# Math 152 Final Exam Review Sheet

**5/10/2023**

Here is an itemized list of the material that the final exam is based on. Make sure you study them carefully. I recommend that you first review your lecture notes, the practice problems on our 3 review sheets, and the midterm problems and solutions. If you have time for extra practice, you can check the worked-out examples in the book and the homework problems and solutions available on WebAssign. The exam is cumulative, with a slight bias toward the material covered in the second half of the semester. They roughly correspond to methods of integration, applications of the integral, differential equations, and infinite series.

- Inverse functions, derivative of the inverse function,

$$\text{if} \quad f(a) = b, \quad \text{then} \quad f^{-1}(b) = a \quad \text{and} \quad (f^{-1})'(b) = \frac{1}{f'(a)}$$

- The natural logarithm $\ln x$ and its basic properties, the derivative formula

$$(\ln x)' = \frac{1}{x} \qquad \text{and more generally} \qquad (\ln u)' = \frac{u'}{u}$$

- The exponential function $e^x$ and its basic properties, the derivative formula

$$(e^x)' = e^x \qquad \text{and more generally} \qquad (e^u)' = e^u \cdot u'$$

- Logarithmic differentiation
- Exponential growth and decay
- The inverse trigonometric functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ and their basic properties, the derivative formulas

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}, \quad (\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}}, \quad (\tan^{-1} x)' = \frac{1}{1+x^2}$$

- L'Hospital's rule, its applications in finding limits of indeterminate forms
- The fundamental theorem of calculus, basic integration formulas (for example, review the formulas 1-8, 10, 12, 16, 17 in the table of integrals on reference page 6 in the back of the book)
- Integration by substitution
- Integration by parts

- Integration by partial fractions
- Improper integrals
- Application of integration in finding areas and volumes: Suppose we have two functions $f, g$ such that $0 \leq g(x) \leq f(x)$ whenever $a \leq x \leq b$. Let $R$ be the region bounded from above by the curve $y = f(x)$, from below by the curve $y = g(x)$, on the left by the line $x = a$ and on the right by the line $x = b$. Then

  □ The area of $R$ is
  $$A = \int_a^b (f(x) - g(x))\, dx.$$
  □ The solid obtained by revolving $R$ about the $x$-axis has volume
  $$V = \pi \int_a^b (f(x)^2 - g(x)^2)\, dx.$$
  □ Assuming $0 \leq a < b$, the solid obtained by revolving $R$ about the $y$-axis has volume
  $$V = 2\pi \int_a^b x\,(f(x) - g(x))\, dx.$$
- In general, the volume of a solid (not necessarily of revolution) is the integral of its cross-sectional area function:
  $$V = \int_a^b A(x)\, dx.$$
- The arc length formula: If $f$ is continuously differentiable, the length of the curve $y = f(x)$ between the points $(a, f(a))$ and $(b, f(b))$ is
  $$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$
- Solving first order differential equations by separating variables.
- Sequences and their limits, the squeeze theorem, increasing and decreasing sequences, bounded monotonic sequences are convergent.
- Two useful limits:
  $$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a \qquad\qquad \lim_{n\to\infty} \sqrt[n]{n} = 1.$$
- The geometric series:
  $$\sum_{n=0}^{\infty} r^n \begin{cases} \text{converges to } \frac{1}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases}$$

- The $p$-series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1. \end{cases}$$

  In particular, the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges.
- Basic divergence test: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- The integral, comparison, and limit comparison tests for series with positive terms.
- The ratio and root tests for series with positive terms.
- The Leibniz test for alternating series, absolute and conditional convergence, an absolutely convergent series is convergent.
- Power series, finding the radius and interval of convergence using the ratio test.
- Manipulating power series, within the interval of convergence a power series can be differentiated and integrated term-by-term.
- Taylor's formula:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

  where the remainder $R_n$ has the form

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad \text{for some } c \text{ between } a \text{ and } x.$$

  If $\lim_{n\to\infty} R_n(x) = 0$ for all $x$ in the interval $(a-R, a+R)$, then $f$ is equal to its Taylor series:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots \end{aligned}$$

  inside the interval $(a-R, a+R)$.
- Taylor series of a few basic functions about $a = 0$ (also known as their Maclaurin series). The most important examples are given below, but other examples can be constructed using these power series, as we discussed in class:

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots && -\infty < x < +\infty \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots && -\infty < x < +\infty \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots && -\infty < x < +\infty \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \cdots && -1 < x < 1 \\
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + x^4 - \cdots && -1 < x < 1 \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots && -1 < x < 1 \\
\tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots && -1 < x < 1 \\
(1+x)^k &= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots && -1 < x < 1.
\end{aligned}
$$

- Applications of Taylor series in approximating values of functions or integrals.

**Practice Problems**

**1.** In each case, find the derivative $y' = dy/dx$:

- $y = \sin^{-1}(x + e^x)$
- $y = \sqrt{\ln(\cos x)}$
- $y = x^{\tan x}$ [Hint: Logarithmic differentiation]

**2.** Verify that the function $f(x) = \ln x + 2x^3$ is strictly increasing and therefore one-to-one in the interval $(0, \infty)$. If $f^{-1}$ denotes the inverse of $f$, find the value of $(f^{-1})'(2)$.

**3.** Find $\lim_{x \to 0} \dfrac{e^{x^2} - \cos x}{x^2}$. [Hint: L'Hospital]

**4.** Evaluate the following integrals:

- $\displaystyle\int_2^{\infty} \frac{dx}{x(\ln x)^2}$ [Hint: Substitution]
- $\displaystyle\int \frac{\ln x}{x^3}\,dx$ [Hint: Integration by parts]

- $\displaystyle\int \frac{3x-1}{x^2+3x-10}\,dx$ [Hint: Partial fractions]

**5.** Let $R$ be the region in the plane bounded by the curve $y = e^x$ and the lines $y = x+1$ and $x = 2$.

(i) Sketch $R$ and find its area.
(ii) Find the volume of the solid obtained by rotating $R$ about the $x$-axis.
(iii) Find the volume of the solid obtained by rotating $R$ about the $y$-axis.

**6.** Set up an integral for the arc length of $y = \tan x$ for $0 \le x \le \pi/3$. Then use your calculator to estimate this length to four decimal places.

**7.** Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{1+x}{xy^2}$$

where $x > 0$. Then find the solution that satisfies the condition $y(1) = 3$.

**8.** Determine the convergence or divergence of the following series:

- $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{7n+3}$
- $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{n}{n^2+1}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{11^n\, n!}{4^n\, n^n}$

**9.** Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{5^n}{n^2}(x+3)^n.$$

**10.** Find the first four terms in the Maclaurin series of the following functions:

- $x^2\cos(5x)$
- $\displaystyle\int \frac{dx}{2+x^3}$

**11.** Find the Maclaurin series of $\sin(x^2)$ and use it to write the integral

$$\int_0^{0.5} \sin(x^2)\,dx$$

as an alternating series. Use this series to estimate the value of the above integral to four decimal places.