Linear Programming

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Chapter 2

Geometric interpretation of linear programming

Mathematical Program

Given a domain $\mathcal{D} \subset \mathbb{R}^n$ and a function $f : \mathcal{D} \to \mathbb{R}$.

A *Mathematical Program* (MP) is a problem that consists in finding an element $x^* \in \mathcal{D}$ (if such an element exists) that maximizes or minimizes the function f. In other words :

1. $\forall x \in \mathcal{D}, f(x) \geq f(x^*)$ (minimization problem : $\min\limits_{x \in \mathcal{D}} f(x)$),
   or
2. $\forall x \in \mathcal{D}, f(x) \leq f(x^*)$ (maximization problem : $\max\limits_{x \in \mathcal{D}} f(x)$).

Vocabulary

Let $(P) : \max\limits_{x \in \mathcal{D}} f(x)$ be a mathematical program. Then :

- A point x is called a *solution* of (P).
- The set $\mathcal{D}$ is called the *feasible region* of (P).
- A point $x \in \mathcal{D}$ is called a *feasible solution* of (P).
- The function f is called the *objective function* of (P).
- A point $x^* \in \mathcal{D}$ is called an *optimal solution* of (P), i.e., a feasible solution whose value is the greatest possible for (P).

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3. If $\mathcal{D} \neq \emptyset$ and f is bounded over $\mathcal{D}$, then (P) :

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   - may *not admit an optimal solution* : the function is bounded but does not attain a maximum.

     Example : $\max\limits_{\{x<0\}}(x)$

General Form of a Linear Program

$$
(P)\begin{cases}
\min \text{ or } \max(Z) = \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i, & \forall i \in I_1 \\
\sum_{j=1}^{n} a_{ij} x_j = b_i, & \forall i \in I_2 \\
\sum_{j=1}^{n} a_{ij} x_j \ge b_i, & \forall i \in I_3 \\
x_j \ge 0, \quad \forall j \in J_1 \\
x_j \le 0, \quad \forall j \in J_2 \\
x_j \in \mathbb{R}, \quad \forall j \in J_3
\end{cases}
$$

- $I = I_1 \cup I_2 \cup I_3$ with $|I| = m$: number of constraints.
- $J = J_1 \cup J_2 \cup J_3$ with $|J| = n$: number of variables.

A linear program (LP) can be represented in matrix form as follows :

$$(P)\begin{cases} \max(Z) = & CX & \\ \text{s.t.} & AX & \leq b \\ & X & \geq 0 \end{cases}$$

where :

- $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$
  Vector of variables $(n \times 1)$
- $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$
  Constraint matrix $(m \times n)$
- $C = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix}$
  Cost vector $(1 \times n)$
- $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$
  Right-hand side vector $(m \times 1)$

Once the linear program is formulated, the next step is to solve the model.

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Mathematical Techniques for Solving a Linear Program

- Graphical method ($n = 2$ or $n = 3$).
- Simplex method ($n \geq 2$).

Convex Set

A set $C \subset \mathbb{R}^n$ is said to be *convex* if

$$\forall x, y \in C,\ \forall \lambda \in [0,1],\ \lambda x + (1-\lambda)y \in C.$$

In other words, a set C is convex if the line segment joining any two of its points lies entirely within the set C.

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Illustration

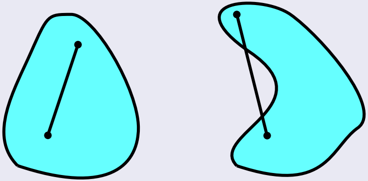

FIGURE – Convex set – Non-convex set

Convex Combination

Let $x_1, \ldots, x_k \in \mathbb{R}^n$. A *convex combination* of $x_1, \ldots, x_k$ is a vector x such that :

$$x = \sum_{i=1}^{k} \lambda_i x_i, \quad \lambda_i \geq 0, \quad \sum_{i=1}^{k} \lambda_i = 1$$

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Example of a Convex Combination

Let $x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and $x_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ be two points in $\mathbb{R}^2$:

Assume : $\lambda_1 = 0.4,\ \lambda_2 = 0.6.$ with $\displaystyle\sum_{i=1}^{2} \lambda_i = 0.4 + 0.6 = 1.$

then :

$x = 0.4x_1 + 0.6x_2 = 0.4 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 0.6 \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2.2 \\ 3.2 \end{pmatrix}$ is a convex combination of the points x_1 and x_2.

Hyperplane

Let a be a nonzero vector in $\mathbb{R}^n$ and let b be a scalar.

A *hyperplane* $\mathcal{H}$ is the set of all points $x \in \mathbb{R}^n$ that satisfy $a^\mathsf{T} x = b$.

$$\mathcal{H} = \{x \in \mathbb{R}^n \mid \ a^\mathsf{T} x = b\}$$

This hyperplane partitions the space $\mathbb{R}^n$ into two half-spaces :

Half-Spaces of the Hyperplane

1. Positive closed half-space $\mathcal{H}^+$:

$$\mathcal{H}^+ = \{x \in \mathbb{R}^n \mid a^\mathsf{T} x \geq b\}$$

2. Negative closed half-space $\mathcal{H}^-$:

$$\mathcal{H}^- = \{x \in \mathbb{R}^n \mid a^\mathsf{T} x \leq b\}$$

Polyhedron

A Polyhedron $\mathcal{P}$ in $\mathbb{R}^n$ is the intersection of finitely many halfspaces. It can be equivalently defined to be the set

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

where $A \in \mathbb{R}^{m \times n}$ is a real matrix, and $b \in \mathbb{R}^m$ is a real vector.

Fact

The feasible region of any linear program can be described as a polyhedron.

Bounded Set

A set $S \subseteq \mathbb{R}^n$ is said to be *bounded* if there exists a constant $k > 0$ such that the absolute value of every component of every element of S is at most k. In other words, $\exists k \in \mathbb{R}^+, \; \|x\| \leq k, \quad \forall x \in S$

Polytope

A nonempty bounded polyhedron is called a *polytope*.

Theorems

1. A hyperplane is convex.
2. The closed half-spaces $\mathcal{H}^+$ and $\mathcal{H}^-$ are convex.
3. Any intersection of convex sets is still convex.
4. Every polyhedron is an intersection of halfspaces, and convex.

Proof of Theorem 1 : A hyperplane is convex.

Let $\mathcal{H} = \{x \in \mathbb{R}^n \mid a^\mathsf{T} x = b\}$ be a hyperplane.

Suppose $u, v \in \mathcal{H}$, then

$$a^\mathsf{T} u = b \quad \text{and} \quad a^\mathsf{T} v = b.$$

Let $\lambda \in [0,1]$ and define $w = \lambda u + (1-\lambda)v$.

Then,

$$a^\mathsf{T} w = a^\mathsf{T}(\lambda u + (1-\lambda)v) = \lambda a^\mathsf{T} u + (1-\lambda) a^\mathsf{T} v = \lambda b + (1-\lambda) b = b.$$

Hence, $w \in \mathcal{H}$, so $\mathcal{H}$ is convex.

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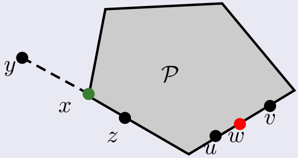
Extreme Point

Let $\mathcal{P} \subset \mathbb{R}^n$ be a polyhedron. A point $x \in \mathcal{P}$ is called an *extreme point* of $\mathcal{P}$ if there do not exist two distinct points $y, z \in \mathcal{P}$ such that x is a convex combination of them :

$$\nexists\, y, z \in \mathcal{C},\ \lambda \in]0,1[\text{ such that } x = \lambda y + (1-\lambda) z.$$

Example

- w is not an extreme point because $\exists u, v \in \mathcal{P}$ such that $w \in [uv]$.
- x is an extreme point because if $x = \lambda y + (1-\lambda)z$, with $\lambda \in]0,1[$ and $y, z \neq x$, then either $y \notin \mathcal{P}$ or $z \notin \mathcal{P}$.

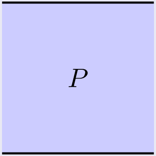


Definition

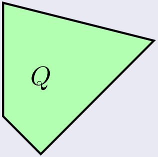
A polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$ is said to *contain a line* if there exist a point $x \in \mathbb{R}^n$ and a nonzero vector $d \in \mathcal{P}$ such that :

$$x + \lambda d \in P \quad \text{for all } \lambda \in \mathbb{R}.$$

Example



P contain a line



Q does not contain a line

Theorem 2 : Existence of Extreme Points

Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polyhedron. The following are equivalent :

1. $\mathcal{P}$ does not contain a line.
2. $\mathcal{P}$ has at least one extreme point.

Corollary

Every bounded polyhedron (i.e., every polytope) has at least one extreme point.

Theorem 3 : Optimality of Extreme Point

If a feasible region $\mathcal{D}$ of a linear program (P) has at least one extreme point, and the LP (P) has an optimal solution, then at least one optimal solution is an extreme point of $\mathcal{D}$.

Proof of Theorem 3

- Let $\mathcal{D} = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a feasible region, and let x^* be an optimal solution to the LP with optimal value $v = c^\mathsf{T} x^*$

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- Let x' be an extreme point of Q. Suppose that x' is not an extreme point of $\mathcal{D}$. Then $\exists y \neq x', z \neq x', \lambda \in]0,1[$ s.t $x' = \lambda y + (1-\lambda) z$

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- $c^\mathsf{T} x' = \lambda c^\mathsf{T} y + (1-\lambda) c^\mathsf{T} z = v$. Since v is the optimal value, and $y, z \in \mathcal{D}$, we have : $c^\mathsf{T} y \leq v \quad \text{and} \quad c^\mathsf{T} z \leq v$

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- $c^\mathsf{T} x' = \lambda c^\mathsf{T} y + (1-\lambda) c^\mathsf{T} z = v$. Since v is the optimal value, and $y, z \in \mathcal{D}$, we have : $c^\mathsf{T} y \leq v \quad \text{and} \quad c^\mathsf{T} z \leq v$
- But the equality above implies : $c^\mathsf{T} y = c^\mathsf{T} z = v \quad \Rightarrow \quad y, z \in Q$ Hence, x' is not an extreme point of Q. Contradiction.

$\blacksquare$

Properties

Let (P) be a linear program (LP) and let $\mathcal{D}$ be its feasible region.

Then :

1. If $\mathcal{D}$ is a non-empty bounded polyhedron, then :
    1. (P) admits a unique optimal solution, and this solution is an extreme point ; or
    2. (P) admits an infinite number of optimal solutions, which are convex combinations of a finite number of extreme points of $\mathcal{D}$.
2. If $\mathcal{D}$ is a non-empty unbounded polyhedron, then :
    1. Cases (1) and (2) may still occur ;
    2. (P) may not have a finite optimal solution, i.e., $\max(Z) \to +\infty$ or $\min(Z) \to -\infty$.
3. If $\mathcal{D} = \emptyset$, then (P) has no feasible solution. As a consequence, (P) has no optimal solution.

Algorithm 1: Graphical Method Algorithm

(1) Draw an orthonormal coordinate system $(O, \vec{x_1}, \vec{x_2})$;

(2) Plot the constraints (both functional and non-negativity) ;

(3) Determine the closed half-plane that satisfies each constraint ;

(4) Draw the set of feasible solutions $(\mathcal{D})$;

(5) **if** $(\mathcal{D})$ is bounded **then** An optimal solution exists, go to (7) ;

(6) **else if** $(\mathcal{D})$ *is unbounded* **then**

  if the problem is a maximization **then** No optimal solution ;

  else if the problem is a minimization **then** An optimal solution exists, go to (7) ;

(7) Find all the extreme points of $(\mathcal{D})$ and select the optimal point using one of the two methods :

- Extreme point enumeration approach ;
- Gradient method

Production Problem

A manufacturer produces two types of strawberry yogurt, labeled A and B, using strawberries, milk, and sugar. Producing one pot of yogurt A requires 1 kg of strawberries, 2 kg of milk, and 1 kg of sugar. For one pot of yogurt B, the recipe requires 2 kg of strawberries, and 1 kg of milk.

The total available quantities of ingredients in stock are 800 kg of strawberries, 700 kg of milk, and 300 kg of sugar. Yogurt A is sold at 4 DA per pot, and yogurt B at 5 DA per pot.

The manufacturer aims to maximize profit.

$$
(P)\begin{cases}
\max(Z) = & 4x_1 + 5x_2 & \\
s.t & x_1 + 2x_2 & \leq 800 \\
 & 2x_1 + x_2 & \leq 700 \\
 & x_1 & \leq 300 \\
 & x_1 \geq 0, x_2 \geq 0 &
\end{cases}
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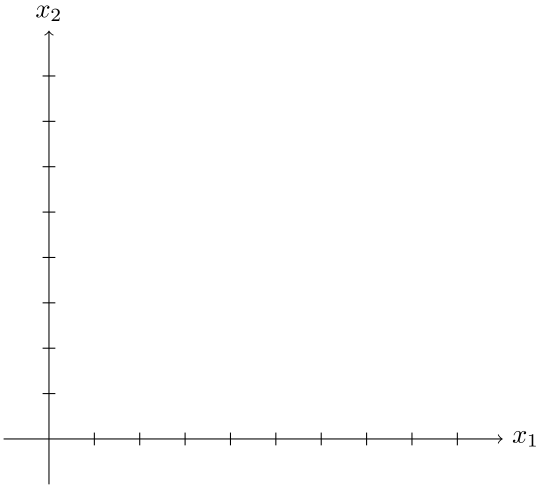

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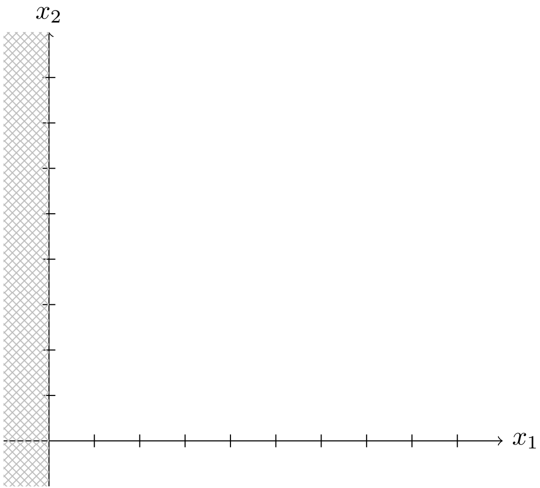

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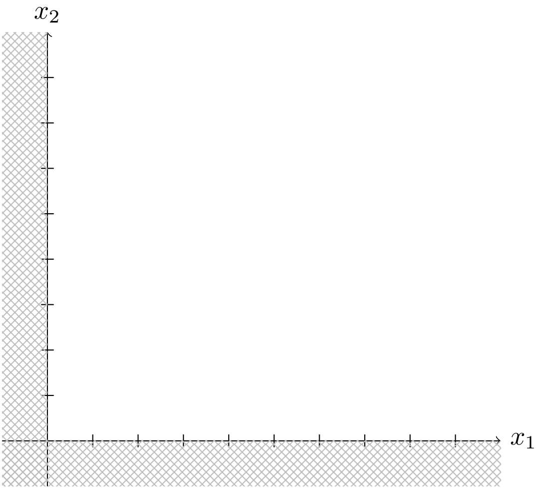

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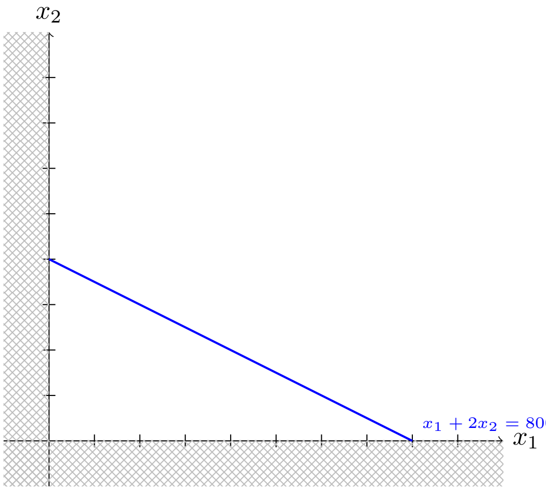

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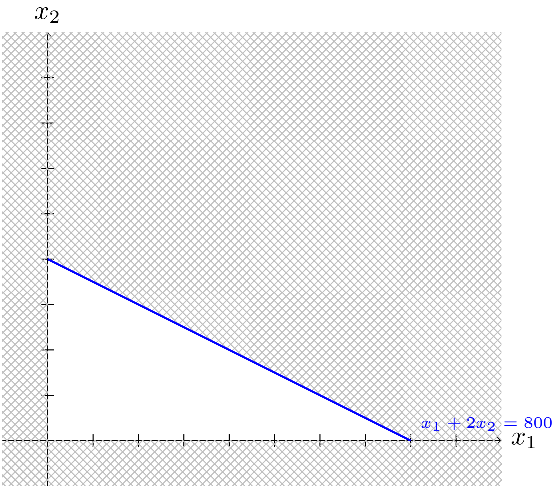

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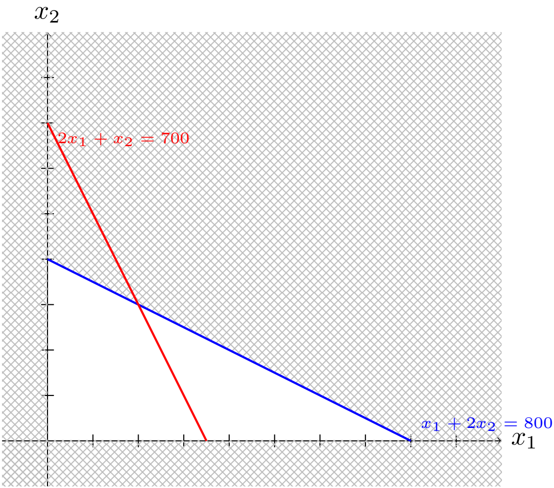

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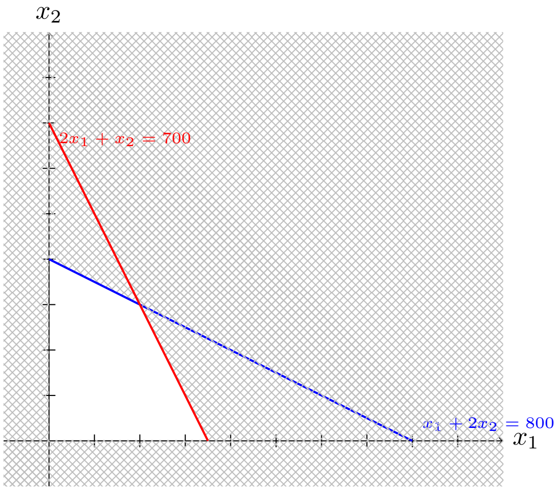

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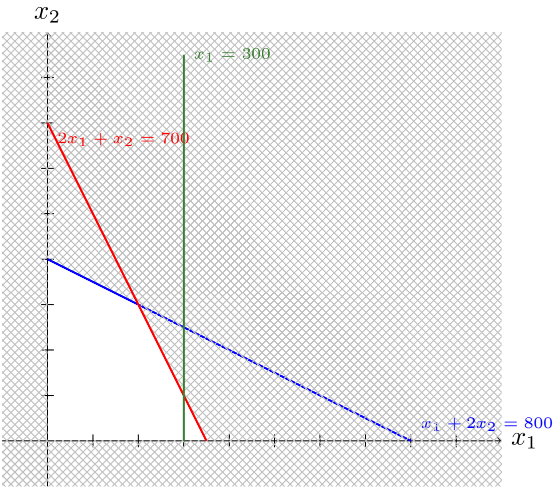

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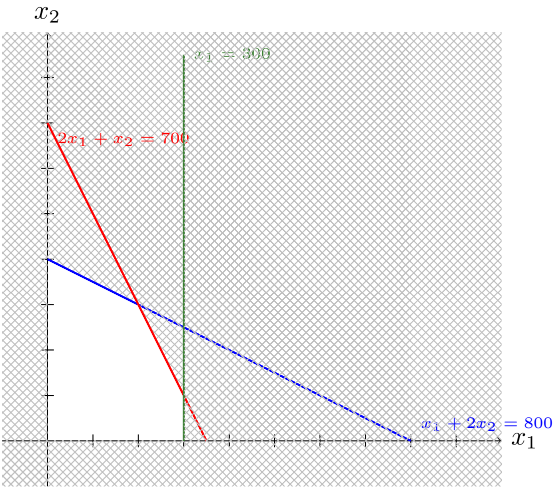

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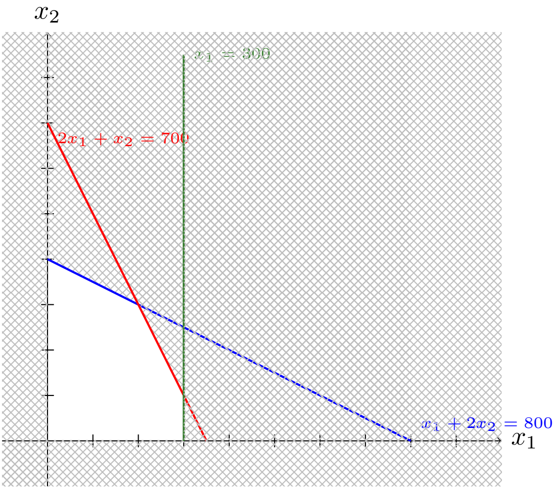


$$(P)\begin{cases} \max(Z) = & 4x_1 + 5x_2 & \\ s.t & x_1 + 2x_2 & \leq 800 \\ & 2x_1 + x_2 & \leq 700 \\ & x_1 & \leq 300 \\ & x_1 \geq 0, x_2 \geq 0 & \end{cases}$$

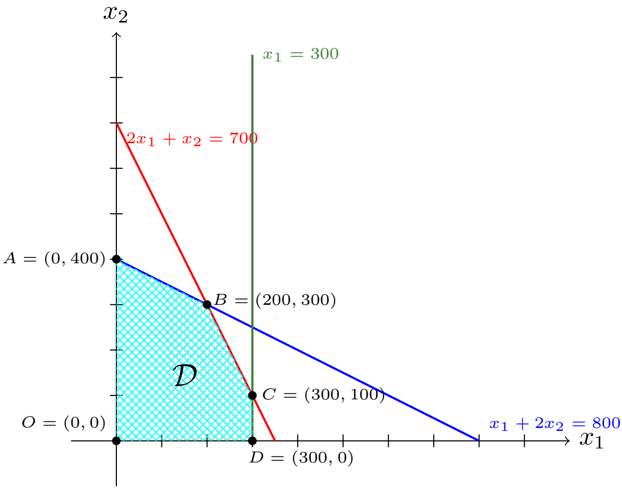


FIGURE – Feasible Region of the Production Problem

1st Method : Extreme point enumeration approach

The feasible region $\mathcal{D}$ of (P) is a convex polytope and the *optimal solution* occurs at one of the extreme points of this polytope.

Method Description

- Enumerate all extreme points of the feasible region.
- Evaluate the objective function at each extreme point.
- Select the extreme point with the best (maximum or minimum) objective value.

1st Method : Extreme point enumeration approach

Vertex	Coordinates	Value of $\max(Z) = 4x_1 + 5x_2$
O	$(0, 0)$	0
A	$(0, 400)$	2000
B	$(200, 300)$	2300
C	$(300, 100)$	1700
D	$(300, 0)$	1200

Conclusion : The optimal solution is $X^* = \begin{pmatrix} 200 \\ 300 \end{pmatrix}$, where $Z^* = 2300$.

Note

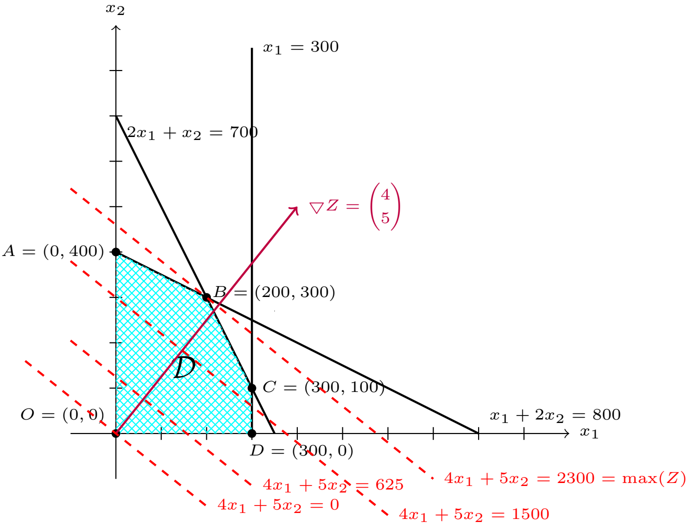
This approach is effective for small problems but becomes computationally expensive in higher dimensions.

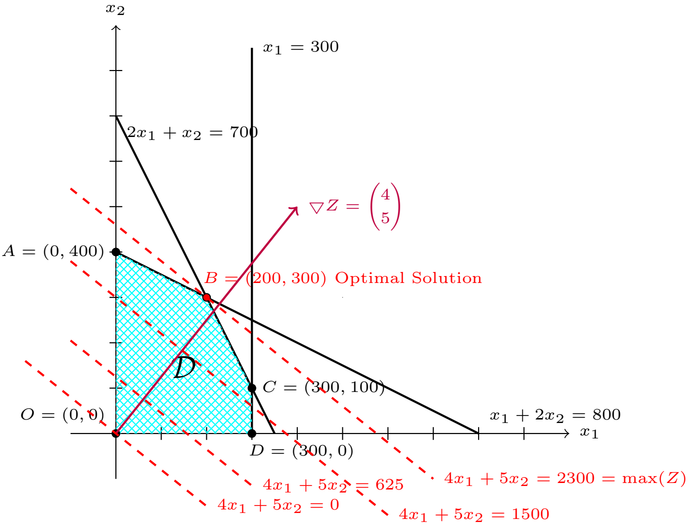
2nd Method : Gradient-Based Geometric Approach

The vector (c_1, c_2) is the gradient of the linear objective function :
$f(x_1, x_2) = c_1 x_1 + c_2 x_2, \quad$ with $c_1, c_2 \in \mathbb{R}$.

Method Description

- The gradient vector (c_1, c_2) is perpendicular to the level lines defined by : $c_1 x_1 + c_2 x_2 = Z, \quad$ for all $Z \in \mathbb{R}$.
- Increasing Z corresponds to shifting these lines in the direction of the gradient (c_1, c_2), without changing their orientation.
- Geometrically, this means "sliding" the line outward, parallel to itself, until it last touches the feasible region.
- This point of contact represents the optimal solution, where the objective function reaches its maximum (or minimum) over the feasible region.

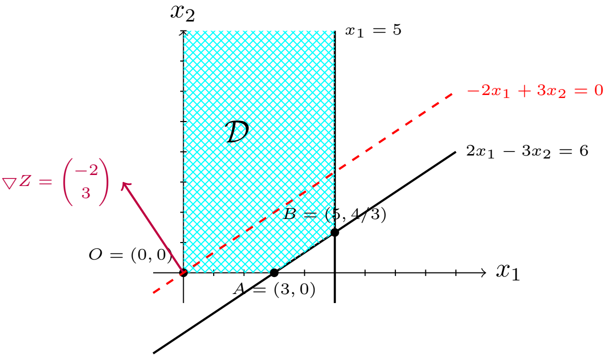

x_2
$x_1 = 300$
$2x_1 + x_2 = 700$
$\nabla Z = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$
$A = (0, 400)$
$B = (200, 300)$
D
$C = (300, 100)$
$O = (0, 0)$
$x_1 + 2x_2 = 800$
x_1
$D = (300, 0)$
$4x_1 + 5x_2 = 2300 = \max(Z)$
$4x_1 + 5x_2 = 625$
$4x_1 + 5x_2 = 0$
$4x_1 + 5x_2 = 1500$


x_2
$x_1 = 300$
$2x_1 + x_2 = 700$
$\nabla Z = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$
$A = (0, 400)$
$B = (200, 300)$ Optimal Solution
D
$C = (300, 100)$
$O = (0, 0)$
$x_1 + 2x_2 = 800$
x_1
$D = (300, 0)$
$4x_1 + 5x_2 = 2300 = \max(Z)$
$4x_1 + 5x_2 = 625$
$4x_1 + 5x_2 = 0$
$4x_1 + 5x_2 = 1500$

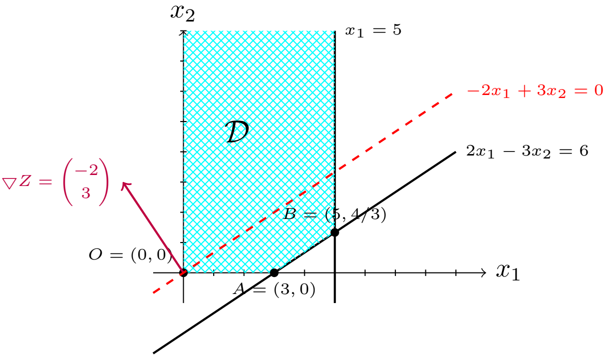
Some examples are given of how to graphically solve linear problems in various possible scenarios.

$$(P)\begin{cases} \max(Z) = & -2x_1 + 3x_2 \\ s.t & x_1 \leq 5 \\ & 2x_1 - 3x_2 \leq 6 \\ & x_1 \geq 0, x_2 \geq 0 \end{cases}$$

$$(P)\begin{cases} \max(Z) = & -2x_1 + 3x_2 \\ s.t & x_1 \leq 5 \\ & 2x_1 - 3x_2 \leq 6 \\ & x_1 \geq 0, x_2 \geq 0 \end{cases}$$



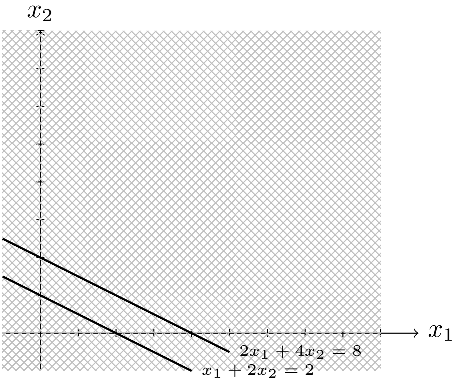
$$(P)\begin{cases} \max(Z) = & -2x_1 + 3x_2 \\ s.t & x_1 \leq 5 \\ & 2x_1 - 3x_2 \leq 6 \\ & x_1 \geq 0, x_2 \geq 0 \end{cases}$$



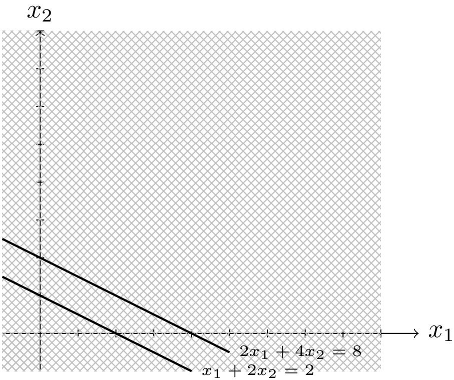
It is clear that the value of Z can be increased in the direction of the gradient vector of the objective function, indicating that the solution is unbounded (i.e., it tends to infinity).

$$(P)\begin{cases} \max(Z) = & 3x_1 + 3x_2 \\ s.t & x_1 + 2x_2 \leq 2 \\ & 2x_1 + 4x_2 \geq 8 \\ & x_1 \geq 0, x_2 \geq 0 \end{cases}$$

$$(P)\begin{cases} \max(Z) = & 3x_1 + 3x_2 \\ s.t & x_1 + 2x_2 \leq 2 \\ & 2x_1 + 4x_2 \geq 8 \\ & x_1 \geq 0, x_2 \geq 0 \end{cases}$$



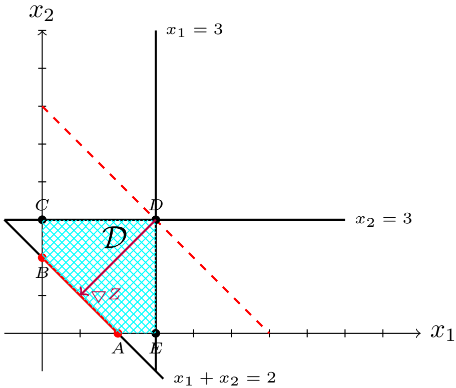
$$
(P)\begin{cases}
\max(Z) = & 3x_1 + 3x_2 \\
s.t & x_1 + 2x_2 \leq 2 \\
 & 2x_1 + 4x_2 \geq 8 \\
 & x_1 \geq 0, x_2 \geq 0
\end{cases}
$$



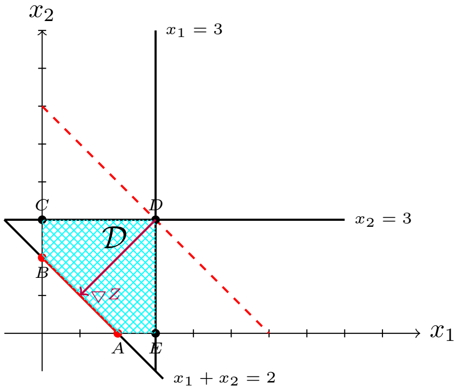
Since the feasible region is empty ($\mathcal{D} = \emptyset$), the problem (P) admits no solution.

$$(P)\begin{cases} \min(Z) = & 2x_1 + 2x_2 \\ s.t & x_1 + x_2 \geq 2 \\ & x_1 \leq 3 \\ & x_2 \leq 3 \\ & x_1 \geq 0, x_2 \geq 0 \end{cases}$$

$$(P)\begin{cases} \min(Z) = & 2x_1 + 2x_2 \\ s.t & x_1 + x_2 \geq 2 \\ & x_1 \leq 3 \\ & x_2 \leq 3 \\ & x_1 \geq 0, x_2 \geq 0 \end{cases}$$



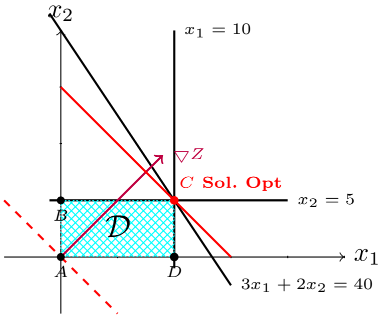
$$(P)\begin{cases} \min(Z) = & 2x_1 + 2x_2 \\ s.t & x_1 + x_2 \geq 2 \\ & x_1 \leq 3 \\ & x_2 \leq 3 \\ & x_1 \geq 0, x_2 \geq 0 \end{cases}$$



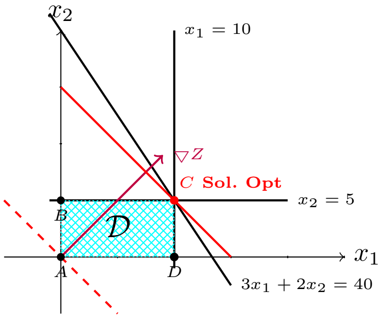
The points on the segment $[AB]$ represent the optimal solutions of (P).
$X^* = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $X^{**} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $Z^* = 4$.

$$(P)\begin{cases} \max(Z) = & x_1 + x_2 \\ s.t & 3x_1 + 2x_2 \leq 40 \\ & x_1 \leq 10 \\ & x_2 \leq 5 \\ & x_1 \geq 0, x_2 \geq 0 \end{cases}$$

$$(P)\begin{cases} \max(Z) = & x_1 + x_2 \\ s.t & 3x_1 + 2x_2 \leq 40 \\ & x_1 \leq 10 \\ & x_2 \leq 5 \\ & x_1 \geq 0, x_2 \geq 0 \end{cases}$$



$$(P)\begin{cases} \max(Z) = & x_1 + x_2 \\ s.t & 3x_1 + 2x_2 \leq 40 \\ & x_1 \leq 10 \\ & x_2 \leq 5 \\ & x_1 \geq 0, x_2 \geq 0 \end{cases}$$



The optimal solution is $X^* = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$, with $Z^* = 15$. A solution X^* is called *degenerate* if at least three constraints are active (i.e., intersect) at that point.