Math 704 midterm exam solutions

Problem 1. Let
$$f(z) = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n^z + n}\right),$$
where, as usual, $n^z = e^{z \log n}$.

(i) Prove that the infinite product converges compactly in the right half-plane $U = \{z \in \mathbb{C} : \mathrm{Re}(z) > 1\}$, so $f \in \mathcal{O}(U)$.

Fix an arbitrary $a > 1$. Suppose $z = x + iy \in U$ with $x = \mathrm{Re}(z) > a$. We have $|n^z| = |e^{z \log n}| = e^{x \log n} = n^x > n^a$, so
$$|n^z + n| \geq |n^z| - n > n^a - n \geq \tfrac{1}{2} n^a$$
for all large enough n (depending on a). Since $\sum 1/n^a$ converges, the Weierstrass M-test shows that $\sum 1/|n^z + n|$ converge uniformly in the half-plane $\{z : \mathrm{Re}(z) > a\}$. Since a was arbitrary, it follows that $\sum 1/|n^z + n|$ converges compactly in U. Theorem 8.15 now shows that the infinite product $f(z) = \prod(1 + 1/(n^z + n))$ converges compactly in U, hence $f \in \mathcal{O}(U)$.

(ii) Show that the vertical line $\partial U = \{z \in \mathbb{C} : \mathrm{Re}(z) = 1\}$ is the natural boundary of f by verifying that every point of ∂U is an accumulation point of zeros of f.

Let us find the zeros of f explicitly. By Theorem 8.15, $f(z) = 0$ if and only if $1/(n^z + n) = -1$ or $n^z = -(n+1)$ for some $n \geq 1$. Writing $z = x + iy$ and taking the real and imaginary parts of each side of $n^z = -(n+1)$ gives the pair of equations
$$\begin{cases} n^x \cos(y \log n) = -(n+1) \\ n^x \sin(y \log n) = 0. \end{cases}$$
There is no solution in x if $n = 1$, so let us assume $n \geq 2$. In this case the second equation implies $\sin(y \log n) = 0$ so $y = k\pi/\log n$ for some $k \in \mathbb{Z}$. Plugging this into the first equation gives $n^x(-1)^k = -(n+1)$ or $n^x = (-1)^{k+1}(n+1)$, which implies k is odd and $x = \log(n+1)/\log n$. Thus, the set of zeros of f consists of all complex numbers of the form
$$z_{n,k} = \frac{\log(n+1)}{\log n} + i\, \frac{(2k+1)\pi}{\log n} \quad \text{with} \quad n \geq 2, k \in \mathbb{Z}.$$
Since $\mathrm{Re}(z_{n,k}) = \log(n+1)/\log n$ tends to 1 as $n \to \infty$, and since the vertical separation between $z_{n,k}$ and $z_{n,k+1}$ which is $2\pi/\log n$ tends to 0 as $n \to \infty$, it follows that every point of the line $\mathrm{Re}(z) = 1$ is accumulated by a sequence of zeros $z_{n,k}$. Explicitly (if you prefer a strictly formal argument), given $y_0 \in \mathbb{R}$ there is a sequence $\{k_n\}$ of integers such that $(2k_n + 1)\pi/\log n \to y_0$ as $n \to \infty$: simply choose k_n to

be the integer part of $y_0 \log n/(2\pi)$. Then the subsequence $\{z_{n,k_n}\}$ of the zeros will converge to $1 + iy_0$ as $n \to \infty$.

Problem 2. Does there exist a bounded $f \in \mathcal{O}(\mathbb{D})$ such that

$$f\Big(1 - \frac{1}{n}\Big) = \frac{(-1)^n}{n} \qquad \text{for all integers } n \geq 1? \tag{1}$$

The answer is no. Suppose there is such f. Then the function $g(z) = f(z) + z - 1$ is bounded and holomorphic in $\mathbb{D}$ and by (1),

$$g\Big(1 - \frac{1}{2k}\Big) = f\Big(1 - \frac{1}{2k}\Big) - \frac{1}{2k} = 0 \qquad \text{for all } k \geq 1.$$

Since $\sum 1/(2k) = \infty$, Theorem 8.34 (see also Example 8.35) implies that g is identically zero, that is, $f(z) = 1 - z$ everywhere in $\mathbb{D}$. But then

$$f\Big(1 - \frac{1}{2k+1}\Big) = \frac{1}{2k+1} \qquad \text{for all } k \geq 1,$$

which contradicts (1) for odd n.

Problem 3. Let f be a meromorphic function in $\mathbb{C}$ with no poles on the real line. Show that we can decompose f as $g + h$, where g, h are meromorphic in $\mathbb{C}$, with all poles of g in the upper half-plane and all poles of h in the lower half-plane. To what extent is such decomposition unique?

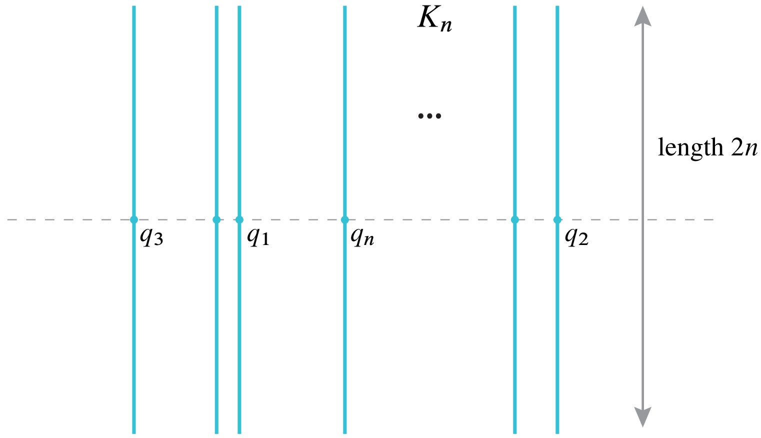
Let E^+ and E^- be the set of poles of f in the upper and lower half-planes, respectively. By Mittag-Leffler's Theorem 9.1, there is a meromorphic function g in the plane which has poles precisely along E^+, with the principal parts equal to those of f. The meromorphic function $h = f - g$ has removable singularities at every point of E^+ and has poles precisely along E^- with the principal parts equal to those of f since g is holomorphic in the lower half-plane.

It is clear that if g, h give such decomposition of f, so do $g + \phi, h - \phi$ for any $\phi \in \mathcal{O}(\mathbb{C})$. Conversely, suppose g, h and $\tilde{g}, \tilde{h}$ are two such decompositions of f. Then $g + h = f = \tilde{g} + \tilde{h}$, so $\tilde{g} - g = h - \tilde{h}$ everywhere in the plane. Since the left side is holomorphic in the lower half-plane and the right side is holomorphic in the upper half-plane, each side is an entire function. Thus, $\tilde{g} = g + \phi, \tilde{h} = h - \phi$ for some $\phi \in \mathcal{O}(\mathbb{C})$.

Problem 4. Let $\mathbb{Q} \subset \mathbb{R}$ denote the countable set of rational numbers. Show that there exists a sequence $\{P_n\}$ of polynomials such that for every $q \in \mathbb{Q}$, $P_n \to q$ uniformly on compact subsets of the line $\mathrm{Re}(z) = q$ as $n \to \infty$.

Let $\{q_n\}_{n \geq 1}$ be an enumeration of $\mathbb{Q}$. For each integer $n \geq 1$, let K_n be the union of closed vertical segments in $\mathbb{C}$ of length $2n$ centered at $q_1, \ldots, q_n$:

$$K_n = \{q_j + it : |t| \leq n \text{ and } 1 \leq j \leq n\}.$$



Evidently each K_n is a full compact set and $K_n \subset K_{n+1}$. Define $f_n : K_n \to \mathbb{C}$ by setting it equal to the constant function q_j on the vertical segment in K_n centered at q_j:

$$f_n(q_j + it) = q_j \qquad \text{if } |t| \le n \text{ and } 1 \le j \le n.$$

Then f_n extends holomorphically to an open set containing K_n (simply take disjoint open neighborhoods of each vertical segment in K_n and extend f_n as a constant function in each neighborhood). By the special case of Runge's Theorem, Corollary 9.19, there is a polynomial P_n such that $\sup_{z \in K_n} |P_n(z) - f_n(z)| < 1/n$. The sequence $\{P_n\}$ obtained this way has the desired property. In fact, fix some $q \in \mathbb{Q}$ and a compact subset K of the vertical line $\mathrm{Re}(z) = q$. Take $N \ge 1$ large enough to guarantee $q \in \{q_1, \ldots, q_N\}$ and $K \subset \{q + it : |t| \le N\}$. Then $K \subset K_n$ for all $n \ge N$. Hence, for all $z \in K$ and all $n \ge N$,

$$|P_n(z) - q| = |P_n(z) - f_n(z)| < \frac{1}{n}.$$

This shows $P_n \to q$ uniformly on K as $n \to \infty$.

Problem 5. Consider the first quadrant $U = \{x + iy \in \mathbb{C} : x > 0, y > 0\}$. Suppose $f \in \mathcal{O}(U)$ has the property that $|f(z)| \to 1$ as z tends to any point of the boundary ∂U. Show that f extends to a meromorphic function in $\mathbb{C}$ which is even and satisfies $f(z)\overline{f(\overline{z})} = 1$ for all $z \in \mathbb{C}$. Conclude that $|f(0)| = 1$ and $\mathrm{Re}(f''(0)/f(0)) = 0$.

We use Schwarz reflection twice to extend f to all of $\mathbb{C}$. Since $|f(z)| \to 1$ as z tends to any point of $(0, +\infty)$, we can apply the Schwarz reflection principle to extend f to the right half-plane $V = \{x + iy \in \mathbb{C} : x > 0\}$ by using the reflection $z \mapsto \overline{z}$ in the domain and $z \mapsto 1/\overline{z}$ in the target. Hence this extension (still denoted by f) satisfies

$$f(\overline{z}) = \frac{1}{\overline{f(z)}} \qquad \text{for all } z \in V. \tag{2}$$

If the original f has a zero at $p \in U$, the extended f has a pole at $\overline{p}$. Thus, f is meromorphic in V, or can be thought of as a holomorphic map $f : V \to \widehat{\mathbb{C}}$.

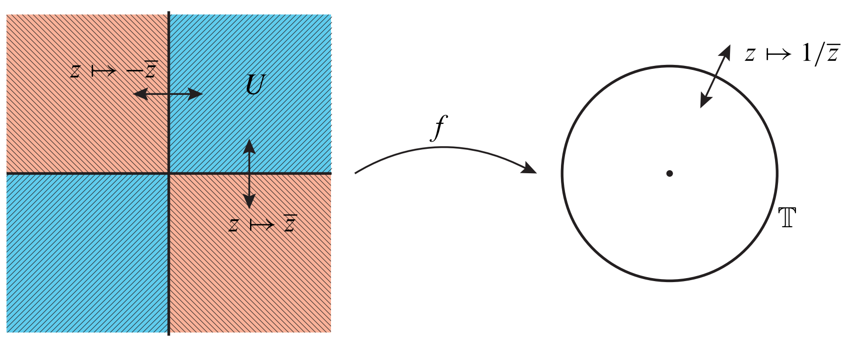
Now the map $f : V \to \widehat{\mathbb{C}}$ still has the property that $|f(z)| \to 1$ when z tends to any point of $\partial V = \{x + iy \in \mathbb{C} : x = 0\}$. Thus, we can apply the Schwarz reflection principle once more to extend f to the whole plane, using $z \mapsto -\overline{z}$ in the domain and $z \mapsto 1/\overline{z}$ in the target. The resulting extension $f : \mathbb{C} \to \widehat{\mathbb{C}}$ satisfies

$$f(-\overline{z}) = \frac{1}{\overline{f(z)}} \qquad \text{for all } z \in \mathbb{C}. \tag{3}$$

Observe that since $f : V \to \widehat{\mathbb{C}}$ satisfies (2) in V, the extension $f : \mathbb{C} \to \widehat{\mathbb{C}}$ must satisfy (2) in all of $\mathbb{C}$ by the identity theorem:

$$f(\overline{z}) = \frac{1}{\overline{f(z)}} \qquad \text{for all } z \in \mathbb{C}. \tag{4}$$

Evidently (3) and (4) imply that $f(-z) = f(z)$ and $f(z)\overline{f(\overline{z})} = 1$ for all z.



Let us verify the remaining claim. Since f is holomorphic in a neighborhood of the origin, it has a power series representation $f(z) = a_0 + a_2 z^2 + a_4 z^4 + \cdots$ will all odd powers missing because f is an even function. Here $a_n = f^{(n)}(0)/n!$ and we need to verify that $|a_0| = 1$ and $\operatorname{Re}(a_2/a_0) = 0$. Imposing the condition $f(z)\overline{f(\overline{z})} = 1$ gives

$$(a_0 + a_2 z^2 + a_4 z^4 + \cdots)(\overline{a_0} + \overline{a_2} z^2 + \overline{a_4} z^4 + \cdots) = 1$$

for all z sufficiently close to 0. In particular,

$$a_0\overline{a_0} = 1 \qquad \text{and} \qquad a_0\overline{a_2} + a_2\overline{a_0} = 0.$$

The first relation gives $|a_0| = 1$ (of course we could have deduced this from the fact that the extended f has to send $0 \in \partial U$ to some point on the unit circle). The second relation gives

$$\operatorname{Re}(a_2\overline{a_0}) = \operatorname{Re}\left(\frac{a_2}{a_0}\right) = 0,$$

as required.