- A full rank matrix, , is one in which has *N* linearly independent column vectors (also rows). Two key implications arise from this:
  - Transforming the vectors space spanning , the resultant set also spans . So we say that the matrix .
  - A logical extension is that ***A*** is a unique mapping, i.e. has a unique solution, ***x***, for all ***y*** .
- Matrices with rank < N are referred to as ***singular***.

Matrix rank

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$$AB = \sum_{k=0}^{M-1} a_{i,k} b_{k,j}$$

$$\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 7 \\ 0.5 & 5 & 15.5 \\ 1 & 3 & 10 \end{bmatrix}$$

$$\mathbb{R}^N$$

$$\mathbb{R}^N$$

$$\mathbf{A} \in \mathbb{R}^{N \times N}$$

$\mathbf{A}$ spans $\mathbb{R}^N$

$$\mathbf{Ax} = \mathbf{y}$$

Singular matrix?

$$\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix} = \begin{bmatrix} 1 & 6 & 7 \\ 0.5 & 5 & 15.5 \\ 1 & 3 & 10 \end{bmatrix}$$

Singular matrix?

$\in \mathbb{R}^N$

$$1A_{:,1} + 3A_{:,2} = A_{:,3}$$

Singular. 3rd col is linear combination of first two

Full rank, all columns are linearly independent

Matrix rank

- A full rank matrix, $\mathbf{A} \in \mathbb{R}^{N \times N}$, is one in which has *N* linearly independent column vectors (also rows). Two key implications arise from this:
  - Transforming the vectors space spanning $\mathbb{R}^N$, the resultant set also spans $\mathbb{R}^N$. So we say that the matrix $\mathbf{A}$ spans $\mathbb{R}^N$.
  - A logical extension is that ***A*** is a unique mapping, i.e. $\mathbf{Ax} = \mathbf{y}$ has a unique solution, ***x***, for all ***y*** $\in \mathbb{R}^N$.
- Matrices with rank < N are referred to as ***singular***.

Singular matrix?

$$\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 7 \\ 0.5 & 5 & 15.5 \\ 1 & 3 & 10 \end{bmatrix}$$

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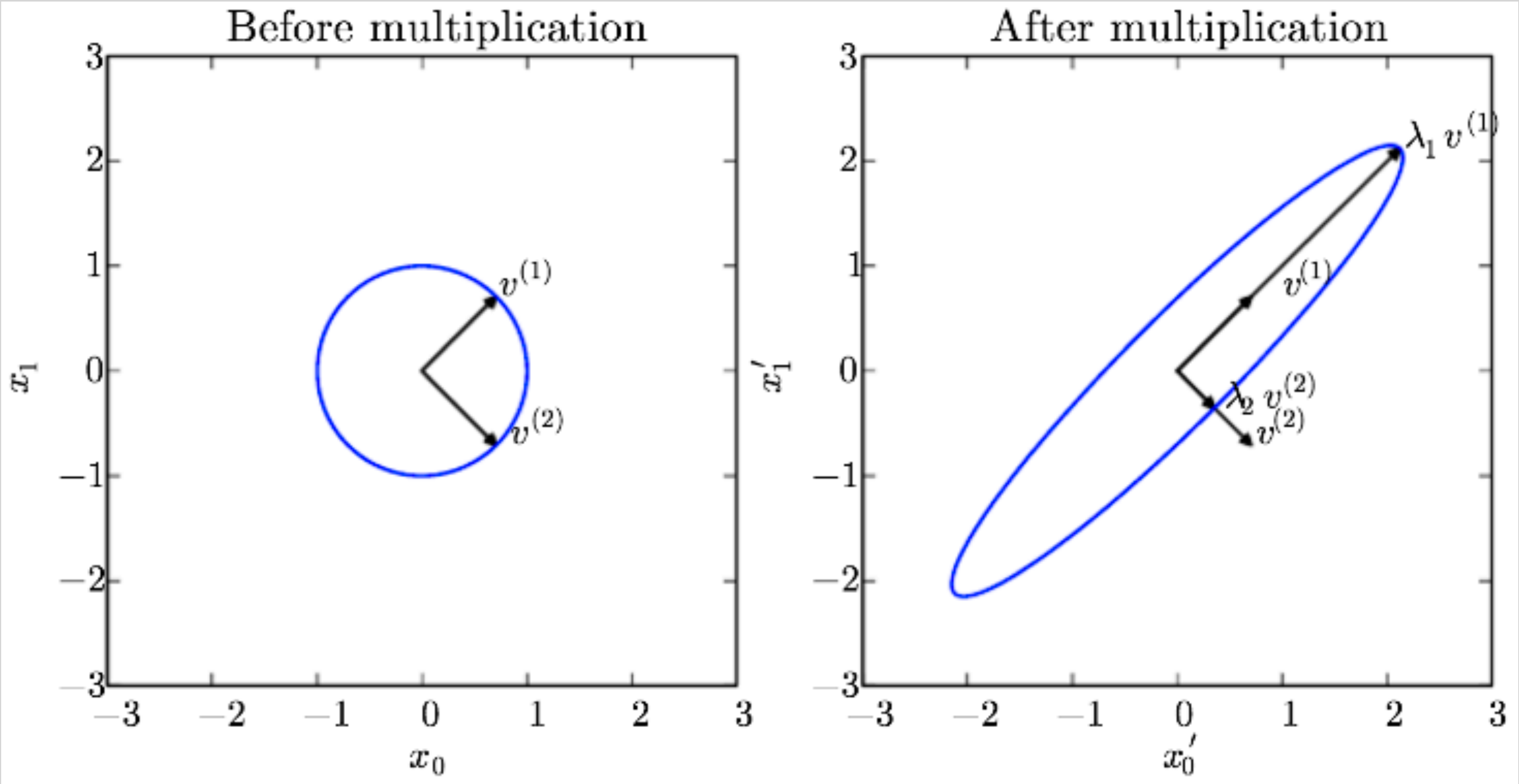
Full rank, all columns are linearly independent

Eigendecomposition

- Definition: $\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$
- Decomposition: $\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^T$ *where* $\mathbf{Q} = \begin{bmatrix} v_1^{(1)} & \dots & v_1^{(N)} \\ \vdots & \vdots & \vdots \\ v_N^{(1)} & \dots & v_N^{(N)} \end{bmatrix}$ *and* $\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_N \end{bmatrix}$
- Applies only to square matrices



Goodfellow et al., 2016, *Deep Learning*