Show that a graph has a unique MST if all edges have distinct weights [duplicate]

This question already has an answer here:
Show that there's a minimum spanning tree if all edges have different costs *2 answers*

> Prove that if all edge-costs are different, then there is only one cheapest tree (minimum spanning tree or MST). (Use contradiction and make sure to keep track of the costs of the different trees involved.)

Here is my attempt:

Proof by contradiction

Assume that there is more than one cheapest tree, A and B. Assume that tree A and B are different and have the same weight. Let says that the smallest difference in edge cost between A and B is 1. Now, change one of the random edge (not contain in B) in A by half. This does not affect the path or order. However, A is cheaper than B now. This is contradiction. Thus, the original statement still hold.

Is this the correct logic? How do I write it out formally?

(optimization) (proof-verification) (trees)

edited Aug 7 '15 at 13:34
nbro
2,041 11 34

asked Sep 8 '14 at 1:55
holidayeveryday
121 1 7

marked as duplicate by N. F. Taussig, TravisJ, Asaf Karagila, Daniel W. Farlow, wythagoras Aug 7 '15 at 16:59

This question has been asked before and already has an answer. If those answers do not fully address your question, please ask a new question.

What does "the smallest difference in edge cost between A and B is 1" mean? And, whatever it means, what if it isn't true? and how do you "change an edge by half"? and what do you mean by "This does not affect the path or order"? I don't find the argument compelling, or even comprehensible. – Gerry Myerson Sep 8 '14 at 7:20

2 Answers

Suppose there are two minimum trees, A and B. Let e be the edge in just one of A, B with the smallest cost. Suppose it is in A but not B. Suppose e is the edge PQ. Then B must contain a path from P to Q which is not simply the edge e. So if we add e to B, then we get a cycle. If all the other edges in the cycle were in A, then A would contain a cycle, which it cannot. So the cycle must contain an edge f not in A. Hence, by the definition of e (and the fact that all edge-costs are different) the cost of f must be greater than the cost of e. So if we replace f by e we get a spanning tree with smaller total cost. Contradiction.

answered Sep 8 '14 at 18:01

almagest

11.2k 10 25

The question is about showing that the minimal spanning tree is unique if all the edges have different weights. If one goes through any of the greedy algorithms (Prim, Kruskal..) for finding a minimal spanning tree one notices that the weights do not need to be added, just compared, that is, the weights should be elements of a totally ordered set. Moreover, the minimal spanning tree T obtained by the algorithm has the property that for any other spanning tree T', if we consider the weight in increasing order for T, respectively T'

$$(w_1, \ldots w_{n-1})$$

and

$$(w'_1, \ldots w'_{n-1})$$

we have

$$w_i \leq w_i'$$

for all i. This is due to the fact that we can get from T to T' by a sequence of intermediate trees so that at each step the next tree is obtained from the previous one by substituting an edge with another edge of $\geq$ weight.

If all the edges have distinct weights then T thus obtained is strictly better than any other tree since at each step we substitute an edge with another one of a strictly large weight (the weights cannot be the same). Hence the uniqueness.

edited Sep 8 '14 at 10:53

answered Sep 8 '14 at 7:59
orangeskid
17.1k 1 12 31

I might be wrong, but this only proves that the claim is true for Prim's and Kruskal's algorithm. Some other algorithm can find an MST of same weight, right? I mean this doesn't disprove that some algorithm can find an MST of same weight. – taninamdar Sep 8 '14 at 9:06

Since the weights for each edge are different one cannot find another tree with same weight vector. – orangeskid Sep 8 '14 at 9:17

You don't need to use a particular algorithm to show that, if a graph has unique weighted edges, then it has a unique MST. You can simply use the general algorithm of finding a MST, reasoning with cuts and paths... – nbro Aug 7 '15 at 13:30