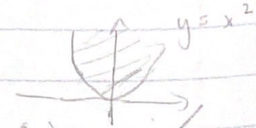
ANLY 580 - Assignment 1

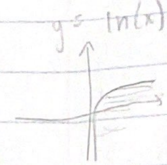
1) Establish the convexity of the following functions:



(a) $f(x) = x^2$

$f'(x) = 2x$ convex.

$\boxed{f''(x) = 2} \Rightarrow f''(x) \geq 0 \quad \forall x \in dom(f)$ ✓



(b) $f(x) = \ln(x)$

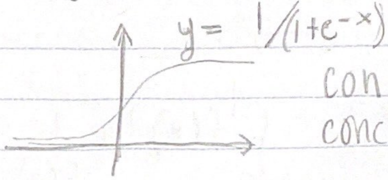
$f'(x) = 1/x$

$f''(x) = -1/x^2 \Rightarrow f''(x) < 0 \therefore$ neither convex or concave but over a "finite" interval $(0, \infty)$ is concave.

(c) $f(x) = 1/(1+e^{-x})$

$f'(x) = e^{-x}/(1+e^{-x})^2$

$f''(x) = \dfrac{2e^{-2x}}{(1+e^{-x})^3} - \dfrac{e^{-x}}{(1+e^{-x})^2}$



convex on $(-\infty, 0]$
concave on $[0, \infty)$

2) $F(x) = \dfrac{1}{\theta - 0} = \dfrac{1}{\theta}$ uniform dist. on $[0, \theta]$

(a) $E_x[x] = \dfrac{1}{2(a+b)} = \dfrac{1}{2\theta}$

Proof: $\displaystyle\int_0^\theta \frac{x}{\theta - 0}\, dx$

$= \dfrac{1}{\theta} \displaystyle\int_0^\theta x\, dx = \dfrac{1}{\theta}\left[\dfrac{x^2}{2}\Big|_0^\theta\right]$

$= \dfrac{1}{\theta}\dfrac{\theta^2}{2} - 0 = \dfrac{1}{2\theta}$ ✓

-QED

(b) $\mathrm{var}(X) = \frac{1}{12}(b-a)^2 = \frac{1}{12}(\theta - 0)^2 = \frac{\theta^2}{12}$

Proof: $\mathrm{var}(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$

$$= \int_0^\theta \frac{x^2}{\theta}\,dx - \left[\int_0^\theta \frac{x}{\theta}\,dx\right]^2$$

$$= \frac{1}{\theta}\left(\frac{x^3}{3}\right)\Big|_0^\theta - \left[\frac{1}{\theta}\,\frac{x^2}{2}\Big|_0^\theta\right]^2$$

$$= \frac{\theta^3}{3\theta} - \left(\frac{\theta^2}{2\theta}\right)^2 = \frac{\theta^2}{3} - \frac{\theta^4}{4\theta^2} = \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{4\theta^2}{12} - \frac{3\theta^2}{12}$$

$$= \frac{\theta^2}{12}$$ ✓

- QED

(c) $H(X) = \int_S f(x) \ln \frac{1}{f(x)}\,dx$ (because we are working with continuous dist)

where $f(x) = \begin{cases} \frac{1}{b-a} & \text{for } x \in [a,b] \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{\theta - 0} & x \in [0,\theta] \\ 0 & \text{otherwise} \end{cases}$

Thus, $H(X) = \int_0^\theta \frac{1}{\theta} \ln\left(\frac{1}{\frac{1}{\theta}}\right) dx = \int_0^\theta \frac{1}{\theta} \ln(\theta)\,dx$

$$= \frac{1}{\theta}\ln(\theta)\int_0^\theta dx = \frac{1}{\theta}\ln\theta\,[1]\Big|_0^\theta = \frac{1}{\theta}\ln\theta \cdot \theta$$

$$= \ln(\theta)$$ ✓

- QED

③ $X \sim \mathrm{Uni}(0, \theta)$

We know $f(x|\theta) = \frac{1}{\theta}$ for $x \in [0, \theta]$. Let's first look at the likelihood function, $\ell(\theta)$. We have M independent samples, so

$$\ell(\theta) = \prod_{i=1}^{M} f(X_i|\theta) = \frac{1}{\theta^M} \quad \text{for } X_1, \ldots, X_M \in [0, \theta]$$

We can then say from the above statement that:

$$\ell(\theta) = 0 \quad \text{if } \theta < \max(X_1, \ldots, X_M)$$

AND

$$\ell(\theta) = \frac{1}{\theta^M} \quad \text{if } \theta \geq \max(X_1, \ldots, X_M)$$

Since the likelihood function is decreasing, we see that $\hat{\theta} = \max(X_1, \ldots, X_M)$.

④ Monty Hall Problem:

[Bayes' Theorem]

- $P(A)$ = probability your choice is \$1 MIL
- $P(B)$ = prob has crickets behind it

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)} = \frac{P(\text{crickets} \mid \text{million})\,P(\text{million})}{P(\text{crickets}|\text{million})\,P(\text{million}) + P(\text{crickets}|\sim\text{MILLION}) \cdot P(\sim\text{MILLION})}$$

$P(\text{crickets} \mid \text{MILLION}) = 1$

$P(\text{MILLION}) = 1/3$

$P(\text{crickets} \mid \sim\text{MILLION}) = 1$

$P(\sim\text{MILLION}) = 1 - P(\text{MILLION}) = 2/3$

$$P(A|B) = \frac{1 \cdot 1/3}{1 \cdot 1/3 + 1 \cdot 2/3} = \frac{1/3}{1} = 1/3$$

Probability your first choice is \$1 MILLION GIVEN the door shown has crickets behind is 1/3.

Thus you should switch because that probability is $1 - P(A|B) = 2/3$. There is a 2/3 chance the switched door has \$1 MIL given crickets shown.

covariance Matrix of a random vector X

(5) $\Sigma \in \mathbb{R}^{N \times N}$, $X \in \mathbb{R}^N$

$$\Sigma = \mathbb{E}_X\left[(X - \mathbb{E}_X[X])(X - \mathbb{E}_X[X])^T\right]$$

M positive semi-definite iff $x^T M_x \geq 0 \ \forall x \in \mathbb{R}^n$

For any random vector we can write

$$x^T \Sigma x = x^T \cdot \mathbb{E}_X\left[(x - \mathbb{E}_X[X])(x - \mathbb{E}_X[X])^T\right] \cdot x$$

$$= \mathbb{E}_x\left[\left[x^T \cdot [x - \mathbb{E}_x[x]]\right][x - \mathbb{E}_x[x]]^T \cdot x\right]$$

$$= \mathbb{E}_x\left[(x^T(x - \mathbb{E}_x[x]))((x - \mathbb{E}_x[x])x)\right]$$

$$= \mathbb{E}_x\left[(x - \mathbb{E}_x[x])\mathbb{E}_x[x]\right]^2$$

$$= \sigma_s^2$$

Thus the square of any real numbers in X of size n will be greater than or equal to 0.

$$x^T \Sigma x = \sigma_s^2 \geq 0$$

$\therefore \Sigma$ is positive semi-definite by proving the given above statement. ✓

-QED