Math 310 Problem Set 3 Solutions

1. Let $\mathbb{F}$ be an ordered field. Use the axioms to verify the following statements:

(i) $1 > 0$.

Since $1 \neq 0$, it suffices to rule out $1 < 0$ because then, by the trichotomy law O1, the only remaining possibility would be $1 > 0$. Assume by way of contradiction that $1 < 0$. By O3, $1 + (-1) < 0 + (-1)$ or $0 < -1$. Multiplying each side by -1 and using O4, we obtain $-1 \cdot 0 < -1 \cdot -1$ or $0 < 1$, which contradicts our assumption.

(ii) $\forall\, x \in \mathbb{F},\ x > 0 \Longleftrightarrow -x < 0$.

Suppose $x > 0$. By O3, $x + (-x) > 0 + (-x)$ which gives $-x < 0$. Conversely, suppose $-x < 0$. By O3, $-x + x < 0 + x$ which gives $x > 0$.

(iii) $\forall\, x \in \mathbb{F},\ x > 0 \Longleftrightarrow 1/x > 0$.

Suppose $x > 0$. To show $1/x > 0$, let us rule out $1/x = 0$ and $1/x < 0$. If $1/x = 0$, then $1 = x \cdot 1/x = x \cdot 0 = 0$, which is impossible. If $1/x < 0$, we can multiply each side by x and use O4 to obtain $x \cdot 1/x < x \cdot 0$ or $1 < 0$, which is also impossible by part (i). Thus, by the trichotomy law O1, the only remaining possibility is $1/x > 0$.

Conversely, suppose $1/x > 0$. Then, by what we just proved (reciprocal of every positive element is positive), $1/(1/x) > 0$ so $x > 0$.

(iv) $\forall\, x, y \in \mathbb{F},\ 0 < x < y \Longrightarrow 0 < 1/y < 1/x$.

Suppose $0 < x < y$. By part (iii) we have $1/x > 0$ and $1/y > 0$. So it is enough to check the inequality $1/y < 1/x$. To this end, multiply each side of $x < y$ by $1/x$ and use O4 to obtain $x \cdot 1/x < y \cdot 1/x$ or $1 < y \cdot 1/x$. Then multiply each side of the last inequality by $1/y$ and use O4 and associativity of multiplication to obtain

$$\frac{1}{y} \cdot 1 < \frac{1}{y} \cdot (y \cdot \frac{1}{x}) \Longrightarrow \frac{1}{y} < (\frac{1}{y} \cdot y) \cdot \frac{1}{x} \Longrightarrow \frac{1}{y} < 1 \cdot \frac{1}{x} \Longrightarrow \frac{1}{y} < \frac{1}{x}.$$

(v) $\forall\, x, y, z \in \mathbb{F},\ (x < y \text{ and } z < 0) \Longrightarrow xz > yz$.

Suppose $x < y$ and $z < 0$. Then $-z > 0$ by part (ii). Hence, by O4, $x \cdot -z < y \cdot -z$ or $-xz < -yz$. By O3, $-xz + xz < -yz + xz$ or $0 < -yz + xz$. Another application of O3 then gives $yz + 0 < yz + (-yz + xz)$ or $yz < xz$, as required.

(vi) $\forall\, x \in \mathbb{F},\ x^2 + 1 > 0$. (Those of you familiar with the field $\mathbb{C}$ of complex numbers will notice that (vi) shows $\mathbb{C}$ cannot be made into an ordered field, since $i^2 + 1 = 0$.)

We consider three cases:

- If $x = 0$, then $x^2 + 1 = 0 \cdot 0 + 1 = 0 + 1 = 1 > 0$ by part (i).

• If $x > 0$, then $x \cdot x > 0 \cdot x$ or $x^2 > 0$ by O4. Hence $x^2 + 1 > 0 + 1$ or $x^2 + 1 > 1$ by O3. Since $1 > 0$, transitivity O2 implies $x^2 + 1 > 0$.
• If $x < 0$, then $-x > 0$ by part (ii). Hence, by the previous case, $x^2 + 1 = (-x)^2 + 1 > 0$.

2. Show that for all real numbers x and y,

$$\big|\,|x| - |y|\,\big| \leq |x - y|.$$

This says that the distance between $|x|$ and $|y|$ is no more than the distance between x and y. (Hint: You need to check the inequalities $-|x-y| \leq |x| - |y| \leq |x-y|$. Both follow from the triangle inequality $|a+b| \leq |a| + |b|$ for suitable choices of a and b.)

By the triangle inequality,

$$|x| = |(x-y) + y| \leq |x-y| + |y| \Longrightarrow |x| - |y| \leq |x-y|.$$

Similarly,

$$|y| = |(y-x) + x| \leq |y-x| + |x| = |x-y| + |x| \Longrightarrow -|x-y| \leq |x| - |y|.$$

Putting the two results together then shows that

$$-|x-y| \leq |x| - |y| \leq |x-y| \Longrightarrow \big||x| - |y|\big| \leq |x-y|.$$

3. In each case, find all the upper bounds (if any) and the least upper bound of $S \subset \mathbb{R}$:

- $S = \{a, b, c\}$, where $a > b > c$

  The set of all upper bounds for S is $[a, +\infty)$. The least upper bound is $\sup(S) = a$.

- $S = \{n + (-1)^n : n \in \mathbb{N}\}$

  Notice that

  $$n + (-1)^n = \begin{cases} n - 1 & \text{if } n \text{ is odd} \\ n + 1 & \text{if } n \text{ is even.} \end{cases}$$

  This shows

  $$S = \{0, 3, 2, 5, 4, 7, 6, \ldots\}$$

  contains 0 and all natural numbers except 1. Thus, S is unbounded above, i.e., the set of all upper bounds for S is empty. By convention, we write $\sup(S) = +\infty$.

- $S = \{x \in [0, \sqrt{10}] : x \text{ is rational}\}$

  Clearly every number in $[\sqrt{10}, +\infty)$ is an upper bound for S. On the other hand, if $t < \sqrt{10}$, by the density of $\mathbb{Q}$ in $\mathbb{R}$, there are rational numbers in the interval $(t, \sqrt{10})$. This shows t cannot be an upper bound for S. Thus, the set of all upper bounds for S is precisely $[\sqrt{10}, +\infty)$. The least upper bound is $\sup(S) = \sqrt{10}$.

- $S = \{-\frac{3}{n} : n \in \mathbb{N}\}$

  Clearly every number in $[0, +\infty)$ is an upper bound for S. If $t < 0$ we can find a large enough $n \in \mathbb{N}$ such that $t < -3/n < 0$ (just take n larger than the positive number $-3/t$). This shows t cannot be an upper bound for S. Thus, the set of all upper bounds for S is precisely $[0, +\infty)$. The least upper bound is $\sup(S) = 0$.

4. Are the following statements true or false? Justify your answers by a brief proof or counterexample.

- If $\sup(S) = \sup(T)$ and $\inf(S) = \inf(T)$, then $S = T$.

  FALSE. $S = [0, 1)$ and $T = (0, 1]$ both have supremum 1 and infimum 0, yet they are not the same set.

- If $a = \inf(S)$ and $a < a'$, then there is an $x \in S$ such that $a < x < a'$.

  FALSE. For example, consider $S = \{0\} \cup [1, 2]$ with $a = \inf(S) = 0$ and let $a' = 1/2$. Then there is no $x \in S$ with $a < x < a'$.

  What is generally true though is that there is an $x \in S$ such that $a \leq x < a'$ (allowing the possibility $x = a$). This follows from the fact that a' is not a lower bound for S (because a is the greatest lower bound), so there must be an $x \in S$ such that $x < a'$. Since $a \leq x$, this gives $a \leq x < a'$.

- If b is an upper bound for S and $b \in S$, then $b = \sup(S)$.

  TRUE. Since b is an upper bound and $\sup(S)$ is the least upper bound for S, we have $\sup(S) \leq b$. Since $b \in S$, we have $b \leq \sup(S)$. It follows that $b = \sup(S)$.

- If S, T are bounded and $S \subset T$, then $\sup(S) \leq \sup(T)$ and $\inf(S) \geq \inf(T)$.

  TRUE. We have $x \leq \sup(T)$ for all $x \in T$ and in particular for all $x \in S$ (since $S \subset T$). This means $\sup(T)$ is an upper bound for S. Since $\sup(S)$ is the least upper bound for S, it follows that $\sup(S) \leq \sup(T)$.

  Similarly, we have $x \geq \inf(T)$ for all $x \in T$ and in particular for all $x \in S$. This means $\inf(T)$ is a lower bound for S. Since $\inf(S)$ is the greatest lower bound for S, it follows that $\inf(S) \geq \inf(T)$.

5. The field $\mathbb{R}$ of real numbers has the *Archimedean property (AP)*, which can be formulated as follows:

> "For every real number $x > 0$, there exists an $n \in \mathbb{N}$ such that $n > x$."

This looks ridiculously obvious, but only because we are so used to our intuition of numbers that we just take it for granted. Amazingly, there are (incomplete) ordered fields in which this property fails. The following outline shows that (AP) is a consequence of the completeness axiom for $\mathbb{R}$. Fill in the blanks:

Suppose (AP) fails. Then we can find some $\boxed{\text{real number } x > 0}$ such that $\boxed{n \leq x}$ for all $n \in \mathbb{N}$. In other words, the real number x is an $\boxed{\text{upper bound}}$ for $\mathbb{N}$. By the $\boxed{\text{completeness axiom}}$, $\mathbb{N}$ must have the least upper bound $b \in \mathbb{R}$. Now the number $b - 1$ cannot be an upper bound for $\mathbb{N}$ because $\boxed{b - 1 < b}$, so there must be an $n \in \mathbb{N}$ such that $\boxed{b - 1 < n}$. But this implies $\boxed{b < n + 1}$ which contradicts the definition of $b = \sup(\mathbb{N})$. This contradiction shows that (AP) must hold.