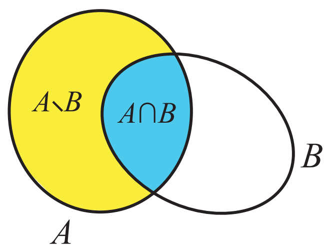
Math 310 Problem Set 2 Solutions

1. Let A, B, C be sets.

(i) Under what condition does the equality $A \smallsetminus (A \smallsetminus B) = B$ hold? Guess the answer using a diagram and then prove it carefully.

It is easy to see that if we take away $A \smallsetminus B$ from A, we are left with $A \cap B$ (see the figure below), so always $A \smallsetminus (A \smallsetminus B) = A \cap B$. Thus, the equality $A \smallsetminus (A \smallsetminus B) = B$ is equivalent to $A \cap B = B$, which holds precisely when $B \subset A$.



Here is a precise reasoning for $A \smallsetminus (A \smallsetminus B) = A \cap B$, using the identity $X \smallsetminus Y = X \cap Y^c$:

$$\begin{aligned} A \smallsetminus (A \smallsetminus B) &= A \cap (A \smallsetminus B)^c = A \cap (A \cap B^c)^c \\ &= A \cap (A^c \cup B) && \text{(De Morgan's law)} \\ &= (A \cap A^c) \cup (A \cap B) && \text{(distribute)} \\ &= \varnothing \cup (A \cap B) = A \cap B. \end{aligned}$$

(ii) Show that $A \smallsetminus (B \cup C) = (A \smallsetminus B) \cap (A \smallsetminus C)$.

We can argue as follows:

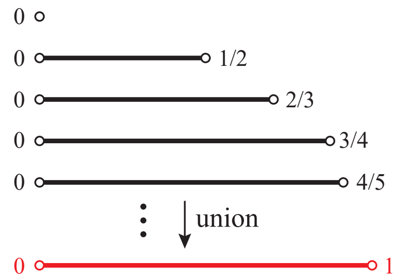
$$\begin{aligned} A \smallsetminus (B \cup C) &= A \cap (B \cup C)^c = A \cap (B^c \cap C^c) \\ &= (A \cap B^c) \cap (A \cap C^c) = (A \smallsetminus B) \cap (A \smallsetminus C). \end{aligned}$$

2. Find the following union and intersection of intervals:

$$\bigcup_{n=1}^{\infty} \left(0, 1 - \frac{1}{n}\right) \qquad \text{and} \qquad \bigcap_{n=1}^{\infty} \left(0, 1 + \frac{1}{n}\right).$$

The intervals $(0, 1 - 1/n)$ all start at 0 but their other end $1 - 1/n = (n-1)/n$ gets closer and closer to 1 from the left (note that for $n = 1$ the interval reduces to the empty set; see the figure below). This suggests that

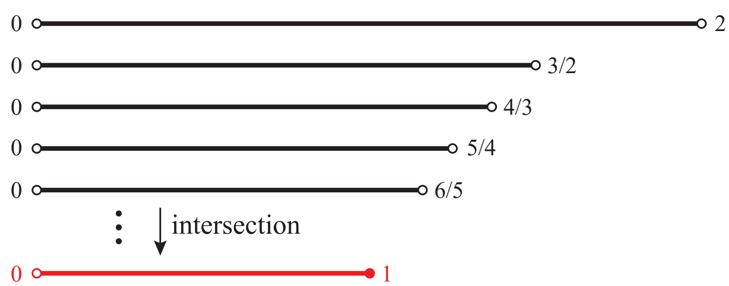
$$\bigcup_{n=1}^{\infty} \left(0, 1 - \frac{1}{n}\right) = (0, 1). \tag{1}$$



Let us prove (1) carefully. Since for every $n \in \mathbb{N}$ the interval $(0, 1 - 1/n)$ is a subset of $(0, 1)$, we clearly have $\bigcup_{n=1}^{\infty}(0, 1 - 1/n) \subset (0, 1)$. To show the reverse inclusion $(0, 1) \subset \bigcup_{n=1}^{\infty}(0, 1 - 1/n)$, we need to demonstrate that if $x \in (0, 1)$ then $x \in (0, 1 - 1/n)$ for some $n \in \mathbb{N}$. This is easy: take any $x \in (0, 1)$ and choose $n \in \mathbb{N}$ larger than the positive number $1/(1 - x)$. Then $1/n < 1 - x$ or $x < 1 - 1/n$, hence $x \in (0, 1 - 1/n)$. This completes the proof of (1).

Similarly, the intervals $(0, 1 + 1/n)$ all start at 0 but their other end $1 + 1/n = (n + 1)/n$ gets closer and closer to 1 from the right (see the figure below). This suggests that

$$\bigcap_{n=1}^{\infty} \left(0, 1 + \frac{1}{n}\right) = (0, 1]. \tag{2}$$



Here is a rigorous proof of (2). For every $n \in \mathbb{N}$ the interval $(0, 1]$ is a subset of $(0, 1 + 1/n)$, so $(0, 1] \subset \bigcap_{n=1}^{\infty}(0, 1 + 1/n)$. To show the reverse inclusion $\bigcap_{n=1}^{\infty}(0, 1 + 1/n) \subset (0, 1]$, we need to demonstrate that if $x \in \bigcap_{n=1}^{\infty}(0, 1 + 1/n)$ then $x \in (0, 1]$. Take any x in this intersection, so $0 < x < 1 + 1/n$ for every $n \in \mathbb{N}$. Assume by way of contradiction that $x \notin (0, 1]$. Since we already know x is positive, we must have $x > 1$. Choose $n \in \mathbb{N}$ larger than the positive number $1/(x - 1)$. Then $1/n < x - 1$ or $1 + 1/n < x$. This implies $x \notin (0, 1 + 1/n)$, which contradicts our assumption that x belonged to the intersection.

3. Recall that the Cartesian product $\mathbb{Z} \times \mathbb{Z}$ is the set of all ordered pairs (m, n) where both m and n are integers. Let $f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be the function defined by $f(n) = (n^2, n)$. Is f injective (one-to-one)? Is it surjective (onto)?

If $f(n) = f(k)$, then $(n^2, n) = (k^2, k)$, so $n = k$. Thus, f is injective. However, f is not surjective because the range of f consists of all ordered pairs $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ for which $m = n^2$, i.e., the first factor is the square of the second factor. This implies that any pair in $\mathbb{Z} \times \mathbb{Z}$ that doesn't have this property (such as $(0,1)$ or $(2,17)$) won't be in the range of f.

4. Let S be the set of all polynomials of the form $p(x) = ax^2 + bx + c$ and T be the set of all polynomials of the form $q(x) = rx + s$. Here the coefficients a, b, c, r, s can be any real numbers. Let $D : S \to T$ be the "differentiation" mapping defined by $D(p(x)) = p'(x)$. Is D injective? Is it surjective? What is the preimage $D^{-1}(\{x\})$?

The map D is not injective because two polynomials in S have the same derivative if and only if they differ by a constant. Thus, if $p(x) \in S$, if c is a non-zero constant, and if $p_2(x) = p_1(x) + c$, then $p_2(x) \neq p_1(x)$ but $D(p_2(x)) = D(p_1(x))$.

On the other hand, D is surjective because given any polynomial $q(x) \in T$ we can find a polynomial $p(x) \in S$ with $D(p(x)) = q(x)$ by taking anti-derivatives. Explicitly, if $q(x) = rx + s$, choose $p(x) = (r/2)x^2 + sx + c$ (c any constant). Then

$$D(p(x)) = D(\tfrac{r}{2}x^2 + sx + c) = rx + s = q(x).$$

Finally, the preimage $D^{-1}(\{x\})$ is the set of all polynomials is S whose derivative is x. Thus,

$$D^{-1}(\{x\}) = \{\tfrac{1}{2}x^2 + c : c \in \mathbb{R}\}.$$

5. Suppose $f : A \to B$ and $g : B \to C$ are functions such that the composition $g \circ f : A \to C$ is injective. Is f necessarily injective? What about g? Justify your answers by a proof or counterexample.

We prove that f must be injective. Suppose $f(a) = f(a')$ for some $a, a' \in A$. We need to show that $a = a'$. To this end, apply g on each side of the above equality to obtain $g(f(a)) = g(f(a'))$ or $(g \circ f)(a) = (g \circ f)(a')$. Since by the assumption $g \circ f$ is injective, we conclude that $a = a'$.

On the other hand, g need not be injective in general. To show this, it suffices to find one example where $g \circ f$ is injective but g is not. Let $A = \{a\}$, $B = \{b, b'\}$, $C = \{c\}$ (here a, b, b', c are distinct objects) and define $f : A \to B$ by $f(a) = b$ and $g : B \to C$ by $g(b) = g(b') = c$ (see the figure below). Then $g \circ f : A \to C$ is automatically injective since A has only one member, but g is not injective.