Proof of the Cauchy-Schwarz Inequality

Prove the Cauchy-Schwarz inequality:

$$|a \cdot b| \leq \|a\|\|b\|$$

with equality if and only if a and b are scalar multiples of each other.

Lemma 1. For any z in R^n, $\|z\|^2 = z \cdot z$.

Lemma 2. For t in R and z in R^n, $tz \cdot tz = t^2(z \cdot z)$.

Lemma 3. $\|z\|^2 = 0$ if and only if $z = 0$.

Proof.

This is reminiscent of a quadratic equation of the form $xt^2 + yt + z$ where $x = \|b\|^2$, $y = 2(a \cdot b)$, and $z = \|a\|^2$, with a general equation of the discriminant $b^2 - 4ac$. The discriminant of this quadratic equation is $(2a \cdot b)^2 - 4\|a\|^2\|b\|^2$.

$\|a + tb\|^2$ (the non-expanded form of the quadratic equation) is 0 if and only if $a + tb = 0$ (they have the same roots) by Lemma 3. Thus, we want to check to find solutions to $a + tb = 0$. As this is degree 1, it has at most 1 root. Alternatively, it could be reasoned that $\|a + tb\|^2$ is nonnegative, so thus cannot make more than 1 root. Because of this, the discriminant cannot be positive, as that would indicate it has 2 roots. If there was 1 solution to $a + tb = 0$, that would mean there exists some t s.t. $a = -tb$. This is possible if and only if a and b are scalar multiples of each other (linearly dependent).

The 0 solutions case is valid for linearly independent vectors. Thus, we can say the discriminant is nonpositive and 0 iff a and b are scalar multiples of each other. Now, we can make the discriminant expression into an inequality, as we know it is nonnegative.

$$(2a \cdot b)^2 - 4\|a\|^2\|b\|^2 \leq 0$$

Continuing from this equation with Lemma 1 and 2:

$$(2a \cdot b)^2 - 4\|a\|^2\|b\|^2 \iff (2a \cdot b)^2 \leq 4\|a\|^2\|b\|^2$$

$$(2a \cdot b)^2 = (2a \cdot b)(2a \cdot b) = 4(a \cdot b)^2$$

So, $4(a \cdot b)^2 \leq 4\|a\|^2\|b\|^2 \iff (a \cdot b)^2 \leq \|a\|^2\|b\|^2 \iff |a \cdot b| \leq \|a\|\|b\|$ Thus, $|a \cdot b| \leq \|a\|\|b\|$ and $|a \cdot b| = \|a\|\|b\|$ if and only if $a = cb$ for some $c \in R$.

Proof of Lemmas

Lemma 1. For any $z \in R^n$, $\|z\|^2 = z \cdot z$.

Proof. $\|z\|^2 = \sqrt{z_1^2 + z_2^2 + \ldots + z_n^2}^2 = z_1^2 + z_2^2 + \ldots + z_n^2$.

$z \cdot z = z_1 \cdot z_1 + z_2 \cdot z_2 + \ldots + z_n \cdot z_n = z_1^2 + z_2^2 + \ldots + z_n^2$

Thus, $\|z\|^2 = z \cdot z$.

Lemma 2. For $t \in R$ and $z \in R^n$, $tz \cdot tz = t^2(z \cdot z)$.

Proof. Using a property of the inner product, that $c\langle u, v\rangle = \langle cu, v\rangle = \langle u, cv\rangle$, two factors of t can be pulled out.

$$\langle tz, tz\rangle = t\langle z, tz\rangle = t^2\langle z, z\rangle$$

Alternatively,

$$tz \cdot tz = tz_1 \cdot tz_1 + tz_2 \cdot tz_2 + \ldots + tz_n \cdot tz_n = t^2z_1^2 + t^2z_2^2 + \ldots + t^2z_n^2 = t^2(z \cdot z)$$

Given $a, b \in R^n$, and using Lemma 1 and Lemma 2.

$\|a+tb\|^2 = (a+tb)\cdot(a+tb) = a \cdot a + 2(a \cdot tb) + tb \cdot tb = a \cdot a + 2t(a \cdot b) + t^2(b \cdot b) = \|a\|^2 + 2(a \cdot b) + t^2\|b\|^2$

Lemma 3. $\|z\|^2 = 0$ if and only if $z = 0$.

Proof. $\|z\|^2 = \sqrt{z_1^2 + \ldots + z_n^2}^2 = z_1^2 + \ldots + z_n^2$. If z was the 0 vector, then each z_i would be 0 and the norm would be 0. If $\|z\|^2 = 0$, knowing that z_i^2 are nonnegative implies each z_i is nonnegative, so each z_i must be 0.