1) Establish the convexity of the following functions

(a) $f(x) = x^2 \rightarrow f'(x) = 2x \rightarrow f''(x) = 2 > 0$ so $\boxed{\text{convex}}$

(b) $f(x) = \ln(x) \rightarrow f'(x) = \frac{1}{x} \rightarrow f''(x) = -\frac{1}{x^2} < 0$ so $\boxed{\text{concave}}$

(c) $f(x) = \dfrac{1}{1+e^{-x}} \rightarrow f'(x) = \dfrac{e^{-x}}{(1+e^{-x})^2} \rightarrow f''(x) = \dfrac{e^{-2x}(-e^{x}+1)}{(1+e^{-x})^3}$

convex when $-\infty < x < 0$ and concave when $0 < x < \infty$

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2) Consider a continuous random variable X that is drawn from an uniform distribution b/t 0 and θ.

$$P(X=x) = \begin{cases} \frac{1}{\theta}, & 0 < x < \theta \\ 0, & \text{elsewhere} \end{cases}$$

(a) $E_x[X] = \int x \cdot P(x)\,dx = \int_0^\theta x \cdot \frac{1}{\theta}\,dx = \frac{1}{\theta} \cdot \left(\frac{x^2}{2}\right)_0^\theta$

$= \frac{1}{\theta} \cdot \frac{\theta^2}{2} = \boxed{\frac{\theta}{2}}$

(b) $\text{var}(x) = E(x^2) - [E(x)]^2 = \int x^2 \cdot p(x)\,dx - \left[\frac{\theta}{2}\right]^2$

$= \int_0^\theta x^2 \cdot \frac{1}{\theta}\,dx - \left[\frac{\theta}{2}\right]^2 = \frac{1}{\theta} \cdot \left[\frac{x^3}{3}\right]_0^\theta = \frac{1}{\theta} \cdot \frac{\theta^3}{3} - \left[\frac{\theta}{2}\right]^2$

$= \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{4\theta^2}{12} - \frac{3\theta^2}{12} = \frac{4\theta^2 - 3\theta^4}{12} = \boxed{\frac{\theta^2}{12}}$

(c) $H(X)$ denotes the entropy of X

$H(x) = -\int p(x) \cdot \log p(x)\,dx$

$= -\int_0^\theta \frac{1}{\theta} \cdot \log\left(\frac{1}{\theta}\right) dx = -\frac{1}{\theta} \cdot \log\left(\frac{1}{\theta}\right) \cdot \int_0^\theta 1 \cdot dx$

$= -\frac{1}{\theta} \cdot \log\left(\frac{1}{\theta}\right) \cdot [x]_0^\theta = -\frac{1}{\theta} \cdot \log\left(\frac{1}{\theta}\right) \cdot \theta$

$= \boxed{-\log\left(\frac{1}{\theta}\right)}$

3) Compute maximum likelihood estimate of θ, $\hat{\theta}$

$$\prod_{i=1}^{m} f(x_i;\theta,\hat{\theta}) = \prod_{i=1}^{n} \frac{1}{\hat{\theta}-\theta} = \frac{1}{(\hat{\theta}-\theta)^m}$$

$$\log \prod_{i=1}^{m} f(x_i;\theta,\hat{\theta}) = \log \prod_{i=1}^{m} \frac{1}{\hat{\theta}-\theta} = \log((\hat{\theta}-\theta)^{-m}) = -m\log(\hat{\theta}-\theta)$$

To find the values for θ and $\hat{\theta}$ that maximize the log-likelihood

take the derivative of respect to θ and $\hat{\theta}$

$$\theta \rightarrow \frac{m}{(\hat{\theta}-\theta)} \qquad \hat{\theta} = -\frac{m}{(\hat{\theta}-\theta)}$$

So, the mle of θ would be when θ is largest : $\min(X_1, X_2, \ldots X_m)$

and, the mle of $\hat{\theta}$ would be when $\hat{\theta}$ is smallest ; $\max(X_1, X_2 \ldots X_m)$

$$\boxed{\begin{aligned} \theta_{MLE} &= \min(X_1, X_2, \ldots X_m) \\ \hat{\theta}_{MLE} &= \max(X_1, X_2, \ldots X_m) \end{aligned}}$$

4) 3 sound proof doors. Behind one door is \$1M, others crickets.

Use Bayes' Rule to determine whether to switch doors.

A: event the \$1M is behind door #1

B: event you open up door #2 to show cricket

$$P(A|B) = \frac{P(B|A)\cdot P(A)}{P(B)} = \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{3}\cdot\frac{1}{2}+\frac{1}{3}\cdot 1} = \frac{1}{3}$$

The probability of the \$1M being behind door #1 is $1/3$, so prob. of the \$1M being behind door #3 is $1-\frac{1}{3} = \boxed{2/3}$

Therefore, yes switch doors

5) Consider covariance matrix, $\Sigma \in \mathbb{R}^{N\times N}$ & random vector $X \in \mathbb{R}^N$.

Show that Σ is a posititive semidefinite matrix

co.var X $\quad \Sigma = E_X[(X - E_X[X])(X - E_X[X])^T]$

$$y^T\Sigma = y^T(E_X[(X - E_X[X])(X - E_X[X])^T)x = \Sigma(X - E_X y)^2 \geq 0$$

which implies, Σ is a positive semidefinite matrix

because $x\Sigma x \geq 0$ for all vectors x

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