**Practice problems.**

(1) Suppose $f, g \in \mathcal{O}(\mathbb{D})$ are non-vanishing and

$$\frac{f'(1/n)}{f(1/n)} = \frac{g'(1/n)}{g(1/n)} \qquad \text{for all large } n \in \mathbb{N}.$$

How are $f, g$ related?

(2) Determine all $f \in \mathcal{O}(\mathbb{C})$ which satisfy the bound $|f(z)| \le e^{xy}$ for every $z = x + iy \in \mathbb{C}$.

(3) Show that there is a holomorphic function $f$ defined near the origin with $f(0) = 0$ such that $f(z)\cos(f(z)) = z$ for all $z$ near 0. Find the first 3 non-zero terms in the power series of $f$.

(4) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{O}(\mathbb{D})$. Suppose $|a_k| = 2^k \sup_{|z|=1/2} |f(z)|$ for some $k \ge 0$. Prove that $f(z) = a_k z^k$.

(5) Suppose $f \in \mathcal{O}(\mathbb{D})$ satisfies $f(0) = 0$ and $|f(z) + zf'(z)| < 1$ for all $z \in \mathbb{D}$. Show that $|f(z)| \le |z|/2$ for all $z \in \mathbb{D}$.

(6) Determine all bounded $f \in \mathcal{O}(\mathbb{H})$ such that $f(i/n) = e^{-n}$ for all $n$.

(7) (i) Show that every odd entire function $f : \mathbb{C} \to \mathbb{C}$ is surjective unless $f$ is the constant function 0.

(ii) Does there exist a non-affine entire function $f : \mathbb{C} \to \mathbb{C}$ such that $f$ is surjective but $f'$ is not?

(8) Suppose $f$ is non-vanishing and holomorphic in the punctured disk $\mathbb{D}^*$. Prove that there are $n \in \mathbb{Z}$ and $g \in \mathcal{O}(\mathbb{D}^*)$ such that $f(z) = z^n \exp(g(z))$ for all $z \in \mathbb{D}^*$.

(9) Suppose $f = u + iv \in \mathcal{O}(\mathbb{C})$ with $|uv|$ bounded. Show that $f$ is constant. What if we replace the condition with $u(z) \ne v(z)$ for all $z \in \mathbb{C}$?

(10) Let $f \in \mathcal{O}(\mathbb{C})$ and $f(0) = 0, f'(0) = 1$. Suppose the sequence of iterates $f^{\circ n} = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}$ is normal in some neighborhood of 0. Show that $f = \mathrm{id}$.

(11) Suppose $f$ is bounded and holomorphic in $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$ and $\mathrm{Im}(f(z)) \to 0$ as $z$ tends to any point on the boundary lines $\mathrm{Re}(z) = 0$ and $\mathrm{Re}(z) = 1$. Show that $f$ is constant.

(12) Suppose $f$ is holomorphic in a neighborhood of $\overline{\mathbb{D}}$, with $f(0) = 4$ and $\sup_{|z|=1} |f(z)| \leq e^2$. Find the largest possible number of zeros of $f$ in the disk $\mathbb{D}(0, 2/3)$.

(13) Show that for every $c \in \mathbb{C}$ the equation $\sin z = cz$ has infinitely many solutions in $z$.

(14) Let $p, q \in \mathbb{D}$ and $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(p) = q$. What are the largest and smallest $|f'(p)|$ could be? Justify your answers.

(15) Let $f_n \in \mathcal{O}(\mathbb{D})$ and suppose $f(z) = \lim_{n\to\infty} f_n(z)$ exists for every $z \in \mathbb{D}$. If $|f_n - 1| \geq 1$ in $\mathbb{D}$ for every $n$, show that $f_n \to f$ compactly in $\mathbb{D}$ (in particular, the pointwise limit $f$ must be holomorphic in $\mathbb{D}$).

**Hints.**

(1) Use the identity theorem.

(2) Apply Liouville's theorem to a suitable function.

(3) The function $g(w) = w \cos w$ has non-zero derivative at $w = 0$.

(4) This is the case of equality in Cauchy's estimates. Parseval's formula may be helpful.

(5) Consider $g(z) = zf(z)$. It smells like the Schwarz lemma, doesn't it?

(6) Use the characterization of zeros of bounded holomorphic functions in $\mathbb{D}$, or equivalently in $\mathbb{H}$ (compare Theorem 8.34, Example 8.35, and problem 6 in HW 2).

(7) For (i), use Picard's theorem. The answer to (ii), which is affirmative, should be easy in view of (i).

(8) Assuming this representation for $f$ and considering $f'/f$ gives you a clue as to what $n$ should be. Looking at the function $f(z)/z^n$ then shows you how to find $g$.

(9) Use Liouville's theorem or Picard's theorem.

(10) If $f \neq \mathrm{id}$, it must have a power series expansion $f(z) = z + az^k + O(z^{k+1})$ for some $k \geq 2$ and $a \neq 0$. In this case what would be the power series of $f^{\circ n}(z)$?

(11) Use repeated Schwarz reflections.

(12) Bound the number of zeros using Jensen's formula (see Corollary 8.31) or use Blaschke products (see problems 17/18 in chapter 4).

(13) Apply Picard's great theorem to a suitable function.

(14) Reduce to the Schwarz lemma.

(15) Postcompose the $f_n$ with a suitable Möbius transformation and apply Montel's theorem.