Post-Newtonian Approximation

—— *Its Foundation and Applications* ——

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We discuss various aspects of the post-Newtonian approximation in general relativity. After presenting the foundation based on the Newtonian limit, we use the (3+1) formalism to formulate the post-Newtonian approximation for the perfect fluid. As an application we show the method for constructing the equilibrium configuration of nonaxisymmetric uniformly rotating fluid. We also discuss the gravitational waves including tail from post-Newtonian systems.

§1. Introduction

The motion and associated emission of gravitational waves (GW) of self gravitating systems have been a main research interest in general relativity. The problem is complicated conceptually as well as mathematically because of the nonlinearity of Einstein's equations. There is no hope in any foreseeable future to have exact solutions describing motions of arbitrary shaped, massive bodies so that we have to adopt some sort of approximation scheme for solving Einstein's equations to study such problems. In the past years many types of approximation schemes have been developed depending on the nature of the system under consideration. Here we shall focus on a particular scheme called the post-Newtonian (PN) approximation. There are many systems in astrophysics where Newtonian gravity is dominant, but general relativistic gravity plays also important roles in their evolution. For such systems it would be nice to have an approximation scheme which gives Newtonian description in the lowest order and general relativistic effects as higher order perturbations. The post-Newtonian approximation is perfectly suited for this purpose. Historically Einstein computed first the post-Newtonian effects, the precession of the perihelion [36)], but a systematic study of the post-Newtonian approximation was not made until the series of papers by Chandrasekhar and associates [21)-25)]. Now it is widely known that the post-Newtonian approximation is important in analyzing a number of relativistic problems, such as the equation of motion of binary pulsar [19),40),53),31)], solar-system tests of general relativity [97),98)], and gravitational radiation reaction [25),20)].

Any approximation scheme nesseciates a small parameter(s) characterizing the nature of the system under consideration. Typical parameter which most of schemes adopt is the magnitude of the metric deviation away from a certain background metric. In particular if the background is Minkowski spacetime and there is no other parameter, the scheme is sometimes called as the post-Minkowskian approximation in the sense that the constructed spacetime reduces to Minkowski spacetime in the

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limit as the parameter tends to zero. This limit is called as the weak field limit. In the case of the post-Newtonian approximation the background spacetime is also Minkowski spacetime, but there is another small parameter, that is, the typical velocity of the system divided by the speed of light which we call ϵ henceforth. These two parameters (the deviation away from the flat metric and the velocity) have to have a certain relation in the following sense. As the gravitational field gets weaker, all velocities and forces characteristic of the material systems become smaller, in order to permit the weakening gravity to remain an important effect on the system's dynamics. For example in case of a binary system, the typical velocity would be the orbital velocity $v/c \sim \epsilon$ and the deviation from the flat metric would be the Newtonian potential, say Φ. Then these are related by $\Phi/c^2 \sim v^2/c^2 \sim \epsilon^2$ which guarantees that the system is bounded by its own gravity.

In the post-Newtonian approximation, the equations in general relativity take the form of Newton's equations in an appropriate limit as $\epsilon \to 0$. Such a limit is called as the Newtonian limit and it will be the bases of constructing the post-Newtonian approximation. However the limit is not in any sense trivial since the limit may be thought of two limits tied together as just described. It is also worth noting that the Newtonian limit cannot be uniform everywhere for all time. For example any compact binary systems, no matter how weak the gravity between components and slow its orbital motion, will eventually spiral together due to backreaction of the emission of gravitational waves. As the result the effects of its Newtonian gravity will be swamped by those of its gravitational waves. This will mean that higher order effect of the post-Newtonian approximation eventually dominates the lowest order Newtonian dynamics and thus if the post-Newtonian approximation is not carefully constructed, this effect can lead to many formal problems, such as divergent integrals [35)]. It has been shown that such divergences may be avoided by carefully defining Newtonian limit [44)]. Moreover, the use of such limit provides us a strong indication that the post-Newtonian hierarchy is an asymptotic approximation to general relativity [47)]. Therefore we shall first discuss in this paper the Newtonian limit and how to construct the post-Newtonian hierarchy before attacking practical problems in later sections.

Before going into the details, we mention the reason for the growth of interest in the post-Newtonian approximation in recent years. Certainly the discovery of the binary neutron star system PSR 1913+16 was a strong reason to have renewed interest in the post-Newtonian approximation since it is the first system in which general relativitistic gravity plays fundamental roles in its evolution [53)]. Particularly indirect discovery of gravitational wave by the observation of the period shortening led to many fruitful studies of the equation of motion with gravitational radiation emission in 1980's [27), 82), 93)]. The effect of radiation reaction appears as the form of the potential force at the order of ϵ^5 higher than Newtonian force in the equation of motion. There have been various attempts to show the validity of the so-called quadrupole formula for the radiation reaction [2), 44), 54) - 58), 81), 94), 95)].

At that time, however, no serious attempts had been made for the study of higher order effects in the equation of motion. The situation changed gradually in 1990's because of the increasing expectation of direct detection of gravitational waves by Kilometer-size interferometric gravitational wave detectors, such as LIGO [1], VIRGO [18] and TAMA [61] now under construction. Coalescing binary neutron stars are the most promising candidate of sources of gravitational waves for such detectors. The reasons are that (1) we expect to detect the signal of coalescence of binary neutron stars about several times per year [76], and (2) the waveform from coalescing binaries can be predicted with a high accuracy compared with other sources [1, 91, 98]. Informations carried by gravitational waves tell us not only various physical parameters of neutron star [90, 91], but also the cosmological parameters [83, 66, 42, 96] if and only if we can make a detailed comparison between the observed signal with theoretical prediction during the epoch of the so-called inspiraling phase where the orbital separation is much larger than the radius of component stars [26]. This is the place where the post-Newtonian approximation may be applied to make a theoretical template for gravitational waves. The problem is that in order to make any meaningful comparison between theory and observation we need to know the detailed waveforms generated by the motion up to 4PN order which is of order ϵ^8 higher than the Newtonian order [88]. This request from gravitational wave astronomy forces us to construct higher order post-Newtonian approximation.

It should be mentioned that Blanchet and Damour have developed a systematic scheme to calculate the waveform at higher order where the post-Minkowskian approximation is used to construct the external field and the post-Newtonian approximation is used to construct the field near the material source [9-13]. Blanchet has obtained the waveform up to the 2.5 PN order which is of order ϵ^5 higher than the lowest qudraupole wave [14, 7, 8] by using the equation of motion up to that order [28-30, 50, 60]. We shall also study the waveform from different point of view because of its importance. We obtain essentially the same results including the tail term with them.

There is another phase in the evolution of the binary neutron star which we can withdraw useful informations from the observed signals of gravitational waves. That is the intermediate phase between the inspiralling phase and the event of merging. In the inspiralling phase the components are usually treated as point particles, while the extendedness becomes important in the intermediate phase and the merging. Full general relativistic hydrodynamic simulation is needed for the understanding of the merging. The material initial data for the simulation will be the density and the velocity distribution of the fluid. The post-Newtonian approximation will be used to construct the data. Furthermore the post-Newtonian approximation is able to take into account of the tidal effect which is important for the orbital instability. Recently, Lai, Rasio and Shapiro [62, 63] have pointed out that such a tidal coupling of binary neutron stars is very important for their evolution in the final merging phase because the tidal effect causes the instability of the circular motion even in the Newtonian gravity. Also important is the general relativistic gravity because in the intermediate phase, the orbital separation is about ten times as small as the Schwarzschild radius of the system. Thus, we need not only a hydrodynamic treatment, but also general relativistic one in order to study the final stage of the evolution of binary neutron stars. Since the usual formulation of the post-Newtonian approximation is based on the particular gauge condition which is not so convenient for numerical purposes, we shall reformulate it by the (3+1) formalism often used in numerical approach of general relativity.

This paper is organized as follows. In section 2 we introduce the Newtonian limit in general relativity and present how to construct the post-Newtonian hierarchy. There we mention how to avoid divergent integrals which appears at higher order in the previous treatments, and we also discuss how to incorporate strong internal gravity in the post-Newtonian approximation. Next we reformulate the post-Newtonian approximation appropriate for numerical treatment in section 3. There we shall adopt the (3+1) formalism which is frequently used in numerical relativity. Based on the formalism developed in section 3, we present a formulation for constructing numerically equilibrium solutions of uniformly rotating fluid in the 2PN approximation in section 4. We shall also discuss the propagation of gravitational waves from slow motion systems in section 5.

§2. Foundation of the post-Newtonian approximation

Since the Newtonian limit is the basis of the post-Newtonian approximation, we shall first formulate the Newtonian limit. We shall follow the formulation by Futamase and Schutz [47]. We will not mention other formulation of Newtonian limit by Ehlers [38),39], because it has not yet used to construct the post-Newtonian approximation.

2.1. *Newtonian limit along a regular asymptotic Newtonian sequence*

The formulation is based on the observation that any asymptotic approximation of any theory need a sequence of solutions of the basic equations of the theory [87),92]. Namely, if we write the equations in abstract form as

$$E(g) = 0, \tag{2·1}$$

for an unknown function g, one would like to have a one-parameter (or possibly multiparameter) family of solutions,

$$E(g(\lambda)) = 0, \tag{2·2}$$

where λ is some parameter. Asymptotic approximation then says that function $f(\lambda)$ approximates $g(\lambda)$ to order λ^p if $|f(\lambda) - g(\lambda)|/\lambda^p \to 0$ as $\lambda \to 0$. We choose the sequence of solutions with appropriate properties in such a way that the properties reflect the character of the system under consideration.

We shall formulate the post-Newtonian approximation according to the general idea just described. As stated in the introduction, we would like to have an approximation which applies the systems whose motion is described almost by Newtonian theory. Thus we need a sequence of solutions of Einstein's equations parameterized by ϵ (the typical velocity of the system divided by the speed of light) which has the Newtonian character as $\epsilon \to 0$.

The Newtonian character is most conveniently described by the following scaling law. The Newtonian equations involve six variables, density (ρ), pressure (P), gravitational potential (Φ), and velocity ($v^i, i = 1, 2, 3$):

$$\nabla^2 \Phi - 4\pi\rho = 0, \tag{2·3}$$

$$\partial_t \rho + \nabla_i(\rho v^i) = 0, \tag{2·4}$$

$$\rho\partial_t v^i + \rho v^j \nabla_j v^i + \nabla^i P + \rho \nabla^i \Phi = 0, \tag{2·5}$$

supplemented by an equation of state. For simplicity we have considered perfect fluid. Here we have set $G = 1$.

It can be seen that the variables $\{\rho(x^i, t), P(x^i, t), \Phi(x^i, t), v^i(x^j, t)\}$ obeying the above equations satisfy the following scaling law.

$$\begin{aligned}
\rho(x^i, t) &\to \epsilon^2 \rho(x^i, \epsilon t),\\
P(x^i, t) &\to \epsilon^4 P(x^i, \epsilon t),\\
v^i(x^k, t) &\to \epsilon v^i(x^k, \epsilon t),\\
\Phi(x^i, t) &\to \epsilon^2 \Phi(x^i, \epsilon t).
\end{aligned} \tag{2·6}$$

One can easily understand the meaning of this scaling by noticing that ϵ is the magnitude of typical velocity (divided by the speed of light). Then the magnitude of the gravitational potential will be of order ϵ^2 because of the balance between gravity and centrifugal force. The scaling of the time variable expresses the fact that the weaker the gravity ($\epsilon \to 0$) the longer the time scale.

Thus we wish to have a sequence of solutions of Einstein's equations which has the above scaling as $\epsilon \to 0$. We shall also take the point of view that the sequence of solutions is determined by the appropriate sequence of initial data. This has a practical advantage because there will be no solution of Einstein's equations which satisfy the above scaling (2·6) even as $\epsilon \to 0$. This is because Einstein's equations are nonlinear in the field variables so it will not be possible to enforce the scaling everywhere in spacetime. We shall therefore impose it only on the initial data for the solution of the sequence.

Here we shall first make a general discussion on the formulation of the post-Newtonian approximation independent of any initial value formalism and then present the concrete treatment in the harmonic coordinate. The condition is used because of its popularity and some advantages in the generalization to the systems with strong internal gravity.

As the initial data for the matter we shall take the same data set in Newtonian case, namely, the density ρ, the pressure P, and the coordinate velocity v^i. In most of the application, we usually assume a simple equation of state which relates the pressure to the density. The initial data for the gravitational field are $g_{\mu\nu}, \partial g_{\mu\nu}/\partial t$. Since general relativity is overdetermined system, there will be constraint equations among the initial data for the field. We shall write the free data for the field as (Q_{ij}, P_{ij}) whose explicit forms depend on the coordinate condition one assumes. In any coordinate we shall assume these data for the field vanish since we are interested in the evolution of an isolated system by its own gravitational interaction. It is expected that these choice corresponds to the absence of radiation far away from the source. Thus we choose the following initial data which is indicated by Newtonian scaling:

$$
\begin{aligned}
\rho(t=0,x^i,\epsilon) &= \epsilon^2 a(x^i),\\
P(t=0,x^i,\epsilon) &= \epsilon^4 b(x^i),\\
v^i(t=0,x^k,\epsilon) &= \epsilon c^i(x^k),\\
Q_{ij}(t=0,x^i,\epsilon) &= 0,\\
P_{ij}(t=0,x^i,\epsilon) &= 0,
\end{aligned}
\tag{2·7}
$$

where the functions a, b , and c^i are C^∞ functions that have compact support contained entirely within a sphere of a finite radius.

Corresponding to the above data, we have a one-parameter set of spacetime parameterized by ϵ. It may be helpful to visualize the set as a fiber bundle, with base space R the real line (coordinate ϵ) and fiber R^4 the spacetime (coordinates t, x^i). The fiber $\epsilon = 0$ is Minkowski spacetime since it is defined by zero data. In the following we shall assume that the solutions are sufficiently smooth functions of ϵ for small $\epsilon \neq 0$. We wish to take the limit $\epsilon \to 0$ along the sequence. The limit is, however, not unique and is defined by giving a smooth nowhere vanishing vector field on the fiber bundle which is nowhere tangent to each fiber [51),87)]. The integral curves of the vector field give a correspondence between points in different fibers, namely events in different spacetime with different values of ϵ. Remembering the Newtonian scaling of the time variable in the limit, we introduce the Newtonian dynamical time:

$$
\tau = \epsilon t, \tag{2·8}
$$

and define the integral curve as the curve on which τ and x^i stay constant. In fact if we take the limit $\epsilon \to 0$ along this curve, the orbital period of the binary system with $\epsilon = 0.01$ is 10 times of that of the system with $\epsilon = 0.1$ as expected from the Newtonian scaling. This is what we define as the Newtonian limit. Notice that this map never reaches the fiber $\epsilon = 0$ (Minkowski spacetime). There is no pure vacuum Newtonian limit as expected.

In the following we assume the existence of such a sequence of solutions constructed by the initial data satisfying the above scaling with respect to ϵ. We shall call such a sequence is a regular asymptotically Newtonian sequence. We have to make further mathematical assumptions about the sequence to make explicit calculations. We will not go into details of them partly because to prove the assumptions we need a deep understanding of the existence and uniqueness properties of Cauchy problem of Einstein's equations with perfect fluids of compact support which are not available at present.

2.2. *Post-Newtonian Hierarchy*

We shall now define the Newtonian, post-Newtonian and higher approximations of various quantities as the appropriate higher tangents of the corresponding quantities to the above integral curve at $\epsilon = 0$. For example the hierarchy of approximations for the spacetime metrics can be expressed as follows:

$$
g_{\mu\nu}(\epsilon,\tau,x^i) = g_{\mu\nu}(0,\tau,x^i) + \epsilon(\mathcal{L}_V g_{\mu\nu})(0,\tau,x^i)
$$

$$+\frac{1}{2}\epsilon^2(\mathcal{L}_V^2 g_{\mu\nu})(0,\tau,x^i) + ... + \frac{\epsilon^n}{n!}(\mathcal{L}_V^n g_{\mu\nu})(0,\tau,x^i) + R_{n+1}, \quad (2{\cdot}9)$$

where $\mathcal{L}_V$ is the Lie derivative with respect to the tangent vector of the curves defined above, and the remainder term $R^{\mu\nu}_{n+1}$ is

$$R^{\mu\nu}_{n+1} = \frac{\epsilon^{n+1}}{(n+1)!}\int_0^1 d\ell(1-\ell)^{n+1}(\mathcal{L}_V^{n+1} g_{\mu\nu})(\ell\epsilon,\tau,x^i). \quad (2{\cdot}10)$$

Taylor's theorem guarantees that the series is asymptotic expansion about $\epsilon = 0$ under certain assumptions mentioned above. It might be useful to point out that the above definition of the approximation scheme may be formulated purely geometrically in terms of jet bundle.

The above definition of the post-Newtonian hierarchy gives us asymptotic series in which each term in the series is manifestly finite. This is based on the ϵ dependence of the domain of dependence of the field point (τ, x^k). The region is finite with finite values of ϵ, and the diameter of the region increases as ϵ^{-1} as $\epsilon \to 0$. Without this linkage of the region with the expansion parameter ϵ, the post-Newtonian approximation leads to divergences in the higher orders. This is closely related with the retarded expansion. Namely, it is assumed that the slow motion assumption enabled one to Taylor expand the retarded integrals in retarded time such as

$$\int drf(\tau-\epsilon r, ...) = \int drf(\tau, ...) - \epsilon\int drrf(\tau, ...)_{,\tau} + .. \quad (2{\cdot}11)$$

and assign the second term to a higher order because of its explicit ϵ in front. This is incorrect because $r \to \epsilon^{-1}$ as $\epsilon \to 0$ and thus ϵr is not uniformly small in the Newtonian limit. Only if the integrand falls off sufficiently fast, can retardation be ignored. This happens in the lower-order PN terms. But at some higher order there appear many terms which do not fall off sufficiently fast because of the nonlinearity of Einstein's equations. This is the reason that the formal PN approximation produces the divergent integrals. It turns out that such divergence appears at 3PN order indicating the breakdown of the PN approximation in the harmonic coordinate. This sort of divergence may be eliminated if we remember that the upper bound of integration does depend on ϵ as ϵ^{-1}. Thus we would get something like $\epsilon^n \ln\epsilon$ instead of $\epsilon^n \ln\infty$ in the usual approach. This shows that the asymptotic Newtonian sequence is not differentiable in ϵ at $\epsilon = 0$, but there are no divergence in the expansion and it has still an asymptotic approximation in ϵ that involves logarithms.

2.3. *Explicit calculation in Harmonic coordinate*

Here we shall use the above formalism to make explicit calculation in the harmonic coordinate. The reduced Einstein's equation in the harmonic condition is written as

$$\tilde{g}^{\alpha\beta}\tilde{g}^{\mu\nu}{}_{,\alpha\beta} = 16\pi\Theta^{\mu\nu} - \tilde{g}^{\mu\alpha}{}_{,\beta}\tilde{g}^{\nu\beta}{}_{,\alpha}, \quad (2{\cdot}12)$$

$$\partial_\mu[\tilde{g}^{\mu\nu}\partial_\nu x^\alpha] = 0, \quad (2{\cdot}13)$$

where

$$\tilde{g}^{\mu\nu} = (-g)^{1/2}g^{\mu\nu}, \quad (2{\cdot}14)$$

$$\Theta^{\alpha\beta}=(-g)(T^{\alpha\beta}+t_{LL}^{\alpha\beta}), \tag{2·15}$$

where $t_{LL}^{\mu\nu}$ is the Landau-Lifshitz pseudotensor [64]. In the following we shall choose an isentropic perfect fluid for $T^{\alpha\beta}$ which is enough for most of applications.

$$T^{\alpha\beta}=(\rho+\rho\Pi+P)u^\alpha u^\beta+Pg^{\alpha\beta}, \tag{2·16}$$

where ρ is the rest mass density, Π the internal energy, P the pressure, u^μ the four velocity of the fluid with normalization;

$$g_{\alpha\beta}u^\alpha u^\beta=-1. \tag{2·17}$$

The conservation of the energy and momentum is expressed as

$$\nabla_\beta T^{\alpha\beta}=0. \tag{2·18}$$

Defining the gravitational field variable as

$$\bar{h}^{\mu\nu}=\eta^{\mu\nu}-(-g)^{1/2}g^{\mu\nu}, \tag{2·19}$$

where $\eta^{\mu\nu}$ is the Minkowski metric, the reduced Einstein's equation (2·12) and the gauge condition (2·13) take the following form;

$$(\eta^{\alpha\beta}-\bar{h}^{\alpha\beta})\bar{h}^{\mu\nu}{}_{,\alpha\beta}=-16\pi\Theta^{\mu\nu}+\bar{h}^{\mu\alpha}{}_{,\beta}\bar{h}^{\nu\beta}{}_{,\alpha}, \tag{2·20}$$

$$\bar{h}^{\mu\nu}{}_{,\nu}=0. \tag{2·21}$$

Thus the characteristics is determined by the operator $(\eta^{\alpha\beta}-\bar{h}^{\alpha\beta})\partial_\alpha\partial_\beta$, and thus the light cone deviates from that in the flat spacetime. We shall use this form of the reduced Einstein's equations in the calculation of wave form far away from the source because the deviation plays a fundamental role there. However in the study of the gravitational field near the source it is not neccesary to consider the deviation of the light cone away from the flat one and thus it is convenient to use the following form of the reduced Einstein's equations [2].

$$\eta^{\mu\nu}\bar{h}^{\alpha\beta}{}_{,\mu\nu}=-16\pi\Lambda^{\alpha\beta}, \tag{2·22}$$

where

$$\Lambda^{\alpha\beta}=\Theta^{\alpha\beta}+\chi^{\alpha\beta\mu\nu}{}_{,\mu\nu}, \tag{2·23}$$

$$\chi^{\alpha\beta\mu\nu}=(16\pi)^{-1}(\bar{h}^{\alpha\nu}\bar{h}^{\beta\mu}-\bar{h}^{\alpha\beta}\bar{h}^{\mu\nu}). \tag{2·24}$$

Equations (2·21) and (2·22) together imply the conservation law

$$\Lambda^{\alpha\beta}_{,\beta}=0. \tag{2·25}$$

We shall take as our variables the set $\{\rho,P,v^i,\bar{h}^{\alpha\beta}\}$, with the definition

$$v^i=u^i/u^0. \tag{2·26}$$

The time component of 4-velocity u^0 is determined from Eq.(2·17). To make a well-defined system of equations we must add the conservation law for number density n which is some function of the density ρ and pressure P:

$$\nabla_\alpha(nu^\alpha) = 0. \tag{2·27}$$

Equations (2·25) and (2·27) imply that the flow is adiabatic. The role of the equation of state is played by the arbitrary function $n(\rho, p)$.

Initial data for the above set of equations are $\bar{h}^{\alpha\beta}, \bar{h}^{\alpha\beta}_{\ \ ,0}, \rho, P$, and v^i, but not all these data are independent because of the existence of constraint equations. Equations.(2·21) and (2·22) imply the four constraint equations among the initial data for the field.

$$\Delta\bar{h}^{\alpha 0} + 16\pi\Lambda^{\alpha 0} - \delta^{ij}\bar{h}^{\alpha\ \ ,0}_{i,j} = 0, \tag{2·28}$$

where Δ is the Laplacian in the flat space. We shall choose $\bar{h}^{ij}$ and $\bar{h}^{ij}_{\ ,0}$ as free data and solve Eq.(2·28) for $\bar{h}^{\alpha 0}(\alpha = 0, .., 3)$ and Eq.(2·21) for $\bar{h}^{\alpha 0}_{\ \ ,0}$. Of course these constraints cannot be solved explicitly, since $\Lambda^{\alpha 0}$ contains $\bar{h}^{\alpha 0}$, but they can be solved iteratively as explained below. As discussed above, we shall assume that the free data $\bar{h}^{ij}$ and $\bar{h}^{ij}_{\ ,0}$ for the field vanish. One can show that such initial data satisfy the O'Murchadha and York criterion for the absence of radiation far away from the source [75].

In the actual calculation, it is convenient to use an expression with explicit dependence of ϵ. The harmonic condition allows us to have such expression in terms of the retarded integral.

$$\bar{h}^{\mu\nu}(\epsilon, \tau, x^i) = 4\int_{C(\epsilon,\tau,x^i)} d^3y\Lambda^{\mu\nu}(\tau - \epsilon r, y^i, \epsilon)/r + h^{\mu\nu}_H(\epsilon, \tau, x^i), \tag{2·29}$$

where $r = |y^i - x^i|$ and $C(\epsilon, \tau, x^i)$ is the past flat light cone of event (τ, x^i) in the spacetime given by ϵ, truncated where it intersects with the initial hypersurface $\tau = 0$. $\bar{h}^{\mu\nu}_H$ is the unique solution of the homogeneous equation

$$\eta^{\alpha\beta}\bar{h}^{\mu\nu}_{\ \ ,\alpha\beta} = 0. \tag{2·30}$$

We shall henceforce ignore the homogeneous solution because they play no important roles. Because of the ϵ dependence of the integral region, the domains of integration are finite as long as $\epsilon \neq 0$ and their diameter increases as ϵ^{-1} as $\epsilon \to 0$.

Given the formal expression (2·29) in terms of initial data (2·7), we can take the Lie derivative and evaluate these derivatives at $\epsilon = 0$. The Lie derivative is nothing but partial derivative with respect to ϵ in the coordinate system for the fiber bundle given by (ϵ, τ, x^i). Accordingly one should convert all the time indices to τ indices. For example, $T^{\tau\tau} = \epsilon^2 T^{tt}$ which is of order ϵ^4 since $T^{tt} \sim \rho$ is of order ϵ^2. Similarly the other components of stress energy tensor $T^{\tau i} = \epsilon T^{ti}$ and T^{ij} are of order ϵ^4 as well. Thus we expect the first non-vanishing derivative in (2·29) will be the forth-derivatives. In fact we find

$${}_{(4)}\bar{h}^{\tau\tau}(\tau, x^i) = 4\int_{R^3} \frac{{}_{(2)}\rho(\tau, y^i)}{r} d^3y, \tag{2·31}$$

$$ {}_{(4)}\bar{h}^{\tau i}(\tau, x^i) = 4\int_{R^3} \frac{{}_{(2)}\rho(\tau, y^k){}_{(1)}v^i(\tau, x^k)}{r} d^3y, \tag{2·32} $$

$$ {}_{(4)}\bar{h}^{ij}(\tau, x^k) = 4\int_{R^3} \frac{{}_{(2)}\rho(\tau, y^k){}_{(1)}v^i(\tau, y^k){}_{(1)}v^j(\tau, y^k) + {}_{(4)}t^{ij}_{LL}(\tau, y^k)}{r} d^3y, \tag{2·33} $$

where we have adopted the notation

$$ {}_{(n)}f(\tau, x^i) = \frac{1}{n!}\lim_{\epsilon\to 0}\frac{\partial^n}{\partial\epsilon^n} f(\epsilon, \tau, x^i), \tag{2·34} $$

and

$$ {}_{(4)}t^{ij}_{LL} = \frac{1}{64\pi}({}_{(4)}\bar{h}^{\tau\tau,i}{}_{(4)}\bar{h}^{\tau\tau,j} - \frac{1}{2}\delta^{ij}{}_{(4)}\bar{h}^{\tau\tau,k}{}_{(4)}\bar{h}^{\tau\tau}_{\ \ ,k}). \tag{2·35} $$

In the above calculation we have taken the point of view that $\bar{h}^{\mu\nu}$ is a tensor field, defined by giving its components in the assumed harmonic coordinate as the difference between the tensor density $\sqrt{-g}g^{\mu\nu}$ and $\eta^{\mu\nu}$.

The conservation law (2·25) also has its first nonvanishing derivative at this order, which are

$$ {}_{(2)}\rho(\tau, x^k)_{,\tau} + ({}_{(2)}\rho(\tau, x^k){}_{(1)}v^i(\tau, x^k))_{,i} = 0, \tag{2·36} $$

$$ ({}_{(2)}\rho_{(1)}v^i)_{,\tau} + ({}_{(2)}\rho_{(1)}v^i{}_{(1)}v^j)_{,j} + {}_{(4)}P^{,i} - \frac{1}{4}{}_{(2)}\rho_{(4)}\bar{h}^{\tau\tau,i} = 0. \tag{2·37} $$

Equations (2·31), (2·36), and (2·37) consist of Newtonian theory of gravity. Thus the lowest non-vanishing derivative with respect to ϵ is indeed Newtonian theory, and the 1PN and 2PN equations emerge from sixth and eighth derivatives, respectively, in the conservation law (2·25). At the next derivative, the quadrupole radiation reaction term comes out.

2.4. *Strong point particle limit*

If we wish to apply the post-Newtonian approximation to the inspiraling phase of binary neutron stars, the strong internal gravity must be taken into account. The usual post-Newtonian approximation assumes explicitly the weakness of gravitational field everywhere including inside the material source. It is argued by appealing the strong equivalence principle that the external gravitational field which governs the orbital motion of the binary system is independent of internal structure of the components up to tidal interaction. Thus it is expected that the results obtained under the assumption of weak gravity also apply for the case of neutron star binary. Experimental evidence for the strong equivalence principle is obtained only for the system with weak gravity [97),98)], but no experiment is available in case with strong internal gravity at present.

In theoretical aspect, the theory of extended object in general relativity [34)] is still in preliminary stage for the application to realistic systems. Matched asymptotic expansion technique has been used to treat the system with strong gravity in certain situations [32),33),55),56),27)]. Another way to handle with strong internal gravity is by the use of Dirac's delta function type source with a fixed mass [37]. However, this makes Einstein's equations mathematically meaningless because of their nonlinearity. Physically, there is no such source in general relativity because of the existence of black holes. Before a body shrinks to a point, it forms a black hole whose size is fixed by its mass. For this reason, it has been claimed that no point particle exists in general relativity.

This conclusion is not correct, however. We can shrink the body keeping the compactness (M/R), i.e., the strength of the internal field fixed. Namely we should scale the mass M just like the radius R. This can be fitted nicely into the concept of regular asymptotic Newtonian sequence defined above because there the mass also scales along the sequence of solutions. In fact, if we take the masses of two stars as M, and the separation between two stars as L, then $\Phi \sim M/L$. Thus the mass M scales as ϵ^2 if we fix the separation. In the above we have assumed that the density scales as ϵ^2 to guarantee this scaling for the mass understanding the size of the body fixed. Now we shrink the size as ϵ^2 to keep the compactness of each component. Then the density should scale as ϵ^{-4}. We shall call such a scheme as strong point particle limit since the limit keeps the strength of internal gravity. The above consideration suggests the following initial data to define a regular asymptotic Newtonian sequence which describes nearly Newtonian system with strong internal gravity [45]. The initial data are two uniformly rotating fluids with compact spatial support whose stress-energy tensor and size scales as ϵ^{-4} and ϵ^2, respectively. We also assume that each of these fluid configurations would be a stationary equilibrium solution of Einstein's equation if other were absent. This is neccesary for the suppression of irrelevant internal motions of each star. Any remaining motions are the tidal effects caused by the other body, which will be of order ϵ^6 smaller than the internal self-force. This data allows us to use the Newtonian time $\tau = \epsilon t$ as a natural time coordinate everywhere including inside the stars.

This choice of the data leads naturally to the introduction of the body zones B_A and the body zone coordinates $\bar{x}^k_A$ defined by

$$B_A = \{x^k; |x^k - \xi^k_A| < \epsilon R\}, \tag{2·38}$$

$$\bar{x}^k_A = \epsilon^{-2}(x^k - \xi^k_A), \tag{2·39}$$

where R is some constant, and $\xi^k_A(\tau), A = I, II$ are the coordinates of the origin of the two stars, where we have used letters with bar to distinguish the body-zone coordinates from their counterparts. The scaling by ϵ^{-2} means that as the star shrinks with respect to the coordinate x^k, it remains of fixed size in the body-zone coordinate $\bar{x}^k$. The boundary of the body zone shrinks to a point with respect to x^k, and expands to infinity with respect to $\bar{x}^k$ as $\epsilon \to 0$. This makes a clean separation of the body zone from the exterior geometry generated by other star.

$$g_{\mu\nu} = (g_B)_{\mu\nu} + (g_{C-B})_{\mu\nu}, \tag{2·40}$$

where $(g_B)_{\mu\nu}$ is the contribution from the body zones, $(g_{C-B})_{\mu\nu}$ from elsewhere.

Actual calculations are most easily performed in the harmonic coordinate. Take two stationary solutions of Einstein's equations for the perfect fluid $\{T^{\mu\nu}_A(x^i), g^{\mu\nu}_A(x^i)\}$

as our initial data in the body zone. As explained above every component of the stress-energy tensor $T^{\mu\nu}_A$ has the same ϵ^{-4} scaling in the inertia coordinates (t, x^k) for a rapidly rotating star. In the body zone coordinates $(\tau, \bar{x}^k)$, these have the following scalings:

$$\bar{T}^{\tau\tau}_A = \epsilon^2 T^{tt}_A \sim \epsilon^{-2}, \tag{2·41}$$

$$\bar{T}^{\tau i}_A = \epsilon^{-1} T^{ti}_A \sim \epsilon^{-5}, \tag{2·42}$$

$$\bar{T}^{ij}_A = \epsilon^{-4} T^{ij}_A \sim \epsilon^{-8}. \tag{2·43}$$

Transformed to the near zone coordinates (τ, x^k), $x^k = \xi^i_A + \epsilon^2 \bar{x}^k$, they take the form

$$T^{\tau\tau}_N = \bar{T}^{\tau\tau}_A, \tag{2·44}$$

$$T^{\tau i}_N = \epsilon^2 \bar{T}^{\tau i}_A + v^i_A \bar{T}^{\tau\tau}_A, \tag{2·45}$$

$$T^{ij}_N = \epsilon^4 \bar{T}^{ij}_A + 2\epsilon^2 v^{(i}_A \bar{T}^{j)\tau}_A + v^i_A v^j_A \bar{T}^{\tau\tau}_A, \tag{2·46}$$

where $v^i_A = d\xi^i_A/d\tau$ is the velocity of the origin of the body A. If there were only one body, these data would produce a stationary solution in the body zone, which moves with uniform velocity v^i_A in the near-zone coordinates. Now we know the ordering of the source, we can solve Eq.(2·29),iteratively as in the weak field case. The difference is that we transform the integration variables to the body zone coordinates in the expression of the fields whose contributions come from the body zone.

$$\bar{h}^{\mu\nu}_B(\epsilon; \tau, x^i) = 4\epsilon^6 \sum_A \int_{B_A} d^3\bar{y}_A |x_A - \epsilon^2 \bar{y}_A|^{-1} \Lambda^{\mu\nu}(\tau - \epsilon|x_A - \epsilon^2 \bar{y}|, \bar{y}^i, \epsilon), \tag{2·47}$$

where $x^i_A = x^i - \xi^i_A$. Expanding these expressions in terms of ϵ, we obtain

$$\bar{h}^{\tau\tau}_B(\epsilon; \tau, x^i) = 4\epsilon^4 \sum_A \frac{M_A}{|x_A|} + O(\epsilon^6), \tag{2·48}$$

$$\bar{h}^{\tau i}_B(\epsilon; \tau, x^i) = 4\epsilon^3 \sum_A \frac{J_A}{|x_A|} + 4\epsilon^4 \sum_A \frac{M_A v^i_A}{|x_A|} + O(\epsilon^5), \tag{2·49}$$

$$\bar{h}^{ij}_B(\epsilon; \tau, x^k) = 4\epsilon^2 \sum_A \frac{Z^{ij}_A}{|x_A|} + 8\epsilon^3 \sum_A \frac{v^{(i}_A J^{j)}_A}{|x_A|} + 4\epsilon^4 \sum_A \frac{M_A v^i_A v^j_A}{|x_A|} + O(\epsilon^5), \tag{2·50}$$

where

$$M_A = \lim_{\epsilon \to 0} \epsilon^2 \int_{B_A} d^3\bar{y} \Lambda^{\tau\tau}_A(\epsilon; \tau, \bar{y}), \tag{2·51}$$

$$J^i_A = \lim_{\epsilon \to 0} \epsilon^3 \int_{B_A} d^3\bar{y} \Lambda^{\tau i}_A(\epsilon; \tau, \bar{y}), \tag{2·52}$$

$$Z^{ij}_A = \lim_{\epsilon \to 0} \epsilon^4 \int_{B_A} d^3\bar{y} \Lambda^{ij}_A(\epsilon; \tau, \bar{y}). \tag{2·53}$$

The M_A defined above is the conserved ADM mass the body A would have if it were isolated. Without loss of generality one can set the linear momentum J^i_A of the overall internal motion of each body to zero by choosing appropriate origin of the coordinates. If we did not assume the internal stationarity in the initial data, then Z_A^{ij} would be finite and no approximation would be possible. However under the condition of internal stationarity one can show that Z_A^{ij} vanishes [45]. Above expressions for the body zone contribution are used to calculate the pseudotensor in the near zone and then to evaluate the contribution $\bar{h}^{\mu\nu}_{C-B}$ outside the body zone. As the result we shall obtain the metric variables up to $O(\epsilon^6)$.

$$\bar{h}^{\tau\tau}(\tau, x^i) = 4\epsilon^4 \sum_A \frac{M_A}{|x_A|} + O(\epsilon^6), \tag{2·54}$$

$$\bar{h}^{\tau i}(\tau, x^i) = 4\epsilon^4 \sum_A \frac{M_A v_A^i}{|x_A|} + +2\epsilon^5 \sum_A \frac{x_A^k}{|x_A|} M_A^{ki} + O(\epsilon^6), \tag{2·55}$$

$$\bar{h}^{ij}(\tau, x^k) = 4\epsilon^4 \sum_A \frac{M_A v_A^i v_A^j}{|x_A|} - 2\epsilon^5 I_{orb}^{(3)ij} + O(\epsilon^6), \tag{2·56}$$

where

$$M_A^{ij} = \lim_{\epsilon \to 0} \epsilon^4 2 \int_{B_A} d^3\bar{y}\bar{y}^{[i} \Lambda_A^{j]\tau}(\epsilon; \tau, \bar{y}), \tag{2·57}$$

$$I_{orb}^{ij} = \sum_A \xi_A^i \xi_A^j M_A J_A^i. \tag{2·58}$$

These are the angular momentum of the body A and the quadrupole moment for the orbital motion, respectively. This kind of calculation is also performed at 2.5PN order to get the standard quadrupole formula except that the mass in the definition of the quadrupole moment is replaced with the ADM mass [45]. The above method has been also applied for the calculation of spin precession and the same form of the equation in the case of weak gravity is obtained. Thus we can extend the strong equivalence principle for bodies with strong internal gravity to the generation of gravitational waves and the spin precession. It is an open question if the strong point particle limit may be taken to lead a well defined equation of motion at higher post-Newtonian orders.

§3. Post-Newtonian Approximation in the (3+1) Formalism

In the evolution of binary neutron stars, the orbital separation eventually becomes comparable to the radius of the neutron star due to radiation reaction. Then, the point particle picture does not apply and each star of the binary begins to behave as a hydrodynamic object because of tidal effect. As described in the introduction we need not only a hydrodynamic treatment, but also general relativistic one in order to study the final stage of the evolution of binary neutron stars. Full general relativistic simulation will be a direct way to answer such a question, but it is one of the hardest problem in astrophysics. Although there is much progress in this direction [69], it will take a long time until numerical relativistic calculations become reliable. One of the main reasons for this would be that we do not know the behavior of geometric variables in the strong gravitational field around coalescing binary neutron stars. Owing to this, we do not know what sort of gauge condition is useful and how to give an appropriate general relativistic initial condition for coalescing binary neutron stars.

The other reason is a technical one: In the case of coalescing binary neutron stars, the wavelength is of order of $\lambda \sim \pi R^{3/2} M^{-1/2}$, where R and M are the orbital radius and the total mass of binary, respectively. Thus, in numerical simulations, we need to cover a region $L > \lambda \propto R^{3/2}$ with numerical grids in order to attain good accuracy. This is in contrast with the case of Newtonian and/or PN simulations, in which we only need to cover a region $\lambda > L > R$. Since the circular orbit of binary neutron stars becomes unstable at $R \leq 10M$ owing to the tidal effects [62), 63)] and/or the strong general relativistic gravity [59)], we must set an initial condition of binary at $R \geq 10M$. For such a case the grid must cover a region $L > \lambda \sim 100M$ in numerical simulations to perform an accurate simulation. When we assume to cover each neutron star of its radius $\sim 5M$ with ~ 30 homogeneous grids [72) - 74), 85), 86)], we need to take grids of at least $\sim 500^3$, but it seems impossible to take such a large amount of mesh points even for the present power of supercomputer. At present, we had better search other methods to prepare a precise initial condition for binary neutron stars.

In the case of PN simulations, the situation is completely different because we do not have to treat gravitational waves explicitly in numerical simulations, and as a result, only need to cover a region at most $L \sim 20 - 30M$. In this case, it seems that $\sim 200^3$ grid numbers are enough. Furthermore, we can take into account general relativistic effects with a good accuracy: In the case of coalescing binary neutron stars, the error will be at most $\sim M/R \sim$ a few $\times$ 10% for the first PN approximation, and $\sim (M/R)^2 \sim$ several % for 2PN approximation. Hence, if we can take into account up through 2PN terms, we will be able to give a highly accurate initial condition (the error $\leq$ several %).

For these reasons, we present in this section the 2.5PN hydrodynamic equations including the 2.5PN radiation reaction potential in such a way that one can apply the formulation directly in numerical simulations. As for the PN hydrodynamic equation, Blanchet, Damour and Schäfer [15)] have already obtained the (1+2.5) PN equations. In their formulation, the source terms of all Poisson equations take nonvanishing values only on the matter, like in the Newtonian hydrodynamics. Although their formulation is very useful for PN hydrodynamic simulations including the radiation reaction [72) - 74), 85), 86), 78)], they did not take into account 2PN terms. In their formulation, they also fixed the gauge conditions to the ADM gauge, but in numerical relativity, it has not been known yet what sort of gauge condition is suitable for simulation of the coalescing binary neutron stars and estimation of gravitational waves from them. First, we develop the formalism for the hydrodynamics using the PN approximation. In particular, we use the (3+1) formalism of general relativity so that we can adopt more general class of slice conditions [46), 4)]. Next, we present methods to obtain numerically terms at the 2PN order [4)].

3.1. *(3+1) Formalism for the post-Newtonian Approximation*

According to the above discussion, we shall apply the (3+1) formalism to formulate the PN approximation. In the (3+1) formalism [3), 92), 68)], spacetime is foliated by a family of spacelike 3D hypersurfaces whose normal one-form is taken as

$$\hat{n}_\mu = (-\alpha,\ \mathbf{0}). \tag{3·1}$$

Then the line element takes the following form

$$ds^2 = -(\alpha^2 - \beta_i\beta^i)dt^2 + 2\beta_i dt dx^i + \gamma_{ij}dx^i dx^j, \tag{3·2}$$

where α, β^i and γ_{ij} are the lapse function, shift vector and metric on the 3D hypersurface, respectively. Using the (3+1) formalism, the Einstein's equation

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{3·3}$$

is also split into the constraint equations and the evolution equations. The formers are the so-called Hamiltonian and momentum constraints which respectively become

$$\mathrm{tr}R - K_{ij}K^{ij} + K^2 = 16\pi\rho_H, \tag{3·4}$$

$$D_i K^i{}_j - D_j K = 8\pi J_j, \tag{3·5}$$

where K_{ij}, K, $\mathrm{tr}R$ and D_i are the extrinsic curvature, the trace part of K_{ij}, the scalar curvature of 3D hypersurface and the covariant derivative with respect of γ_{ij}. ρ_H and J_j are defined as

$$\rho_H = T_{\mu\nu}\hat{n}^\mu\hat{n}^\nu, \tag{3·6}$$

$$J_j = -T_{\mu\nu}\hat{n}^\mu\gamma^\nu{}_j. \tag{3·7}$$

The Evolution equations for the spatial metric and the extrinsic curvature are respectively

$$\frac{\partial}{\partial t}\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i, \tag{3·8}$$

$$\begin{aligned}\frac{\partial}{\partial t}K_{ij} = {} & \alpha(R_{ij} + KK_{ij} - 2K_{il}K^l{}_j) - D_iD_j\alpha + (D_j\beta^m)K_{mi} + (D_i\beta^m)K_{mj} \\ & +\beta^m D_m K_{ij} - 8\pi\alpha\Big(S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - S^l{}_l)\Big),\end{aligned} \tag{3·9}$$

$$\frac{\partial}{\partial t}\gamma = 2\gamma(-\alpha K + D_i\beta^i), \tag{3·10}$$

$$\frac{\partial}{\partial t}K = \alpha(\mathrm{tr}R + K^2) - D^iD_i\alpha + \beta^j D_j K + 4\pi\alpha(S^l{}_l - 3\rho_H), \tag{3·11}$$

where R_{ij}, γ and S_{ij} are, respectively, the Ricci tensor with respect of γ_{ij}, the determinant of γ_{ij} and

$$S_{ij} = T_{kl}\gamma^k{}_i\gamma^l{}_j. \tag{3·12}$$

Hereafter we use the conformal factor $\psi = \gamma^{1/12}$ instead of γ for simplicity.

To distinguish the wave part from the non-wave part (for example, Newtonian potential) in the metric, we use $\tilde{\gamma}_{ij} = \psi^{-4}\gamma_{ij}$ instead of γ_{ij}. Then $\det(\tilde{\gamma}_{ij}) = 1$ is satisfied. We also define $\tilde{A}_{ij}$ as

$$\tilde{A}_{ij} \equiv \psi^{-4}A_{ij} \equiv \psi^{-4}\Big(K_{ij} - \frac{1}{3}\gamma_{ij}K\Big). \tag{3·13}$$

We should note that in our notation, indices of $\tilde{A}_{ij}$ are raised and lowered by $\tilde{\gamma}_{ij}$, so that the relations, $\tilde{A}^i{}_j = A^i{}_j$ and $\tilde{A}^{ij} = \psi^4 A^{ij}$, hold. Using these variables, the evolution equations (3·8)-(3·11) can be rewritten as follows;

$$\frac{\partial}{\partial n}\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + \tilde{\gamma}_{il}\frac{\partial\beta^l}{\partial x^j} + \tilde{\gamma}_{jl}\frac{\partial\beta^l}{\partial x^i} - \frac{2}{3}\tilde{\gamma}_{ij}\frac{\partial\beta^l}{\partial x^l}, \tag{3·14}$$

$$\begin{aligned}\frac{\partial}{\partial n}\tilde{A}_{ij} = \frac{1}{\psi^4}\Big[&\alpha\Big(R_{ij} - \frac{1}{3}\gamma_{ij}\mathrm{tr}R\Big) - \Big(\tilde{D}_i\tilde{D}_j\alpha - \frac{1}{3}\tilde{\gamma}_{ij}\tilde{\Delta}\alpha\Big) \\ &-\frac{2}{\psi}\Big(\psi_{,i}\alpha_{,j} + \psi_{,j}\alpha_{,i} - \frac{2}{3}\tilde{\gamma}_{ij}\tilde{\gamma}^{kl}\psi_{,k}\alpha_{,l}\Big)\Big] \\ &+\alpha(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l{}_j) + \frac{\partial\beta^m}{\partial x^i}\tilde{A}_{mj} + \frac{\partial\beta^m}{\partial x^j}\tilde{A}_{mi} - \frac{2}{3}\frac{\partial\beta^m}{\partial x^m}\tilde{A}_{ij} \\ &-8\pi\frac{\alpha}{\psi^4}\Big(S_{ij} - \frac{1}{3}\gamma_{ij}S^l{}_l\Big),\end{aligned} \tag{3·15}$$

$$\frac{\partial}{\partial n}\psi = \frac{\psi}{6}\Big(-\alpha K + \frac{\partial\beta^i}{\partial x^i}\Big), \tag{3·16}$$

$$\frac{\partial}{\partial n}K = \alpha\Big(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\Big) - \frac{1}{\psi^4}\tilde{\Delta}\alpha - \frac{2}{\psi^5}\tilde{\gamma}^{kl}\psi_{,k}\alpha_{,l} + 4\pi\alpha(S^i{}_i + \rho_H), \tag{3·17}$$

where $\tilde{D}_i$ and $\tilde{\Delta}$ are the covariant derivative and Laplacian with respect to $\tilde{\gamma}_{ij}$ and

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial t} - \beta^i\frac{\partial}{\partial x^i}. \tag{3·18}$$

The Hamiltonian constraint equation then takes the following form.

$$\tilde{\Delta}\psi = \frac{1}{8}\mathrm{tr}\tilde{R}\psi - 2\pi\rho_H\psi^5 - \frac{\psi^5}{8}\Big(\tilde{A}_{ij}\tilde{A}^{ij} - \frac{2}{3}K^2\Big), \tag{3·19}$$

where $\mathrm{tr}\tilde{R}$ is the scalar curvature with respect to $\tilde{\gamma}_{ij}$. The momentum constraint is also written as

$$\tilde{D}_j(\psi^6\tilde{A}^j{}_i) - \frac{2}{3}\psi^6\tilde{D}_iK = 8\pi\psi^6 J_i. \tag{3·20}$$

Now let us consider R_{ij} in Eq.(3·15), which is one of the main source terms of the evolution equation for $\tilde{A}_{ij}$. First we split R_{ij} into two parts as

$$R_{ij} = \tilde{R}_{ij} + R^{\psi}_{ij}, \tag{3·21}$$

where $\tilde{R}_{ij}$ is the Ricci tensor with respect to $\tilde{\gamma}_{ij}$ and R^{ψ}_{ij} is defined as

$$R^{\psi}_{ij} = -\frac{2}{\psi}\tilde{D}_i\tilde{D}_j\psi - \frac{2}{\psi}\tilde{\gamma}_{ij}\tilde{D}^k\tilde{D}_k\psi + \frac{6}{\psi^2}(\tilde{D}_i\psi)(\tilde{D}_j\psi) - \frac{2}{\psi^2}\tilde{\gamma}_{ij}(\tilde{D}_k\psi)(\tilde{D}^k\psi). \tag{3·22}$$

Using $\det(\tilde{\gamma}_{ij}) = 1$, $\tilde{R}_{ij}$ is written as

$$\begin{aligned}\tilde{R}_{ij} = \frac{1}{2}\Big[&-h_{ij,kk} + h_{jl,li} + h_{il,lj} + f^{kl}{}_{,k}(h_{lj,i} + h_{li,j} - h_{ij,l}) \\ &+ f^{kl}(h_{kj,il} + h_{ki,jl} - h_{ij,kl})\Big] - \tilde{\Gamma}^l_{kj}\tilde{\Gamma}^k_{li}\ ,\end{aligned} \tag{3·23}$$

where $_{,i}$ denotes $\partial/\partial x^i$, $\tilde{\Gamma}^k_{ij}$ is the Christoffel symbol, and we split $\tilde{\gamma}_{ij}$ and $\tilde{\gamma}^{ij}$ as $\delta_{ij}+h_{ij}$ and $\delta^{ij}+f^{ij}$, by writing the flat metric as δ_{ij}.

We shall consider only the linear order in h_{ij} and f_{ij} assuming $|h_{ij}|$, $|f_{ij}| \ll 1$. (As a result, $h_{ij}=-f^{ij}+O(h^2)$.) Such a treatment is justified because h_{ij} turns out to be 2PN quantity in our choice of gauge condition (see below). Here, to clarify the wave property of $\tilde{\gamma}_{ij}$, we impose a kind of transverse gauge to h_{ij} as

$$h_{ij,j}=0, \tag{3·24}$$

which was first proposed by Nakamura in relation to numerical relativity [69]. Hereafter, we call this condition merely the transverse gauge. The equation (3·14) shows that this condition is guaranteed if β^i satisfies

$$-\beta^k_{,\,j}\tilde{\gamma}_{ij,k}=\Big(-2\alpha\tilde{A}_{ij}+\tilde{\gamma}_{il}\beta^l_{\ ,j}+\tilde{\gamma}_{jl}\beta^l_{\ ,i}-\frac{2}{3}\tilde{\gamma}_{ij}\beta^l_{\ ,l}\Big)_{,j}. \tag{3·25}$$

Using the transverse gauge, Eq.(3·23) becomes

$$\tilde{R}_{ij}=-\frac{1}{2}\Delta h_{ij}+O(h^2), \tag{3·26}$$

where Δ is the Laplacian with respect to δ_{ij}. Note that $\mathrm{tr}\tilde{R}=O(h^2)$ is guaranteed in the transverse gauge because the traceless property of h_{ij} holds in the linear order.

We show the equations for the isentropic perfect fluid (2·16). The conservation law of mass density, (2·27), may be written as

$$\frac{\partial\rho_*}{\partial t}+\frac{\partial(\rho_* v^i)}{\partial x^i}=0, \tag{3·27}$$

where ρ_* is the conserved density defined as

$$\rho_*=\alpha\psi^6\rho u^0. \tag{3·28}$$

The equation of motion and the energy equation are obtained from the conservation law (2·18) which takes the following forms.

$$\frac{\partial S_i}{\partial t}+\frac{\partial(S_i v^j)}{\partial x^j}=-\alpha\psi^6 P_{,i}-\alpha\alpha_{,i}S^0+S_j\beta^j_{\ ,i}-\frac{1}{2S^0}S_jS_k\gamma^{jk}_{\ \ ,i}, \tag{3·29}$$

and

$$\frac{\partial H}{\partial t}+\frac{\partial(Hv^j)}{\partial x^j}=-P\Big(\frac{\partial(\alpha\psi^6u^0)}{\partial t}+\frac{\partial(\alpha\psi^6u^0v^j)}{\partial x^j}\Big), \tag{3·30}$$

where

$$\begin{aligned}
S_i&=\alpha\psi^6(\rho+\rho\Pi+P)u^0u_i=\rho_*\Big(1+\Pi+\frac{P}{\rho}\Big)u_i(=\psi^6J_i),\\
S^0&=\alpha\psi^6(\rho+\rho\Pi+P)(u^0)^2\Big(=\frac{(\rho_H+P)\psi^6}{\alpha}\Big),\\
H&=\alpha\psi^6\rho\Pi u^0=\rho_*\Pi,\\
v^i&\equiv\frac{u^i}{u^0}=-\beta^i+\frac{\gamma^{ij}S_j}{S^0}.
\end{aligned} \tag{3·31}$$

Finally, we note that in the above equations, only β^i appears, and β_i does not, so that, in the subsequent section, we only consider the PN expansion of β^i, not of β_i.

3.2. *Post-Newtonian approximation in the (3+1) formalism*

Next, we consider the PN approximation of the above set of equations. First of all, we review the PN expansion of the variables. Each metric variable may be expanded up to the relevant order as

$$\begin{aligned}
\psi &= 1 + {}_{(2)}\psi + {}_{(4)}\psi + {}_{(6)}\psi + {}_{(7)}\psi + \dots, \\
\alpha &= 1 + {}_{(2)}\alpha + {}_{(4)}\alpha + {}_{(6)}\alpha + {}_{(7)}\alpha + \dots, \\
&= 1 - U + \left(\frac{U^2}{2} + X\right) + {}_{(6)}\alpha + {}_{(7)}\alpha + \dots, \\
\beta^i &= {}_{(3)}\beta^i + {}_{(5)}\beta^i + {}_{(6)}\beta^i + {}_{(7)}\beta^i + {}_{(8)}\beta^i + \dots, \\
h_{ij} &= {}_{(4)}h_{ij} + {}_{(5)}h_{ij} + \dots, \\
\tilde{A}_{ij} &= {}_{(3)}\tilde{A}_{ij} + {}_{(5)}\tilde{A}_{ij} + {}_{(6)}\tilde{A}_{ij} + \dots, \\
K &= {}_{(3)}K + {}_{(5)}K + {}_{(6)}K + \dots,
\end{aligned} \tag{3·32}$$

where subscripts denote the PN order(ϵ^n) and U is the Newtonian potential satisfying

$$\Delta U = -4\pi\rho. \tag{3·33}$$

X depends on the slice condition, and in the standard PN gauge [21], we usually use $\Phi = -X/2$, which satisfies

$$\Delta\Phi = -4\pi\rho\left(v^2 + U + \frac{1}{2}\Pi + \frac{3}{2}\frac{P}{\rho}\right). \tag{3·34}$$

Note that the terms relevant to the radiation reaction appear in ${}_{(7)}\psi$, ${}_{(7)}\alpha$, ${}_{(8)}\beta^i$ and ${}_{(5)}h_{ij}$, and the quadrupole formula is derived from ${}_{(7)}\alpha$ and ${}_{(5)}h_{ij}$.

The four velocity is expanded as

$$\begin{aligned}
u^\tau &= \epsilon\left[1 + \epsilon^2\left(\frac{1}{2}v^2 + U\right) + \epsilon^4\left(\frac{3}{8}v^4 + \frac{5}{2}v^2U + \frac{1}{2}U^2 + {}_{(3)}\beta^i v^i - X\right) + O(\epsilon^6)\right], \\
u_\tau &= -\epsilon^{-1}\left[1 + \epsilon^2\left(\frac{1}{2}v^2 - U\right) + \epsilon^4\left(\frac{3}{8}v^4 + \frac{3}{2}v^2U + \frac{1}{2}U^2 + X\right) + O(\epsilon^6)\right], \\
u^i &= v^i\left[1 + \epsilon^2\left(\frac{1}{2}v^2 + U\right) + \epsilon^4\left(\frac{3}{8}v^4 + \frac{5}{2}v^2U + \frac{1}{2}U^2 + {}_{(3)}\beta^i v^i - X\right)\right] + O(\epsilon^7), \\
u_i &= v^i + \epsilon^3\left\{{}_{(3)}\beta^i + v^i\left(\frac{1}{2}v^2 + 3U\right)\right\} + \epsilon^5\left[{}_{(5)}\beta^i + {}_{(3)}\beta^i\left(\frac{1}{2}v^2 + 3U\right) + {}_{(4)}h_{ij}v^j\right. \\
&\quad \left. + v^i\left(\frac{3}{8}v^4 + \frac{7}{2}v^2U + 4U^2 - X + 4{}_{(4)}\psi + {}_{(3)}\beta^j v^j\right)\right] + \epsilon^6\left({}_{(6)}\beta^i + {}_{(5)}h_{ij}v^j\right) \\
&\quad + O(\epsilon^7),
\end{aligned} \tag{3·35}$$

where $v^i = O(\epsilon)$ and $v^2 = v^i v^i$. From $u^\mu u_\mu = -1$, we obtain the useful relation

$$\begin{aligned}
(\alpha u^\tau)^2 &= 1 + \gamma^{ij}u_i u_j \\
&= 1 + \epsilon^2 v^2 + \epsilon^4\left(v^4 + 4v^2U + 2{}_{(3)}\beta^i v^i\right) + O(\epsilon^6).
\end{aligned} \tag{3·36}$$

Thus ρ_H, J_i and S_{ij} are respectively expanded as

$$
\begin{aligned}
\rho_H &= \epsilon^2\rho\Big[1+\epsilon^2\Big(v^2+\Pi\Big)+\epsilon^4\Big\{v^4+v^2\Big(4U+\Pi+\frac{P}{\rho}\Big)+2_{(3)}\beta^i v^i\Big\}\Big]+O(\epsilon^8),\\
J_i &= \epsilon^3\rho\Big[v^i\Big(1+\epsilon^2\Big(v^2+3U+\Pi+\frac{P}{\rho}\Big)\Big)+\epsilon^3{}_{(3)}\beta^i\Big]+O(\epsilon^7),\\
S_{ij} &= \epsilon^4\rho\Big[\Big(v^iv^j+\frac{P}{\rho}\delta_{ij}\Big)+\epsilon^2\Big\{\Big(v^2+6U+\Pi+\frac{P}{\rho}\Big)v^iv^j+v^i{}_{(3)}\beta^j+v^j{}_{(3)}\beta^i\\
&\qquad +2\frac{UP}{\rho}\delta_{ij}\Big\}\Big]+O(\epsilon^8),\\
S_l{}^l &= \epsilon^4\rho\Big[v^2+\frac{3P}{\rho}+\epsilon^2\Big\{2_{(3)}\beta^iv^i+v^2\Big(v^2+4U+\Pi+\frac{P}{\rho}\Big)\Big\}\Big]+O(\epsilon^8). \qquad (3\cdot37)
\end{aligned}
$$

The conformal factor ψ (and α in the conformal slice) is determined by the Hamiltonian constraint. In the PN approximation, the Laplacian with respect to $\tilde{\gamma}^{ij}$ for the scalar is expanded as

$$\tilde{\Delta}=\Delta-(\epsilon^4{}_{(4)}h_{ij}+\epsilon^5{}_{(5)}h_{ij})\partial_i\partial_j+O(\epsilon^6). \qquad (3\cdot38)$$

At the lowest order, the Hamiltonian constraint becomes

$$\Delta_{(2)}\psi=-2\pi\rho. \qquad (3\cdot39)$$

Thus, ${}_{(2)}\alpha=-2_{(2)}\psi=-U$ is satisfied in this paper. At the 2PN and 3PN orders, the Hamiltonian constraint equation becomes, respectively,

$$\Delta_{(4)}\psi=-2\pi\rho\Big(v^2+\Pi+\frac{5}{2}U\Big), \qquad (3\cdot40)$$

and

$$
\begin{aligned}
\Delta_{(6)}\psi &= -2\pi\rho\Big\{v^4+v^2\Big(\Pi+\frac{P}{\rho}+\frac{13}{2}U\Big)+2_{(3)}\beta^iv^i+\frac{5}{2}\Pi U+\frac{5}{2}U^2+5_{(4)}\psi\Big\}\\
&\quad +\frac{1}{2}{}_{(4)}h_{ij}U_{,ij}-\frac{1}{8}\Big({}_{(3)}\tilde{A}_{ij\,(3)}\tilde{A}_{ij}-\frac{2}{3}{}_{(3)}K^2\Big). \qquad (3\cdot41)
\end{aligned}
$$

The term relevant to the radiation reaction first appears in ${}_{(7)}\psi$ whose equation becomes

$$\Delta_{(7)}\psi=\frac{1}{2}{}_{(5)}h_{ij}U_{,ij}. \qquad (3\cdot42)$$

Hence, ${}_{(7)}\alpha$ may be also relevant to the radiation reaction, depending on the slice condition.

Let us now derive equations for β^i. From Eq.(3·25), the relation between ${}_{(3)}\tilde{A}_{ij}$ and ${}_{(3)}\beta^i$ becomes

$$-2_{(3)}\tilde{A}_{ij}+{}_{(3)}\beta^i_{,j}+{}_{(3)}\beta^j,i-\frac{2}{3}\delta_{ij\,(3)}\beta^l_{,l}=0. \qquad (3\cdot43)$$

where we used the boundary condition that ${}_{(3)}\tilde{A}_{ij}$ and ${}_{(3)}\beta^i$ vanishes at the spatial infinity. ${}_{(3)}\tilde{A}_{ij}$ must also satisfy the momentum constraint. Since ${}_{(3)}\tilde{A}_{ij}$ does not contain the transverse-traceless (TT) part and only contains the longitudinal part, it can be written as

$$ {}_{(3)}\tilde{A}_{ij} = {}_{(3)}W_{i,j} + {}_{(3)}W_{j,i} - \frac{2}{3}\delta_{ij\,(3)}W_{k,k} \ , \tag{3·44} $$

where ${}_{(3)}W_i$ is a vector on the 3D hypersurface and satisfies the momentum constraint at the first PN order as follows;

$$ \Delta_{(3)}W_i + \frac{1}{3}{}_{(3)}W_{j,ji} - \frac{2}{3}{}_{(3)}K_{,i} = 8\pi\rho v^i. \tag{3·45} $$

From Eq.(3·43), the relation,

$$ {}_{(3)}\beta^i = 2_{(3)}W_i \ , \tag{3·46} $$

holds and at the first PN order, Eq.(3·16) becomes

$$ 3\dot{U} = -{}_{(3)}K + {}_{(3)}\beta^l,l \ , \tag{3·47} $$

where $\dot{U}$ denotes the derivative of U with respect to time. Thus Eq.(3·45) is rewritten as

$$ \Delta_{(3)}\beta^i = 16\pi\rho v^i + \Big({}_{(3)}K_{,i} - \dot{U}_{,i}\Big). \tag{3·48} $$

This is the equation for the vector potential at the first PN order.

At the higher order, ${}_{(n)}\beta^i$ is also determined by the gauge condition, $h_{ij,j} = 0$. The equation for ${}_{(5)}\beta^i$ is obtained by using the momentum constraint and the 2PN order of Eq.(3·16) as follows,

$$ \begin{aligned} \Delta_{(5)}\beta^i = 16\pi\rho\Big[v^i\Big(v^2 + 2U + \Pi + \frac{P}{\rho}\Big) + {}_{(3)}\beta^i\Big] - 8U_{,j\,(3)}\tilde{A}_{ij} \\ + {}_{(5)}K_{,i} - U_{(3)}K_{,i} + \frac{1}{3}U_{,i(3)}K - 2_{(4)}\dot{\psi}_{,i} + \frac{1}{2}(U\dot{U})_{,i} + ({}_{(3)}\beta^l U_{,l})_{,i} \ . \end{aligned} \tag{3·49} $$

The equation for ${}_{(6)}\beta^i$ takes the purely geometrical form since the material contribution J_i at the 1.5PN order vanishes.

$$ \Delta_{(6)}\beta^i = {}_{(6)}K_{,i} \ . \tag{3·50} $$

Then, let us consider the wave equation for h_{ij}. From Eqs.(3·14), (3·15), (3·21) and (3·26), the wave equation for h_{ij} is written as

$$ \begin{aligned} \Box h_{ij} = &\Big(1 - \frac{\alpha^2}{\psi^4}\Big)\Delta h_{ij} + \Big(\frac{\partial^2}{\partial n^2} - \frac{\partial^2}{\partial t^2}\Big)h_{ij} \\ &+ \frac{2\alpha}{\psi^4}\Big[-\frac{2\alpha}{\psi}\Big(\tilde{D}_i\tilde{D}_j - \frac{1}{3}\tilde{\gamma}_{ij}\tilde{\Delta}\Big)\psi + \frac{6\alpha}{\psi^2}\Big(\tilde{D}_i\psi\tilde{D}_j\psi - \frac{1}{3}\tilde{\gamma}_{ij}\tilde{D}_k\psi\tilde{D}^k\psi\Big) \\ &\quad - \Big(\tilde{D}_i\tilde{D}_j - \frac{1}{3}\tilde{\gamma}_{ij}\tilde{\Delta}\Big)\alpha - \frac{2}{\psi}\Big(\tilde{D}_i\psi\tilde{D}_j\alpha + \tilde{D}_j\psi\tilde{D}_i\alpha - \frac{2}{3}\tilde{\gamma}_{ij}\tilde{D}^k\psi\tilde{D}_k\alpha\Big)\Big] \\ &+ 2\alpha^2\Big(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l{}_j\Big) + 2\alpha\Big(\beta^m{}_{,i}\tilde{A}_{mj} + \beta^m{}_{,j}\tilde{A}_{mi} - \frac{2}{3}\beta^m{}_{,m}\tilde{A}_{ij}\Big) \\ &- 16\pi\frac{\alpha^2}{\psi^4}\Big(S_{ij} - \frac{1}{3}\gamma_{ij}S^l{}_l\Big) - \frac{\partial}{\partial n}\Big(\beta^m{}_{,i}\tilde{\gamma}_{mj} + \beta^m{}_{,j}\tilde{\gamma}_{mi} - \frac{2}{3}\beta^m{}_{,m}\tilde{\gamma}_{ij}\Big) + 2\frac{\partial\alpha}{\partial n}\tilde{A}_{ij} \\ \equiv\ &\tau_{ij}, \end{aligned} \tag{3·51} $$

where $\Box$ is the flat spacetime wave operator defined as

$$\Box = -\frac{\partial^2}{\partial t^2} + \Delta. \tag{3·52}$$

We should note that ${}_{(4)}\tau_{ij}$ has the TT property, i.e., ${}_{(4)}\tau_{ij,j} = 0$ and ${}_{(4)}\tau_{ii} = 0$. This is a natural consequence of the transverse gauge, $h_{ij,j} = 0$ and $h_{ii} = O(h^2)$. Thus ${}_{(4)}h_{ij}$ is determined from

$$\Delta_{(4)}h_{ij} = {}_{(4)}\tau_{ij}. \tag{3·53}$$

Since $O(h^2)$ turns out to be $O(\epsilon^8)$, it is enough to consider only the linear order of h_{ij} in the case when we perform the PN approximation up to the 3.5PN order. We can obtain ${}_{(5)}h_{ij}$ by evaluating

$${}_{(5)}h_{ij}(\tau) = \frac{1}{4\pi}\frac{\partial}{\partial\tau}\int {}_{(4)}\tau_{ij}(\tau, \mathbf{y})d^3y, \tag{3·54}$$

and the quadrupole mode of gravitational waves in the wave zone is written as

$$h^{rad}_{ij}(\tau, \mathbf{x}) = -\frac{1}{4\pi}\lim_{|\mathbf{x}|\to\infty}\int \frac{{}_{(4)}\tau_{ij}(\tau - \epsilon|\mathbf{x}-\mathbf{y}|, \mathbf{y})}{|\mathbf{x}-\mathbf{y}|}d^3y. \tag{3·55}$$

Later, we shall derive the quadrupole radiation-reaction metric in the near zone using Eq.(3·54).

Finally, we show the evolution equation for K. Since we adopt slice conditions which do not satisfy $K = 0$ (i.e. the maximal slice condition), the evolution equation for K is necessary. The evolution equations appear at the 1PN, 2PN and 2.5PN orders which become respectively

$$\frac{\partial}{\partial\tau}{}_{(3)}K = 4\pi\rho\left(2v^2 + \Pi + 2U + 3\frac{P}{\rho}\right) - \Delta X, \tag{3·56}$$

$$\begin{aligned}\frac{\partial}{\partial\tau}{}_{(5)}K = {} & 4\pi\rho\left[2v^4 + v^2\left(6U + 2\Pi + 2\frac{P}{\rho}\right) - \left(\Pi + \frac{3P}{\rho}\right)U - 4U^2 + 4{}_{(4)}\psi + X\right.\\ & \left. + 4{}_{(3)}\beta^i v^i\right] + {}_{(3)}\tilde{A}_{ij}{}_{(3)}\tilde{A}_{ij} + \frac{1}{3}{}_{(3)}K^2 - {}_{(4)}h_{ij}U_{,ij} + {}_{(3)}\beta^i{}_{(3)}K_{,i}\\ & -\frac{3}{2}UU_{,k}U_{,k} - U_{,k}X_{,k} + 2U_{,k(4)}\psi_{,k} - \Delta_{(6)}\alpha + 2U\Delta X, \end{aligned} \tag{3·57}$$

$$\frac{\partial}{\partial\tau}{}_{(6)}K = -\Delta_{(7)}\alpha - {}_{(5)}h_{ij}U_{,ij}. \tag{3·58}$$

We note that for the PN equations of motion up to the 2.5PN order, we need ${}_{(2)}\alpha$, ${}_{(4)}\alpha$, ${}_{(6)}\alpha$, ${}_{(7)}\alpha$, ${}_{(2)}\psi$, ${}_{(4)}\psi$, ${}_{(3)}\beta^i$, ${}_{(5)}\beta^i$, ${}_{(6)}\beta^i$, ${}_{(4)}h_{ij}$, ${}_{(5)}h_{ij}$, ${}_{(3)}K$, ${}_{(5)}K$ and ${}_{(6)}K$. Therefore, if we solve the above set of the equations, we can obtain the 2.5 PN equations of motion. Up to the 2.5PN order, the hydrodynamic equations become

$$\begin{aligned}\frac{\partial S_i}{\partial\tau} + \frac{\partial(S_i v^j)}{\partial x^j} = {} & -\left(1 + 2U + \frac{5}{4}U^2 + 6{}_{(4)}\psi + X\right)P_{,i}\\ & +\rho_*\left[U_{,i}\left\{1 + \Pi + \frac{P}{\rho} + \frac{3}{2}v^2 - U + \frac{5}{8}v^4 + 4v^2U\right.\right.\end{aligned}$$

$$+\left(\frac{3}{2}v^2-U\right)\left(\Pi+\frac{P}{\rho}\right)+3_{(3)}\beta^j v^j\Big\}$$
$$-X_{,i}\left(1+\Pi+\frac{P}{\rho}+\frac{v^2}{2}\right)+2v^2{}_{(4)}\psi_{,i}-{}_{(6)}\alpha_{,i}-{}_{(7)}\alpha_{,i}$$
$$+v^j\Big\{{}_{(3)}\beta^j{}_{,i}\left(1+\Pi+\frac{P}{\rho}+\frac{v^2}{2}+3U\right)+{}_{(5)}\beta^j{}_{,i}+{}_{(6)}\beta^j_{,i}\Big\}$$
$$+{}_{(3)}\beta^j{}_{(3)}\beta^j_{,i}+\frac{1}{2}v^jv^k({}_{(4)}h_{jk,i}+{}_{(5)}h_{jk,i})\Big]+O(\epsilon^8), \quad (3{\cdot}59)$$
$$\frac{\partial H}{\partial\tau}+\frac{\partial(Hv^j)}{\partial x^j}=-P\left[v^j{}_{,j}+\frac{\partial}{\partial t}\left(\frac{1}{2}v^2+3U\right)+\frac{\partial}{\partial x^j}\Big\{\left(\frac{1}{2}v^2+3U\right)v^j\Big\}+O(\epsilon^5)\right], \quad (3{\cdot}60)$$

where we have omitted ϵ^n coefficients for the sake of simplicity, and used relations

$$\alpha S^0=\rho_*\left[1+\Pi+\frac{P}{\rho}+\frac{v^2}{2}+\frac{v^2}{2}\left(\Pi+\frac{P}{\rho}\right)+\frac{3}{8}v^4+2v^2U+{}_{(3)}\beta^jv^j)\right]+O(\epsilon^6),$$
$$S_i=\rho_*\left[v^i\left(1+\Pi+\frac{P}{\rho}+\frac{v^2}{2}+3U\right)+{}_{(3)}\beta^i\right]+O(\epsilon^5). \quad (3{\cdot}61)$$

3.3. *Strategy to obtain 2PN tensor potential*

Although the 2PN tensor potential is formally solved as

$${}_{(4)}h_{ij}(\tau,\mathbf{x})=-\frac{1}{4\pi}\int\frac{{}_{(4)}\tau_{ij}(\tau,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}d^3y, \quad (3{\cdot}62)$$

it is difficult to estimate this integral numerically since ${}_{(4)}\tau_{ij}\to O(r^{-3})$ for $r\to\infty$ and the integral is taken all over the space. Thus it is desirable to replace this equation by some tractable forms in numerical evaluation. We shall present two methods to do so. One is to change Eq.(3·62) into the form in which the integration is performed only over the matter distribution like as in the Newtonian potential. The other is to solve Eq.(3·62) as the boundary value problem [4].

Strategy1: Direct integration method

The explicit form of ${}_{(4)}\tau_{ij}$ is

$${}_{(4)}\tau_{ij}=-2\hat{\partial}_{ij}(X+2{}_{(4)}\psi)+U\hat{\partial}_{ij}U-3U_{,i}U_{,j}+\delta_{ij}U_{,k}U_{,k}-16\pi\left(\rho v^iv^j-\frac{1}{3}\delta_{ij}\rho v^2\right)$$
$$-\left({}_{(3)}\dot\beta^i_{,j}+{}_{(3)}\dot\beta^j_{,i}-\frac{2}{3}\delta_{ij}{}_{(3)}\dot\beta^k_{,k}\right), \quad (3{\cdot}63)$$

where

$$\hat{\partial}_{ij}\equiv\frac{\partial^2}{\partial x^i\partial x^j}-\frac{1}{3}\delta_{ij}\Delta. \quad (3{\cdot}64)$$

Although ${}_{(4)}\tau_{ij}$ looks as if it depends on the slice condition, the independence is shown as follows. Eq.(3·48) is solved formally as

$${}_{(3)}\beta^i=p_i-\frac{1}{4\pi}\left(\int\frac{{}_{(3)}K}{|\mathbf{x}-\mathbf{y}|}d^3y\right)_{,i}, \quad (3{\cdot}65)$$

where

$$p_i = -4\int \frac{\rho v^i}{|\mathbf{x}-\mathbf{y}|}d^3y - \frac{1}{2}\Big(\int \dot\rho|\mathbf{x}-\mathbf{y}|d^3y\Big)_{,i} . \qquad (3\cdot66)$$

From Eqs.(3·40) and (3·57), we obtain

$${}_{(3)}\dot K = -\Delta(X + 2_{(4)}\psi) + 4\pi\rho\Big(v^2 + 3\frac{P}{\rho} - \frac{U}{2}\Big). \qquad (3\cdot67)$$

Combining Eq.(3·65) with Eq.(3·67), the equation for ${}_{(3)}\dot\beta^i$ is written as

$${}_{(3)}\dot\beta^i = \dot p_i - (X + 2_{(4)}\psi)_{,i} - \Big[\int \frac{\Big(\rho v^2 + 3P - \rho U/2\Big)}{|\mathbf{x}-\mathbf{y}|}d^3y\Big]_{,i} . \qquad (3\cdot68)$$

Using this relation, the source term, ${}_{(4)}\tau_{ij}$, is split into five parts

$${}_{(4)}\tau_{ij} = {}_{(4)}\tau_{ij}^{(S)} + {}_{(4)}\tau_{ij}^{(U)} + {}_{(4)}\tau_{ij}^{(C)} + {}_{(4)}\tau_{ij}^{(\rho)} + {}_{(4)}\tau_{ij}^{(V)}, \qquad (3\cdot69)$$

where we introduced the following notations.

$$\begin{aligned}
{}_{(4)}\tau_{ij}^{(S)} &= -16\pi\Big(\rho v^i v^j - \frac{1}{3}\delta_{ij}\rho v^2\Big),\\
{}_{(4)}\tau_{ij}^{(U)} &= UU_{,ij} - \frac{1}{3}\delta_{ij}U\Delta U - 3U_{,i}U_{,j} + \delta_{ij}U_{,k}U_{,k},\\
{}_{(4)}\tau_{ij}^{(C)} &= 4\frac{\partial}{\partial x^j}\int\frac{(\rho v^i)^{\cdot}}{|\mathbf{x}-\mathbf{y}|}d^3y + 4\frac{\partial}{\partial x^i}\int\frac{(\rho v^j)^{\cdot}}{|\mathbf{x}-\mathbf{y}|}d^3y - \frac{8}{3}\delta_{ij}\frac{\partial}{\partial x^k}\int\frac{(\rho v^k)^{\cdot}}{|\mathbf{x}-\mathbf{y}|}d^3y,\\
{}_{(4)}\tau_{ij}^{(\rho)} &= \hat\partial_{ij}\int\ddot\rho|\mathbf{x}-\mathbf{y}|d^3y,\\
{}_{(4)}\tau_{ij}^{(V)} &= 2\hat\partial_{ij}\int\frac{\Big(\rho v^2 + 3P - \rho U/2\Big)}{|\mathbf{x}-\mathbf{y}|}d^3y.
\end{aligned} \qquad (3\cdot70)$$

Thus it becomes clear that ${}_{(4)}h_{ij}$ and ${}_{(5)}h_{ij}$ as well as ${}_{(4)}\tau_{ij}$ are expressed in terms of matter variables only and thus their forms do not depend on slicing conditions, though values of matter variables depend on gauge conditions.

Then, we define ${}_{(4)}h_{ij}^{(S)} = \Delta^{-1}{}_{(4)}\tau_{ij}^{(S)}$, ${}_{(4)}h_{ij}^{(U)} = \Delta^{-1}{}_{(4)}\tau_{ij}^{(U)}$, ${}_{(4)}h_{ij}^{(C)} = \Delta^{-1}{}_{(4)}\tau_{ij}^{(C)}$, ${}_{(4)}h_{ij}^{(\rho)} = \Delta^{-1}{}_{(4)}\tau_{ij}^{(\rho)}$ and ${}_{(4)}h_{ij}^{(V)} = \Delta^{-1}{}_{(4)}\tau_{ij}^{(V)}$, and consider each term separately. First, since ${}_{(4)}\tau_{ij}^{(S)}$ is a compact source, we immediately obtain

$${}_{(4)}h_{ij}^{(S)} = 4\int\frac{\Big(\rho v^i v^j - \frac{1}{3}\delta_{ij}\rho v^2\Big)}{|\mathbf{x}-\mathbf{y}|}d^3y. \qquad (3\cdot71)$$

Second, we consider the following equation

$$\Delta G(\mathbf{x},\mathbf{y_1},\mathbf{y_2}) = \frac{1}{|\mathbf{x}-\mathbf{y_1}||\mathbf{x}-\mathbf{y_2}|}. \qquad (3\cdot72)$$

It is possible to write ${}_{(4)}h_{ij}^{(U)}$ using integrals over the matter if this function, G, is used. Eq.(3·72) has solutions [43), 71)],

$$G(\mathbf{x}, \mathbf{y_1}, \mathbf{y_2}) = \ln(r_1 + r_2 \pm r_{12}), \tag{3·73}$$

where

$$r_1 = |\mathbf{x} - \mathbf{y_1}|,\ r_2 = |\mathbf{x} - \mathbf{y_2}|,\ r_{12} = |\mathbf{y_1} - \mathbf{y_2}|. \tag{3·74}$$

Note that $\ln(r_1 + r_2 - r_{12})$ is not regular on the interval between $\mathbf{y_1}$ and $\mathbf{y_2}$, while $\ln(r_1 + r_2 + r_{12})$ is regular on the matter. Thus we use $\ln(r_1 + r_2 + r_{12})$ as a kernel. Using this kernel, $UU_{,ij}$ and $U_{,i}U_{,j}$ are rewritten as

$$\begin{aligned}
UU_{,ij} &= \left[\frac{\partial^2}{\partial x^i \partial x^j}\left(\int \frac{\rho(\mathbf{y_1})}{|\mathbf{x} - \mathbf{y_1}|} d^3y_1\right)\right]\left(\int \frac{\rho(\mathbf{y_2})}{|\mathbf{x} - \mathbf{y_2}|} d^3y_2\right) \\
&= \int d^3y_1 d^3y_2 \rho(\mathbf{y_1})\rho(\mathbf{y_2}) \frac{\partial^2}{\partial y_1^i \partial y_1^j}\left(\frac{1}{|\mathbf{x} - \mathbf{y_1}||\mathbf{x} - \mathbf{y_2}|}\right) \\
&= \Delta \int d^3y_1 d^3y_2 \rho(\mathbf{y_1})\rho(\mathbf{y_2}) \frac{\partial^2}{\partial y_1^i \partial y_1^j} \ln(r_1 + r_2 + r_{12}), \\
U_{,i}U_{,j} &= \left(\frac{\partial}{\partial x^i}\int \frac{\rho(\mathbf{y_1})}{|\mathbf{x} - \mathbf{y_1}|} d^3y_1\right)\left(\frac{\partial}{\partial x^j}\int \frac{\rho(\mathbf{y_2})}{|\mathbf{x} - \mathbf{y_2}|} d^3y_2\right) \\
&= \int d^3y_1 d^3y_2 \rho(\mathbf{y_1})\rho(\mathbf{y_2}) \frac{\partial^2}{\partial y_1^i \partial y_2^j}\left(\frac{1}{|\mathbf{x} - \mathbf{y_1}||\mathbf{x} - \mathbf{y_2}|}\right) \\
&= \Delta \int d^3y_1 d^3y_2 \rho(\mathbf{y_1})\rho(\mathbf{y_2}) \frac{\partial^2}{\partial y_1^i \partial y_2^j} \ln(r_1 + r_2 + r_{12}).
\end{aligned} \tag{3·75}$$

Thus we can express ${}_{(4)}h_{ij}^{(U)}$ using the integral over the matter as

$$\begin{aligned}
{}_{(4)}h_{ij}^{(U)} = \int d^3y_1 d^3y_2 \rho(\mathbf{y_1})\rho(\mathbf{y_2}) &\left[\left(\frac{\partial^2}{\partial y_1^i \partial y_1^j} - \frac{1}{3}\delta_{ij}\triangle_1\right) - 3\left(\frac{\partial^2}{\partial y_1^i \partial y_2^j} - \frac{1}{3}\delta_{ij}\triangle_{12}\right)\right] \\
&\times \ln(r_1 + r_2 + r_{12}),
\end{aligned} \tag{3·76}$$

where we introduced

$$\triangle_1 = \frac{\partial^2}{\partial y_1^k \partial y_1^k},\ \triangle_{12} = \frac{\partial^2}{\partial y_1^k \partial y_2^k}. \tag{3·77}$$

Using relations $\Delta|\mathbf{x} - \mathbf{y}| = 2/|\mathbf{x} - \mathbf{y}|$ and $\Delta|\mathbf{x} - \mathbf{y}|^3 = 12|\mathbf{x} - \mathbf{y}|$, ${}_{(4)}h_{ij}^{(C)}$, ${}_{(4)}h_{ij}^{(\rho)}$ and ${}_{(4)}h_{ij}^{(V)}$ are solved as

$${}_{(4)}h_{ij}^{(C)} = 2\frac{\partial}{\partial x^i}\int (\rho v^j)^{\cdot}|\mathbf{x} - \mathbf{y}|d^3y + 2\frac{\partial}{\partial x^j}\int (\rho v^i)^{\cdot}|\mathbf{x} - \mathbf{y}|d^3y + \frac{4}{3}\delta_{ij}\int \ddot{\rho}|\mathbf{x} - \mathbf{y}|d^3y, \tag{3·78}$$

$${}_{(4)}h_{ij}^{(\rho)} = \frac{1}{12}\frac{\partial^2}{\partial x^i \partial x^j}\int \ddot{\rho}|\mathbf{x} - \mathbf{y}|^3 d^3y - \frac{1}{3}\delta_{ij}\int \ddot{\rho}|\mathbf{x} - \mathbf{y}|d^3y, \tag{3·79}$$

$$
{}_{(4)}h_{ij}^{(V)} = \frac{\partial^2}{\partial x^i \partial x^j} \int \left(\rho v^2 + 3P - \frac{\rho U}{2}\right)|\mathbf{x}-\mathbf{y}| d^3y - \frac{2}{3}\delta_{ij} \int \frac{\left(\rho v^2 + 3P - \rho U/2\right)}{|\mathbf{x}-\mathbf{y}|} d^3y. \tag{3·80}
$$

Thus we find that the 2PN tensor potentials can be expressed as the integrals only over the matter.

Strategy2: Partial use of boundary value approach

Although the above expression for ${}_{(4)}h_{ij}$ is quite interesting and might play an important role in some theoretical applications, it will take a very long time to evaluate *double* integration numerically. Therefore, we propose another strategy where Eq.(3·53) is solved as the boundary value problem. Here, we would like to emphasize that the boundary condition should be imposed at $r(=|\mathbf{x}|) \gg |\mathbf{y_1}|, |\mathbf{y_2}|$, but r does not have to be greater than λ, where λ is a typical wave length of gravitational waves. We only need to impose $r > R$ (a typical size of matter). This means that we do not need a large amount of grid numbers compared with the case of fully general relativistic simulations, in which we require $r > \lambda \gg R$.

First of all, we consider the equation

$$
\Delta\left({}_{(4)}h_{ij}^{(S)} + {}_{(4)}h_{ij}^{(U)}\right) = {}_{(4)}\tau_{ij}^{(S)} + {}_{(4)}\tau_{ij}^{(U)}. \tag{3·81}
$$

Since the source ${}_{(4)}\tau_{ij}^{(U)}$ behaves as $O(r^{-6})$ at $r \to \infty$, this equation can be accurately solved under the boundary condition at $r > R$ as

$$
\begin{aligned}
{}_{(4)}h_{ij}^{(S)} + {}_{(4)}h_{ij}^{(U)} = {}& \frac{2}{r}\left(\ddot{I}_{ij} - \frac{1}{3}\delta_{ij}\ddot{I}_{kk}\right) \\
& + \frac{2}{3r^2}\left(n^k \ddot{I}_{ijk} - \frac{1}{3}\delta_{ij} n^k \ddot{I}_{llk} + 2n^k(\dot{S}_{ikj} + \dot{S}_{jki}) - \frac{4}{3}\delta_{ij} n^k \dot{S}_{lkl}\right) \\
& + O(r^{-3}),
\end{aligned} \tag{3·82}
$$

where

$$
\begin{aligned}
I_{ijk} &= \int \rho x^i x^j x^k d^3x, \\
S_{ijk} &= \int \rho(v^i x^j - v^j x^i) x^k d^3x.
\end{aligned} \tag{3·83}
$$

Next, we consider the equations for ${}_{(4)}h_{ij}^{(C)}$, ${}_{(4)}h_{ij}^{(\rho)}$ and ${}_{(4)}h_{ij}^{(V)}$. Using the identity,

$$
\ddot{\rho} = -(\rho v^i)^{\cdot}_{,i} = (\rho v^i v^j)_{,ij} + \Delta P - (\rho U_{,i})_{,i} \ , \tag{3·84}
$$

we find the following relations;

$$
\begin{aligned}
\int \ddot{\rho} |\mathbf{x}-\mathbf{y}| d^3y &= -\int d^3y \frac{x^i - y^i}{|\mathbf{x}-\mathbf{y}|}(\rho v^i)^{\cdot}, \\
\int \ddot{\rho} |\mathbf{x}-\mathbf{y}|^3 d^3y &= 3\int d^3y \Bigg[\rho v^i v^j \frac{(x^i - y^i)(x^j - y^j)}{|\mathbf{x}-\mathbf{y}|} \\
& \qquad + \left(4P + \rho v^2 - \rho U_{,i}(x^i - y^i)\right)|\mathbf{x}-\mathbf{y}|\Bigg].
\end{aligned} \tag{3·85}
$$

Using Eqs.(3·85), ${}_{(4)}h_{ij}^{(C)}$, ${}_{(4)}h_{ij}^{(\rho)}$ and ${}_{(4)}h_{ij}^{(V)}$ in Eqs.(3·78)-(3·80) can be written as

$$
{}_{(4)}h_{ij}^{(C)} = 2\int(\rho v^j)^{\cdot}\frac{x^i-y^i}{|\mathbf{x}-\mathbf{y}|}d^3y + 2\int(\rho v^i)^{\cdot}\frac{x^j-y^j}{|\mathbf{x}-\mathbf{y}|}d^3y - \frac{4}{3}\delta_{ij}\int(\rho v^k)^{\cdot}\frac{x^k-y^k}{|\mathbf{x}-\mathbf{y}|}d^3y, \tag{3·86}
$$

$$
\begin{aligned}
{}_{(4)}h_{ij}^{(\rho)} = {} & \frac{1}{4}\frac{\partial^2}{\partial x^i\partial x^j}\int \rho v^k v^l \frac{(x^k-y^k)(x^l-y^l)}{|\mathbf{x}-\mathbf{y}|}d^3y + \frac{1}{3}\delta_{ij}\int(\rho v^k)^{\cdot}\frac{x^k-y^k}{|\mathbf{x}-\mathbf{y}|}d^3y \\
& + \frac{1}{2}\left\{\frac{\partial}{\partial x^i}\int P'\frac{(x^j-y^j)}{|\mathbf{x}-\mathbf{y}|}d^3y + \frac{\partial}{\partial x^j}\int P'\frac{(x^i-y^i)}{|\mathbf{x}-\mathbf{y}|}d^3y\right\} \\
& - \frac{1}{8}\left\{2\int \rho\frac{U_{,j}(x^i-y^i)+U_{,i}(x^j-y^j)}{|\mathbf{x}-\mathbf{y}|}d^3y \right. \\
& \left. + x^k\frac{\partial}{\partial x^i}\int \rho\frac{U_{,k}(x^j-y^j)}{|\mathbf{x}-\mathbf{y}|}d^3y + x^k\frac{\partial}{\partial x^j}\int \rho\frac{U_{,k}(x^i-y^i)}{|\mathbf{x}-\mathbf{y}|}d^3y\right\},
\end{aligned} \tag{3·87}
$$

and

$$
\begin{aligned}
{}_{(4)}h_{ij}^{(V)} = {} & \frac{1}{2}\left[\frac{\partial}{\partial x^i}\int\left(\rho v^2+3P-\frac{\rho U}{2}\right)\frac{x^j-y^j}{|\mathbf{x}-\mathbf{y}|}d^3y \right. \\
& \left. + \frac{\partial}{\partial x^j}\int\left(\rho v^2+3P-\frac{\rho U}{2}\right)\frac{x^i-y^i}{|\mathbf{x}-\mathbf{y}|}d^3y\right] \\
& - \frac{2}{3}\delta_{ij}\int\frac{\left(\rho v^2+3P-\rho U/2\right)}{|\mathbf{x}-\mathbf{y}|}d^3y,
\end{aligned} \tag{3·88}
$$

where $P' = P + \rho v^2/4 + \rho U_{,l}y^l/4$. Furthermore they are rewritten in terms of the potentials as follows.

$$
{}_{(4)}h_{ij}^{(C)} = 2(x^i{}_{(3)}\dot{P}^j + x^j{}_{(3)}\dot{P}^i - Q_{ij}) + \frac{4}{3}\delta_{ij}\left(\frac{Q_{kk}}{2} - x^k{}_{(3)}\dot{P}^k\right),
$$

$$
\begin{aligned}
{}_{(4)}h_{ij}^{(\rho)} = {} & \frac{1}{4}\frac{\partial^2}{\partial x^i\partial x^j}\left(V_{kl}^{(\rho v)}x^kx^l - 2V_k^{(\rho v)}x^k + V^{(\rho v)}\right) + \frac{1}{3}\delta_{ij}\left(x^k{}_{(3)}\dot{P}_k - \frac{Q_{kk}}{2}\right) \\
& + \frac{1}{2}\left\{\frac{\partial}{\partial x^i}\left(V^{(P)}x^j - V_j^{(P)}\right) + \frac{\partial}{\partial x^j}\left(V^{(P)}x^i - V_i^{(P)}\right)\right\} \\
& - \frac{1}{8}\left\{2\left(x^iV_j^{(\rho U)} + x^iV_i^{(\rho U)} - V_{ij}^{(\rho U)} - V_{ji}^{(\rho U)}\right)\right. \\
& \left. + x^k\frac{\partial}{\partial x^i}\left(x^jV_k^{(\rho U)} - V_{kj}^{(\rho U)}\right) + x^k\frac{\partial}{\partial x^j}\left(x^iV_k^{(\rho U)} - V_{ki}^{(\rho U)}\right)\right\},
\end{aligned}
$$

$$
{}_{(4)}h_{ij}^{(V)} = \frac{1}{2}\left(Q_{,j}^{(I)}x^i + Q_{,i}^{(I)}x^j - Q_{i,j}^{(I)} - Q_{j,i}^{(I)}\right) + \frac{1}{3}Q^{(I)}\delta_{ij}, \tag{3·89}
$$

where the potentials are defined as

$$
\begin{aligned}
\Delta_{(3)}P_i &= -4\pi\rho v^i,\\
\Delta Q_{ij} &= -4\pi\left(x^j(\rho v^i)^{\cdot} + x^i(\rho v^j)^{\cdot}\right),\\
\Delta Q^{(I)} &= -4\pi\left(\rho v^2 + 3P - \frac{1}{2}\rho U\right),\\
\Delta Q_i^{(I)} &= -4\pi\left(\rho v^2 + 3P - \frac{1}{2}\rho U\right)x^i,\\
\Delta V_{ij}^{(\rho v)} &= -4\pi\rho v^i v^j,\\
\Delta V_i^{(\rho v)} &= -4\pi\rho v^i v^j x^j,\\
\Delta V^{(\rho v)} &= -4\pi\rho(v^j x^j)^2,\\
\Delta V^{(P)} &= -4\pi P',\\
\Delta V_i^{(P)} &= -4\pi P' x^i,\\
\Delta V_i^{(\rho U)} &= -4\pi\rho U_{,i},\\
\Delta V_{ij}^{(\rho U)} &= -4\pi\rho U_{,i}x^j.
\end{aligned}
\tag{3·90}
$$

It should be noted that these Poisson equations have compact sources.

In this strategy ${}_{(4)}h_{ij}^{(S)}$ and ${}_{(4)}h_{ij}^{(U)}$ are solved as the boundary value problem, while other parts are obtained by the same method as in Newtonian gravity.

Strategy 3: Boundary Value approach

Instead of the above procedure, we may solve the Poisson equation for ${}_{(4)}h_{ij}$ as a whole carefully imposing the boundary condition for $r \gg R$ as

$$
\begin{aligned}
{}_{(4)}h_{ij} = \frac{1}{r}\Bigg\{&\frac{1}{4}I_{ij}^{(2)} + \frac{3}{4}n^k\left(n^i I_{kj}^{(2)} + n^j I_{ki}^{(2)}\right)\\
&- \frac{5}{8}n^i n^j I_{kk}^{(2)} + \frac{3}{8}n^i n^j n^k n^l I_{kl}^{(2)} + \frac{1}{8}\delta_{ij}I_{kk}^{(2)} - \frac{5}{8}\delta_{ij}n^k n^l I_{kl}^{(2)}\Bigg\}\\
+\frac{1}{r^2}\Bigg[\Bigg\{&-\frac{5}{12}n^k I_{ijk}^{(2)} - \frac{1}{24}(n^i I_{jkk}^{(2)} + n^j I_{ikk}^{(2)}) + \frac{5}{8}n^k n^l(n^i I_{jkl}^{(2)} + n^j I_{ikl}^{(2)})\\
&- \frac{7}{8}n^i n^j n^k I_{kll}^{(2)} + \frac{5}{8}n^i n^j n^k n^l n^m I_{klm}^{(2)} + \frac{11}{24}\delta_{ij}n^k I_{kll}^{(2)} - \frac{5}{8}\delta_{ij}n^k n^l n^m I_{klm}^{(2)}\Bigg\}\\
&+ \Bigg\{\frac{2}{3}n^k(\dot S_{ikj} + \dot S_{jki}) - \frac{4}{3}(n^i\dot S_{jkk} + n^j\dot S_{ikk})\\
&+ 2n^k n^l(n^i\dot S_{jkl} + n^j\dot S_{ikl}) + 2n^i n^j n^k\dot S_{kll} + \frac{2}{3}\delta_{ij}n^k\dot S_{kll}\Bigg\}\Bigg] + O(r^{-3}).
\end{aligned}
\tag{3·91}
$$

It is verified that $O(r^{-1})$ and $O(r^{-2})$ parts satisfy the traceless and divergence-free conditions respectively. It should be noted that ${}_{(4)}h_{ij}$ obtained in this way becomes meaningless at the far zone because Eq.(3·53), from which ${}_{(4)}h_{ij}$ is derived, is valid only in the near zone.

3.4. *The Radiation Reaction due to Quadrupole Radiation*

This topic has been already investigated by using some gauge conditions in previous papers [25), 79), 15)]. However, if we use the combination of the conformal slice and the transverse gauge, calculations are simplified.

3.4.1. conformal slice

In combination of the conformal slice [84)] and the transverse gauge, Eq.(3·54) becomes [4)]

$$
\begin{aligned}
{}_{(5)}h_{ij}(\tau) = \frac{1}{4\pi}\frac{\partial}{\partial\tau}\int\Big[&-16\pi\Big(\rho v^i v^j - \frac{1}{3}\delta_{ij}\rho v^2\Big) \\
&+\Big(UU_{,ij} - \frac{1}{3}\delta_{ij}U\Delta U - 3U_{,i}U_{,j} + \delta_{ij}U_{,k}U_{,k}\Big)\Big]d^3y \\
&+\frac{1}{4\pi}\frac{\partial}{\partial\tau}\int\Big({}_{(3)}\dot{\beta}^i,j + {}_{(3)}\dot{\beta}^j_{,i} - \frac{2}{3}\delta_{ij}\,{}_{(3)}\dot{\beta}^k_{,k}\Big)d^3y. \qquad (3\cdot92)
\end{aligned}
$$

From a straightforward calculation, we find [4)] that the sum of the first and second lines becomes $-2\not{I}^{(3)}_{ij}$ and the third line becomes $6\not{I}^{(3)}_{ij}/5$, where $\not{I}^{(3)}_{ij} = d^3\not{I}_{ij}/dt^3$ and

$$
\not{I}_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}I_{kk}. \qquad (3\cdot93)
$$

Thus, ${}_{(5)}h_{ij}$ in the near zone becomes

$$
{}_{(5)}h_{ij} = -\frac{4}{5}\not{I}^{(3)}_{ij}. \qquad (3\cdot94)
$$

Since h_{ij} has the transverse and traceless property, it is likely that ${}_{(5)}h_{ij}$ remains the same for other slices. However it is not clear whether the TT property of h_{ij} is satisfied even after the PN expansion is taken in the near zone and, as a result, whether ${}_{(5)}h_{ij}$ is independent of slicing conditions or not. The fact that slicing conditions never affect ${}_{(5)}h_{ij}$ is understood on the ground that ${}_{(4)}\tau_{ij}$ does not depend on slices, as already shown in Eq.(3·69).

Then the Hamiltonian constraint at the 2.5PN order, Eq.(3·42), turns out to be

$$
\Delta_{(7)}\psi = -\frac{2}{5}\not{I}^{(3)}_{ij}U_{,ij} = \frac{1}{5}\not{I}^{(3)}_{ij}\Delta\chi_{,ij}\ , \qquad (3\cdot95)
$$

where χ is the superpotential [21)] defined as

$$
\chi = -\int \rho|\mathbf{x}-\mathbf{y}|d^3y, \qquad (3\cdot96)
$$

which satisfies the relation $\Delta\chi = -2U$. From this, we find ${}_{(7)}\psi$ takes the following form,

$$
\begin{aligned}
{}_{(7)}\psi &= -\frac{1}{5}\not{I}^{(3)}_{ij}\int\rho_{,i}\frac{(x^j-y^j)}{|\mathbf{x}-\mathbf{y}|}d^3y \\
&= \frac{1}{5}\not{I}^{(3)}_{ij}\Big(-x^jU_{,i} + \int\frac{\rho_{,i}y^j}{|\mathbf{x}-\mathbf{y}|}d^3y\Big). \qquad (3\cdot97)
\end{aligned}
$$

Therefore, the lapse function at the 2.5PN order, ${}_{(7)}\alpha = -2{}_{(7)}\psi$, is derived from U and U_r, where U_r satisfies [15)]

$$\Delta U_r = -4\pi \not{I}^{(3)}_{ij} \rho_{,i} x^j. \tag{3·98}$$

Since the right-hand side of Eq.(3·58) cancels out, ${}_{(6)}K$ disappears if the ${}_{(6)}K$ does not exist on the initial hypersurface, which seems reasonable under the condition that there are no initial gravitational waves. Also, ${}_{(6)}\beta^i$ vanishes according to Eq.(3·50). Hence, the quadrupole radiation reaction metric has the same form as that derived in the case of the maximal slice [79), 15)].

From Eq.(3·29), the PN equation of motion becomes

$$\dot{v}^i + v^j v^i_{\ ,\,j} = -\frac{P_{,i}}{\rho} + U_{,i} + F_i^{1PN} + F_i^{2PN} + F_i^{2.5PN} + O(\epsilon^8), \tag{3·99}$$

where F_i^{1PN} and F_i^{2PN} are, respectively, the 1PN and 2PN forces. Since the radiation reaction potentials, ${}_{(5)}h_{ij}$ and ${}_{(7)}\alpha$, are the same as those by Schäfer (1985) and Blanchet, Damour and Schäfer (1990) in which they use the ADM gauge, the radiation reaction force per unit mass, $F_i^{2.5PN} \equiv F_i^r$, is the same as their force and

$$\begin{aligned} F_i^r &= -\Big(({}_{(5)}h_{ij} v^j)^{\cdot} + v^k v^j_{,k\,(5)} h_{ij} + {}_{(7)}\alpha_{,i} \Big) \\ &= \left[\frac{4}{5}(\not{I}^{(3)}_{ij} v^j)^{\cdot} + \frac{4}{5}\not{I}^{(3)}_{ij} v^k v^j_{,k} + \frac{2}{5}\not{I}^{(3)}_{kl} \frac{\partial}{\partial x^i} \int \rho(t,\mathbf{y}) \frac{(x^k - y^k)(x^l - y^l)}{|\mathbf{x}-\mathbf{y}|^3} d^3y \right]. \end{aligned} \tag{3·100}$$

Since the work done by the force (3·100) is given by

$$\begin{aligned} W &\equiv \int \rho F_i^r v^i d^3x \\ &= \frac{4}{5}\frac{d}{d\tau}\left(\not{I}^{(3)}_{ij} \int \rho v^i v^j d^3x \right) - \frac{1}{5}\not{I}^{(3)}_{ij} \not{I}^{(3)ij}, \end{aligned} \tag{3·101}$$

we obtain the so-called quadrupole formula of the energy loss by averaging Eq.(3·101) with respect to time as

$$\left\langle \frac{dE_N}{d\tau} \right\rangle = -\frac{1}{5}\left\langle \not{I}^{(3)}_{ij} \not{I}^{(3)ij} \right\rangle + O(\epsilon^6). \tag{3·102}$$

3.4.2. Radiation reaction in other slice conditions

In this subsection, we do not specify the slice condition. In this case, the reaction force takes the following form

$$F_i^r = \frac{4}{5}(\not{I}^{(3)}_{ij} v^j)^{\cdot} + \frac{4}{5}\not{I}^{(3)}_{ij} v^k v^j_{,k} - {}_{(7)}\alpha_{,i} - {}_{(6)}\dot{\beta}^i + v^j {}_{(6)}\beta^j{}_{,i} - v^j {}_{(6)}\beta^i{}_{,j}. \tag{3·103}$$

Here, ${}_{(7)}\alpha$ corresponds to the slice condition. From Eq.(3·103), we obtain the work done by the reaction force as

$$W \equiv \int \rho F_i^r v^i d^3x$$

$$= \int d^3x\rho v^i \Big[\frac{4}{5}\Big(\rlap{-}I^{(3)}_{ij} v^j \Big)^{\cdot} + \frac{4}{5} \rlap{-}I^{(3)}_{ij} v^j{}_{,l} v^l$$

$$- {}_{(7)}\alpha_{,i} - {}_{(6)}\dot{\beta}^i + v^j{}_{(6)}\beta^j{}_{,i} - v^j{}_{(6)}\beta^i{}_{,j} \Big]. \qquad (3\cdot104)$$

Explicit calculations are done separately: For the first and second terms of Eq.(3·104), we obtain

$$\frac{4}{5}\int d^3x\rho\Big(\rlap{-}I^{(3)}_{ij} v^j \Big)^{\cdot} v^i = \frac{4}{5}\frac{d}{dt}\Big[\rlap{-}I^{(3)}_{ij} \int d^3x\rho v^i v^j \Big] - \frac{1}{5}\rlap{-}I^{(3)}_{ij}\rlap{-}I^{(3)}_{ij}$$

$$-\frac{2}{5}\rlap{-}I^{(3)}_{ij}\int d^3x\rho(x)v^k \frac{\partial}{\partial x^k}\int d^3y \frac{\rho(y)(x^i-y^i)(x^j-y^j)}{|\mathbf{x}-\mathbf{y}|^3}$$

$$-\frac{2}{5}\rlap{-}I^{(3)}_{ij}\int d^3x\dot{\rho}v^i v^j, \qquad (3\cdot105)$$

and

$$\frac{4}{5}\int d^3x\rho \rlap{-}I^{(3)}_{ij} v^j_{,l} v^l v^i = \frac{2}{5}\rlap{-}I^{(3)}_{ij}\int d^3x\dot{\rho}v^i v^j. \qquad (3\cdot106)$$

As for the fourth term in the integral of Eq.(3·104), we find the relation

$${}_{(6)}\dot{\beta}^i = -{}_{(7)}\alpha_{,i} - \frac{2}{5}\rlap{-}I^{(3)}_{kl}\frac{\partial}{\partial x^i}\int d^3y\rho\frac{(x^k-y^k)(x^l-y^l)}{|\mathbf{x}-\mathbf{y}|^3}, \qquad (3\cdot107)$$

which is given by Eqs.(3·50), (3·58) and (3·94).

Using Eqs.(3·104), (3·105), (3·106) and (3·107), we obtain

$$W = \frac{4}{5}\frac{d}{d\tau}\Big(\rlap{-}I^{(3)}_{ij}\int \rho v^i v^j d^3x \Big) - \frac{1}{5}\rlap{-}I^{(3)}_{ij}\rlap{-}I^{(3)ij}. \qquad (3\cdot108)$$

This expression for W does not depend on the slice condition. However, this never means that the value of W is invariant under the change of the slice condition, since the meaning of the time derivative depends on the slice condition.

3.5. *Conserved quantities*

The conserved quantities are gauge-invariant so that, in general relativity, they play important roles to characterize various systems described in different gauge conditions. From the practical point of view, these are also useful for checking the numerical accuracy in simulations. Thus, we show several conserved quantities in the 2PN approximation.

Conserved Mass And Energy

In general relativity, we have the following conserved mass;

$$M_* = \int \rho_* d^3x. \qquad (3\cdot109)$$

In the PN approximation, ρ_* defined by Eq.(3·28) is expanded as [4]

$$\rho_* = \rho\Big[1 + \Big(\frac{1}{2}v^2 + 3U\Big)$$

$$+\Big(\frac{3}{8}v^4 + \frac{7}{2}v^2U + \frac{15}{4}U^2 + 6_{(4)}\psi + {}_{(3)}\beta^i v^i\Big) + {}_{(6)}\delta_*\Big] + O(\epsilon^7), \qquad (3\cdot110)$$

where ${}_{(6)}\delta_*$ denotes the 3PN contribution to ρ_*. This term ${}_{(6)}\delta_*$ will be calculated later.

Next, we consider the ADM mass which is conserved. Since the asymptotic behavior of the conformal factor becomes

$$\psi = 1 + \frac{M_{ADM}}{2r} + O\left(\frac{1}{r^2}\right), \tag{3·111}$$

the ADM mass in the PN approximation becomes

$$\begin{aligned} M_{ADM} &= -\frac{1}{2\pi}\int \Delta\psi d^3x \\ &= \int d^3x\rho\Bigg[\bigg\{1 + \left(v^2 + \Pi + \frac{5}{2}U\right) + \bigg(v^4 + \frac{13}{2}v^2U + v^2\Pi + \frac{P}{\rho}v^2 + \frac{5}{2}U\Pi \\ &\qquad + \frac{5}{2}U^2 + 5_{(4)}\psi + 2_{(3)}\beta^iv^i\bigg)\bigg\} + \frac{1}{16\pi\rho}\left({}_{(3)}\tilde{A}_{ij\,(3)}\tilde{A}_{ij} - \frac{2}{3}{}_{(3)}K^2\right) \\ &\qquad + {}_{(6)}\delta_{ADM} + O(\epsilon^7)\Bigg], \end{aligned} \tag{3·112}$$

where ${}_{(6)}\delta_{ADM}$ denotes the 3PN contribution. This term ${}_{(6)}\delta_{ADM}$ will be calculated later.

Using these two conserved quantities, we can define the conserved energy as follows;

$$\begin{aligned} E &\equiv M_{ADM} - M_* \\ &= \int d^3x\rho\Bigg[\bigg\{\left(\frac{1}{2}v^2 + \Pi - \frac{1}{2}U\right) \\ &\qquad + \left(\frac{5}{8}v^4 + 3v^2U + v^2\Pi + \frac{P}{\rho}v^2 + \frac{5}{2}U\Pi - \frac{5}{4}U^2 - {}_{(4)}\psi + {}_{(3)}\beta^iv^i\right)\bigg\} \\ &\qquad + \frac{1}{16\pi\rho}\left({}_{(3)}\tilde{A}_{ij\,(3)}\tilde{A}_{ij} - \frac{2}{3}{}_{(3)}K^2\right) + \left({}_{(6)}\delta_{ADM} - {}_{(6)}\delta_*\right) + O(c^{-7})\Bigg] \\ &\equiv E_N + E_{1PN} + E_{2PN} + \cdots. \end{aligned} \tag{3·113}$$

We should note that the following equation holds

$$\int {}_{(3)}\tilde{A}_{ij\,(3)}\tilde{A}_{ij}d^3x = -8\pi\int \rho v^i{}_{(3)}\beta^i d^3x + \int\left(\frac{2}{3}{}_{(3)}K^2 + 2\dot{U}_{(3)}K\right)d^3x, \tag{3·114}$$

where we used Eqs.(3·47) and (3·48). Then, we obtain the Newtonian and the first PN energies as

$$E_N = \int \rho\left(\frac{1}{2}v^2 + \Pi - \frac{1}{2}U\right)d^3x, \tag{3·115}$$

and

$$\begin{aligned} E_{1PN} = \int d^3x\bigg[&\rho\left(\frac{5}{8}v^4 + \frac{5}{2}v^2U + v^2\Pi + \frac{P}{\rho}v^2 + 2U\Pi - \frac{5}{2}U^2 + \frac{1}{2}{}_{(3)}\beta^iv^i\right) \\ &+ \frac{1}{8\pi}\dot{U}_{(3)}K\bigg]. \end{aligned} \tag{3·116}$$

E_{1PN} can be rewritten immediately in the following form used by Chandrasekhar [23];

$$E_{1PN} = \int d^3x\rho\biggl[\frac{5}{8}v^4 + \frac{5}{2}v^2U + v^2\Bigl(\Pi + \frac{P}{\rho}\Bigr) + 2U\Pi - \frac{5}{2}U^2 - \frac{1}{2}v^iq_i\biggr], \quad (3\cdot117)$$

where q_i is the first PN shift vector in the standard PN gauge [23], which turns out to be ${}_{(3)}K = 0$ in the (3+1) formalism and it satisfies

$$\Delta q_i = -16\pi\rho v^i + \dot{U}_{,i}. \quad (3\cdot118)$$

The total energy at the 2PN order E_{2PN} is calculated from the 3PN quantities ${}_{(6)}\delta_*$ and ${}_{(6)}\delta_{ADM}$. In a straightforward manner, we obtain

$$_{(6)}M_* = \int \rho_{(6)}\delta_* d^3x, \quad (3\cdot119)$$

where

$$\begin{aligned}{}_{(6)}\delta_* = {}& \frac{5}{16}v^6 + \frac{33}{8}v^4U + v^2\Bigl(5_{(4)}\psi + \frac{93}{8}U^2 + \frac{3}{2}{}_{(3)}\beta^iv^i - X\Bigr) + 6_{(6)}\psi + 15U_{(4)}\psi \\ & + \frac{5}{2}U^3 + 7_{(3)}\beta^iv^iU + \frac{1}{2}{}_{(4)}h_{ij}v^iv^j + \frac{1}{2}{}_{(3)}\beta^i{}_{(3)}\beta^i + {}_{(5)}\beta^iv^i. \end{aligned} \quad (3\cdot120)$$

Next, we consider ${}_{(6)}\delta_{ADM}$. The Hamiltonian constraint at $O(\epsilon^8)$ becomes

$$\begin{aligned}&\Delta_{(8)}\psi - {}_{(4)}h_{ij\,(4)}\psi_{,ij} - \frac{1}{2}{}_{(6)}h_{ij}U_{,ij} \\ &= -\frac{1}{32}(2_{(4)}h_{kl,m\,(4)}h_{km,l} + {}_{(4)}h_{kl,m\,(4)}h_{kl,m}) - 2\pi_{(6)}\rho_\psi \\ &-\frac{1}{4}\Bigl({}_{(3)}\tilde{A}_{ij\,(5)}\tilde{A}_{ij} - \frac{2}{3}{}_{(3)}K_{(5)}K\Bigr) - \frac{1}{16}U\Bigl({}_{(3)}\tilde{A}_{ij\,(3)}\tilde{A}_{ij} - \frac{2}{3}{}_{(3)}K^2\Bigr), \end{aligned} \quad (3\cdot121)$$

where we define ${}_{(6)}\rho_\psi$ as

$$\begin{aligned}{}_{(6)}\rho_\psi = \rho\biggl[& v^6 + v^4\Bigl(\Pi + \frac{P}{\rho} + \frac{21}{2}U\Bigr) + v^2\Bigl\{\frac{13}{2}U\Bigl(\Pi + \frac{P}{\rho}\Bigr) + 9_{(4)}\psi - 2X + 20U^2\Bigr\} \\ & + \Pi\Bigl(5_{(4)}\psi + \frac{5}{2}U^2\Bigr) + 5_{(6)}\psi + 10U_{(4)}\psi + \frac{5}{4}U^3 + {}_{(4)}h_{ij}v^iv^j \\ & + 2_{(3)}\beta^iv^i\Bigl\{2v^2 + \Pi + \frac{P}{\rho} + \frac{13}{2}U\Bigr\} + 2_{(5)}\beta^iv^i + {}_{(3)}\beta^i{}_{(3)}\beta^i\biggr]. \end{aligned} \quad (3\cdot122)$$

Making use of the relations ${}_{(4)}h_{ij,j} = 0$ and ${}_{(6)}h_{ij,j} = 0$, we obtain

$$\begin{aligned}{}_{(6)}M_{ADM} = {}& \int d^3x_{(6)}\rho_\psi + \frac{1}{8\pi}\int d^3y\Bigl({}_{(3)}\tilde{A}_{ij\,(5)}\tilde{A}_{ij} - \frac{2}{3}{}_{(3)}K_{(5)}K\Bigr) \\ & + \frac{1}{32\pi}\int d^3yU\Bigl({}_{(3)}\tilde{A}_{ij\,(3)}\tilde{A}_{ij} - \frac{2}{3}{}_{(3)}K^2\Bigr), \end{aligned} \quad (3\cdot123)$$

where we assumed ${}_{(6)}h_{ij} \to O(r^{-1})$ as $r \to \infty$. Although this assumption must be verified by performing the 3PN expansions which have not been done here, it seems reasonable in the asymptotically flat spacetime. From ${}_{(6)}M_{ADM}$ and ${}_{(6)}M_*$, we obtain the conserved energy at the 2PN order

$$\begin{aligned}
E_{2PN} &= {}_{(6)}M_{ADM} - {}_{(6)}M_* \\
&= \int d^3x\rho\Big[\frac{11}{16}v^6 + v^4\Big(\Pi + \frac{P}{\rho} + \frac{47}{8}U\Big) \\
&\qquad + v^2\Big\{4_{(4)}\psi - X + 6U\Big(\Pi + \frac{P}{\rho}\Big) + \frac{41}{8}U^2 + \frac{5}{2}{}_{(3)}\beta^i v^i\Big\} \\
&\qquad + \Pi\Big(5_{(4)}\psi + \frac{5}{4}U^2\Big) - \frac{15}{2}U_{(4)}\psi - \frac{5}{2}U^3 \\
&\qquad + \frac{1}{2}{}_{(4)}h_{ij}v^iv^j + 2_{(3)}\beta^i v^i\Big\{\Big(\Pi + \frac{P}{\rho}\Big) + 5U\Big\} + {}_{(5)}\beta^i v^i + \frac{1}{2}{}_{(3)}\beta^i{}_{(3)}\beta^i\Big] \\
&\quad + \frac{1}{8\pi}\int d^3y\Big({}_{(4)}h_{ij}UU_{,ij} + {}_{(3)}\tilde{A}_{ij\,(5)}\tilde{A}_{ij} - \frac{2}{3}{}_{(3)}K_{(5)}K\Big).
\end{aligned} \qquad (3\cdot124)$$

Here we used Eq.(3·41) and the relation, $\int d^3x\rho_{(6)}\psi = -\frac{1}{4\pi}\int d^3xU\Delta_{(6)}\psi$, in order to eliminate ${}_{(6)}\psi$.

Conserved linear momentum

When we use the center of mass system as usual, the linear momentum of the system should vanish. However, it may arise from numerical errors in numerical calculations. Since it is useful to check the numerical accuracy, we mention the linear momentum derived from

$$\begin{aligned}
P_i &= \frac{1}{8\pi}\lim_{r\to\infty}\oint\Big(K_{ij}n^j - Kn^i\Big)dS \\
&= \frac{1}{8\pi}\lim_{r\to\infty}\oint\Big(\psi^4\tilde{A}_{ij}n^j - \frac{2}{3}Kn^i\Big)dS,
\end{aligned} \qquad (3\cdot125)$$

where the surface integrals are taken over a sphere of constant r. Since the asymptotic behavior of $\tilde{A}_{ij}$ is determined by

$${}_{(3)}\tilde{A}_{ij} = \frac{1}{2}\Big({}_{(3)}\beta^i_{,j} + {}_{(3)}\beta^j_{,i} - \frac{2}{3}\delta_{ij\,(3)}\beta^l_{,l}\Big) + O(r^{-3}), \qquad (3\cdot126)$$

and

$${}_{(5)}\tilde{A}_{ij} = \frac{1}{2}\Big({}_{(5)}\beta^i_{,j} + {}_{(5)}\beta^j_{,i} - \frac{2}{3}\delta_{ij\,(5)}\beta^l_{,l}\Big) + {}_{(5)}\tilde{A}^{TT}_{ij} + O(r^{-3}), \qquad (3\cdot127)$$

the leading term of the shift vector is necessary. Using the asymptotic behavior derived from Eq.(3·48)

$${}_{(3)}\beta^i = -\frac{7}{2}\frac{l_i}{r} - \frac{1}{2}\frac{n^in^jl_j}{r} + O(r^{-2}), \qquad (3\cdot128)$$

we obtain

$$\int\Big({}_{(3)}\beta^i_{,j} + {}_{(3)}\beta^j_{,i} - \frac{2}{3}\delta_{ij\,(3)}\beta^l_{,l}\Big)n^jdS = 16\pi l_i, \qquad (3\cdot129)$$

where we defined $l_i = \int \rho v^i d^3x$. Therefore the Newtonian linear momentum is

$$P_N{}^i = \int d^3x \rho v^i. \tag{3·130}$$

Similarly the first PN linear momentum is obtained as

$$P_{1PN}{}^i = \int d^3x \rho \Big[v^i \Big(v^2 + \Pi + 6U + \frac{P}{\rho} \Big) + {}_{(3)}\beta^i \Big]. \tag{3·131}$$

We obtain $P_{2PN}{}^i$ by the similar procedure as

$$P_{2PN}{}^i = \int d^3x \rho v^i \Big[2_{(3)}\beta^i v^i + 10_{(4)}\psi + \Big(6U + v^2 \Big) \Big(\Pi + \frac{P}{\rho} \Big) + \frac{67}{4} U^2 + 10Uv^2 + v^4 - X \Big]. \tag{3·132}$$

§4. Formulation for Nonaxisymmetric Uniformly Rotating Equilibrium Configurations

We now consider the construction of the spacetime involving a close binary neutron stars as an application of PN approximation. It is assumed that the binary stars are regarded as uniformly rotating equilibrium configurations. Here, we mention the importance of this investigation. To interpret the implication of the signal of gravitational waves, we need to understand the theoretical mechanism of merging in detail. When the orbital separation of binary neutron stars is $\lesssim 10GM/c^2$, where M is the total mass of binary neutron stars, they move approximately in circular orbits because the timescale of the energy loss due to gravitational radiation is much longer than the orbital period. However, when the orbital separation becomes $6-10GM/c^2$, the circular orbit cannot be maintained because of instabilities due to the GR gravity [59] or the tidal field [62), 63]. As a result of such instabilities, the circular orbit of binary neutron stars changes into the plunging orbit to merge. Gravitational waves emitted at this transition may present us important information about the structure of NS's since the location where the instability occurs will depend sensitively on the equation of state (EOS) of NS [62), 63), 102]. Thus, it is very important to investigate the location of the innermost stable circular orbit (ISCO) of binary neutron stars.

In order to search the ISCO, we can take the following procedure: First, neglecting the evolution due to gravitational radiation, we construct equilibrium configurations. Next, we take into account the radiation reaction as a correction to the equilibrium configurations. The ISCO is the orbit, where the dynamical instability for the equilibrium configurations occurs. Hence we shall present a formalism to obtain equilibrium configurations of uniformly rotating fluid in 2PN order as a first step [5], though in reality due to the conservation of the circulation for gravitational radiation reaction, tidally locked configuration such as uniformly rotating fluid is not a good approximation.

4.1. *Formulation*

We shall use in this section the maximal slice condition $K_i{}^i = 0$. As the spatial gauge condition, we adopt the transverse gauge $\tilde{\gamma}_{ij,j} = 0$ in order to remove the gauge modes from $\tilde{\gamma}_{ij}$. In this case, up to the 2PN approximation, each metric variable is expanded as[4)]

$$\psi = 1 + \frac{1}{c^2}\frac{U}{2} + \frac{1}{c^4}{}_{(4)}\psi + O(c^{-6}), \tag{4·1}$$

$$\alpha = 1 - \frac{1}{c^2}U + \frac{1}{c^4}\left(\frac{U^2}{2} + X\right) + \frac{1}{c^6}{}_{(6)}\alpha + O(c^{-7}), \tag{4·2}$$

$$\beta^i = \frac{1}{c^3}{}_{(3)}\beta^i + \frac{1}{c^5}{}_{(5)}\beta^i + O(c^{-7}), \tag{4·3}$$

$$\tilde{\gamma}_{ij} = \delta_{ij} + \frac{1}{c^4}h_{ij} + O(c^{-5}). \tag{4·4}$$

In this section, we use $1/c$ instead of ϵ as the expansion parameter because of its convenience in numerical applications.

For simplicity, we assume that the matter obeys the polytropic equation of state (EOS);

$$P = (\Gamma - 1)\rho\Pi = K\rho^\Gamma, \tag{4·5}$$

where Γ and K are the polytropic exponent and polytropic constant, respectively. Up to the 2PN order, the four velocity is given by Eq.(3·35)[24),4),5)]. Since we need u^0 up to 3PN order to obtain the 2PN equations of motion, we derive it here. Using Eq.(3·35), we can calculate $(\alpha u^0)^2$ up to 3PN order as

$$\begin{aligned}(\alpha u^0)^2 &= 1 + \psi^{-4}\tilde{\gamma}^{ij}u_i u_j \\ &= 1 + \frac{v^2}{c^2} + \frac{1}{c^4}\left(2_{(3)}\beta^j v^j + 4Uv^2 + v^4\right) + \frac{1}{c^6}\Big\{{}_{(3)}\beta^j{}_{(3)}\beta^j + 8_{(3)}\beta^j v^j U \\ &\quad + h_{ij}v^i v^j + 2_{(5)}\beta^i v^i + \left(4_{(3)}\beta^j v^j + 4_{(4)}\psi + \frac{15}{2}U^2 - 2X\right)v^2 + 8Uv^4 + v^6\Big\} \\ &\quad + O(c^{-7}),\end{aligned} \tag{4·6}$$

where we used $\tilde{\gamma}^{ij} = \delta_{ij} - c^{-4}h_{ij} + O(c^{-5})$. Thus, we obtain u^0 up to the 3PN order as

$$\begin{aligned}u^0 &= 1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 + U\right) + \frac{1}{c^4}\left(\frac{3}{8}v^4 + \frac{5}{2}v^2 U + \frac{1}{2}U^2 + {}_{(3)}\beta^i v^i - X\right) \\ &\quad + \frac{1}{c^6}\Big\{-{}_{(6)}\alpha + \frac{1}{2}\left({}_{(3)}\beta^j{}_{(3)}\beta^j + h_{ij}v^i v^j\right) + {}_{(5)}\beta^j v^j + 5_{(3)}\beta^j v^j U - 2UX \\ &\qquad + \left(\frac{3}{2}{}_{(3)}\beta^j v^j + 2_{(4)}\psi + 6U^2 - \frac{3}{2}X\right)v^2 + \frac{27}{8}Uv^4 + \frac{5}{16}v^6\Big\} \\ &\quad + O(c^{-7}).\end{aligned} \tag{4·7}$$

Substituting PN expansions of metric and matter variables into the Einstein equation, and using the polytropic EOS, we find that the metric variables obey the following Poisson equations[4)];

$$\Delta U = -4\pi\rho, \tag{4·8}$$

$$\Delta X = 4\pi\rho\left(2v^2 + 2U + (3\Gamma - 2)\Pi\right), \tag{4·9}$$

$$\Delta_{(4)}\psi = -2\pi\rho\Big(v^2 + \Pi + \frac{5}{2}U\Big), \tag{4·10}$$

$$\Delta_{(3)}\beta^i = 16\pi\rho v^i - \dot{U}_{,i}, \tag{4·11}$$

$$\begin{aligned}\Delta_{(5)}\beta^i = 16\pi\rho\Big[v^i\Big(v^2 + 2U + \Gamma\Pi\Big) + {}_{(3)}\beta^i\Big] - 4U_{,j}\Big({}_{(3)}\beta^i_{,j} + {}_{(3)}\beta^j_{,i} - \frac{2}{3}\delta_{ij}{}_{(3)}\beta^k_{,k}\Big) \\ - 2{}_{(4)}\dot{\psi}_{,i} + \frac{1}{2}(U\dot{U})_{,i} + ({}_{(3)}\beta^l U_{,l})_{,i},\end{aligned} \tag{4·12}$$

$$\begin{aligned}\Delta h_{ij} = \Big(UU_{,ij} - \frac{1}{3}\delta_{ij}U\Delta U - 3U_{,i}U_{,j} + \delta_{ij}U_{,k}U_{,k}\Big) - 16\pi\Big(\rho v^i v^j - \frac{1}{3}\delta_{ij}\rho v^2\Big) \\ -\Big({}_{(3)}\dot{\beta}^i_{,j} + {}_{(3)}\dot{\beta}^j_{,i} - \frac{2}{3}\delta_{ij}{}_{(3)}\dot{\beta}^k_{,k}\Big) - 2\Big((X + 2{}_{(4)}\psi)_{,ij} - \frac{1}{3}\delta_{ij}\Delta(X + 2{}_{(4)}\psi)\Big),\end{aligned} \tag{4·13}$$

$$\begin{aligned}\Delta_{(6)}\alpha = 4\pi\rho\Big[2v^4 + 2v^2\Big(5U + \Gamma\Pi\Big) + (3\Gamma - 2)\Pi U + 4{}_{(4)}\psi + X + 4{}_{(3)}\beta^i v^i\Big] \\ - h_{ij}U_{,ij} - \frac{3}{2}UU_{,l}U_{,l} + U_{,l}(2{}_{(4)}\psi - X)_{,l} \\ + \frac{1}{2}{}_{(3)}\beta^i_{,j}\Big({}_{(3)}\beta^i_{,j} + {}_{(3)}\beta^j_{,i} - \frac{2}{3}\delta_{ij}{}_{(3)}\beta^k_{,k}\Big).\end{aligned} \tag{4·14}$$

Here, we consider *the uniformly rotating fluid around z-axis* with the angular velocity Ω, i.e.,

$$v^i = \epsilon_{ijk}\Omega^j x^k = (-y\Omega, x\Omega, 0), \tag{4·15}$$

where we choose $\Omega^j = (0, 0, \Omega)$ and ϵ_{ijk} is the completely anti-symmetric unit tensor. In this case, the following relations hold;

$$\Big(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\varphi}\Big)Q = \Big(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\varphi}\Big)Q_i = \Big(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\varphi}\Big)Q_{ij} = 0, \tag{4·16}$$

where Q, Q_i and Q_{ij} are arbitrary scalars, vectors, and tensors, respectively. Then, the conservation law (2·18) can be integrated as [65]

$$\int\frac{dP}{\rho c^2 + \rho\Pi + P} = \ln u^0 + C, \tag{4·17}$$

where C is a constant. For the polytropic EOS, Eq.(4·17) becomes

$$\ln\Big[1 + \frac{\Gamma K}{c^2(\Gamma - 1)}\rho^{\Gamma-1}\Big] = \ln u^0 + C. \tag{4·18}$$

Using Eq.(4·7), the 2PN approximation of Eq.(4·18) is written as

$$\begin{aligned}H - \frac{H^2}{2c^2} + \frac{H^3}{3c^4} = \frac{v^2}{2} + U + \frac{1}{c^2}\Big(2Uv^2 + \frac{v^4}{4} - X + {}_{(3)}\beta^i v^i\Big) \\ +\frac{1}{c^4}\Big(-{}_{(6)}\alpha + \frac{1}{2}{}_{(3)}\beta^i{}_{(3)}\beta^i + 4{}_{(3)}\beta^i v^i U - \frac{U^3}{6} + {}_{(3)}\beta^i v^i v^2 + 2{}_{(4)}\psi v^2\end{aligned}$$

$$+\frac{15}{4}U^2v^2 + 2Uv^4 + \frac{1}{6}v^6 - UX - v^2X + {}_{(5)}\beta^i v^i + \frac{1}{2}h_{ij}v^iv^j\Big)$$
$$+C, \tag{4·19}$$

where $H = \Gamma K\rho^{\Gamma-1}/(\Gamma-1)$, $v^2 = R^2\Omega^2$ and $R^2 = x^2 + y^2$. Note that Eq.(4·19) can be also obtained from the 2PN Euler equation like in the first PN case[22]. If we solve the coupled equations (4·8)-(4·14) and (4·19), we can obtain equilibrium configurations of the non-axisymmetric uniformly rotating body.

In the above Poisson equations for metric variables, the source terms in the Poisson equations for ${}_{(3)}\beta^i$, ${}_{(5)}\beta^i$, and h_{ij} fall off slowly as $r \to \infty$ because these terms behave as $O(r^{-3})$ at $r \to \infty$. These Poisson equations do not take convenient forms when we try to solve them numerically as the boundary value problem. Hence in the following, we rewrite them into convenient forms.

As for h_{ij}, first of all, we split the equation into three parts as[4]

$$\Delta h_{ij}^{(U)} = U\Big(U_{,ij} - \frac{1}{3}\delta_{ij}\Delta U\Big) - 3U_{,i}U_{,j} + \delta_{ij}U_{,k}U_{,k} \equiv -4\pi S_{ij}^{(U)}, \tag{4·20}$$

$$\Delta h_{ij}^{(S)} = -16\pi\Big(\rho v^iv^j - \frac{1}{3}\delta_{ij}\rho v^2\Big), \tag{4·21}$$

$$\begin{aligned}\Delta h_{ij}^{(G)} = &-\Big({}_{(3)}\dot\beta^i_{,j} + {}_{(3)}\dot\beta^j_{,i} - \frac{2}{3}\delta_{ij}\,{}_{(3)}\dot\beta^k_{,k}\Big)\\ &- 2\Big((X + 2_{(4)}\psi)_{,ij} - \frac{1}{3}\delta_{ij}\Delta(X + 2_{(4)}\psi)\Big).\end{aligned} \tag{4·22}$$

The equation for $h_{ij}^{(S)}$ has a compact source, and also the source term of $h_{ij}^{(U)}$ behaves as $O(r^{-6})$ at $r \to \infty$, so that Poisson equations for them are solved easily as the boundary value problem. On the other hand, the source term of $h_{ij}^{(G)}$ behaves as $O(r^{-3})$ at $r \to \infty$, so that it seems troublesome to solve the equation as the boundary value problem. In order to solve the equation for $h_{ij}^{(G)}$ as the boundary value problem, we had better rewrite the equation into useful forms. As shown by Asada, Shibata and Futamase[4], Eq.(4·22) is integrated to give

$$\begin{aligned}h_{ij}^{(G)} = &\,2\frac{\partial}{\partial x^i}\int (\rho v^j)^{\dot{}}|\mathbf{x}-\mathbf{y}|d^3y + 2\frac{\partial}{\partial x^j}\int (\rho v^i)^{\dot{}}|\mathbf{x}-\mathbf{y}|d^3y + \delta_{ij}\int \ddot\rho|\mathbf{x}-\mathbf{y}|d^3y\\ &+\frac{1}{12}\frac{\partial^2}{\partial x^i\partial x^j}\int \ddot\rho|\mathbf{x}-\mathbf{y}|^3d^3y + \frac{\partial^2}{\partial x^i\partial x^j}\int\Big(\rho v^2 + 3P - \frac{\rho U}{2}\Big)|\mathbf{x}-\mathbf{y}|d^3y\\ &-\frac{2}{3}\delta_{ij}\int\frac{\Big(\rho v^2 + 3P - \rho U/2\Big)}{|\mathbf{x}-\mathbf{y}|}d^3y.\end{aligned} \tag{4·23}$$

Using the relations

$$\begin{aligned}&\ddot\rho = -(\rho v^j)^{\dot{}}_{,j} + O(c^{-2}),\\ &\dot v^i = 0,\\ &v^ix^i = 0,\end{aligned} \tag{4·24}$$

Eq.(4·23) is rewritten as

$$h_{ij}^{(G)} = \frac{7}{4}\left(x^i{}_{(3)}\dot{P}_j + x^j{}_{(3)}\dot{P}_i - \dot{Q}_{ij}^{(T)} - \dot{Q}_{ji}^{(T)}\right) - \delta_{ij}x^k{}_{(3)}\dot{P}_k$$
$$-\frac{1}{8}x^k\left[\frac{\partial}{\partial x^i}\left(x^j{}_{(3)}\dot{P}_k - \dot{Q}_{kj}^{(T)}\right) + \frac{\partial}{\partial x^j}\left(x^i{}_{(3)}\dot{P}_k - \dot{Q}_{ki}^{(T)}\right)\right]$$
$$+\frac{1}{2}\left[\frac{\partial}{\partial x^i}\left(x^j Q^{(I)} - Q_j^{(I)}\right) + \frac{\partial}{\partial x^j}\left(x^i Q^{(I)} - Q_i^{(I)}\right)\right] - \frac{2}{3}\delta_{ij}Q^{(I)}, \quad (4·25)$$

where

$$\Delta_{(3)}P_i = -4\pi\rho v^i, \quad (4·26)$$
$$\Delta Q_{ij}^{(T)} = -4\pi\rho v^i x^j, \quad (4·27)$$
$$\Delta Q^{(I)} = -4\pi\left(\rho v^2 + 3P - \frac{1}{2}\rho U\right), \quad (4·28)$$
$$\Delta Q_i^{(I)} = -4\pi\left(\rho v^2 + 3P - \frac{1}{2}\rho U\right)x^i. \quad (4·29)$$

Therefore, $h_{ij}^{(G)}$ can be deduced from variables which satisfy the Poisson equations with compact sources.

The source terms in the Poisson equations (4·11) and (4·12) for ${}_{(3)}\beta^i$ and ${}_{(5)}\beta^i$ also fall off slowly. However, we can rewrite them as [4]

$${}_{(3)}\beta^i = -4{}_{(3)}P_i - \frac{1}{2}\left(x^i\dot{U} - \dot{q}_i\right), \quad (4·30)$$
$${}_{(5)}\beta^i = -4{}_{(5)}P_i - \frac{1}{2}\left(2x^i{}_{(4)}\dot{\psi} - \dot{\eta}_i\right), \quad (4·31)$$

where

$$\Delta q_i = -4\pi\rho x^i, \quad (4·32)$$
$$\Delta_{(5)}P_i = -4\pi\rho\left[v^i\left(v^2 + 2U + \Gamma\Pi\right) + {}_{(3)}\beta^i\right] + U_{,j}\left({}_{(3)}\beta^i_{,j} + {}_{(3)}\beta^j_{,i} - \frac{2}{3}\delta_{ij}{}_{(3)}\beta^k_{,k}\right)$$
$$-\frac{1}{8}(\dot{U}U)_{,i} - \frac{1}{4}({}_{(3)}\beta^l U_{,l})_{,i}, \quad (4·33)$$
$$\Delta\eta_i = -4\pi\rho\left(v^2 + \Pi + \frac{5}{2}U\right)x^i. \quad (4·34)$$

Thus ${}_{(3)}\beta^i$ and ${}_{(5)}\beta^i$ can be obtained by solving the Poisson equations (4·32)-(4·34) whose source terms fall off fast enough, $O(r^{-5})$, for numerical calculation.

For later convenience, using the relation ${}_{(3)}P_i = \epsilon_{izk}q_k\Omega$ and Eq.(4·16), ${}_{(3)}\beta^i$ and ${}_{(5)}\beta^i$ may be written in the form with explicit Ω dependence as

$${}_{(3)}\beta^i = \Omega\left[-4\epsilon_{izk}q_k + \frac{1}{2}\left(x^i U_{,\varphi} - q_{i,\varphi}\right)\right] \equiv \Omega{}_{(3)}\hat{\beta}^i, \quad (4·35)$$
$${}_{(5)}\beta^i = \Omega\left[-4{}_{(5)}\hat{P}_i + \frac{1}{2}\left(2x^i{}_{(4)}\psi_{,\varphi} - \eta_{i,\varphi}\right)\right], \quad (4·36)$$

where

$$\Delta_{(5)}\hat{P}_i = -4\pi\rho\left[\epsilon_{izk}x^k\left(v^2+2U+\Gamma\Pi\right)+{}_{(3)}\hat{\beta}^i\right]+U_{,j}\left({}_{(3)}\hat{\beta}^i_{,j}+{}_{(3)}\hat{\beta}^j_{,i}-\frac{2}{3}\delta_{ij\,(3)}\hat{\beta}^k_{,k}\right)$$
$$+\frac{1}{8}(UU_{,\varphi})_{,i}-\frac{1}{4}({}_{(3)}\hat{\beta}^kU_{,k})_{,i}. \tag{4·37}$$

4.2. *Basic equations appropriate for numerical approach*

Although equilibrium configurations can be formally obtained by solving Eq.(4·19) as well as metric potentials, U, X, ${}_{(4)}\psi$, ${}_{(6)}\alpha$, ${}_{(3)}\beta^i$, ${}_{(5)}\beta^i$ and h_{ij}, they do not take convenient forms for numerical calculation. Thus, we here change Eq.(4·19) into forms appropriate to obtain numerically equilibrium configurations.

In numerical calculation, the standard method to obtain equilibrium configurations is as follows [52), 72)];
(1) We give a trial density configuration for ρ.
(2) We solve the Poisson equations.
(3) Using Eq.(4·19), we give a new density configuration.
These procedures are repeated until a sufficient convergence is achieved. Here, at (3), we need to specify unknown constants, Ω and C. In standard numerical methods [52), 72)], these are calculated during iteration by fixing densities at two points; i.e., if we put ρ_1 at x_1 and ρ_2 at x_2 into Eq.(4·19), we obtain two equations for Ω and C. Solving these two equations give us Ω and C. However, the procedure is not so simple in the PN case: Ω is included in the source of the Poisson equations for the variables such as X, ${}_{(4)}\psi$, ${}_{(6)}\alpha$, η_i, ${}_{(5)}\hat{P}_i$, $h^{(S)}_{ij}$, $Q^{(T)}_{ij}$, $Q^{(I)}$ and $Q^{(I)}_i$. Thus, if we use Eq.(4·19) as it is, equations for Ω and C become implicit equations for Ω. In such a situation, the convergence to a solution is very slow. Therefore, we transform those equations into other forms in which the potentials as well as Eq.(4·19) become explicit polynomial equations in Ω.

First of all, we define q_2, q_{2i}, q_4, q_u, q_e and q_{ij} which satisfy

$$\Delta q_2 = -4\pi\rho R^2, \tag{4·38}$$
$$\Delta q_{2i} = -4\pi\rho R^2 x^i, \tag{4·39}$$
$$\Delta q_4 = -4\pi\rho R^4, \tag{4·40}$$
$$\Delta q_u = -4\pi\rho U, \tag{4·41}$$
$$\Delta q_e = -4\pi\rho \Pi, \tag{4·42}$$
$$\Delta q_{ij} = -4\pi\rho x^i x^j. \tag{4·43}$$

Then, X, ${}_{(4)}\psi$, $Q^{(I)}$, $Q^{(I)}_i$, η_i, ${}_{(5)}\hat{P}_i$, $Q^{(T)}_{ij}$, and $h^{(S)}_{ij}$ are written as

$$X = -2q_2\Omega^2 - 2q_u - (3\Gamma-2)q_e, \tag{4·44}$$
$${}_{(4)}\psi = \frac{1}{2}\left(q_2\Omega^2 + q_e + \frac{5}{2}q_u\right), \tag{4·45}$$
$$Q^{(I)} = q_2\Omega^2 + 3(\Gamma-1)q_e - \frac{1}{2}q_u \equiv q_2\Omega^2 + Q^{(I)}_0, \tag{4·46}$$
$$Q^{(I)}_i = q_{2i}\Omega^2 + Q^{(I)}_{0i}, \tag{4·47}$$

$$\eta_i = q_{2i}\Omega^2 + \eta_{0i}, \tag{4·48}$$

$$_{(5)}\hat{P}_i = \epsilon_{izk}q_{2k}\Omega^2 + {}_{(5)}P_{0i}, \tag{4·49}$$

$$Q^{(T)}_{ij} = \epsilon_{izl}q_{lj}\Omega, \tag{4·50}$$

$$h^{(S)}_{ij} = 4\Omega^2\Big(\epsilon_{izk}\epsilon_{jzl}q_{kl} - \frac{1}{3}\delta_{ij}q_2\Big), \tag{4·51}$$

where $Q^{(I)}_{0i}$, η_{0i} and $_{(5)}P_{0i}$ satisfy

$$\Delta Q^{(I)}_{0i} = -4\pi\Big(3P - \frac{1}{2}\rho U\Big)x^i = -4\pi\rho\Big(3(\Gamma-1)\Pi - \frac{1}{2}U\Big)x^i, \tag{4·52}$$

$$\Delta\eta_{0i} = -4\pi\rho\Big(\Pi + \frac{5}{2}U\Big)x^i, \tag{4·53}$$

$$\begin{aligned}\Delta_{(5)}P_{0i} = &-4\pi\rho\Big[\epsilon_{izk}x^k\Big(2U + \Gamma\Pi\Big) + {}_{(3)}\hat{\beta}^i\Big] + U_{,j}\Big({}_{(3)}\hat{\beta}^i_{,j} + {}_{(3)}\hat{\beta}^j_{,i} - \frac{2}{3}\delta_{ij\,(3)}\hat{\beta}^k_{,k}\Big) \\ &+ \frac{1}{8}(UU_{,\varphi})_{,i} - \frac{1}{4}({}_{(3)}\hat{\beta}^k U_{,k})_{,i} \equiv -4\pi S^{(P)}_i.\end{aligned} \tag{4·54}$$

Note that $_{(5)}\beta^i$ and $h^{(G)}_{ij}$ are the cubic and quadratic equations in Ω, respectively, as

$$_{(5)}\beta^i = {}_{(5)}\beta^{i\,(A)}\Omega + {}_{(5)}\beta^{i\,(B)}\Omega^3, \tag{4·55}$$

$$h^{(G)}_{ij} = h^{(A)}_{ij} + h^{(B)}_{ij}\Omega^2, \tag{4·56}$$

where

$$_{(5)}\beta^{i\,(A)} = -4_{(5)}P_{0i} + \frac{1}{2}\Big\{x^i\Big(q_e + \frac{5}{2}q_u\Big)_{,\varphi} - \eta_{0i,\varphi}\Big\}, \tag{4·57}$$

$$_{(5)}\beta^{i\,(B)} = -4\epsilon_{izk}q_{2k} + \frac{1}{2}\Big(x^i q_{2,\varphi} - q_{2i,\varphi}\Big), \tag{4·58}$$

$$\begin{aligned} h^{(A)}_{ij} &= \frac{1}{2}\Big[\frac{\partial}{\partial x^j}\Big(x^i Q^{(I)}_0 - Q^{(I)}_{0i}\Big) + \frac{\partial}{\partial x^i}\Big(x^j Q^{(I)}_0 - Q^{(I)}_{0j}\Big) - \frac{4}{3}\delta_{ij}Q^{(I)}_0\Big], \\ h^{(B)}_{ij} &= \Big[\frac{1}{2}\Big\{\frac{\partial}{\partial x^j}\Big(x^i q_2 - q_{2i}\Big) + \frac{\partial}{\partial x^i}\Big(x^j q_2 - q_{2j}\Big) - \frac{4}{3}\delta_{ij}q_2\Big\} \\ &\quad -\frac{7}{4}\Big(x^i\epsilon_{jzk}q_{k,\varphi} + x^j\epsilon_{izk}q_{k,\varphi} - \epsilon_{izk}q_{kj,\varphi} - \epsilon_{jzk}q_{ki,\varphi}\Big) + \delta_{ij}x^k\epsilon_{kzl}q_l \\ &\quad +\frac{1}{8}x^k\Big\{\frac{\partial}{\partial x^i}\Big(x^j\epsilon_{kzl}q_{l,\varphi} - \epsilon_{kzl}q_{lj,\varphi}\Big) + \frac{\partial}{\partial x^j}\Big(x^i\epsilon_{kzl}q_{l,\varphi} - \epsilon_{kzl}q_{li,\varphi}\Big)\Big\}\Big]. \end{aligned} \tag{4·59}$$

Finally, we write $_{(6)}\alpha$ as

$$_{(6)}\alpha = {}_{(6)}\alpha_0 + {}_{(6)}\alpha_2\Omega^2 - 2q_4\Omega^4, \tag{4·60}$$

where $_{(6)}\alpha_0$ and $_{(6)}\alpha_2$ satisfy

$$\Delta_{(6)}\alpha_0 = 4\pi\rho\Big[\Big(3\Gamma - 2\Big)\Pi U - \Big(3\Gamma - 4\Big)q_e + 3q_u\Big]$$

$$-\Big(h_{ij}^{(U)}+h_{ij}^{(A)}\Big)U_{,ij}-\frac{3}{2}UU_{,l}U_{,l}+U_{,l}\frac{\partial}{\partial x^l}\Big(\frac{9}{2}q_u+(3\Gamma+1)q_e\Big)$$

$$\equiv -4\pi S^{(\alpha_0)}, \tag{4·61}$$

$$\Delta_{(6)}\alpha_2=8\pi\rho R^2\Big(5U+\Gamma\Pi+2{}_{(3)}\hat{\beta}_\varphi\Big)-\Big(4\epsilon_{izk}\epsilon_{jzl}q_{kl}-\frac{4}{3}\delta_{ij}q_2+h_{ij}^{(B)}\Big)U_{,ij}$$

$$+3q_{2,l}U_{,l}+\frac{1}{2}{}_{(3)}\hat{\beta}^i_{,j}\Big({}_{(3)}\hat{\beta}^i_{,j}+{}_{(3)}\hat{\beta}^j_{,i}-\frac{2}{3}\delta_{ij}{}_{(3)}\hat{\beta}^k_{,k}\Big)$$

$$\equiv -4\pi S^{(\alpha_2)}. \tag{4·62}$$

Using the above quantities, Eq.(4·19) is rewritten as

$$H-\frac{H^2}{2c^2}+\frac{H^3}{3c^4}=A+B\Omega^2+D\Omega^4+\frac{R^6}{6c^4}\Omega^6+C, \tag{4·63}$$

where

$$A=U+\frac{1}{c^2}\Big(2q_u+(3\Gamma-2)q_e\Big)+\frac{1}{c^4}\Big\{-{}_{(6)}\alpha_0-\frac{U^3}{6}+U\Big(2q_u+(3\Gamma-2)q_e\Big)\Big\},$$

$$B=\frac{R^2}{2}+\frac{1}{c^2}\Big(2R^2U+2q_2+{}_{(3)}\hat{\beta}_\varphi\Big)+\frac{1}{c^4}\Big\{-{}_{(6)}\alpha_2+\frac{1}{2}{}_{(3)}\hat{\beta}^i{}_{(3)}\hat{\beta}^i+4{}_{(3)}\hat{\beta}_\varphi U$$

$$+(3\Gamma-1)q_eR^2+\frac{9}{2}q_uR^2+\frac{15}{4}U^2R^2+2q_2U+{}_{(5)}\beta_\varphi^{(A)}+\frac{1}{2}\Big(h_{\varphi\varphi}^{(U)}+h_{\varphi\varphi}^{(A)}\Big)\Big\},$$

$$D=\frac{R^4}{4c^2}+\frac{1}{c^4}\Big\{2q_4+{}_{(3)}\hat{\beta}_\varphi R^2+\frac{7}{3}q_2R^2+2UR^4+{}_{(5)}\beta_\varphi^{(B)}+\frac{1}{2}\Big(h_{\varphi\varphi}^{(B)}+4R^2q_{RR}\Big)\Big\}. \tag{4·64}$$

Note that in the above, we use the following relations which hold for arbitrary vector Q_i and symmetric tensor Q_{ij},

$$Q_\varphi=-yQ_x+xQ_y,$$

$$Q_{\varphi\varphi}=y^2Q_{xx}-2xyQ_{xy}+x^2Q_{yy},$$

$$R^2Q_{RR}=x^2Q_{xx}+2xyQ_{xy}+y^2Q_{yy}. \tag{4·65}$$

We also note that source terms of Poisson equations for variables which appear in A, B and D do not depend on Ω explicitly. Thus, Eq.(4·63) takes the desired form for numerical calculation.

In this formalism, we need to solve 29 Poisson equations for U, q_x, q_y, q_z, ${}_{(5)}P_{0x}$, ${}_{(5)}P_{0y}$, η_{0x}, η_{0y}, $Q_{0x}^{(I)}$, $Q_{0y}^{(I)}$, $Q_{0z}^{(I)}$, q_2, q_{2x}, q_{2y}, q_{2z}, q_u, q_e, $h_{xx}^{(U)}$, $h_{xy}^{(U)}$, $h_{xz}^{(U)}$, $h_{yy}^{(U)}$, $h_{yz}^{(U)}$, q_{xx}, q_{xy}, q_{xz}, q_{yz}, ${}_{(6)}\alpha_0$, ${}_{(6)}\alpha_2$ and q_4. In Table 1, we show the list of the Poisson equations to be solved. In Table 2, we also summarize what variables are needed to calculate the metric variables U, X, ${}_{(4)}\psi$, ${}_{(6)}\alpha$, ${}_{(3)}\beta^i$, ${}_{(5)}\beta^i$, $h_{ij}^{(U)}$, $h_{ij}^{(S)}$, $h_{ij}^{(A)}$ and $h_{ij}^{(B)}$. Note that we do not need ${}_{(5)}P_{0z}$, η_{0z}, and q_{zz} because they do not appear in any equation. Also, we do not have to solve the Poisson equations for $h_{zz}^{(U)}$ and q_{yy} because they can be calculated from $h_{zz}^{(U)}=-h_{xx}^{(U)}-h_{yy}^{(U)}$ and $q_{yy}=q_2-q_{xx}$.

In order to derive U, q_i, q_2, q_{2i}, q_4, q_e and q_{ij}, we do not need any other potential because only matter variables appear in the source terms of their Poisson equations. On the other hand, for q_u, $Q^{(I)}_{0i}$, η_{0i} and $h^{(U)}_{ij}$, we need the Newtonian potential U, and for ${}_{(5)}P_{0i}$, ${}_{(6)}\alpha_0$ and ${}_{(6)}\alpha_2$, we need the Newtonian as well as PN potentials. Thus, U, q_i, q_2, q_{2i}, q_4, q_e and q_{ij} must be solved first, and then q_u, $Q^{(I)}_{0i}$, η_{0i}, $h^{(U)}_{ij}$, ${}_{(5)}P_{0i}$ and ${}_{(6)}\alpha_2$ should be solved. ${}_{(6)}\alpha_0$ must be solved after we obtain q_u because its Poisson equation involves q_u in the source term. In Table 1 we also list potentials which are included in the source terms of the Poisson equations for other potentials.

The configuration which we are most interested in and would like to obtain is the equilibrium state for BNS's of equal mass. Hence, we show the boundary condition at $r \to \infty$ for this problem. When we consider equilibrium configurations for BNS's where the center of mass for each NS is on the x-axis, boundary conditions for potentials at $r \to \infty$ become

$$\begin{aligned}
U &= \frac{1}{r}\int \rho d^3x + O(r^{-3}), & q_x &= \frac{n^x}{r^2}\int \rho x^2 d^3x + O(r^{-4}),\\
q_2 &= \frac{1}{r}\int \rho R^2 d^3x + O(r^{-3}), & q_y &= \frac{n^y}{r^2}\int \rho y^2 d^3x + O(r^{-4}),\\
q_e &= \frac{1}{r}\int \rho \Pi d^3x + O(r^{-3}), & q_z &= \frac{n^z}{r^2}\int \rho z^2 d^3x + O(r^{-4}),\\
q_u &= \frac{1}{r}\int \rho U d^3x + O(r^{-3}), & q_4 &= \frac{1}{r}\int \rho R^4 d^3x + O(r^{-3}),
\end{aligned} \tag{4·66}$$

$$\begin{aligned}
{}_{(5)}P_{0x} &= \frac{n^x}{r^2}\int S^{(P)}_x x d^3x + \frac{n^y}{r^2}\int S^{(P)}_y y d^3x + O(r^{-3}),\\
{}_{(5)}P_{0y} &= \frac{n^x}{r^2}\int S^{(P)}_y x d^3x + \frac{n^y}{r^2}\int S^{(P)}_y y d^3x + O(r^{-3}),
\end{aligned} \tag{4·67}$$

$$\begin{aligned}
\eta_{0x} &= \frac{n^x}{r^2}\int \rho x^2 \left(\Pi + \frac{5}{2}U\right) d^3x + O(r^{-4}),\\
\eta_{0y} &= \frac{n^y}{r^2}\int \rho y^2 \left(\Pi + \frac{5}{2}U\right) d^3x + O(r^{-4}),
\end{aligned} \tag{4·68}$$

$$\begin{aligned}
Q^{(I)}_{0x} &= \frac{n^x}{r^2}\int \rho x^2 \left(3(\Gamma-1)\Pi - \frac{1}{2}U\right) d^3x + O(r^{-4}), & q_{2x} &= \frac{n^x}{r^2}\int \rho R^2 x^2 d^3x + O(r^{-4}),\\
Q^{(I)}_{0y} &= \frac{n^y}{r^2}\int \rho y^2 \left(3(\Gamma-1)\Pi - \frac{1}{2}U\right) d^3x + O(r^{-4}), & q_{2y} &= \frac{n^y}{r^2}\int \rho R^2 y^2 d^3x + O(r^{-4}),\\
Q^{(I)}_{0z} &= \frac{n^y}{r^2}\int \rho z^2 \left(3(\Gamma-1)\Pi - \frac{1}{2}U\right) d^3x + O(r^{-4}), & q_{2z} &= \frac{n^z}{r^2}\int \rho R^2 z^2 d^3x + O(r^{-4}),
\end{aligned} \tag{4·69}$$

$$h^{(U)}_{xx} = \frac{1}{r}\int S^{(U)}_{xx} d^3x + O(r^{-3}), \qquad h^{(U)}_{xy} = \frac{3n^x n^y}{r^3}\int S^{(U)}_{xy} xy d^3x + O(r^{-5}),$$

$$h^{(U)}_{yy} = \frac{1}{r}\int S^{(U)}_{yy} d^3x + O(r^{-3}), \qquad h^{(U)}_{xz} = \frac{3n^x n^z}{r^3}\int S^{(U)}_{xz} xz d^3x + O(r^{-5}),$$
$$h^{(U)}_{yz} = \frac{3n^y n^z}{r^3}\int S^{(U)}_{yz} yz d^3x + O(r^{-5}), \tag{4·70}$$

$$q_{xx} = \frac{1}{r}\int \rho x^2 d^3x + O(r^{-3}), \qquad q_{xy} = \frac{3n^x n^y}{r^3}\int \rho x^2 y^2 d^3x + O(r^{-5}),$$
$$q_{xz} = \frac{3n^x n^z}{r^3}\int \rho x^2 z^2 d^3x + O(r^{-5}), \qquad q_{yz} = \frac{3n^y n^z}{r^3}\int \rho y^2 z^2 d^3x + O(r^{-5}), \tag{4·71}$$

$$_{(6)}\alpha_0 = \frac{1}{r}\int S^{(\alpha_0)} d^3x + O(r^{-3}), \quad _{(6)}\alpha_2 = \frac{1}{r}\int S^{(\alpha_2)} d^3x + O(r^{-3}), \tag{4·72}$$

where $n^i = x^i/r$. We note that $S^{(P)}_i \to O(r^{-5})$, $S^{(U)}_{ij} \to O(r^{-6})$, $S^{(\alpha_0)} \to O(r^{-4})$ and $S^{(\alpha_2)} \to O(r^{-4})$ as $r \to \infty$, so that all the above integrals are well defined.

We would like to emphasize that from the 2PN order, the tensor part of the 3-metric, $\tilde{\gamma}_{ij}$, cannot be neglected even if we ignore gravitational waves. Recently, Wilson and Mathews [100)], Wilson, Mathews and Marronetti [101)] presented numerical equilibrium configurations of binary neutron stars using a semi-relativistic approximation, in which they assume the spatially conformal flat metric as the spatial 3-metric, i.e., $\tilde{\gamma}_{ij} = \delta_{ij}$. Thus, in their method, a 2PN term, h_{ij}, was completely neglected. However, it should be noted that this tensor potential plays an important role at the 2PN order: This is because they appear in the equations to determine equilibrium configurations as shown in previous sections and they also contribute to the total energy and angular momentum of systems. This means that if we performed the stability analysis ignoring the tensor potentials, we might reach an incorrect conclusion. If we hope to obtain a general relativistic equilibrium configuration of binary neutron stars with a better accuracy (say less than 1%), we should take into account the tensor part of the 3-metric.

In our formalism, we extract terms depending on the angular velocity Ω from the integrated Euler equation and Poisson equations for potentials, and rewrite the integrated Euler equation as an explicit equation in Ω. This reduction will improve the convergence in numerical iteration procedure. As a result, the number of Poisson equations we need to solve in each step of iteration reaches 29. However, source terms of the Poisson equations decrease rapidly enough, at worst $O(r^{-4})$, in the region far from the source, so that we can solve accurately these equations as the boundary value problem like in the case of the first PN calculations.

The formalism presented here will be useful to obtain equilibrium configurations for synchronized BNS's or the Jacobi ellipsoid. In this context, Bonazzola, Frieben and Gourgoulhon (1996) obtained an approximate nonaxisymmetric neutron star by perturbing a stationary axisymmetric configuration. However, they do not solve the exact 3D Einstein's equation. Thus, it is interesting to examine their conclusion on the transition between equilibrium configurations, which are approximately ellipsoids, by our PN methods.

§5. Gravitational Waves from post-Newtonian sources

In the post-Minkowskian approximation, the background geometry is the Minkowski spacetime where linearized gravitational waves propagate [89), 11), 12)]. The corrections to propagation of gravitational waves can be taken into account if one performs the post-Minkowskian approximation up to higher orders. In fact, Blanchet and Damour obtained the tail term of gravitational waves as the integral over the past history of the source [12), 13)]. They introduced an unphysical complex parameter B, and used the analytic continuation as a mathematical device in order to evaluate the so-called log term in the tail contribution. The method is powerful, but it is not easy to see the origin of the tail term. Will and Wiseman (1996) have also obtained the tail term from the mass quadrupole moment, by improving the Epstein-Wagoner formalism [41)]. Nakamura [70)] and Schäfer [80)] have obtained the quadrupole energy loss formula including the contribution from the tail term by studying the wave propagation in the Coulomb type potential. This indicates clearly that the origin of the tail term is due to the Coulomb type potential generated by the mass of the source. In the following treatment we calculate the waveform from the slow motion source and show explicitly how the tail term originates from the difference between the flat light cone and the true one which is due to the mass of the source GM/c^2 as the lowest order correction.

5.1. *Gravitational waves in Coulomb type potential*

We wish to clarify that the main part of the tail term is due to the propagation of gravitational waves on the light cone which deviates slightly from the flat light cone owing to the mass of the source. For this purpose, we shall work in the harmonic coordinate since the deviation may be easily seen in the reduced Einstein's equation in this coordinates.

$$(\eta^{\alpha\beta}-\bar{h}^{\alpha\beta})\bar{h}^{\mu\nu}{}_{,\alpha\beta}=-16\pi\Theta^{\mu\nu}+\bar{h}^{\mu\alpha}{}_{,\beta}\bar{h}^{\nu\beta}{}_{,\alpha}. \tag{5·1}$$

Since we look for $1/r$ part of the solution, the spatial derivative is not relevant in the differential operator so that it is more convenient for our purpose to transform Eq.(5·1) into the following form

$$(\Box-\bar{h}^{00}\partial_0\partial_0)\bar{h}^{\mu\nu}=-16\pi\mathcal{S}^{\mu\nu}, \tag{5·2}$$

where we defined $\mathcal{S}^{\mu\nu}$ as

$$\mathcal{S}^{\mu\nu}=\Theta^{\mu\nu}-\frac{1}{16\pi}(\bar{h}^{\mu\alpha}{}_{,\beta}\bar{h}^{\nu\beta}{}_{,\alpha}-2\bar{h}^{0i}\bar{h}^{\mu\nu}{}_{,0i}-\bar{h}^{ij}\bar{h}^{\mu\nu}{}_{,ij}). \tag{5·3}$$

Substituting the lowest order expression for $\bar{h}^{00}$ derived in section 2, we finally obtain our basic equation to be solved.

$$\left[-\left(1+\frac{4M}{r}\right)\frac{\partial^2}{\partial t^2}+\Delta\right]\bar{h}^{\mu\nu}=\tilde{\tau}^{\mu\nu}, \tag{5·4}$$

where the effective source $\tilde{\tau}^{\mu\nu}$ is defined as

$$\tilde{\tau}^{\mu\nu}=\mathcal{S}^{\mu\nu}-\frac{1}{16\pi}\left(\bar{h}^{00}-\frac{4M}{r}\right)\bar{h}^{\mu\nu}{}_{,00}. \tag{5·5}$$

At the lowest order, we obtain ${}_{(4)}\tilde{\tau}_{\mu\nu} = {}_{(4)}\Theta_{\mu\nu}$.

The solution for (5·4) may be written down by using the retarded Green's function in the following form.

$$\bar{h}^{\mu\nu}(x) = \int dx' G_M^{(+)}(x,x')\tilde{\tau}^{\mu\nu}(x'). \tag{5·6}$$

The retarded Green's function is defined as satisfying the equation;

$$\Box_M G_M^{(+)}(x,y) = \delta^4(x-y), \tag{5·7}$$

with an appropriate boundary condition. We also defined the following symbol for the differential operator appearing in our basic equation.

$$\Box_M = \left[-\left(1+\frac{4M}{r}\right)\frac{\partial^2}{\partial t^2} + \Delta\right], \tag{5·8}$$

where Δ is the Laplacian in the flat space.

The Green's function G_0 for $M = 0$ i.e. in the Minkowski spacetime is well known. We shall present the detailed derivation of the Green function in appendix B, and present here the result

$$\begin{aligned} G_M^{(+)}(x,x') = \sum_{lm} e^{i\sigma_l} \int d\omega sgn(\omega)\Big(&\Psi^{+\,\omega\,l\,m}(x)\Psi^{S\,\omega\,l\,m\,*}(x')\theta(r-r') \\ &+ \Psi^{S\,\omega\,l\,m}(x)\Psi^{+\,\omega\,l\,m\,*}(x')\theta(r'-r)\Big), \end{aligned} \tag{5·9}$$

where $*$ denotes the complex conjugate, $sgn(\omega)$ denotes a sign of ω and we defined $\Psi^{+\,\omega\,l\,m}(x)$ and $\Psi^{S\,\omega\,l\,m}(x)$ as

$$\begin{aligned} \Psi^{+\omega\,l\,m}(x) &= \sqrt{\frac{|\omega|}{2\pi}} e^{-i\omega t}\rho^{-1}u_l^{(+)}(\rho;\gamma)Y_{l\,m}, \\ \Psi^{S\,\omega\,l\,m}(x) &= \sqrt{\frac{|\omega|}{2\pi}} e^{-i\omega t}\rho^{-1}F_l(\rho;\gamma)Y_{l\,m}, \end{aligned} \tag{5·10}$$

where $\rho = \omega r$ and $u_l^{(+)}$ is a spherical Coulomb function satisfying the outgoing wave condition at null infinity, and F_ℓ is also a spherical Coulomb function which is regular at the origin.

Since we study an asymptotic form of the waves, we use Eqs.(B·7) and (B·8) in appendix B to obtain the asymptotic form of the Green function as

$$\begin{aligned} G_M^{(+)}(x,x') = \frac{1}{r}\sum_{lm}\frac{(-i)^l}{2\pi}\int d\omega c_l e^{i\sigma_l} e^{-i\omega(t-r-2M\ln 2M\omega - t')}(\omega r')^l Y_{lm}(\Omega)Y^*_{lm}(\Omega') \\ +O(r^{-2}) \quad \text{for} \quad r\to\infty. \end{aligned} \tag{5·11}$$

In addition, for slow motion sources, we evaluate the asymptotic form of the Green function up to $O(M\omega)$ as

$$G_M^{(+)}(x,x') = \frac{1}{r}\sum_{lm}\frac{(-i)^l}{2\pi(2l+1)!!}\int d\omega\left\{1+\pi M\omega + 2iM\omega\left(\ln 2M\omega - \sum_{s=1}^{l}\frac{1}{s} + C\right)\right.$$

$$+O(M^2\omega^2)\Big\}e^{-i\omega(t-r-t')}(\omega r')^l Y_{lm}(\Omega)Y^*_{lm}(\Omega')+O(r^{-2})$$
$$=\frac{1}{r}\sum_{lm}\frac{(-i)^l}{2\pi(2l+1)!!}\int d\omega\Big[1+2M\omega\Big\{i\Big(-\sum_{s=1}^{l}\frac{1}{s}+\ln 2M\Big)+\frac{\pi}{2}sgn(\omega)$$
$$+i(\ln|\omega|+C)\Big\}+O(M^2\omega^2)\Big]e^{-i\omega(t-r-t')}(\omega r')^l Y_{lm}(\Omega)Y^*_{lm}(\Omega')$$
$$+O(r^{-2}), \tag{5·12}$$

where C is Euler's number. In deriving the above expression we have used the following expansion for c_l and σ_l in $M\omega$.

$$c_l=\frac{1+\pi M\omega+O(M^2\omega^2)}{(2l+1)!!},$$
$$\sigma_l=2M\omega\Big(C-\sum_{s=1}^{l}\frac{1}{s}\Big)+O(M^2\omega^2). \tag{5·13}$$

Now we apply the formula[49), 16)]

$$\omega\int_0^1 dv e^{i\omega v}\ln v+i\int_1^\infty\frac{dv}{v}e^{i\omega v}=-\frac{\pi}{2}sgn(\omega)-i(\ln|\omega|+C), \tag{5·14}$$

to Eq.(5·12), then we obtain

$$G_M^{(+)}(x,x')=\frac{1}{r}\sum_{lm}\frac{(-i)^l}{2\pi(2l+1)!!}$$
$$\times\int d\omega\Big\{1-2M\Big(-\sum_{s=1}^{l}\frac{1}{s}+\ln 2M\Big)\frac{d}{dt}+2M\Big(\int_0^1 dve^{i\omega v}\ln v\Big)\frac{d^2}{dt^2}$$
$$+2M\Big(\int_1^\infty\frac{dv}{v}e^{i\omega v}\Big)\frac{d}{dt}+O(M^2\omega^2)\Big\}$$
$$\times e^{-i\omega(t-r-t')}(\omega r')^l Y_{lm}(\Omega)Y^*_{lm}(\Omega')+O(r^{-2}). \tag{5·15}$$

It is worthwhile to point out that $\ln 2M$ in Eq.(5·15) can be removed by using the freedom to time translation.

Here, we assume the no incoming radiation condition on the initial hypersurface so that we may take

$$\lim_{v\to\infty}e^{-i\omega(t-r-v-t')}\ln v\to 0. \tag{5·16}$$

Thus we can make the following replacement

$$\int_1^\infty\frac{dv}{v}e^{i\omega v}\to -i\omega\int_1^\infty dve^{i\omega v}\ln v=\Big(\int_1^\infty dve^{i\omega v}\ln v\Big)\frac{d}{dt}. \tag{5·17}$$

Inserting Eq.(5·17) into Eq.(5·15), we finally obtain the desired expression for the retarded Green's function

$$G_M^{(+)}(x,x')=\frac{1}{r}\sum_{lm}\frac{(-i)^l}{2\pi(2l+1)!!}\int d\omega\Big[1+2M\Big(\sum_{s=1}^{l}\frac{1}{s}-\ln 2M\Big)\frac{d}{dt}$$

$$
\begin{aligned}
&\qquad +2M\left(\int_0^\infty dv e^{i\omega v}\ln v\right)\frac{d^2}{dt^2}+O(M^2\omega^2)\Big]\\
&\times e^{-i\omega(t-r-t')}(\omega r')^l Y_{lm}(\Omega)Y^*_{lm}(\Omega')+O(r^{-2})\\
=&\ \frac{1}{r}\text{ part of}\\
&\left[G_0(x,x')+2M\frac{d^2}{dt^2}\sum_{lm}\int dv\left\{\ln\left(\frac{v}{2M}\right)+\sum_{s=1}^{l}\frac{1}{s}\right\}G_0^{lm}(t-v,\mathbf{x},x')\right.\\
&\left.+O(M^2)\right]+O(r^{-2}),
\end{aligned}\tag{5·18}
$$

where we defined the spherical harmonic expansion coefficient of the flat Green's function as follows.

$$
G_0^{lm}(x,x')=Y_{lm}(\Omega)\int d\Omega' G_0(x,x')Y_{lm}(\Omega'). \tag{5·19}
$$

As a result, we obtain the waveform generated by the linear part of the effective source which corresponds to (C1) by Blanchet [7].

$$
\begin{aligned}
h_{ij}^{TT}=\frac{4}{r}P_{ijpq}\sum_{l=2}^{\infty}\frac{1}{l!}\Big[&n_{L-2}\left\{\tilde{M}_{pqL-2}(t-r)+2M\frac{d^2}{dt^2}\int dv\left\{\ln\left(\frac{v}{2M}\right)+\sum_{s=1}^{l-2}\frac{1}{s}\right\}\right.\\
&\left.\times\tilde{M}_{pqL-2}(t-r-v)\right\}-\frac{2l}{l+1}n_{aL-2}\left\{\epsilon_{ab(p}\tilde{S}_{q)bL-2}(t-r)\right.\\
&\left.+2M\frac{d^2}{dt^2}\int dv\left\{\ln\left(\frac{v}{2M}\right)+\sum_{s=1}^{l-1}\frac{1}{s}\right\}\epsilon_{ab(p}\tilde{S}_{q)bL-2}(t-r-v)\right\}\\
&+O(M^2)\Big]+O(r^{-2}),
\end{aligned}\tag{5·20}
$$

where P_{ijpq} is the transverse and traceless projection tensor and parentheses denote symmetrization. $\tilde{M}_{pqL-2}$ and $\tilde{S}_{pqL-2}$ are the mass and current multipole moments generated by the full nonlinear effective source $\tilde{\tau}_{\mu\nu}$. They take the same forms as in Thorne (1980).

5.2. *Comparison with the previous work*

By using the post-Minkowskian approximation, Blanchet obtained the radiative mass moment and radiative current moment as [7]

$$
U_L^{(l)}(u)=M_L^{(l)}(u)+2GM\int_0^\infty dv M_L^{(l+2)}(u-v)\left\{\ln\left(\frac{v}{P}\right)+\kappa_l\right\}+O(G^2M^2), \tag{5·21}
$$

and

$$
V_L^{(l)}(u)=S_L^{(l)}(u)+2GM\int_0^\infty dv S_L^{(l+2)}(u-v)\left\{\ln\left(\frac{v}{P}\right)+\kappa'_l\right\}+O(G^2M^2), \tag{5·22}
$$

where their moments M_L and S_L do not contain the nonlinear contribution outside the matter, P is a constant with temporal dimension, and κ_l and κ_l' are defined as

$$\kappa_l = \sum_{s=1}^{l-2} \frac{1}{s} + \frac{2l^2+5l+4}{l(l+1)(l+2)}, \tag{5·23}$$

and

$$\kappa_l' = \sum_{s=1}^{l-1} \frac{1}{s} + \frac{l-1}{l(l+1)}. \tag{5·24}$$

In black hole perturbation, the same tail corrections with multipole moments induced by linear perturbations have been obtained [77]. Compared with Eqs.(5·21)-(5·24), Eq.(5·20) shows that *log* term and $\sum 1/s$ in the radiative moments (5·21) and (5·22) originate from propagation on the slightly curved light cone determined by Eq.(5·8). That is to say, *log* term is produced by propagation of gravitational waves in the Coulomb type potential in the external region. It is worthwhile to point out the following fact: Only the *log* term has a hereditary property expressed as the integral over the past history of the source, since the constants κ_l and κ_l' represent merely instantaneous parts after performing the integral under the assumption that the source approaches static as the past infinity.

However, Eq.(5·20) does not apparently agree with Eqs.(5·21) and (5·22), because the definition of the moments $\{\tilde{M}_L, \tilde{S}_L\}$ and $\{M_L, S_L\}$ are different. We shall show the equivalence between our expression and that of Blanchet (1995), by calculating the contribution from nonlinear terms like $M \times M_L$ or $M \times S_L$. It is noteworthy that the luminosity of gravitational waves obtained by the present approach agrees at the tail term i.e. $O(c^{-3})$ with that by the post-Minkowskian approximation.

$$\mathcal{L} = \sum_{l=2}^{\infty} \frac{(l+1)(l+2)}{(l-1)l} \frac{1}{l!(2l+1)!!} < \overset{(l+1)}{U_L} \overset{(l+1)}{U_L} > + \sum_{l=2}^{\infty} \frac{4l(l+2)}{(l-1)} \frac{1}{(l+1)!(2l+1)!!} < \overset{(l+1)}{V_L} \overset{(l+1)}{V_L} > . \tag{5·25}$$

5.3. *Contributions from the nonlinear sources*

In order to evaluate the waveform produced by the nonlinear sources in $\tilde{\tau}_{\mu\nu}$, it is enough to use the flat Green's function

$$G_0(x,x') = -i \sum_{lm} \int d\omega \frac{\omega}{\pi} \Big(e^{-i\omega t} h_l(\omega r) Y_{lm}(\Omega) e^{i\omega t'} j_l(\omega r') Y^*_{lm}(\Omega') \theta(r-r') + e^{-i\omega t} j_l(\omega r) Y_{lm}(\Omega) e^{i\omega t'} h_l(\omega r') Y^*_{lm}(\Omega') \theta(r'-r) \Big). \tag{5·26}$$

Since the nonlinear sources have the form of either $M \times M_L$ or $M \times S_L$, we have to treat the following type of retarded integral

$$\Box^{-1} \Big[\partial_i (\frac{1}{r}) \hat{\partial}_Q \Big(\frac{F(t-r)}{r} \Big) \Big] = (-)^{q+1} \sum_{j=0}^{q} \frac{(q+j)!}{2^j j!(q-j)!}$$

$$\times\Box^{-1}\left[\left(\hat{n}_{iQ}+\frac{q}{2q+1}\delta_{i<a_q}\hat{n}_{Q-1>}\right)\frac{1}{r^{j+3}}\overset{(q-j)}{F}(t-r)\right]. \tag{5·27}$$

The evaluation can be made by using the formula which will be proved in appendix C.

$$\Box^{-1}\left[\frac{\hat{n}_L}{r^k}F(t-r)\right] = -2^{l+1}\lim_{\lambda\to 0}\left(\sum_{n=0}^{\infty}\frac{(l+n+1)!\Gamma(-k+l+3+2n-\lambda)}{n!(2l+2n+2)!}\right)$$
$$\times\frac{\hat{n}_L}{r}\overset{(k-3)}{F}(t-r)+O(r^{-2}). \tag{5·28}$$

Although some terms of nonlinear sources may produce disastrous divergence in Eq.(5·28), we expect these divergent parts cancel out in total, which is explicitly demonstrated in the appendix.

Using the above formula we find

$$\Box^{-1}\left[\partial_i(\frac{1}{r})\hat{\partial}_Q\left(\frac{F(t-r)}{r}\right)\right]$$
$$=\frac{(-)^q}{2(q+1)}\left[\hat{n}_{iQ}-\frac{q+1}{2q+1}\delta_{i<a_1}\hat{n}_{Q-1>}\right]\frac{1}{r}\overset{(q)}{F}(t-r)+O(r^{-2}), \tag{5·29}$$

which is same with the expression (C2) obtained by Blanchet [7].

Applying Eqs.(5·29) to the nonlinear source $\tilde{\tau}_{\mu\nu}$, we can evaluate its contribution to the waveform. Together with Eq.(5·20), we obtain the total waveform as

$$h_{ij}^{TT}=\frac{4}{r}P_{ijpq}\sum_{l=2}^{\infty}\frac{1}{l!}\left[n_{L-2}\left\{M_{pqL-2}(t-r)+2M\frac{d^2}{dt^2}\int dv\left\{\ln\left(\frac{v}{2M}\right)+\kappa_l\right\}\right.\right.$$
$$\left.\times M_{pqL-2}(t-r-v)\right\}-\frac{2l}{l+1}n_{aL-2}\left\{\epsilon_{ab(p}S_{q)bL-2}(t-r)\right.$$
$$\left.+2M\frac{d^2}{dt^2}\int dv\left\{\ln\left(\frac{v}{2M}\right)+\kappa_l'\right\}\epsilon_{ab(p}S_{q)bL-2}(t-r-v)\right\}$$
$$\left.+O(M^2)\right]+O(r^{-2}). \tag{5·30}$$

In this section, we have derived the formula (5·30) for gravitational waves including tail by using the formula (5·20) and (5·29). We would like to emphasize two points on the derivation: First, in deriving Eq.(5·20), spherical Coulomb functions are used, since we use the wave operator (5·8) which take account of the Coulomb-type potential M/r. As a consequence, $\ln(v/2M)$ appears naturally in Eq.(5·30). This is in contrast with Blanchet and Damour's method, where an arbitrary constant with temporal dimension P appears in the form of $\ln(v/P)$. Our derivation shows that the main part of the tail, which needs the past history of the source only through $\ln v$, is produced by propagation in the Coulomb-type potential. Our method might make it easy to clarify conditions for wave formula including tail. This remains as a future work.

The second point relates with physical application: At the starting point of our derivation, the Fourier representation in frequency space has been used. Such a representation seems to simplify the calculation of gravitational waveforms from compact binaries in the quasi-circular orbit, since such a system can be described by a characteristic frequency. Applications to physical systems will be also done in the future.

§6. Conclusion

We have discussed various aspects on the post-Newtonian approximation mainly based on our own work. After presenting the basic structure of the PN approximation in the framework of Newtonian limit along a regular asymptotic Newtonian sequence, we reformulated it in the appropriate form for numerical approach. For this purpose we have adopted the (3+1) formalism in general relativity. Although we restricted ourselves within the transverse gauge in this paper, we can use any gauge condition and investigate its property relatively easily in the (3+1) formalism, compared with in the standard PN approximation performed so far [21)-25)]. Using the developed formalism, we have written down the hydrodynamic equation up to 2.5PN order. For the sake of an actual numerical simulation, we consider carefully methods to solve the various metric quantities, especially, the 2PN tensor potential ${}_{(4)}h_{ij}$. We found it possible to solve them by using the numerical methods familiar in Newtonian gravity. Thus, the formalism discussed in this paper will be useful in numerical applications. As an example of the application, we have presented the formalism for constructing the equilibrium configuration of nonaxisymmetric uniformly rotating fluid. We have also discussed the propagation of gravitational waves explicitly taking into account of the deviation of the light cone from that of the flat spacetime and obtained essentially the same result with that of Blanchet and Damour for the waveform from post-Newtonian systems.

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Appendix A

—— Conserved quantities for 2PN approximation of uniformly rotating fluid ——

The conserved quantities in the 2PN approximation will be useful to investigate the stability property of equilibrium solutions obtained in numerical calculations. Since the definitions of the conserved quantities are given in section 3, we shall present only the results in this appendix.

(1)Conserved mass [4];

$$M_* \equiv \int \rho_* d^3x, \tag{A·1}$$

where

$$\rho_* = \rho\left[1+\frac{1}{c^2}\left(\frac{1}{2}v^2+3U\right)+\frac{1}{c^4}\left(\frac{3}{8}v^4+\frac{13}{2}v^2U+\frac{45}{4}U^2+3U\Pi+{}_{(3)}\beta^i v^i\right)+O(c^{-6})\right]. \tag{A·2}$$

(2)ADM mass [3), 92), 4];

$$M_{ADM} = -\frac{1}{2\pi}\int \Delta\psi d^3x \equiv \int \rho_{ADM} d^3x, \tag{A·3}$$

where

$$\rho_{ADM} = \rho\left[1+\frac{1}{c^2}\left(v^2+\Pi+\frac{5}{2}U\right)\right.$$
$$\left.+\frac{1}{c^4}\left(v^4+9v^2U+\Gamma\Pi v^2+5U\Pi+\frac{35}{4}U^2+\frac{3}{2}{}_{(3)}\beta^i v^i\right)+O(c^{-6})\right]. \tag{A·4}$$

(3)Total energy, which is calculated from $M_{ADM}-M_*$ in the third PN order [4];

$$E \equiv \int \rho_E d^3x, \tag{A·5}$$

where

$$\rho_E = \rho\left[\left(\frac{1}{2}v^2+\Pi-\frac{1}{2}U\right)+\frac{1}{c^2}\left(\frac{5}{8}v^4+\frac{5}{2}v^2U+\Gamma v^2\Pi+2U\Pi-\frac{5}{2}U^2+\frac{1}{2}{}_{(3)}\beta^i v^i\right)\right.$$
$$+\frac{1}{c^4}\left\{\frac{11}{16}v^6+v^4\left(\Gamma\Pi+\frac{47}{8}U\right)+v^2\left(4{}_{(4)}\psi+6\Gamma\Pi U+\frac{41}{8}U^2+\frac{5}{2}{}_{(3)}\beta^i v^i-X\right)\right.$$
$$-\frac{5}{2}U^3+2\Gamma{}_{(3)}\beta^i v^i\Pi+5\Pi{}_{(4)}\psi+5U{}_{(3)}\beta^i v^i-\frac{15}{2}U{}_{(4)}\psi+\frac{5}{4}U^2\Pi$$
$$+\frac{1}{2}h_{ij}v^iv^j+\frac{1}{2}{}_{(3)}\beta^i{}_{(3)}\beta^i$$
$$\left.\left.+\frac{U}{16\pi\rho}\left(2h_{ij}U_{,ij}+{}_{(3)}\beta^i_{,j}\left({}_{(3)}\beta^i_{,j}+{}_{(3)}\beta^j_{,i}-\frac{2}{3}\delta_{ij}{}_{(3)}\beta^k_{,k}\right)\right)\right\}+O(c^{-6})\right]. \tag{A·6}$$

It is noteworthy that terms including ${}_{(5)}\beta^i$ cancel out in total.

(4)Total linear and angular momenta: In the case $K_i{}^i = 0$, these are calculated from (Wald 1984)

$$\begin{aligned}
P_i &= \frac{1}{8\pi} \lim_{r\to\infty} \oint K_{ij} n^j dS \\
&= \frac{1}{8\pi} \int (\psi^6 K_i{}^j)_{,j} d^3x \\
&= \int \left(J_i + \frac{1}{16\pi} \psi^4 \tilde{\gamma}_{jk,i} K^{jk} \right) \psi^6 d^3x,
\end{aligned} \tag{A·7}$$

where $J_i = (\rho c^2 + \rho \Pi + P)\alpha u^0 u_i$. Up to the 2PN order, the second term in the last line of Eq.(A·7) becomes

$$\begin{aligned}
&\frac{1}{16\pi} \int h_{jk,i(3)} \beta^j_{,k} d^3x \\
&= \frac{1}{16\pi} \int \left[\left(h_{jk,i(3)} \beta^j \right)_{,k} - h_{jk,ik(3)} \beta^j \right] d^3x \\
&= \frac{1}{16\pi} \lim_{r\to\infty} \oint h_{jk,i(3)} \beta^j n^k dS = 0,
\end{aligned} \tag{A·8}$$

where we use $h_{jk} \to O(r^{-1})$ and ${}_{(3)}\beta^j \to O(r^{-2})$ at $r \to \infty$, and the gauge condition $h_{jk,k} = 0$. Thus, in the 2PN approximation, P_i becomes

$$P_i \equiv \int p_i d^3x, \tag{A·9}$$

where

$$\begin{aligned}
p_i = \rho \biggl[v^i &+ \frac{1}{c^2} \left\{ v^i \left(v^2 + \Gamma \Pi + 6U \right) + {}_{(3)}\beta^i \right\} + \frac{1}{c^4} \biggl\{ h_{ij} v^j + {}_{(5)}\beta^i \\
&+ {}_{(3)}\beta^i \left(v^2 + 6U + \Gamma \Pi \right) + v^i \biggl(2{}_{(3)}\beta^i v^i + 10{}_{(4)}\psi + 6\Gamma \Pi U \\
&+ \frac{67}{4} U^2 + \Gamma \Pi v^2 + 10 U v^2 + v^4 - X \biggr) \biggr\} + O(c^{-5}) \biggr].
\end{aligned} \tag{A·10}$$

The total angular momentum J becomes

$$J = \int p_\varphi d^3x, \tag{A·11}$$

where $p_\varphi = -y p_x + x p_y$.

Appendix B
—— *Construction of Green's function* ——

In this appendix we shall present a detailed derivation of Green's function for $\Box_M$. The following procedure is similar to that by Thorne (1980) for the Minkowski background spacetime[89].

The Green's function satisfying Eq.(5·7) can be constructed by using the homogeneous solutions for the equation

$$\Box_M \Psi = 0. \qquad \text{(B·1)}$$

The homogeneous solution for Eq.(B·1) takes a form of

$$e^{-i\omega t} f_l(\rho) Y_{lm}(\theta, \phi), \qquad \text{(B·2)}$$

where we defined

$$\rho = \omega r. \qquad \text{(B·3)}$$

Then the radial function $\tilde{f}_l(\rho) \equiv \rho f_l(\rho)$ satisfies

$$\left(\frac{d^2}{d\rho^2} + 1 + \frac{4M\omega}{\rho} - \frac{l(l+1)}{\rho^2} \right) \tilde{f}_l(\rho) = 0, \qquad \text{(B·4)}$$

so that Eq.(B·2) is a solution for Eq.(B·1). Thus we can obtain homogeneous solutions for Eq.(B·1) by choosing $\tilde{f}_l(\rho)$ as one of spherical Coulomb functions; $u_l^{(\pm)}(\rho;\gamma)$, $F_l(\rho;\gamma)$ and $G_l(\rho;\gamma)$ with $\gamma = -2M\omega$. Here, we adopted the following definition of the spherical Coulomb function [67] as

$$\begin{aligned} F_l(\rho;\gamma) &= c_l e^{i\rho} \rho^{l+1} F(l+1+i\gamma|2l+2|-2i\rho), \\ u_l^{(\pm)}(\rho;\gamma) &= \pm 2ie^{\mp i\sigma_l} c_l e^{\pm i\rho} \rho^{l+1} W_1(l+1 \pm i\gamma|2l+2| \mp 2i\rho), \\ G_l(\rho;\gamma) &= \frac{1}{2}(u_l^{(+)} e^{i\sigma_l} + u_l^{(-)} e^{-i\sigma_l}), \end{aligned} \qquad \text{(B·5)}$$

where c_l and σ_l are defined as

$$\begin{aligned} c_l &= 2^l e^{-\pi\gamma/2} \frac{|\Gamma(l+1+i\gamma)|}{(2l+1)!}, \\ \sigma_l &= \arg\Gamma(l+1+i\gamma). \end{aligned} \qquad \text{(B·6)}$$

Here, F and W_1 are the confluent hypergeometric function and the Whittaker's function respectively. These spherical Coulomb functions have asymptotic behavior as

$$\begin{aligned} F_l &\sim \sin\left(\rho - \gamma \ln 2\rho - \frac{1}{2} l\pi + \sigma_l \right), \\ G_l &\sim \cos\left(\rho - \gamma \ln 2\rho - \frac{1}{2} l\pi + \sigma_l \right), \\ u^{(\pm)} &\sim \exp\left[\pm i \left(\rho - \gamma \ln 2\rho - \frac{1}{2} l\pi \right) \right] \quad \text{for} \quad r \to \infty, \end{aligned} \qquad \text{(B·7)}$$

and

$$\begin{aligned} F_l &\sim c_l \rho^{l+1}, \\ G_l &\sim \frac{1}{(2l+1)c_l} \rho^{-l} \quad \text{for} \quad r \to 0. \end{aligned} \qquad \text{(B·8)}$$

In terms of spherical Coulomb functions introduced above, we can construct the Green's function in the standard way.

$$G_M^{(\epsilon)}(x,x') = \sum_{lm} e^{i\sigma_l} \int d\omega sgn(\omega) \Big(\Psi^{\epsilon\,\omega\,l\,m}(x) \Psi^{S\,\omega\,l\,m\,*}(x') \theta(r-r') + \Psi^{S\,\omega\,l\,m}(x) \Psi^{\epsilon\,\omega\,l\,m\,*}(x') \theta(r'-r) \Big), \tag{B·9}$$

where $*$ denotes the complex conjugate, $sgn(\omega)$ denotes a sign of ω and we defined $\Psi^{\epsilon\,\omega\,l\,m}(x)$ and $\Psi^{S\,\omega\,l\,m}(x)$ as

$$\begin{aligned} \Psi^{\epsilon\omega\,l\,m}(x) &= \sqrt{\frac{|\omega|}{2\pi}} e^{-i\omega t} \rho^{-1} u_l^{(+)}(\rho;\gamma) Y_{l\,m}, \\ \Psi^{S\,\omega\,l\,m}(x) &= \sqrt{\frac{|\omega|}{2\pi}} e^{-i\omega t} \rho^{-1} F_l(\rho;\gamma) Y_{l\,m}. \end{aligned} \tag{B·10}$$

The retarded Green's function is obtained for $\epsilon = +$.

Appendix C

—— *Derivation of the formula* (5.28) ——

We evaluate the asymptotic form of the retarded integral as

$$\begin{aligned} \Box^{-1}\left[\frac{\hat{n}_L}{r^k} F(t-r)\right] &= \int d^4x' G_0(x,x') \left[\frac{\hat{n}'_L}{r'^k} F(t'-r')\right] \\ &\to -i \int d\omega \omega e^{-i\omega t} h_l(\omega r) F_\omega \hat{n}_L \int_0^r r'^2 dr' j_l(\omega r') \frac{e^{i\omega r'}}{r'^k} \\ &\qquad \text{for large r,} \end{aligned} \tag{C·1}$$

where we defined F_ω as

$$F_\omega = \frac{1}{2\pi} \int dt\, e^{i\omega t} F(t). \tag{C·2}$$

Since the Bessel function is defined as

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-)^n (z/2)^{\nu+2n}}{n! \Gamma(\nu+n+1)}, \tag{C·3}$$

we obtain formally

$$\begin{aligned} \int_0^\infty dy J_\nu(by) e^{-ay} y^{\mu-1} &= \sum_{n=0}^{\infty} \frac{(-)^n (b/2)^{\nu+2n}}{n! \Gamma(\nu+n+1)} \int_0^\infty dy e^{-ay} y^{\mu+\nu+2n-1} \\ &= \sum_{n=0}^{\infty} \frac{(-)^n (b/2)^{\nu+2n} \Gamma(\mu+\nu+2n)}{n! a^{\mu+\nu+2n} \Gamma(\nu+n+1)}. \end{aligned} \tag{C·4}$$

Putting $a = -i$, $b = 1$, $\mu = -k + 5/2 - \lambda$ and $\nu = l + 1/2$ in Eq.(C·4), we obtain where we introduced small $\lambda \in C$ in order to avoid the pole of $\Gamma(-k + l + 3 + 2n)$

for $k \geq l+3$. Thus we obtain

$$\int_0^r r'^2 dr' j_l(\omega r') \frac{e^{i\omega r'}}{r'^{k+\lambda}} = \omega^{k-3}\left(\sqrt{\frac{\pi}{2}} \frac{i^{-k+l+3}}{2^{l+\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{\Gamma(-k+l+3+2n-\lambda)}{2^{2n} n! \Gamma(l+n+\frac{3}{2})} + O(r^{-1})\right). \tag{C·5}$$

From Eqs.$(C{\cdot}1)$, $(C{\cdot}2)$ and $(C{\cdot}5)$, we obtain

$$\Box^{-1}\left[\frac{\hat{n}_L}{r^k} F(t-r)\right] = -\lim_{\lambda\to 0} \frac{\sqrt{\pi}}{2^{l+1}} \left(\sum_{n=0}^{\infty} \frac{\Gamma(-k+l+3+2n-\lambda)}{2^{2n} n! \Gamma(l+n+\frac{3}{2})}\right) \frac{\hat{n}_L}{r^k} \overset{(k-3)}{F}(t-r) + O(r^{-2}). \tag{C·6}$$

This equation is valid for $k \geq 3$. For $3 \leq k \leq l+2$, we obtain

$$\Box^{-1}\left[\frac{\hat{n}_L}{r^k} F(t-r)\right] = -2^{k-3} \frac{(k-3)!(l-k+2)!}{(l+k-2)!} \frac{\hat{n}_L}{r} \overset{(k-3)}{F}(t-r) + O(r^{-2}) \quad \text{for} \quad 3 \leq k \leq l+2, \tag{C·7}$$

since

$$\frac{\sqrt{\pi}}{2^{l+1}} \left(\sum_{n=0}^{\infty} \frac{\Gamma(-k+l+3+2n)}{2^{2n} n! \Gamma(l+n+\frac{3}{2})}\right) = 2^{k-3} \frac{(k-3)!(l-k+2)!}{(l+k-2)!} \quad \text{for} \quad 3 \leq k \leq l+2. \tag{C·8}$$

This formula for $3 \leq k \leq l+2$ has been derived by Blanchet and Damour (1988).

Appendix D

—— *Derivation of* (5.29) ——

From Eq.(5·27), we shall derive Eq.(5·29) in this appendix. First, for the first term in the right hand side of Eq.(5·27), we use Eq.(C·7) to obtain

$$\begin{aligned} &(-)^{q+1} \sum_{j=0}^{q} \frac{(q+j)!}{2^j j!(q-j)!} \Box^{-1}\left[\hat{n}_{iQ} \frac{1}{r^{j+3}} \overset{(q-j)}{F}(t-r)\right] \\ &= \frac{(-)^q}{2(q+1)} \frac{\hat{n}_{iQ}}{r} \overset{(q)}{F}(t-r) + O(r^{-2}). \end{aligned} \tag{D·1}$$

Next, the second term in the right hand side of Eq.(5·27) is calculated as

$$\begin{aligned} &(-)^{q+1} \frac{q}{2q+1} \sum_{j=0}^{q-2} \frac{(q+j)!}{2^j j!(q-j)!} \Box^{-1}\left[\delta_{i<a_q} \hat{n}_{Q-1>} \frac{1}{r^{j+3}} \overset{(q-j)}{F}(t-r)\right] \\ &= (-)^q \frac{q-1}{2q+1} \frac{\delta_{i<a_q} \hat{n}_{Q-1>}}{r} \overset{(q)}{F}(t-r) + O(r^{-2}), \end{aligned} \tag{D·2}$$

and

$$(-)^{q+1} \frac{q}{2q+1} \sum_{j=q-1}^{q} \frac{(q+j)!}{2^j j!(q-j)!} \Box^{-1}\left[\delta_{i<a_q} \hat{n}_{Q-1>} \frac{1}{r^{j+3}} \overset{(q-j)}{F}(t-r)\right]$$

$$
\begin{aligned}
&= (-)^q \frac{q(2q)!}{(2q+1)q!} \frac{\delta_{i<a_q} \hat{n}_{Q-1>}}{r} \overset{(q)}{F}(t-r) \\
&\quad \times \lim_{\lambda \to 0} \left[\sum_{n=1}^{\infty} \left\{ \frac{(q+n)!}{n!(2q+2n)!} \Big(\Gamma(2n-\lambda) + \Gamma(2n-1-\lambda) \Big) \right. \right. \\
&\qquad \left. \left. + \frac{q!}{(2q)!} \Big(\Gamma(-\lambda) + \Gamma(-1-\lambda) \Big) \right\} \right] + O(r^{-2}) \\
&= (-)^{q+1} \frac{2q-1}{2(2q+1)} \frac{\delta_{i<a_q} \hat{n}_{Q-1>}}{r} \overset{(q)}{F}(t-r) + O(r^{-2}). \qquad \text{(D·3)}
\end{aligned}
$$

Here we used Eq.(5·28),

$$
\sum_{n=1}^{\infty} \frac{(q+n)!}{n!(2q+2n)!} \Big(\Gamma(2n) + \Gamma(2n-1) \Big) = \frac{q!}{2q(2q)!}, \qquad \text{(D·4)}
$$

and

$$
\lim_{\lambda \to 0} \Big(\Gamma(-\lambda) + \Gamma(-1-\lambda) \Big) = \lim_{\lambda \to 0} (-\lambda) \Gamma(-1-\lambda) = -1, \qquad \text{(D·5)}
$$

which, in fact, means *the cancellation of the poles.*

From Eqs.(5·27) and $(D{\cdot}1)$-$(D{\cdot}3)$, we obtain

$$
\begin{aligned}
&\Box^{-1} \left[\partial_i \left(\frac{1}{r}\right) \hat{\partial}_Q \left(\frac{F(t-r)}{r} \right) \right] \\
&= \frac{(-)^q}{2(q+1)} \left[\hat{n}_{iQ} - \frac{q+1}{2q+1} \delta_{i<a_1} \hat{n}_{Q-1>} \right] \frac{1}{r} \overset{(q)}{F}(t-r) + O(r^{-2}), \qquad \text{(D·6)}
\end{aligned}
$$

which equals to (C2) obtained by Blanchet [7]. As for Eq.$(D{\cdot}6)$, we used unphysical but small parameter λ in order to avoid the pole of the integrals. However, in the limit $\lambda \to 0$, we showed explicitly the cancellation of poles and obtained the finite values in total.

Table 1

List of potentials to be solved (column 1), Poisson equations for them (column 2), and other potential variables which appear in the source term of the Poisson equation (column 3). Note that i and j run x, y, z. Also, note that we do not have to solve η_{0z}, ${}_{(5)}P_{0z}$, q_{yy}, q_{zz} and $h_{zz}^{(U)}$.

Pot.	Eq.	Needed pots.	Pot.	Eq.	Needed pots.
U	(2.11)	None	q_{ij}	(4.6)	None
q_i	(3.14)	None	$Q_{0i}^{(I)}$	(4.15)	U
q_2	(4.1)	None	η_{0i}	(4.16)	U
q_{2i}	(4.2)	None	${}_{(5)}P_{0i}$	(4.17)	$U,\ q_i$
q_4	(4.3)	None	${}_{(6)}\alpha_0$	(4.21)	$U,\ q_e,\ q_u,\ h_{ij}^{(U)},\ Q_{0i}^{(I)}$
q_u	(4.4)	U	${}_{(6)}\alpha_2$	(4.22)	$U,\ q_2,\ q_i,\ q_{2i},\ q_{ij}$
q_e	(4.5)	None	$h_{ij}^{(U)}$	(3.1)	U

Table 2

Variables to be solved in order to obtain the original metric variables.

Metric	Variables to be solved	see Eq.
U	U	(2.11)
${}_{(3)}\beta^i$	q_i, U	(3.17)
X	$q_2,\ q_u,\ q_e$	(4.7)
${}_{(4)}\psi$	$q_2,\ q_u,\ q_e$	(4.8)
${}_{(5)}\beta^{i\,(A)}$	${}_{(5)}P_{0i},\ \eta_{0i},\ q_u,\ q_e$	(4.18)
${}_{(5)}\beta^{i\,(B)}$	$q_{2i},\ q_2$	(4.18)
${}_{(6)}\alpha$	${}_{(6)}\alpha_0,\ {}_{(6)}\alpha_2,\ q_4$	(4.20)
$h_{ij}^{(U)}$	$h_{ij}^{(U)}$	(3.1)
$h_{ij}^{(S)}$	q_{ij}, q_2	(4.14)
$h_{ij}^{(A)}$	$Q_{0i}^{(I)},\ q_u,\ q_e$	(4.19)
$h_{ij}^{(B)}$	$q_{ij},\ q_2,\ q_{2i},\ q_i$	(4.19)