one can proceed to form the *collection* thereof; the latter identifies the identicality between a collection of a's and a collection of collections of a's. Thus, returning to Putnam's hypothetical, we need only *one* initial object, and can produce not merely seven but any quantity n of ostensibly *'new'* objects (as illustrated by the proposition $[Aa] : A\varepsilon cl(a). \equiv .A\varepsilon \overbrace{cl(\ldots cl}^{n}(a))$; one may even produce collections *ad infinitum*).

But perhaps it is disingenuous to suggest that the successive iteration of the *collection* predicate causes the production of any *'new'* ontological object in the manner of the *fusion* operation. The fusion $\sum(a,b)$ is *compositionally* quite distinct from either a or b; the same cannot be said of the collection $cl(cl(a))$ with respect to $cl(a)$.

Proceeding to formalisations, we consider the (essential) axioms for manifold theory, preserving the original logistic and presenting them as claims. The sole *necessary* meta-axiom for identity is as follows:[14]

$$\vdash a \simeq a$$

The neutral identity predicate $\simeq$ holds between its respective argumental terms (singular or plural) if and only if they designate the *same manifold*. Fundamentally, there *is no empty manifold;* for, if there are to exist manifolds, there must exist individuals to comprise them. A *plurality* is precisely a manifold containing more than one member – it is just as easily said that any individual is the manifold containing only itself[15].

Existence in the theory is defined by virtue of identity and quantification –

$$Ea := \exists u(u \simeq a)$$

– thus, a distinction between the predicate of *existence, E,* and the standard quantifier $\exists$ is made. The distinction is of varying triviality, contingent on the ontological significance one bestows the predicate $\simeq$ with; in our theory, we take $\simeq$ to be interchangeable with the Leśniewskian ε[16], itself often conflated with the equivalence relation[17]. Therefore, employing

13 The definition relies on elementary notions of set theory; see **??** for elaboration.

14 The functor $\vdash$ corresponds to Simons' meta-axioms for predicate logic:

If A is a tautology of propositional calculus, $\vdash A$

$\vdash \forall z(A \supset B) \supset .\forall zA \supset \forall zB$

$\vdash A \supset \forall zA$, *where z is any variable not free in A*

$\vdash \forall zA \supset A(y/z)$, *where z is free in A, and y is of the same subcategory as A*

Generally, axioms initiated with $\vdash$ can be considered identical in the absence thereof; *.e.g,* $\vdash a \simeq b$ is syntactically identical to $a \simeq b$.
The original text notes also an addition identity meta-axom:
CL. M:1.2: $\vdash a \simeq b \supset .A \supset A(b//a)$

15 The distinction between *manifold* and *set* becomes further apparent. The individual not contained in a set (the *urelement*) and the set containing *only* such an individual (the *singleton*) are not, set-theoretically, identical. Immediate ontological problems arise thus; see **??**.

16 Recalling the earlier Leśniewskian definition of existence –

$$\forall x[Ex(x) \iff \exists y(y\varepsilon x)]$$

– it can be easily observed that both the Leśniewskian and manifold-theoretic definitions of existence are essentially identical.

17 We recall that *equality* defined by virtue of the primitive Leśniewskian copula ε takes the following form:

$$x = y \iff (x\varepsilon y \wedge y\varepsilon x).$$

this interchangeability, the existence definition in manifold theory is reduced to the utmost of trivialities (and congruent with the Hintikkan maxim *to be is to be identical to something*):

$$Ea := \exists u(u\varepsilon a).$$

Appreciating the term u as designatory of a manifold, however, the definition is expansible by way of primitive mereological considerations. Simons' definition for *singular* existence is almost syntactically identical, different only by the replacement of the manifold term u with the individual term x:[18]

$$E!a := \exists x(x \simeq a).$$

Mereological considerations are invoked precisely when we make explicit the *individuality* of the variable x without necessity for prior knowledge thereof. By definition, an individual is the sum of its atoms, and it is easily inferrable that

$$E!a := \exists \sum_{i=1}^{n} Atom(x_i)(x \simeq a)$$

where x is taken to be an abbreviation of the prior explicated fusion. Clearly, it is this *singular* existence that is of relevance to the set-theoretic notion of the element[19]. With an existential predicate at hand, we proceed to axiomatise *inclusion*, and invoke a further predicate of great importance; the predicate of non-empty neutral inclusion, denoted $\ni$[20].

$$\vdash a \ni b \equiv Ea \,\&\, \forall x(x \ni a \supset x \ni b)$$

$$\vdash \forall x(x \ni a \equiv x \ni b) \supset a \simeq b$$

The above expressions capture the intended interpretation of the primitive '$\ni$'; $a \ni b$ is true if and only if a is non-empty and every individual designated by a is also designated by b. The *extensionality principle* detailed by CL. M:2.2 details the neutral identity of manifolds composed of the same members; thus, manifold theory is largely reflective of class theory in the original Leśniewskian spirit. Again, the epsilon connective ε is substitutable for all instances of the

18 The definition is, of course, rewriteable in the local logistic, as previously shown.

19 The scope of existential quantifiers can, in some instances, serve as a convenience in situations concerning plural quantification. The existential quantifier denoting that there exists *at least one* is trivially defined as follows:

$$\exists_1 zA := \exists zA.$$

The form zA is precisely the form of a description; z is a variable, and A is a well-formed formula. Similarly, the existential quantifier denoting that there exists *at most one* is definable, and therefore the unique existential quantifier (denoted $\exists!$ in the local logistic) is simply the conjunction of the lower and upper quantificatory bounds:

$$\exists^1 zA := \forall z \forall y(A \,\&\, A(y/z) \supset y \simeq z) \quad \textit{where } y \textit{ is of the same subcategory as } z$$

$$\exists_1^1 zA := \exists_1 zA \,\&\, \exists^1 zA.$$

Three further existential axioms for inclusive were outlined by Simons:

CL. M:3.3.1: $\vdash \exists_1^1 x(x \ni w)$

CL. M:3.3.2: $\vdash \exists_2 x(x \ni h)$

CL. M:3.3.3: $\vdash \exists x(x \ni u)$

In order to understand CL. M:3.3.2, the numerical quantifier $\exists_2$ must be defined in the obvious sense:

$$\exists_2 zA := \exists z \exists y(A \,\&\, A(y/z) \,\&\, \sim (y \simeq z)), \quad \textit{where } y \textit{ is of the same subcategory as } z.$$

20 The symbol as employed by Simons differs slightly, but has been altered for typographical considerations.

identity predicate $\simeq$. Herefrom, the two fundamental set-theoretic notions of inclusion can be defined:[21]

$$a \in b := E!a \,\&\, a \ni b$$

$$a \subset b := \sim Ea \vee a \ni b.$$

Whilst the functor $\ni$ is taken as primitive in the manifold system, its intended interpretation is indeed expressed (as previously noted), albeit informally. One observes an initially innocuous condition for the validity of the sentence $a \ni b$; namely, that a is non-empty. The latter metatheorem, whilst valid in Simons' manifold system, will later be shown as false when inclusion is defined.

The null set – the very factor which we may regard as distiguishing set membership from mere inclusion – is fundamentally Lewisian when construed nominalistically[22]. The following definition is offered:

$$\exists x(\sim \exists u(x \lhd u) \,\&\, \forall w(\sim \exists u(w \lhd u) \supset x = w)).$$

Where the relation $\lhd$, read as *'is representative of'*, is taken as primitive. However, this definition is unsuitable precisely due to the Leśniewskian Mereological principle that *any individual forms a class*.

Leśniewski rejected the existence of the null set outrightly[23]; but the nominalist need not necessarily follow in this trend. Whilst unable to accept the null set as the 'convenient fiction' it is often regarded[24], one *can* accept a nominalistically sound – yet radically counter-intuitive – Lewisian characterisation.

21 A noteworthy metatheorem can be considered:

$$\vdash a \in b \equiv .\exists u(u \in a) \,\&\, \forall u(u \in a \supset u \in b) \,\&\, \forall uv(u \in a \,\&\, v \in a \supset u \in v).$$

The metatheorem

$$a \in b \iff \exists u(u \in a) \wedge \forall u[(u \in a) \implies (u \in b)] \wedge \forall u \forall v\{[(u \in a) \wedge (v \in a)] \implies (u \in v)\}$$

becomes false; expand and illustrate.

22 'The empty set is that individual which is not representative of any manifold' ([75]).

23 Whilst the null set as traditionally conceived does not have a direct Leśniewskian counterpart, conceptual parallels can be noted between the null set and Leśniewski's Ontology. Słupecki [76] *(cf. p. 69)* observes the correspondence between a triad of Kuratowski-Mostowski set-theoretic propositions (introduced in Kuratowski and Mostowski [36]) and their purported Leśniewskian correlates:

$$\begin{gathered}
Z(x) \equiv (x = 0) + \sum(v \in x) \\
R(x) \equiv Z(x) \cdot \prod_y [(y \in x) \to Z(y)] \\
\prod_x \{R(X) \to \sum_y [Z(Y) \cdot \prod_x ((x \in Y) \equiv \sum_z [(x \in z) \cdot (z \in X)])]\} \\
Z/X/ \equiv (X \underset{Z}{=} \Lambda \vee [\exists v]\{v\varepsilon X\}) \\
R/X/ \equiv Z/X/ \wedge [y]\{y\varepsilon X \supset Z/y/\} \\
[X]\{R/X/ \supset [\exists Y]\{Z/Y/ \wedge [x]\{x\varepsilon Y \equiv [\exists z]\{x\varepsilon z \wedge z\varepsilon Y\}\}\}\}
\end{gathered}$$

The latter three expressions are, of course, the Leśniewskian counterparts to the former three; whilst the initial two expressions are definitive, the respective third is presented as an axiom. By inspection, one can see that the direct Leśniewskian correlate of the Kuratowski-Mostowskian $x = 0$ (where, in respect to the original notation, 0 is the symbol for the empty set) is the expression $X \underset{Z}{=} \Lambda$. The expression bears direct resemblance to Leśniewski's definition of the *empty name,* which Slupecki considers the Ontological counterpart to the empty set:

$$x\varepsilon\Lambda =_{df} x\varepsilon x \wedge \neg(x\varepsilon x).$$

24 *cf.* Hausdorff [30]. Note also that many set-theorists (and many mathematicians in general) do not regard the null set as a 'convenient fiction', or a fiction in any sense; such mathematical platonists may see no reason to exclude the null set from the realm of existence whilst admitting innumerable other mathematical abstracta.

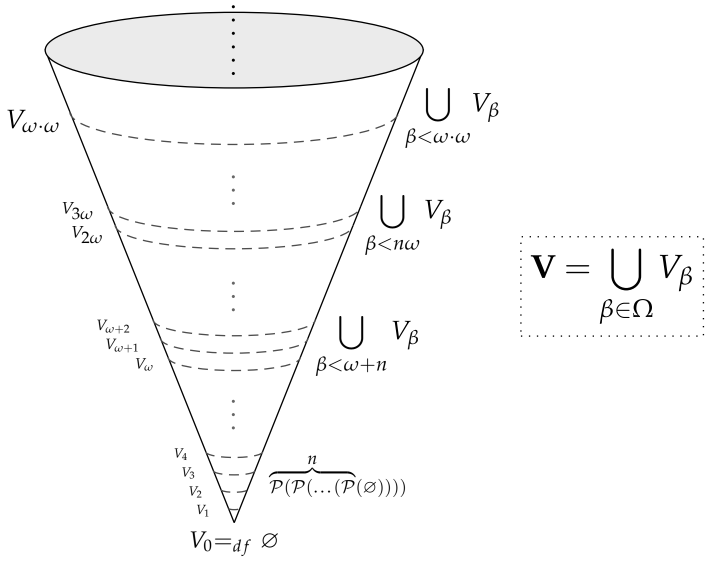


Figure 10: The von Neumann Universe **V**

The canonical conception of set theory takes the form of a *set-theoretic hierarchy;* that is, an inverted cone proceeding upwards with successive iterations of the *power set* and *union,* of which the null set acts as the vertex. The upward progression of such a hierarchy is indexed by the von Neumann ordinals; to each ordinal α corresponds a *'slice of sets'*, inferrably of *ordinal rank* α. Von Neumann, subsequent to outlining the structure of the cumulative hierarchy, proceeded to define the *set-theoretic universe* as the union of all iterations V_β, for any ordinal β as an *element* of the *class of all ordinals* Ω. While perhaps ostensibly of little ontological substance, the null set can thus be seen to be of irreducible fundamentality; all mathematical structures – the tensor fields, the differential manifolds, the Euclidean and non-Euclidean spaces, *and so on* – are contained within the immense vastness of the hierarchy.

But the nominalist of any degree of austerity is, for obvious reasons, unable in good conscience to allow the null set as *traditionally conceived* to form as the basis of such a hierarchy, irrespective of its demonstrable utility. Lewis et al. [51] expresses a similar sentiment – but not without nominalistic recourse.

In *Parts of Classes,* Lewis claims one need not be 'ontologically serious' in asserting a concrete representation of the null set[25]. Lewis contemplates a number of ontological alternatives, both mereological and metaphysical, to the null set; the definition ultimately proposed is both logically preferable and intuitively expedient to such alternatives. We observe the definition in the following passage:

> '[. . .] I prefer not to leave the arbitrary choice unmade. Instead, I make it – arbitrarily! – as follows. Instead of the unhelpful definition of the null set as the set without members, I adopt this
>
> *Redefinition:* The *null set* is the fusion of all individuals.
>
> That's an easy selection to specify; and it's guaranteed to select an individual to be the null set if there exists any individuals at all' (Lewis et al. [51], *p. 14*).

Lewis thus commands a *complete reversal* of the customary ontological nature of the null set; not only is the set not *nothing,* it is not merely *something;* rather, it is *everything.* Such a definition, an undoubted offense to the intuition of many (those who may wish to conceive the null set as irreducibly exiduous and nondescript), holds many advantages over the immediately obvious

25 See Lewis et al. [51], *p. 13:* 'Must we then accept the null set as a most extraordinary individual, a little speck of sheer nothingness, a sort of black hole in the fabric of Reality itself? Not really. We needn't be ontologically serious about the null set. It is useful to have a name that is guaranteed to denote some individual, but it needn't be a special individual with a whiff of nothingness about it.'

mereological alternative; that is, designating an individual, however nugatory, to serve the role of the null set.

Foremostly, under the above Lewisian definition, there is *no one individual* which one may identify as the null set – in this regard, the intuitive form of the null set's traditional conception is most closely matched, for in concern to what is said to consist therein, the null set is non-discriminatory[26]. Lewis passingly notes also a comforting advantage for the modal realist – should one hold to a thesis of *plurality of worlds,* that is, the thesis that all *possible worlds* are extant in causal isolation from our own *actual world,* it is the case that such an individual designated as the null set could have *failed to exist* altogether. Should that be so, must the foundations of mathematics fall on the contigency of such an individual, however trivial its individuality?

Thus, recourse is granted to those unsettled by the null set's reliance on arbitrary individual designation; for, as Lewis notes, if any multitude of individuals exist, the fusion thereof is the null set. Quite simply, if anything at all exists, the null set does.

In *Podstawy Ogj Teoryi Mnogości I* [40], Leśniewski informally noted the following theorem[27]:

'*Th. XLIV.* The class of non-contradictory objects is the universe' (Leśniewski [40], *p. 31*).

The formalisation of the above proposition that most likely remains true to the original Leśniewskian form can be observed in the notes of Sobociński and Kreczmar, which detail Leśniewski's class theory system from 1915–1923. We observe in Srzednicki and Stachniak [84], *p. 70:*

$$[P] : P\varepsilon Ws \equiv .P\varepsilon Kl(obj)$$

Where Ws abbreviates the Polish *wszechświatem (universe).* Thus, $P\varepsilon Ws$ has the intuitive reading *'P is the universe',* such that P is defined as the *class of objects.* Leśniewski's Lecture Notes, p. 49. We observe the following definition of an object[28]:

$$[A] : A\varepsilon V. \equiv .A\varepsilon A.$$

We observe the following definition from Sobociński's mereology[29]:

$$[A] A\varepsilon Kl(V). \equiv .A\varepsilon \mathbb{U}.$$

As before, the expression $A\varepsilon Kl(V)$ reads, intuitively, *'A is the class of all objects'.* Sobociński thus progresses to note the following formula[30]:

$$[A] :: A\varepsilon \mathbb{U}. \equiv . \cdot .A\varepsilon A. \cdot .[a] : a \subset extr(A). \supset .a \subset \Lambda$$

Where $extr$ is X and Λ is in accordance with X. The theorem shows that *A is the universe if and only if everything outside of it is non-existent.*

The Leśniewskian definition of *universe* does not heretofore suffice also as a definition of the Lewisian *null set*, for there are indeed discrepancies to be made manifest. We recall the definition of *mereological fusion:*

$$z\varepsilon\sum(x,y) =_{df} \forall w\{(w \circ z) \implies [(w \circ x) \vee (w \circ y)]\}$$

Whilst Leśniewski did not make explicit a notion of *mereological fusion,* there exists, as noted by Sobociński, a notion of *sum,* defined by virtue of the aforementioned *exterior* primitive:

26 'For what it's worth, it respects our 'intuition' that the null set is no place in particular, no more one place than another' (Lewis et al. [51], *p. 14*).

27 My translation. See Leśniewski [40], *p. 31:*

'*Twierdzenie XLIV.* Klasa przedmiotów niesprzecznych jest wszechświatem.'

28 Leśniewski frequently used $A\varepsilon obj$ in place of $A\varepsilon V$.

29 Urbaniak also proposes a definition of the Leśniewskian universe, different yet from those previously examined:

$$\forall a(a\varepsilon\mathbb{U} \equiv a\varepsilon a \wedge a\varepsilon cl(obj)).$$

One notices that, in contrast to the both the Sobocińskian definition and the definition observed in X's notes, the Ontological expression $a\varepsilon a$ is not ommitted.

30 Leśniewski was never made aware of it.

$$[Aa] : \cdot : A\varepsilon Kl(a) :: [BC]. \cdot .B\varepsilon a.C\varepsilon a. \supset: B = C. \vee .B\varepsilon extr(C) ::\equiv .A\varepsilon Sm(a)$$

Where the expression $A\varepsilon Sm(a)$ *('A is the sum of a's')* expresses the condition that the a's are discrete with respect to one another and that A is the class of them. Thus, whereas set-theoretic (or class-theoretic) notions are need not be introduced in order to define mereological fusion, Leśniewski's definition indeed does employ this notion in doing so. In order to present a local definition – that is, a definition of *universe* relying only the mereological primitive of discretion – it is opportune to reconstruct Leśniewskian class theory in order to illustrate the precise difference between the null set and the universe.

DF. 2:4.1.1: $a \in b =_{df} \exists u(u\varepsilon a) \wedge \forall x[(x \leq_p a) \implies (x \leq_p b)]$

The Leśniewskian expansion takes the form:

$$A\varepsilon\mathbb{U} \iff A\varepsilon A \wedge \forall C(C\varepsilon V \implies C \in A) \wedge \forall D[D \in A \implies \exists E\exists F(E\varepsilon V \wedge F \in D \wedge F \in E)]$$

$$\begin{aligned} A\varepsilon\mathbb{U} \iff A\varepsilon A \wedge \forall C\big(C\varepsilon V \implies \{\exists u(u\varepsilon C) \wedge \forall x[(x \leq_p C) \implies (x \leq_p A)]\}\big) \\ \wedge\forall D\big\{\exists v\{(v\varepsilon D) \wedge \forall x[(x \leq_p D) \implies (x \leq_p A)]\} \implies \exists E\exists F\big[E\varepsilon V \\ \wedge\exists w\big((w\varepsilon F) \wedge \forall x\{(x \leq_p F) \implies [(x \leq_p D) \wedge (x \leq_p E)]\}\big)\big]\big\} \end{aligned}$$

Whereas, using the aforementioned notion of mereological fusion, we define the *fusion of all individuals* as follows, similarly reducing the definition to the elementary notion of *parthood*:

$$\begin{aligned} z\varepsilon\sum V =_{df} \forall w\forall u\big[[(u \leq_p z) \wedge (u \leq_p w)] \iff \forall x\forall y\big([(x\varepsilon V) \wedge (y\varepsilon V)] \\ \implies \{[(u \leq_p w) \wedge (u \leq_p x)] \vee [(u \leq_p w) \wedge (u \leq_p y)]\}\big)\big] \end{aligned}$$

Following the Lewisian definition literally, the null set is defined takes the following form –

DF. 2:4.1.2: $A\varepsilon\varnothing =_{df} A\varepsilon\sum V$ *Null Set*

– and thus, on a stringently Mereological basis, we have offered some recourse for Modal Structuralist developments which require Neumannian set-theoretic hierarchies.

APPENDIX

4

4.1 THE PROTOTHETIC SYSTEM $\mathfrak{S}5$

Rule of Definition. A is a well–formed definition if the following *desiderata* are satisfied:

> 'When the *definiens* contains one or more real variables, the *definiendum* must also contain them. For in the case we have a function of the real variables, and the *definiendum* must have the same meaning as the *definiens* for all values of these variables, which requires that the symbol which is the *definiendum* should contain the letters representing the real variables. [...] In the definitions [...] of "$p.q$" and "$p \supset q$" and "$p \equiv q$," p and q are real variables, and therefore appear on both sides of the definition' (Russell and Whitehead [68], *p. 19*).

> 'Even though definitions are required for certain proofs, all of the terms introduced by definition are eliminable–any statement containing the defined term can be replaced by one which "means the same thing" but which does not contain the defined term' (Kearns [33], *p. 70*).

Distribution of Quantifier. We appreciate the following inferrence to be Protothetically valid:

$$\forall p \forall q \forall r[p \implies (q \implies r)] \iff (p \wedge q) \implies r$$
$$\vdots$$
$$\forall p \forall q \forall r[p \implies (q \implies r)] \iff \forall p \forall q \forall r[(p \wedge q) \implies r]$$

Detachment. It can be inferred from expressions of the type $\phi(\alpha\beta)$ that α and β. It cannot be inferred, however:

$$\llcorner pq \lrcorner \ulcorner \phi\Big(\phi(pp)\phi(\phi(pq)\phi(pq))\Big)\urcorner \qquad \llcorner p \lrcorner \ulcorner \phi(pp)\urcorner$$

$$\llcorner pq \lrcorner \ulcorner \phi(\phi(pq)\phi(qp))\urcorner$$

Substitution. Accomplished by definition: $\llcorner p \lrcorner \ulcorner \phi(\phi(pp)Vr(p))\urcorner$[1]. For f in $\llcorner fp \lrcorner \ulcorner \Phi((f(p))\urcorner$ substitute Vr to get $\llcorner p \lrcorner \ulcorner \Phi(Vr(p))\urcorner$. Thus, derive $\llcorner p \lrcorner \ulcorner \phi\Big(\Phi(Vr(p))\Phi(\phi(pp))\Big)\urcorner$. Detach and distribute to get the desired $\llcorner p \lrcorner \ulcorner \Phi(\phi(pp))\urcorner$.

Extensionality. Law of extensionality for sentences[2]:

$$[pq]. \cdot .p \equiv q. \equiv: [f] : f(p). \equiv .f(q).$$

1 We note that Vr is given by Tarski in Peano-Russellian form as $Vr \equiv .[p].p \equiv p$; thus, the more familiar (and more immediately trivial) expansion of the Protothetical definition for substitution is

$$[p] : p \equiv p. \equiv .p \equiv p.$$

2 Notably, once Prothetic enforces extensionality, conjunction as previously defined –

$$[pq] :: p.q. \equiv . \cdot .[f]. \cdot .p \equiv: [r].p \equiv f(r). \equiv .[r].q \equiv f(r)$$

– can be reformulated as the more succinct

4.2 AXIOMS A AND V ARE EQUIVALENT TO THE SYSTEM $\mathfrak{B}$

The following Ontological theorem is referenced throughout.

TH. 1:3.4: $\forall A \forall a (A\varepsilon a \implies A\varepsilon A)$

PF. $\forall A \forall a$

1.	$A\varepsilon a \implies \exists B(B\varepsilon A \wedge B\varepsilon a)$	*Ontology*
	$\forall B \forall C$	
2.	$(A\varepsilon A \wedge B\varepsilon A \wedge C\varepsilon A) \implies B\varepsilon C$	*Ontology*
3.	$\{a\varepsilon A \wedge [(B\varepsilon A \wedge C\varepsilon A) \implies B\varepsilon C]\} \implies A\varepsilon A$	*Th., Subs.*
3.1.	$a\varepsilon A \wedge a\varepsilon B \wedge \forall D[(C\varepsilon A \wedge D\varepsilon A) \implies C\varepsilon D] \implies A\varepsilon B$	*Ontology*
4.	$A\varepsilon A$	3

We show that the system consisting of the axioms (**A**)

$$\forall A \forall B \Big[A\varepsilon pt(B) \iff \Big(B\varepsilon B \wedge \forall C \forall a \{ \forall D \big[D\varepsilon C \iff \big(\forall E[E\varepsilon a \implies E\varepsilon pt(D)] \wedge \forall E\{E\varepsilon pt(D) \implies \exists F \exists G[F\varepsilon a \wedge G\varepsilon pt(F) \wedge G\varepsilon pt(E)]\}\big)\big] \wedge [B\varepsilon pt(B) \wedge B\varepsilon a \implies A\varepsilon pt(C)]\} \Big) \Big]$$

and (**V**)

$$\forall A \big(A\varepsilon A \implies \exists B\{B\varepsilon pt(A) \wedge \forall C[C\varepsilon pt(B) \implies C = B]\}\big)$$

is inferrentially equivalent to the system

$$\forall A[A\varepsilon at(B) \implies B\varepsilon B]$$
$$\forall A \forall B \forall C\{[A\varepsilon at(B) \wedge C\varepsilon at(A)] \implies C = A\}$$
$$\forall A \forall B \big(\{A\varepsilon A \wedge B\varepsilon B \wedge \forall C[C\varepsilon at(A) \iff C\varepsilon at(B)]\} \implies A = B\big)$$
$$\forall A \forall a \big[A\varepsilon a \implies \exists B\big(\exists E([E\varepsilon at(B)] \wedge \forall C\{C\varepsilon at(B) \iff \exists D[C\varepsilon at(D) \wedge D\varepsilon a]\}\big)\big].$$

TH. 1:3.5.1.1: $\forall A \forall B[A\varepsilon at(B) \implies B\varepsilon B]$

PF. $\forall A \forall B$

1.	$A\varepsilon at(B)$	*Supp.*
2.	$A\varepsilon pt(B)$	DF. 2:2.2, 1
3.	$B\varepsilon B$	**A**

TH. 1:3.5.1.2: $\forall A \forall B \forall C\{[A\varepsilon at(B) \wedge C\varepsilon at(A)] \implies C = A\}$

PF. $\forall A \forall B \forall C$

1.	$C\varepsilon at(A)$	*Supp.*
2.	$C\varepsilon pt(A)$	DF. 2:2.2, 1
3.	$A\varepsilon at(B)$	*Supp.*
4.	$C = A$	DF. 2:2.1*, 1, 3

DF. 1:3.5.2.1: $\forall A \forall a \big(A\varepsilon Kl(a) \iff A\varepsilon A \wedge \forall B[B\varepsilon a \implies B\varepsilon pt(A)] \wedge \forall B\{B\varepsilon pt(A) \implies \exists E \exists F[E\varepsilon a \wedge F\varepsilon pt(E) \wedge F\varepsilon pt(B)]\}\big)$

TH. 1:3.5.2.2: $\forall A\{A\varepsilon A \implies A\varepsilon Kl[at(A)]\}$

$$[pq]. \cdot .p.q. \equiv [f] : p \equiv .f(p) \equiv f(q).$$

PF. $\forall A$

1.	$A \varepsilon A$	*Supp.*
2.	$\exists C[C\varepsilon at(A)]$	**V**, 1
	$\exists B$	
3.	$B\varepsilon Kl[at(a)]$	**A**, 2
	$\forall D$	
4.	$D\varepsilon pt(B) \implies \exists F[F\varepsilon pt(D) \wedge F\varepsilon pt(A)]$	*Th.*, 3
5.	$D\varepsilon pt(A) \implies \exists F[F\varepsilon pt(D) \wedge F\varepsilon pt(B)]$	*Th.*, 3
6.	$B\varepsilon pt(A)$	*Th.*, **A**, 3, 4
7.	$A\varepsilon pt(B)$	**A**, 1, 5
8.	$A = B$	**A**, 7, 6
9.	$A\varepsilon Kl[at(a)]$	3, 8

TH. 1:3.5.3.1: $\forall a \forall b\{\forall A(A\varepsilon a \iff A\varepsilon b) \iff [\forall \varphi(\varphi\{a\} \iff \varphi\{b\})]\}$

TH. 1:3.5.3.2: $\forall A \forall B\big(\{A\varepsilon A \wedge B\varepsilon B \wedge \forall C[C\varepsilon at(A) \iff C\varepsilon at(B)]\} \implies A = B\big)$

PF. $\forall A \forall B$

1.	$A\varepsilon A$	*Supp.*
2.	$B\varepsilon B$	*Supp.*
3.	$\forall C[C\varepsilon at(A) \iff C\varepsilon at(B)]$	*Supp.*
4.	$A\varepsilon Kl[at(A)]$	*Th.*, 1
5.	$B\varepsilon Kl[at(B)]$	*Th.*, 2
6.	$B\varepsilon Kl[at(A)]$	*Th.*, *Subs*, 3, 5
7.	$A = B$	**A**, 4, 6

TH. 1:3.4.1: $\forall A \forall B \forall C\{[A\varepsilon at(B) \wedge B\varepsilon pt(C)] \implies A\varepsilon at(C)\}$

TH. 1:3.4.2: $\forall A \forall C \forall a\{[A\varepsilon Kl(a) \wedge C\varepsilon at(A)] \implies \exists D[C\varepsilon at(D) \wedge D\varepsilon a]\}$

PF. $\forall A \forall C \forall a$

1.	$A\varepsilon Kl(a)$	*Supp.*
2.	$C\varepsilon at(A)$	*Supp.*
3.	$C\varepsilon pt(A)$	DF. 2:2.2, 2
4.	$C\varepsilon atm$	DF. 2:2.2, 2
	$\exists E \exists F$	
5.	$E\varepsilon a$	1, 3
6.	$F\varepsilon pt(E)$	1, 3
7.	$F\varepsilon pt(C)$	1, 3
8.	$F = C$	DF. 2:2.1*, 4, 7
9.	$C\varepsilon pt(E)$	6, 8
10.	$\exists D[C\varepsilon at(D) \wedge D\varepsilon a]$	DF. 2:2.2, 4, 5, 9

TH. 1:3.4.3: $\forall A \forall C \forall D \forall a\{[A\varepsilon Kl(a) \wedge C\varepsilon at(D) \wedge D\varepsilon a] \implies C\varepsilon at(A)\}$

PF. $\forall A \forall C \forall D \forall a$

1.	$A\varepsilon Kl(a)$	*Supp.*
2.	$C\varepsilon at(D)$	*Supp.*
3.	$D\varepsilon a$	*Supp.*
4.	$D\varepsilon pt(A)$	1, 3
5.	$C\varepsilon at(A)$	*Th.*, 2, 4

TH. 1:3.5.1: $\forall A \forall B \forall D \forall a\big[\big(\forall C\{[C\varepsilon at(D) \wedge D\varepsilon a] \implies C\varepsilon at(A)\} \wedge B\varepsilon a \wedge D\varepsilon pt(B)\big) \implies \exists F[F\varepsilon pt(D) \wedge F\varepsilon pt(A)]\big]$

PF. $\forall A\forall B\forall D\forall a$

1.	$\forall C\{[C\varepsilon at(D)\wedge D\varepsilon a]\implies C\varepsilon at(A)\}$	*Supp.*
2.	$B\varepsilon a$	*Supp.*
3.	$D\varepsilon pt(B)$	*Supp.*
	$\exists F$	
4.	$F\varepsilon at(D)$	*Th.*, **V**, 3
5.	$F\varepsilon at(B)$	*Th.*, 3, 4
6.	$F\varepsilon at(A)$	1, 2, 5
7.	$\exists F[F\varepsilon pt(D)\wedge F\varepsilon pt(A)]$	DF. 2:2.2, 4, 6

TH. 1:3.5.2.1: $\forall A\forall B\forall a\big[\big(\forall C\forall D\{[C\varepsilon at(D)\wedge D\varepsilon a]\implies C\varepsilon at(A)\}\wedge B\varepsilon a\big)\implies B\varepsilon pt(A)\big]$

TH. 1:3.5.2.2: $\forall A\forall B\forall a\big[\big(\forall C\{C\varepsilon at(A)\implies\exists D[C\varepsilon at(D)\wedge D\varepsilon a]\}\wedge B\varepsilon pt(A)\big)\implies\exists E\exists F[E\varepsilon a\wedge F\varepsilon pt(E)\wedge F\varepsilon pt(B)]\big]$

PF. $\forall A\forall B\forall a$

1.	$\forall C\{C\varepsilon at(A)\implies\exists D[C\varepsilon at(D)\wedge D\varepsilon a]\}$	*Supp.*
2.	$B\varepsilon pt(A)$	*Supp.*
	$\exists F$	
3.	$F\varepsilon at(B)$	*Th.*, **V**, 2
4.	$F\varepsilon at(A)$	*Th.*, 2, 3
	$\exists E$	
5.	$F\varepsilon at(E)$	1, 4
6.	$F\varepsilon pt(E)$	DF. 2:2.2, 5
7.	$E\varepsilon a$	1, 4
8.	$F\varepsilon pt(B)$	DF. 2:2.2, 3
9.	$\exists E\exists F[E\varepsilon a\wedge F\varepsilon pt(E)\wedge F\varepsilon pt(B)]$	6–8

TH. 1:3.5.3: $\forall A\forall a\big(A\varepsilon Kl(a)\iff A\varepsilon A\wedge\forall C\{C\varepsilon at(A)\iff\exists D[C\varepsilon at(D)\wedge D\varepsilon a]\}\big)$

TH. 1:3.5.4: $\forall A\forall a\big[A\varepsilon a\implies\exists B\big(\exists E[E\varepsilon at(B)]\wedge\forall C\{C\varepsilon at(A)\iff\exists D[C\varepsilon at(D)\wedge D\varepsilon a]\}\big)\big]$

PF. $\forall A\forall a$

1.	$A\varepsilon a$	*Supp.*
	$\exists B$	
2.	$B\varepsilon Kl(a)$	**A**, 1
3.	$\forall C\{C\varepsilon at(B)\iff\exists D[C\varepsilon at(D)\wedge D\varepsilon a]\}$	*Th.*, 2
4.	$\exists E[E\varepsilon at(B)]$	*Th.*, **V**, 2
5.	$\exists B\big(\exists E[E\varepsilon at(B)]\wedge\forall C\{C\varepsilon at(A)\iff\exists D[C\varepsilon at(D)\wedge D\varepsilon a]\}\big)$	3, 4

TH. 1:3.5.5: $\forall A\forall B\big(A\varepsilon pt(B)\iff\{A\varepsilon A\wedge\forall C[C\varepsilon at(A)\implies C\varepsilon at(B)]\}\big)$

TH. 1:3.5.6: $\forall A[A\varepsilon at(A)\iff A\varepsilon atm]$

4.3 A DERIVATION OF THE *higher epsilon* ANALOGUE OF THE AXIOM OF ONTOLOGY

TH. 27: $\forall\Phi\forall\varphi\forall\psi\{\Phi\overset{\varepsilon}{\underset{\infty}{}}\varphi\iff\exists a[\Phi\{a\}\wedge\varphi\{a\}\wedge Qua(\Phi)\wedge Num(\Phi)\wedge Num(\varphi)]\}$

TH. 28: $\forall\Phi\forall\varphi(\Phi\overset{\varepsilon}{\underset{\infty}{}}\varphi\implies\Phi\overset{\varepsilon}{\underset{\infty}{}}\Phi)$

TH. 29: $\forall\Phi\forall\Psi\forall\varphi[(\Phi\overset{\varepsilon}{\underset{\infty}{}}\varphi\wedge\Psi\overset{\varepsilon}{\underset{\infty}{}}\Phi)\implies\Psi\overset{\varepsilon}{\underset{\infty}{}}\varphi]$

PF. $\forall\Phi\forall\Psi\forall\varphi$

1.	$\Phi\overset{\varepsilon}{\underset{\infty}{}}\varphi$	*Supp.*
2.	$\Psi\overset{\varepsilon}{\underset{\infty}{}}\Phi$	*Supp.*
3.	$Qua(\Phi) \wedge Num(\Phi)$	TH. 27, 1
4.	$Qua(\Psi) \wedge Num(\Psi)$	TH. 27, 2
	$\exists a$	
5.	$\Phi\{a\} \wedge \varphi\{a\}$	TH. 27, 1
	$\exists b$	
6.	$\Psi\{b\} \wedge \Phi\{b\}$	TH. 27, 2
7.	$a\infty b$	DF. 6, 3, 5, 6
8.	$\varphi\{b\}$	DF. 4, 3, 5, 7
9.	$\Psi\overset{\varepsilon}{\underset{\infty}{}}\varphi$	TH. 27, 3, 4, 6, 8

TH. 30: $\forall\Phi\forall\Psi\forall A\forall\varphi[(\Phi\overset{\varepsilon}{\underset{\infty}{}}\varphi \wedge \Psi\overset{\varepsilon}{\underset{\infty}{}}\Phi \wedge A\overset{\varepsilon}{\underset{\infty}{}}\Phi) \implies \Psi\overset{\varepsilon}{\underset{\infty}{}}A]$

PF. $\forall A\forall\Phi\forall\Psi\forall A\forall\varphi$

1.	$\Phi\overset{\varepsilon}{\underset{\infty}{}}\varphi$	*Supp.*
2.	$\Psi\overset{\varepsilon}{\underset{\infty}{}}\Phi$	*Supp.*
3.	$A\overset{\varepsilon}{\underset{\infty}{}}\Phi$	*Supp.*
4.	$Qua(\Phi)$	TH. 27, 1
5.	$Qua(\Psi) \wedge Num(\Psi)$	TH. 27, 2
6.	$Num(A)$	TH. 27, 3
	$\exists a$	
7.	$\Phi\{a\} \wedge \Psi\{a\}$	TH. 27, 2
	$\exists b$	
8.	$A\{b\} \wedge \Phi\{b\}$	TH. 27, 3
9.	$a\infty b$	DF. 6, 4, 7, 8
10.	$\Psi\{b\}$	DF. 4, 5, 7, 9
11.	$\Psi\overset{\varepsilon}{\underset{\infty}{}}A$	TH. 27, 5, 6, 8, 10

TH. 31: $\forall\varphi\forall a[(\varphi\{a\} \wedge Num(\varphi) \iff \infty\langle a\rangle\overset{\varepsilon}{\underset{\infty}{}}\varphi]$

TH. 32: $\forall\varphi\forall a\forall b[a\infty b \implies (\infty\langle a\rangle\overset{\varepsilon}{\underset{\infty}{}}\varphi \iff \infty\langle b\rangle\overset{\varepsilon}{\underset{\infty}{}}\varphi)]$

TH. 33: $\forall\Phi\forall\Psi\forall a\forall b\big(\{\Psi\overset{\varepsilon}{\underset{\infty}{}}\Phi \wedge \forall A[(\Psi\overset{\varepsilon}{\underset{\infty}{}}\Phi \wedge A\overset{\varepsilon}{\underset{\infty}{}}\Phi) \implies \Psi\overset{\varepsilon}{\underset{\infty}{}}A] \wedge \Phi\{a\} \wedge \Phi\{b\}\} \implies a\infty b\big)$

PF. $\forall\Phi\forall\Psi\forall a\forall b$

1.	$\Psi\overset{\varepsilon}{\underset{\infty}{}}\Phi$	*Supp.*
2.	$\forall A[(\Psi\overset{\varepsilon}{\underset{\infty}{}}\Phi \wedge A\overset{\varepsilon}{\underset{\infty}{}}\Phi) \implies \Psi\overset{\varepsilon}{\underset{\infty}{}}A]$	*Supp.*
3.	$\Phi\{a\}$	*Supp.*
4.	$\Phi\{b\}$	*Supp.*
5.	$Num(\Phi)$	TH. 27, 1
6.	$\infty\langle a\rangle\overset{\varepsilon}{\underset{\infty}{}}\Phi$	TH. 2X, 3, 5
7.	$\infty\langle b\rangle\overset{\varepsilon}{\underset{\infty}{}}\Phi$	TH. 2X, 4, 5
8.	$\infty\langle a\rangle\overset{\varepsilon}{\underset{\infty}{}}\infty\langle b\rangle$	2, 6, 7
9.	$\exists c(\infty\langle a\rangle\{c\}\overset{\varepsilon}{\underset{\infty}{}}\infty\langle b\rangle\{c\})$	TH. 27, 8
10.	$a\infty b$	DF. 2X, TH. 2X, TH. 2X, 9

TH. 34: $\forall\Phi\forall\Psi\big(\{\Psi\overset{\varepsilon}{\underset{\infty}{}}\Phi \wedge \forall A[(\Psi\overset{\varepsilon}{\underset{\infty}{}}\Phi \wedge A\overset{\varepsilon}{\underset{\infty}{}}\Phi) \implies \Psi\overset{\varepsilon}{\underset{\infty}{}}A]\} \implies Qua(\Phi)\big)$

TH. 35: $\forall\Phi\forall\Psi\forall\varphi\big(\{\Psi\overset{\varepsilon}{\underset{\infty}{}}\Phi \wedge \forall A[(\Psi\overset{\varepsilon}{\underset{\infty}{}}\Phi \wedge A\overset{\varepsilon}{\underset{\infty}{}}\Phi) \implies \Psi\overset{\varepsilon}{\underset{\infty}{}}A] \wedge (\Psi\overset{\varepsilon}{\underset{\infty}{}}\Phi \implies \Psi\overset{\varepsilon}{\underset{\infty}{}}\varphi)\} \implies \Phi\overset{\varepsilon}{\underset{\infty}{}}\varphi\big)$

PF. $\forall\Phi\forall\Psi\forall\varphi$

1.	$\Psi \overset{\varepsilon}{\infty} \Phi$	*Supp.*
2.	$\forall A[(\Psi \overset{\varepsilon}{\infty} \Phi \wedge A \overset{\varepsilon}{\infty} \Phi) \implies \Psi \overset{\varepsilon}{\infty} A]$	*Supp.*
3.	$\Psi \overset{\varepsilon}{\infty} \Phi \implies \Psi \overset{\varepsilon}{\infty} \varphi$	*Supp.*
4.	$Num(\Phi)$	TH. 27, 1
5.	$Qua(\Phi)$	TH. 2X, 1, 2
	$\exists a$	
6.	$\Phi\{a\}$	TH. 2X, 1
7.	$\Phi \overset{\varepsilon}{\infty} \Phi$	TH. 2X, 4, 5, 6
8.	$\Phi \overset{\varepsilon}{\infty} \varphi$	3, 7

TH. 36: $\forall\Phi\forall\varphi[\Phi \overset{\varepsilon}{\infty} \varphi \iff (\exists\Psi(\Psi \overset{\varepsilon}{\infty} \Phi) \wedge \forall\Psi\{\forall A[(\Psi \overset{\varepsilon}{\infty} \Phi \wedge A \overset{\varepsilon}{\infty} \Phi) \implies \Psi \overset{\varepsilon}{\infty} A] \wedge (\Psi \overset{\varepsilon}{\infty} \Phi \implies \Psi \overset{\varepsilon}{\infty} \varphi)\})]$

4.4 NUMERICAL FUNCTORS HAVE ONTOLOGICAL ANALOGUES

TH. 44: $\forall\Phi\forall\varphi\{[\Phi \overset{\varepsilon}{\infty} \Phi \wedge Num(\varphi) \wedge \forall a(\Phi\{a\} \implies \varphi\{a\})] \implies \Phi \overset{\varepsilon}{\infty} \varphi\}$

TH. 45: $\forall\Phi\forall\varphi\{\Phi \overset{\varepsilon}{\infty} \varphi \implies [\Phi \overset{\varepsilon}{\infty} \Phi \wedge Num(\varphi) \wedge \forall a(\Phi\{a\} \implies \varphi\{a\})]\}$

PF. $\forall\Phi\forall\varphi$

1.	$\Phi \overset{\varepsilon}{\infty} \varphi$	*Supp.*
2.	$\Phi \overset{\varepsilon}{\infty} \Phi \wedge Qua(\Phi) \wedge Num(\varphi)$	TH. 2X, 1
	$\exists b$	
3.	$\Phi\{b\} \wedge \varphi\{b\}$	TH. 2X, 1
	$\forall a$	
4.	$\Phi\{a\} \wedge b \infty a$	DF. 2X, 2, 3
5.	$b \infty a \implies \Phi\{a\}$	DF. 2X, 2, 3
6.	$\forall a(\Phi\{a\} \implies \varphi\{a\})$	4, 5
7.	$\Phi \overset{\varepsilon}{\infty} \Phi \wedge Num(\varphi) \wedge \forall a(\Phi\{a\} \implies \varphi\{a\})$	2, 6

TH. 46: $\forall\Phi\forall\varphi\{\Phi \overset{\varepsilon}{\infty} \varphi \iff [\Phi \overset{\varepsilon}{\infty} \Phi \wedge Num(\varphi) \wedge \forall a(\Phi\{a\} \implies \varphi\{a\})]\}$

TH. 47: $\forall\Phi\forall\varphi\forall\psi\forall\mu(\forall a\{[\mu\langle\varphi\rangle(\infty\langle a\rangle) \iff \psi\langle\varphi\rangle\{a\}] \wedge \Phi \overset{\varepsilon}{\infty} \psi\langle\varphi\rangle\} \implies \mu\langle\varphi\rangle(\Phi))$

PF. $\forall\Phi\forall\varphi\forall\psi\forall\mu$

	$\forall a$	
1.	$[\mu\langle\varphi\rangle(\infty\langle a\rangle) \iff \psi\langle\varphi\rangle\{a\}]$	*Supp.*
2.	$\Phi \overset{\varepsilon}{\infty} \psi\langle\varphi\rangle$	*Supp.*
	$\forall b$	
3.	$(\Phi\{a\} \wedge \Phi\{b\}) \implies a \infty b$	DF. 2X, TH. 2X, 2
4.	$(\Phi\{a\} \wedge a \infty b) \implies \Phi\{b\}$	DF. 2X, TH. 2X, 2
	$\exists a$	
5.	$\Phi\{a\} \wedge \psi\langle\varphi\rangle\{a\}$	TH. 2X, 2
	$\forall b$	
6.	$\Phi\{b\} \implies a \infty b$	3, 5
7.	$a \infty b \implies \Phi\{b\}$	4, 5
8.	$\Phi\{b\} \iff \infty\langle a\rangle\{b\}$	DF. 2X, 6, 7
9.	$\mu\langle\varphi\rangle(\Phi) \iff \mu\langle\varphi\rangle(\infty\langle a\rangle)$	TH. 2X, 8
10.	$\mu\langle\varphi\rangle(\Phi)$	1, 5, 9

4.5 THE ONTOLOGICAL PRINCIPLE OF EXTENSIONALITY HAS A *numerical epsilon* ANALOGUE

TH. 4:2.1: $\forall\varphi\forall\psi\forall\mu\{[\mu(\varphi) \iff \mu(\psi)] \implies \forall\Phi(\Phi \overset{\varepsilon}{\infty} \varphi \iff \Phi \overset{\varepsilon}{\infty} \psi)\}$

TH. 4:2.2: $\forall\varphi\forall\psi\{[Num(\varphi) \wedge Num(\psi) \wedge \forall\Phi(\Phi \overset{\varepsilon}{\infty} \varphi \iff \Phi \overset{\varepsilon}{\infty} \psi)] \implies \forall\mu[\mu(\varphi) \iff \mu(\psi)]\}$

PF. $\forall\forall\varphi\forall\psi$

1.	$Num(\varphi)$	*Supp.*
2.	$Num(\psi)$	*Supp.*
3.	$\forall\Phi(\Phi \overset{\varepsilon}{\underset{\infty}{}} \varphi \iff \Phi \overset{\varepsilon}{\underset{\infty}{}} \psi)$	*Supp.*
	$\forall a$	
4.	$\infty\langle a\rangle \overset{\varepsilon}{\underset{\infty}{}} \varphi \iff \infty\langle a\rangle \overset{\varepsilon}{\underset{\infty}{}} \psi$	3
5.	$[\varphi\{a\} \wedge Num(\varphi)] \iff [\psi\{a\} \wedge Num(\psi)]$	TH. 2X, 4
6.	$\varphi\{a\} \iff \psi\{a\}$	1, 2, 5
7.	$\forall\mu[\mu(\varphi) \iff \mu(\psi)]$	TH. 2X, 6

TH. 4:2.3: $\forall\varphi\forall\psi\big([Num(\varphi) \wedge Num(\psi)] \implies \{\forall\Phi(\Phi \overset{\varepsilon}{\underset{\infty}{}} \varphi \iff \Phi \overset{\varepsilon}{\underset{\infty}{}} \psi) \iff \forall\mu[\mu(\varphi) \iff \mu(\psi)]\}\big)$

DF. 4:2.4: $\forall\Phi\forall\Psi[(\Phi \overset{\varepsilon}{\underset{\infty}{}} \Psi \wedge \Psi \overset{\varepsilon}{\underset{\infty}{}} \Phi) \iff \Phi \overset{=}{\underset{\infty}{}} \Psi$

TH. 4:2.5: $\forall\Phi\forall\Psi\{\Phi \overset{=}{\underset{\infty}{}} \Psi \implies \forall\mu[\mu(\Phi) \iff \mu(\Psi)]\}$

PF. $\forall\Phi\forall\Psi$

1.	$\Phi \overset{=}{\underset{\infty}{}} \Psi$	*Supp.*
2.	$\Phi \overset{\varepsilon}{\underset{\infty}{}} \Psi \wedge \Psi \overset{\varepsilon}{\underset{\infty}{}} \Phi$	DF. 40, 1
3.	$Num(\Phi) \wedge Num(\Psi)$	TH. 2X, 2
4.	$\forall A(A \overset{\varepsilon}{\underset{\infty}{}} \Phi \iff A \overset{\varepsilon}{\underset{\infty}{}} \Psi)$	TH. 2X, 2
5.	$\forall\mu[\mu(\Phi) \iff \mu(\Psi)]$	TH. 2X, 3, 4

TH. 4:2.6: $\forall\Phi\forall\Psi\big(\{\forall\mu[\mu(\Phi) \iff \mu(\Psi)] \wedge \Phi \overset{\varepsilon}{\underset{\infty}{}} \Phi \wedge \Psi \overset{\varepsilon}{\underset{\infty}{}} \Psi\} \implies \Phi \overset{=}{\underset{\infty}{}} \Psi\big)$

PF. $\forall\Phi\forall\Psi$

1.	$\forall\mu[\mu(\Phi) \iff \mu(\Psi)]$	*Supp.*
2.	$\Phi \overset{\varepsilon}{\underset{\infty}{}} \Phi$	*Supp.*
3.	$\Psi \overset{\varepsilon}{\underset{\infty}{}} \Psi$	*Supp.*
4.	$Num(\Phi)$	TH. 2X, 2
5.	$Num(\Psi)$	TH. 2X, 3
6.	$\forall A(A \overset{\varepsilon}{\underset{\infty}{}} \Phi \iff A \overset{\varepsilon}{\underset{\infty}{}} \Psi)$	TH. 2X, 1, 4, 5
7.	$\Phi \overset{\varepsilon}{\underset{\infty}{}} \Psi$	2, 6
8.	$\Psi \overset{\varepsilon}{\underset{\infty}{}} \Phi$	3, 6
9.	$\Phi \overset{=}{\underset{\infty}{}} \Psi$	7, 8

TH. 4:2.7: $\forall\Phi\forall\Psi\big(\Phi \overset{=}{\underset{\infty}{}} \Psi \iff \{\Phi \overset{\varepsilon}{\underset{\infty}{}} \Phi \wedge \Psi \overset{\varepsilon}{\underset{\infty}{}} \Psi \wedge \forall\mu[\mu(\Phi) \iff \mu(\Psi)]\}\big)$

$$\forall A\forall B\{A = B \iff [A\varepsilon A \wedge B\varepsilon B \wedge \forall\varphi(\varphi\{A\} \iff \varphi\{B\})]\}.$$

We obtain an exact analogue of the *principle of extensionality* for identical names.

4.6 A MEREOLOGICAL INTERPRETATION OF THE ONTOLOGICAL ε-CONNECTIVE *(the functor η)*

We develop a functorial model for Leśniewski's Mereology, wherein the primitive epsilon copula of Ontology is locally intepreted. We draw largely from Clay [12, 15].

$$\forall a[!\{a\} \iff \exists A(A\varepsilon a)]$$

$$0 \neq 1$$

$$0 \neq 1 \iff [0\varepsilon 0 \wedge 1\varepsilon 1 \wedge \neg(0\varepsilon 1 \wedge 1\varepsilon 0)]$$

DF. 2:3.1: $\forall\sigma\big(\mathfrak{M}\langle\sigma\rangle \iff \forall a\{[!\{a\} \iff \exists A(A\varepsilon\sigma\{a\})] \wedge \forall A[A\varepsilon\sigma\{a\}$
$\implies (A = 0 \vee A = 1)] \wedge \forall A\forall B[(A\varepsilon\sigma\{a\} \wedge B\varepsilon\sigma\{a\})$

$\implies A = B]\}$

DF. 2:3.2: $\forall\pi\big(\mathfrak{N}\langle\pi\rangle \iff \{\mathfrak{M}\langle\pi\rangle \wedge \forall a\forall b[(1\varepsilon\pi\{a\} \wedge 1\varepsilon\pi\{b\}) \implies a \circ b]$
$\wedge\exists a(1\varepsilon\pi\{a\}) \wedge \exists a(1\varepsilon\pi\{a\})\}\big)$

DF. 2:3.3: $\forall\pi\forall\sigma\{\pi\eta\sigma \iff [\mathfrak{N}\langle\pi\rangle \wedge \mathfrak{M}\langle\sigma\rangle \wedge \forall a(1\varepsilon\pi\{a\} \implies 1\varepsilon\sigma\{a\})]\}$

TH. 2:4.1: $\forall\pi(\pi\eta\pi \iff \mathfrak{N}\langle\pi\rangle)$ 3.3, 3.2

TH. 2:4.2: $\forall\pi\forall\sigma(\pi\eta\sigma \implies \pi\eta\pi)$ 3.3, 4.1

TH. 2:4.3: $\forall\pi\forall\sigma\forall a[(\mathfrak{N}\langle\pi\rangle \wedge \mathfrak{M}\langle\sigma\rangle \wedge 1\varepsilon\pi\{a\} \wedge 1\varepsilon\sigma\{a\}) \implies \pi\eta\sigma]$

PF. $\forall\pi\forall\sigma\forall a$

1.	$\mathfrak{N}\langle\pi\rangle$	*Supp.*
2.	$\mathfrak{M}\langle\sigma\rangle$	*Supp.*
3.	$1\varepsilon\pi\{a\}$	*Supp.*
4.	$1\varepsilon\sigma\{a\}$	*Supp.*
	$\forall b$	
5.	$\forall c[(1\varepsilon\pi\{b\} \wedge 1\varepsilon\pi\{c\}) \implies b \circ c]$	DF. 2:3.2, 1
6.	$1\varepsilon\pi\{b\} \implies 1\varepsilon\sigma\{b\}$	3, 4, 5
7.	$\pi\eta\sigma$	DF. 2:3.3, 1, 2, 6

TH. 2:4.4: $\forall\pi\forall\sigma[\pi\eta\sigma \implies \exists a(1\varepsilon\sigma\{a\} \wedge 1\varepsilon\pi\{a\})]$

PF. $\forall\pi\forall\sigma\forall a$

1.	$\pi\eta\sigma$	*Supp.*
2.	$\mathfrak{N}\langle\pi\rangle$	DF. 2:3.3, 1
	$\exists a$	
3.	$1\varepsilon\pi\{a\}$	DF. 2:3.2, 1
4.	$1\varepsilon\sigma\{a\}$	DF. 2:3.3, 1, 3
7.	$\exists a(1\varepsilon\sigma\{a\} \wedge 1\varepsilon\pi\{a\})$	3, 4

TH. 2:4.5: $\forall\pi\forall\sigma\{\pi\eta\sigma \implies [\mathfrak{N}\langle\pi\rangle \wedge \mathfrak{M}\langle\sigma\rangle \wedge \exists a(1\varepsilon\sigma\{a\} \wedge 1\varepsilon\pi\{a\})]\}$

TH. 2:4.6: $\forall\pi\forall\varphi\forall\sigma[(\pi\eta\sigma \wedge \varphi\eta\pi) \implies \varphi\eta\sigma]$

PF. $\forall\pi\forall\sigma\forall a$

1.	$\pi\eta\sigma$	*Supp.*
2.	$\varphi\eta\pi$	*Supp.*
3.	$\mathfrak{N}\langle\varphi\rangle$	DF. 2:3.3, 2
4.	$\forall a(1\varepsilon\varphi\{a\} \implies 1\varepsilon\pi\{a\})$	DF. 2:3.3, 2
5.	$\mathfrak{M}\langle\sigma\rangle$	DF. 2:3.3, 1
	$\forall a$	
6.	$1\varepsilon\pi\{a\} \implies 1\varepsilon\sigma\{a\}$	DF. 2:3.3, 1
7.	$1\varepsilon\varphi\{a\} \implies 1\varepsilon\sigma\{a\}$	4, 6
8.	$\varphi\eta\sigma$	DF. 2:3.3, 3, 5, 7

TH. 2:4.7: $\forall\pi\forall\varphi\forall\psi\forall\sigma[(\pi\eta\sigma \wedge \varphi\eta\pi \wedge \psi\eta\pi) \implies \varphi\eta\psi]$

PF. $\forall\pi\forall\sigma\forall a$

1.	$\pi\eta\sigma$	*Supp.*
2.	$\varphi\eta\pi$	*Supp.*
3.	$\psi\eta\pi$	*Supp.*
4.	$\mathfrak{N}\langle\pi\rangle$	DF. 2:3.3, 1
5.	$\mathfrak{N}\langle\varphi\rangle$	DF. 2:3.3, 2
6.	$\mathfrak{N}\langle\psi\rangle$	DF. 2:3.3, 3
7.	$\mathfrak{M}\langle\psi\rangle$	DF. 2:3.2, 6
	$\exists a$	
8.	$1\varepsilon\varphi\{a\}$	TH. 2:4.4, 2
9.	$1\varepsilon\pi\{a\}$	TH. 2:4.4, 2
	$\exists b$	
10.	$1\varepsilon\varphi\{b\}$	TH. 2:4.4, 3
11.	$1\varepsilon\pi\{b\}$	TH. 2:4.4, 3
12.	$a \circ b$	DF. 2:3.2, 4, 9, 11
13.	$1\varepsilon\varphi\{a\}$	10, 12
14.	$\varphi\eta\psi$	TH. 2:4.3, 5, 7, 8, 13

DF. 2:4.8: $\forall A\forall a\forall b\big[A\varepsilon\nabla\lbag a\rbag(b) \iff \big(A\varepsilon A\wedge !\{a\}\wedge !\{b\} \wedge \{(A=1\wedge a\circ b)\vee[A=0\wedge\neg(a\circ b)]\}\big)\big]$

TH. 2:4.9: $\forall A\forall a\forall b\big[A\varepsilon\nabla\lbag a\rbag(b) \iff \big(!\{a\}\wedge !\{b\} \wedge \{(A=1\wedge a\circ b)\vee[A=0\wedge\neg(a\circ b)]\}\big)\big]$

TH. 2:4.10: $\forall a\forall b(1\varepsilon\nabla\lbag a\rbag(b) \iff !\{a\}\wedge a\circ b)$

TH. 2:4.11: $\forall a(1\varepsilon\nabla\lbag a\rbag(a) \iff !\{a\})$

TH. 2:4.12: $\forall a\forall b[!\{a\} \implies (1\varepsilon\nabla\lbag a\rbag(b) \iff a\circ b)]$

TH. 2:4.13: $\forall A\forall a\forall b[A\varepsilon\nabla\lbag a\rbag(b) \implies (A=0\vee A=1)]$

TH. 2:4.14: $\forall A\forall B\forall a\forall b\{[A\varepsilon\nabla\lbag a\rbag(b)\wedge B\varepsilon\nabla\lbag a\rbag(b)] \implies A=B\}$

PF. $\forall A\forall B\forall a\forall b$

1.	$A\varepsilon\nabla\lbag a\rbag(b)$	*Supp.*
2.	$B\varepsilon\nabla\lbag a\rbag(b)$	*Supp.*
3.	$(A=1\wedge a\circ b)\vee[A=0\wedge\neg(a\circ b)]$	TH. 2:4.9, 1
4.	$(B=1\wedge a\circ b)\vee[B=0\wedge\neg(a\circ b)]$	TH. 2:4.9, 2
5.	$A=B$	3, 4

TH. 2:4.15: $\forall a\forall b\{(!\{a\}\wedge !\{b\}) \implies \exists A[A\varepsilon\nabla\lbag a\rbag(b)]\}$

PF. $\forall a\forall b$

1.	$!\{a\}$	*Supp.*
2.	$!\{b\}$	*Supp.*
3.	$1\varepsilon\nabla\lbag a\rbag(b)\vee 0\varepsilon\nabla\lbag a\rbag(b)$	TH. 2:4.9, 1, 2
4.	$\exists A[A\varepsilon\nabla\lbag a\rbag(b)]$	3

TH. 2:4.16: $\forall a\big(!\{a\} \implies \{!\{b\} \iff \exists A[A\varepsilon\nabla\lbag a\rbag(b)]\}\big)$

TH. 2:4.17: $\forall a(!\{a\} \implies \mathfrak{M}\langle\nabla\lbag a\rbag\rangle)$

TH. 2:4.18: $\forall a(!\{a\} \implies \mathfrak{N}\langle\nabla\lbag a\rbag\rangle)$

TH. 2:4.19: $\forall a(\mathfrak{N}\langle\nabla\lbag a\rbag\rangle \implies !\{a\})$

PF. $\forall a$

1.	$\mathfrak{N}\langle\nabla\lbag a\rbag\rangle$	*Supp.*
2.	$\mathfrak{M}\langle\nabla\lbag a\rbag\rangle$	DF. 2:3.2, 1
3.	$\exists b[1\varepsilon\nabla\lbag a\rbag(b)]$	DF. 2:3.2, 1
4.	$!\{a\}$	TH. 2:4.9, 3

TH. 2:4.20: $\forall a(!\{a\} \iff \mathfrak{N}\langle\nabla\lbag a\rbag\rangle)$

TH. 2:4.21: $\forall\pi\forall a\forall b\big(\{\mathfrak{M}\langle\pi\rangle \wedge 1\varepsilon\pi\{a\} \wedge 1\varepsilon\pi\{b\} \wedge \forall\varphi\forall\psi[(\varphi\eta\pi \wedge \psi\eta\pi) \implies \varphi\eta\psi]\} \implies a \circ b\big)$

PF. $\forall\pi\forall a\forall b$

1.	$\mathfrak{M}\langle\pi\rangle$	*Supp.*
2.	$1\varepsilon\pi\{a\}$	*Supp.*
3.	$1\varepsilon\pi\{b\}$	*Supp.*
4.	$\forall\varphi\forall\psi[(\varphi\eta\pi \wedge \psi\eta\pi) \implies \varphi\eta\psi]$	*Supp.*
5.	$!\{a\}$	DF. 2:3.1, 1, 2
6.	$!\{b\}$	DF. 2:3.1, 1, 3
7.	$1\varepsilon\nabla\mathord{\not(}a\mathord{\not)}(a)$	TH. 2:4.11, 5
8.	$1\varepsilon\nabla\mathord{\not(}b\mathord{\not)}(b)$	TH. 2:4.11, 6
9.	$\mathfrak{N}\langle\nabla\mathord{\not(}a\mathord{\not)}\rangle$	TH. 2:4.20, 5
10.	$\mathfrak{N}\langle\nabla\mathord{\not(}b\mathord{\not)}\rangle$	TH. 2:4.20, 6
11.	$\nabla\mathord{\not(}a\mathord{\not)}\eta\pi$	TH. 2:4.3, 1, 2, 7, 9
12.	$\nabla\mathord{\not(}b\mathord{\not)}\eta\pi$	TH. 2:4.3, 1, 3, 8, 10
13.	$\nabla\mathord{\not(}a\mathord{\not)}\eta\nabla\mathord{\not(}b\mathord{\not)}$	4, 11, 12
14.	$1\varepsilon\nabla\mathord{\not(}b\mathord{\not)}(a)$	DF. 2:3.3, 13, 7
15.	$a \circ b$	DF. 2:3.2, 8, 10, 14

TH. 2:4.22: $\forall\theta\forall\pi\forall\sigma\big[\big(\theta\eta\pi \wedge \forall\varphi\{(\varphi\eta\pi \implies \varphi\eta\sigma) \wedge \forall\psi[(\varphi\eta\pi \wedge \psi\eta\pi) \implies \varphi\eta\psi]\}\big) \implies \pi\eta\sigma\big]$

PF. $\forall\theta\forall\pi\forall\sigma$

1.	$\theta\eta\pi$	*Supp.*
	$\forall\varphi$	
2.	$\varphi\eta\pi \implies \varphi\eta\sigma$	*Supp.*
3.	$\forall\psi[(\varphi\eta\pi \wedge \psi\eta\pi) \implies \varphi\eta\psi]$	*Supp.*
4.	$\mathfrak{M}\langle\pi\rangle$	DF. 2:3.3, 1
5.	$\forall a\forall b[(1\varepsilon\pi\{a\} \wedge 1\varepsilon\pi\{b\}) \implies a \circ b]$	TH. 2:4.21, 3, 4
6.	$\mathfrak{N}\langle\pi\rangle$	DF. 2:3.2, 4, 5
7.	$\pi\eta\pi$	TH. 2:4.1, 6
8.	$\pi\eta\sigma$	2, 7

A Mereological analagoue of a the single axiom of Ontology is given above.

4.7 THE *numerical epsilon* ANALOGUE OF THE PRINCIPLE OF MATHEMATICAL INDUCTION

DF. 4:7.1.1: $\forall\Phi\forall\varphi\forall\psi[(\Phi \mathrel{\substack{\varepsilon\\ \infty}} \varphi \wedge \Phi \mathrel{\substack{\varepsilon\\ \infty}} \psi) \iff \bigcap\limits_{\infty}\langle\varphi\psi\rangle(\Phi)]$

DF. 4:7.1.2: $\forall\varphi\forall\psi\forall a[(\infty\langle a\rangle \mathrel{\substack{\varepsilon\\ \infty}} \varphi \wedge \infty\langle a\rangle \mathrel{\substack{\varepsilon\\ \infty}} \psi) \iff \bigcap\limits_{\infty}\langle\varphi\psi\rangle\{a\}]$

TH. 4:7.2.1: $\forall\varphi\forall\psi\forall a[\bigcap\limits_{\infty}\langle\varphi\psi\rangle(\infty\langle a\rangle) \iff \bigcap\limits_{\infty}\langle\varphi\psi\rangle\{a\}]$

TH. 4:7.2.2: $\forall\varphi\forall\psi\forall a\forall b[(\bigcap\limits_{\infty}\langle\varphi\psi\rangle\{a\} \wedge a\infty b) \implies \bigcap\limits_{\infty}\langle\varphi\psi\rangle\{b\}]$

PF. $\forall\varphi\forall\psi\forall a\forall b$

1.	$\bigcap\limits_{\infty}\langle\varphi\psi\rangle\{a\}$	*Supp.*
2.	$a\infty b$	*Supp.*
3.	$\infty\langle a\rangle \mathrel{\substack{\varepsilon\\ \infty}} \varphi \wedge \infty\langle a\rangle \mathrel{\substack{\varepsilon\\ \infty}} \psi$	*Th.*, 1
4.	$\infty\langle b\rangle \mathrel{\substack{\varepsilon\\ \infty}} \varphi \wedge \infty\langle ab\rangle \mathrel{\substack{\varepsilon\\ \infty}} \psi$	*Th.*, 2, 3
5.	$\bigcap\limits_{\infty}\langle\varphi\psi\rangle\{b\}$	*Th.*, 4

TH. 4:7.3: $\forall\varphi\forall\psi[Num(\bigcap_{\infty}\langle\varphi\psi\rangle)]$

TH. 4:7.4.1: $\forall\Phi\forall\varphi\forall\psi[\Phi\overset{\varepsilon}{\underset{\infty}{}}\bigcap_{\infty}\langle\varphi\psi\rangle \iff (\Phi\overset{\varepsilon}{\underset{\infty}{}}\varphi \wedge \Phi\overset{\varepsilon}{\underset{\infty}{}}\psi \wedge \Phi\overset{\varepsilon}{\underset{\infty}{}}\Phi)]$

TH. 4:7.4.2: $\forall\Phi\forall\varphi\forall\psi[\Phi\overset{\varepsilon}{\underset{\infty}{}}\bigcap_{\infty}\langle\varphi\psi\rangle \iff (\Phi\overset{\varepsilon}{\underset{\infty}{}}\varphi \wedge \Phi\overset{\varepsilon}{\underset{\infty}{}}\psi)]$

TH. 4:7.5: $\forall A\forall\varphi\forall b\{[0\overset{\varepsilon}{\underset{\infty}{}}\varphi \wedge \forall\Psi(\Psi\overset{\varepsilon}{\underset{\infty}{}}\varphi \implies \mathbf{s}\langle\Psi\rangle\overset{\varepsilon}{\underset{\infty}{}}\varphi) \wedge A\varepsilon A \wedge \varphi\{b\} \wedge \neg(A\varepsilon b)] \implies \varphi\{b\cup A\}\}$

PF. $\forall A\forall\varphi\forall b$

1.	$0\overset{\varepsilon}{\underset{\infty}{}}\varphi$	*Supp.*
2.	$\forall\Psi(\Psi\overset{\varepsilon}{\underset{\infty}{}}\varphi \implies \mathbf{s}\langle\Psi\rangle\overset{\varepsilon}{\underset{\infty}{}}\varphi)$	*Supp.*
3.	$A\varepsilon A$	*Supp.*
4.	$\varphi\{b\}$	*Supp.*
5.	$\neg(A\varepsilon b)$	*Supp.*
6.	$Num(\varphi)$	*Th.*, 1
7.	$\infty\langle b\rangle\overset{\varepsilon}{\underset{\infty}{}}\varphi$	*Th.*, 4, 6
8.	$\mathbf{s}\langle\infty\langle b\rangle\rangle\overset{\varepsilon}{\underset{\infty}{}}\varphi$	2, 7
9.	$\forall a(\mathbf{s}\langle\infty\langle b\rangle\rangle\{a\} \implies \varphi\{a\})$	*Th.*, 8
10.	$A\varepsilon b\cup A$	*Th.*, 3
11.	$(b\cup A) - A \circ b$	*Th.*, 3, 5
12.	$\infty\langle b\rangle\{(b\cup A) - A\}$	*Th.*, 11
13.	$\mathbf{s}\langle\infty\langle b\rangle\rangle\{b\cup A\}$	*Th.*, 10, 12
14.	$\varphi\{b\cup A\}$	9, 13

TH. 4:7.6: $\forall\varphi\{[0\overset{\varepsilon}{\underset{\infty}{}}\varphi \wedge \forall\Psi(\Psi\overset{\varepsilon}{\underset{\infty}{}}\varphi \implies \mathbf{s}\langle\Psi\rangle\overset{\varepsilon}{\underset{\infty}{}}\varphi)] \implies \forall A\forall b[(A\varepsilon A \wedge \varphi\{b\}) \implies \varphi\{b\cup A\}]\}$

TH. 4:7.7: $\forall\Phi\forall\varphi\{[0\overset{\varepsilon}{\underset{\infty}{}}\varphi \wedge \forall\Psi(\Psi\overset{\varepsilon}{\underset{\infty}{}}\varphi \implies \mathbf{s}\langle\Psi\rangle\overset{\varepsilon}{\underset{\infty}{}}\varphi) \wedge \Phi\overset{\varepsilon}{\underset{\infty}{}}Finite] \implies \Phi\overset{\varepsilon}{\underset{\infty}{}}\varphi\}$

PF. $\forall\Phi\forall\varphi$

1.	$0\overset{\varepsilon}{\underset{\infty}{}}\varphi$	*Supp.*
2.	$\forall\Psi(\Psi\overset{\varepsilon}{\underset{\infty}{}}\varphi \implies \mathbf{s}\langle\Psi\rangle\overset{\varepsilon}{\underset{\infty}{}}\varphi)$	*Supp.*
3.	$\Phi\overset{\varepsilon}{\underset{\infty}{}}Finite$	*Supp.*
4.	$\forall A\forall b[(A\varepsilon A \wedge \varphi\{b\}) \implies \varphi\{b\cup A\}]$	*Th.*, 1, 2
5.	$Num(\varphi)$	*Th.*, 1
6.	$Num(\Phi) \wedge Qua(\Phi)$	*Th.*, 3
7.	$\exists b(0\{b\} \wedge \varphi\{b\})$	*Th.*, 1
8.	$\varphi\{\bigwedge\}$	*Th.*, 7
	$\exists a$	
9.	$\Phi\{a\} \wedge Finite\{a\}$	*Th.*, 3
10.	$\forall A\forall b[(A\varepsilon b \wedge \Phi\{b\}) \implies \varphi\{b\cup A\}]$	*Th.*, 4
11.	$\varphi\{a\}$	*Th.*, 8, 9, 10
12.	$\Phi\overset{\varepsilon}{\underset{\infty}{}}\varphi$	*Th.*, 5, 6, 9, 11

TH. 4:7.8: $\forall\Phi\forall\varphi(\{0\overset{\varepsilon}{\underset{\infty}{}}\varphi \wedge \forall\Psi[(\Psi\overset{\varepsilon}{\underset{\infty}{}}Finite \wedge \Psi\overset{\varepsilon}{\underset{\infty}{}}\varphi) \implies \mathbf{s}\langle\Psi\rangle\overset{\varepsilon}{\underset{\infty}{}}\varphi] \wedge \Phi\overset{\varepsilon}{\underset{\infty}{}}Finite\} \implies \Phi\overset{\varepsilon}{\underset{\infty}{}}\varphi)$

PF. $\forall \Phi \forall \varphi$

1.	$0 \mathrel{\substack{\varepsilon \\ \infty}} \varphi$	*Supp.*
2.	$\forall \Psi [(\Psi \mathrel{\substack{\varepsilon \\ \infty}} Finite \wedge \Psi \mathrel{\substack{\varepsilon \\ \infty}} \varphi) \implies \mathbf{s}\langle \Psi \rangle \mathrel{\substack{\varepsilon \\ \infty}} \varphi]$	*Supp.*
3.	$\Phi \mathrel{\substack{\varepsilon \\ \infty}} Finite$	*Supp.*
4.	$Num(\varphi)$	*Th.*, 1
5.	$0 \mathrel{\substack{\varepsilon \\ \infty}} \bigcap_{\infty} \langle \varphi Finite \rangle$	*Th.*, 1
6.	$\forall \Psi [\Psi \mathrel{\substack{\varepsilon \\ \infty}} \bigcap_{\infty} \langle \varphi Finite \rangle \implies \mathbf{s}\langle \Psi \rangle \mathrel{\substack{\varepsilon \\ \infty}} \bigcap_{\infty} \langle \varphi Finite \rangle$	*Th.*, 2
7.	$\Phi \mathrel{\substack{\varepsilon \\ \infty}} \bigcap_{\infty} \langle \varphi Finite \rangle$	*Th.*, 3, 5, 6
8.	$\Phi \mathrel{\substack{\varepsilon \\ \infty}} \varphi$	*Th.*, 7

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