Vector spaces, inner & outer products

- In this class we represent data in a vector space
  - This means a data point lies in a grid (2D), cube (3D), or hypercube (4D+)

  … in the general case it's *N*-dimensional
  - A point in space is represented by a vector
    - vector: $\mathbf{x} \in \mathbb{R}^N$
    - its scalar components: $x_i \in \mathbb{R} \quad \textit{where} \quad 0 \leq i < N$
    - concretely: $\mathbf{x} = [x_0, x_1, \ldots., x_{N-1}]^T$

- Inner product (dot product in this class): $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=0}^{N-1} a_i \, b_i \quad \textit{where} \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^N$

- Outer product: $\mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b}^T = \begin{bmatrix} a_1 b_1 & \ldots & a_1 b_N \\ \vdots & \ddots & \vdots \\ a_N b_1 & \ldots & a_N b_N \end{bmatrix}$

Tensors

- Formal definition: context dependent
- Definition in this class: A linear transformation within or between vector spaces
- Example: $\mathbf{Ax} = \begin{bmatrix} A_{1,1} & \dots & A_{1,N} \\ \vdots & \ddots & \vdots \\ A_{N,1} & \dots & A_{N,N} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} A_{1,i}\ x_i \\ \vdots \\ \sum_{i=1}^{N} A_{N,i}\ x_i \end{bmatrix}$
- Matrix product (general form of op above): $\mathbf{AB} = \sum_{k=0}^{N-1} A_{i,k} B_{k,j}$
- Hadamard (a.k.a. element-wise, a.k.a. Schur) product: $\left(\mathbf{A} \circ \mathbf{B}\right)_{i,j} = (A_{i,j})(B_{i,j})$

$$\mathbf{A} \circ \mathbf{B} = \begin{bmatrix} A_{1,1} & \dots & A_{1,N} \\ \vdots & \ddots & \vdots \\ A_{N,1} & \dots & A_{N,N} \end{bmatrix} \cdot \begin{bmatrix} B_{1,1} & \dots & B_{1,N} \\ \vdots & \ddots & \vdots \\ B_{N,1} & \dots & B_{N,N} \end{bmatrix} = \begin{bmatrix} A_{1,1}B_{1,1} & \dots & A_{1,N}B_{1,N} \\ \vdots & \ddots & \vdots \\ A_{N,1}B_{N,1} & \dots & A_{N,N}B_{N,N} \end{bmatrix}$$