Meth Gauss choix du pivot (de fault / partiel / total)

Gauss Jordan: Au lieu d'éliminer la variable dans la ligne qui suit seulement, on l'élimine dans toutes les lignes pour avoir une mat diag

$\frac{2n^3}{3}$ $O(n^3)$

(On fait les m̂ trans à I_n jusqu'à ce que $A \to I_n$ alors $I_n \to A^{-1}$)

Fac LU:

M_k : A after $(k-1)$ iterations, pour les $k-1$ prem colonne, les val au dessous de diag sont nulle

hyp: $(M_k)_{k,k} \neq 0$ (c'est lui qui va jouer le pivot) k colonne

$$(M_k)_{i,k} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \neq 0 \\ \vdots \\ \alpha_r \end{pmatrix} \Rightarrow E_k = \begin{pmatrix} 1 & & & & (0) \\ & \ddots & & & \\ & & 1 & & \\ & & -\alpha_{k+1}/\alpha_k & 1 & \\ (0) & & \vdots & (0) & \ddots \\ & & -\alpha_n/\alpha_n & & 1 \end{pmatrix}$$

k+1 ème ligne

$$E_k^{-1} = \begin{pmatrix} 1 & & & (0) \\ & \ddots & & \\ & & 1 & \\ & & L_{k+1,k} & \ddots \\ (0) & & \vdots & (0) \\ & & & 1 \end{pmatrix}$$

~~$L_{k+1,k} =$~~ $L_{i,k} = \alpha_i/\alpha_k$

$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{i,j}|$

$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{i,j}|$

$$L = \begin{pmatrix} 1 & & (0) \\ L_{2,1} & \ddots & \\ \vdots & & \\ L_{n,1} & \cdots & L_{n,n-1} & 1 \end{pmatrix} \Rightarrow A = LU$$

$\hookrightarrow$ (gauss 1)

Cholesky:

A sym def positive $B = \begin{pmatrix} b_{11} & & (0) \\ \vdots & \ddots & \\ b_{n,1} & \cdots & b_{nn} \end{pmatrix}$

alors $A = B B^T$

$\to$ 1ème colonne: $b_{11} = \sqrt{a_{11}}$ / $b_{i1} = \frac{a_{i1}}{b_{11}}$ $2 \le i \le n$

$\to$ $2 \le j \le n$: $b_{jj} = \sqrt{a_{jj} - \sum_{k=1}^{j-1} b_{jk}^2}$

$i \in [\![j+1, n]\!]$: $b_{i,j} = \frac{a_{ij} - \sum_{k=1}^{j-1} b_{ik} b_{jk}}{b_{jj}}$

n° d'op $\approx n^3/3$ $O(n^3)$

Jacobi:

$$J_{pq}(\theta) = \begin{array}{c} \begin{array}{ccc} & \text{p colom} & \text{q colo} \end{array} \\ \begin{pmatrix} 1 & & & & (0) \\ & \ddots & & & \\ & & c_\theta & & s_\theta \\ & & & \ddots & \\ & & -s_\theta & & c_\theta \\ (0) & & & & \ddots & 1 \end{pmatrix} \begin{array}{l} \leftarrow \text{p row} \\ \\ \leftarrow \text{q row} \end{array} \end{array}$$

Alg J_{pq}:

1) set $t_0 = \min \left| \frac{a_{pp} - a_{qq}}{2 a_{pq}} \pm \frac{\sqrt{\left(\frac{a_{pp} - a_{qq}}{a_{pq}}\right)^2 + 4}}{2} \right|$ fullfils (smallest one) ~~$t^2 + 2xt + 1 = 0$~~ ~~$x = \frac{a_{pp} - a_{qq}}{2 a_{pq}}$~~

2) $c_\theta = \frac{1}{\sqrt{1 + t_0^2}}$, $s_\theta = c_\theta t_0$

3) Return J_{pq}

$off(A) = \sum_{i=1}^{n} \sum_{j \neq i} a_{i,j}$

Alg Jacobi
- $k = 0$ $A_0 = A$
- while $off(A) > \varepsilon$ do:
  - i) choose p, q s.t $|a_{p,q}| = \max_{i \neq j} |a_{i,j}|$
  - ii) get $J_{p,q}$
  - iii) set $A_{k+1} = J_{pq}^T A_k J_{pq}$ set $k = k+1$
- return A_k.

$$B = J_{pq}^T A J_{pq} \Leftrightarrow \begin{cases} i \notin \{p,q\} \text{ and } j \notin \{p,q\} & b_{ij} = b_{ji} = a_{ij} \\ i \notin \{p,q\} & b_{pi} = b_{ip} = c a_{pi} - s a_{qi} \\ i \notin \{p,q\} & b_{qi} = b_{iq} = s a_{pi} + c a_{qi} \\ b_{pp} = a_{pp} - t a_{pq} \\ b_{qq} = a_{qq} + t a_{pq} \\ b_{pq} = b_{qp} = 0 \end{cases}$$

$\Omega = \Omega_1 \cdots \Omega_m$

1ère étape ${}^t\Omega_1 A \Omega_1 = B$

2ème ${}^t\Omega_2 B \Omega_2 = C \ldots$

Lin Prog / Simplex

LP: $\max_{x \in \mathbb{R}^n} c^T x$ $\begin{cases} Ax \leq b \\ x \geq 0 \end{cases}$ $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^n$

feasible region. $P = \{x \in \mathbb{R}^n : Ax \leq b, \ x \geq 0\}$

Th: one (exactly) of the following holds
- i) $P = \emptyset$ infeasible
- ii) $\exists$ sequence $x^{(m)} \in P$ s.t $\lim_{m \to +\infty} c^T x = +\infty$ (unbounded)
- iii) max achieved at some vertex of P

Simplex alg: Dictionary:

each basic variable
$$\begin{cases} x_{n+1} = b_1 - \sum a_{1,j} x_j \\ x_{n+2} = b_2 - \sum a_{2,j} x_j \\ \vdots \\ x_{n+m} = b_m - \sum a_{m,j} x_j \end{cases}$$

$z = \sum_{j=1}^{n} c_j x_j$ $x_1, \ldots, x_n$ non basic variable

Any n-tuple $x^* = (x_1^*, \ldots, x_n^*)$ satisfying the constraints $Ax^* \leq b$ and $x^* \geq 0$ is called feasible.
A solution $(x_1^*, \ldots x_{n+m}^*)$ is feasible iff all its values are non neg. A feasible sol that max z (objective funct) is called optimal

- unbounded z can take arbitrarily large val

basic sol, giving all non basic var $= 0$

End when $c_j < 0 \ \forall j$

obligation: $\forall i \in [\![1, n+m]\!]: x_i^* \geq 0$

who enters / leaves : n variables / m constraints : (var as the entering var if it has positive coeff)

1) Dantzig's 1st rule: choose largest positive coeff in z $O(n)$

2) Dantzig's 2st rule: choose the variable that increases z the most $O(mn)$

3) Bland's rule : choose the entering / leaving variables with the smallest index $O(n)$ (If there are more than one, we apply one of these)

Dict is degenerate:
if the basic sol has some basic var that are null

If dict is degenerate, just apply Bland's rule, you are sure that you are not cycling.

Dic feasible: all var have non neg val in the basic sol

Duality:

$$\text{Primal } \max c^T x \quad \text{s.t } Ax \leq b, \ x \geq 0 \longrightarrow \text{Dual } \min y^T b \quad \text{s.t } A^T y \geq c, \ y \geq 0$$

Th (weak duality) $x^* \in \mathbb{R}^n$ feasible sol of prim, $y^* \in \mathbb{R}^n$ ——— dual

then $c^T x \leq y^T b$

Th (strong duality) If either (P) or (D) is feasible then $z_P^* = z_D^*$

and if the final dict of (P) look like $z = z^* + \sum_{i=1}^{n+m} d_i x_i$

then $y_i = -d_{n+i}$ is optimal for (D)

Th (Complementary slackness)

A feasible point x^* of (P) is optimal iff
$\exists$ a feas pnt y^* of (D) s.t

Th: $\max z = \sum_{j=1}^{n} c_j x_j$ s.t $\begin{cases} \forall i \in [1, m] & \sum_{j=1}^{n} a_{i,j} x_j \leq b_i \\ \forall j \in [1, n] & x_j \geq 0 \end{cases}$ (P)

assume (P) not degenerate, variations δb_i of b_i

$\Rightarrow (P_\delta)$: $\max z = \sum_{j=1}^{n} c_j x_j$ s.t $\begin{cases} \forall i \in [1, m] & \sum_{j=1}^{n} a_{i,j} x_j \leq b_i + \delta b_i \\ \forall j \in [1, n] & x_j \geq 0 \end{cases}$

assume δb_i small enough so that the optimal basis for (P) is still feasible for (P_δ). the variation of the optimum value of z is: $\sum_{i=1}^{m} \delta b_i y_i^*$ where $(y_1^*, \ldots, y_m^*)$ opt sol for dual prob of (P)

Aux prob: when one of the ineq has a neg second mem, so the initial dic is infeasible. If we want to check the feasibility of the prob, we add to that sec mem x_0 to relax the constraint, the goal of this first phase is to max $z' = -x_0$ ($x_0 \geq$ … we start by ~~swappin~~ x_0 entering, and the basic var with the most neg val in basic sol leaves and we cont normaly. once we have a feasible basic sol with $z' = 0$ we are done, the initial dict is feasible and we put back the init obj func z, and we give $x_0 = 0$ and solve the

If some b_i's are neg (~~pro~~ dic infeasible) and all c_j are neg then we can use the dual

Th^m (complementary slackness)

1) for $1 \le i \le n$ if $\sum_{j=1}^{n} a_{i,j} x_j < b_i$ (i^{th} constraint without $=$)

then $y_i^* = 0$

2) for $1 \le j \le n$ if $x_j^* > 0$ then $\sum_{i=1}^{m} y_i^* a_{i,j} = c_j$

3) x^* and y^* opti

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define x_{min}, x_{max} for which

define step $h > 0$

-) if $f'(0) < 0$, do:

--) $x_{min} \leftarrow 0$

--) as long as $f'(h) < 0$ do:

* $x_{min} \leftarrow h$
* $h \leftarrow 2h$

--) $x_{max} \leftarrow h$

-) else if $f'(0) > 0$, do:

--) $h \leftarrow -h$

--) $x_{max} \leftarrow 0$

--) as long as $f'(h) > 0$, do:

* $x_{max} \leftarrow h$
* $h \leftarrow 2h$

--) $x_{min} \leftarrow h$.

Newton method

f class C^2

$\nabla^2 f(x^k)$ def positive

$$x^{k+1} = x^k - (\nabla^2 f(x^k))^{-1} \nabla f(x^k)$$

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x^0 chosen ~~closely~~ sufficiently close to x^*

(x^k) has quadratic convergence

A constant matrix
$$\begin{cases} \nabla(u^t A) = \nabla(u^t) A \\ \nabla(u^t v) = \nabla(u^t) v + \nabla(v^t) u. \end{cases}$$

$f\ C^1$ in x^0 $\quad f(x) = f(x^0) + (x - x^0).\nabla f(x^0) + \|x - x^0\| \varepsilon(x)$

$$g'(s) = d^t \nabla f(x^0 + s.d) \quad (g(s) = f(x^0 + s.d))$$

$f\ C^2$ in x^0 $\quad f(x) = f(x^0) + (x - x^0)^t \nabla f(x^0) + \frac{1}{2}(x - x^0)^t \nabla^2 f(x^0)(x - x^0) + \|x - x^0\|^2 \varepsilon(x)$