Math 363/663 Homework 2 Solutions

Problem 1.

(i) Find the general solution of the first order linear PDE

$$u_x + u_y + u = e^{x+2y}.$$

The change of variables

$$\begin{cases} z &= x \\ w &= x - y \end{cases} \iff \begin{cases} x &= z \\ y &= z - w \end{cases}$$

transforms our PDE into

$$u_z + u = e^{3z-2w}$$

with missing u_w term, which can be solved as a first order linear ODE in z:

$$\begin{aligned} u_z + u = e^{3z-2w} &\implies e^z u_z + e^z u = e^{4z-2w} \implies \frac{\partial}{\partial z}(e^z u) = e^{4z-2w} \\ &\implies e^z u = \int e^{4z-2w}\, dz = \frac{1}{4}e^{4z-2w} + K(w) \\ &\implies u = \frac{1}{4}e^{3z-2w} + K(w)e^{-z}. \end{aligned}$$

Going back to the original variables (x,y), we obtain the general solution

$$u(x,y) = \frac{1}{4}e^{x+2y} + K(x-y)e^{-x},$$

where $K(\cdot)$ is an arbitrary C^1 function of one variable.

(ii) Find the solution which satisfies the side condition $u(x,0) = 0$.

We have

$$u(x,0) = \frac{1}{4}e^x + K(x)e^{-x} = 0 \implies K(x) = -\frac{1}{4}e^{2x},$$

which shows

$$u(x,y) = \frac{1}{4}e^{x+2y} - \frac{1}{4}e^{2(x-y)}e^{-x} = \frac{1}{4}e^{x+2y} - \frac{1}{4}e^{x-2y}.$$

(iii) How would the answer change if the side condition were $u(x,x) = 0$? What about $u(x,x) = \frac{1}{4}e^{3x}$?

Suppose the side condition were $u(x,x) = 0$. This would imply

$$u(x,x) = \frac{1}{4}e^{3x} + K(0)e^{-x} = 0 \implies K(0) = -\frac{1}{4}e^{4x},$$

which is impossible to hold for all x since the left side is a constant. So there is no solution subject to the side condition $u(x,x) = 0$.

Now suppose the side condition were $u(x,x) = \frac{1}{4}e^{3x}$. Then

$$u(x,x) = \frac{1}{4}e^{3x} + K(0)e^{-x} = \frac{1}{4}e^{3x} \Longrightarrow K(0)e^{-x} = 0 \Longrightarrow K(0) = 0.$$

It follows that if $K(\cdot)$ is any C^1 function with $K(0) = 0$, the solution u satisfies the side condition $u(x,x) = \frac{1}{4}e^{3x}$. Thus, there are infinitely many solutions subject to this side condition.

Problem 2. Solve the equation

$$3u_y + u_{xy} = 0.$$

Set $v = u_y$. Then $3v + v_x = 3u_y + 3u_{yx} = 3u_y + 3u_{xy} = 0$ (here we have used the equality $u_{xy} = u_{yx}$ of the mixed partial derivatives, which is known to hold when u is C^1). Thus, v satisfies the first order linear homogeneous PDE $v_x + 3v = 0$ which can be easily solved as an ODE in x:

$$v_x + 3v = 0 \Longrightarrow e^{3x}v_x + 3e^{3x}v = 0 \Longrightarrow \frac{\partial}{\partial x}(e^{3x}v) = 0$$
$$\Longrightarrow e^{3x}v = K(y) \Longrightarrow v = u_y = K(y)e^{-3x}.$$

Here $K(\cdot)$ can be any continuous function (continuity of K is enough because the PDE on v does not require differentiability of v in y). Integrating with respect to y and denoting the antiderivative $\int K(y)\,dy$ by $G(y)$ (which is now C^1) yields

$$u(x,y) = G(y)e^{-3x} + H(x),$$

where $G(\cdot)$ and $H(\cdot)$ are arbitrary C^1 functions.

Problem 3. Solve the first order linear PDE

$$y\,u_x - 4x\,u_y = 2xy$$

which satisfies the side condition $u(x,0) = x^4$. What do the characteristic curves of this PDE look like?

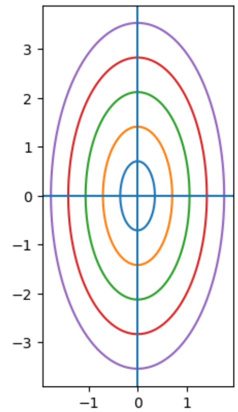
By definition, the characteristic curves are the solutions of the ODE

$$\frac{dy}{dx} = -\frac{4x}{y}.$$

This is a first order separable ODE, so it is easily solvable:

$$y\,dy = -4x\,dx \Longrightarrow \int y\,dy = -4\int x\,dx$$
$$\Longrightarrow \frac{1}{2}y^2 = -2x^2 + \text{const.} \Longrightarrow 4x^2 + y^2 = \text{const.}$$

This shows that the characteristic curves form a family of ellipses (see the figure). Since the function $h(x,y) = 4x^2 + y^2$ is constant along the characteristic curves, we are led to consider the change of variables



$$\begin{cases} z &= x \\ w &= 4x^2 + y^2 \end{cases} \quad \Longleftrightarrow \quad \begin{cases} x &= z \\ y &= \pm\sqrt{w - 4z^2} \end{cases}$$

(notice that this change of variables is not one-to-one on the whole plane but, as seen below, that won't be an issue). This transforms our PDE into

$$y\, u_z = 2xy \quad \text{or} \quad u_z = 2z,$$

whose solution is $u = z^2 + K(w)$. Thus, the general solution is

$$u(x, y) = x^2 + K(4x^2 + y^2),$$

where $K(\cdot)$ is an arbitrary C^1 function.

Let us now impose the side condition:

$$u(x, 0) = x^4 \Longrightarrow x^2 + K(4x^2) = x^4 \Longrightarrow K(4x^2) = x^4 - x^2.$$

Calling $t = 4x^2$, this gives

$$K(t) = \left(\frac{t}{4}\right)^2 - \frac{t}{4} = \frac{t^2}{16} - \frac{t}{4}.$$

Thus,

$$u(x, y) = x^2 + \frac{1}{16}(4x^2 + y^2)^2 - \frac{1}{4}(4x^2 + y^2) = x^4 + \frac{1}{16}y^4 + \frac{1}{2}x^2y^2 - \frac{1}{4}y^2.$$