Math 310 Problem Set 1 Solutions

1. Write (in words) the negation of each of the following statements:

(i) Jack and Jill are good drivers.

"Either Jack or Jill is a bad driver" (this includes the possibility that they both are bad drivers).

(ii) All roses are red.

"Some roses aren't red."

(iii) Some real numbers do not have a square root.

"All real numbers have square roots."

(iv) If you are rich and famous, you are happy.

"You can be rich and famous but unhappy."

2. Provide a counterexample to each of the following statements:

(i) For every real number x, if $x^2 > 4$, then $x > 2$.

$x = -3$ is a counterexample because $(-3)^2 = 9 > 4$ but $-3 < 2$.

(ii) For every positive integer n, $n^2 + n + 41$ is a prime number.

If you try successive values of n from $n = 1$ to $n = 40$, you notice that $n^2 + n + 41$ is indeed a prime number! But clearly for $n = 41$ we obtain a composite number because

$$41^2 + 41 + 41 = 41(41 + 1 + 1) = 41 \times 43.$$

Thus, $n = 41$ is a counterexample.

(iii) No real number x satisfies $x + 1/x = -2$.

$x = -1$ is a counterexample: $(-1) + 1/(-1) = -2$.

3. Recall from calculus that a function f defined on the real line $\mathbb{R}$ is *increasing* provided that $x < y$ implies $f(x) < f(y)$. In symbols,

$$\forall\, x, y \in \mathbb{R},\ (x < y \Longrightarrow f(x) < f(y)).$$

(i) State precisely what it means for a function f *not* to be increasing.

f is not increasing when there exists a pair of real numbers x, y such that $x < y$ but $f(x) \geq f(y)$. In symbols,

$$\exists\, x, y \in \mathbb{R} : (x < y) \wedge (f(x) \geq f(y))$$

(ii) Using (i), show that the function $f(x) = x^3 - 3x$ is not increasing.

It suffices to find a pair of numbers x, y such that $x < y$ but $f(x) \geq f(y)$. Examining the formula or graph of f, it is easy to find such pairs. For example, the choice $x = -1, y = 1$ does the job because $-1 < 1$ but $f(-1) = 2 > f(1) = -2$.

4. Recall that $n! = 1 \cdot 2 \cdots (n-1)n$. Use mathematical induction to show that

$$n! > 2^n$$

for all integers $n \geq 4$.

The inequality $n! > 2^n$ does not hold for $n = 1, 2, 3$ because

$$1! < 2^1, \qquad 2! < 2^2, \qquad 3! < 2^3.$$

That's why the induction base starts at $n = 4$:

$$4! = 24 \quad \text{and} \quad 2^4 = 16, \quad \text{so} \quad 4! > 2^4.$$

Now assume $k! > 2^k$ for some $k \geq 4$ (the induction hypothesis). We need to show that $(k+1)! > 2^{k+1}$ (the induction step). To this end, simply notice that

$$\begin{aligned}
(k+1)! = (k+1)k! &\geq 5k! && \text{(because } k+1 \geq 5\text{)} \\
&> 2k! \\
&> 2 \cdot 2^k && \text{(by the induction hypothesis)} \\
&= 2^{k+1}.
\end{aligned}$$

5. Use mathematical induction to show that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for all integers $n \geq 1$. Can you find a direct, induction-free proof of this? (See if you can come up with a "trick" to simplify the sum on the left.)

Call the given formula $P(n)$. Clearly $P(1)$ is true:

$$\frac{1}{1 \cdot 2} = \frac{1}{1+1}.$$

Assume $P(k)$ is true for some $k \geq 1$. We need to verify that $P(k+1)$ is also true, that is, we need to check that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}.$$

This is quite easy. Simply write, using the induction hypothesis,

$$\begin{aligned}
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}.
\end{aligned}$$

Here is an alternative proof of the formula without any reference to induction. Note that each term $\frac{1}{k(k+1)}$ in the sum can be decomposed as a difference:

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Thus,

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

On the right side, the terms $-1/2, 1/2, -1/3, 1/3, \ldots, -1/n, 1/n$ all cancel in pairs and we are left with the first and last terms $1/1, -1/(n+1)$ which don't have a canceling partner (this is an example of a *telescoping sum*). It follows that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1} = \frac{n}{n+1},$$

as required.