Quiz 2 Solutions

January 22, 2014

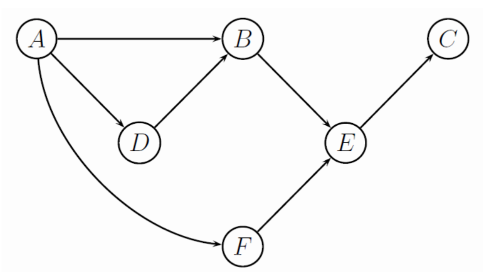
1. True or False: For **any** DAG, there must exist at least one topological ordering.

   Answer: True
   A topological ordering must always exist. We will show this inductively, based on the number of nodes in a DAG, as follows.
   Base Case: $n = 1$
   Since there is only one node, the only topological ordering is this single node. Thus the claim is true for $n = 1$.
   Induction hypothesis: Claim is true for all DAGs with at most k nodes.
   Induction Step: Let G be any DAG with $k + 1$ nodes. Since G is a DAG, there is at least one sink node v in G. Remove v from G to get another graph G' with k nodes. Since, G' has a topological ordering (by Induction Hypothesis), we can get a topological ordering for G by placing node v at the end of G's topological ordering.

2. An arbitrary DAG $G = (V, E)$ is given as an adjacency list. Which of the following can be completed in linear time? You will get credit ONLY for marking all valid choices (and no other choices).
   A. Reverse all edges of G
   B. Identify all SCCs in G
   C. Find all source and sink nodes in G
   D. Topologically sort the vertices of G

   Answer: ABCD. All of the above can be completed in linear time.

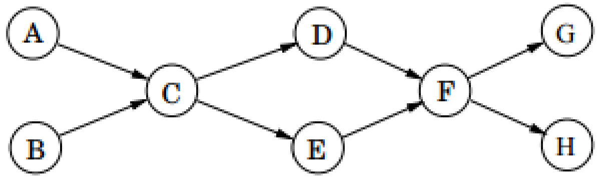
3. Which of the following are possible topological orderings of the vertices of this graph? Mark ALL answers that give valid topological orderings. You will get credit ONLY for marking all valid topological orderings (and no other orderings).



A. ADFBEC
B. AFDBEC
C. AFEDBC
D. ADBFEC

Answer: The correct options are **A**, **B** and **D**.
Option **C** is not a topological ordering because in **C**, node E is placed before node B, but there is an edge from node B to node E.

4. How many topological orderings does the following directed graph have?



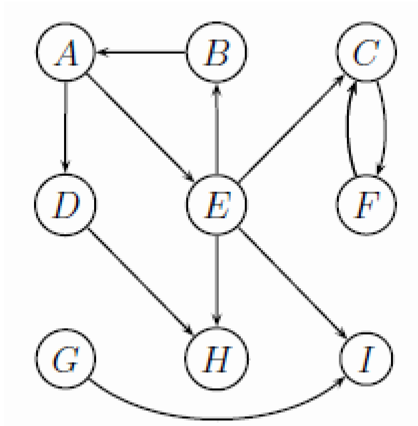
(Credit: Dasgupta et al., page 106.)

Answer: 8. Topological ordering proceeds by choosing one of the source vertices present in the graph, removing all outgoing edges from that vertex, and repeating. In this graph, any of the two initial sources A and B can be chosen, but only after both are chosen can C be added to the ordering. Similarly, after C is chosen, D and E can be chosen in any order, followed by F, and then G and H. Therefore, the total number of orderings is given by $2 \times 2 \times 2 = 8$.

5. Given a directed graph $G = (V, E)$ with $|V| = n$ vertices. What is the minimum number of edges possible in the edge set E, given that all n vertices of V belong to the same strongly connected component?
A. 1
B. $n - 1$
C. $\log n$
D. n
E. $n/2$

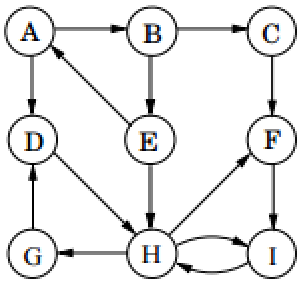
Answer: The correct option is **D**.
By taking the example of a cyclic graph $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{v_1v_2, v_2v_3, \ldots, v_nv_1\}$, we see that it is possible to get a strongly connected component of n nodes with only n edges.
We can see that at least n edges are required, as follows. Since G is an SCC, for every pair of nodes u, v, there exists a path from u to v and a path from v to u. Consider any pair of nodes u, v. Since v is reachable from u, v must have an in-degree of at least one. Similarly, since there is a path from v to u, the out-degree of v is at least one. Thus we get that for every node v, both the in-degree and the out-degree are at least one. Summing the in-degrees and out-degrees across all n nodes, we get that $\sum_{i=1}^{n}[\text{in-degree}(v_i) + \text{out-degree}(v_i)] \geq 2n$. Since $\sum_{i=1}^{n}[\text{in-degree } (v_i) + \text{ out-degree}(v_i)] = 2|E| \geq 2n$. We get that if G is strongly connected then $|E| \geq n$.

6. Let $G = (V, E)$ be the graph given below. Let M be the number of vertices in the meta-graph constructed from G, and N be the total number of vertices in all the sink SCCs of the graph G. What is the value of $M \times N$? (Enter a positive integer.)



Answer: 24 ($M = 6, N = 4$)
The meta-graph will consist of the components $\{(C, F), (G), (H), (I), (D), (A, B, E)\}$. Thus $M = 6$. The sink SCCs are $(C, F), (H)$ and (I). Thus $N = 4$.

7. What is the minimum number of edges you must add to the following directed graph to make it strongly connected?



(Credit: Dasgupta et al., page 107.)

Answer: 1
We can add the edge (D, E) to make the graph strongly connected. We can think of the meta-graph and try to make the meta-graph a strongly connected component. Doing this, we see that any edge directed from any node in the SCC (D, G, H, I) to any node in the SCC (A, B, E) will make the whole graph connected.

8. The runtime of a divide-and-conquer algorithm is described by the following recurrence: $T(n) = 3T(n/2) + O(1)$. How many subproblems will we have at the 5^{th} level of recursion if the top level is considered to be the 0^{th} level?

Answer: $3^5 = 243$
From the recurrence relation, we note that the branching factor $a = 3$. At level 0, there is only one problem. For every subsequent level, the number of subproblems is multiplied by a. Therefore, at the 5^{th} level of recursion, the total number of subproblems is given by: $a^5 = 3^5 = 243$.

9. The runtime of a divide-and-conquer algorithm is described by the following recurrence: $T(n) = 4T(n/2) + n$. What is the total contribution of the "$+ n$" term over all subproblems at the third level of the recursion?
A. n
B. $2n$
C. $4n$
D. $8n$
E. $16n$

Answer: The correct option is **D**.
The "$+ n$" term indicates that the amount of work done at each level is linear in the problem size. The total amount of work done at a particular level is given by the total problem size of the subproblems at each level. From the recurrence relation, we can observe that at the third level of recursion, there will be $4^3 = 64$ subproblems of size $n/2^3 = n/8$ each. Hence, the total amount of work done is $64 \times n/8 = 8n$.

10. Suppose that a certain divide and conquer (DQ) algorithm is described by the following recurrence relation: $T(n) = 9T(n/3) + O(n)$. Applying Master theorem, what is the time complexity of the above mentioned algorithm?
A. $T(n) = O(n^3)$
B. $T(n) = O(n\ log(n))$
C. $T(n) = O(n^2)$
D. None of the above

Answer: The correct option is **C**.
In the given recurrence equation, $a = 9, b = 3$ and $d = 1$. Since $d = 1 < \log_b a = \log_3 9 = 2$, we obtain $T(n) = O(n^{\log_3 9}) = O(n^2)$.

11. True or False: You are given an array A of size n, whose elements are in sorted order (ascending or descending), and a number k. Every algorithm that takes A and k as inputs and returns true if k is an element in A, and false otherwise, requires $\Omega(n)$ time in the worst case.

Answer: False
Binary search allows us to search in $O(\log n)$ time in the worst case.