Math 704 Problem Set 5 Solutions

Problem 1. Let K be a compact subset of the unit circle $\mathbb{T}$. If $K = \mathbb{T}$, every polynomial P satisfies $|P(0)| \leq \sup_{z \in K} |P(z)|$ by the maximum principle. If $K \neq \mathbb{T}$, this can fail dramatically. Show that in this case for every $\varepsilon > 0$ there is a polynomial P with $P(0) = 1$ such that $\sup_{z \in K} |P(z)| < \varepsilon$.

If K is a proper subset of $\mathbb{T}$, then $\widehat{\mathbb{C}} \smallsetminus K$ is clearly path-connected, hence connected. In other words, K is full. By the special case of Runge's theorem (Corollary 9.19), the function $f(z) = 1/z$ in $\mathcal{O}(\mathbb{C}^*)$ can be uniformly approximated on K by polynomials. Thus, for any $\varepsilon > 0$ there is a polynomial Q such that $\sup_{z \in K} |1/z - Q(z)| < \varepsilon$. Since $|z| = 1$ for every $z \in K$, it follows that $\sup_{z \in K} |1 - zQ(z)| < \varepsilon$. The polynomial $P(z) = 1 - zQ(z)$ satisfies the desired properties.

Problem 2. Show that there is a sequence $\{P_n\}$ of polynomials such that

$$\lim_{n \to \infty} P_n(z) = \begin{cases} 1 & \text{if } \operatorname{Im}(z) > 0 \\ 0 & \text{if } \operatorname{Im}(z) = 0 \\ -1 & \text{if } \operatorname{Im}(z) < 0. \end{cases}$$

Can you achieve the additional property

$$|P_n(z)| \leq 1 \qquad \text{for every } z \in \mathbb{D} \text{ and } n \geq 1?$$

For $n \geq 2$ define

$$\begin{aligned} X_n &= \{z \in \mathbb{C} : |z| \leq n \text{ and } \operatorname{Im}(z) \geq 1/n\} \\ Y_n &= \{z \in \mathbb{C} : |z| \leq n \text{ and } \operatorname{Im}(z) \leq -1/n\} \\ K_n &= X_n \cup Y_n \cup [-n, n]. \end{aligned}$$

Each K_n is compact and full, with $K_n \subset K_{n+1}$ and $\bigcup_{n \geq 1} K_n = \mathbb{C}$. Define $f_n : K_n \to \mathbb{C}$ by

$$f_n = \begin{cases} 1 & \text{on } X_n \\ 0 & \text{on } [-n, n] \\ -1 & \text{on } Y_n. \end{cases}$$

Since f_n extends holomorphically to a neighborhood of K_n, Runge's theorem shows that there is a polynomial P_n such that $\sup_{z \in K_n} |f_n(z) - P_n(z)| < 1/n$. Evidently the sequence $\{P_n\}$ has the desired property and even more: $P_n \to 1$ compactly in the upper half-plane $\mathbb{H}$, $P_n \to -1$ compactly in the lower half-plane $-\mathbb{H}$, and $P_n \to 0$ compactly in $\mathbb{R}$.

We can never find such a sequence $\{P_n\}$ which satisfies the uniform bound $|P_n(z)| \leq 1$ for all $z \in \mathbb{D}$ and all $n \geq 1$. If we could, Montel's theorem would imply that a subsequence of $\{P_n\}$ converges compactly in $\mathbb{D}$ to a holomorphic function. This would be a contradiction since such a limit, being 1 in $\mathbb{D} \cap \mathbb{H}$ and -1 in $\mathbb{D} \cap -\mathbb{H}$, is discontinuous in $\mathbb{D}$.

Problem 3. Is there a sequence of polynomials which tends to 0 compactly in the upper half-plane but does not have a limit at any point of the lower half-plane?

The answer is yes. With X_n, Y_n as in the previous problem, define $K_n = X_n \cup Y_n$. Again, each K_n is compact and full, with $K_n \subset K_{n+1}$ and $\bigcup_{n \geq 1} K_n = \mathbb{C} \smallsetminus \mathbb{R}$. Define $f_n : K_n \to \mathbb{C}$ by

$$f_n = \begin{cases} 0 & \text{on } X_n \\ n & \text{on } Y_n. \end{cases}$$

Since f_n extends holomorphically to a neighborhood of K_n, Runge's theorem shows that there is a polynomial P_n such that $\sup_{z \in K_n} |f_n(z) - P_n(z)| < 1/n$. It is easy to see that $P_n \to 0$ compactly in the upper half-plane, and $P_n \to \infty$ in the lower half-plane.

We have the flexibility to make the behavior in the lower half-plane even more erratic. As an example, take any countable dense set $\{w_n\}$ in $\mathbb{C}$, define $f_n : K_n \to \mathbb{C}$ to be 0 on X_n and w_n on Y_n, and find polynomials P_n such that $\sup_{z \in K_n} |f_n(z) - P_n(z)| < 1/n$. Then, at each point z in the lower half-plane, the sequence $\{P_n(z)\}$ is dense in $\mathbb{C}$.

Problem 4. Deduce Mittag-Leffler's Theorem 9.4 for an open set U from Runge's Theorem 9.18 by completing the following outline: Let $\emptyset = K_0 \subset K_1 \subset K_2 \subset \cdots$ be a nice exhaustion of U. For $n \geq 1$, let Q_n be the finite sum of the principal parts $P_k(1/(z - z_k))$ over all k such that $z_k \in K_n \smallsetminus K_{n-1}$. For each $n \geq 2$, find a rational function R_n with poles outside U such that $|Q_n - R_n| \leq 2^{-n}$ on K_{n-1}. Show that $f = Q_1 + \sum_{n=2}^{\infty}(Q_n - R_n)$ is a meromorphic function in U with the principal part $P_k(1/(z - z_k))$ at each z_k, and with no other poles.

Following the suggested outline, consider a nice exhaustion $\{K_n\}$ of U and define Q_n accordingly. Notice that Q_n is holomorphic in a neighborhood of K_{n-1} since its poles are outside K_{n-1}. By Runge's theorem, for each $n \geq 2$ there is a rational function R_n with poles outside K_{n-1} such that $|Q_n - R_n| \leq 2^{-n}$ on K_{n-1}. Since every component of $\widehat{\mathbb{C}} \smallsetminus K_{n-1}$ contains a component of $\widehat{\mathbb{C}} \smallsetminus U$, we can arrange the poles of R_n to be outside U. Set $f = Q_1 + \sum_{n=2}^{\infty}(Q_n - R_n)$. For any $N \geq 2$, the rational function Q_n, hence $Q_n - R_n$, is holomorphic in a neighborhood of K_N whenever $n \geq N + 1$. Since

$$\sum_{n=N+1}^{\infty} |Q_n - R_n| \leq \sum_{n=N+1}^{\infty} 2^{-n} < +\infty \qquad \text{on } K_N,$$

the Weierstrass M-test shows that the series $\sum_{n=N+1}^{\infty}(Q_n - R_n)$ converges to a holomorphic function g in the interior of K_N and $f = g + Q_1 + \sum_{n=2}^{N}(Q_n - R_n)$ is meromorphic there. Now $f - \sum_{n=1}^{N} Q_n = g - \sum_{n=2}^{N} R_n$ is holomorphic in the interior of K_N, so the principal parts of f in this interior are those given by $Q_1, \ldots, Q_N$. Since this is true for every $N \geq 2$, we conclude that f is meromorphic in U with the prescribed principal parts.

Problem 5. Let $K \subset \mathbb{C}$ be compact and connected. Show that every connected component of $\widehat{\mathbb{C}} \setminus K$ is simply connected.

For simplicity we denote the complement $\widehat{\mathbb{C}} \setminus X$ by X^c. Let Ω be a component of K^c. We need to prove that Ω is simply connected. By Theorem 9.27 it suffices to show that Ω^c is connected. If Ω is the only component of K^c, then $\Omega^c = K$ is connected and we are done. Otherwise, let $\Omega_1, \Omega_2, \ldots$ be the (finite or countably infinite collection of) components of K^c other than Ω. Since $\partial\Omega_j \subset K$, we have

$$\Omega^c = K \cup \bigcup_j \Omega_j = K \cup \bigcup_j \overline{\Omega_j}.$$

Each closure $\overline{\Omega_j}$ is connected since Ω_j is, and $\overline{\Omega_j} \cap K = \partial\Omega_j \neq \emptyset$. It follows inductively that the unions

$$K, \ K \cup \overline{\Omega_1}, \ K \cup \overline{\Omega_1} \cup \overline{\Omega_2}, \ \ldots$$

are all connected. So the same must be true of their union Ω^c.[1]

Problem 6. Let $U, V \subset \mathbb{C}$ be simply connected domains with $U \cap V \neq \emptyset$. Show that every connected component of $U \cap V$ is simply connected.

This can be easily reduced to the previous problem by setting $K = U^c \cup V^c$. Notice that in the previous problem we assumed K to be a compact subset of $\mathbb{C}$, but the statement holds when K is any compact subset of $\widehat{\mathbb{C}}$: simply rotate the sphere to place ∞ off of K.

So let $K = U^c \cup V^c$, which is a compact subset of $\widehat{\mathbb{C}}$ containing ∞. By simple connectivity of U and V, both U^c and V^c are connected and $\infty \in U^c \cap V^c$, so the union $K = U^c \cup V^c$ must be connected. Thus, by the previous problem, every component of $K^c = U \cap V$ must be simply connected.

Alternative proof. Let Ω be a component of $U \cap V$. By Theorem 9.27 it suffices to prove that $H_1(\Omega) = 0$. Take an arbitrary cycle γ in Ω and any $p \in \mathbb{C} \setminus \Omega$. We need to show that $\mathrm{W}(\gamma, p) = 0$. If $p \in U^c \cup V^c$, then $\mathrm{W}(\gamma, p) = 0$ since $H_1(U) = H_1(V) = 0$ by simple connectivity of U and V. If p is in a component Ω_j of $U \cap V$ other than Ω, we can find a $q \in \partial\Omega_j$ in the same component of $\mathbb{C} \setminus |\gamma|$ as p; this is because the connected set $\overline{\Omega_j}$ lies in a component of $\mathbb{C} \setminus |\gamma|$. Thus, $\mathrm{W}(\gamma, p) = \mathrm{W}(\gamma, q) = 0$ by the first case since $q \in \partial\Omega_j \subset U^c \cup V^c$.

[1] Here we have used an elementary result in topology: If $\{X_\alpha\}$ is any collection of connected sets in a topological space with $\bigcap_\alpha X_\alpha \neq \emptyset$, then $\bigcup_\alpha X_\alpha$ is connected.