Math 363/663 Homework 1 Solutions

Problem 1.

(i) Find the general solution $u = u(x)$ of the first order linear ODE

$$x^2\,u' + x\,u = 1.$$

Then write a formula for the solution which satisfies the condition $u(1) = -3$.

Rewrite the ODE as $u' + (1/x)u = 1/x^2$ to make sure the coefficient of u' is 1. The integrating factor is

$$\mu(x) = \exp\left(\frac{1}{x}\,dx\right) = \exp(\ln(x)) = x.$$

Multiplying the ODE by $\mu(x)$ and integrating then gives the general solution:

$$xu' + u = \frac{1}{x} \implies (xu)' = \frac{1}{x} \implies xu = \ln|x| + C \implies u = \frac{\ln|x| + C}{x}.$$

If $u(1) = -3$, then $C = -3$ and we obtain the particular solution

$$u = \frac{\ln x - 3}{x} \qquad (x > 0).$$

(ii) Find the general solution $u = u(x)$ of the second order linear ODE

$$u'' + 2u' + 2u = 0.$$

Then write a formula for the solution which satisfies the conditions $u(0) = u'(0) = 1$.

The characteristic equation is $\lambda^2 + 2\lambda + 2 = 0$, which by the quadratic formula has the roots $\lambda = -1 \pm i$. Thus, the general solution of the ODE is

$$u = C_1 e^{-x}\cos x + C_2 e^{-x}\sin x.$$

If $u(0) = 1$, then $C_1 = 1$, so $u = e^{-x}\cos x + C_2 e^{-x}\sin x$. If $u'(0) = 1$, then

$$\left(-e^{-x}\cos x + e^{-x}\sin x - C_2 e^{-x}\sin x + C_2 e^{-x}\cos x\right)\Big|_{x=0} = -1 + C_2 = 1,$$

so $C_2 = 2$. Thus, the particular solution is

$$u = e^{-x}\cos x + 2e^{-x}\sin x.$$

Problem 2. Find the general solution $u = u(x)$ of the homogeneous Cauchy-Euler equation

$$x^2\,u'' - 3x\,u' + 4u = 0.$$

The characteristic equation is $\lambda^2 - 4\lambda + 4 = 0$, which by the quadratic formula has the repeated root $\lambda = 2$. Thus, the general solution is

$$u = C_1 x^2 + C_2 x^2 \ln x.$$

Problem 3. Verify that each of the following functions satisfies the given PDE:

(i) $u(x,y) = 3x^2y - y^3; \qquad u_{xx} + u_{yy} = 0$

$$\begin{cases} u_x = 6xy \\ u_y = 3x^2 - 3y^2 \end{cases} \implies \begin{cases} u_{xx} = 6y \\ u_{yy} = -6y \end{cases} \implies u_{xx} + u_{yy} = 0.$$

(ii) $u(x,t) = \sin(x - ct); \qquad u_{tt} = c^2\, u_{xx}$ (c is a constant)

$$\begin{cases} u_x = \cos(x - ct) \\ u_t = -c\cos(x - ct) \end{cases} \implies \begin{cases} u_{xx} = -\sin(x - ct) \\ u_{tt} = -c^2\sin(x - ct) \end{cases} \implies u_{tt} = c^2 u_{xx}.$$

(iii) $u(x,t) = \dfrac{1}{\sqrt{t}} \exp\left(-\dfrac{x^2}{4kt}\right); \qquad u_t = k\, u_{xx}$ (k is a constant)

$$\begin{aligned} u_t &= \frac{-t^{-3/2}}{2} \exp\left(-\frac{x^2}{4kt}\right) + t^{-1/2} \exp\left(-\frac{x^2}{4kt}\right) \cdot \frac{x^2}{4kt^2} \\ &= \frac{-t^{-3/2}}{2} \exp\left(-\frac{x^2}{4kt}\right)\left[1 - \frac{x^2}{2kt}\right] \qquad (*) \end{aligned}$$

On the other hand,

$$\begin{aligned} u_x &= t^{-1/2} \exp\left(-\frac{x^2}{4kt}\right) \cdot \frac{-x}{2kt} = -\frac{xt^{-3/2}}{2k} \exp\left(-\frac{x^2}{4kt}\right) \\ \implies u_{xx} &= -\frac{t^{-3/2}}{2k} \exp\left(-\frac{x^2}{4kt}\right) - \frac{xt^{-3/2}}{2k} \exp\left(-\frac{x^2}{4kt}\right) \cdot \frac{-x}{2kt} \\ &= -\frac{t^{-3/2}}{2k} \exp\left(-\frac{x^2}{4kt}\right)\left[1 - \frac{x^2}{2kt}\right] \qquad (**) \end{aligned}$$

Comparing $(*)$ and $(**)$, we obtain $u_t = ku_{xx}$.