DATA 468: Applied Stochastic Process

by
Dr. Zakir

Department of Mathematics University of Arizona

Email: zakir@arizona.edu

Outline

- ❖ Stochastic Process
  - ➢ Continuous-time Stochastic process
  - ➢ Discrete-time Stochastic process

- ❖ Stochastic Processes as Random Functions

3 ❖ Mean Function of a Random Process

- ❖ Autocorrelation and Auto-covariance

4

- ❖ Supplementary reading

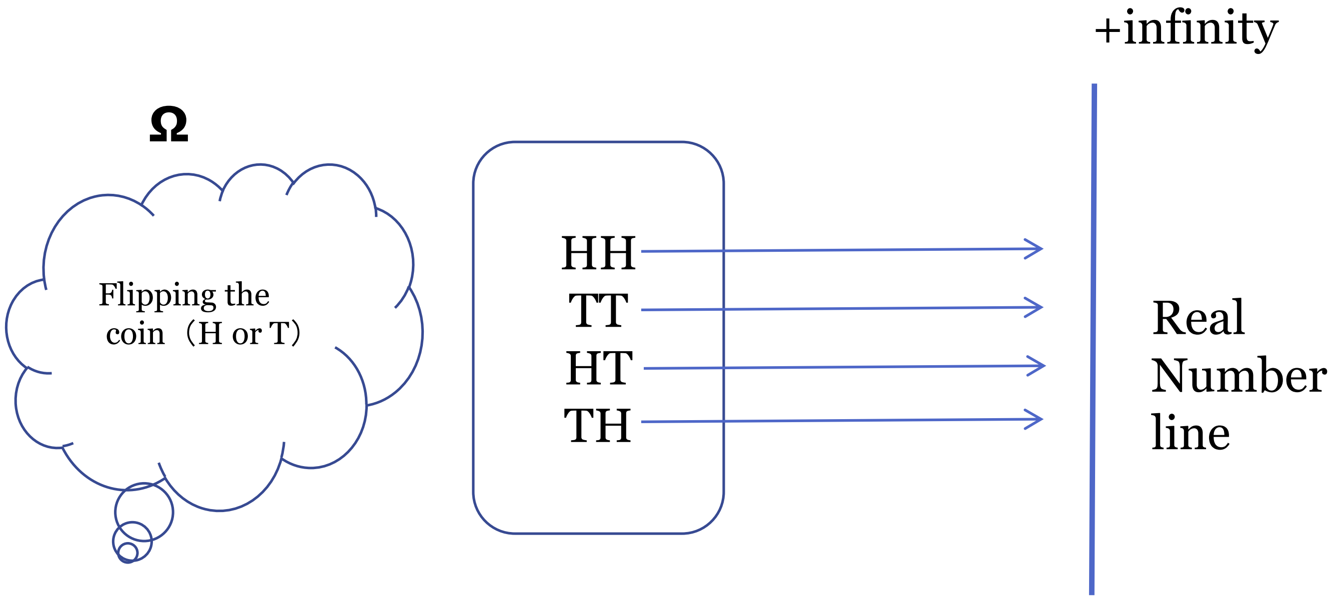
Random Variables

❖ **Random experiment**: physical situation whose outcome cannot be predicted until it is observed.

❖**Random Variable : X : S → R**. For every event in **Ω,** ***X*** is a function that maps **Ω** to a real number line **R**.

❖Consider flipping a fair coin 2 times.

Sample Space (S)	Random Variable (X)
HH	0
HT	1
TH	2
TT	3



-infinity

Outcome

$\Omega=\{\omega_1, \omega_2, \cdots, \omega_n\}$ or $\Omega=\{\omega_1, \omega_2, \cdots\}$, Event: $E=\{\omega_1, \omega_2\}$.

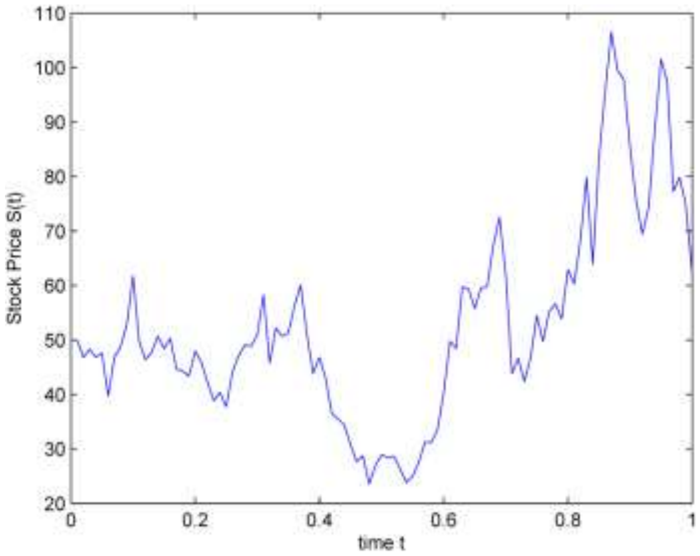
Stochastic/Random Process

❖Stochastic process assigns a function of time to each outcome, a dynamic, evolving system of random variables.

❖A set of random variables indexed by time $\{X_t\}$ where $t \subseteq R$, is called stochastic process or random process.

Examples:

- ➢Let X_t or $X(t)$ be the temperature in Beijing at time $t \in [0, \infty)$.
- ➢Let $W(t)$ be the thermal noise generated across a resistor in an electric circuit at time t, for $t \in [0, \infty)$.
- ➢Let $X(t)$ be the stock price at time t for $t \in [0, \infty)$.



Discrete and continuous-time random process

❖**Continuous-time:** $\{X(t), t \in \mathbb{R}\}$, **t** is uncountable, i.e., interval on the real line , $[-1, 1]$, $[0,\infty)$, $(-\infty, \infty)$ etc

❖**Discrete-time:** $\{X(t), t \in \mathbb{N}\}$, sequence of random variables, discrete-time random processes are sometimes referred to as random sequences.

Random Variable vs Random Process

Feature	Random Variable	Random Process
Definition	A variable whose value is subject to randomness	A collection of random variables indexed by time (or space)
Notation	X	$X(t)$or $X(t,\omega)$
Input	Probability space	Probability space and time (or space index)
Output	Single random outcome	A function over time (a stochastic signal)
Examples	Outcome of a die roll	Temperature over time at a location
Application	Probability theory, statistics	Signal processing, control systems, finance

Probability Space (Ω F P)

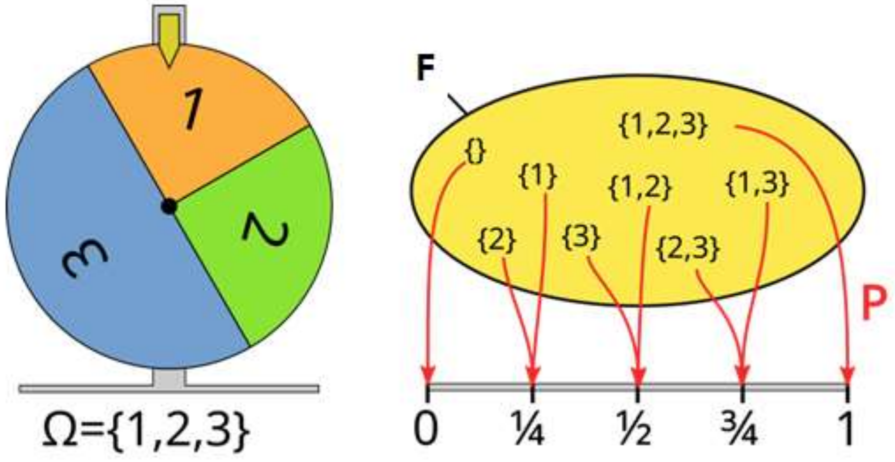
❖ Each random variable(X_t) is defined on a common probability space (Ω, F, P).

❖ (i) Sample Space/State Space(**Ω**): which lists all possible outcomes, where an outcomes is written as **ω**.

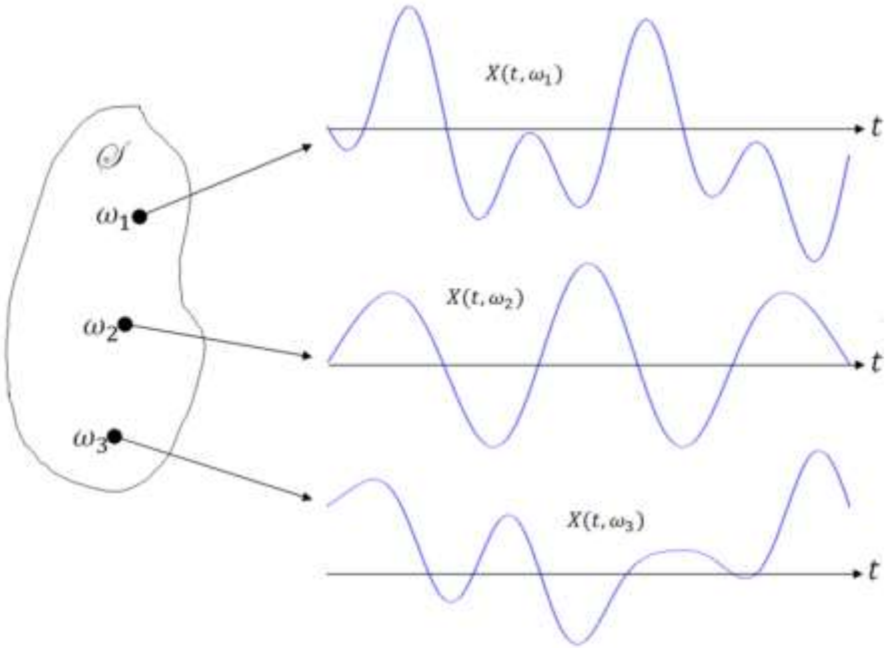
❖ (ii) **σ**-algebra (**F**): A collection of subsets of **Ω** that define events.

- If some set A is in F, then so is its complement, X\A.
- If $A_1, A_2, A_3, \ldots$ are in F, then so is $A = A_1 \cup A_2 \cup A_3 \cup \cdots$.

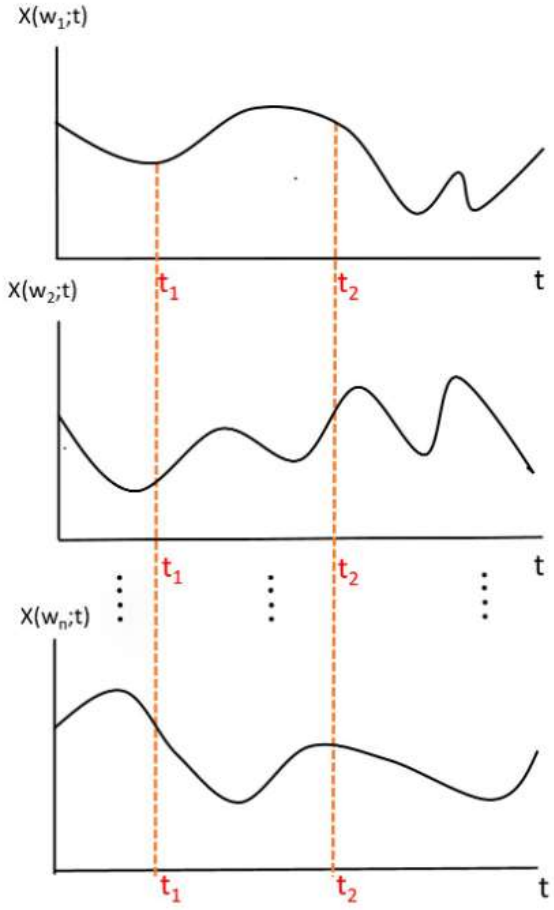
❖ (iii) **Probability Measure (P)**: A function that assigns probabilities to events in F, $\mathbf{P : F \rightarrow [0, 1]}$.



For fixed ω: $X(\omega_i, t)$ it is called a sample path or a sample function or a sample realization of $X(t)$.



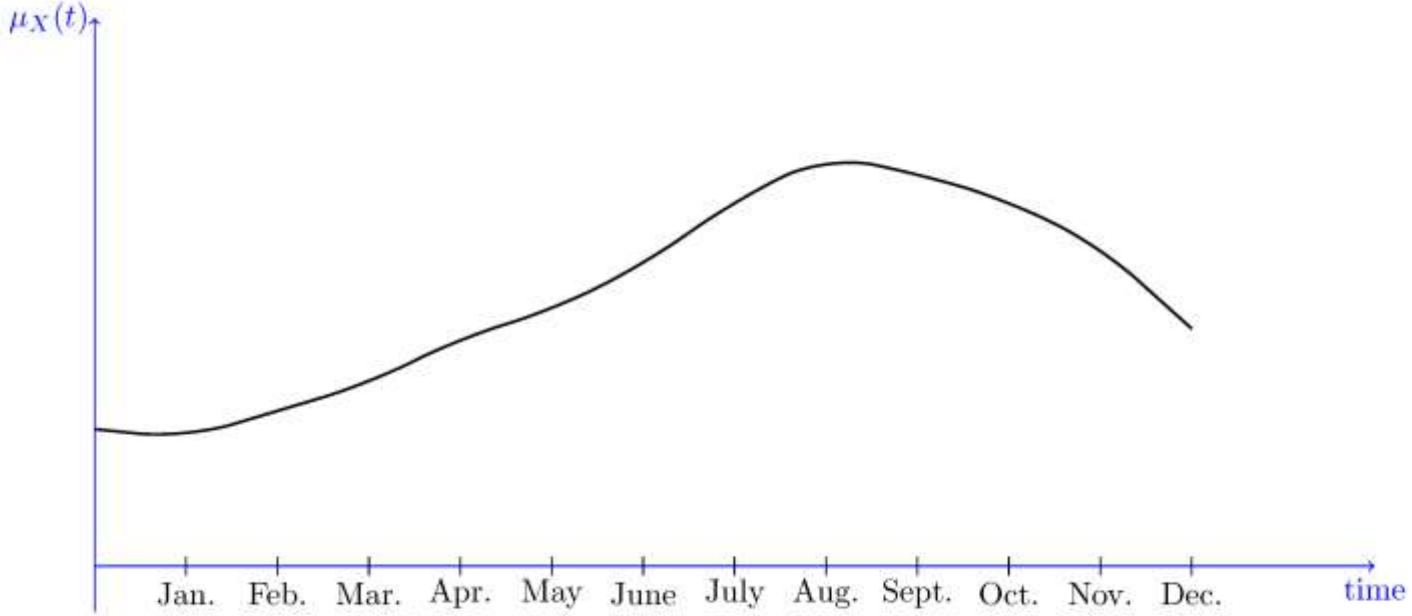
For fixed t



(1): $\mathbf{X(t, \omega) = X(t))}$: random process.
(2): $X(t_o, \omega)$: random variable for fixed t_o.
(3): $\mathbf{X(t, \omega_o)}$: real-valued function of **t** for fixed $\boldsymbol{\omega_o}$.
(4): $\mathbf{X(t_o, \omega_o)}$ is a real number for fixed $\mathbf{t_o}$ and $\boldsymbol{\omega_o}$.

Properties of Random Process: mean

- A random process is described by some properties such as the **mean**, **autocorrelation**, **cross-correlation, auto-covariance etc**.
- For a random process $\{X(t), t\in J\}$, the **mean function** $\mu_X(t): J\rightarrow R$, is defined as $\mu_X(t)=E[X(t)]$, where at each t, $X(t)$ is a random variable with its own expected value.
  - If $X(t)$ is the temperature in a certain city, the mean function $\mu_{X(t)}$ might look like the function shown in the following Figure. As we see, the expected value of $X(t)$ is lowest in the winter and highest in summer.



Example1: Mean

- You have 1000 dollars to put in an account with interest rate R, compounded annually. That is, if Xn is the value of the account at year n, then $Xn=1000(1+R)^n$, for n=0,1, 2, ⋯. The value of R is a random variable that is determined when you put the money in the bank, but it does not change after that. In particular, assume that R∼Uniform(0.04,0.05).
- Find all possible sample functions for the random process {Xn, n=0,1,2,...}.
- For any Specific value of R=r, $Xn=1000(1+r)^n$, for $r\in[0.04, 0.05]$, and $n\geq 0$.
- Sample Function of X_n is $f(n)= 1000(1+r)^n$
- Find the expected value of your account at year three($E[X_3]$=?).

$$\mu_X(n)=E[X_n]$$

$$=1000E[Y^n] \quad \text{(where } Y=1+R\sim\text{Uniform}(1.04,1.05))$$

$$=1000\int_{1.04}^{1.05} 100y^n dy$$

$$=\frac{10^5}{n+1}[y^{n+1}]_{1.04}^{1.05}$$

$$=\frac{10^5}{n+1}[(1.05)^{n+1}-(1.04)^{n+1}], \quad \text{for all } n\in\{0,1,2,3...\}.$$

Since R~Uniform(0.04,0.05) is uniform its pdf is:

$$f_R(r)=\frac{1}{1.05-1.04}=100,\ 1.04\leq n\leq 1.05$$

Example 2: Mean

- Let $\{X(t), t\in[0,\infty)\}$ be defined as $X(t)=A+Bt$, for all $t\in[0,\infty)$, A and B are independent normal $N(1,1)$ random variables.
- (1): Find all possible sample functions for this random process.
- (2): Define the random variable $Y=X(1)$. Find the PDF of Y.

Solution:

- Lets A=a and B=b, then **$X(t)=a+bt$** (sample functions , $t\geq 0$).

- $Y=X(1)=A+B$, since A and B are independent $N(1,1)=N(\mu, \sigma^2)$ random variables, $Y=A+B$ is also normal with $E(Y)=E(A+B)=E(A)+E(B)=1+1=2$
- $Var(Y)=Var(A+B)=Var(A)+Var(B)$(since A and B are independent)$=1+1=2$.

$$f_Y(y)=\frac{1}{\sqrt{4\pi}}e-\frac{(y-2)^2}{4}$$

$$f_Y(y)=\frac{1}{\sqrt{2\pi\sigma^2}}e-\frac{(y-\mu)^2}{2\sigma^2}$$

General expression of the Normal the pdf with normal distribution.

Example2: Mean

❖Let also $Z=X(2)$.

❖Find $E[YZ]$

$Var(A)=E[A^2]-(E[A])^2$

❖**Solution:**

$E[A^2]=Var(A)+(E[A])^2=2$

$E[YZ]=E[(A+B)(A+2B)]$
$=E[A^2+3AB+2B^2]$
$=E[A^2]+3E[AB]+2E[B^2]$
$=2+3E[A]E[B]+2\cdot 2$ (since A and B are independent)$=9$.

Auto-correlation

❖The correlation of the random process X(t) with itself at different points in time.

$$R_X(t_1,\ t_2) = E[X(t_1)X(t_2)]$$

$$R_{\mathrm{X}}\left(m,n\right)=\mathrm{E}[\mathrm{X}^{m}X^{n}]$$

$$=1000\mathrm{E}[\mathrm{Y}^{m}\mathrm{Y}^{\mathrm{n}}] \quad (\text{where } \mathrm{Y}=1+\mathrm{R}\sim\mathrm{Uniform}(1.04,1.05))$$

$$=10^{6}\int_{1.04}^{1.05}100\mathrm{y}^{\mathrm{m+n}}\mathrm{dy}$$

$$=\frac{10^{8}}{\mathrm{m+n+1}}[\mathrm{y}^{m+n+1}]_{1.04}^{1.05}$$

$$=\frac{10^{8}}{\mathrm{m+n+1}}[(1.05)^{\mathrm{m+n+1}}-(1.04)^{\mathrm{m+n+1}}], \quad \text{for all } \mathrm{m,n}\in\{0,1,2,3...\}.$$

Auto-covariance

❖The covariance of the random process X(t) with itself at different points in time.

$$C_X(t_1,\ t_2) = Cov(X(t_1),\ X(t_2)) = E[(X(t_1) - E[X(t_1)])\ (X(t_2) - E[X(t_2)])]$$
$$=\ E[X(t_1),\ X(t_2)] - E[X(t_1)]E[X(t_2)]$$

$$C_{\mathrm{X}}(m,n) = \mathrm{R}_{\mathrm{X}}(\mathrm{m},\ \mathrm{n}) - \mathrm{E}[\mathrm{X}_m]E[X_n]$$

$$= \frac{10^8}{\mathrm{m+n+1}}[(1.05)^{\mathrm{m+n+1}} - (1.04)^{\mathrm{m+n+1}}]$$
$$- \frac{10^{10}}{(\mathrm{m+1})(\mathrm{n+1})}[(1.05)^{\mathrm{m+1}} - (1.04)^{\mathrm{n+1}}][(1.05)^{\mathrm{m+1}} - (1.04)^{\mathrm{n+1}}]$$

PDF of a random process

❖ A **PDF** describes the **likelihood** of a continuous random variable taking on a particular value.

❖ For a continuous variable X, the PDF is a function f(x) such that:

$$P(a \le X \le b) = \int_a^b f(x)dx \qquad f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e - \frac{(y-\mu)^2}{2\sigma^2}$$

❖ The PDF itself **does not give probabilities directly** — the probability that X is in an interval is the **area under the curve** of f(x) over that interval.

❖ Random processes are characterized by joint probability distributions, which describe the probabilities of the variables at different time points.

PDF: Example

- ❖Consider the random process $\{X_n, n=0,1,2,\cdots\}$, in which X_i's are i.i.d. **standard normal random variables**. Write down $f_{X_n}(x)$ for $n=0, 1, 2, \cdots$.
- ❖Write down $f_{X_m X_n}(x_1, x_2)$ for $m \neq n$, Since $X_n \sim N(0,1)$, we have

$$f_{Xn}(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \quad for \ all \ x \in R$$

If $m \neq n$, then X_m and X_n are independent (because of the i.i.d. assumption), so

$$f_{Xm\,Xn}(x_1, x_2) = f_{Xm}(x_1) f_{Xn}(x_2) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x_1^2}{2}} . \frac{1}{\sqrt{2\pi}} e^{\frac{-x_2^2}{2}} = \frac{1}{2\pi} \mathrm{e}\{-\frac{x_1^2 + x_2^2}{2}\}$$

CDF of a Random process

❖ Consider the random process $\{X(t), t\in J\}$. For any $t_0 \in J$, $X(t_0)$ is a random variable, so we can write its CDF.

$$F_{X(t_0)}(x) = P\left(X(t_0) \le x\right) \qquad F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt, \quad for\, x \in R.$$

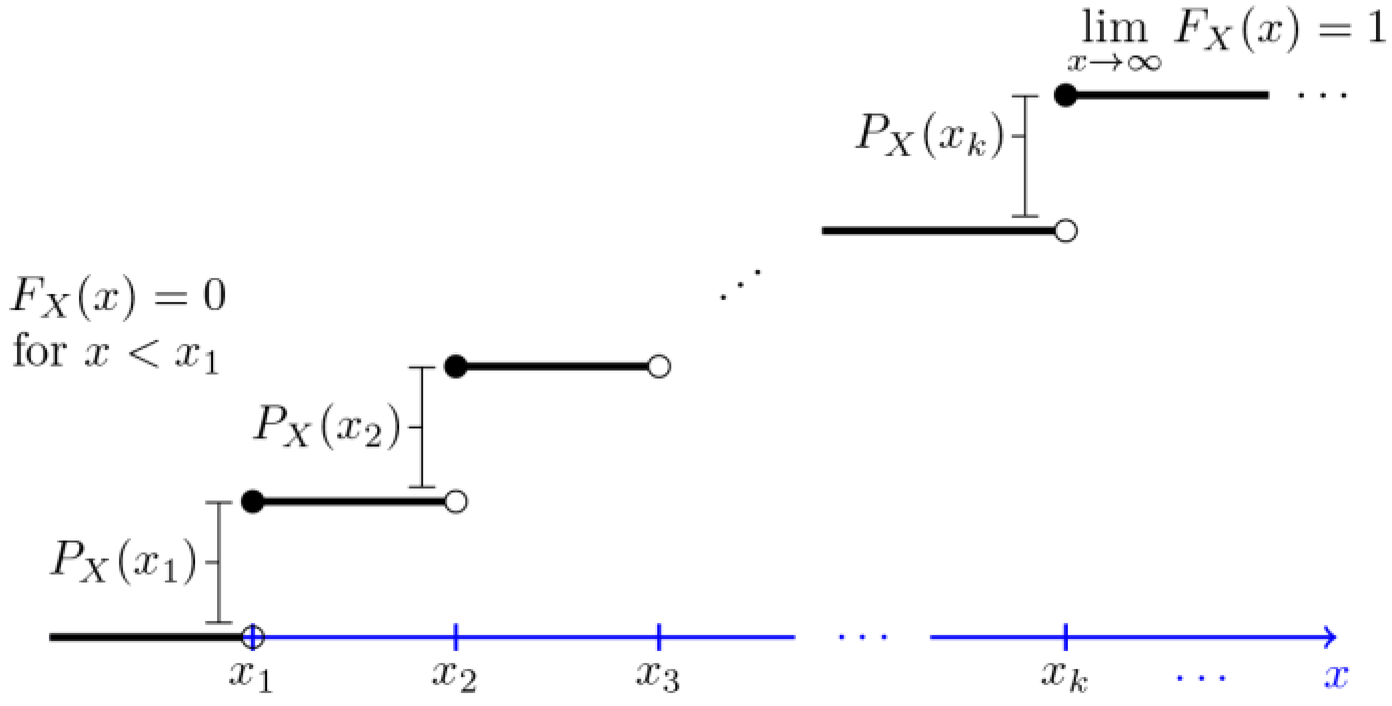
❖ If $t_1, t_2 \in J$, and $X(t_1)$ and $X(t_2)$, then a random process has a **family of CDFs**, one for each time t.

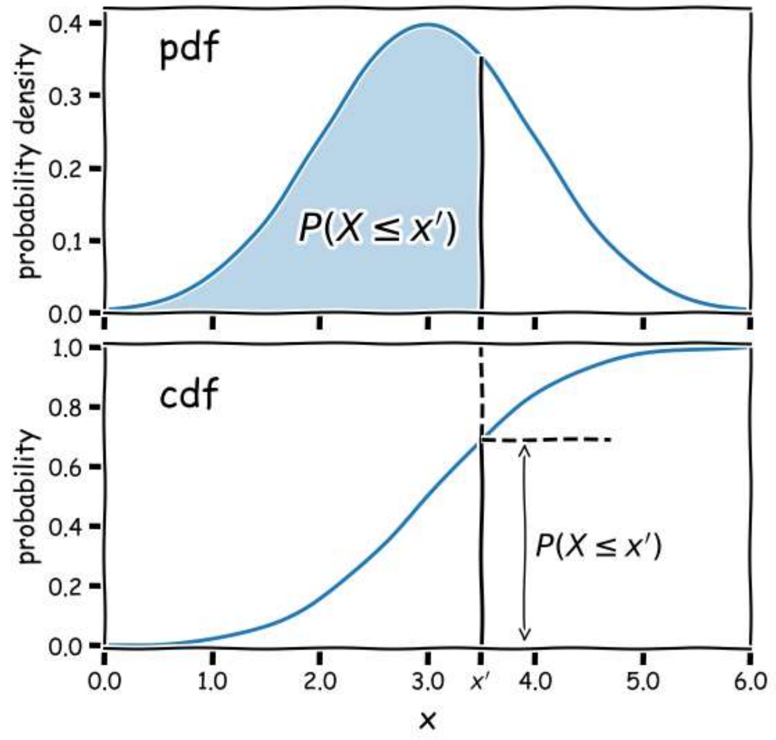
$$F_{X(t_1)X(t_2)}(x_1, x_2) = P\left(X(t_1) \le x_1, X(t_2) \le x_2\right) = P\left(X(t_1) \le x_1\right). P\left(X(t_2) \le x_2\right)$$

❖ For $t_1, t_2, \cdots, t_n \in J$, we can write $F_{X(t1)\, X(t2)\cdots,\, X(tn)}(x_1, x_2,\cdots, x_n)=P(X(t_1)\le x_1, X(t_2)\le x_2, \cdots, X(t_n)\le x_n)$.

Supplementary readings: PDF, CDF & PMF

$$CDF : F(x) = \int_{-\infty}^{x} f(x)dx, \; PDF : f(x) = \frac{d}{dx}F(x)$$

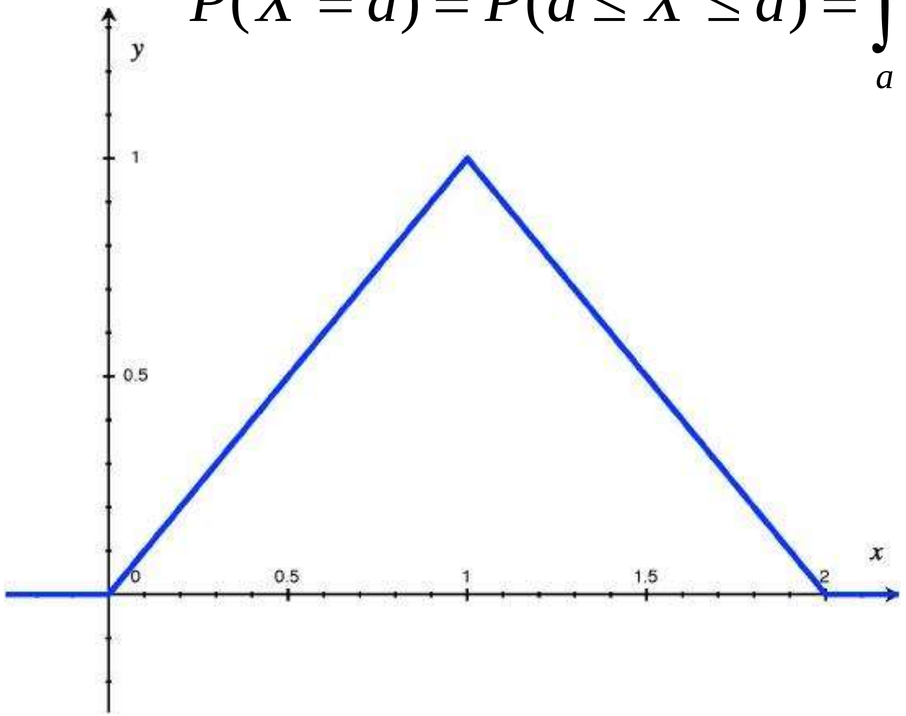




Supplementary readings: Example

❖ Let X denote the time a person waits for an elevator to arrive. Suppose the longest wait for is 2 minutes, so that the possible values of X (in minutes) are given by the interval [0,2]. Calculate the probability that a person waits less than 30 seconds for the elevator to arrive P(0≤X≤0.5).

$$P(X=a)=P(a\le X\le a)=\int_a^a f(x)dx=0 \quad \int_{-\infty}^{\infty} f(x)dx=\int_0^2 xdx=\int_0^1 xdx+\int_1^2 (2-x)dx=1$$



$$f(x)=\begin{cases} x, & for\, 0\le x\le 1 \\ 2-x, & for\, 1<x\le 2 \\ 0, & \text{otherwise} \end{cases}$$

Supplementary **readings: CDF Examples**

Outcome	Probability	Cumulative Probability
2	1/36	1/36
3	2/36	3/36
4	3/36	6/36
5	4/36	10/36
6	5/36	15/36
7	6/36	21/36
8	5/36	26/36
9	4/36	30/36
10	3/36	33/36
11	2/36	35/36
12	1/36	36/36=1

Supplementary reading: **Famous Probability Distributions**