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Rotation Sets and Complex Dynamics

Preface

For an integer $d \geq 2$, let $m_d : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ denote the multiplication by d map of the circle defined by $m_d(t) = dt \pmod{\mathbb{Z}}$. A ***rotation set*** for m_d is a compact subset of $\mathbb{R}/\mathbb{Z}$ on which m_d acts in an order-preserving fashion and therefore has a well-defined rotation number. Rotation sets for the doubling map m_2 seem to have first appeared under the disguise of Sturmian sequences in a 1940 paper of Morse and Hedlund on symbolic dynamics [MH] (the equivalence with the rotation set condition was later shown by Gambaudo et al. [GLT] and Veerman [V1]). Fertile ground for their come back was provided half a century later by the resurgence of the field of holomorphic dynamics. For example, in the early 1990's Goldberg [G] and Goldberg and Milnor [GM] studied rational rotation sets in their work on fixed point portraits of complex polynomials. The main result of [G] was later extended by Goldberg and Tresser to irrational rotation sets [GT]. Around the same time, Bullett and Sentenac investigated rotation sets for the doubling map and their connection with the Douady-Hubbard theory of the Mandelbrot set [BS]. Aspects of this work was generalized to arbitrary degrees a decade later by Blokh et al. who in particular gave recipes for constructing a rotation set for m_{d+1} from one for m_d and vice versa [BMMOP]. More recently, Bonifant, Buff and Milnor used rotation sets for the tripling map m_3 in their work on antipode-preserving cubic rational maps [BBM]. In an entirely different context, rational rotation sets appear in McMullen's study of the space of proper holomorphic maps of the unit disk [Mc2]; they play a role analogous to simple closed geodesics on compact hyperbolic surfaces.

This monograph presents the first systematic treatment of the theory of rotation sets for m_d in both rational and irrational cases. Our approach, partially inspired by the ideas in [BBM], has a rather geometric flavor and yields several new results on the structure of rotation sets, their gap dynamics, maximal and minimal rotation sets, rigidity, and continuous dependence on parameters. This "abstract" part is supplemented with a "concrete" part which explains how rotation sets arise in the dynamical plane of complex polynomial maps and how suitable parameter spaces of such polynomials provide a complete catalog of all rotation sets of a given degree.

Here is an outline of the material presented in this monograph:

Chapter 1 provides background material on the dynamics of degree 1 monotone maps of the circle. Given such a map $g : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, its Poincaré ***rotation number*** $\rho(g)$ is constructed using a Dedekind cut approach that quickly leads to basic properties of the rotation number and how it essentially determines the asymptotic behavior of the orbits of g. These orbits converge to a cycle if $\rho(g)$ is rational, and to a unique minimal Cantor set if $\rho(g)$ is irrational. A key tool in understanding this dichotomy is the semiconjugacy between g and the rigid rotation $r_\theta : t \mapsto t + \theta \pmod{\mathbb{Z}}$ by the angle $\theta = \rho(g)$. This semiconjugacy is also utilized in studying the existence and uniqueness of invariant probability measures for g: If $\rho(g)$ is rational, every such measure is a convex combination of Dirac measures supported on the cycles of g, while if $\rho(g)$ is irrational, there is a unique such measure supported on the minimal Cantor set of g.

Chapter 2 introduces rotation sets for the map m_d and develops their basic properties. A rotation set for m_d is a non-empty compact set $X \subset \mathbb{R}/\mathbb{Z}$, with $m_d(X) = X$, such that the restriction $m_d|_X$ extends to a degree 1 monotone map of the circle. The rotation number of X, denoted by $\rho(X)$, is defined as the rotation number of any such extension. We refer to X as a rational or irrational rotation set according as $\rho(X)$ is rational or irrational. Understanding X is facilitated by studying the dynamics of the complementary intervals of X called its ***gaps***. A gap I is labeled ***minor*** or ***major*** according as $m_d|_I : I \to m_d(I)$ is or is not a homeomorphism, and the ***multiplicity*** of I is the number of times the covering map m_d wraps I around the circle. Counting multiplicities, X has $d-1$ major gaps, a statement reminiscent of the fact that a degree d polynomial has $d-1$ critical points. Major gaps completely determine a rotation set and the pattern of how they are mapped around can be captured in a combinatorial object called the ***gap graph***.

Next, we study maximal and minimal rotation sets. Maximal rotation sets for m_d are characterized as having $d-1$ distinct major gaps of length $1/d$. A rational rotation set may well be contained in infinitely many maximal rotation sets. By contrast, we show that an irrational rotation set for m_d is contained in at most $(d-1)!$ maximal rotation sets. Minimal rotation sets are cycles in the rational case and Cantor sets in the irrational case. We prove that a rational rotation set contains at most as many minimal rotation sets as the number of its distinct major gaps. As a special case, we recover Goldberg's result in [G] according to which a rational rotation set for m_d contains at most $d-1$ cycles. On the other hand, every irrational rotation set is easily shown to contain a unique minimal rotation set.

Chapter 3 offers a more in-depth study of minimal rotation sets by presenting a unified treatment of the deployment theorem of Goldberg and Tresser. Suppose X is a minimal rotation set for m_d with the rotation number $\rho(X) = \theta \neq 0$. Then X is a q-cycle (i.e., a cycle of length q) if $\theta = p/q$ in lowest terms, and a Cantor set if θ is irrational. The ***natural measure*** on X is the unique m_d-invariant Borel probability measure μ supported on X. The ***canonical semiconjugacy*** associated with X is a degree 1 monotone map $\varphi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, whose plateaus are precisely the gaps of X, which satisfies $\varphi \circ m_d = r_\theta \circ \varphi$ on X. It is related to the natural measure by $\varphi(t) = \mu[0,t] \pmod{\mathbb{Z}}$. The covering map m_d has $d-1$ fixed points $u_i = i/(d-1) \pmod{\mathbb{Z}}$. The ***deployment vector*** of X is the probability vector $\delta(X) = (\delta_1, \ldots, \delta_{d-1})$ where $\delta_i = \mu[u_{i-1}, u_i)$. Note that $q\delta(X) \in \mathbb{Z}^{d-1}$ if θ is rational of the form p/q.

The deployment theorem asserts that given any θ and any probability vector $\delta \in \mathbb{R}^{d-1}$ that satisfies $q\delta \in \mathbb{Z}^{d-1}$ if $\theta = p/q$, there exists a unique minimal rotation set $X = X_{\theta,\delta}$ for m_d with $\rho(X) = \theta$ and $\delta(X) = \delta$. The rational case of this theorem that appears in [G] and its irrational case proved in [GT] are treated using very different arguments. By contrast, we provide a proof that reveals the nearly identical nature of the two cases. The key tool in our unified treatment is the ***gap measure***

$$\nu = \sum_{i=1}^{d-1} \sum_{k=0}^{\infty} d^{-(k+1)} \mathbb{1}_{\sigma_i - k\theta},$$

where $\sigma_i = \delta_1 + \cdots + \delta_i$ and $\mathbb{1}_x$ denotes the unit mass at x. This is an atomic measure supported on the union of at most $d-1$ backward orbits of the rotation r_θ. The general idea is that the gap measure can be used to construct the "inverse" of the canonical semiconjugacy of X, and therefore X itself (this measure makes a brief cameo in an appendix of [GT], but its real power is not nearly utilized there). In addition to its theoretical role, the gap measure turns out to be a highly effective computational gadget.

Chapter 3 also includes a fairly detailed discussion of finite rotation sets, namely unions of cycles that have a well-defined rotation number. Let $\mathscr{C}_d(p/q)$ denote the collection of all q-cycles under m_d with rotation number p/q. According to the deployment theorem, $\mathscr{C}_d(p/q)$ can be identified with a finite subset of the simplex $\Delta^{d-2} = \{(x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1} : x_i \geq 0 \text{ and } \sum_{i=1}^{d-1} x_i = 1\}$ with $\binom{q+d-2}{q}$ elements. A collection of cycles in $\mathscr{C}_d(p/q)$ are ***compatible*** if their union forms a rotation set. In [Mc2], McMullen proposes that $\mathscr{C}_d(p/q)$ can be identified with the vertices of a simplicial subdivision Δ_q^{d-2} of Δ^{d-2}, where each collection of compatible cycles corresponds to the vertices of a simplex in Δ_q^{d-2}. We provide a justification for this geometric model; in particular, for each $x \in \Delta^{d-2}$ our proof gives an explicit algorithm for finding a simplex in Δ_q^{d-2} that contains x. The subdivision Δ_q^{d-2} is different from (and in a sense simpler than) the standard barycentric subdivision and could perhaps be of independent interest in applications outside dynamics.

In chapter 4 we give sample applications of the results of chapters 2 and 3, especially the deployment theorem. For example, we show that every admissible graph without loops can be realized as the gap graph of an irrational rotation set. We also study the dependence of the minimal rotation set $X_{\theta,\delta}$ on the parameter (θ, δ). We prove that the map $(\theta, \delta) \mapsto X_{\theta,\delta}$ is lower semicontinuous in the Hausdorff topology, and it is continuous at some parameter (θ_0, δ_0) if and only if X_{θ_0,δ_0} is ***exact*** in the sense that it is both minimal and maximal. We provide a characterization of exactness which shows that the set of such parameters has full measure in $(\mathbb{R}/\mathbb{Z}) \times \Delta^{d-2}$.

As another application, we use the gap measure to compute the ***leading angle*** ω of $X = X_{\theta,\delta}$, that is, the smallest angle when X is identified with a subset of $(0, 1)$:

$$\omega = \frac{1}{d-1}\, \nu(0, \theta] + \frac{N_0}{d-1} = \frac{1}{d-1} \sum_{i=1}^{d-1} \sum_{0<\sigma_i - k\theta \leq \theta} d^{-(k+1)} + \frac{N_0}{d-1}.$$

Here, $N_0 \geq 0$ is the number of indices i with $\sigma_i = 0$. The formula gives an explicit algorithm for computing the base-d expansion of the angle $(d-1)\omega$, which has an itinerary interpretation in the context of polynomial dynamics. We exploit the leading angle formula in the low-degree cases $d = 2, 3$ to carry out a detailed analysis of the structure of minimal rotation sets under the doubling and tripling maps.

Chapter 5 explores the link between rotation sets and complex polynomial maps. After a brief review of the basic definitions in polynomial dynamics, we explain how an indifferent fixed point of a polynomial of degree d determines a rotation set under m_d. More precisely, the angles of the dynamic rays that land on a parabolic point or on the boundary of a "good" Siegel disk define a rotation set X with $\rho(X) = \theta$, where $e^{2\pi i\theta}$ is the multiplier of the parabolic point or the center of the Siegel disk. In the parabolic case, this statement is well known and goes back to the work of Goldberg and Milnor [GM]. The Siegel case, while similar in spirit, is trickier because of the possibility of rays accumulating but not landing on the boundary. The "good" Siegel disk assumption refers to a limb decomposition hypothesis, similar to Milnor's in [M3], that allows us to prove the required landing statements (this hypothesis is weaker than local connectivity of the Julia set and presumably holds for Lebesgue almost every θ). The deployment vector $\delta(X)$ can be recovered from the internal angles of the marked roots on the boundary of the Siegel disk, as seen from its center.

These general remarks are illustrated in greater detail in two low-degree families of polynomial maps. According to Douady and Hubbard, the combinatorial structure of the Mandelbrot set (specifically, the boundary of the main cardioid and the limbs growing from it) catalogs all rotation sets under the doubling map m_2 (see [DH] and [M2]). We give a brief account of this in a section on the quadratic family, setting the stage for the simplest higher-degree example, namely, the family of cubic polynomials with an indifferent fixed point of a given rotation number. This one-dimensional slice was studied in [Z1] in the irrational case and has been the subject of investigations by others (see for example [BH]). There are indeed intriguing connections between rotation sets under the tripling map m_3 and this cubic family.

Fix $0 < \theta < 1$ and consider the space of monic cubic polynomials with a fixed point of multiplier $e^{2\pi i\theta}$ at the origin. Each such cubic has the form $f_a : z \mapsto e^{2\pi i\theta} z + az^2 + z^3$ for some $a \in \mathbb{C}$, where f_a and f_{-a} are affinely conjugate under the involution $z \mapsto -z$. The ***connectedness locus***

$$\mathscr{M}_3(\theta) = \{a \in \mathbb{C} : \text{The Julia set } J(f_a) \text{ is connected}\}$$

is compact, connected and full (compare Figures 5.8 and 5.10 right). Outside $\mathscr{M}_3(\theta)$ exactly one critical point of f_a escapes to ∞ and the Böttcher coordinate of the escaping co-critical point gives a conformal isomorphism $\mathbb{C} \smallsetminus \mathscr{M}_3(\theta) \to \mathbb{C} \smallsetminus \overline{\mathbb{D}}$ which can be used to define the ***parameter rays*** of $\mathscr{M}_3(\theta)$.

When θ is rational of the form p/q in lowest terms, the set X_a of angles of dynamic rays that land at the parabolic point 0 is a rotation set under tripling with $\rho(X_a) = p/q$. There are $2q + 1$ possibilities for X_a parametrized by their deployment vectors $(i/(2q), 1 - i/(2q))$ for $i = 0, \ldots, 2q$. Here X_a is a q-cycle if i is even and the union of two compatible q-cycles of i is odd. We briefly describe the locus of each possibility in the a-plane as a piece of $\mathscr{M}_3(\theta)$ cut off by a pair of parameter rays.

Next we assume θ is an irrational of bounded type. In this case each f_a has a Siegel disk Δ_a centered at 0 whose topological boundary $\partial\Delta_a$ is a quasicircle containing at least one critical point of f_a. According to [Z1], there is an embedded arc $\Gamma \subset \mathscr{M}_3(\theta)$ containing $a = 0$ and having endpoints at $a = \pm\sqrt{3e^{2\pi i\theta}}$ with the property that $a \in \Gamma$ if and only if $\partial\Delta_a$ contains both critical points of f_a (Fig. 5.10 left). This arc is parametrized by a well-defined choice $\tau_a \in [0, 1]$ of the conformal angle between the two critical points, as seen from the center 0 of the disk Δ_a. For each $a \in \Gamma$ the set of angles of dynamic rays that land on the boundary $\partial\Delta_a$ contains a unique minimal rotation set X_a under tripling with $\rho(X_a) = \theta$. If $(\delta_a, 1-\delta_a)$ is the deployment vector of X_a, then $\delta_a = \tau_a$. Thus, as a moves along Γ from one end to the other, δ_a assumes all values between 0 and 1 monotonically. In particular, every minimal rotation set for m_3 with rotation number θ occurs exactly once in the family $\{X_a\}_{a\in\Gamma}$.

The connectedness locus $\mathscr{M}_3(\theta)$ has a limb structure much like the Mandelbrot set, where the role of the boundary of the main cardioid is being played by Γ. The analysis of the rotation sets under tripling in chapter 4 allows us to give a combinatorial description of the limbs growing from Γ and the associated wakes $\pm\mathscr{W}_n$ corresponding to the parameters $\pm a_n \in \Gamma$ where $\delta_{\pm a_n} = \pm n\theta \pmod{\mathbb{Z}}$ for $n \geq 0$. We show that the angles of the parameter rays bounding these wakes are all transcendental but depend rationally on a single ***base angle***

$$\omega = \sum_{0 < -k\theta \leq \theta} 3^{-(k+1)}$$

which is just the leading angle of the minimal rotation set X under tripling with $\rho(X) = \theta$ and $\delta(X) = (1, 0)$. Explicit computations are given for the golden mean $\theta = (\sqrt{5} - 1)/2$, where $\omega \approx 0.12809959$. This description remains combinatorial at the moment as we do not address the issue of landing of these parameter rays.

To make sense of the rotation set X_a for $a \notin \Gamma$, one possibility is to verify the limb decomposition hypothesis for all Julia sets $J(f_a)$, but this is not yet verified for every $a \in \mathscr{M}_3(\theta)$ (although it is known to hold for many parameters). We take an alternative route by approaching $\mathscr{M}_3(\theta)$ from outside, allowing disconnected Julia sets and bifurcated rays. Using the fact that outside $\mathscr{M}_3(\theta)$ the cubic f_a has a quadratic-like restriction hybrid equivalent to $z \mapsto e^{2\pi i\theta}z + z^2$, we define the rotation set X_a for $a \notin \mathscr{M}_3(\theta)$, describe its deployment vector in terms of the external angle of a, and show that it remains unchanged within each open set $\pm\mathscr{W}_n \smallsetminus \mathscr{M}_3(\theta)$. A holomorphic motion argument then allows extending this result to the entire wakes $\pm\mathscr{W}_n$.

Understanding the combinatorial structure of the limbs growing from Γ is related to the question of whether $\mathscr{M}_3(\theta)$ contains a homeomorphic copy of the filled Julia set of the quadratic $z \mapsto e^{2\pi i\theta}z + z^2$ in which the Siegel disk is collapsed into an arc. A similar statement has been proved for the attracting perturbations of these maps in [PT] and there is every indication that the phenomenon persists in the indifferent case, at least when θ is of bounded type.

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