Here are the solutions to practice problems 1-4 that we didn't have time to go over today:

1. In each case, find the derivative $y' = dy/dx$:

- $y = \sin^{-1}(x + e^x)$

  By the chain rule,

  $$y' = \frac{1}{\sqrt{1-(x+e^x)^2}} \cdot (x+e^x)' = \frac{1+e^x}{\sqrt{1-(x+e^x)^2}}.$$

- $y = \sqrt{\ln(\cos x)}$

  Writing $y = (\ln(\cos x))^{1/2}$ and applying the chain rule twice, we obtain

  $$\begin{aligned} y' &= \frac{1}{2}(\ln(\cos x))^{-1/2} \cdot (\ln(\cos x))' = \frac{1}{2}(\ln(\cos x))^{-1/2} \cdot \frac{1}{\cos x} \cdot (\cos x)' \\ &= \frac{1}{2}(\ln(\cos x))^{-1/2} \cdot \frac{-\sin x}{\cos x} = \frac{-\tan x}{2\sqrt{\ln(\cos x)}}. \end{aligned}$$

- $y = x^{\tan x}$

  We use logarithmic differentiation:

  $$\ln y = \tan x \cdot \ln x \Longrightarrow \frac{y'}{y} = \sec^2 x \cdot \ln x + \tan x \cdot \frac{1}{x}.$$

  It follows that

  $$y' = \left(\sec^2 x \cdot \ln x + \frac{\tan x}{x}\right) y = \left(\sec^2 x \cdot \ln x + \frac{\tan x}{x}\right) x^{\tan x}.$$

2. Verify that the function $f(x) = \ln x + 2x^3$ is strictly increasing and therefore one-to-one in the interval $(0, \infty)$. If f^{-1} denotes the inverse of f, find the value of $(f^{-1})'(2)$.

First note that for $x > 0$,

$$f'(x) = \frac{1}{x} + 6x^2 > 0.$$

Since the derivative f' is positive, f must be strictly increasing and therefore one-to-one in the interval $(0, \infty)$. To find the derivative of the inverse function f^{-1} at 2, note that $f(1) = 2$ so $f^{-1}(2) = 1$. Hence

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{7}.$$

3. Find $\lim_{x\to 0} \dfrac{e^{x^2} - \cos x}{x^2}$.

Method 1. L'Hospital's rule:

$$\lim_{x\to 0} \frac{e^{x^2} - \cos x}{x^2} = \lim_{x\to 0} \frac{2x\, e^{x^2} + \sin x}{2x} = \lim_{x\to 0} \frac{2\, e^{x^2} + 2x \cdot 2x\, e^{x^2} + \cos x}{2} = \frac{3}{2}.$$

Method 2. Power series:

$$\begin{aligned}
\lim_{x\to 0} \frac{e^{x^2} - \cos x}{x^2} &= \lim_{x\to 0} \frac{\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)}{x^2} \\
&= \lim_{x\to 0} \frac{\left(1 + \frac{1}{2!}\right)x^2 + \left(\frac{1}{2!} - \frac{1}{4!}\right)x^4 + \left(\frac{1}{3!} + \frac{1}{6!}\right)x^6 + \cdots}{x^2} \\
&= \lim_{x\to 0} \left(1 + \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{4!}\right)x^2 + \left(\frac{1}{3!} + \frac{1}{6!}\right)x^4 + \cdots \\
&= \left(1 + \frac{1}{2!}\right) = \frac{3}{2}.
\end{aligned}$$

4. Evaluate the following integrals:

- $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^2}$

  we use the substitution $u = \ln x$, $du = \dfrac{dx}{x}$ to write

  $$\int \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{\ln x}.$$

  Thus,

  $$\begin{aligned}
  \int_2^\infty \frac{dx}{x(\ln x)^2} &= \lim_{R\to\infty} \int_2^R \frac{dx}{x(\ln x)^2} = \lim_{R\to\infty} -\frac{1}{\ln x}\Big|_2^R \\
  &= \lim_{R\to\infty} \left(-\frac{1}{\ln R} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.
  \end{aligned}$$

- $\displaystyle\int \frac{\ln x}{x^3}\, dx$

  We use integration by parts with

  $$\begin{cases} u = \ln x & v' = \dfrac{1}{x^3} \\ u' = \dfrac{1}{x} & v = -\dfrac{1}{2x^2} \end{cases}$$

to obtain

$$\int \underbrace{\frac{\ln x}{x^3}}_{uv'}\,dx = \underbrace{-\frac{\ln x}{2x^2}}_{uv} - \int \underbrace{\frac{-1}{2x^3}}_{u'v}\,dx$$

$$= -\frac{\ln x}{2x^2} + \frac{1}{2}\int \frac{1}{x^3}\,dx = -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C.$$

- $\displaystyle\int \frac{3x-1}{x^2+3x-10}\,dx$

  We use partial fractions. Write

  $$\frac{3x-1}{x^2+3x-10} = \frac{3x-1}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2},$$

  so the identity

  $$3x-1 = A(x-2) + B(x+5)$$

  must hold for all x. Setting $x = -5$ gives $-16 = -7A$ or $A = 16/7$. Setting $x = 2$ gives $5 = 7B$ or $B = 5/7$. Thus,

  $$\frac{3x-1}{x^2+3x-10} = \frac{16}{7}\frac{1}{x+5} + \frac{5}{7}\frac{1}{x-2},$$

  which shows

  $$\int \frac{3x-1}{x^2+3x-10}\,dx = \frac{16}{7}\int \frac{1}{x+5}\,dx + \frac{5}{7}\int \frac{1}{x-2}\,dx$$

  $$= \frac{16}{7}\ln|x+5| + \frac{5}{7}\ln|x-2| + C.$$