Tensors

- Formal definition: context dependent

- Definition in this class: A linear transformation within or between vector spaces

- Example: $\mathbf{Ax} = \begin{bmatrix} A_{1,1} & \dots & A_{1,N} \\ \vdots & \ddots & \vdots \\ A_{N,1} & \dots & A_{N,N} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} A_{1,i} \ x_i \\ \vdots \\ \sum_{i=1}^{N} A_{N,i} \ x_i \end{bmatrix}$

- Matrix product (general form of op above): $\mathbf{AB} = \sum_{k=0}^{N-1} A_{i,k} B_{k,j}$

- Hadamard (a.k.a. element-wise, a.k.a. Schur) product: $\left(\mathbf{A} \circ \mathbf{B}\right)_{i,j} = (A_{i,j})(B_{i,j})$

$$\mathbf{A} \circ \mathbf{B} = \begin{bmatrix} A_{1,1} & \dots & A_{1,N} \\ \vdots & \ddots & \vdots \\ A_{N,1} & \dots & A_{N,N} \end{bmatrix} \cdot \begin{bmatrix} B_{1,1} & \dots & B_{1,N} \\ \vdots & \ddots & \vdots \\ B_{N,1} & \dots & B_{N,N} \end{bmatrix} = \begin{bmatrix} A_{1,1}B_{1,1} & \dots & A_{1,N}B_{1,N} \\ \vdots & \ddots & \vdots \\ A_{N,1}B_{N,1} & \dots & A_{N,N}B_{N,N} \end{bmatrix}$$

Vector and tensor norms

- Measure of the size, or magnitude of a vector or tensor

- Definition: The following criteria qualify $f(\cdot)$ as a norm:
  - Positive definite: $f(\mathbf{x}) = 0 \quad iff \quad \mathbf{x} = \mathbf{0}$
  - Triangle inequality: $f(\mathbf{x}+\mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$
  - Homogeneity: $\forall \alpha \in \mathbb{R} : f(\alpha \mathbf{x}) = |\alpha| f(\mathbf{x})$

- The ones we care about are:
  - Lp norm: $\|\mathbf{x}\|_p = \left(\sum_i |x_i|^p \right)^{1/p}$
  - Frobenius norm: $\|\mathbf{A}\|_F = \sqrt{\sum_{i,j} A_{i,j}^2}$