Math 704 Problem Set 9 Solutions

Problem 1. Prove that every $f \in \mathcal{O}(\mathbb{D}^*)$ with a pole or essential singularity at 0 has arbitrarily large unramified disks.

Consider the function $g(z) = f(1/z)$ which is holomorphic in $\{z : |z| > 1\}$. The computation

$$g'(z) = -\frac{1}{z^2} f'\left(\frac{1}{z}\right)$$

shows that $zg'(z) = -wf'(w)$, where $w = 1/z$. The origin $w = 0$ is a pole or essential singularity of f, so it is a pole of order ≥ 2 or an essential singularity of f' and therefore a pole or essential singularity of $wf'(w)$. Thus, $wf'(w)$ cannot be bounded in any punctured neighborhood of $w = 0$. Equivalently, $zg'(z)$ cannot be bounded in $\{z : |z| > r\}$ for any $r > 0$.

Now take a sequence $z_n \to \infty$ such that $|z_n g'(z_n)| \to +\infty$. Fix any $R > 0$ less than Bloch's constant $\mathfrak{B}$. Applying Corollary 11.3 of Bloch's theorem to the restriction of g to the disk $\mathbb{D}(z_n, |z_n|/2)$ shows that g has an unramified disk of radius $R_n = R(|z_n|/2)|g'(z_n)|$. Since $R_n \to +\infty$, we conclude that g, and therefore f, has arbitrarily large unramified disks.

Problem 2. Verify that Picard's little theorem is equivalent to the statement that there are no non-constant entire functions f and g which satisfy the equation $e^f + e^g = 1$.

If $e^f + e^g = 1$ for some $f, g \in \mathcal{O}(\mathbb{C})$, then e^f and e^g are entire functions that omit the values 0 and 1. By Picard's little theorem, e^f and e^g are both constant. It easily follows that f and g are constant.

Conversely, assume the equation $e^f + e^g = 1$ has no non-constant entire solutions in f, g. Take an $h \in \mathcal{O}(\mathbb{C})$ that omits the values 0 and 1. Then h and $1 - h$ are non-vanishing entire functions. Since $\mathbb{C}$ is simply connected, we have $h = e^f, 1 - h = e^g$ for some $f, g \in \mathcal{O}(\mathbb{C})$. It follows that both f and g are constant, so h must be constant.

Problem 3. Suppose f is a periodic entire function in the sense that $f(z + \omega) = f(z)$ for some $\omega \neq 0$. Show that f has a fixed point.

Assume f has no fixed point, so the entire function $g(z) = z - f(z)$ omits the value 0. The relation

$$g(z + \omega) = (z + \omega) - f(z + \omega) = z + \omega - f(z) = g(z) + \omega$$

then shows that g omits the value ω as well. By Picard's little theorem, g must be constant. This is a contradiction since a constant function cannot commute with the translation $z \mapsto z + \omega$.

Problem 4. Let f be an entire function such that $f \circ f$ has no fixed point (i.e., $f(f(z)) \neq z$ for all $z \in \mathbb{C}$). Prove that $f(z) = z + c$ for some $c \neq 0$.

Since $f \circ f$ has no fixed point, neither does f. Hence

$$g(z) = \frac{f(f(z)) - z}{f(z) - z}$$

is an entire function. Since $f(f(z)) \neq z$ for all z, g omits the value 0 and since $f(f(z)) \neq f(z)$ for all z, g omits the value 1. By Picard's little theorem, g must be constant, so

$$f(f(z)) - z = \lambda(f(z) - z) \qquad \text{for some constant } \lambda \in \mathbb{C} \smallsetminus \{0, 1\}.$$

Differentiation of this equation gives

$$f'(f(z))f'(z) - 1 = \lambda(f'(z) - 1) \quad \text{or} \quad f'(z)(f'(f(z)) - \lambda) = 1 - \lambda,$$

which shows $f'(z) \neq 0$ and $f'(f(z)) \neq \lambda$ for all z. In particular, $f'(f(z))$ omits the values $0, \lambda$. Again by Picard's little theorem, $f' \circ f$ must be constant. Since f itself is not constant (otherwise it would have a fixed point), $f(\mathbb{C})$ is open and f' is constant on $f(\mathbb{C})$, hence constant everywhere by the identity theorem. Thus, $f(z) = bz + c$ for some non-zero constants $b, c \in \mathbb{C}$. Since f has no fixed points, $b = 1$ and $f(z) = z + c$.

Problem 5. Let f be a non-constant entire function which omits the value q, and P be a polynomial which is not identically q. Prove that the equation $f(z) = P(z)$ has infinitely many solutions.

Consider the entire function

$$g(z) = \frac{P(z) - q}{f(z) - q},$$

which is non-constant since f, having an omitted value, is not a polynomial. As P is not identically q, g takes the value 0 finitely many times. It follows from Picard's great theorem that g takes the value 1 infinitely often. This complete the proof since $g(z) = 1$ if and only if $f(z) = P(z)$.

Problem 6.

(i) Give an example of a family of holomorphic functions $\mathbb{C} \to \mathbb{C} \smallsetminus \{0\}$ that fails to be normal.

Consider the sequence $f_n(z) = \exp(nz)$. For $\mathrm{Re}(z) > 0$, $f_n(z) \to \infty$ and for $\mathrm{Re}(z) < 0$, $f_n(z) \to 0$. This shows that $\{f_n\}$ cannot be normal in $\mathbb{C}$. (Note however that the restrictions of $\{f_n\}$ to each of the half-planes $\{z : \mathrm{Re}(z) > 0\}$ and $\{z : \mathrm{Re}(z) < 0\}$ is indeed normal.)

(ii) Let $f_1(z) = z + z^2$ and define $\{f_n\}$ inductively by $f_n = f_1 \circ f_{n-1}$ for $n \geq 2$. Show that $\{f_n\}$ is not normal in any neighborhood of 0.

Normality in some neighborhood U of 0 would imply that, after passing to a subsequence, $\{f_n\}$ converges compactly in U to some $f \in \mathcal{O}(U)$ (the alternative $f_n \to \infty$ cannot occur since $f_n(0) = 0$ for all n). It would then follow that $f_n'' \to f''$ compactly in U, and in particular $\{f_n''(0)\}$ is bounded. Thus, to prove the failure of normality, it suffices to show that $f_n''(0) \to \infty$ as $n \to \infty$.

Differentiation of $f_n = f_1 \circ f_{n-1}$ gives

$$f_n' = (f_1' \circ f_{n-1})\, f_{n-1}' \tag{$*$}$$

which leads to the recursion

$$f_n'(0) = f_1'(0)\, f_{n-1}'(0).$$

Since $f_1'(0) = 1$, we obtain $f_n'(0) = 1$ for all n. Differentiating $(*)$ now gives

$$f_n'' = (f_1'' \circ f_{n-1})\, (f_{n-1}')^2 + (f_1' \circ f_{n-1})\, f_{n-1}''.$$

This leads to the recursion

$$f_n''(0) = f_1''(0)(f_{n-1}'(0))^2 + f_1'(0) f_{n-1}''(0) = f_1''(0) + f_{n-1}''(0).$$

Since $f_1''(0) = 2$, we obtain $f_n''(0) = 2n$ for all n. In particular, $f_n''(0) \to \infty$ as $n \to \infty$.