Math 310 Problem Set 9 Solutions

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that is *decreasing* (in the sense that $x < x' \implies f(x) > f(x')$). Use the intermediate value theorem to show that f has a unique fixed point, i.e., there is a unique $c \in \mathbb{R}$ such that $f(c) = c$.

Uniqueness of such c is clear: Suppose c, c' are two distinct fixed points and assume $c < c'$. Then $f(c) > f(c')$ because f is decreasing. But since $f(c) = c$ and $f(c') = c'$, this gives $c > c'$, which contradicts our assumption.

It remains to show the existence of a fixed point. Consider the continuous function $g(x) = f(x) - x$. We want to show $g(c) = 0$ for some c. By the intermediate value theorem, it suffices to check that g takes both positive and negative values. Observe that

$$x < 0 \overset{f \text{ decreasing}}{\implies} f(x) > f(0) \implies f(x) - x > f(0) - x \implies g(x) > f(0) - x.$$

In particular, for x negative and less than $f(0)$, we have $g(x) > 0$. Similarly,

$$x > 0 \overset{f \text{ decreasing}}{\implies} f(x) < f(0) \implies f(x) - x < f(0) - x \implies g(x) < f(0) - x.$$

In particular, for x positive and greater than $f(0)$, we have $g(x) < 0$. This shows g takes both positive and negative values, as required.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ -x + 2 & \text{if } x > 1 \end{cases}$$

Prove that f is not differentiable at $x = 1$.

Recall that by definition

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}.$$

We show that this limit does not exist by checking that the left and right limits are different:

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{(-x + 2) - 1}{x - 1} = \lim_{x \to 1^+} \frac{-x + 1}{x - 1} = -1$$

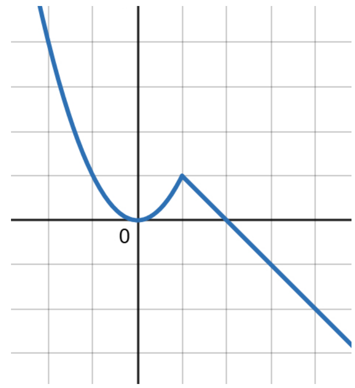
but

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1^-} (x + 1) = 2.$$

Thus,

$$f'(x) = \begin{cases} 2x & \text{if } x < 1 \\ \text{DNE} & \text{if } x = 1 \\ -1 & \text{if } x > 1. \end{cases}$$

In fact, every point in the open interval $(-\infty, 1)$ has a neighborhood in which $f(x) = x^2$, so $f'(x) = 2x$ in $(-\infty, 1)$. Similarly, every point in the open interval $(1, +\infty)$ has a neighborhood in which $f(x) = -x + 2$, so $f'(x) = -1$ in $(1, +\infty)$. But neither of the two formulas alone describes $f(x)$ in some neighborhood of 1. That's why we went back to the limit definition of derivative to investigate the existence of $f'(1)$.



3. Let $f(x) = x|x|$. At what x does $f'(x)$ exist?

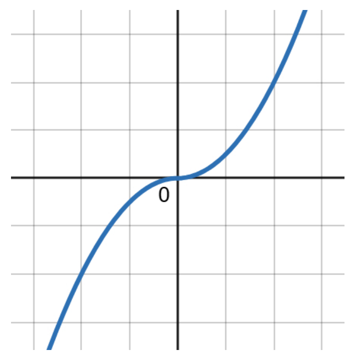
In the open interval $(-\infty, 0)$ we have $f(x) = -x^2$, so $f'(x) = -2x$. Similarly, in the open interval $(0, +\infty)$ we have $f(x) = x^2$, so $f'(x) = 2x$.

To examine differentiability at 0, we use the limit definition of the derivative:

$$f'(0) = \lim_{x \to 0} \frac{x|x| - 0}{x} = \lim_{x \to 0} |x| = 0.$$

Thus, $f'(x)$ exists everywhere and

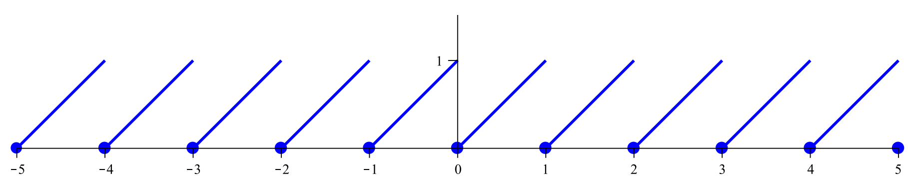
$$f'(x) = \begin{cases} -2x & \text{if } x \leq 0 \\ 0 & \text{if } x = 0 \\ 2x & \text{if } x > 0. \end{cases}$$



4. Let $f(x) = x - \lfloor x \rfloor$ (as usual, $\lfloor x \rfloor$ is the integer part of x). At what x does $f'(x)$ exist?

Let $n \in \mathbb{Z}$. Then $f(x) = x - n$ for $n < x < n + 1$. It follows that $f'(x) = 1$ for $n < x < n + 1$. However, $f'(n)$ does not exist because f is not even continuous at $x = n$:

$$\lim_{x \to n^+} f(x) = \lim_{x \to n^+} (x - n) = 0 \quad \text{while} \quad \lim_{x \to n^-} f(x) = \lim_{x \to n^-} (x - (n - 1)) = 1.$$



Thus,

$$f'(x) = \begin{cases} 1 & \text{if } x \notin \mathbb{Z} \\ \text{DNE} & \text{if } x \in \mathbb{Z}. \end{cases}$$

5. Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be three functions that are differentiable everywhere. Find formulas for the derivative of the product fgh and the composition $f \circ g \circ h$. Can you generalize your formulas to the products and compositions of n functions?

Applying the product rule $(fg)' = f'g + fg'$ twice gives

$$(fgh)' = f'(gh) + f(gh)' = f'gh + f(g'h + gh') = f'gh + fg'h + fgh'.$$

Similarly, applying the chain rule $(f \circ g)' = (f' \circ g)g'$ twice gives

$$(f \circ g \circ h)' = (f' \circ (g \circ h))(g \circ h)' = (f' \circ (g \circ h))(g' \circ h)h'.$$

The generalization to n functions $f_1, f_2, \ldots, f_n$ is straightforward and can be easily proved by induction on n. The result for the product is

$$(f_1 f_2 \cdots f_n)' = f_1' f_2 \cdots f_n + f_1 f_2' \cdots f_n + \cdots + f_1 f_2 \cdots f_n'$$

and for the composition is

$$(f_1 \circ f_2 \circ \cdots \circ f_n)' = (f_1' \circ f_2 \circ \cdots \circ f_n)(f_2' \circ \cdots \circ f_n) \cdots f_n'.$$

For example, for $n = 4$,

$$(f_1 f_2 f_3 f_4)' = f_1' f_2 f_3 f_4 + f_1 f_2' f_3 f_4 + f_1 f_2 f_3' f_4 + f_1 f_2 f_3 f_4'$$

and

$$(f_1 \circ f_2 \circ f_3 \circ f_4)' = (f_1' \circ f_2 \circ f_3 \circ f_4)(f_2' \circ f_3 \circ f_4)(f_3' \circ f_4) f_4'.$$