ON SIEGEL DISKS OF A CLASS OF ENTIRE MAPS

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To the memory of Adrien Douady (1935-2006)

Contents

1. Introduction

Let f be an entire map of the complex plane or a rational map of the Riemann sphere. Suppose $f(0) = 0$ and $f'(0) = e^{2\pi i\theta}$, where the *rotation number* $0 < \theta < 1$ is irrational. We say f is *locally linearizable* at the fixed point 0 if there exists a holomorphic change of coordinates near 0 which conjugates f to its linear part $R_\theta : z \mapsto e^{2\pi i\theta} z$. The maximal region $\Delta = \Delta_f$ on which f is conjugate to R_θ is a simply-connected domain called the *Siegel disk* of f centered at 0. Thus f acts as an irrational rotation in Δ. However, understanding the topology and geometry of the boundary $\partial\Delta$, and the dynamics of f on it, is often quite difficult.

This paper will study Siegel disks in the family $\mathcal{E}^{p,q}$ of non-linear entire maps of the form

$$f : z \mapsto P(z) \exp(Q(z)),$$

where P and Q are polynomials of degree p and q, respectively. We consider the subfamily $\mathcal{E}^{p,q}(\theta) \subset \mathcal{E}^{p,q}$ of maps which have a Siegel disk of rotation number θ centered at the origin, so $P(0) = Q(0) = 0$ and $P'(0) = f'(0) = e^{2\pi i\theta}$. There are good reasons to view these entire maps as close relatives of polynomials. For example, they have finitely many zeros and critical points and, in the transcendental case $q > 0$, a

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single (finite) asymptotic value at the origin. They belong to the Speiser class $\mathcal{S}$ of entire maps with finitely many singular values, or more generally to the Eremenko-Lyubich class $\mathcal{B}$ of entire maps with a bounded set of singular values, which are known to share many of their dynamical properties with polynomials (see [**EL**] and compare [**MNTU**] in which such maps are called "decorated exponential"). Our primary focus will of course be on the transcendental case $q > 0$, but the analysis will cover the polynomial case $q = 0$ as well.

Expand the rotation number θ into its continued fraction $[a_1, a_2, a_3, \ldots]$, where each a_n is a positive integer. Recall that θ is of *bounded type* if $\{a_n\}$ is a bounded sequence. It is well-known that in this case f is locally linearizable at the origin.

Main Theorem. *Let $0 < \theta < 1$ be an irrational number of bounded type and $f \in \mathcal{E}^{p,q}(\theta)$. Then the Siegel disk of f centered at the origin is bounded by a quasicircle in the plane which contains at least one critical point of f.*

Compare Fig. 1.

This generalizes and unifies several results obtained over the past 20 years by various authors. These include Douady-Herman-Swiatek's for quadratic polynomials [**D**], this author's for cubic polynomials [**Z1**], Shishikura's for polynomials of arbitrary degree [**S2**], Geyer's for the map $z \mapsto e^{2\pi i\theta} z\, e^z$ [**Ge**], and Keen-Zhang's for the maps of the form $z \mapsto (e^{2\pi i\theta} z + az^2)\, e^z$ [**KZ**]. See also Chéritat's examples of "simple" entire maps in [**C**].

It is important to realize that the choice of normalization for the family $\mathcal{E}^{p,q}(\theta)$ in the transcendental case $q > 0$ is not a matter of convenience. In fact, when $q > 0$, unlike the polynomial case, the space $\mathcal{E}^{p,q}$ is not invariant under affine conjugations that move the origin. As a result, the Main Theorem does not imply anything about bounded type Siegel disks in $\mathcal{E}^{p,q}$ that are centered at points other than 0. This is not entirely a fault of our method: for example, if θ is an irrational of bounded type and $\lambda = e^{2\pi i\theta}$, the boundary of the Siegel disk centered at λ of the map $z \mapsto \lambda\, e^{z-\lambda}$ in $\mathcal{E}^{0,1}$ contains ∞, hence fails to be a Jordan curve on the sphere (see [**H2**] and compare Fig. 2). On the other hand, it is unknown whether the Main Theorem holds for bounded type Siegel disks of arbitrary rational maps of the sphere.

Our strategy of proof is strongly inspired by Shishikura's unpublished work for Siegel disks of polynomials. Let $\zeta = \zeta_f : \mathbb{D} \to \Delta$ be the unique conformal isomorphism that satisfies $\zeta(0) = 0, \zeta'(0) > 0$. It follows from Schwarz Lemma that ζ *linearizes* f in the sense that

$$f \circ \zeta = \zeta \circ R_\theta \qquad \text{in } \mathbb{D}.$$

Following Shishikura, we show that the invariant curves $\gamma_r = \gamma_{f,r} := \zeta(\{z : |z| = r\})$ in Δ are K-quasicircles for a constant $K > 1$ independent of the radius $0 < r < 1$. A simple compactness argument then proves that $\partial\Delta$ is a quasicircle (Theorem 2.3).

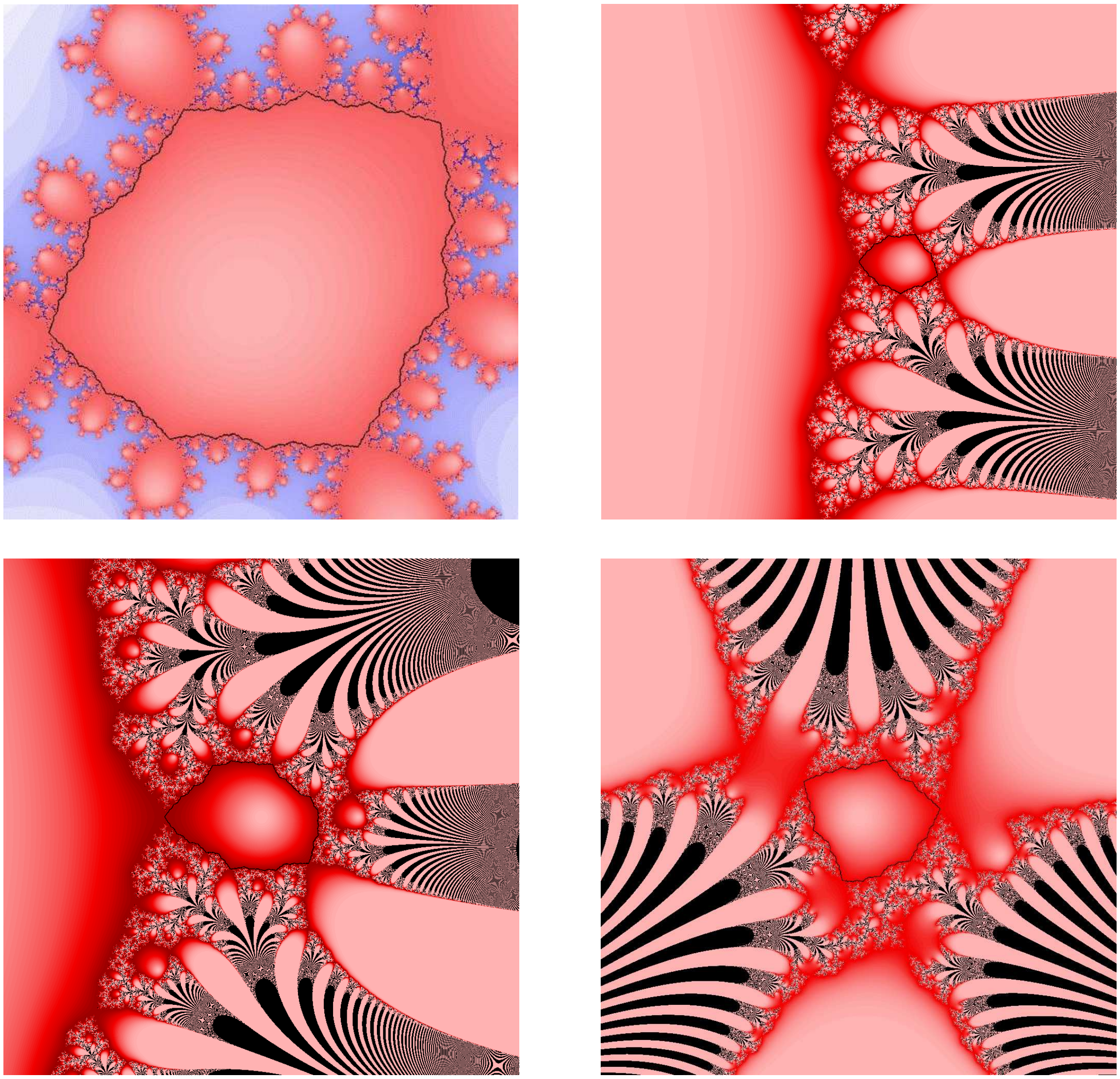

FIGURE 1. *Julia sets of four maps in $\mathcal{E}^{p,q}(\theta)$. In each case, the boundary of the Siegel disk Δ is the quasicircle delineated in black at the center of picture. Upper left: $z \mapsto \lambda z + z^2$; upper right: $z \mapsto \lambda\, z\, e^z$; lower left: $z \mapsto \lambda\, z\,(1 - 2z/3)\, e^z$; lower right: $z \mapsto \lambda\, z\,(1-(11+3i)z/13)\, e^{iz^3}$. Here $\lambda = e^{\pi i(\sqrt{5}-1)}$ corresponds to the golden mean rotation number.*

That $\partial\Delta$ must contain a critical point follows from a standard argument (Theorem 2.8).

Let us give a rough outline of the proof: Fix $0 < r < 1$ and take a suitable quasiconformal reflection $I : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which swaps 0 and ∞ and keeps the invariant

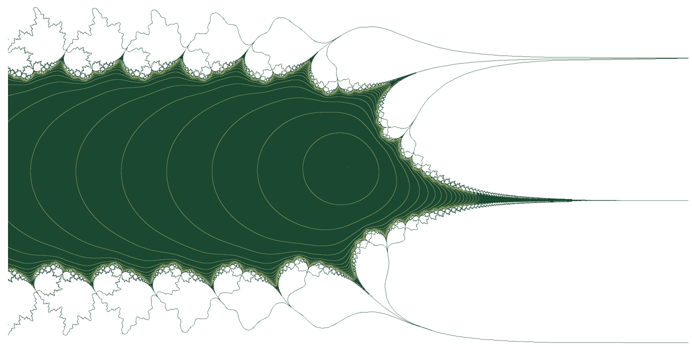

Figure 2. *(Courtesy of A. Chéritat) Invariant curves in the Siegel disk of the map* $z \mapsto \lambda e^{z-\lambda}$ *centered at* λ*. As before,* $\lambda = e^{\pi i(\sqrt{5}-1)}$*. This Siegel disk is unbounded and its boundary fails to be a Jordan curve.*

curve $\gamma_r \subset \Delta$ fixed pointwise. Use I to "symmetrize" f about γ_r in order to produce a quasiregular map $F : \mathbb{C}^* \to \widehat{\mathbb{C}}$ which commutes with I. This replaces the Siegel disk of f by a "quasiconformal Herman ring" for F. The sphere admits a conformal structure μ of bounded dilatation which is invariant under both F and I. Straightening μ by an appropriately normalized quasiconformal map $\xi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ shows that the conjugate map $G := \xi \circ F \circ \xi^{-1} : \mathbb{C}^* \to \widehat{\mathbb{C}}$ is holomorphic and commutes with the reflection $z \mapsto 1/\overline{z}$ across the unit circle $\mathbb{T} = \xi(\gamma_r)$. The map G has a genuine Herman ring having $\mathbb{T}$ as an invariant curve. Note, however, that the maximal dilatation of ξ generally depends on r and *a priori* can grow large as $r \to 1$.

By analyzing the explicit form of G and estimating the location of its poles, we show that there is an $\varepsilon > 0$, depending only on p, q, such that G is holomorphic in the annulus $\{z : 1 - \varepsilon < |z| < 1 + \varepsilon\}$ (Theorem 5.5). This step is rather easy for polynomials but requires work in the transcendental case. Since the rotation number θ is assumed to be of bounded type, the theorem of Herman-Swiatek (Theorem 2.7) shows that the restriction of G to $\mathbb{T}$ is k-quasisymmetrically conjugate to R_θ for a constant $k > 1$ which only depends on p, q, θ. Extend this conjugacy to a K-quasiconformal map $\mathbb{D} \to \mathbb{D}$, with $K > 1$ independent of r, and use it to modify the action of G on $\mathbb{D}$ into a K-quasiconformal rotation by angle θ. Intuitively, we paste a "quasiconformal Siegel disk" on $\mathbb{D}$ to produce a new quasiregular dynamics $\hat{G} : \mathbb{C} \to \mathbb{C}$. The map $\hat{G}$ admits an invariant conformal structure ν of bounded dilatation. Straightening ν by an appropriately normalized K-quasiconformal map $\psi : \mathbb{C} \to \mathbb{C}$ gives an entire map $g := \psi \circ \hat{G} \circ \psi^{-1} : \mathbb{C} \to \mathbb{C}$. It is easily verified that $g \in \mathcal{E}^{p,q}(\theta)$ and that the K-quasicircle $\psi(\mathbb{T})$ is just the invariant curve $\gamma_{g,r} = \zeta_g(\{z : |z| = r\})$ in the Siegel disk Δ_g. If we could show that g is the same map f that we started with, it would follow that $\gamma_{f,r}$ is a K-quasicircle for a K independent of r, which would prove the Main Theorem.

Unfortunately, the rigidity property $g = f$ is too good to be true for a general $f \in \mathcal{E}^{p,q}(\theta)$. The procedure

$$f \quad \overset{\text{symmetrize}}{\longmapsto} \quad F \quad \overset{\text{straighten}}{\longmapsto} \quad G \quad \overset{\text{modify}}{\longmapsto} \quad \hat{G} \quad \overset{\text{straighten}}{\longmapsto} \quad g$$

defines a ***surgery map*** $\mathcal{S}_r : \mathcal{E}^{p,q}(\theta) \to \mathcal{E}^{p,q}(\theta)$ which is far from the identity. In fact, the map $g = \mathcal{S}_r(f)$ may not be quasiconformally conjugate to f, and even when it is, one may not be able to promote the conjugacy to a conformal one. The difficulty arises when f has a critical point that is ***captured*** by its Siegel disk, in the sense that its forward orbit eventually hits Δ. Let us call the first point of hitting a ***capture spot*** of f in Δ. Then, a necessary and sufficient condition for the existence of a conformal conjugacy between f and $g = \mathcal{S}_r(f)$ is that the capture spots of f and g have the same conformal positions in their respective Siegel disks Δ_f and Δ_g (Theorem 6.1). This can hardly be guaranteed in the above surgery.

To circumvent this problem, we separate the argument depending on the number and position of capture spots of f in Δ:

Case 1. The only capture spot of f, if any at all, is the fixed point 0. In other words, every critical orbit is either disjoint from Δ or else lands at the origin. This case is easy to handle since a standard pull-back argument shows that f is rigid, so $\mathcal{S}_r(f) = f$ (Corollary 6.2).

Case 2. There is precisely one non-zero capture spot of f in Δ. In other words, there is an $\omega \in \Delta \smallsetminus \{0\}$ such that the forward orbit of every captured critical point hits Δ for the first time at ω or else at 0. In this case, we produce a holomorphic one-parameter family $\{f_t\}_{t \in \mathbb{D}^*}$ of quasiconformal deformations of f in $\mathcal{E}^{p,q}(\theta)$ with the property that the capture spot ω_t of f_t has conformal position t in Δ_t (Theorem 7.1). We use holomorphic motions to show that there is a constant K independent of r such that the invariant curve $\gamma_{t,r} := \zeta_{f_t}(\{z : |z| = r\}) \subset \Delta_t$ is a K-quasicircle whenever $0 < |t| < 1/2$ or $r < |t| < 1$ (Lemma 7.4 and Corollary 7.6). The case of the intermediate values of $|t|$ is then covered by applying the Maximum Modulus Principle to a suitable cross-ratio function $\mathbb{D}^* \to \mathbb{C}$ (Theorem 7.7).

Case 3. For the general case, let U be an iterated preimage of Δ which contains m critical points counting multiplicities. We modify the dynamics of f on a compact subset of U so that the new map $U \to f(U)$ is a smooth branched covering with a single critical point of order m. We apply this modification to all such U, making sure that the resulting branched points eventually map to 0 or some designated point $\omega \in \Delta$. Straightening the resulting action, we obtain a map $g \in \mathcal{E}^{p,q}(\theta)$ which falls into one of the categories covered by the cases (i) or (ii) above. The maps f and g are not topologically conjugate, but there is a quasiconformal homeomorphism of the plane which maps $\partial\Delta_f$ to $\partial\Delta_g$. Since $\partial\Delta_g$ is a quasicircle by the cases (i) or (ii), it follows that $\partial\Delta_f$ is a quasicircle as well, which completes the proof of the Main Theorem.

2. Preliminaries

Throughout the paper we will adopt the following notations:

- $\mathbb{C}$ is the complex plane and $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere.
- $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ is the upper half-plane.
- $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ and $\mathbb{D} := \mathbb{D}_1$.
- $\mathbb{T}_r := \{z \in \mathbb{C} : |z| = r\}$ and $\mathbb{T} := \mathbb{T}_1$.
- $\mathbb{C}^* := \mathbb{C} \smallsetminus \{0\}$ and $\mathbb{D}^* := \mathbb{D} \smallsetminus \{0\}$.
- $\mathbb{A}_{r,s} := \{z \in \mathbb{C} : r < |z| < s\}$.

We assume the reader is familiar with the basic theory of quasiconformal mappings in the plane, as in [**A**] or [**LV**].

2.1. Quasisymmetric maps.

An orientation-preserving homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is k-*quasisymmetric* if

$$k^{-1} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq k \tag{2.1}$$

for all $x \in \mathbb{R}$ and $t > 0$. It is well-known that this condition is equivalent to f having a K-quasiconformal extension $\hat{f} : \mathbb{H} \to \mathbb{H}$. Moreover, the constants K and k depend only on each other and not on the choice of f.

According to Douady and Earle, each orientation-preserving homeomorphism $f : \mathbb{R} \to \mathbb{R}$ can be extended to a homeomorphism $DE(f) : \mathbb{H} \to \mathbb{H}$ in such a way that the assignment $f \mapsto DE(f)$ is *conformally natural*, i.e.,

$$DE(\alpha \circ f \circ \beta) = \alpha \circ DE(f) \circ \beta \qquad \text{for all } \alpha, \beta \in \ \mathrm{Aut}(\mathbb{H}).$$

Moreover, if f is k-quasisymmetric, the extension $DE(f)$ is K-quasiconformal for some $K = K(k) > 1$ [**DE**]. We use this result to define *standard* extensions of circle homeomorphisms to disks and annuli as follows. Let $f : \mathbb{T} \to \mathbb{T}$ be an orientation-preserving homeomorphism and $g : \mathbb{R} \to \mathbb{R}$ be a lift of f under the covering map $x \mapsto e^{2\pi i x}$. Then g is uniquely determined up to an additive integer and commutes with the unit translation $x \mapsto x + 1$. By definition, f is k-quasisymmetric if g is k-quasisymmetric is the sense of (2.1). In this case, the Douady-Earle extension $DE(g) : \mathbb{H} \to \mathbb{H}$ is K-quasiconformal for some $K = K(k)$ and by conformal naturalness commutes with $z \mapsto z+1$, so it descends under the covering map $z \mapsto e^{2\pi i z}$ to a K-quasiconformal homeomorphism $\hat{f} : \mathbb{D} \to \mathbb{D}$ which fixes the origin.

The following lemma gives a similar construction for the annulus:

Lemma 2.1. *Suppose $f : \partial \mathbb{A}_{r,s} \to \partial \mathbb{A}_{r,s}$ restricts to k-quasisymmetric maps on each of the circles $\mathbb{T}_r$ and $\mathbb{T}_s$, with $f(r) = r$ and $f(s) = s$. Then f can be extended to a K-quasiconformal homeomorphism $\hat{f} : \mathbb{A}_{r,s} \to \mathbb{A}_{r,s}$, where $K = K(k, s/r)$.*

Proof. After a radially affine stretch, we may assume $r = 1, s = e^{2\pi^2}$ and construct a K-quasiconformal extension of f with $K = K(k)$. Under the covering map from the strip $S := \{z : 0 < \mathrm{Im}(z) < \pi\}$ to $\mathbb{A}_{r,s}$ defined by $z \mapsto e^{-2\pi i z}$, the map f lifts to a homeomorphism $h : \partial S \to \partial S$ which satisfies $h(0) = 0, h(\pi i) = \pi i$, and commutes with $z \mapsto z+1$. Moreover, h restricts to k-quasisymmetric maps on each of the lines $\mathrm{Im}(z) = 0, \mathrm{Im}(z) = \pi$. Under the conformal isomorphism $\mathbb{H} \to S$ defined by $z \mapsto \log z$, the map h induces a homeomorphism $g : \mathbb{R} \to \mathbb{R}$ which satisfies $g(0) = 0, g(1) = 1, g(-1) = -1$, and commutes with $z \mapsto ez$. It is not hard to see that g is k'-quasisymmetric for some k' depending on k. The Douady-Earle extension $DE(g) : \mathbb{H} \to \mathbb{H}$ also commutes with $z \mapsto ez$, and it is K-quasiconformal for some K depending only on k', hence on k. The induced K-quasiconformal map $\hat{h} : S \to S$ commutes with $z \mapsto z+1$, so it descends to a K-quasiconformal extension $\hat{f} : \mathbb{A}_{r,s} \to \mathbb{A}_{r,s}$, as required. □

2.2. **Quasicircles.** A Jordan curve $\gamma \subset \widehat{\mathbb{C}}$ is called a K-*quasicircle* if there is a K-quasiconformal map $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\gamma = \varphi(\mathbb{T})$. We call γ a *quasicircle* if it is a K-quasicircle for some $K \geq 1$.

The following lemma is standard:

Lemma 2.2. *Let $\gamma \subset \widehat{\mathbb{C}}$ be a K-quasicircle, U be a component of $\widehat{\mathbb{C}} \smallsetminus \gamma$ and $\zeta : \mathbb{D} \to U$ be a conformal isomorphism. Then ζ extends to a K^2-quasiconformal map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.*

Proof. Take a K-quasiconformal map $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $\gamma = \varphi(\mathbb{T})$. The reflection $\iota : z \mapsto 1/\overline{z}$ across $\mathbb{T}$ induces a K^2-quasiconformal reflection $j := \varphi \circ \iota \circ \varphi^{-1}$ fixing γ pointwise. The homeomorphism

$$\hat{\zeta} := \begin{cases} \zeta & \text{inside } \mathbb{D} \\ j \circ \zeta \circ \iota & \text{outside } \mathbb{D} \end{cases}$$

is easily seen to be a K^2-quasiconformal extension of ζ. □

Theorem 2.3. *Let $\zeta : \mathbb{D} \to U \subset \widehat{\mathbb{C}}$ be a conformal isomorphism. Then, the following conditions are equivalent:*

(i) *The boundary ∂U is a quasicircle.*

(ii) *The Jordan curves* $\gamma_r := \zeta(\mathbb{T}_r)$ *are* K*-quasicircles for some* K *independent of* $0 < r < 1$.

Proof. First suppose ∂U is a K-quasicircle. By Lemma 2.2, $\zeta : \mathbb{D} \to U$ extends to a K^2-quasiconformal map $\hat{\zeta}$ of the sphere. Since γ_r is the image of $\mathbb{T}$ under $z \mapsto \hat{\zeta}(rz)$, it follows that γ_r is a K^2-quasicircle for every $0 < r < 1$.

Conversely, suppose there is a K such that γ_r is a K-quasicircle for every $0 < r < 1$. By Lemma 2.2, the conformal isomorphisms $\zeta_r : \mathbb{D} \to \zeta(\mathbb{D}_r)$ defined by $\zeta_r(z) := \zeta(rz)$ extend to K^2-quasiconformal maps of the sphere. By compactness, there is a sequence $r_n \to 1$ such that ζ_{r_n} tends locally uniformly to a K^2-quasiconformal map φ, and $\varphi = \zeta$ in $\mathbb{D}$. It follows that $\partial U = \varphi(\mathbb{T})$ is a K^2-quasicircle. $\square$

Corollary 2.4. *In the situation of Theorem 2.3, assume that* $\zeta(0) = 0$ *and that every* γ_r *is the image of* $\mathbb{T}$ *under a* K*-quasiconformal map* $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *which fixes* 0 *and* ∞*. Then the quasicircle* ∂U *does not pass through* ∞*, so* U *is a bounded domain in* $\mathbb{C}$.

Proof. By the proof of Lemma 2.2, every ζ_r extends to a K^2-quasiconformal map of the sphere which fixes 0 and ∞. Hence the limit $\varphi = \lim_{n\to\infty} \zeta_{r_n}$ in the proof of Theorem 2.3 will also have 0 and ∞ as fixed points. In particular, $\partial U = \varphi(\mathbb{T})$ does not pass through ∞. $\square$

We will need the following geometric characterization of quasicircles is terms of cross-ratios, which is equivalent to Ahlfors's "bounded turning condition" [**A**]. Define the *cross-ratio* of a quadruple (a, b, c, d) of distinct points in $\widehat{\mathbb{C}}$ by

$$\mathbf{Cr}(a, b, c, d) := \frac{(a-b)(c-d)}{(a-c)(b-d)}. \tag{2.2}$$

It is easily checked that $\mathbf{Cr}$ is invariant under the action of the Möbius group $\mathrm{Aut}(\widehat{\mathbb{C}})$. In particular, $0 < \mathbf{Cr}(a, b, c, d) < 1$ whenever the points a, b, c, d lie on a circle (in this cycle order).

Theorem 2.5. *The following conditions on a Jordan curve* $\gamma \subset \widehat{\mathbb{C}}$ *are equivalent:*

(i) γ *is a* K*-quasicircle.*

(ii) *There is a constant* $M > 0$ *such that for every quadruple of distinct points* $a, b, c, d \in \gamma$ *(in this cyclic order),*

$$|\,\mathbf{Cr}(a, b, c, d)| \leq M.$$

The constants K *and* M *depend only on each other and not on the choice of* γ.

2.3. **Linearization of circle maps.** It will be convenient in the following discussion to identify the unit circle $\mathbb{T}$ with the quotient $\mathbb{R}/\mathbb{Z}$. Let $f : \mathbb{T} \to \mathbb{T}$ be an orientation-preserving homeomorphism and $\hat{f} : \mathbb{R} \to \mathbb{R}$ be any lift of f. The limit

$$\lim_{n\to\infty} \frac{\hat{f}^{\circ n}(x)}{n} \qquad (\mathrm{mod}\ \mathbb{Z})$$

exists and is independent of the choice of $x \in \mathbb{R}$ and the lift $\hat{f}$. We call this residue class the *rotation number* of f and often identify it with its unique representative in the interval $[0, 1)$. It is a basic invariant of the conjugacy class of f.

Suppose $f : \mathbb{T} \to \mathbb{T}$ has an irrational rotation number $0 < \theta < 1$. We say f is *topologically linearizable* if there exists a homeomorphism $h : \mathbb{T} \to \mathbb{T}$ which conjugates f to the rigid rotation $R_\theta : x \mapsto x + \theta$ (mod $\mathbb{Z}$):

$$f \circ h = h \circ R_\theta \qquad \text{on } \mathbb{T}.$$

Such linearizing map h is unique when normalized so that $h(0) = 0$. The map f is quasisymmetrically (resp. smoothly, analytically) linearizable if its normalized linearizing map is quasisymmetric (resp. smooth, real-analytic).

An irrational number $0 < \theta < 1$ is *Diophantine of exponent* $\nu \geq 2$ if there is a constant $C > 0$ such that

$$\left| \theta - \frac{p}{q} \right| > \frac{C}{q^\nu}$$

for all rational numbers p/q with $q > 0$. We say θ is of *bounded type* if it is Diophantine of exponent $\nu = 2$. Equivalently, θ is bounded type if the integers a_n in the continued fraction expansion

$$\theta = [a_1, a_2, a_3, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

form a bounded sequence.

The most basic result on linearization of real-analytic diffeomorphisms is the following theorem of Herman [**H1**]:

Theorem 2.6 (Herman)**.** *Every real-analytic circle diffeomorphism with a Diophantine rotation number is analytically linearizable.*

In the presence of critical points, however, the situation is much more subtle. Let us call a quadruple (a, b, c, d) of points in $\mathbb{T}$ *sorted* if $a < b < c < d < a + 1$ or $a > b > c > d > a - 1$ (mod $\mathbb{Z}$). The cross-ratio of a sorted quadruple is defined by (2.2) and satisfies $0 < \mathbf{Cr}(a, b, c, d) < 1$. Given an orientation-preserving homeomorphism $f : \mathbb{T} \to \mathbb{T}$ and an interval $I \subset \mathbb{T}$, define the *cross-ratio distortion of f on I* by

$$\mathcal{D}(f, I) := \sup\ \log \frac{\mathbf{Cr}(f(a), f(b), f(c), f(d))}{\mathbf{Cr}(a, b, c, d)},$$

where the supremum is taken over all sorted quadruples (a, b, c, d) of points in I. For a collection $\mathcal{C}$ of intervals in $\mathbb{T}$, we define the *thickness* $\tau(\mathcal{C})$ as the maximum number of overlapping intervals in $\mathcal{C}$. Equivalently,

$$\tau(\mathcal{C}) = \sup_{\mathbb{T}} \sum_{I \in \mathcal{C}} \chi_I,$$

where χ_I is the characteristic function of the interval I. Finally, define the *cross-ratio distortion norm* of f by

$$\mathcal{D}(f) := \sup_{\mathcal{C}} \frac{1}{\tau(\mathcal{C})} \sum_{I \in \mathcal{C}} \mathcal{D}(f, I),$$

where the supremum is taken over all $\mathcal{C}$ with finite thickness.

The following theorem of Herman and Swiatek addresses the linearization problem of real-analytic circle homeomorphisms, allowing the presence of critical points (see [**H3**] and [**S1**]):

Theorem 2.7 (Herman-Swiatek). *Let $f : \mathbb{T} \to \mathbb{T}$ be an orientation-preserving homeomorphism whose rotation number θ is an irrational of bounded type.*

(i) *If the cross-ratio distortion norm $\mathcal{D}(f)$ is finite, then f is k-quasisymmetrically linearizable, where k depends only on $\mathcal{D}(f)$ and θ.*

(ii) *If f is real-analytic, then $\mathcal{D}(f)$ is finite and depends only on the modulus m of the largest annular neighborhood of $\mathbb{T}$ on which f extends holomorphically. As a result, f is k-quasisymmetrically linearizable, with k depending only on m and θ.*

2.4. **Siegel disks.** Let $0 < \theta < 1$ be an irrational number and f be a holomorphic map defined in a neighborhood of the origin, with $f(0) = 0$ and $f'(0) = e^{2\pi i\theta}$. We say f is *locally linearizable* at the origin if there exists a holomorphic change of coordinates near 0 which conjugates f to its linear part $R_\theta : z \mapsto e^{2\pi i\theta} z$. The largest neighborhood of 0 in which f is conjugate to R_θ is a simply-connected domain $\Delta = \Delta_f$ called the *Siegel disk* of f centered at 0. There is a unique conformal isomorphism $\zeta = \zeta_f : \mathbb{D} \to \Delta$ such that $\zeta(0) = 0$ and $\zeta'(0) > 0$. The number $\zeta'(0)$ is called the *conformal radius* of Δ. Applying the Schwarz Lemma to $\zeta^{-1} \circ f \circ \zeta$, we see that ζ conjugates f to R_θ:

$$f \circ \zeta = \zeta \circ R_\theta \qquad \text{in } \mathbb{D}.$$

We often refer to ζ as the *linearizing map* of f in Δ.

According to Siegel [**Si**], when θ is Diophantine, every holomorphic map f with $f(0)=0$ and $f'(0)=e^{2\pi i\theta}$ is locally linearizable at 0. In particular, f has a Siegel disk centered at 0 if the rotation number θ is of bounded type.

The following result, originally due to Ghys [**Gh**], will be used in the proof of the Main Theorem:

Theorem 2.8. *Let $f:\mathbb{C}\to\mathbb{C}$ be an entire map with $f(0)=0$ and $f'(0)=e^{2\pi i\theta}$, where $0<\theta<1$ is Diophantine. Suppose the Siegel disk boundary $\partial\Delta$ is a Jordan curve in $\mathbb{C}$. Then $\partial\Delta$ contains a critical point of f.*

Proof. Assume there are no critical points on $\partial\Delta$. Then f is univalent in a neighborhood of the closed disk $\overline{\Delta}$. Take a conformal isomorphism $\varphi:\mathbb{C}\smallsetminus\overline{\Delta}\to\mathbb{C}\smallsetminus\overline{\mathbb{D}}$. The map $g:=\varphi\circ f\circ\varphi^{-1}$ is well-defined and holomorphic in an outer neighborhood of the unit circle $\mathbb{T}$. By Schwarz reflection, g extends holomorphically to an annular neighborhood of $\mathbb{T}$. In particular, $g:\mathbb{T}\to\mathbb{T}$ is a real-analytic diffeomorphism and its rotation number is easily seen to be θ. Since θ is assumed Diophantine, Herman's Theorem 2.6 shows that g is analytically conjugate to R_θ on the circle. Extending this conjugacy to a neighborhood of $\mathbb{T}$, we see that g is conjugate to R_θ in a neighborhood of $\mathbb{T}$. Pulling this neighborhood back by φ, it follows that f is conjugate to R_θ in an outer neighborhood of $\partial\Delta$. This contradicts the maximality of the Siegel disk Δ. $\square$

Remark 2.9. The assumptions that f is entire and $\partial\Delta$ is a Jordan curve are not essential. In fact, the theorem holds if we only assume that $\partial\Delta$ is a compact subset of the plane on which f acts injectively. This was first shown by Herman [**H2**], but it can also be proved by an argument similar to the above, based on the theory of "hedgehogs" introduced by Perez-Marco (see [**P**] and compare [**Z2**]).

3. The families $\mathcal{E}^{p,q}$ and $\mathcal{E}^{p,q}(\theta)$

3.1. **Generalities.** First consider the family $\mathcal{E}^{p,q}$ of all non-constant entire maps of the form

$$f(z)=P(z)\,\exp(Q(z)), \tag{3.1}$$

where P and Q are polynomials of degree p and q. Thus, f is polynomial if $q=0$ and transcendental if $q>0$. Counting multiplicities, f has p zeros and $p+q-1$ critical points (the roots of the polynomial equation $P'+PQ'=0$).

It will be useful to have simple characterizations for the entire maps in the family $\mathcal{E}^{p,q}$. Recall that the *growth order* of an entire map $f:\mathbb{C}\to\mathbb{C}$ is defined by

$$\limsup_{r\to+\infty}\frac{\log\log M(f,r)}{\log r},$$

where $M(f,r):=\sup_{|z|=r}|f(z)|$. For example, the growth order of every map in $\mathcal{E}^{p,q}$ is easily seen to be q.

Lemma 3.1. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire map of finite growth order, with p zeros and $p + q - 1$ critical points counting multiplicities. Then $f \in \mathcal{E}^{p,q}$.*

Proof. Let P be a polynomial of degree p with the same zeros of the same multiplicities as f. The singularities of f/P are removable and the resulting entire map is nowhere vanishing. It follows that $f = P\,\exp(Q)$ for some entire function Q.

The growth order of f and f/P are clearly the same, so $\exp(Q)$ is of finite growth order. It easily follows that Q must be a polynomial of some degree d. The number $p + d - 1$ of critical points of f is by the assumption equal to $p + q - 1$. Hence $d = q$ and $f \in \mathcal{E}^{p,q}$, as required. □

Corollary 3.2. *Suppose $g : \mathbb{C} \to \mathbb{C}$ is entire and there are quasiconformal maps $\varphi, \hat{\varphi} : \mathbb{C} \to \mathbb{C}$ which fix 0 such that $\varphi^{-1} \circ g \circ \hat{\varphi} \in \mathcal{E}^{p,q}$. Then $g \in \mathcal{E}^{p,q}$.*

Proof. Clearly g has p zeros and $p + q - 1$ critical points counting multiplicities. By Lemma 3.1 it suffices to check that g has finite growth order. Let $f := \varphi^{-1} \circ g \circ \hat{\varphi}$. As quasiconformal maps, φ and $\hat{\varphi}$ satisfy a Hölder condition

$$C_1|z|^{1/K} \leq |\hat{\varphi}(z)| \text{ and } |\varphi(z)| \leq C_2|z|^K \qquad \text{near } \infty,$$

where $C_1, C_2 > 0$ and $K > 1$ are constants. It follows from $\varphi \circ f = g \circ \hat{\varphi}$ that

$$M(g, r) \leq C_2(M(f, C_3 r^K))^K \qquad \text{for large } r,$$

where $C_3 = C_1^{-K}$. This shows that the growth order of g is at most K times that of f, that is at most Kq. □

We will need a few basic facts about general mapping properties of elements of $\mathcal{E}^{p,q}$. We begin with the following version of the "monodromy theorem," a standard result which is included here for convenience.

Theorem 3.3. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a non-constant entire map, V is a domain containing no asymptotic value of f, and U is a connected component of $f^{-1}(V)$ containing no critical point of f. Then $f : U \to V$ is a covering map.*

Recall that $v \in \mathbb{C}$ is an *asymptotic value* of f if there is a path $\eta : [0, 1) \to \mathbb{C}$ such that $\lim_{t\to 1} \eta(t) = \infty$ and $\lim_{t\to 1} f(\eta(t)) = v$.

Proof. The map $f : U \to f(U)$ is a local homeomorphism with the curve lifting property, so it must be a covering. If $f(U) \neq V$, choose a path $\eta : [0, 1) \to f(U)$ such that $v := \lim_{t\to 1} \eta(t) \in \partial f(U) \cap V$, and let $\hat{\eta} : [0, 1) \to U$ be any lift of η. We claim that as $t \to 1$, $\hat{\eta}(t)$ cannot have an accumulation point in $\mathbb{C}$. In fact, if $\hat{v}$ is such a point, then $\hat{v} \in \overline{U}$ and $f(\hat{v}) = v$. On the other hand, $\hat{v} \notin U$ since f is an open map and $\hat{v} \notin \partial U$ since $f(\partial U) \subset \partial V$. This proves the claim and shows that $\lim_{t\to 1} \hat{\eta}(t) = \infty$. But then v will be an asymptotic value of f in V, which contradicts our assumption. □

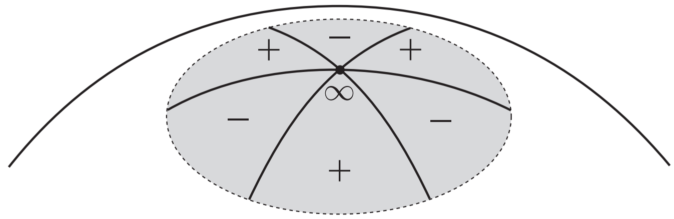

FIGURE 3. *Positive and negative sectors near ∞ for the case $q = 3$.*

Corollary 3.4. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a non-constant entire map, V is a domain containing no asymptotic value of f, and U is a connected component of $f^{-1}(V)$ containing at most finitely many critical points of f. Then $f(U) = V$.*

Proof. Let C be the finite (possibly empty) set of critical points of f in U. By Theorem 3.3, $f : U \smallsetminus f^{-1}(f(C)) \to V \smallsetminus f(C)$ is a covering map. In particular, f maps $U \smallsetminus f^{-1}(f(C))$ onto $V \smallsetminus f(C)$, from which it follows that $f(U) = V$. $\square$

Now let $f = P \exp(Q) \in \mathcal{E}^{p,q}$ with $q = \deg Q > 0$. To study the behavior of the transcendental map f near infinity, it will be convenient to introduce the following notion. The polynomial Q acts like $z \mapsto z^q$ in suitable coordinates near ∞, so there are $2q$ equally spaced rays coming together at ∞ along which $\mathrm{Re}(Q) = 0$. We call these the *neutral directions* of f at infinity. They divide a punctured neighborhood of ∞ into q *positive sectors* in which $\mathrm{Re}(Q) > 0$ interjected with q *negative* sectors in which $\mathrm{Re}(Q) < 0$ (see Fig. 3).

Theorem 3.5. *Every $f \in \mathcal{E}^{p,q}$ with $q > 0$ has a unique asymptotic value at 0.*

Proof. Clearly 0 is an asymptotic value. Suppose by way of contradiction that $v \neq 0$ is an asymptotic value and choose a path $\eta : [0, 1) \to \mathbb{C}$ such that $\eta(t) \to \infty$ and $f(\eta(t)) \to v$ as $t \to 1$. Since $f(z) \to \infty$ as $z \to \infty$ on the closure of each positive sector, we see that $\eta(t)$ must be contained in a negative sector for all t close to 1. Hence, there is a continuous branch of the path $\log(P(\eta(t)))$ whose imaginary part remains bounded. Since

$$f(\eta(t)) = P(\eta(t)) \exp(Q(\eta(t))) \to v \neq 0$$

as $t \to 1$, it follows that

$$\log P(\eta(t)) + Q(\eta(t))$$

has a well-defined limit as $t \to 1$. Hence,

$$\frac{Q(\eta(t))}{\log P(\eta(t))} \to -1 \quad \text{as } t \to 1.$$

This is impossible because as $t \to 1$, the size of the numerator is comparable to $|\eta(t)|^q$ while the denominator, having bounded imaginary part, has size comparable to $\log |P(\eta(t))|$, which in turn is comparable to $\log |\eta(t)|$. $\square$

Corollary 3.6. *Suppose $f \in \mathcal{E}^{p,q}$ with $q > 0$. If $f(0) = 0$, then each iterate $f^{\circ k}$ has a unique asymptotic value at* 0.

3.2. **Covering properties of maps in $\mathcal{E}^{p,q}$.** We continue assuming $f = P \exp(Q) \in \mathcal{E}^{p,q}$ with $q > 0$. Let c be a critical point of f such that $f(c) = 0$. Then c is a root of the equations $P' + PQ' = P = 0$, that is, a common root of P and P'. Hence the number k of such critical points is at most $p - 1$. Since there are $p + q - 1$ critical points altogether, it follows that f has $p + q - 1 - k \geq q > 0$ critical points that are not mapped to 0. Let $v_1, \ldots, v_n$ be the corresponding distinct critical values of f in $\mathbb{C}^*$. For each $1 \leq j \leq n$, take a smooth ray L_j in $\mathbb{C}^*$ from v_j to ∞, and arrange that $L_1, \ldots, L_n$ be disjoint. Evidently, each component of $f^{-1}(L_j)$ is either a *non-critical ray*, i.e., a ray from a non-critical preimage of v_j to ∞, or a "bouquet" of $d > 1$ *critical rays* from a critical preimage c of v_j to infinity, where d is the local degree of f at c. Set $L := \bigcup_{j=1}^{n} L_j \cup \{0\}$ and $W := \mathbb{C} \smallsetminus f^{-1}(L)$. It is easy to see that W decomposes into $p + q - k$ unbounded connected components $W_1, \ldots, W_{p+q-k}$. By Theorem 3.3,

$$f : W_j \to \mathbb{C} \smallsetminus L \tag{3.2}$$

is a covering map for each $1 \leq j \leq p + q - k$. As $\mathbb{C} \smallsetminus L$ is conformally isomorphic to the punctured disk, it follows that each W_j is isomorphic to the punctured disk or to the upper half-plane.

- *Case 1.* The degree d of (3.2) is finite. Then W_j is conformally isomorphic to the punctured disk. Setting $\pi_d(z) := z^d$, it follows that there is a covering space isomorphism

$$\begin{tikzcd} W_j \arrow[r, "\varphi"] \arrow[dr, "f"'] & \mathbb{C} \smallsetminus \pi_d^{-1}(L) \arrow[d, "\pi_d"] \\ & \mathbb{C} \smallsetminus L \end{tikzcd} \tag{3.3}$$

which induces a homeomorphism between ∂W_j and $\pi_d^{-1}(L)$ except that a critical ray pair in ∂W_j is identified under φ with a single ray in $\pi_d^{-1}(L)$. In particular, W_j is bounded by finitely many rays and is punctured at a unique preimage of 0 where the local degree of f is d.

- *Case 2.* The degree of (3.2) is infinite. Then W_j is simply-connected, hence conformally isomorphic to the upper half-plane. Setting $E(z) := \exp(z)$, it follows that there is a covering space isomorphism

$$\begin{tikzcd} W_j \arrow[r, "\varphi"] \arrow[dr, "f"'] & \mathbb{C} \smallsetminus E^{-1}(L) \arrow[d, "E"] \\ & \mathbb{C} \smallsetminus L \end{tikzcd} \tag{3.4}$$

which induces a homeomorphism between ∂W_j and $E^{-1}(L)$ except that a critical ray pair in ∂W_j is identified under φ with a single ray in $E^{-1}(L)$. In particular, W_j is bounded by countably many rays and does not contain a preimage of 0.

Thus, there is a one-to-one correspondence between the preimages of 0 and the W_j's of type (3.3). As 0 has $p-k$ distinct preimages, it follows that $p-k$ of the W_j's are of type (3.3) and q of them are of type (3.4).

The above covering properties are used in the proof of the following two lemmas which will be need in §8:

Lemma 3.7. *Let $f \in \mathcal{E}^{p,q}$ with $q > 0$. Suppose V is a simply-connected domain in $\mathbb{C}^*$ and U is a connected component of $f^{-1}(V)$. Then $f : U \to V$ is a proper map.*

Proof. By Corollary 3.4, $f(U) = V$. Choose the rays L_j in the above construction so that $V \smallsetminus L$ is simply-connected. Then, using the fact that $f : W_j \to \mathbb{C} \smallsetminus L$ fits into one of the isomorphisms (3.3) or (3.4), we see that

$$f : U \cap W_j \to V \smallsetminus L$$

is a conformal isomorphism whenever $U \cap W_j \neq \emptyset$. Since there are finitely many of the W_j, it easily follows that $f : U \to V$ must be proper. $\square$

Lemma 3.8. *Let $f \in \mathcal{E}^{p,q}$ with $q > 0$ and η be a Jordan curve which winds around the origin. Then $f^{-1}(\eta)$ has finitely many components. Furthermore,*

- (i) *If η avoids the critical values of f, each component of $f^{-1}(\eta)$ is either a Jordan curve or a simple open arc both ends of which tend to ∞. In the latter case, each end is eventually in some negative sector and asymptotic to a neutral direction of f at infinity.*
- (ii) *If η does contain critical values of f, each component of $f^{-1}(\eta)$ is of the form cited in (i) except that we must allow finitely many self-intersections at the critical points.*

Proof. Arrange that each ray L_j intersect η in at most one point. Then, either of the covering space isomorphisms (3.3) or (3.4) implies that $f^{-1}(\eta) \cap \overline{W_j}$ consists of finitely many curves for each j. Since there are finitely many of the W_j, this shows finiteness of the number of components of $f^{-1}(\eta)$.

Now let η avoid the critical values of f and $\hat{\eta}$ be a component of $f^{-1}(\eta)$. Then $\hat{\eta}$ maps locally biholomorphically under f to η. Hence $\hat{\eta}$ is locally a simple arc. It follows that $\hat{\eta}$ is a Jordan curve if it is bounded and a simple open arc going to ∞ in both directions if it is unbounded. Suppose we are in the latter case. Since $f(z) \to \infty$ as $z \to \infty$ on the closure of a positive sector, the ends of $\hat{\eta}$ must eventually lie in a negative sector. They should be asymptotic to neutral directions, for otherwise their image would tend to 0.

The case where η contains critical values follows easily by a similar argument. $\square$

3.3. **The family $\mathcal{E}^{p,q}(\theta)$.** Now let $0 < \theta < 1$ be an irrational of bounded type and consider the subfamily $\mathcal{E}^{p,q}(\theta) \subset \mathcal{E}^{p,q}$ of the entire maps of the form $f = P \exp(Q)$ such that $P(0) = Q(0) = 0$ and $P'(0) = f'(0) = e^{2\pi i\theta}$. Each $f \in \mathcal{E}^{p,q}(\theta)$ has a Siegel disk $\Delta = \Delta_f$ centered at the origin, with the linearizing map $\zeta = \zeta_f : \mathbb{D} \to \Delta$ which satisfies $\zeta(0) = 0$ and $\zeta'(0) > 0$.

We will always normalize maps in $\mathcal{E}^{p,q}(\theta)$ so that the conformal radius of their Siegel disk equal to 1. To this end, note the conformal conjugacy class of f in $\mathcal{E}^{p,q}(\theta)$ consists of all maps $f_\alpha(z) = f(\alpha z)/\alpha$ for $\alpha \in \mathbb{C}^*$. Since

$$\zeta_{f_\alpha}(z) = \frac{1}{\alpha}\, \zeta_f\Big(\frac{\alpha}{|\alpha|} z\Big),$$

we can choose a representative f in each conjugacy class such that $\zeta_f'(0) = 1$. Any two such representatives will then be conjugate by a rotation.

For each $0 < r < 1$, we define

$$\begin{aligned} \Delta_r = \Delta_{f,r} &:= \zeta(\mathbb{D}_r) \\ \gamma_r = \gamma_{f,r} &:= \zeta(\mathbb{T}_r) \\ \Omega_r = \Omega_{f,r} &:= \bigcup_{n \geq 0} f^{-n}(\Delta_r). \end{aligned} \tag{3.5}$$

Thus, Δ_r is an invariant subdisk of Δ bounded by the real-analytic invariant curve γ_r, and Ω_r is the smallest totally invariant set containing Δ_r.

4. Main constructions

4.1. **A quasiconformal reflection.** Fix a map $f \in \mathcal{E}^{p,q}(\theta)$ and a radius $0 < r < 1$. Consider the radii

$$0 < a := r^{3/2} < r < b := r^{1/2} < 1$$

and the open f-invariant annuli

$$\begin{aligned} A_a &:= \zeta(\mathbb{A}_{a,r}) \\ A_b &:= \zeta(\mathbb{A}_{r,b}) \\ A &:= \zeta(\mathbb{A}_{a,b}) = A_a \cup \gamma_r \cup A_b \end{aligned}$$

(see Fig. 4). Note that as $r \to 1$, the modulus of A_a, A_b and A tends to zero.

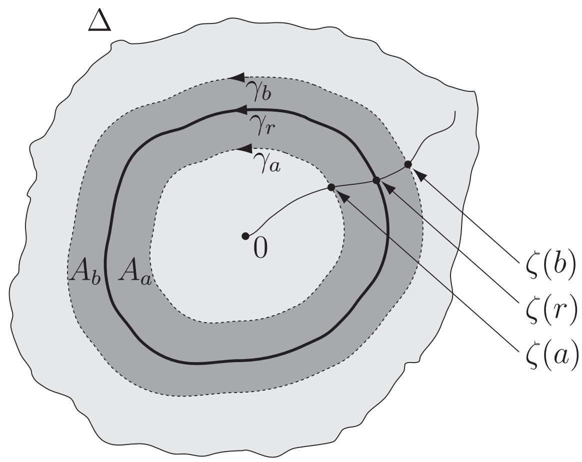

FIGURE 4. *Some invariant curves and annuli in the Siegel disk* Δ.

The main construction begins with the choice of an orientation-reversing quasiconformal involution $I : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with the following properties:

- $I|_{\gamma_r} = \mathrm{id}$ and $I(\Delta_r) = \widehat{\mathbb{C}} \smallsetminus \overline{\Delta_r}$;
- $I : \Delta_a \to \widehat{\mathbb{C}} \smallsetminus \overline{\Delta_b}$ is the unique anti-conformal map normalized by the conditions $I(0) = \infty$ and $I(\zeta(a)) = \zeta(b)$;
- $I(A_a) = A_b$ and hence $I(A) = A$;

A priori, these conditions may force I to have a big dilatation (depending on r) inside the annulus A. But, as it turns out, this will not be a cause for concern.

The choice of such I, of course, is far from unique. In order to have something explicit to work with, we will adopt the following construction. Suppose $\hat{\zeta} : \{z : |z| > b\} \to \mathbb{C} \smallsetminus \overline{\Delta_b}$ is the unique conformal isomorphism such that $\hat{\zeta}(b) = \zeta(b)$. By Lemma 2.2, both ζ and $\hat{\zeta}$ have K_b^2-quasiconformal extensions to the sphere, where K_b is the quasicircle constant of the invariant curve γ_b. Hence the restriction of $\zeta^{-1} \circ \hat{\zeta}$ to $\mathbb{T}_b$ is k-quasisymmetric with k depending only on K_b. Let $\varphi : \partial\mathbb{A}_{r,b} \to \partial\mathbb{A}_{r,b}$ be the homeomorphism which restricts to the identity on $\mathbb{T}_r$ and to $\zeta^{-1} \circ \hat{\zeta}$ on $\mathbb{T}_b$. Use Lemma 2.1 to extend φ to a K-quasiconformal map $\mathbb{A}_{r,b} \to \mathbb{A}_{r,b}$, where $K = K(K_b, r)$. This allows us to extend $\hat{\zeta}$ to a K-quasiconformal map $\{z : |z| > r\} \to \mathbb{C} \smallsetminus \overline{\Delta_r}$ by setting it equal to $\zeta \circ \varphi$ on $\mathbb{A}_{r,b}$. Denoting by ι the reflection $z \mapsto r^2/\overline{z}$, we can now define $I = \hat{\zeta} \circ \iota \circ \zeta^{-1}$ in Δ_r and set $I = I^{-1}$ elsewhere.

Corollary 4.1. *The dilatation of the quasiconformal reflection I constructed above depends only on r and the quasicircle constant of the invariant curve γ_b.*

4.2. **Symmetrizing f.** Next we construct a quasiregular dynamics $F : \mathbb{C}^* \to \widehat{\mathbb{C}}$ by symmetrizing f about the invariant curve γ_r using the reflection I. Explicitly, define

$$F := \begin{cases} f & \text{outside } \Delta_r \\ I \circ f \circ I & \text{in } \Delta_r \smallsetminus \{0\}. \end{cases}$$

Note that F has a "quasiconformal Herman ring" $\Delta \cap I(\Delta)$ containing the annulus A.

Theorem 4.2. *The map F is*

(i) *holomorphic outside $\overline{\Delta_r} \cap F^{-1}(\overline{A})$;*

(ii) *symmetric about γ_r in the sense that $F \circ I = I \circ F$.*

Proof. Outside $\overline{\Delta_r}$, $F = f$ is clearly holomorphic. In the open set $\overline{\Delta_r} \smallsetminus F^{-1}(\overline{A})$, $F = I \circ f \circ I$ is a composition of one holomorphic and two anti-holomorphic maps, hence is itself holomorphic. This proves (i).

For (ii), observe that if $z \notin \Delta_r$,

$$(F \circ I)(z) = (I \circ f \circ I \circ I)(z) = (I \circ f)(z) = (I \circ F)(z),$$

and if $z \in \Delta_r$,

$$(F \circ I)(z) = (f \circ I)(z) = (I \circ I \circ f \circ I)(z) = (I \circ F)(z).$$ □

4.3. **Straightening F.** Below we show that the symmetric map F constructed above is quasiconformally conjugate to a holomorphic map.

Theorem 4.3. *There exists a measurable conformal structure μ of bounded dilatation on $\widehat{\mathbb{C}}$ which is invariant under the action of both F and I.*

In general, the dilatation of μ depends on the dilatation I, and *a priori* it could grow large as $r \to 1$.

Proof. Define μ on A by setting $\mu := \mu_0$ on $A_b \cup \gamma_r$ and $\mu := I^*(\mu_0)$ on A_a. Here μ_0 denotes the standard conformal structure of the plane represented by the zero Beltrami differential. Since F is holomorphic in A_b and $F \circ I = I \circ F$, μ is invariant under $F : A \to A$. Spread μ along the backward orbit of A by using iterates of F, i.e., define

$$\mu := (F^{\circ n})^*(\mu) \qquad \text{on } F^{-n}(A).$$

On the rest of $\widehat{\mathbb{C}}$, set $\mu = \mu_0$. It is clear from the definition that μ is F-invariant. Using the symmetry relation $F \circ I = I \circ F$ again, we see that the conformal structure $I^*(\mu)$ must also be F-invariant. Since $I^*(\mu) = \mu$ holds in A, it should hold everywhere, which means μ is I-invariant.

Finally, μ has bounded dilatation on A since I is quasiconformal. The first pull-back of μ to $F^{-1}(A) \smallsetminus A$ is likely to increases the dilatation because the branch of F used for pulling back need not be holomorphic. However, all the subsequent pull-backs are taken using the branches of F which are holomorphic by Theorem 4.2, so they will not increase the dilatation further. $\square$

According to the Measurable Riemann Mapping Theorem [**A**], there exists a quasi-conformal map $\xi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which solves the Beltrami equation $\xi^*(\mu_0) = \mu$. Moreover, ξ is unique if normalized so that

$$\xi(0) = 0, \quad \xi(\infty) = \infty \quad \text{and} \quad \xi(\zeta(r)) = 1.$$

Theorem 4.4. *The homeomorphism ξ*

(i) *is conformal off Ω_r, in the sense that $\overline{\partial}\xi = 0$ a.e. on the closed set $\mathbb{C} \smallsetminus \Omega_r$;*

(ii) *conjugates I to the reflection $\iota : z \mapsto 1/\overline{z}$;*

(iii) *maps γ_r homeomorphically to the unit circle.*

Proof. The forward f-orbit of every point z outside Ω_r is either disjoint from A or else lands in $A_b \cup \gamma_r$. In either case, it follows from the construction that $\mu(z) = 0$, proving (i).

The quasiconformal map $\hat{\xi} := \iota \circ \xi \circ I : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ solves the Beltrami equation $\hat{\xi}^*(\mu_0) = \mu$ also:

$$\hat{\xi}^*(\mu_0) = (I^* \circ \xi^* \circ \iota^*)(\mu_0) = (I^* \circ \xi^*)(\mu_0) = I^*(\mu) = \mu.$$

Since $\hat{\xi}$ fixes 0 and ∞ and sends $\zeta(r)$ to 1, we have $\hat{\xi} = \xi$ by uniqueness. This proves (ii).

The assertion (iii) follows immediately since each of these Jordan curves is characterized as the fixed point set of the corresponding reflection. $\square$

Now consider the conjugate quasiregular map $G : \mathbb{C}^* \to \widehat{\mathbb{C}}$ defined by

$$G := \xi \circ F \circ \xi^{-1}. \tag{4.1}$$

Theorem 4.5. *The map G*

(i) *is holomorphic;*

(ii) *commutes with the reflection $\iota : z \mapsto 1/\overline{z}$, hence preserves the unit circle $\mathbb{T}$;*

(iii) *restricts to an orientation-preserving real-analytic diffeomorphism of $\mathbb{T}$ with rotation number θ. In particular, it has a Herman ring $\xi(\Delta \cap I(\Delta))$ of rotation number θ containing $\mathbb{T}$;*

(iv) *has $p-1$ zeros, all in $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$, and $p-1$ poles, all in $\mathbb{D}^*$.*

Recall that the number p in (iv) is the degree of the polynomial P in the representation $f = P\,\exp(Q) \in \mathcal{E}^{p,q}(\theta)$. If $p = 1$, (iv) is understood as saying that G has no poles or zeros.

Proof. For (i), note that $F^*(\mu) = \mu = \xi^*(\mu_0)$, hence $G^*(\mu_0) = \mu_0$. Thus, as a quasiregular map preserving the standard conformal structure, G must be holomorphic. Assertion (ii) follows from Theorems 4.2 and 4.4:

$$\begin{aligned} G \circ \iota &= \xi \circ F \circ \xi^{-1} \circ \iota = \xi \circ F \circ I \circ \xi^{-1} \\ &= \xi \circ I \circ F \circ \xi^{-1} = \iota \circ \xi \circ F \circ \xi^{-1} = \iota \circ G. \end{aligned}$$

Part (iii) easily follows from the corresponding property of F. For (iv), observe that by the definition of F and the normalization $\xi(0) = 0$,

$$G^{-1}(0) = \xi(F^{-1}(0)) = \xi(f^{-1}(0) \smallsetminus \{0\}) = \xi(P^{-1}(0) \smallsetminus \{0\}).$$

Since 0 is a simple root of P, it follows that G has $p-1$ zeros in $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$, counting multiplicities. The assertion about the number of poles follows by symmetry. $\square$

4.4. **Surgery.** We now perform a surgery on G to turn it back into an entire function, its Herman ring back into a Siegel disk. The idea is roughly to "cut out" $\mathbb{D}$, "glue in" a quasiconformal Siegel disk instead, and straighten the resulting action in order to realize it as an entire map $g \in \mathcal{E}^{p,q}(\theta)$.

By Theorem 4.5, $G : \mathbb{T} \to \mathbb{T}$ is a real-analytic diffeomorphism with rotation number θ, which is assumed to be an irrational of bounded type. By Herman-Swiatek's Theorem 2.7, the normalized linearizing map $h : \mathbb{T} \to \mathbb{T}$ of G is quasisymmetric. Let $H : \mathbb{D} \to \mathbb{D}$ be the standard quasiconformal extension of h constructed in §2.1 which satisfies $H(0) = 0, H(1) = 1$. Define the modified quasiregular map $\hat{G} : \mathbb{C} \to \mathbb{C}$ by

$$\hat{G} := \begin{cases} G & \text{outside } \mathbb{D} \\ H^{-1} \circ R_\theta \circ H & \text{in } \mathbb{D}. \end{cases}$$

We claim that $\hat{G}$ admits an invariant conformal structure ν of bounded dilatation. In fact, since R_θ is holomorphic, $\nu := H^*(\mu_0)$ is clearly $\hat{G}$-invariant in $\mathbb{D}$ (as before, μ_0 denotes the standard conformal structure of the plane). We spread ν along the backward orbit of $\mathbb{D}$ by setting

$$\nu := (\hat{G}^{\circ n})^*(\nu) \qquad \text{on } \hat{G}^{-n}(\mathbb{D}).$$

On the rest of $\mathbb{C}$, we set $\nu = \mu_0$. By the very construction, ν is $\hat{G}$-invariant. Moreover, since the branches of $\hat{G} = G$ used to spread ν around are all holomorphic, the dilatation of ν on $\mathbb{C}$ is the same as its dilatation on $\mathbb{D}$, which is bounded since H is quasiconformal.

By the Measurable Riemann Mapping Theorem, there is a quasiconformal map $\psi : \mathbb{C} \to \mathbb{C}$ which fixes the origin and solves the Beltrami equation $\psi^*(\mu_0) = \nu$. Consider the conjugate map $g : \mathbb{C} \to \mathbb{C}$ defined by

$$g := \psi \circ \hat{G} \circ \psi^{-1}.$$

Since $\hat{G}^*(\nu) = \nu = \psi^*(\mu_0)$, the definition of g shows that $g^*(\mu_0) = \mu_0$, which implies g is an entire function. It clearly has a Siegel disk Δ_g centered at the origin which contains $\psi(\mathbb{D})$ as a proper invariant subdisk.

To make ψ and hence g unique, we choose the following normalization: Since both H and ψ pull μ_0 back to ν on $\mathbb{D}$, the composition $\psi \circ H^{-1}$ is conformal. Hence, we can find a unique quasiconformal solution of $\psi^*(\mu_0) = \nu$ which satisfies

$$\psi(0) = 0 \qquad \text{and} \qquad (\psi \circ H^{-1})'(0) = r. \tag{4.2}$$

Theorem 4.6. *The quasiconformal map $\varphi := \psi \circ \xi : \mathbb{C} \to \mathbb{C}$ has the following properties:*

(i) $\varphi \circ f = g \circ \varphi$ *off* $\Delta_{f,r}$.

(ii) φ *is conformal off* $\Omega_{f,r}$.

(iii) $\varphi(\gamma_{f,r}) = \gamma_{g,r}$.

(iv) $\varphi = \zeta_g \circ \zeta_f^{-1}$ *on* $\gamma_{f,r}$.

Proof. For (i), simply note that on $\mathbb{C} \smallsetminus \Delta_{f,r}$,

$$\begin{aligned} \varphi \circ f &= \psi \circ \xi \circ f \\ &= \psi \circ \xi \circ F \\ &= \psi \circ G \circ \xi \\ &= \psi \circ \hat{G} \circ \xi \\ &= g \circ \psi \circ \xi = g \circ \varphi, \end{aligned}$$

where the fourth equality holds since by Theorem 4.4, ξ maps the complement of $\Delta_{f,r}$ to the complement of $\mathbb{D}$.

Next, ξ is conformal off $\Omega_{f,r}$ by Theorem 4.4 and ψ is conformal off the image $\xi(\Omega_{f,r}) = \bigcup_{n\geq 0} \hat{G}^{-n}(\mathbb{D})$ by the construction of ν. This proves (ii).

Since $\xi(\gamma_{f,r}) = \mathbb{T}$, (iii) is equivalent to showing that $\psi(\mathbb{T}) = \gamma_{g,r}$. Observe that $\psi(\mathbb{T})$ is a g-invariant curve in the Siegel disk Δ_g, hence $\psi(\mathbb{T}) = \gamma_{g,s}$ for some $0 < s < 1$. Since the annulus $\Delta_f \smallsetminus \overline{\Delta_{f,r}}$ is disjoint from $\Omega_{f,r}$, part (ii) shows that $\varphi : \Delta_f \smallsetminus \overline{\Delta_{f,r}} \to \Delta_g \smallsetminus \overline{\Delta_{g,s}}$ is a conformal isomorphism. Hence the two annuli have the same modulus and $r = s$.

Finally, the composition $z \mapsto (\psi \circ H^{-1})(z/r)$ maps $\mathbb{D}_r$ conformally to $\Delta_{g,r}$, fixes the origin and has derivative 1 there by (4.2). The linearizing map ζ_g has the same

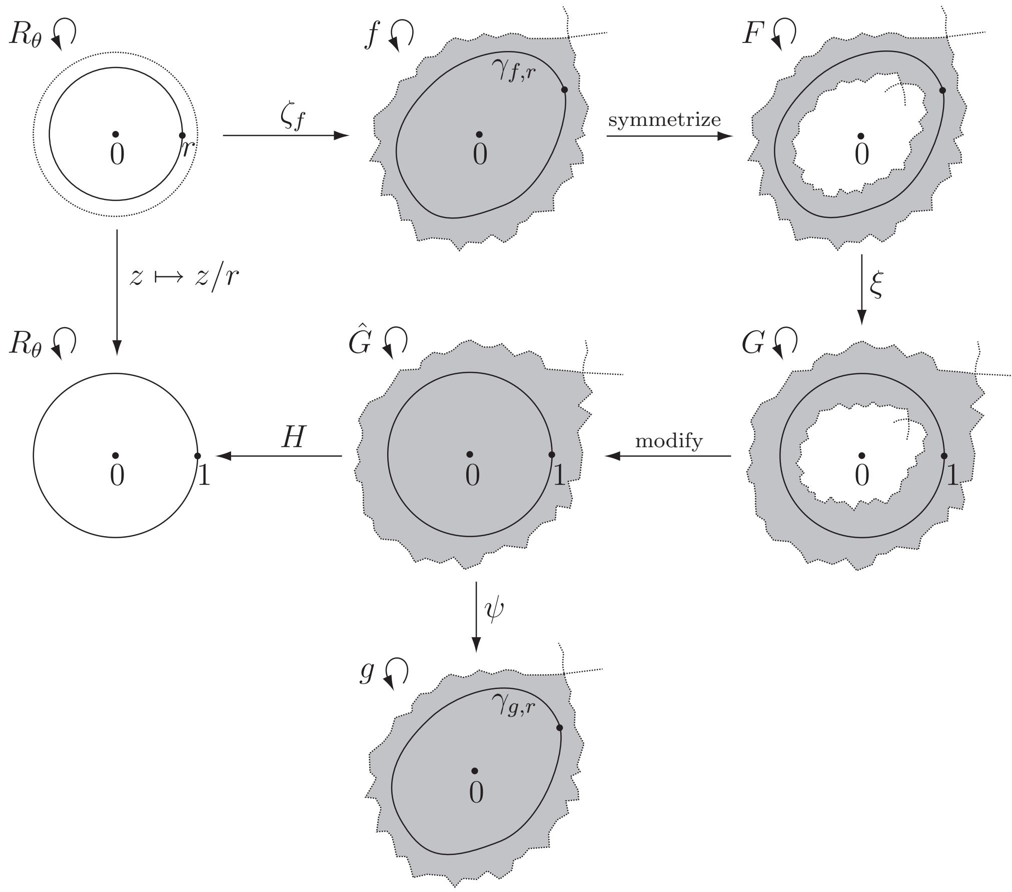

FIGURE 5. *Various steps in the construction of the surgery map $f \mapsto g$.*

properties, so by uniqueness $\zeta_g(z) = (\psi \circ H^{-1})(z/r)$ whenever $|z| \leq r$. On the other hand, $(1/r)(\xi \circ \zeta_f)^{-1} : \mathbb{T} \to \mathbb{T}$ conjugates G to R_θ and fixes 1. By uniqueness, $(1/r)(\xi \circ \zeta_f)^{-1} = H$ on the unit circle. It follows that when $|z| = r$,

$$(\varphi \circ \zeta_f)(z) = (\psi \circ \xi \circ \zeta_f)(z) = (\psi \circ H^{-1})(z/r) = \zeta_g(z),$$

which proves (iv). $\square$

The following fact was shown in the course of the above proof:

Corollary 4.7. $\zeta_g(z) = (\psi \circ H^{-1})(z/r)$ *whenever* $|z| \leq r$*. In particular, the conformal radius of* Δ_g *is* 1.

Theorem 4.8. $g \in \mathcal{E}^{p,q}(\theta)$.

Proof. Define

$$\hat{\varphi} := \begin{cases} \varphi & \text{off } \Delta_{f,r} \\ g^{-1} \circ \varphi \circ f & \text{on } \Delta_{f,r}. \end{cases}$$

Then $\hat{\varphi}$ is quasiconformal and $\varphi \circ f = g \circ \hat{\varphi}$. It follows from Corollary 3.2 that $g \in \mathcal{E}^{p,q}$. Since g has a Siegel disk of rotation number θ and conformal radius 1 centered at 0, we have $g \in \mathcal{E}^{p,q}(\theta)$. □

Remark 4.9. The map φ is not a conjugacy between f and g inside $\Delta_{f,r}$ unless the extension H of h is chosen so that

$$H = \frac{1}{r}(\xi \circ \zeta_f)^{-1} \qquad \text{in } \mathbb{D}.$$

The reason we did not choose this extension is the dilatation issue: *a priori*, the dilatation of ξ, hence that of $(1/r)(\xi \circ \zeta_f)^{-1}$, depends on r while our argument is heavily based on the fact that there is a quasiconformal extension H whose dilatation is independent of r (see §5).

Definition 4.10. Let $0 < r < 1$. The *surgery map* $\mathcal{S}_r : \mathcal{E}^{p,q}(\theta) \to \mathcal{E}^{p,q}(\theta)$ is the one which assigns to each f the entire function g constructed above.

Fig. 5 is a schematic summary of various steps in the construction of the surgery map $\mathcal{S}_r$.

5. A priori estimate of the dilatation

Let $f \in \mathcal{E}^{p,q}(\theta)$ and $g := \mathcal{S}_r(f) \in \mathcal{E}^{p,q}(\theta)$ be the result of surgery on f, as described in §4. In this section we prove that the invariant curve $\gamma_{g,r}$ is a K-quasicircle for some $K > 1$ independent of the choice of f and r (Corollary 5.7). This uniformity will be at the heart of the proof of the Main Theorem.

5.1. **The explicit form of G.** Consider the holomorphic map $G : \mathbb{C}^* \to \widehat{\mathbb{C}}$ constructed in §4.3. By Theorem 4.5, G has $p-1$ zeros $\{z_1, \ldots, z_{p-1}\}$, where $|z_j| > 1$ and each root is repeated according to its multiplicity. By symmetry, there are $p-1$ poles at $\{1/\overline{z_1}, \ldots, 1/\overline{z_{p-1}}\}$. Consider the finite Blaschke product

$$B(z) := \prod_{j=1}^{p-1} \left(\frac{z - z_j}{1 - \overline{z_j}\, z} \right) \tag{5.1}$$

which has the same zeros and poles of the same multiplicities as G. When $p = 1$, G has no zeros or poles and we agree to set $B = 1$. In either case, the quotient $S(z) := G(z)/B(z)$ extends to a holomorphic map $\mathbb{C}^* \to \mathbb{C}^*$ without zeros or poles.

Lemma 5.1. *The map $S : \mathbb{C}^* \to \mathbb{C}^*$ has the form*

$$S(z) = \lambda\, z^n \exp(\alpha(z) - \overline{\alpha(1/\overline{z})}),$$

where $|\lambda| = 1$, n is an integer, and $\alpha : \mathbb{C} \to \mathbb{C}$ is an entire function with $\alpha(0) = 0$.

Proof. As a holomorphic map $\mathbb{C}^* \to \mathbb{C}^*$, S has a *unique* representation

$$S(z) = \lambda\, z^n \exp(\alpha(z) + \beta(1/z)), \tag{5.2}$$

where $\lambda \neq 0$, n is an integer, and α, β are entire functions with $\alpha(0) = \beta(0) = 0$. Since G and B commute with the reflection $\iota : z \mapsto 1/\overline{z}$, so does their ratio S. Imposing this condition on the representation (5.2), we obtain

$$\lambda\, z^n \exp(\alpha(z) + \beta(1/z)) = \frac{1}{\overline{\lambda}}\, z^n \exp(-\overline{\alpha(1/\overline{z})} - \overline{\beta(\overline{z})})$$

for all z. Hence, by uniqueness,

$$|\lambda| = 1 \qquad \text{and} \qquad \beta(z) = -\overline{\alpha(\overline{z})}. \qquad \square$$

Lemma 5.2. *The exponent n in Lemma 5.1 is equal to p.*

Proof. Apply the Argument Principle to the function

$$G(z) = B(z)S(z) = \lambda\, z^n\, B(z)\, \exp(\alpha(z) - \overline{\alpha(1/\overline{z})})$$

on the unit circle:

$$\frac{1}{2\pi i}\int_{|z|=1} \frac{G'(z)}{G(z)}\, dz = n + \frac{1}{2\pi i}\int_{|z|=1} \frac{B'(z)}{B(z)}\, dz + \frac{1}{2\pi i}\int_{|z|=1} \frac{d}{dz}(\alpha(z) - \overline{\alpha(1/\overline{z})})\, dz.$$

The left side is equal to 1 since $G : \mathbb{T} \to \mathbb{T}$ is an orientation-preserving homeomorphism. The middle term on the right is $-(p-1)$ since the Blaschke product B has $p-1$ poles and no zeros in $\mathbb{D}$. The term on the far right is zero since the integrand has a holomorphic primitive in a neighborhood of $\mathbb{T}$. Thus, $n = p$ as required. $\square$

Lemma 5.3. *The entire function α of Lemma 5.1 is a polynomial of degree q.*

Proof. When $q = 0$, the map $F = f$ is a polynomial in a neighborhood of infinity, so ∞ is a pole of G. In this case α vanishes identically and the lemma holds. Let us then assume $q > 0$. Since $F = f$ in a neighborhood of infinity, the growth order of F is q, hence the quasiconformally conjugate map G must have finite positive growth order (compare the proof of Corollary 3.2). It follows that $\exp(\alpha)$ is an entire function of finite order, so α is a polynomial of some degree $d > 0$. The number $2(p + q - 1)$ of critical points of F must match the number $2(p + d - 1)$ for G, hence $d = q$. $\square$

Corollary 5.4. *The holomorphic map $G : \mathbb{C}^* \to \widehat{\mathbb{C}}$ has the form*

$$G(z) = \lambda\, z^p\, B(z)\, \exp(\alpha(z) - \overline{\alpha(1/\overline{z})}), \tag{5.3}$$

where $|\lambda| = 1$, B is a degree $p-1$ Blaschke product as in (5.1) *with all zeros in $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$ (constant function* 1 *if $p = 1$) and α is a polynomial of degree q with $\alpha(0) = 0$.*

5.2. **Linearizing G on the unit circle.** The restriction $G : \mathbb{T} \to \mathbb{T}$ is an orientation-preserving real-analytic diffeomorphism of bounded type rotation number θ. By Theorem 2.7, the linearizing map $h : \mathbb{T} \to \mathbb{T}$ of G is k-quasisymmetric, where k depends only on θ and the modulus of the largest annular neighborhood of $\mathbb{T}$ in which G is holomorphic. By the explicit form (5.3), the size of this annulus is controlled by how close the poles (equivalently zeros) of the Blaschke product B are to the unit circle. Since this is not an issue in the case $p = 1$ where $B = 1$, we assume throughout the following discussion that $p > 1$.

Theorem 5.5. *There exists a constant $\varepsilon = \varepsilon(p, q) > 0$ such that the zeros $\{z_j\}$ of the Blaschke product B in (5.3) satisfy $|z_j| > 1 + \varepsilon$.*

Proof. First assume $q > 0$ and take an integer $1 \le k \le q$. Set $R(z) := \alpha(z) - \overline{\alpha(1/\overline{z})}$. Logarithmic differentiation of

$$G(z) = \lambda\, z^p\, B(z)\, \exp(R(z))$$

yields

$$\frac{z^k G'(z)}{G(z)} = p\, z^{k-1} + \frac{z^k B'(z)}{B(z)} + z^k R'(z). \tag{5.4}$$

Integrating over the unit circle, we obtain

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{z^k G'(z)}{G(z)}\, dz = \frac{1}{2\pi i} \int_{|z|=1} \frac{z^k B'(z)}{B(z)}\, dz + \frac{1}{2\pi i} \int_{|z|=1} z^k R'(z)\, dz. \tag{5.5}$$

The Argument Principle applied to the formula of B in (5.1) yields

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{z^k B'(z)}{B(z)}\, dz = -\sum_{j=1}^{p-1} \frac{1}{\overline{z_j}^{\,k}}. \tag{5.6}$$

To compute the integral of $z^k R'(z)$ over the unit circle, let $\alpha(z) = a_1\, z + \cdots + a_q\, z^q$, so

$$R(z) = -\frac{\overline{a_q}}{z^q} - \cdots - \frac{\overline{a_1}}{z} + a_1\, z + \cdots + a_q\, z^q,$$

and

$$z^k R'(z) = q\, \overline{a_q}\, z^{k-q-1} + \cdots + \overline{a_1}\, z^{k-2} + a_1\, z^k + \cdots + q\, a_q\, z^{k+q-1}. \tag{5.7}$$

Thus,

$$\frac{1}{2\pi i} \int_{|z|=1} z^k R'(z)\, dz = k\, \overline{a_k}. \tag{5.8}$$

Substituting (5.6) and (5.8) into (5.5) and using the fact that $|z_i| > 1$, we obtain

$$k\, |a_k| = \left| \frac{1}{2\pi i} \int_{|z|=1} \frac{z^k G'(z)}{G(z)}\, dz + \sum_{j=1}^{p-1} \frac{1}{\overline{z_j}^{\,k}} \right| \le \frac{1}{2\pi} \int_0^{2\pi} |G'(e^{it})|\, dt + p - 1. \tag{5.9}$$

Since $G : \mathbb{T} \to \mathbb{T}$ is an orientation-preserving diffeomorphism, we have $zG'(z)/G(z) = d(\log G(z))/d(\log z) > 0$ on the unit circle. This implies

$$\frac{zG'(z)}{G(z)} = \left| \frac{zG'(z)}{G(z)} \right| = |G'(z)| \qquad \text{whenever } |z| = 1. \tag{5.10}$$

Hence, by another application of the Argument Principle,

$$\frac{1}{2\pi} \int_0^{2\pi} |G'(e^{it})| \, dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} G'(e^{it})}{G(e^{it})} \, dt = \frac{1}{2\pi i} \int_{|z|=1} \frac{G'(z)}{G(z)} \, dz = 1. \tag{5.11}$$

Putting (5.9) and (5.11) together, we obtain the estimate

$$k \, |a_k| \leq p \qquad \text{for all} \;\; 1 \leq k \leq q. \tag{5.12}$$

This immediately gives an L^∞ estimate for $zR'(z) = 2 \operatorname{Re}(z\alpha'(z))$ on the unit circle. In fact, by (5.7) with $k = 1$ and (5.12),

$$\sup_{|z|=1} |zR'(z)| \leq 2 \sum_{j=1}^{q} j \, |a_j| \leq 2 \, p \, q. \tag{5.13}$$

This, in turn, allows an L^∞ estimate for the logarithmic derivative $zB'(z)/B(z)$ on the unit circle: Start with (5.4) with $k = 1$:

$$\frac{zG'(z)}{G(z)} = p + \frac{zB'(z)}{B(z)} + zR'(z).$$

On the unit circle, each term in this identity is real, with the left side being positive by (5.10) and the absolute value of the term on the far right being bounded by $2pq$ by (5.13). Hence,

$$-\frac{zB'(z)}{B(z)} \leq p \, (2q + 1) \qquad \text{whenever } |z| = 1.$$

A brief computation using (5.1) shows that on the unit circle,

$$-\frac{zB'(z)}{B(z)} = \sum_{j=1}^{p-1} \frac{|z_j|^2 - 1}{|z - z_j|^2}.$$

It follows that for each $1 \leq j \leq p - 1$,

$$\frac{|z_j| + 1}{|z_j| - 1} = \sup_{|z|=1} \frac{|z_j|^2 - 1}{|z - z_j|^2} \leq \sup_{|z|=1} \frac{-zB'(z)}{B(z)} \leq p \, (2q + 1).$$

This gives $|z_j| \geq 1 + \varepsilon$, with

$$\varepsilon = \varepsilon(p, q) := \frac{2}{2pq + p - 1} > 0.$$

In the polynomial case where $q = 0$, the rational function R is identically zero, so the same argument shows that $\varepsilon = 2/(p - 1)$ will work. $\square$

Herman-Swiatek's Theorem 2.7 now implies:

Corollary 5.6. *The normalized linearizing map $h : \mathbb{T} \to \mathbb{T}$ of G is k-quasisymmetric for a constant k depending only on p, q, θ. Hence, its standard extension $H : \mathbb{D} \to \mathbb{D}$ is K-quasiconformal, where K depends only on p, q, θ.*

Corollary 5.7. *Suppose $f \in \mathcal{E}^{p,q}(\theta)$ and $g := \mathcal{S}_r(f)$. Then, the g-invariant curve $\gamma_{g,r}$ is a K-quasicircle for some K which depends only on p, q, θ.*

Proof. By Theorem 4.6, $\gamma_{g,r} = \psi(\mathbb{T})$, where the maximal dilatation of the quasiconformal map ψ is the same as that of the standard extension $H : \mathbb{D} \to \mathbb{D}$. Hence the result follows from Corollary 5.6. $\square$

6. Maps with no free capture spots

Ideally, one would hope that the surgery on $f \in \mathcal{E}^{p,q}(\theta)$ as described in §4 would produce an entire map $\mathcal{S}_r(f)$ which is conformally conjugate to f. However, this type of "rigidity" for a general f is wishful thinking. In reality, f and $\mathcal{S}_r(f)$ may not be even quasiconformally conjugate. The problem arises when f has critical orbits which hit its Siegel disk Δ_f.

6.1. **Captured critical points.** A critical point c of $f \in \mathcal{E}^{p,q}(\theta)$ is said to be *captured* by Δ_f if its forward orbit eventually hits Δ_f. In this case, there is a smallest integer $n \geq 1$ such that $\hat{c} := f^{\circ n}(c) \in \Delta_f$. We call $\hat{c}$ a *capture spot* of f in Δ_f; if $\hat{c} \neq 0$, we call it a *free* capture spot. In this terminology, f has no free capture spot if the forward orbit of each critical point of f is either disjoint from Δ_f or lands directly at the fixed point 0.

Recall that $\zeta_f : \mathbb{D} \to \Delta_f$ is the unique linearizing map of f which is normalized so that $\zeta_f'(0) = 1$. By the *conformal position* of $z \in \Delta_f$ we mean the point $\zeta_f^{-1}(z)$ in the unit disk.

6.2. **Rigidity.** The following theorem shows that the conformal positions of the capture spots are the only obstructions to promoting a quasiconformal conjugacy to a conformal one along the backward orbit of the Siegel disk.

Theorem 6.1. *Suppose $f, g \in \mathcal{E}^{p,q}(\theta)$, $0 < r < 1$ and $\varphi : \mathbb{C} \to \mathbb{C}$ is a quasiconformal map such that*

(i) $\varphi \circ f = g \circ \varphi$ *on* $\mathbb{C} \smallsetminus \Delta_{f,r}$.

(ii) $\overline{\partial}\varphi = 0$ *a.e. on* $\mathbb{C} \smallsetminus \Omega_{f,r}$.

(iii) $\varphi = \zeta_g \circ \zeta_f^{-1}$ *on* $\gamma_{f,r}$.

Let $\{\hat{c}_1, \ldots, \hat{c}_m\}$ be the capture spots of f in $\Delta_{f,r}$ and $\{\hat{e}_1, \ldots, \hat{e}_m\}$ be the corresponding capture spots of g in $\Delta_{g,r}$. If $\hat{c}_j$ and $\hat{e}_j$ have the same conformal position for each j, then $f = g$.

The capture spot $\hat{e}_j$ "corresponds" to $\hat{c}_j$ in the following sense: if $\hat{c}_j = f^{\circ n}(c_j)$ for some critical point c_j of f, then $\hat{e}_j = g^{\circ n}(e_j)$, where $e_j = \varphi(c_j)$.

Proof. We will modify φ along $\Omega_{f,r}$ in order to promote it to a conformal conjugacy Φ between f and g. Define $\Phi := \varphi$ on $\mathbb{C} \smallsetminus \Omega_{f,r}$ and set

$$\Phi := \zeta_g \circ \zeta_f^{-1} \qquad \text{in } \Delta_{f,r}. \tag{6.1}$$

Clearly $\Phi : \Delta_{f,r} \to \Delta_{g,r}$ is a conformal conjugacy between f and g, which, by the condition (iii), is continuous along the invariant curve $\gamma_{f,r}$.

We extend Φ to the remaining part of $\Omega_{f,r}$ as follows. For each non-zero $\hat{c}_j$, consider the radial segment in $\mathbb{D}$ from $\zeta_f^{-1}(\hat{c}_j)$ out to the boundary $\mathbb{T}$ and let J be the union of all such segments together with the segment $[0,1] \subset \mathbb{R}$. Set $L := \zeta_f(J)$. Do the same for g, i.e., consider the radial segments from $\zeta_g^{-1}(\hat{e}_j)$ to $\mathbb{T}$ for each non-zero $\hat{e}_j$, let J' be the union of all such segments together with $[0,1]$, and set $L' := \zeta_g(J')$. Since $\hat{c}_j$ and $\hat{e}_j$ have the same conformal position for each j, we have $J = J'$ and so $L' = \Phi(L)$.

Now let $n \geq 1$ and U be a connected component of $f^{-n}(\Delta_{f,r}) \smallsetminus f^{-n+1}(\Delta_{f,r})$. The slit disk $\Delta_{f,r} \smallsetminus L$ is simply-connected and contains no critical or, by Corollary 3.6, asymptotic value of the iterate $f^{\circ n} : U \to \Delta_{f,r}$. It follows from Theorem 3.3 that the components $\{V_i\}$ of $U \smallsetminus f^{-n}(L)$ are all simply-connected and $f^{\circ n} : V_i \to \Delta_{f,r} \smallsetminus L$ is a conformal isomorphism for each i. Let $U' := \varphi(U)$ and denote by $\{V_i'\}$ the components of $U' \smallsetminus g^{-n}(L')$, where the labeling is chosen so that $\partial V_i' \cap \partial U' = \varphi(\partial V_i \cap \partial U) = \Phi(\partial V_i \cap \partial U)$. Then, by the same reasoning, $g^{\circ n} : V_i' \to \Delta_{g,r} \smallsetminus L'$ is a conformal isomorphism for each i. Since $\Phi(L) = L'$, we can define $\Phi : V_i \to V_i'$ unambiguously by

$$\Phi := g^{-n} \circ \Phi \circ f^{\circ n}.$$

Putting these partially defined maps $V_i \to V_i'$ together, we obtain a homeomorphism $\Phi : U \smallsetminus f^{-n}(L) \to U' \smallsetminus g^{-n}(L')$. Using continuity of Φ along L and $\gamma_{f,r}$, it is easily seen that Φ extends to a homeomorphism $U \to U'$ which is compatible with the boundary map $\Phi = \varphi : \partial U \to \partial U'$. Moreover, since this homeomorphism is conformal off the removable set $f^{-n}(L) \cap U$ of analytic arcs, it must be conformal.

Repeating this process for all components of $f^{-n}(\Delta_{f,r}) \smallsetminus f^{-n+1}(\Delta_{f,r})$ for all $n \geq 1$, we obtain a global conjugacy $\Phi : \mathbb{C} \to \mathbb{C}$ between f and g which by the condition (ii) is conformal off the union $\bigcup_{n \geq 0} f^{-n}(\gamma_{f,r})$. Since this is clearly a removable set, Φ must be a conformal automorphism of the plane. As Φ fixes the origin, it must have the form $\Phi(z) = \alpha z$ for some $\alpha \in \mathbb{C}^*$. Since both Δ_f and Δ_g have conformal radius 1, we obtain $\alpha = \Phi'(0) = \zeta_g'(0)/\zeta_f'(0) = 1$. $\square$

Corollary 6.2. *Suppose $f \in \mathcal{E}^{p,q}(\theta)$ has no free capture spot in $\Delta_{f,r}$ for some $0 < r < 1$. Then $\mathcal{S}_r(f) = f$.*

Proof. Apply Theorem 6.1 to f, $g := \mathcal{S}_r(f)$, and the quasiconformal map φ given by Theorem 4.6. $\square$

Proof of the Main Theorem when f has no free capture spots. By Corollary 5.7 and Corollary 6.2, for every $0 < r < 1$ the f-invariant curve $\gamma_{f,r}$ is a K-quasicircle for some K independent of r. Hence $\partial\Delta_f$ is a quasicircle by Theorem 2.3 and contains a critical point of f by Theorem 2.8. $\square$

7. Maps with one free capture spot

7.1. **A one-dimensional deformation space.** Now consider the case where $f \in \mathcal{E}^{p,q}(\theta)$ has precisely one free capture spot. Recall that this means there is a point $\omega \in \Delta_f \smallsetminus \{0\}$ such that the forward orbit of every captured critical point of f hits Δ_f for the first time at ω or at the fixed point 0.

Theorem 7.1. *For each $t \in \mathbb{D}^*$ there exists an entire map $f_t \in \mathcal{E}^{p,q}(\theta)$ with the following properties:*

(i) *f_t is conjugate to f by a quasiconformal map $\varphi : \mathbb{C} \to \mathbb{C}$ which satisfies $\varphi = \zeta_{f_t} \circ \zeta_f^{-1}$ on $\gamma_{f,r}$ and $\overline{\partial}\varphi = 0$ off $\Omega_{f,r}$ for some $0 < r < 1$.*

(ii) *The free capture spot $\omega_t = \varphi(\omega) \in \Delta_{f_t}$ has the conformal position t.*

The map f_t with these properties is unique. Moreover, the family $\{f_t\}_{t \in \mathbb{D}^*}$ depends holomorphically on t.

Proof. To show existence of f_t, let $t_0 := \zeta_f^{-1}(\omega) \in \mathbb{D}^*$ and fix a small ε and a radius r so that

$$0 < \varepsilon < \min\{|t|, |t_0|\} \leq \max\{|t|, |t_0|\} < r < 1.$$

Let $\beta : \mathbb{D} \to \mathbb{D}$ be a quasiconformal map such that

$$\beta(t_0) = t,$$

$$\beta \circ R_\theta = R_\theta \circ \beta, \tag{7.1}$$

and

$$\beta = \mathrm{id} \quad \text{in } \mathbb{D}_\varepsilon \cup \mathbb{A}_{r,1}. \tag{7.2}$$

The conformal structure $\mu := \beta^*(\mu_0)$ on $\mathbb{D}$ is R_θ-invariant and has bounded dilatation. Define a conformal structure ν on $\mathbb{C}$ by first setting $\nu := (\zeta_f^{-1})^*(\mu)$ on Δ_f, then spreading it along the iterated preimages of Δ_f using appropriate branches of f^{-n} and letting $\nu = \mu_0$ elsewhere. Evidently, ν is f-invariant and of bounded dilatation, and $\nu = \mu_0$ off the iterated preimages of $\Delta_r \smallsetminus \overline{\Delta_\varepsilon}$. Let $\varphi : \mathbb{C} \to \mathbb{C}$ be the unique quasiconformal solution of $\varphi^*(\mu_0) = \nu$ normalized by the conditions $\varphi(0) = 0$ and $\varphi'(0) = 1$. The conjugate map $f_t := \varphi \circ f \circ \varphi^{-1}$ is holomorphic, hence it belongs to $\mathcal{E}^{p,q}$ by Corollary 3.2. Moreover, f_t has a Siegel disk $\Delta_{f_t} = \varphi(\Delta_f)$ of rotation number θ centered at 0. The composition $\varphi \circ \zeta_f \circ \beta^{-1} : \mathbb{D} \to \Delta_{f_t}$ preserves μ_0, conjugates R_θ

to f_t, and has derivative 1 at the origin. Hence $\zeta_{f_t} = \varphi \circ \zeta_f \circ \beta^{-1}$ and the conformal radius of Δ_{f_t} is 1. Thus, $f_t \in \mathcal{E}^{p,q}(\theta)$ and

$$\zeta_{f_t}^{-1}(\omega_t) = (\beta \circ \zeta_f^{-1})(\omega) = \beta(t_0) = t,$$

which means the conformal position of ω_t is t. Uniqueness of f_t follows from Theorem 6.1.

It remains to show that f_t depends holomorphically on t. Fix $t = t_1 \in \mathbb{D}^*$, suppose $t_1 \neq t_0$, and construct the maps β, φ and the conformal structures μ, ν as above. Consider the conformal structure $s\mu$ on $\mathbb{D}$ for $|s| < 1+\delta$, where $\delta > 0$ is small enough to guarantee $s\mu$ has bounded dilatation. Let $\beta_s : \mathbb{D} \to \mathbb{D}$ be the unique solution of the Beltrami equation $\beta_s^*(\mu_0) = s\mu$ subject to the normalization $\beta_s(0) = 0$ and $\beta_s(1) = 1$. Then $\beta_0 = \mathrm{id}$, $\beta_1 = \beta$, and by the Measurable Riemann Mapping Theorem β_s depends holomorphically on s. We have $R_\theta^*(s\mu) = s\mu$ since R_θ is holomorphic. Hence $\beta_s \circ R_\theta \circ \beta_s^{-1}$ is a conformal automorphism of the disk, which can only be the rotation R_θ itself. It follows that β_s commutes with R_θ, so

$$\beta_s(e^{2\pi i n\theta} z) = e^{2\pi i n\theta} \beta_s(z)$$

for all integers n. For each $z \neq 0$ choose a sequence $\{n_k\}$ such that $e^{2\pi i n_k \theta} \to |z|/z$ as $k \to \infty$. Substituting $n = n_k$ in the above equation and letting $k \to \infty$ then shows that β_s satisfies the relation

$$\beta_s(|z|) = \frac{|z|}{z}\, \beta_s(z)$$

whenever $0 < |z| < 1$. In other words, $z \mapsto \beta_s(z)/z$ depends only on $|z|$. Since this function is holomorphic in $\mathbb{D}_\varepsilon^* \cup \mathbb{A}_{r,1}$, it must be constant in each of $\mathbb{D}_\varepsilon^*$ and $\mathbb{A}_{r,1}$. The normalization $\beta_s(1) = 1$ gives $\beta_s(z) = z$ in $\mathbb{A}_{r,1}$, while we obtain $\beta_s(z) = a_s z$ in $\mathbb{D}_\varepsilon$, where $a_s \neq 0$ depends holomorphically on s.

Now let $\varphi_s : \mathbb{C} \to \mathbb{C}$ be the unique solution of $\varphi_s^*(\mu_0) = s\nu$ normalized so that $\varphi_s(0) = 0$ and $\varphi_s'(0) = a_s$. Then $\varphi_0 = \mathrm{id}$, $\varphi_1 = \varphi$, and φ_s also depends holomorphically on s. By a similar argument as above, the map $\varphi_s \circ f \circ \varphi_s^{-1}$ belongs to $\mathcal{E}^{p,q}(\theta)$ and its linearizing map is $\varphi_s \circ \zeta_f \circ \beta_s^{-1}$. It follows from the uniqueness of the family $\{f_t\}$ that

$$\varphi_s \circ f \circ \varphi_s^{-1} = f_{\beta_s(t_0)}. \tag{7.3}$$

The non-constant holomorphic function $s \mapsto \beta_s(t_0)$ sends a neighborhood of $s = 1$ onto a neighborhood of $\beta_1(t_0) = t_1$. Let $t \mapsto s(t)$ be a local inverse branch of this map defined on a small slit-disk neighborhood N of t_1. By (7.3), the map $t \mapsto f_t$ from N to $\mathcal{E}^{p,q}(\theta)$ can be written as a composition of holomorphic maps

$$t \mapsto s(t) \mapsto \varphi_{s(t)} \circ f \circ \varphi_{s(t)}^{-1},$$

so is itself holomorphic. Since this is true of every $t_1 \in \mathbb{D}^*$ and every choice of the small slit-disk neighborhood N of t_1, it follows that $t \mapsto f_t$ is holomorphic in $\mathbb{D}^*$. $\square$

For simplicity, we denote the Siegel disk Δ_{f_t} by Δ_t, the invariant curves $\gamma_{f_t,r}$ by $\gamma_{t,r}$, the linearizing map ζ_{f_t} by ζ_t, and so on.

Lemma 7.2. *The family of linearizing maps $\zeta_t : \mathbb{D} \to \Delta_t$ depends holomorphically on $t \in \mathbb{D}^*$.*

Proof. Let $t_0 \in \mathbb{D}^*$. By Theorem 7.1, any two maps in the family $\{f_t\}$ are quasiconformally conjugate. The conjugacy maps repelling (resp. attracting) cycles to repelling (resp. attracting) cycles. It also maps indifferent cycles to indifferent cycles, preserving the multipliers. It follows that the repelling cycles of f_t move holomorphically without collision. Since these cycles are dense in the Julia set, the λ-lemma [**MSS**] implies there is a disk neighborhood N of t_0 over which $J(f_t)$ moves holomorphically. This holomorphic motion restricts to a motion of $\partial\Delta_t$ over N. As Sullivan shows in [**Su**], this implies the existence of a holomorphic family of Riemann maps $\{\chi_t : \mathbb{D} \to \Delta_t\}_{t\in N}$ with $\chi_t(0) = 0$. By Schwarz lemma, $\chi_t(z) = \zeta_t(\lambda_t z)$ for some constant λ_t with $|\lambda_t| = 1$. But

$$\lambda_t = \chi_t'(0) = \frac{1}{2\pi i}\int_{|z|=1/2} \frac{\chi_t(z)}{z^2}$$

depends holomorphically on $t \in N$ as well, so $\lambda_t = \lambda$ is in fact independent of t. It follows that for each fixed z, the map $t \mapsto \zeta_t(z) = \chi_t(\lambda^{-1}z)$ is holomorphic in N. $\square$

Lemma 7.3. *For each $0 < r < 1$ there is a constant $K(r) > 1$ such that the invariant curve $\gamma_{t,r} \subset \Delta_t$ is a $K(r)$-quasicircle whenever $0 < |t| < 1/2$.*

Proof. The family of linearizing maps $\{\zeta_t : \mathbb{D} \to \Delta_t\}_{t\in\mathbb{D}^*}$ is normal, so any limit function of ζ_t as $t \to 0$ is normalized and univalent in $\mathbb{D}$. By Lemma 7.2, for each $z \in \mathbb{D}$ the map $t \mapsto \zeta_t(z)$ is holomorphic in $\mathbb{D}^*$ and stays bounded as $t \to 0$ by normality. Hence $t = 0$ is a removable singularity of this map. Setting $\zeta_0(z) := \lim_{t\to 0}\zeta_t(z)$, it follows that the extended family $\{\zeta_t\}_{t\in\mathbb{D}}$ depends holomorphically on t, and $\zeta_t \to \zeta_0$ locally uniformly in $\mathbb{D}$ as $t \to 0$.

Now fix $0 < r < 1$. By Slodkowski's improved λ-lemma [**Sl**], the holomorphic motion

$$\zeta_t \circ \zeta_0^{-1} : \zeta_0(\mathbb{T}_r) \to \gamma_{t,r}$$

of the Jordan curve $\zeta_0(\mathbb{T}_r)$ extends to a holomorphic motion of the plane $\mathbb{C}$ which is $(1 + |t|)/(1 - |t|)$-quasiconformal. If $K(r)$ is the quasicircle constant of $\zeta_0(\mathbb{T}_r)$, it follows that $\gamma_{t,r}$ is a $3K(r)$-quasicircle whenever $0 < |t| < 1/2$. $\square$

7.2. **Surgery on the family $\{f_t\}$.** We now look at the effect of the surgery map $\mathcal{S}_r$ of §4 on the family $\{f_t\}$. Fix $0 < r < 1$. The quasiconformal map of Theorem 4.6, which initially conjugates f_t to $\mathcal{S}_r(f_t)$ off $\Delta_{t,r}$ only, can be easily modified, first inside $\Delta_{t,r}$ and then along $\Omega_{t,r}$ by pull-backs, to obtain a global quasiconformal conjugacy between f_t and $\mathcal{S}_r(f_t)$. It follows from the uniqueness part of Theorem 7.1 that of

$\mathcal{S}_r(f_t)$ must belong to the family $\{f_t\}$. Thus, at the level of parameters, $\mathcal{S}_r$ induces a map $\sigma_r : \mathbb{D}^* \to \mathbb{D}^*$ so that

$$\mathcal{S}_r(f_t) = f_{\sigma_r(t)}.$$

Lemma 7.4. *For each* $0 < r < 1$,

$$\sigma_r(t) = t \qquad \text{whenever } r < |t| < 1.$$

Hence there is a constant K*, depending only on* p, q, θ*, such that the invariant curve* $\gamma_{t,r}$ *is a* K*-quasicircle whenever* $r < |t| < 1$.

Proof. When $r < |t| < 1$, every critical orbit of f_t hitting $\Delta_{t,r}$ must land at 0. In this case the assumptions of Theorem 6.1 hold for $f = f_t$, $g = \mathcal{S}_r(f_t)$ and the quasiconformal conjugacy $\varphi = \varphi_t$ given by Theorem 4.6. It follows that $\mathcal{S}_r(f_t) = f_t$, as required.

The second assertion follows from Corollary 5.7. □

Lemma 7.5. *For each* $0 < r < 1$,

$$\lim_{t \to 0} \sigma_r(t) = 0.$$

Proof. Recall the various maps involved in the construction of $\mathcal{S}_r(f_t)$: the reflection $I_t : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ (§4.1), the standard extension $H_t : \mathbb{D} \to \mathbb{D}$ and the quasiconformal maps $\xi_t, \psi_t : \mathbb{C} \to \mathbb{C}$ (§4.4). Since by Corollary 4.7, $z \mapsto (\psi_t \circ H_t^{-1})(z/r)$ is the linearizing map for $f_{\sigma_r(t)}$, it is not hard to see that

$$\sigma_r(t) = rH_t(\xi_t(\omega_t)), \tag{7.4}$$

where $\omega_t = \zeta_t(t)$ is the free capture spot of f_t. By Lemma 7.3 and Corollary 4.1, I_t and hence ξ_t can be chosen $K(r)$-quasiconformal whenever $0 < |t| < 1/2$. The map $\xi_t \circ \zeta_t : \mathbb{D}_r \to \mathbb{D}$ is $K(r)$-quasiconformal, hence uniformly Hölder continuous of exponent $1/K(r)$. It follows that

$$\lim_{t \to 0} (\xi_t \circ \zeta_t)(t) = 0.$$

By Corollary 5.6, the standard extension $H_t : \mathbb{D} \to \mathbb{D}$ is K-quasiconformal for some K independent of t and r, so

$$\lim_{t \to 0} H_t(\xi_t(\omega_t)) = 0.$$

This, in view of (7.4), proves the lemma. □

We can now prove the following improvement of Lemma 7.3:

Lemma 7.6. *There exists a constant* K*, depending only on* p, q, θ*, such that the invariant curve* $\gamma_{t,r}$ *is a* K*-quasicircle if* $0 < r < 1$ *and* $0 < |t| < 1/2$.

Proof. Let $0 < r < 1$ and K be the constant given by Corollary 5.7, so the invariant curve $\gamma_{\sigma_r(t),r}$ is a K-quasicircle for all $t \in \mathbb{D}^*$. Letting $t \to 0$ and making use of Lemma 7.5, we see by an argument similar to the proof of Theorem 2.3 that the Jordan curve $\zeta_0(\mathbb{T}_r)$ is a K^2-quasicircle. The last part of the proof of Lemma 7.3 then shows that $\gamma_{t,r}$ is a $3K^2$-quasicircle whenever $0 < |t| < 1/2$. $\square$

Theorem 7.7. *There exists a constant K, depending only on p, q, θ, such that the invariant curve $\gamma_{t,r}$ is a K-quasicircle if $0 < r < 1$ and $t \in \mathbb{D}^*$.*

Proof. Fix $0 < r < 1$ and four distinct points $a, b, c, d \in \mathbb{T}$ (in this cyclic order). Consider the holomorphic map $Z : \mathbb{D}^* \to \mathbb{C}$ defined by

$$Z(t) := \mathbf{Cr}(\zeta_t(ra), \zeta_t(rb), \zeta_t(rc), \zeta_t(rd)),$$

where $\mathbf{Cr}$ is the cross-ratio given by (2.5). By Theorem 2.5, Lemma 7.4 and Lemma 7.6, there exists a constant $M > 1$, depending only on p, q, θ, such that

$$|Z(t)| \leq M \qquad \text{if } r < |t| < 1 \text{ or } 0 < |t| < 1/2.$$

It follows from the Maximum Modulus Principle that $|Z| \leq M$ throughout $\mathbb{D}^*$. Since this holds for every $0 < r < 1$ and every quadruple (a, b, c, d), another application of Theorem 2.5 shows that $\gamma_{t,r}$ is a K-quasicircle for some $K > 1$ which depends only on M, hence only on p, q, θ. $\square$

Proof of the Main Theorem when f has one free capture spot. Embed f in the family $\{f_t\}$ given by Theorem 7.1. By Theorem 7.7 the invariant curve $\gamma_{f,r}$ is a K-quasicircle for some K independent of r. The rest of the argument is as before. $\square$

8. The general case

Now we address the case of an arbitrary $f \in \mathcal{E}^{p,q}(\theta)$ with several free capture spots. We will perform a cut-and-paste surgery on f to construct a new map $g \in \mathcal{E}^{p,q}(\theta)$ with at most one free capture spot. Even though g is no longer conjugate to f, there is a quasiconformal map of the plane which sends $\partial\Delta_f$ to $\partial\Delta_g$. The special cases of the Main Theorem proved in §6 and §7 then show that $\partial\Delta_g$ is a quasicircle passing through a critical point. Hence the same must be true of $\partial\Delta_f$.

8.1. **The preimages of Δ_r.** Fix an $f \in \mathcal{E}^{p,q}(\theta)$. For simplicity, we once again drop the subscript f from our notations. According to [**EL**], all the Fatou components of a transcendental entire map with a bounded set of critical and asymptotic values must be simply-connected. In particular, the connected components of $f^{-n}(\Delta)$ are simply-connected Fatou components, which can be bounded or unbounded (in the non-polynomial case where $q > 0$, unbounded preimages of Δ always exist). Set $\Gamma_0 := \{\Delta\}$ and for $n \geq 1$,

$$\Gamma_n := \text{ the collection of the connected components of } f^{-n}(\Delta) \smallsetminus f^{-n+1}(\Delta).$$

If $U \in \Gamma_n$ for some $n > 1$, Corollary 3.4 shows that $f(U) \in \Gamma_{n-1}$. However, if $U \in \Gamma_1$, the image $f(U)$ could be Δ or $\Delta \smallsetminus \{0\}$.

The *capture radius* of f is the number in $[0,1)$ defined by

$$\kappa := \max\{|\zeta^{-1}(\hat{c})| : \hat{c} \text{ is a capture spot of } f\},$$

where $\zeta : \mathbb{D} \to \Delta$ is the linearizing map for f. Alternatively, κ is the smallest radius r for which the annulus $\Delta \smallsetminus \overline{\Delta_r}$ is disjoint from the critical orbits of f. Note that $\kappa = 0$ if and only if f has no free capture spot.

Let $0 < r < 1$ and $n \geq 1$. For each $U \in \Gamma_n$, define

$$U_r := f^{-n}(\Delta_r) \cap U.$$

Lemma 8.1. *U_r is a simply-connected domain whenever $\kappa < r < 1$.*

The proof shows that for every $0 < r < 1$, each component of U_r is simply-connected. We need the condition $\kappa < r < 1$ only to guarantee U_r is connected.

Proof. Let W be a component of U_r, η be a simple closed curve in W and O be the bounded component of $\mathbb{C} \smallsetminus \eta$. Since $\partial O = \eta \subset W$, we have $f^{\circ n}(z) \in \Delta_r$ if $z \in \partial O$. Since $f^{\circ n}$ is an open mapping, the same holds if $z \in O$, proving $O \subset W$. Thus, W is simply-connected.

To show connectivity of U_r, take any s with $\kappa < s < r$ and note that by Theorem 3.3 the restriction of $f^{\circ n}$ to each component of $U \smallsetminus \overline{U_s}$ is a covering map onto $\Delta \smallsetminus \overline{\Delta_s}$. Hence the radial foliation of $\Delta \smallsetminus \overline{\Delta_s}$ pulls back under f^{-n} to a foliation of $U \smallsetminus \overline{U_s}$ by analytic arcs. Moreover, each leaf of the latter foliation lands at a well-defined point of ∂U_s. Hence we can define a deformation retraction $U \to \overline{U_s}$ by sending each point of $U \smallsetminus \overline{U_s}$ to the landing point of the leaf passing through it, and keeping $\overline{U_s}$ fixed pointwise. Thus $\overline{U_s}$ is connected and by letting $s \to r$ it follows that U_r must be connected also. $\square$

8.2. **Action of f on immediate preimages of Δ_r.** Throughout we fix a radius r such that $\kappa < r < 1$.

Lemma 8.2. *Suppose $U \in \Gamma_1$ and U_r is bounded. Then ∂U_r is a Jordan curve and $f : U_r \to \Delta_r$ is a finite-degree branched covering.*

Proof. By Lemma 3.8, U_r is bounded by a Jordan curve. Evidently $f(U_r) \subset \Delta_r$ and $f(\partial U_r) \subset \partial \Delta_r = \gamma_r$. Hence $f : U_r \to \Delta_r$ is a proper holomorphic map. As such, it must be a finite-degree branched covering. $\square$

Here is the corresponding statement when U_r is unbounded:

Lemma 8.3. *Suppose $U \in \Gamma_1$ and U_r is unbounded. Then ∂U_r is a disjoint union of finitely many simple analytic arcs tending to ∞ in both directions. The map $f : U_r \to \Delta_r$ or $\Delta_r \smallsetminus \{0\}$ is an infinite-degree branched covering with finitely many branched* *points. More precisely, if $\varphi : U_r \to \mathbb{D}$ and $\psi : \Delta_r \to \mathbb{D}$ are conformal isomorphisms with $\psi(0) = 0$, then the induced map $F := \psi \circ f \circ \varphi^{-1} : \mathbb{D} \to \mathbb{D}$ is an inner function of the form*

$$F(z) = B(z)\, \exp\left(\frac{A(z)+1}{A(z)-1}\right).$$

Here $A, B : \mathbb{D} \to \mathbb{D}$ are finite Blaschke products, the degree of A is equal to the number of ends of U_r (equivalently, the number of the components of ∂U_r), and the degree of B is equal to the number of zeros of f in U_r.

Proof. The claim on ∂U_r follows from Theorem 3.8 since $\partial U_r \subset f^{-1}(\gamma_r)$ and $f^{-1}(\gamma_r)$ has finitely many components. None of these components is a Jordan curve since U_r is simply-connected and unbounded.

Let $B : \mathbb{D} \to \mathbb{D}$ be a finite Blaschke product having the same zeros of the same multiplicities as F (set $B \equiv 1$ if F has no zeros). The map $G := F/B : \mathbb{D} \to \mathbb{D}^*$ lifts under the universal covering $\mathbb{D} \to \mathbb{D}^*$ given by $z \mapsto \exp((z+1)/(z-1))$ to a holomorphic map $A : \mathbb{D} \to \mathbb{D}$. We show that A is proper and its degree is the number of ends of U_r.

The conformal isomorphisms φ and ψ extend analytically across the boundaries ∂U_r and $\gamma_r = \partial \Delta_r$. The image $\varphi(\partial U_r)$ is of the form $\mathbb{T} \smallsetminus \{a_1, \ldots, a_k\}$, where k is the number of ends of U_r. Since $f(\partial U_r) = \gamma_r$, it follows that F (hence G) extends analytically across each of the k arcs of $\mathbb{T} \smallsetminus \{a_1, \ldots, a_k\}$, mapping these arcs to $\mathbb{T}$. However, the non-tangential limit of F (hence G) at each a_j is 0.

Lifting under $z \mapsto \exp((z+1)/(z-1))$, we see that A extends analytically across each arc of $\mathbb{T} \smallsetminus \{a_1, \ldots, a_k\}$, mapping these arcs to $\mathbb{T}$. To check properness of A, it is therefore enough to show that every sequence $\{z_i\}$ in $\mathbb{D}$ converging to some a_j has a subsequence for which $A(z_i) \to 1$. This is evident if there is a subsequence of $\{z_i\}$ for which $F(z_i) \to 0$. Otherwise $\{F(z_i)\}$ stays away from 0. After passing to a subsequence, we may assume that all $F(z_i)$ belong to an annulus $\mathbb{A}_{a,b}$ $(0 < a < b < 1)$ containing no critical value of F. Let W be the component of $F^{-1}(\mathbb{A}_{a,b})$ containing the sequence $\{z_i\}$. The map $F : W \to \mathbb{A}_{a,b}$ is easily seen to be a universal covering, hence there is a conformal isomorphism $\hat{F} : W \to \{z : \log a < \mathrm{Re}(z) < \log b\}$ such that $F = \exp(\hat{F})$. Since $\{z_i\}$ tends to a_j, it follows that

$$|\mathrm{Im}(\hat{F}(z_i))| \to \infty \quad \text{as } i \to \infty. \tag{8.1}$$

On the other hand, W is simply connected and B has no zeros in there, so there is a lift $\hat{B} : W \to \mathbb{C}$ satisfying $B = \exp(\hat{B})$. Moreover, since $\{B(z_i)\}$ converges to a well-defined point on $\mathbb{T}$,

$$|\mathrm{Im}(\hat{B}(z_i))| \ \text{ remains bounded as } i \to \infty. \tag{8.2}$$

Now both $(A+1)/(A-1)$ and $\hat{F}-\hat{B}$ are lifts of G in W under the exponential map, so after adding an integer multiple of $2\pi i$ to $\hat{B}$ we may assume that

$$\frac{A+1}{A-1} = \hat{F} - \hat{B} \quad \text{throughout } W.$$

By (8.1) and (8.2), the imaginary part of the left side along $\{z_i\}$ is unbounded. This easily implies $A(z_i) \to 1$, as required.

Thus, as a proper holomorphic map, $A : \mathbb{D} \to \mathbb{D}$ is a finite Blaschke product. The above argument shows that $A^{-1}(1) = \{a_1, \dots, a_k\}$, so the degree of A is k. □

Corollary 8.4. *Suppose $U \in \Gamma_1$ and U_r is unbounded. Let*

$$\begin{aligned} k &:= \text{the number of ends of } U_r \\ \ell &:= \text{the number of zeros of } f \text{ in } U_r \\ m &:= \text{the number of critical points of } f \text{ in } U_r \end{aligned}$$

Then

$$k + \ell - m = 1.$$

Proof. The Blaschke products A and B of Lemma 8.3 have degrees k and ℓ, respectively. The critical points of F are the roots of the rational equation

$$B'(z) - B(z)\frac{2A'(z)}{(A(z)-1)^2} = 0,$$

so there are $2(k+\ell-1)$ of them counting multiplicities. Since F commutes with the reflection $z \mapsto 1/\overline{z}$ and has no critical point on $\mathbb{T}$, precisely half of its critical points must be in $\mathbb{D}$, which shows $m = k + \ell - 1$. □

8.3. **Action of f on further preimages of Δ_r.** We continue assuming $\kappa < r < 1$.

Lemma 8.5. *Suppose $U \in \Gamma_n$ for some $n > 1$, so $V := f(U) \in \Gamma_{n-1}$. Then U_r is bounded if and only if V_r is bounded. The map $f : U_r \to V_r$ is always a finite-degree branched covering.*

Proof. Suppose first that U_r is bounded. Then clearly $V_r = f(U_r)$ is also bounded and ∂U_r and ∂V_r are both analytic Jordan curves. The inclusion $f(\partial U_r) \subset \partial V_r$ shows that $f : U_r \to V_r$ is proper, hence a finite-degree branched covering.

Now suppose U_r is unbounded. We have $V \neq \Delta$ since $n > 1$. By Lemma 3.7, $f : U_r \to V_r$ is a proper map, hence a finite-degree branched covering. This also shows V_r is unbounded, for otherwise a suitably chosen curve in U_r tending to ∞ has its image tending to a point on ∂V_r, which contradicts Theorem 3.5. □

Remark 8.6. We can now conclude that for each $U \in \Gamma_n$, the boundary ∂U_r has finitely many components. For $n = 1$, this follows from Lemma 3.8, and the general case follows from Lemma 8.5 by induction on n.

8.4. **Modifying f on critical preimages of Δ_r.** Suppose now that $f \in \mathcal{E}^{p,q}(\theta)$ has more than one free capture spots and $\max\{\kappa, 1/2\} < r < 1$. We first assign a *center* o_U to each iterated preimage U of the Siegel disk Δ inductively as follows. Set $o_\Delta := 0$ and $\omega := \zeta(1/2) \in \Delta_r$. If $U \in \Gamma_1$ and U_r is bounded, then $f(U_r) = \Delta_r$ (Lemma 8.2) and we choose the center o_U arbitrarily from the finite set $f^{-1}(0) \cap U$. If $U \in \Gamma_1$ and U_r is unbounded, then $f(U_r) \supset \Delta_r \smallsetminus \{0\}$ (Lemma 8.3) and we choose the center o_U from the infinite set $f^{-1}(\omega) \cap U$. Suppose $n > 1$ and we have defined the centers of all the iterated preimages of Δ in Γ_j for $1 \leq j \leq n-1$. If $U \in \Gamma_n$, then $V := f(U) \in \Gamma_{n-1}$ and $f(U_r) = V_r$ (Lemma 8.5), and we choose the center o_U from the finite set $f^{-1}(o_V) \cap U$. This finishes the inductive definition of the centers. Note that the assignment $U \mapsto o_U$ respects the action of f:

$$f(o_U) = o_{f(U)} \qquad \text{for all } U \in \bigcup_{n \geq 2} \Gamma_n.$$

Now suppose $U \in \Gamma_n$ for some $n \geq 1$ and U_r contains at least one critical point of f. We will modify f on U_r so that the new (smooth) map has a single branch point at o_U. We will distinguish three cases:

• *Case 1.* U_r is bounded. If $V := f(U)$, it follows from Lemma 8.5 that V_r is also bounded, both ∂U_r and ∂V_r are analytic Jordan curves and $f : U_r \to V_r$ is a branched covering of some degree $d \geq 2$. Take smooth diffeomorphisms $\varphi : \overline{U_r} \to \overline{\mathbb{D}}$ and $\psi : \overline{V_r} \to \overline{\mathbb{D}}$, with $\varphi(o_U) = \psi(o_V) = 0$, such that

$$f = \psi^{-1} \circ \pi_d \circ \varphi \ \text{ on } \ \partial U_r,$$

where $\pi_d : z \mapsto z^d$. Modify f in U_r as

$$\hat{f} := \psi^{-1} \circ \pi_d \circ \varphi$$

Thus $\hat{f} = f$ on ∂U_r and $\hat{f} : U_r \to V_r$ is a smooth degree d branched covering with a single branch point at o_U which is ramified over o_V.

• *Case 2.* U_r is unbounded and $n > 1$. If $V := f(U)$, it follows from Lemma 8.5 that V_r is unbounded and $f : U_r \to V_r$ is a branched covering of some degree $d \geq 2$. Take conformal isomorphisms $\varphi : U_r \to \mathbb{D}$ and $\psi : V_r \to \mathbb{D}$, with $\varphi(o_U) = \psi(o_V) = 0$. The induced map $B := \psi \circ f \circ \varphi^{-1} : \mathbb{D} \to \mathbb{D}$ is proper, hence a degree d Blaschke product. If $0 < s < 1$ is close to 1, there are smooth diffeomorphisms $\hat{\varphi} : B^{-1}(\overline{\mathbb{D}_s}) \to \overline{\mathbb{D}}$ and $\hat{\psi} : \overline{\mathbb{D}_s} \to \overline{\mathbb{D}}$, both fixing the origin, such that

$$\hat{\psi} \circ B \circ \hat{\varphi}^{-1} = \pi_d \ \text{ on } \ \mathbb{T}.$$

Modify f in U_r as

$$\hat{f} := \begin{cases} \psi^{-1} \circ \hat{\psi}^{-1} \circ \pi_d \circ \hat{\varphi} \circ \varphi & \text{in } \varphi^{-1}(B^{-1}(\mathbb{D}_s)) \\ f & \text{elsewhere in } U_r. \end{cases}$$

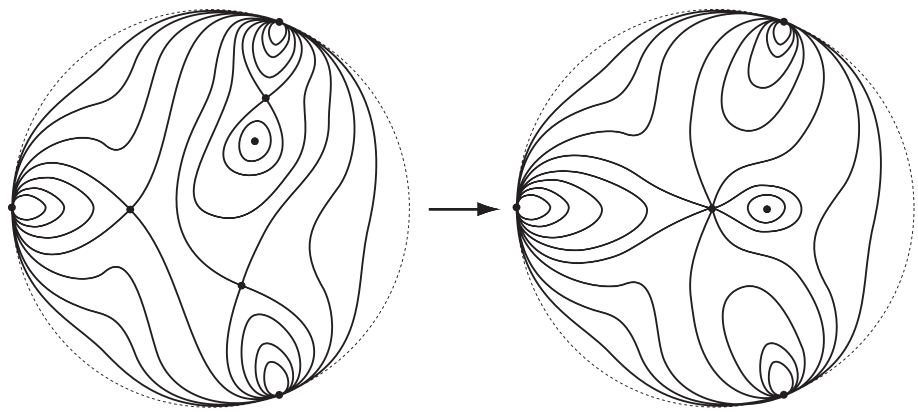

FIGURE 6. *Left: The singular foliation* $|F| = \text{const.}$, *where* $F : \mathbb{D} \to \mathbb{D}$ *is the inner function induced by* $f : U_r \to \Delta_r$. *In this example,* F *has* $m = 3$ *simple critical points and* $\ell = 1$ *zero, and the* $k = 3$ *marked points on* $\partial\mathbb{D}$ *correspond to the ends of* U_r *near infinity, so* $k+\ell-m = 1$ *as in Corollary 8.4. Right: The singular foliation* $|G| = \text{const.}$ *of the modified smooth map* $G : \mathbb{D} \to \mathbb{D}$, *with a single branch point of order* $m = k + \ell - 1 = 3$ *at the origin.*

Note that the simply-connected domain $\varphi^{-1}(B^{-1}(\mathbb{D}_s))$ is compactly contained in U_r, so this modification does not change f near ∞. As in *Case 1*, the map $\hat{f} : U_r \to V_r$ is a smooth degree d branched covering with a single branch point at o_U which is ramified over o_V.

• *Case 3.* U_r is unbounded and $n = 1$. Take a conformal isomorphism $\varphi : U_r \to \mathbb{D}$ with $\varphi(o_U) = 0$ and, as usual, let $\zeta : \mathbb{D} \to \Delta$ be the linearizing map of f. By Lemma 8.3, the induced map $F := \zeta^{-1} \circ f \circ \varphi^{-1} : \mathbb{D} \to \mathbb{D}$ has the form

$$F(z) = B(z) \exp\left(\frac{A(z) + 1}{A(z) - 1}\right),$$

where A, B are finite Blaschke products, $\deg A = k \geq 1$ is the number of ends of U_r and $\deg B = \ell \geq 0$ is the number of zeros of f in U_r. It is not hard to see that there is a smooth, infinite-degree branched covering $G : \mathbb{D} \to \mathbb{D}$, which coincides with F in the annulus $\mathbb{A}_{s,1}$ for some s close to 1, which has a single branch point of order $k + \ell - 1$ at the origin, and which is ramified over $G(0) = 1/2$ (compare Fig. 6).

Modify f in U_r to

$$\hat{f} := \begin{cases} \zeta \circ G \circ \varphi & \text{in } \varphi^{-1}(\mathbb{D}_s) \\ f & \text{elsewhere in } U_r. \end{cases}$$

The map $\hat{f} : U_r \to \Delta_r$ is a smooth, infinite-degree branched covering with a single branch point of order $k + \ell - 1$ at o_U which is ramified over $\omega = \zeta(1/2)$.

Proof of the Main Theorem in the general case. Let $f \in \mathcal{E}^{p,q}(\theta)$ have several free capture spots. Define the modified map $\hat{f}$ on every preimage of Δ_f containing a critical point, as above. Extend $\hat{f}$ to a smooth quasiregular map $\mathbb{C} \to \mathbb{C}$ by setting it equal to f elsewhere. Note in particular that $\hat{f} = f$ in a neighborhood of ∞. It is easy to see that $\hat{f}$ admits an invariant conformal structure μ of bounded dilatation: it suffices to set $\mu = \mu_0$ on Δ_f, define

$$\mu := (\hat{f}^{\circ n})^*(\mu) \qquad \text{on } \hat{f}^{-n}(\Delta_f) = f^{-n}(\Delta_f)$$

for every $n \geq 1$, and set $\mu = \mu_0$ elsewhere. This μ is clearly $\hat{f}$-invariant by the construction. It has bounded dilatation since $\hat{f}$ fails to be holomorphic on at most finitely many of the preimages of Δ_f. Let $\varphi : \mathbb{C} \to \mathbb{C}$ be the unique quasiconformal map which solves the Beltrami equation $\varphi^*(\mu_0) = \mu$ and satisfies $\varphi(0) = 0, \varphi'(0) = 1$. The conjugate map $g := \varphi \circ \hat{f} \circ \varphi^{-1}$ preserves μ_0, hence is entire. In a neighborhood of ∞, g and $\hat{f} = f$ are quasiconformally conjugate, so it follows from an argument similar to the proof of Corollary 3.2 that $g \in \mathcal{E}^{p,q}$. Since g has a Siegel disk of rotation number θ and conformal radius 1 centered at 0, we actually have $g \in \mathcal{E}^{p,q}(\theta)$. By the construction of $\hat{f}$, the map g has at most one free capture spot. It follows from the special cases of the Main Theorem proved in §6 and §7 that $\partial\Delta_g$ is a quasicircle passing through a critical point. Since $\partial\Delta_f = \varphi^{-1}(\partial\Delta_g)$, the same must be true of $\partial\Delta_f$, and this completes the proof. □

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