Math 231 Final Review Sheet, 12/7/2025

The final exam will be on Thursday 12/18 from 1:45 to 3:45PM in Kiely 320. It has five problems (one being a few multiple-choice questions) and is 110 minutes long. You may use a calculator for basic arithmetic, although it won't be something you need. *You are not allowed to use any built-in linear algebra packages on a calculator.* Below is a list of the topics that you are advised to study.

Earlier and background material:

- Vectors in $\mathbb{R}^n$; vector addition and scalar multiplication; the dot product and its basic properties; norm and distance; the angle between two vectors
- Solving systems of linear equations by elimination; reducing matrices to row echelon forms; leading and free variables
- The row space, column space and null space of a matrix; finding bases for these spaces by reducing to row echelon forms; rank and nullity of a matrix; the rank plus nullity theorem for matrices
- Evaluating $\det(A)$ by the cofactor expansion formula; basic properties of the determinant; A is invertible if and only if $\det(A) \neq 0$

More recent material that you should mostly focus on:

- Orthogonal vectors in $\mathbb{R}^n$; the orthogonal complement $W^\perp$ of a subspace W; the relation $(\mathrm{row}(A))^\perp = \mathrm{null}(A)$ and its application in finding bases for orthogonal complements; orthogonal and orthonormal bases
- The orthogonal projection theorem: If W is a subspace of $\mathbb{R}^n$, every $\mathbf{v} \in \mathbb{R}^n$ can be decomposed uniquely as $\mathbf{v} = \mathbf{p} + \mathbf{q}$ where $\mathbf{p} \in W$ and $\mathbf{q} \in W^\perp$. Moreover, if we choose an orthogonal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ for W, we can find $\mathbf{p}, \mathbf{q}$ as follows:
$$\begin{aligned}\mathbf{p} &= \mathbf{proj}_W\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}_1}{\|\mathbf{u}_1\|^2}\,\mathbf{u}_1 + \cdots + \frac{\mathbf{v}\cdot\mathbf{u}_k}{\|\mathbf{u}_k\|^2}\,\mathbf{u}_k \\ \mathbf{q} &= \mathbf{perp}_W\mathbf{v} = \mathbf{v} - \mathbf{p}.\end{aligned}$$
- General vector spaces; basic examples: $\mathbb{R}^n$, $\mathcal{M}_{m,n}$, $\mathcal{F}$, $\mathcal{P}_n$; subspaces of vector spaces; the span of a set of vectors; linear dependence and independence
- Definition of a basis for a vector space; coordinates of a vector relative to a basis; definition of dimension;
$$\dim(\mathbb{R}^n) = n \qquad \dim(\mathcal{M}_{m,n}) = mn \qquad \dim(\mathcal{P}_n) = n+1$$

- Suppose $\dim(V) = n$.

  (i) Any linearly independent set in V has at most n vectors. If it has exactly n vectors, it is a basis for V. If it has less than n vectors, it can be enlarged to a basis for V.

  (ii) Any spanning set for V has at least n vectors. If it has exactly n vectors, it is a basis for V. If it has more than n vectors, it can be reduced to a basis for V.

- Linear maps between vector spaces; linear maps are uniquely determined by their action on basis vectors; every linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is of the form $T(x) = Ax$ for an $m \times n$ matrix A whose *columns* are the vectors $T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)$; composition of two such linear maps corresponds to multiplication of their matrices

- Basic examples of linear maps $\mathbb{R}^2 \to \mathbb{R}^2$ coming from geometry

- Kernel and range of a linear map; the rank plus nullity theorem: If $T : V \to W$ is a linear map, then

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V).$$

Practice problems

1. Consider the subspace $W = \operatorname{span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^3$.

   (i) Find a basis for $W^\perp$.

   (ii) If $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, find $\mathbf{p} = \mathbf{proj}_W(\mathbf{v})$ and $\mathbf{q} = \mathbf{perp}_W(\mathbf{v})$.

2. Give bases for the spaces

$$V = \{p(x) \in \mathcal{P}_2 : p(0) = p'(0) = 0\}$$
$$W = \{A \in \mathcal{M}_{3,3} : A \text{ is lower triangular}\}$$

and use them to find $\dim(V)$ and $\dim(W)$.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map that satisfies

$$T \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \qquad \text{and} \qquad T \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -4 \end{bmatrix}.$$

Find a formula for $T \begin{bmatrix} x \\ y \end{bmatrix}$. What is the standard matrix of T?

4. Consider the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the reflection in the x-axis followed by the 30° counterclockwise rotation around the origin. Find a formula for T.

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal projection on the line $y = mx$. Use the projection formula to show that the standard matrix of T is

$$\begin{bmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2} \end{bmatrix}.$$

6. Let $T : \mathcal{P}_2 \to \mathbb{R}$ be the linear map defined by

$$T(p(x)) = \int_{-1}^{1} p(x)\, dx.$$

(i) Describe $\ker(T)$ and $\operatorname{range}(T)$.
(ii) Find $\operatorname{nullity}(T)$ and $\operatorname{rank}(T)$ and verify that the rank plus nullity theorem holds.

7. True or false?

- The vectors

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$$

form an orthogonal basis for $\mathbb{R}^3$.
- If $V = \{A \in \mathcal{M}_{2,2} : A = -A^T\}$, then $\dim(V) = 2$.
- There are 5 linearly independent polynomials in $\mathcal{P}_3$.
- There is a linear map $T : \mathbb{R}^4 \to \mathbb{R}^3$ with the property $\ker(T) = \{\mathbf{0}\}$.