CYCLIC PERMUTATIONS: DEGREES AND COMBINATORIAL TYPES

SAEED ZAKERI

ABSTRACT. This note will give elementary counts for the number of n-cycles in the permutation group $\mathcal{S}_n$ with a given degree (a variant of the descent number), and studies similar counting problems for the conjugacy classes of n-cycles under the action of the rotation subgroup of $\mathcal{S}_n$. This is achieved by relating such cycles to periodic orbits of an associated dynamical system acting on the circle. It is also shown that the distribution of degree on n-cycles is asymptotically normal as $n \to \infty$.

CONTENTS

1. PRELIMINARIES

Fix an integer $n \geq 2$. We denote by $\mathcal{S}_n$ the group of all permutations of $\{1, \ldots, n\}$ and by $\mathcal{C}_n$ the collection of all n-cycles in $\mathcal{S}_n$. Following the tradition of group theory, we represent $\nu \in \mathcal{C}_n$ by the symbol

$$(1 \ \ \nu(1) \ \ \nu^2(1) \ \ \cdots \ \ \nu^{n-1}(1)).$$

The ***rotation group*** $\mathcal{R}_n$ is the cyclic subgroup of $\mathcal{S}_n$ generated by the n-cycle

$$\rho := (1 \ \ 2 \ \ \cdots \ \ n).$$

Elements of $\mathcal{R}_n \cap \mathcal{C}_n$ are called ***rotation cycles***. Thus, $\nu \in \mathcal{C}_n$ is a rotation cycle if and only if $\nu = \rho^m$ for some integer $1 \leq m < n$ with $\gcd(m, n) = 1$. The reduced fraction m/n is called the ***rotation number*** of ρ^m.

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The rotation group $\mathcal{R}_n$ acts on $\mathcal{C}_n$ by conjugation. We refer to each orbit of this action as a ***combinatorial type*** in $\mathcal{C}_n$. The combinatorial type of an n-cycle ν is denoted by $[\nu]$. It is easy to see that ν is a rotation cycle if and only if $[\nu]$ consists of ν only. In fact, if $\rho\nu\rho^{-1} = \nu$, then $\nu = \rho^m$ where $m = \nu(n)$.

1.1. **The symmetry order.** The combinatorial type of $\nu \in \mathcal{C}_n$ can be explicitly described as follows. Let

$$\mathcal{G}_\nu := \left\{\rho^i : \rho^i \nu \rho^{-i} = \nu\right\}$$

be the stabilizer group of ν under the action of $\mathcal{R}_n$. We call the order of $\mathcal{G}_\nu$ the ***symmetry order*** of ν and denote it by $\operatorname{sym}(\nu)$. If $r := n/\operatorname{sym}(\nu)$, it follows that $\mathcal{G}_\nu$ is generated by the power ρ^r and the combinatorial type of ν is the r-element set

$$[\nu] = \left\{\nu, \rho\nu\rho^{-1}, \ldots, \rho^{r-1}\nu\rho^{-(r-1)}\right\}.$$

Since $\operatorname{sym}(\rho\nu\rho^{-1}) = \operatorname{sym}(\nu)$, we can define the symmetry order of a combinatorial type unambiguously as that of any cycle representing it:

$$\operatorname{sym}([\nu]) := \operatorname{sym}(\nu).$$

Evidently there are no 2- or 3-cycles of symmetry order 1, and there is no 4-cycle of symmetry order 2. By contrast, it is not hard to see that for every $n \geq 5$ and every divisor s of n there is a $\nu \in \mathcal{C}_n$ with $\operatorname{sym}(\nu) = s$.

Of the $(n-1)!$ elements of $\mathcal{C}_n$, precisely $\varphi(n)$ are rotation cycles. Here φ is Euler's totient function defined by

$$\varphi(n) := \#\{m \in \mathbb{Z} : 1 \leq m \leq n \text{ and } \gcd(m,n) = 1\}.$$

If $\nu_1, \ldots, \nu_T$ are representatives of the distinct combinatorial types in $\mathcal{C}_n$, then

$$(n-1)! = \sum_{\nu_i \in \mathcal{R}_n} \#[\nu_i] + \sum_{\nu_i \notin \mathcal{R}_n} \#[\nu_i] = \varphi(n) + \sum_{\nu_i \notin \mathcal{R}_n} \#[\nu_i].$$

When n is a prime number, we have $\varphi(n) = n - 1$ and each $\#[\nu_i]$ in the far right sum is n. In this case the number of distinct combinatorial types in $\mathcal{C}_n$ is given by

$$T = (n-1) + \frac{(n-1)! - (n-1)}{n} = \frac{(n-1)! + (n-1)^2}{n}. \tag{1.1}$$

Observe that T being an integer gives a simple proof of Wilson's theorem according to which $(n-1)! = -1 \pmod{n}$ whenever n is prime.

Example 1.1. The $4! = 24$ cycles in $\mathcal{C}_5$ fall into $(4! + 4^2)/5 = 8$ distinct combinatorial types. The 4 rotation cycles

$$\rho = (1\ 2\ 3\ 4\ 5) \qquad \rho^2 = (1\ 3\ 5\ 2\ 4)$$
$$\rho^3 = (1\ 4\ 2\ 5\ 3) \qquad \rho^4 = (1\ 5\ 4\ 3\ 2)$$

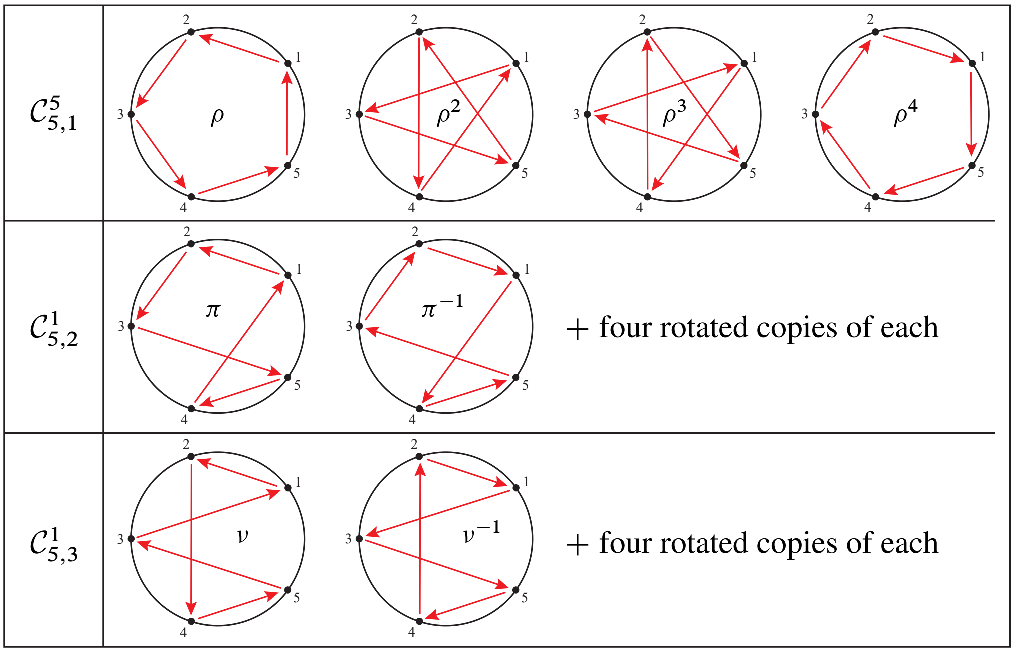

FIGURE 1. *The decomposition of $\mathcal{C}_5$ into subsets $\mathcal{C}^s_{5,d}$ of cycles with degree d and symmetry order s, where the only admissible pairs are $(d,s) = (1,5),(2,1),(3,1)$. See Examples 1.1 and 1.2.*

(of rotation numbers $1/5, 2/5, 3/5, 4/5$) form 4 distinct combinatorial types. The remaining 20 cycles have symmetry order 1, so they fall into 4 combinatorial types each containing 5 elements. These types are represented by

$$\pi = (1\ 2\ 3\ 5\ 4) \qquad \pi^{-1} = (1\ 4\ 5\ 3\ 2)$$
$$\nu = (1\ 2\ 4\ 5\ 3) \qquad \nu^{-1} = (1\ 3\ 5\ 4\ 2).$$

Compare Fig. 1.

1.2. **Descent number vs. degree.** A permutation $\nu \in \mathcal{S}_n$ has a ***descent*** at $i \in \{1,\dots,n-1\}$ if $\nu(i) > \nu(i+1)$. The total number of such i is called the ***descent number*** of ν and is denoted by $\mathrm{des}(\nu)$:

$$\mathrm{des}(\nu) := \#\big\{1 \le i \le n-1 : \nu(i) > \nu(i+1)\big\}$$

Note that $0 \le \mathrm{des}(\nu) \le n-1$. The descent number is a basic tool in enumerative combinatorics (see for example [St]).

In this paper we will be working with a rotationally invariant version of the descent number called ***degree***.[1] It simply amounts to counting $i = n$ as a descent

[1] What we define as the "degree" in this paper is called the "descent number" in [PZ].

if $\nu(n) > \nu(1)$:

$$\deg(\nu) := \begin{cases} \mathrm{des}(\nu) & \text{if } \nu(n) < \nu(1) \\ \mathrm{des}(\nu) + 1 & \text{if } \nu(n) > \nu(1). \end{cases}$$

The terminology comes from the following topological characterization (see [M] and [PZ]): Take any set $\{x_1, \ldots, x_n\}$ of distinct points on the circle in positive cyclic order. Then $\deg(\nu)$ is the minimum degree of a continuous covering map $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ which acts on this set as the permutation ν in the sense that $f(x_i) = x_{\nu(i)}$ for all i.

Example 1.2. The cycle $\nu = (1\ 2\ 4\ 5\ 3) \in \mathcal{C}_5$ has descents at $i = 2$, $i = 4$ and $i = 5$, so $\deg(\nu) = 3$. The eight representative cycles in $\mathcal{C}_5$ described in Example 1.1 have the following degrees:

$$\begin{aligned} &\deg(\rho) = \deg(\rho^2) = \deg(\rho^3) = \deg(\rho^4) = 1, \\ &\deg(\pi) = \deg(\pi^{-1}) = 2, \\ &\deg(\nu) = \deg(\nu^{-1}) = 3. \end{aligned}$$

Compare Fig. 1.

The following statement summarizes the basic properties of the degree for cycles. For a proof, see [PZ].

Theorem 1.3. *Let* $\nu \in \mathcal{C}_n$ *with* $\mathrm{sym}(\nu) = s$ *and* $\deg(\nu) = d$.

(i) $1 \leq d \leq n - 2$ *if* $n \geq 3$.

(ii) $d = 1 \iff s = n \iff \nu$ *is a rotation cycle.*

(iii) s *is a divisor of* $d - 1$.

(iv) $\deg(\rho\nu) = \deg(\nu\rho) = \deg(\rho\nu\rho^{-1}) = d$.

By (iv), the degree of a combinatorial type is well-defined:

$$\deg([\nu]) := \deg(\nu).$$

1.3. **Decompositions of $\mathcal{C}_n$.** Fix $n \geq 3$ and consider the following cross sections of $\mathcal{C}_n$ by the symmetry order and degree:

$$\begin{aligned} \mathcal{C}_n^s &:= \{\nu \in \mathcal{C}_n : \mathrm{sym}(\nu) = s\} \\ \mathcal{C}_{n,d} &:= \{\nu \in \mathcal{C}_n : \deg(\nu) = d\} \\ \mathcal{C}_{n,d}^s &:= \mathcal{C}_n^s \cap \mathcal{C}_{n,d}. \end{aligned}$$

Observe that in our notation the symmetry order always appears as a superscript and the degree as a subscript after n. By Theorem 1.3,

$$\mathcal{C}_n^n = \mathcal{C}_{n,1} = \mathcal{C}_{n,1}^n = \mathcal{C}_n \cap \mathcal{R}_n$$

and we have the decompositions

$$\begin{aligned}
\mathcal{C}_n &= \bigcup_{s|n} \mathcal{C}_n^s = \bigcup_{d=1}^{n-2} \mathcal{C}_{n,d} \\
\mathcal{C}_n^s &= \bigcup_{j=1}^{\lfloor (n-3)/s \rfloor} \mathcal{C}_{n,js+1} && \text{if } s|n,\ s<n \\
\mathcal{C}_{n,d} &= \bigcup_{s|\gcd(n,d-1)} \mathcal{C}_{n,d}^s && \text{if } 2 \le d \le n-2.
\end{aligned}$$

Hence the cardinalities

$$\begin{aligned}
N_n^s &:= \# \mathcal{C}_n^s \\
N_{n,d} &:= \# \mathcal{C}_{n,d} \\
N_{n,d}^s &:= \# \mathcal{C}_{n,d}^s
\end{aligned}$$

satisfy the following relations:

$$N_n^n = N_{n,1} = N_{n,1}^n = \varphi(n)$$

$$(n-1)! = \sum_{s|n} N_n^s = \sum_{d=1}^{n-2} N_{n,d}$$

$$N_n^s = \sum_{j=1}^{\lfloor (n-3)/s \rfloor} N_{n,js+1} \qquad \text{if } s|n,\ s<n \tag{1.2}$$

$$N_{n,d} = \sum_{s|\gcd(n,d-1)} N_{n,d}^s \qquad \text{if } 2 \le d \le n-2. \tag{1.3}$$

Let us also consider the counts for the corresponding combinatorial types

$$\begin{aligned}
T_n &:= \# \{[\nu] : \nu \in \mathcal{C}_n\} \\
T_n^s &:= \# \{[\nu] : \nu \in \mathcal{C}_n^s\} \\
T_{n,d} &:= \# \{[\nu] : \nu \in \mathcal{C}_{n,d}\} \\
T_{n,d}^s &:= \# \{[\nu] : \nu \in \mathcal{C}_{n,d}^s\}.
\end{aligned}$$

Evidently

$$T_{n,d}^s = \frac{s}{n}\, N_{n,d}^s \qquad \text{and} \qquad T_n^s = \frac{s}{n}\, N_n^s$$

and we have the following relations:

$$
\begin{aligned}
T_n^n &= T_{n,1} = T_{n,1}^n = \varphi(n) \\
T_n &= \frac{1}{n} \sum_{s|n} s N_n^s \qquad\qquad (1.4)\\
T_{n,d} &= \frac{1}{n} \sum_{s|\gcd(n,d-1)} s N_{n,d}^s \qquad \text{if } 2 \leq d \leq n-2.
\end{aligned}
$$

Of course knowing the joint distribution $N_{n,d}^s$ would allow us to count all the N's and T's. However, finding an closed formula for $N_{n,d}^s$ seems to be difficult (a sample computation can be found in §3.3). In §2.1 we derive a formula for N_n^s by a direct count which in turn leads to a formula for T_n (see Theorems 2.1 and 2.3). In §3.2 we find a formula for $N_{n,d}$ indirectly by relating cycles in $\mathcal{C}_{n,d}$ to periodic orbits of an associated dynamical system acting on the circle (see Theorem 3.5).

2. THE SYMMETRY ORDER COUNTS

2.1. **The numbers N_n^s.** We begin with the simplest of our counting problems, that is, finding a formula for N_n^s. We will make use of the ***Möbius inversion formula***

$$g(m) = \sum_{k|m} f(k) \quad \Longleftrightarrow \quad f(m) = \sum_{k|m} \mu(k)\, g\Big(\frac{m}{k}\Big) \tag{2.1}$$

on a pair of arithmetical functions f, g. Here μ is the Möbius function uniquely determined by the conditions $\mu(1) := 1$ and $\sum_{k|m} \mu(k) = 0$ for $m > 1$. Applying (2.1) to the relation

$$m = \sum_{k|m} \varphi(k)$$

gives the classical identity

$$\varphi(m) = \sum_{k|m} \frac{m}{k}\, \mu(k) = \sum_{k|m} k \mu\Big(\frac{m}{k}\Big). \tag{2.2}$$

Theorem 2.1. *For every $n \geq 2$ and every divisor s of n,*

$$N_n^s = \frac{1}{n} \sum_{j|\frac{n}{s}} \mu(j)\, \varphi(sj)\, (sj)^{\frac{n}{sj}} \Big(\frac{n}{sj}\Big)! \tag{2.3}$$

When $s = n$ the formula reduces to $N_n^n = (1/n)\mu(1)\varphi(n)n = \varphi(n)$ which agrees with our earlier count.

Proof. Set $r := n/s$. We have $\rho^r \nu \rho^{-r} = \nu$ if and only if $\mathrm{sym}(\nu)$ is a multiple of s if and only if $\nu \in \mathcal{C}_n^{n/j}$ for some $j\,|\,r$. Denoting ν by $(\nu_1\ \nu_2\ \cdots\ \nu_n)$, this condition can be written as

$$(\rho^r(\nu_1)\ \rho^r(\nu_2)\ \cdots\ \rho^r(\nu_n)) = (\nu_1\ \nu_2\ \cdots\ \nu_n),$$

which holds if and only if there is an integer m such that

$$\rho^r(\nu_i) = \nu_{\rho^m(i)} \qquad \text{for all } i. \tag{2.4}$$

The rotations $\rho^r : i \mapsto i + r$ and $\rho^m : i \mapsto i + m \pmod{n}$ have orders $n/\gcd(r,n) = n/r$ and $n/\gcd(m,n)$ respectively. By (2.4), these orders are equal, hence

$$r = \gcd(m,n).$$

Setting $t := m/r$ gives $\gcd(t,s) = 1$, so there are at most $\varphi(s)$ possibilities for t and therefore for m. The action of the rotation ρ^m partitions $\mathbb{Z}/n\mathbb{Z}$ into r disjoint orbits each consisting of s elements and these r orbits are represented by $1,\ldots,r$. In fact, if

$$i + \ell m = i' + \ell' m \pmod{n} \quad \text{for some } 1 \le i, i' \le r \text{ and } 1 \le \ell, \ell' \le s,$$

then $i - i' = m(\ell' - \ell) \pmod{n}$ so $i = i' \pmod{r}$ which gives $i = i'$. Moreover, $\ell m = \ell' m \pmod{n}$ so $\ell t = \ell' t \pmod{s}$. Since $\gcd(t,s) = 1$, this implies $\ell = \ell' \pmod{s}$ which shows $\ell = \ell'$.

Now (2.4) shows that for each of the $\varphi(s)$ choices of m, the cycle ν is completely determined by the integers $\nu_1,\ldots,\nu_r$, and different choices of m lead to different cycles. We may always assume $\nu_1 = 1$. This leaves $n - s$ choices for ν_2 (corresponding to the elements of $\{1,\ldots,n\}$ that are not in the orbit of $\nu_1 = 1$ under ρ^m), $n - 2s$ choices for ν_3, ... and $n - (r-1)s = s$ choices for ν_r. Thus, the total number of choices for ν is

$$\varphi(s)(n-s)(n-2s)\cdots s = \varphi(s)\, s^{r-1}\,(r-1)! = \frac{1}{n}\varphi(s)\, s^r\, r!$$

This proves

$$\sum_{j|r} N_n^{n/j} = \frac{1}{n}\varphi\Big(\frac{n}{r}\Big)\Big(\frac{n}{r}\Big)^r r!$$

An application of the Möbius inversion formula (2.1) then gives

$$\begin{aligned} N_n^s = N_n^{n/r} &= \frac{1}{n}\sum_{j|r} \mu(j)\,\varphi\Big(\frac{nj}{r}\Big)\Big(\frac{nj}{r}\Big)^{\frac{r}{j}}\Big(\frac{r}{j}\Big)! \\ &= \frac{1}{n}\sum_{j|\frac{n}{s}} \mu(j)\,\varphi(sj)\,(sj)^{\frac{n}{sj}}\Big(\frac{n}{sj}\Big)! \end{aligned}$$

□

$n \backslash s$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	0	1													
3	0	–	2												
4	4	0	–	2											
5	20	–	–	–	4										
6	108	6	4	–	–	2									
7	714	–	–	–	–	–	6								
8	4992	40	–	4	–	–	–	4							
9	40284	–	30	–	–	–	–	–	6						
10	362480	380	–	–	16	–	–	–	–	4					
11	3628790	–	–	–	–	–	–	–	–	–	10				
12	39912648	3768	312	60	–	8	–	–	–	–	–	4			
13	479001588	–	–	–	–	–	–	–	–	–	–	–	12		
14	6226974684	46074	–	–	–	–	36	–	–	–	–	–	–	6	
15	87178287120	–	3880	–	192	–	–	–	–	–	–	–	–	–	8

TABLE 1. *The distributions N_n^s for $2 \leq n \leq 15$.*

Table 1 shows the values of N_n^s for $2 \leq n \leq 15$. Notice that $N_2^1 = N_3^1 = N_4^2 = 0$ but all other values are positive. Moreover, as n gets larger the distribution N_n^s appears to be overwhelmingly concentrated at $s = 1$. This is quantified in the following

Theorem 2.2. $N_n^1 \sim (n-1)!$ *as* $n \to \infty$.

This justifies the intuition that the chance of a randomly chosen n-cycle having any non-trivial rotational symmetry tends to zero as $n \to \infty$.

Proof. The formula (2.3) with $s = 1$ gives

$$nN_n^1 = n! + \mu(n)\varphi(n)n + \sum_j \mu(j)\,\varphi(j)\, j^{\frac{n}{j}} \left(\frac{n}{j}\right)!$$

or

$$\frac{N_n^1}{(n-1)!} = 1 + \frac{\mu(n)\varphi(n)}{(n-1)!} + \frac{1}{n!} \sum_j \mu(j)\,\varphi(j)\, j^{\frac{n}{j}} \left(\frac{n}{j}\right)!$$

where the sums are taken over all divisors j of n with $1 < j < n$. We need only check that the term on the far right tends to 0 as $n \to \infty$. If $j|n$ and $1 < j < n$, then $j \leq \lfloor n/2 \rfloor$ and $n/j \leq \lfloor n/2 \rfloor$. Hence,

$$\varphi(j)\, j^{\frac{n}{j}} \left(\frac{n}{j}\right)! \leq j^{\frac{n}{j}+1} \left(\frac{n}{j}\right)! \leq \left\lfloor \frac{n}{2} \right\rfloor^{\lfloor n/2 \rfloor + 1} \left\lfloor \frac{n}{2} \right\rfloor ! \tag{2.5}$$

The Stirling formula $k! \sim \sqrt{2\pi k}\, k^k\, e^{-k}$ gives the elementary estimate

$$\frac{k^k\, k!}{(2k)!} \leq \text{const.} \left(\frac{e}{4}\right)^k.$$

Applying this to (2.5) for $k = \lfloor n/2 \rfloor$, we obtain

$$\frac{1}{n!}\,\varphi(j)\,j^{\frac{n}{j}}\Big(\frac{n}{j}\Big)! \leq \text{const.}\, n\Big(\frac{e}{4}\Big)^{\frac{n}{2}}.$$

Thus,

$$\frac{1}{n!}\left|\sum_{j}\mu(j)\,\varphi(j)\,j^{\frac{n}{j}}\Big(\frac{n}{j}\Big)!\right| \leq \frac{1}{n!}\sum_{j}\varphi(j)\,j^{\frac{n}{j}}\Big(\frac{n}{j}\Big)! \leq \text{const.}\, n^2\Big(\frac{e}{4}\Big)^{\frac{n}{2}},$$

which tends to 0 as $n \to \infty$. □

2.2. **The numbers T_n.** The count (2.3) leads to the following formula for the number of distinct combinatorial types of n-cycles. It turns out that this formula is not new: It appears in the *On-line Encyclopedia of Integer Sequences* as the number of 2-colored patterns of an $n \times n$ chessboard [Sl].

Theorem 2.3. *For every $n \geq 2$,*

$$T_n = \frac{1}{n^2}\sum_{j|n}(\varphi(j))^2\, j^{\frac{n}{j}}\Big(\frac{n}{j}\Big)! \tag{2.6}$$

Observe that for prime n the formula reduces to

$$T_n = \frac{1}{n^2}\big((\varphi(1))^2\, n! + (\varphi(n))^2\, n\big) = \frac{1}{n}((n-1)! + (n-1)^2)$$

which agrees with our derivation in (1.1). Table 2 shows the values of T_n for $2 \leq n \leq 20$.

Proof. By (1.4) and (2.3),

$$T_n = \frac{1}{n}\sum_{s|n} sN_n^s = \frac{1}{n^2}\sum_{s|n}\sum_{j|\frac{n}{s}} s\mu(j)\,\varphi(sj)\,(sj)^{\frac{n}{sj}}\Big(\frac{n}{sj}\Big)!$$

The sum interchange formula

$$\sum_{s|n}\sum_{j|\frac{n}{s}} f(j,s) = \sum_{j|n}\sum_{s|j} f\Big(\frac{j}{s}, s\Big)$$

n	T_n
2	1
3	2
4	3
5	8
6	24
7	108
8	640
9	4492
10	36336
11	329900
12	3326788
13	36846288
14	444790512
15	5811886656
16	81729688428
17	1230752346368
18	19760413251956
19	336967037143596
20	6082255029733168

TABLE 2. *The values of T_n for $2 \leq n \leq 20$.*

then gives

$$\begin{aligned} T_n &= \frac{1}{n^2} \sum_{j|n} \sum_{s|j} s\mu\Big(\frac{j}{s}\Big) \varphi(j)\, j^{\frac{n}{j}} \Big(\frac{n}{j}\Big)! \\ &= \frac{1}{n^2} \sum_{j|n} \Big(\sum_{s|j} s\mu\Big(\frac{j}{s}\Big)\Big)\, \varphi(j)\, j^{\frac{n}{j}} \Big(\frac{n}{j}\Big)! \\ &= \frac{1}{n^2} \sum_{j|n} (\varphi(j))^2\, j^{\frac{n}{j}} \Big(\frac{n}{j}\Big)! \qquad \text{(by (2.2))}, \end{aligned}$$

as required. □

It is evident from Table 2 that the sequence $\{T_n\}$ grows rapidly as $n \to \infty$.

Theorem 2.4. $T_n \sim \dfrac{n!}{n^2} \sim (n-2)!$ *as* $n \to \infty$.

Proof. This is easy to verify. By (2.6),

$$\frac{n^2\, T_n}{n!} = 1 + \frac{(\varphi(n))^2}{(n-1)!} + \frac{1}{n!} \sum_{j} (\varphi(j))^2\, j^{\frac{n}{j}} \Big(\frac{n}{j}\Big)!$$

where the sum is taken over all divisors j of n with $1 < j < n$. The same estimate as in the proof of Theorem 2.2 shows that for such j,

$$\frac{1}{n!}(\varphi(j))^2\, j^{\frac{n}{j}}\Big(\frac{n}{j}\Big)! \le \text{const.}\, n^2\Big(\frac{e}{4}\Big)^{\frac{n}{2}}.$$

Thus,

$$\frac{1}{n!}\sum_j(\varphi(j))^2\, j^{\frac{n}{j}}\Big(\frac{n}{j}\Big)! \le \text{const.}\, n^3\Big(\frac{e}{4}\Big)^{\frac{n}{2}}$$

which tends to 0 as $n \to \infty$. □

Remark 2.5. The ratio $n^2T_n/n!$ tends to 1 at a much faster rate than geometric. In fact, a slightly more careful estimate gives the improved (but not optimal) bound

$$\frac{n^2\, T_n}{n!} = 1 + O\Big(\Big(\frac{3}{n}\Big)^{\frac{n}{2}}\Big) \qquad \text{as } n \to \infty.$$

3. The degree counts

We now turn to the problem of counting n-cycles with a given degree, using the dynamics of a family of covering endomorphisms of the circle.

Conventions 3.1. (i) It will be convenient to extend the definition of $N_{n,d}$ to all $d \ge 1$ by setting $N_{n,d} = 0$ if $d \ge n-1$.

(ii) We follow the customary practice of setting

$$\binom{a}{b} = 0 \qquad \text{if } \ b < 0 \ \text{ or } \ 0 < a < b.$$

3.1. **The circle endomorphisms $\mathbf{m}_k$.** For each integer $k \ge 2$, consider the multiplication-by-k map of the circle $\mathbb{R}/\mathbb{Z}$ defined by

$$\mathbf{m}_k(x) := kx \pmod{\mathbb{Z}}.$$

Let $\mathcal{O} = \{x_1, x_2, \dots, x_n\}$ be a period n orbit of $\mathbf{m}_k$, where the representatives are labeled so that $0 < x_1 < x_2 < \cdots < x_n < 1$. We say that $\mathcal{O}$ ***realizes*** the cycle $\nu \in \mathcal{C}_n$ if

$$\mathbf{m}_k(x_i) = x_{\nu(i)} \qquad \text{for all } i.$$

We say that $\mathcal{O}$ realizes a combinatorial type $[\nu]$ in $\mathcal{C}_n$ if it realizes the cycle $\rho^i\nu\rho^{-i}$ for some i. For example, the periodic orbit

$$\Big\{x_1 = \frac{16}{242}, x_2 = \frac{48}{242}, x_3 = \frac{86}{242}, x_4 = \frac{144}{242}, x_5 = \frac{190}{242}\Big\}$$

of the tripling map $\mathbf{m}_3$ realizes $\nu = (1\ 2\ 4\ 5\ 3) \in \mathcal{C}_5$ and therefore it realizes the combinatorial type $\{\nu, \rho\nu\rho^{-1}, \rho^2\nu\rho^{-2}, \rho^3\nu\rho^{-3}, \rho^4\nu\rho^{-4}\}$.

It follows from the topological interpretation of the degree in §1.2 that if an orbit of $\mathbf{m}_k$ realizes $\nu \in \mathcal{C}_{n,d}$, then necessarily $k \geq d$. Conversely, if $\nu \in \mathcal{C}_{n,d}$ and $k \geq \max\{d, 2\}$, there are always period n orbits of $\mathbf{m}_k$ that realize the combinatorial type $[\nu]$. In fact, by translating the realization problem to finding the steady-state of a regular Markov chain, the following result is proved in [PZ]:

Theorem 3.2. *If $\nu \in \mathcal{C}^s_{n,d}$ and $k \geq \max\{d, 2\}$, there are precisely*

$$\frac{k-1}{s}\binom{n+k-d-1}{n-1}$$

period n orbits of $\mathbf{m}_k$ that realize the combinatorial type $[\nu]$.

The following corollary is immediate:

Corollary 3.3. *For every $k \geq 2$ and $d \geq 1$, the number of period n orbits of $\mathbf{m}_k$ that realize some $\nu \in \mathcal{C}_{n,d}$ is*

$$\frac{k-1}{n}\binom{n+k-d-1}{n-1} N_{n,d}.$$

Proof. The claim is trivial if $d > k$ since in this case the number of such orbits and the binomial coefficient $\binom{n+k-d-1}{n-1}$ are both 0. If $2 \leq d \leq k$, then by Theorem 3.2 for each divisor s of $\gcd(n, d-1)$ there are

$$\frac{k-1}{s}\binom{n+k-d-1}{n-1} T^s_{n,d} = \frac{k-1}{n}\binom{n+k-d-1}{n-1} N^s_{n,d}$$

period n orbits of $\mathbf{m}_k$ that realize some $\nu \in \mathcal{C}^s_{n,d}$. The result then follows from (1.3) by summing over all such s. Finally, since $\mathcal{C}^n_{n,1} = \mathcal{C}_{n,1}$, Theorem 3.2 shows that there are

$$\frac{k-1}{n}\binom{n+k-2}{n-1} T_{n,1} = \frac{k-1}{n}\binom{n+k-2}{n-1} N_{n,1}$$

period n orbits of $\mathbf{m}_k$ that realize some $\nu \in \mathcal{C}_{n,1}$. □

3.2. **The numbers $N_{n,d}$.** For $k \geq 2$ let $P_n(k)$ denote the number of periodic points of $\mathbf{m}_k$ of period n. The periodic points of $\mathbf{m}_k$ whose period is a divisor of n are precisely the $k^n - 1$ solutions of the equation $k^n x = x \pmod{\mathbb{Z}}$. Thus,

$$\sum_{r|n} P_r(k) = k^n - 1 \tag{3.1}$$

and the Möbius inversion formula gives

$$P_n(k) = \sum_{r|n} \mu\Big(\frac{n}{r}\Big)(k^r - 1). \tag{3.2}$$

Introduce the integer-valued quantity

$$\Delta_n(k) := \begin{cases} \dfrac{P_n(k)}{k-1} & \text{if } k \geq 2 \\ \varphi(n) & \text{if } k = 1. \end{cases}$$

By (3.2), for every $k \geq 2$,

$$\Delta_n(k) = \sum_{r|n} \mu\Big(\frac{n}{r}\Big) \frac{k^r - 1}{k-1} = \sum_{r|n} \mu\Big(\frac{n}{r}\Big)\Big(\sum_{j=0}^{r-1} k^j\Big).$$

If $k = 1$, the sum on the far right reduces to $\sum_{r|n} r\mu(n/r)$ which is equal to $\varphi(n)$ by (2.2). It follows that

$$\Delta_n(k) = \sum_{r|n} \mu\Big(\frac{n}{r}\Big)\Big(\sum_{j=0}^{r-1} k^j\Big) \qquad \text{for all } k \geq 1. \tag{3.3}$$

Since $\mathbf{m}_k$ has $P_n(k)/n$ period n *orbits* altogether, Corollary 3.3 shows that for every $k \geq 2$,

$$\frac{k-1}{n} \sum_{d=1}^{n-2} \binom{n+k-d-1}{n-1} N_{n,d} = \frac{P_n(k)}{n}$$

or

$$\sum_{d=1}^{n-2} \binom{n+k-d-1}{n-1} N_{n,d} = \Delta_n(k). \tag{3.4}$$

This is in fact true for every $k \geq 1$ (the case $k = 1$ reduces to $N_{n,1} = \Delta_n(1) = \varphi(n)$).

Remark 3.4. Since the summand in (3.4) is zero unless $1 \leq d \leq \min(n-2, k)$, we can replace the upper bound of the sum by k.

Theorem 3.5. *For every $d \geq 1$,*

$$N_{n,d} = \sum_{i=1}^{d} (-1)^{d-i} \binom{n}{d-i} \Delta_n(i). \tag{3.5}$$

In particular, the theorem claims vanishing of the sum if $d \geq n-1$. Table 3 shows the values of $N_{n,d}$ for $2 \leq n \leq 12$.

$n \backslash d$	1	2	3	4	5	6	7	8	9	10
2	1									
3	2									
4	2	4								
5	4	10	10							
6	2	42	54	22						
7	6	84	336	252	42					
8	4	208	1432	2336	980	80				
9	6	450	5508	16548	14238	3402	168			
10	4	950	19680	99250	153860	77466	11320	350		
11	10	1936	66616	534688	1365100	1233760	389224	36784	682	
12	4	3972	217344	2671560	10568280	15593376	8893248	1851096	116580	1340

TABLE 3. *The distributions* $N_{n,d}$ *for* $2 \leq n \leq 12$.

Proof. This is a form of inversion for binomial coefficients. Use (3.4) to write

$$
\begin{aligned}
&\sum_{i=1}^{d} (-1)^{d-i} \binom{n}{d-i} \Delta_n(i) \\
&= \sum_{i=1}^{d} \sum_{j=1}^{n-2} (-1)^{d-i} \binom{n}{d-i} \binom{n+i-j-1}{n-1} N_{n,j} \\
&= \sum_{i=1}^{d} \sum_{j=1}^{i} (-1)^{d-i} \binom{n}{d-i} \binom{n+i-j-1}{n-1} N_{n,j} \qquad \text{(by Remark 3.4)} \\
&= \sum_{j=1}^{d} \left(\sum_{i=j}^{d} (-1)^{d-i} \binom{n}{d-i} \binom{n+i-j-1}{n-1} \right) N_{n,j}.
\end{aligned}
$$

Thus, (3.5) is proved once we check that

$$
\sum_{i=j}^{d} (-1)^{d-i} \binom{n}{d-i} \binom{n+i-j-1}{n-1} = 0 \qquad \text{for } j < d. \tag{3.6}
$$

Introduce the new variables $a := i - j$ and $b := d - j > 0$ so (3.6) takes the form

$$
\sum_{a=0}^{b} (-1)^{a} \binom{n}{b-a} \binom{n+a-1}{n-1} = 0. \tag{3.7}
$$

The identity

$$\binom{n}{b-a}\binom{n+a-1}{n-1} = \frac{n}{b}\binom{n+a-1}{b-1}\binom{b}{a}$$

shows that (3.7) is in turn equivalent to

$$\sum_{a=0}^{b}(-1)^a\binom{n+a-1}{b-1}\binom{b}{a} = 0. \tag{3.8}$$

To prove (3.8), consider the binomial expansion

$$P(x) := x^{n-1}(x+1)^b = \sum_{a=0}^{b}\binom{b}{a}x^{n+a-1}$$

and differentiate it $b-1$ times with respect to x to get

$$P^{(b-1)}(x) = (b-1)!\sum_{a=0}^{b}\binom{n+a-1}{b-1}\binom{b}{a}x^{n+a-b}.$$

Since P has a zero of order b at $x=-1$, we have $P^{(b-1)}(-1) = 0$ and (3.8) follows. □

As an application of Theorem 3.5, we record the following result which will be invoked in §4:

Theorem 3.6. *The generating function* $G_n(x) := \sum_{d=1}^{n-2} N_{n,d}\, x^d$ *has the expansion*

$$G_n(x) = (1-x)^n \sum_{i\geq 1} \Delta_n(i)\, x^i. \tag{3.9}$$

This should be viewed as an equality between formal power series. It is a true equality for $|x|<1$ where the series on the right converges absolutely.[2]

Proof. For each $d \geq 1$ the coefficient of x^d in the product

$$(1-x)^n \sum_{i\geq 1}\Delta_n(i)\, x^i = \sum_{j=0}^{n}(-1)^j\binom{n}{j}x^j \cdot \sum_{i\geq 1}\Delta_n(i)\, x^i$$

is $\sum_{i=1}^{d}(-1)^{d-i}\binom{n}{d-i}\Delta_n(i)$. This is $N_{n,d}$ by (3.5). □

[2]This is because $\Delta_n(i)$ grows like i^{n-1} for fixed n as $i\to\infty$; compare Lemma 4.5.

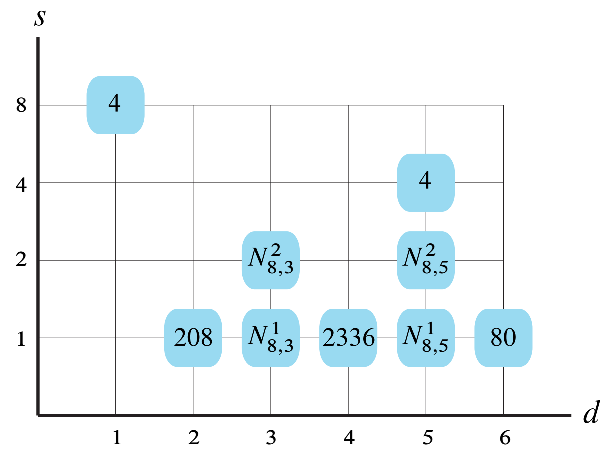

FIGURE 2. *Computation of the joint distribution $N^s_{8,d}$ in Example 3.7.*

3.3. **The numbers $T_{n,d}$.** Our counts for the numbers N^s_n and $N_{n,d}$ lead to the system of linear equations (1.2) and (1.3) on the $N^s_{n,d}$, but such systems are typically under-determined. Thus, additional information is needed to find the $N^s_{n,d}$ and therefore $T_{n,d}$. The following example serves to illustrates this point, where we use the dynamics of $\mathbf{m}_k$ to obtain this additional information.

Example 3.7. For $n = 8$ there are nine admissible pairs

$$(d, s) = (1, 8), (2, 1), (3, 1), (3, 2), (4, 1), (5, 1), (5, 2), (5, 4), (6, 1).$$

We record the values of $N^s_{8,d}$ on a grid as shown in Fig. 2. By (1.2) and (1.3), the values along the s-th row add up to N^s_8 and those along the d-th column add up to $N_{8,d}$, both available from Tables 1 and 3. This immediately gives five of the required nine values:

$$N^8_{8,1} = 4, \quad N^1_{8,2} = 208, \quad N^1_{8,4} = 2336, \quad N^4_{8,5} = 4, \quad N^1_{8,6} = 80.$$

Moreover, it leads to the system of linear equations

$$\begin{cases} N^1_{8,3} + N^2_{8,3} &= 1432 \\ N^1_{8,5} + N^2_{8,5} &= 976 \\ N^1_{8,3} + N^1_{8,5} &= 2368 \\ N^2_{8,3} + N^2_{8,5} &= 40 \end{cases} \tag{3.10}$$

on the remaining four unknowns which has rank 3 and therefore does not determine the solution uniquely. An additional piece of information can be obtained by considering the period 8 orbits of $\mathbf{m}_3$ which realize cycles in $\mathcal{C}^2_{8,3}$ (see [PZ], especially Theorem 6.6, for the results supporting the following claims). Every

$n \backslash d$	1	2	3	4	5	6	7	8	9	10
2	1									
3	2									
4	2	1								
5	4	2	2							
6	2	7	10	5						
7	6	12	48	36	6					
8	4	26	182	292	126	10				
9	6	50	612	1844	1582	378	20			
10	4	95	1978	9925	15408	7753	1138	35		
11	10	176	6056	48608	124100	112160	35384	3344	62	
12	4	331	18140	222654	880848	1299448	741260	154258	9732	113

TABLE 4. *The distributions $T_{n,d}$ for $2 \le n \le 12$. The entries in red cannot be obtained from the sole knowledge of the N_n^s and $N_{n,d}$ in Tables 1 and 3.*

such orbit is ***self-antipodal*** in the sense that it is invariant under the 180° rotation $x \mapsto x + 1/2$ of the circle $\mathbb{R}/\mathbb{Z}$. It follows that x belongs to such orbit if and only if it satisfies

$$3^4 x = x + \frac{1}{2} \pmod{\mathbb{Z}}.$$

This is equivalent to x being rational of the form

$$x = \frac{2j-1}{160} \pmod{\mathbb{Z}} \quad \text{for some } 1 \le j \le 80.$$

Of the 10 orbits of $\mathbf{m}_3$ thus determined, 4 realize rotation cycles in $\mathcal{C}^8_{8,1}$ and the remaining 6 realize cycles in $\mathcal{C}^2_{8,3}$. Moreover, by Theorem 3.2 every combinatorial type in $\mathcal{C}^2_{8,3}$ is realized by a *unique* orbit of $\mathbf{m}_3$. It follows that $N^2_{8,3} = 4T^2_{8,3} = 24$. Now from (3.10) we obtain

$$N^1_{8,3} = 1408, \quad N^2_{8,3} = 24, \quad N^1_{8,5} = 960, \quad N^2_{8,5} = 16$$

and therefore

$$T_{8,1} = 4, \ T_{8,2} = 26, \ T_{8,3} = 182, \ T_{8,4} = 292, \ T_{8,5} = 126, \ T_{8,6} = 10.$$

Observe that $T_8 = \sum_{d=1}^{6} T_{8,d} = 640$, in agreement with the value in Table 2 coming from formula (2.6).

Table 4 shows the result of similar but often more complicated dynamical arguments to determine $T_{n,d}$ for n up to 12. It would be desirable to develop a general method (and perhaps a closed formula) to compute these numbers for arbitrary n.

4. A STATISTICAL VIEW OF THE DEGREE

4.1. **Classical Eulerian numbers.** The numbers $N_{n,d}$ are the analogs of the ***Eulerian numbers*** $A_{n,d}$ which tally the permutations of descent number d in the full symmetric group $\mathcal{S}_n$:[3]

$$A_{n,d} := \#\{\nu \in \mathcal{S}_n : \mathrm{des}(\nu) = d\}.$$

For each n the index d now runs from 0 to $n-1$, with $A_{n,0} = A_{n,n-1} = 1$. The Eulerian numbers occur in many contexts, including areas outside of combinatorics, and have been studied extensively (for an excellent account, see [Pe]). Here are a few of their basic properties:

- *Symmetry*:
$$A_{n,d} = A_{n,n-d-1}.$$
- *Linear recurrence relation*:
$$A_{n,d} = (d+1)A_{n-1,d} + (n-d)A_{n-1,d-1}.$$
- *Worpitzky's identity*:
$$\sum_{d=0}^{n-1} \binom{n+k-d-1}{n} A_{n,d} = k^n \qquad \text{for all } k \geq 1 \tag{4.1}$$
- *Alternating sum formula*:
$$A_{n,d} = \sum_{i=1}^{d+1} (-1)^{d-i+1} \binom{n+1}{d-i+1} i^n. \tag{4.2}$$
- *Carlitz's identity*: The generating function $A_n(x) := \sum_{d=0}^{n-1} A_{n,d}\, x^d$ (also known as the n-th "Eulerian polynomial") satisfies
$$A_n(x) = (1-x)^{n+1} \sum_{i \geq 1} i^n\, x^{i-1}. \tag{4.3}$$

The last three formulas reveal a remarkable similarity between the sequences $N_{n,d}$ and $A_{n-1,d-1}$. In fact, (3.4) is the analog of Worpitzky's identity (4.1) for $A_{n-1,d-1}$ once $\Delta_n(k)$ is replaced with k^{n-1}. Similarly, (3.5) is the analog of the alternating sum formula (4.2) for $A_{n-1,d-1}$ when we replace $\Delta_n(i)$ with i^{n-1}. Finally, (3.9) is the analog of Carlitz's identity (4.3) for $\sum_{d=1}^{n-1} A_{n-1,d-1}\, x^d = x A_{n-1}(x)$, again replacing $\Delta_n(i)$ with i^{n-1}.

There is also an analogy between the $N_{n,d}$ and the restricted Eulerian numbers

$$B_{n,d} := \#\{\nu \in \mathcal{C}_n : \mathrm{des}(\nu) = d\}. \tag{4.4}$$

[3] The numbers $A_{n,d}$ are denoted by $\left\langle {n \atop d} \right\rangle$ in [GKP] and by $A(n, d+1)$ in [St].

In the beautiful paper [DMP] which is motivated by the problem of riffle shuffles of a deck of cards, the authors obtain exact formulas for the distribution of descents in a given conjugacy class of $\mathcal{S}_n$. As a special case, their formulas show that

$$B_{n,d} = \sum_{i=1}^{d+1} (-1)^{d-i+1} \binom{n+1}{d-i+1} f_n(i),$$

where

$$f_n(i) := \frac{1}{n} \sum_{r|n} \mu\Big(\frac{n}{r}\Big) i^r$$

is the number of aperiodic circular words of length n from an alphabet of i letters. One cannot help but notice the similarity between the above formula for $B_{n-1,d-1}$ and (3.5), and between $f_n(i)$ and $\Delta_n(i)$ in (3.3).

4.2. **Asymptotic normality.** The statistical behavior of classical Eulerian numbers is well understood. For example, it is known that the distribution $\{A_{n,d}\}_{0 \le d \le n-1}$ is unimodal with a peak at $d = \lfloor n/2 \rfloor$. Moreover, the descent number of a randomly chosen permutation in $\mathcal{S}_n$ (with respect to the uniform measure) has the mean and variance

$$\tilde{\mu}_n := \frac{1}{n!} \sum_{d=0}^{n-1} d A_{n,d} = \frac{n-1}{2}$$

$$\tilde{\sigma}_n^2 := \frac{1}{n!} \sum_{d=0}^{n-1} (d - \tilde{\mu}_n)^2 A_{n,d} = \frac{n+1}{12}.$$

These computations can be expressed in terms of the generating functions A_n introduced in §4.1:

$$\frac{A_n'(1)}{n!} = \frac{n-1}{2} \tag{4.5}$$

$$\frac{A_n''(1)}{n!} + \frac{A_n'(1)}{n!} - \left(\frac{A_n'(1)}{n!}\right)^2 = \frac{n+1}{12}. \tag{4.6}$$

When rescaled by its mean and variance, the distribution $\{A_{n,d}\}_{0 \le d \le n-1}$ converges to the standard normal distribution as $n \to \infty$ (see [B], [H], [Pi]). This is the central limit theorem for Eulerian numbers. In fact, we have the error bound

$$\sup_{x \in \mathbb{R}} \left| \frac{1}{n!} \sum_{d \le \tilde{\sigma}_n x + \tilde{\mu}_n} A_{n,d} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt \right| = O(n^{-1/2}).$$

Similar results hold for the restricted Eulerian numbers $B_{n,d}$ defined in (4.4). In [F], Fulman shows that the mean and variance of $\mathrm{des}(\nu)$ for a randomly chosen

$\nu \in \mathcal{C}_n$ are also $(n-1)/2$ and $(n+1)/12$ provided that $n \geq 3$ and $n \geq 4$ respectively. More generally, he shows that the k-th moment of $\operatorname{des}(\nu)$ for $\nu \in \mathcal{C}_n$ is equal to the k-th moment of $\operatorname{des}(\nu)$ for $\nu \in \mathcal{S}_n$ provided that $n \geq 2k$. From this result one can immediately conclude that the rescaled distribution $B_{n,d}$ also converges to normal as $n \to \infty$.

Below we will prove corresponding results for the distribution of degree for randomly chosen n-cycles.

Theorem 4.1. *The mean and variance of* $\deg(\nu)$ *for a randomly chosen* $\nu \in \mathcal{C}_n$ *(with respect to the uniform measure) are*

$$\mu_n := \frac{1}{(n-1)!} \sum_{d=1}^{n-2} d N_{n,d} = \frac{n}{2} - \frac{1}{n-1} \qquad (n \geq 3),$$

$$\sigma_n^2 := \frac{1}{(n-1)!} \sum_{d=1}^{n-2} (d-\mu_n)^2 N_{n,d} = \frac{n}{12} + \frac{n}{(n-1)^2(n-2)} \qquad (n \geq 5).$$

Proof. The argument is inspired by the method of [F, Theorem 2]. We begin by using the formula (3.3) for $\Delta_n(i)$ in the equation (3.9) to express the generating function G_n in terms of the Eulerian polynomials A_j in (4.3):

$$\begin{aligned} G_n(x) &= (1-x)^n \sum_{i \geq 1} \sum_{r|n} \sum_{j=0}^{r-1} \mu\Big(\frac{n}{r}\Big) i^j x^i \\ &= (1-x)^n \sum_{i \geq 1} \sum_{j=0}^{n-1} i^j x^i + (1-x)^n \sum_{i \geq 1} \sum_{\substack{r|n \\ r<n}} \sum_{j=0}^{r-1} \mu\Big(\frac{n}{r}\Big) i^j x^i \\ &= \sum_{j=0}^{n-1} x(1-x)^{n-j-1} A_j(x) + \sum_{\substack{r|n \\ r<n}} \sum_{j=0}^{r-1} \mu\Big(\frac{n}{r}\Big) x(1-x)^{n-j-1} A_j(x). \end{aligned} \tag{4.7}$$

If $n \geq 3$, every index j in the double sum in (4.7) is $\leq n-3$, so the polynomial in x defined by this double sum has $(1-x)^2$ as a factor. It follows that for $n \geq 3$,

$$G_n(x) = x A_{n-1}(x) + x(1-x) A_{n-2}(x) + O((1-x)^2)$$

as $x \to 1$. This gives

$$G_n'(1) = A_{n-1}'(1) + A_{n-1}(1) - A_{n-2}(1),$$

so by (4.5)

$$\mu_n = \frac{G_n'(1)}{(n-1)!} = \frac{n-2}{2} + 1 - \frac{1}{n-1} = \frac{n}{2} - \frac{1}{n-1}.$$

Similarly, if $n \geq 5$, every index j in the double sum in (4.7) is $\leq n-4$, so the polynomial defined by this double sum has $(1-x)^3$ as a factor. It follows that for $n \geq 5$,

$$G_n(x) = xA_{n-1}(x) + x(1-x)A_{n-2}(x) + x(1-x)^2 A_{n-3}(x) + O((1-x)^3)$$

as $x \to 1$. This gives

$$G_n''(1) = A_{n-1}''(1) + 2A_{n-1}'(1) - 2A_{n-2}'(1) - 2A_{n-2}(1) + 2A_{n-3}(1).$$

A straightforward computation using (4.5) and (4.6) then shows that

$$\sigma_n^2 = \frac{G_n''(1)}{(n-1)!} + \frac{G_n'(1)}{(n-1)!} - \left(\frac{G_n'(1)}{(n-1)!}\right)^2 = \frac{n}{12} + \frac{n}{(n-1)^2(n-2)},$$

as required. □

Remark 4.2. More generally, the expression (4.7) shows that for fixed k and large enough n,

$$G_n(x) = \sum_{j=0}^{k} x(1-x)^j A_{n-j-1}(x) + O((1-x)^{k+1})$$

as $x \to 1$. Differentiating this k times and evaluating at $x = 1$, we obtain the relation

$$G_n^{(k)}(1) = \sum_{j=0}^{k} (-1)^j \left(\binom{k}{j} j!\, A_{n-j-1}^{(k-j)}(1) + \binom{k}{j+1} (j+1)!\, A_{n-j-1}^{(k-j-1)}(1) \right)$$

which in theory links the moments of $\deg(\nu)$ for $\nu \in \mathcal{C}_n$ to the moments of $\operatorname{des}(\nu)$ for $\nu \in \mathcal{S}_j$ for $n-k \leq j \leq n-1$.

Numerical evidence suggest that the distribution $\{N_{n,d}\}_{1 \leq d \leq n-2}$ is also unimodal and reaches a peak at $d = \lfloor n/2 \rfloor$. Theorem 4.3 below asserts that when rescaled by its mean and variance, the distribution $\{N_{n,d}\}_{1 \leq d \leq n-2}$ converges to normal as $n \to \infty$. In particular, the asymmetry of the numbers $N_{n,d}$ relative to d will asymptotically disappear. These facts are illustrated in Fig. 3.

Consider the sequence of normalized random variables

$$X_n := \frac{1}{\sigma_n}\big(\deg|_{\mathcal{C}_n} - \mu_n\big).$$

Let $\mathcal{N}(0,1)$ denote the normally distributed random variable with the mean 0 and variance 1.

Theorem 4.3. $X_n \to \mathcal{N}(0,1)$ *in distribution as* $n \to \infty$.

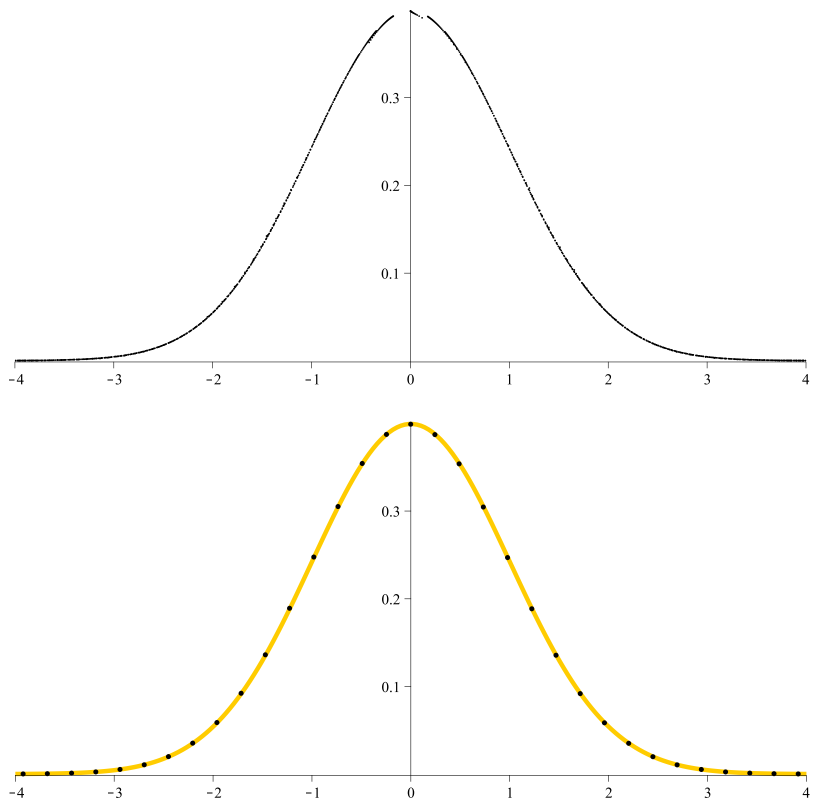

FIGURE 3. *The combined distributions* $\{N_{n,d}\}_{n \leq 100}$ *(top) and the distribution* $N_{200,d}$ *(bottom), rescaled by their mean and variance. The continuous curve in yellow is the standard normal distribution.*

The proof follows the strategy of [FKL] and makes use of the following recent result of [KL] which is a variant of a classical theorem of Curtiss. Recall that the ***moment generating function*** M_X of a random variable X is the expected value of e^{sX}:

$$M_X(s) := \mathbb{E}(e^{sX}) \qquad\qquad (s \in \mathbb{R}).$$

Lemma 4.4 ([KL])**.** *Let* $\{X_n\}_{n \geq 1}$ *and* Y *be random variables and assume that* $\lim_{n \to \infty} M_{X_n}(s) = M_Y(s)$ *for all* s *in some non-empty open interval in* $\mathbb{R}$*. Then* $X_n \to Y$ *in distribution as* $n \to \infty$*.*

The proof of Theorem 4.3 via Lemma 4.4 will depend on two preliminary estimates.

Lemma 4.5. *For every $\varepsilon > 0$ there are constants $n(\varepsilon), i(\varepsilon) > 0$ such that*

$$\Delta_n(i) \begin{cases} \le (1+\varepsilon)\, i^{n-1} & \text{if } n \ge 2 \text{ and } i \ge i(\varepsilon) \\ \ge (1-\varepsilon)\, i^{n-1} & \text{if } n \ge n(\varepsilon) \text{ and } i \ge 2. \end{cases}$$

Proof. By (3.1),

$$\Delta_n(i) \le \sum_{r|n} \Delta_r(i) = \frac{i^n - 1}{i - 1}.$$

The upper bound follows since $(i^n - 1)/(i-1) < (1+\varepsilon) i^{n-1}$ for all n if i is large enough depending on ε.

For the lower bound, first note that the inequality $(i^r - 1)/(i-1) \le 2i^{r-1}$ holds for all $r \ge 1$ and all $i \ge 2$. Thus, by (3.3), we can estimate

$$\begin{aligned} \Delta_n(i) &\ge \frac{i^n - 1}{i-1} - \sum_{\substack{r|n \\ r<n}} \frac{i^r - 1}{i-1} \ge i^{n-1} - \sum_{\substack{r|n \\ r<n}} 2i^{r-1} \\ &\ge i^{n-1} - 2\sum_{r=1}^{\lfloor n/2 \rfloor} i^{r-1} \ge i^{n-1} - 2\,\frac{i^{n/2} - 1}{i-1} \\ &\ge i^{n-1} - 4i^{n/2-1}. \end{aligned}$$

The last term is bounded below by $(1-\varepsilon)i^{n-1}$ for all i if n is large enough depending on ε. $\square$

Lemma 4.6 ([FKL])**.** *For every $0 < x < 1$ and $n \ge 1$,*

$$\frac{(n-1)!\, x}{(\log(1/x))^n} \le \sum_{i \ge 2} i^{n-1} x^i \le \frac{(n-1)!}{x(\log(1/x))^n}.$$

Proof. By elementary calculus,

$$\sum_{i\ge 2} i^{n-1} x^i \le \int_0^\infty u^{n-1} x^{u-1}\, du = \frac{(n-1)!}{x(\log(1/x))^n}$$

and

$$\sum_{i\ge 2} i^{n-1} x^i \ge \int_0^\infty u^{n-1} x^{u+1}\, du = \frac{(n-1)!\, x}{(\log(1/x))^n}. \qquad \square$$

Proof of Theorem 4.3. By Lemma 4.4 it suffices to show that $\lim_{n\to\infty} M_{X_n}(s) = M_{\mathcal{N}(0,1)}(s) = e^{s^2/2}$ for all negative values of s. Fix an $s < 0$ and set $0 < x := e^{s/\sigma_n} < 1$ (we will think of x as a function of n, with $\lim_{n\to\infty} x = 1$). Notice that by Theorem 4.1

$$\mu_n = \frac{n}{2} + O(n^{-1}) \quad \text{and} \quad \sigma_n^2 = \frac{n}{12} + O(n^{-2}) \quad \text{as } n \to \infty. \tag{4.8}$$

Using (3.9), we can write

$$\begin{aligned} M_{X_n}(s) = \mathbb{E}(e^{sX_n}) &= \frac{e^{-s\mu_n/\sigma_n}}{(n-1)!}\, G_n(e^{s/\sigma_n}) = \frac{x^{-\mu_n}}{(n-1)!}\, G_n(x) \\ &= \frac{x^{1-\mu_n}(1-x)^n \varphi(n)}{(n-1)!} + \frac{x^{-\mu_n}(1-x)^n}{(n-1)!} \sum_{i\geq 2} \Delta_n(i) x^i. \end{aligned}$$

As the first term is easily seen to tend to zero, it suffices to show that

$$H_n := \frac{x^{-\mu_n}(1-x)^n}{(n-1)!} \sum_{i\geq 2} \Delta_n(i) x^i \xrightarrow{n\to\infty} e^{s^2/2}. \tag{4.9}$$

By (4.8) we have the estimate

$$1-x = -\frac{s}{\sigma_n} - \frac{s^2}{2\sigma_n^2} + O(n^{-3/2}).$$

This, combined with the basic expansion

$$\log\left(\frac{1-x}{\log(1/x)}\right) = -\frac{1}{2}(1-x) - \frac{5}{24}(1-x)^2 + O((1-x)^3),$$

shows that

$$\left(\frac{1-x}{\log(1/x)}\right)^n = \exp\left(\frac{ns}{2\sigma_n} + \frac{ns^2}{24\sigma_n^2} + O(n^{-1/2})\right). \tag{4.10}$$

Take any $\varepsilon > 0$ and find $n(\varepsilon)$ from Lemma 4.5. Then, if $n \geq n(\varepsilon)$,

$$\begin{aligned} H_n &\geq \frac{x^{-\mu_n}(1-x)^n}{(n-1)!}\,(1-\varepsilon) \sum_{i\geq 2} i^{n-1} x^i \\ &\geq (1-\varepsilon)\, x^{1-\mu_n} \left(\frac{1-x}{\log(1/x)}\right)^n && \text{(by Lemma 4.6)} \\ &= (1-\varepsilon) \exp\left(\frac{s(1-\mu_n)}{\sigma_n} + \frac{ns}{2\sigma_n} + \frac{ns^2}{24\sigma_n^2} + O(n^{-1/2})\right) && \text{(by (4.10))} \\ &= (1-\varepsilon) \exp\left(\frac{s(1+O(n^{-1}))}{\sigma_n} + \frac{s^2}{2+O(n^{-3})} + O(n^{-1/2})\right) && \text{(by (4.8))}. \end{aligned}$$

Taking the lim inf as $n \to \infty$ and then letting $\varepsilon \to 0$, we obtain

$$\liminf_{n\to\infty} H_n \geq e^{s^2/2}.$$

Similarly, take any $\varepsilon > 0$, find $i(\varepsilon)$ from Lemma 4.5 and use the basic inequality $\Delta_n(i) \le (i^n - 1)/(i-1) \le 2i^{n-1}$ for all $n, i \ge 2$ to estimate

$$
\begin{aligned}
H_n &= \frac{x^{-\mu_n}(1-x)^n}{(n-1)!}\left(\sum_{2\le i<i(\varepsilon)} + \sum_{i\ge i(\varepsilon)}\right)\Delta_n(i)x^i \\
&\le \frac{2x^{-\mu_n}(1-x)^n}{(n-1)!}\sum_{2\le i<i(\varepsilon)} i^{n-1}x^i + \frac{(1+\varepsilon)x^{-\mu_n}(1-x)^n}{(n-1)!}\sum_{i\ge i(\varepsilon)} i^{n-1}x^i.
\end{aligned}
$$

The first term is a polynomial in x and is easily seen to tend to zero as $n \to \infty$. The second term is bounded above by

$$
\begin{aligned}
&(1+\varepsilon)\,x^{-1-\mu_n}\left(\frac{1-x}{\log(1/x)}\right)^n && \text{(by Lemma 4.6)} \\
=&(1+\varepsilon)\exp\left(\frac{s(-1-\mu_n)}{\sigma_n} + \frac{ns}{2\sigma_n} + \frac{ns^2}{24\sigma_n^2} + O(n^{-1/2})\right) && \text{(by (4.10))} \\
=&(1+\varepsilon)\exp\left(\frac{s(-1+O(n^{-1}))}{\sigma_n} + \frac{s^2}{2+O(n^{-3})} + O(n^{-1/2})\right) && \text{(by (4.8))}
\end{aligned}
$$

Taking the lim sup as $n \to \infty$ and then letting $\varepsilon \to 0$, we obtain

$$\limsup_{n\to\infty} H_n \le e^{s^2/2}.$$

This verifies (4.9) and completes the proof. □

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DEPARTMENT OF MATHEMATICS, QUEENS COLLEGE OF CUNY, 65-30 KISSENA BLVD., QUEENS, NEW YORK 11367, USA

THE GRADUATE CENTER OF CUNY, 365 FIFTH AVE., NEW YORK, NY 10016, USA
E-mail address: saeed.zakeri@qc.cuny.edu