Math 310 Problem Set 6

10/16/2025

1. Show that
$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$
(Hint: Verify that $0 \le \dfrac{n!}{n^n} \le \dfrac{1}{n}$ for every $n \in \mathbb{N}$ and apply the squeeze theorem.)

2. True or false? Give a brief proof or a counterexample.

- If $x_n \le y_n$ for all n and $\lim_{n\to\infty} y_n = -\infty$, then $\lim_{n\to\infty} x_n = -\infty$.
- If $x_n \neq 0$ for all n and $\lim_{n\to\infty} x_n = 0$, then $\lim_{n\to\infty} 1/x_n = +\infty$.
- If $\{x_n\}$ is increasing and not bounded above, then $\lim_{n\to\infty} x_n = +\infty$.

3. Use the *definition* of limits involving $\pm\infty$ to prove the following statements:

- $\displaystyle\lim_{n\to\infty} 2n^3 = +\infty$
- $\displaystyle\lim_{n\to\infty} \ln\left(\frac{1}{n}\right) = -\infty$

4. When you enter a positive real number into your calculator and keep performing the $\sqrt{}$ function again and again, you see that the outputs get closer and closer to 1, irrespective of your initial choice. Here is an explanation: Define a sequence $\{x_n\}$ by choosing x_1 to be any positive number, and setting $x_{n+1} = \sqrt{x_n}$ for every $n \ge 1$.

- If $0 < x_1 < 1$, show that $\{x_n\}$ is increasing and bounded above, so it has a limit L. Use the relation $x_{n+1} = \sqrt{x_n}$ to show that $L = 1$.
- If $x_1 > 1$, show that $\{x_n\}$ is decreasing and bounded below, so it has a limit L. Conclude as above that $L = 1$.

(Hint: Use the basic calculus fact that $0 < x < \sqrt{x}$ for $0 < x < 1$, while $1 < \sqrt{x} < x$ for $x > 1$.)

5. Let
$$s_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}.$$
Prove that $\{s_n\}$ is not convergent by showing that it is not a Cauchy sequence. (Hint: Verify that the inequality $s_{2n} - s_n > 1$ holds for every $n > 1$.)