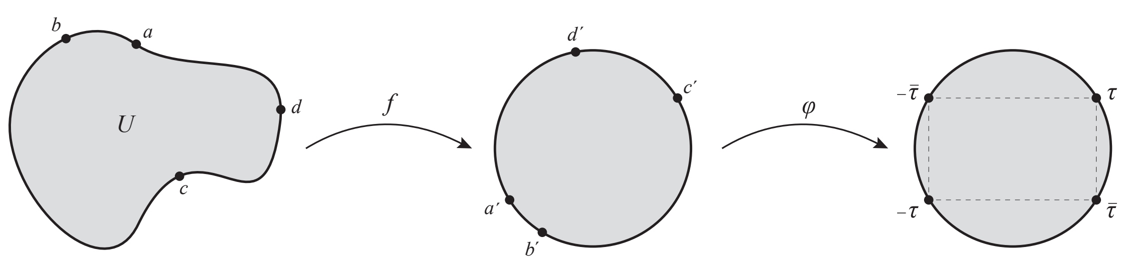
Math 704 Problem Set 8 Solutions

Problem 1. Let $\triangle p_1p_2p_3$ denote the closed triangle (interior and boundary) with vertices p_1, p_2, p_3 labeled counterclockwise. Show that for any two triangles $\triangle p_1p_2p_3$ and $\triangle q_1q_2q_3$ there exists a unique homeomorphism $f : \triangle p_1p_2p_3 \to \triangle q_1q_2q_3$ with $f(p_i) = q_i$ for $i = 1, 2, 3$ which is a biholomorphism between the interiors.

By the Riemann mapping theorem, there are conformal maps g and h sending the interiors of $\triangle p_1p_2p_3$ and $\triangle q_1q_2q_3$, respectively, to $\mathbb{D}$. By Carathéodory's theorem, both maps extend to homeomorphisms $g : \triangle p_1p_2p_3 \to \overline{\mathbb{D}}$ and $h : \triangle q_1q_2q_3 \to \overline{\mathbb{D}}$. The triples $g(p_1), g(p_2), g(p_3)$ and $h(q_1), h(q_2), h(q_3)$ appear in counterclockwise order on the unit circle $\mathbb{T} = \partial\mathbb{D}$, so there is a unique $\varphi \in \mathrm{Aut}(\mathbb{D})$ which maps $g(p_i)$ to $h(q_i)$ for $i = 1, 2, 3$. The composition $f = h^{-1} \circ \varphi \circ g$ satisfies the required properties. Uniqueness of f is immediate since if $\hat{f}$ also has the required properties, then $\hat{\varphi} = h \circ \hat{f} \circ g^{-1}$ will be an element of $\mathrm{Aut}(\mathbb{D})$ mapping $g(p_i)$ to $h(q_i)$ for $i = 1, 2, 3$, so $\hat{\varphi} = \varphi$ and $\hat{f} = f$.

Problem 2. Let $U \subset \mathbb{C}$ be a simply connected domain bounded by a Jordan curve and (a, b, c, d) be an ordered quadruple of points on ∂U chosen in counterclockwise direction. Show that there is a conformal map $f : U \to \mathbb{D}$ which sends (a, b, c, d) to the vertices of a rectangle inscribed in $\mathbb{D}$, and that f is unique up to a rotation of the disk about 0.

By Carathéodory's theorem, any conformal map $U \to \mathbb{D}$ extends to a homeomorphism $\overline{U} \to \overline{\mathbb{D}}$, so it maps (a, b, c, d) to a quadruple (a', b', c', d') on $\mathbb{T}$. It suffices to show that there is a $\varphi \in \mathrm{Aut}(\mathbb{D})$ which sends (a', b', c', d') to a quadruple of the form $(\tau, -\bar{\tau}, -\tau, \bar{\tau})$ with $\tau = e^{i\theta}$ for a unique $0 < \theta < \pi/2$.



Recall from Theorem 4.7 that the cross-ratio

$$[a, b, c, d] = \frac{(c-a)(d-b)}{(b-a)(d-c)}$$

is real if and only if a, b, c, d lie on a circle, and two quadruples can be mapped to each other by a Möbius transformation if and only if they have the same cross-ratio. The cross-ratio of (a', b', c', d') on $\mathbb{T}$ is a real number in $(1, +\infty)$ since (a', b', c', d') can be mapped by a Möbius transformation to $(0, 1, x, \infty)$ on $\hat{\mathbb{R}}$ for some $1 < x < +\infty$ and $[0, 1, x, \infty] = x$.

On the other hand,

$$[\tau, -\bar{\tau}, -\tau, \bar{\tau}] = \frac{(-2\tau)(2\bar{\tau})}{(-\bar{\tau} - \tau)(\bar{\tau} + \tau)} = \frac{-4}{-(2\,\mathrm{Re}(\tau))^2} = \frac{1}{\cos^2\theta}.$$

Hence there is a unique $0 < \theta < \pi/2$ such that $[a', b', c', d'] = [\tau, -\bar{\tau}, -\tau, \bar{\tau}]$. If φ is the unique Möbius transformation that maps (a', b', c', d') to $(\tau, -\bar{\tau}, -\tau, \bar{\tau})$, then $\varphi(\mathbb{T}) = \mathbb{T}$ and it follows by considering orientation that $\varphi(\mathbb{D}) = \mathbb{D}$.

Problem 3. Let S be the unit square $\{x + iy \in \mathbb{C} : 0 < x, y < 1\}$, U be a bounded simply connected domain in $\mathbb{C}$, and $f : S \to U$ be a conformal map. For each $y \in (0, 1)$, let $L(y)$ denote the length of the curve $\gamma_y : (0, 1) \to \mathbb{C}$ defined by $\gamma_y(x) = f(x + iy)$. Use the length-area method to verify the following:

(i) $L(y)$ is finite and therefore γ_y lands on both ends for a.e. $y \in (0, 1)$.

By the Cauchy-Schwarz inequality,

$$L^2(y) = \left(\int_0^1 |f'(x + iy)|\, dx\right)^2 \le \int_0^1 |f'(x + iy)|^2\, dx,$$

hence

$$\int_0^1 L^2(y)\, dy \le \int_0^1 \int_0^1 |f'(x + iy)|^2\, dx\, dy = \mathrm{area}(U) < +\infty.$$

Thus, $L(y) < +\infty$ for a.e. $y \in (0, 1)$. It follows that for each such y both limits $\lim_{x\to 0^+} \gamma_y(x)$ and $\lim_{x\to 1^-} \gamma_y(x)$ exist (Lemma 6.23).

(ii) The majority of the γ_y aren't too long: The measure of the set of $y \in (0, 1)$ for which $L(y) \le \sqrt{2\,\mathrm{area}(U)}$ is at least $1/2$.

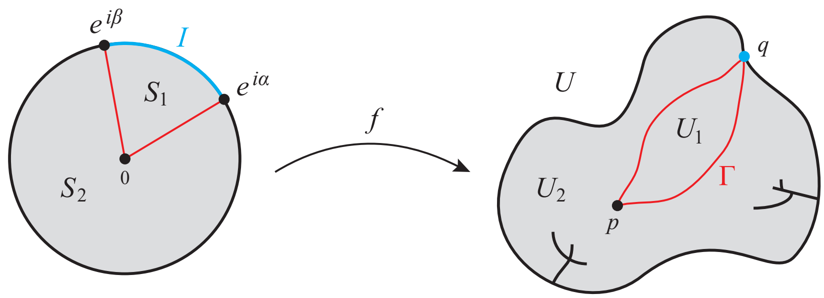
Let $E \subset (0, 1)$ be the set of y for which $L(y) \le \sqrt{2\,\mathrm{area}(U)}$. We want to prove $m(E) \ge 1/2$, where m denotes Lebesgue measure on $\mathbb{R}$. By the inequality in part (i),

$$\mathrm{area}(U) \ge \int_{[0,1]} L^2(y)\, dy \ge \int_{[0,1]\smallsetminus E} L^2(y)\, dy \ge 2\,\mathrm{area}(U)\, m([0, 1] \smallsetminus E).$$

This gives $m([0, 1] \smallsetminus E) \le 1/2$, or $m(E) \ge 1/2$.

Problem 4. Let $f : \mathbb{D} \to U$ be a conformal map. Suppose the radial limits $f^*(e^{i\alpha})$ and $f^*(e^{i\beta})$ exist and are equal for some $0 \le \alpha < \beta < 2\pi$. Show that the domain bounded by the curves $r \mapsto f(re^{i\alpha})$ and $r \mapsto f(re^{i\beta})$ for $0 \le r \le 1$ cannot be contained in U.

The curves $r \mapsto f(re^{i\alpha})$ and $r \mapsto f(re^{i\beta})$ have the same initial point $p = f(0)$ and end point $q = f^*(e^{i\alpha}) = f^*(e^{i\beta})$ and are otherwise disjoint. Thus, their union is a Jordan curve Γ. The disk $\mathbb{D}$ with the two radial segments $[0, e^{i\alpha})$ and $[0, e^{i\beta})$ removed has two connected components (sectors) S_1, S_2 which map conformally to the two connected components U_1, U_2 of $U \smallsetminus \Gamma$. Assume by way of contradiction that the domain in $\mathbb{C}$



bounded by Γ is contained in U. Then we may suppose, after relabeling, that this domain is $U_1 = f(S_1)$. Consider the arc $I = \partial S_1 \cap \mathbb{T}$ on the unit circle. For every sequence $\{z_n\}$ in S_1 which converges to a point of I, the image sequence $\{f(z_n)\}$ is in U_1 but can only accumulate on ∂U. Since $\overline{U_1} \cap \partial U = \Gamma \cap \partial U = \{q\}$, it follows that $f(z_n) \to q$. In other words, f extends continuously to $\mathbb{D} \cup I$ and maps I to the singleton $\{q\}$. By the Schwarz reflection principle, f can be extended holomorphically across I and the identity theorem implies that f is the constant function q. Contradiction!

Problem 5. Show that the function

$$f(z) = \exp\left(\frac{z+1}{z-1}\right)$$

is bounded and holomorphic in $\mathbb{D}$, and $f(z) \to 0$ as $z \to 1$ radially. However, for every $a \in \mathbb{D}$ there exists a sequence $z_n \to 1$ such that $f(z_n) \to a$.

We have $f = \exp \circ \phi$, where the Möbius transformation $\phi(z) = (z+1)/(z-1)$ maps $\mathbb{D}$ conformally onto the left half-plane $U = \{w : \mathrm{Re}(w) < 0\}$. Thus f maps $\mathbb{D}$ holomorphically onto the punctured disk $\exp(U) = \mathbb{D}^*$.

As $r \to 1^-$, $\phi(r) \to -\infty$, so $f(r) \to 0$. On the other hand, let $a \in \mathbb{D}^*$ and take any $w \in U$ with $\exp(w) = a$. Then $w_n = w + 2\pi i n \in U$ satisfies $\exp(w_n) = a$ for every $n \geq 1$. Evidently, the sequence

$$z_n = \phi^{-1}(w_n) = \frac{b + 2\pi i n + 1}{b + 2\pi i n - 1}$$

in $\mathbb{D}$ satisfies $z_n \to 1$ and $f(z_n) = f(\phi^{-1}(w_n)) = \exp(w_n) = a$ for every n.

Comment. Since ϕ extends continuously to $\mathbb{T} \setminus \{1\}$, the radial limit $f^*(e^{it}) = \lim_{r\to 1} f(re^{it})$ exists for *every* t, and $|f^*(e^{it})| = 1$ if $e^{it} \neq 1$. Note also that the sequence $\{z_n\}$ constructed above tends to 1 *tangentially* (in fact, along a circle that is tangent to $\mathbb{T}$ at 1). If we consider any sequence $\{z_n\}$ that tends to 1 *non-tangentially* in the sense that $|1 - z_n|/(1 - |z_n|)$ is bounded (geometrically, it means z_n stays in a cone of angle $< \pi$ at 1), then $\lim_{n\to\infty} f(z_n) = f^*(1) = 0$.

Problem 6. Define $f, g \in \mathcal{O}(\mathbb{D})$ by

$$g(z) = \exp\left(\frac{1+z}{1-z}\right) \qquad \text{and} \qquad f(z) = (1-z)\exp(-g(z)).$$

Prove that the radial limit $f^*(e^{it}) = \lim_{r\to 1} f(re^{it})$ exists everywhere and defines a continuous function on $\mathbb{T}$. However, f is not even bounded in $\mathbb{D}$. Why doesn't this contradict the maximum principle?

First notice that f is indeed holomorphic in $\mathbb{C} \smallsetminus \{1\}$, so it has a continuous extension to $\mathbb{T} \smallsetminus \{1\}$. In particular, if $f^*(e^{it}) = f(e^{it})$ exists if $e^{it} \neq 1$. There is no continuous extension at $z = 1$ since this point is an essential singularity of f. However, the radial limit $f^*(1)$ still exists. In fact, the radial segment $[0, 1)$ maps under $\phi(z) = (1+z)/(1-z)$ to the segment $[1, +\infty)$, so $\lim_{r\to 1} g(r) = +\infty$ and $\lim_{r\to 1} f(r) = 0$, proving that $f^*(1) = 0$. Thus, the radial limit f^* exists everywhere on $\mathbb{T}$.

Continuity of f^* at each point of $\mathbb{T} \smallsetminus \{1\}$ is clear since f has a continuous extension to $\mathbb{T} \smallsetminus \{1\}$. To see continuity of f^* at 1, take any sequence $\{z_n\} \subset \mathbb{T} \smallsetminus \{1\}$ such that $z_n \to 1$. Then $|g(z_n)| = 1$, so $\mathrm{Re}(g(z_n)) \geq -1$. It follows that

$$|f^*(z_n)| = |1 - z_n|\,|e^{-g(z_n)}| = |1 - z_n|\, e^{-\mathrm{Re}(g(z_n))} \leq e\,|1 - z_n|$$

for all n, which shows $\lim_{n\to\infty} f^*(z_n) = 0 = f^*(1)$.

To see that f is not bounded in $\mathbb{D}$, consider the sequence

$$z_n = \frac{n + \pi i - 1}{n + \pi i + 1}$$

in $\mathbb{D}$. We have $g(z_n) = \exp(n + \pi i) = -e^n$, so $f(z_n) = (1 - z_n)\exp(e^n)$. Since $|1 - z_n|$ tends to 0 like $1/n$, we have $f(z_n) \to \infty$.

The result does not contradict the maximum principle since f does not extend continuously to the closed disk $\overline{\mathbb{D}}$.