# CS-230b

# Advanced Algorithms and Applications

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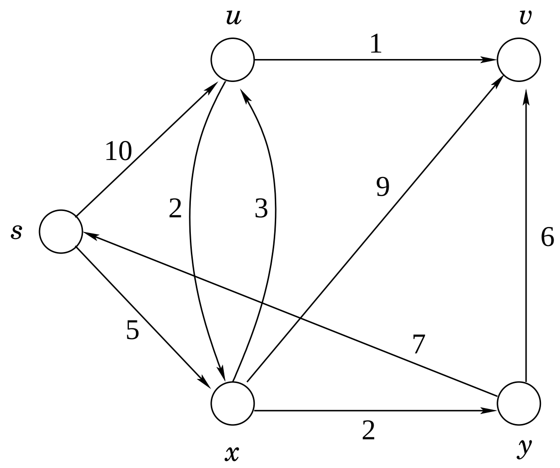
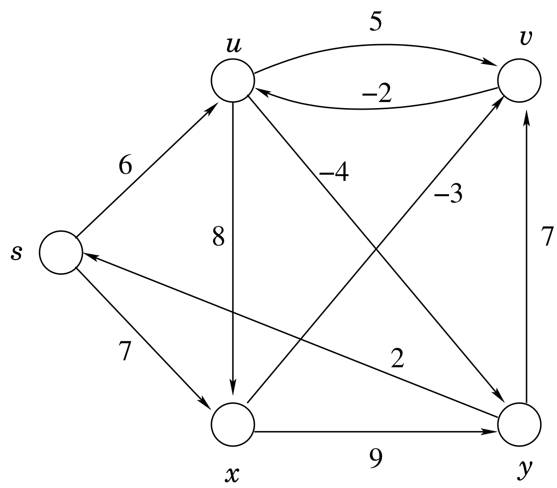
# Shortest Paths



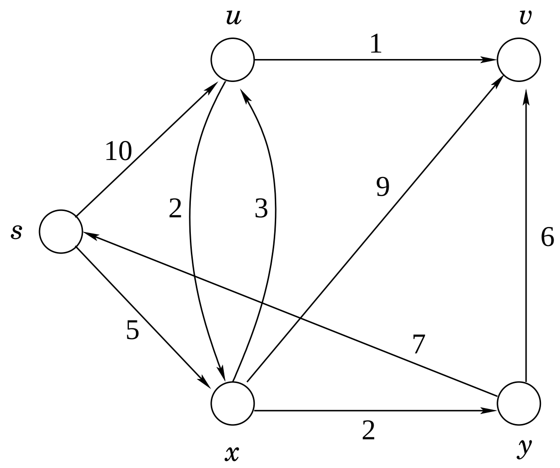

- **Find shortest length path from $s$ to $v$?**
- $s \rightarrow x \rightarrow u \rightarrow v$ **has length** $5 + 3 + 1 = 9$.
- **Many applications. Discussed later.**
- **A network** $G = (V, E)$.
- **Vertices (nodes)** $V = \{1, \ldots, n\}$.
- **Edges (links)** $E = \{e_1, e_2, \ldots, e_m\}$. **Edge** $e_{ij} = (i, j)$ **is directed from** $i$ **to** $j$.
- **Edge** $e_{ij}$ **has cost (weight)** $c_{ij}$. **The costs can be positive or negative!**

# Negative Cost Shortest Paths




- **What's the shortest path from $s$ to $y$?**
- $s \rightarrow x \rightarrow v \rightarrow u \rightarrow y$ **has length** $-2$**.**
- **How can costs be negative?**
- **Examples later. (Arbitrage trading, scientific simulations, matching algorithms, min cost network flows).**
- **More general the formulation, the better.**
- **Any simple way to eliminate negative edges? Adding a constant to all edges?**

# Getting Started




- **Source node $s = v_0$.**

- **Compute SP distances from $s$ to every node $v_j$.**

- **The paths themselves can be recovered from predecessors.**

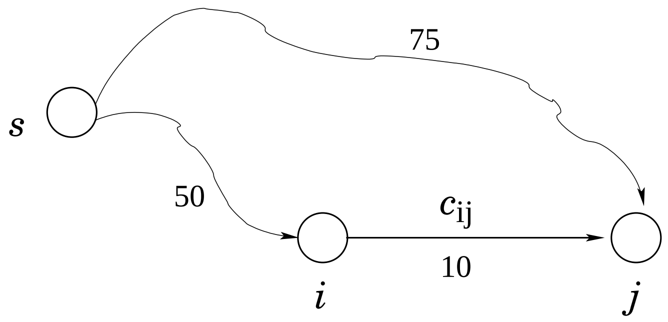
- **Distance labels $d(j)$: best path length to $j$ found so far.**

- **Initially, $d(0) = 0$, and $d(j) = \infty$ for others.**

- **The algorithm improves estimates for all $d(j)$ until SP distances become known.**

# Basic Idea

- **How to improve the distance estimate?**




- **Suppose there is an edge $(i, j)$ such that**

$$d(j) \; > \; d(i) + c_{ij}$$

**then we can improve the estimate of $d(j)$:**

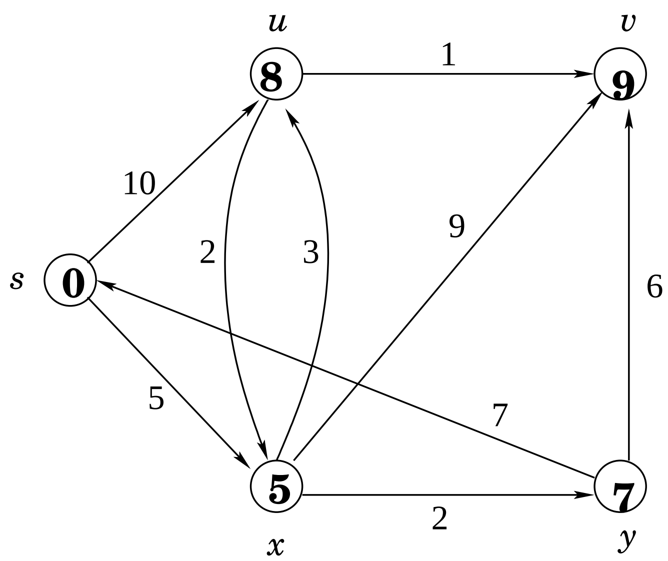
$$d(j) \; = \; d(i) + c_{ij}$$

- **Previously, $d(i) = 50$ and $d(j) = 75$. The relabeling step finds a better path to $j$ via $i$ of cost $60$.**

# Optimality Condition

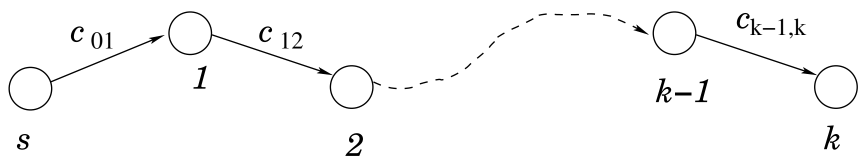
**Theorem: Suppose each $d(j)$ is the length of some feasible path from $s$ to $j$. Then, these $d()$ distances are shortest path distance** *if and only if*

$$d(j) \leq d(i) + c_{ij}, \qquad \textbf{for all}(i,j) \in E$$




*Necessity Proof:* **Suppose $\exists$ an edge $(i,j)$ violating the condition. Then, $d(j) > d(i) + c_{ij}$. But then we can reach $j$ via $i$ at cost $d(i) + c_{ij}$, which is smaller than $d(j)$, contradicting the $d(j)$ is shortest path distance.**

# Sufficiency




1. **Suppose** $d(j) \leq d(i) + c_{ij}$ **holds.**

2. **Let** $k$ **be a node with incorrect distance.**

3. $s \to 1 \to 2 \cdots k - 1 \to k$ **be actual SP.**

4. **So,** $c_{01} + c_{12} + \cdots + c_{k-1,k} < d(k)$. $(*)$

5. **By optimality condition we have**

$$\begin{aligned} d(k) &\leq d(k-1) + c_{k-1,k} \\ d(k-1) &\leq d(k-2) + c_{k-2,k-1} \\ &\vdots \\ d(1) &\leq d(0) + c_{01} \end{aligned}$$

6. **By summing,** $d(k) \leq c_{01} + c_{12} + \cdots + c_{k-1,k}$ **which contradicts** $(*)$**!!!.**

# Generic SP Algorithm

**algorithm Label-Correcting**

**1.** $d(s) = 0$**,** $\text{pred}(s) = 0$**;**

**2.** $d(j) = \infty$**, for** $j = 1, 2, \ldots, n$**.**

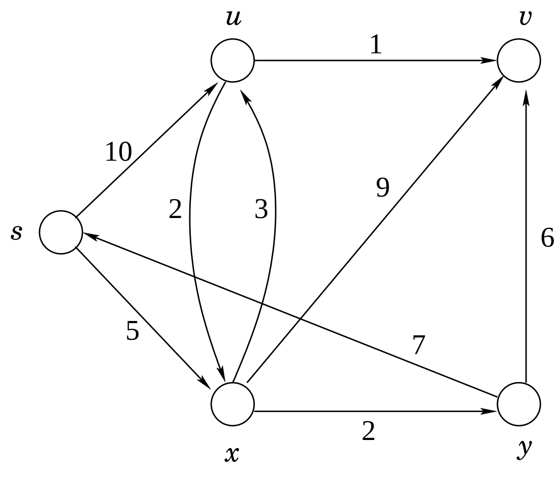
**3. while** $\exists(i, j)$ **with** $d(j) > d(i) + c_{ij}$ **do**

**4.** $\quad d(j) = d(i) + c_{ij}$**;**

**5.** $\quad \text{pred}(j) = i$**;**

**6. end;**

- **Run the algorithm on example.**



# Analysis

- **By optimality theorem, the output is correct IF the algorithm halts.**
- **If costs are integers, each distance update changes the value by $\geq 1$.**
- **If the largest edge cost is $|C|$, then SP length cannot drop below $-nC$. The largest feasible path length is $+nC$.**
- **Total number of relabeling is at most $n^2C$.**
- **So, the algorithm halts, although $C$ can be very large. Algorithm not strongly polynomial.**
- **Alternatively, one can bound the run time by $O(2^n)$.**

# Self-Study

- **Construct bad examples for generic labeling algorithm.**
- **Show termination even when costs are non-integer.**
- **Prove upper bound of $O(2^n)$.**

# Improved Label-Correcting

- **The Generic Label-Correcting does not specify any order for selecting edge violations.**
- **To improve running time, we need a better order.**
- **Many variants of this algorithm, with different complexity and performance.**
- **Bellman-Ford Method: elegant, admits simple analysis, works very well in practice, and has the best theoretical running time.**

# Bellman-Ford Algorithm

**algorithm Bellman-Ford**

**Input: Graph $G = (V, E)$, with source node $s$.**

**1.** $d(s) = 0$**,** $\mathrm{pred}(s) = 0$**;**

**2.** $d(j) = \infty$**, for** $j = 1, 2, \ldots, n$**.**

**3. for** $k = 1$ **to** $|V| - 1$ **do**

**4.**   **for each edge** $(i, j)$ **in** $E$ **do**

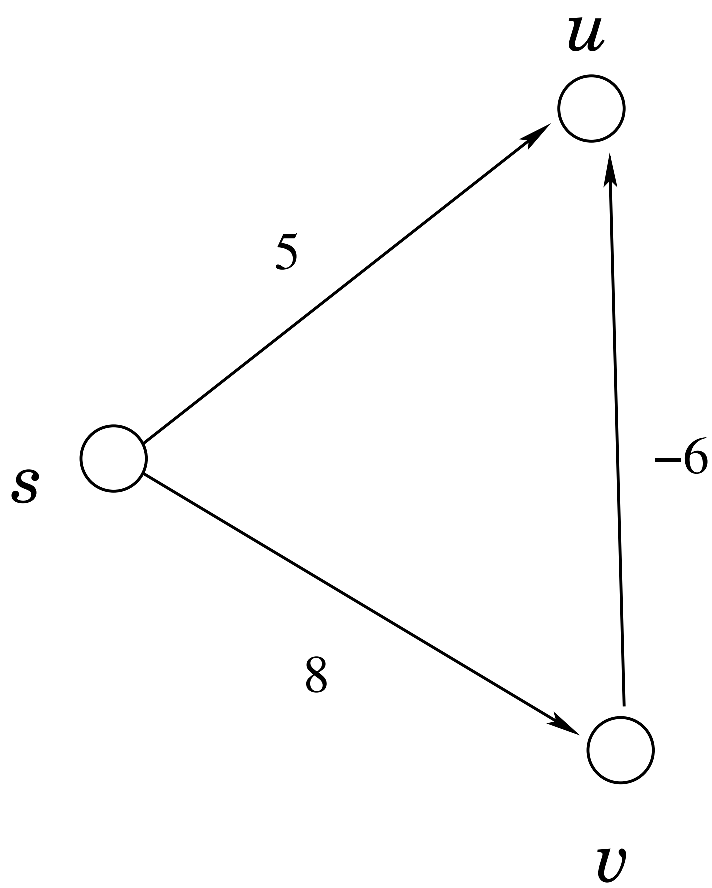
**5.**     **if** $d(j) > d(i) + c_{ij}$**;**

**6.**      $d(j) = d(i) + c_{ij}$**;**

**7.**      $\mathrm{pred}(j) = i$**;**

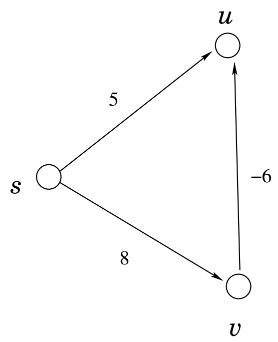
**8. end;**

# Example




- **Use edge order** $(v, u), (s, u), (s, v)$**.**

# Bellman-Ford: Correctness




1. **The algorithm terminates after $O(nm)$ steps, by construction. (Key: SP has $\leq n$ edges.)**

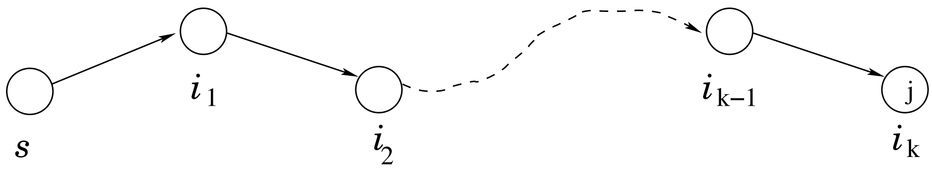
2. **Show that distances computed are correct.**

3. **By Induction.** *If the SP to a node $j$ consists of $k$ edges, then after $k$ iterations of the outer loop, $d(j)$ is correct.*

4. **Basis of induction: $k = 1$. If the shortest path to $j$ has only one edge, then $E$ must contain this edge $(s, j)$. During the $k = 1$ iteration, the inner loop scans this edge at some point, and updates the distance $d(j)$ correctly.**

# Bellman-Ford: Correctness

**General Case of Induction:**




- **Suppose, by IH, that after $k-1$ iterations, nodes whose SP use at most $k-1$ edges are correctly labeled.**
- **Consider a node $j$ whose SP path has exactly $k$ edges. Let $s \rightarrow i_1 \rightarrow i_2 \cdots i_{k-1} \rightarrow i_k = j$ be the SP to $j$.**
- **By induction, $d(i_{k-1})$ is correct.**
- **When the edge $(i_{k-1}, j)$ is scanned, during the $k$th iteration of outer loop, we update $d(j) \leftarrow d(i_{k-1}) + c_{i_{k-1},j}$.**
- **So, $d(j)$ is correct after $k$ iterations.**

# Negative Edges and Cycles

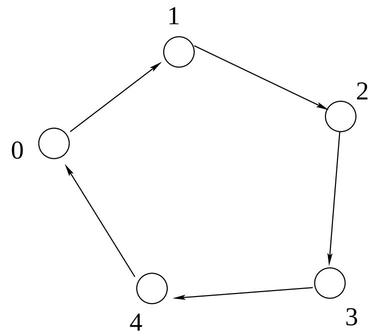
1. **Effect of negative costs on Bellman-Ford?**
2. **The optimality condition is bullet-proof.**
3. **What about the termination condition?**
4. **Suppose $G$ contains a cycle $Z$ whose total cost is negative. By repeatedly going around $Z$ we can continuously lower the path length!**
5. **Theorem: SP distances in $G$ are well defined if and only if $G$ does not contains a negative-cost cycle reachable from $s$. If SP are well-defined, then there is always a simple shortest path from $s$ to each $j$.**
6. **Proof: If all reachable cycles are non-negative, then we can always cut them off.**
7. **Consequently, Bellman-Ford halts after $O(nm)$ steps IF there is no negative cycle reachable from $s$.**

# Detecting Negative Cycles

- **How to tell if $G$ has a negative cycle?**
- **[Idea 1:] We know that $-nC$ is a lower bound on any $d()$. So, if any label drops below $-nC$, we have a negative cycle. You can trace the cycle using** pred **pointers.**
- **[Idea 2:] Use the Bellman-Ford algorithm. At the end, make one more pass over all edges $E$. If you find ANY edge $(i, j)$ for which $d(j) > d(i) + c_{ij}$, then $G$ must have a negative cycle.**
- **Call it the Bellman-Ford Post Processing check.**

# Detecting Negative Cycles

**Theorem:** $G$ **has a negative cycle if and only Bellman-Ford Post Processing check fails.**




1. **No negative cycle means all distances are correct. So, by optimality condition,** $d(j) \leq d(i) + c_{ij}$**.**

2. **Now, suppose there is a negative cycle.** **Ex:** $c_{0,1} + c_{1,2} + c_{2,3} + c_{3,4} + c_{4,0} < 0$**.**

3. **Suppose the Post Processing check passes. Then,** $d(j) \leq d(j-1) + c_{j-1,j}$**, for all** $j$**.**

4. **Add them up, and cancel the common term. We get** $c_{0,1} + c_{1,2} + c_{2,3} + c_{3,4} + c_{4,0} \geq 0$**, which contradicts (2)!!!**

# Applications of Shortest Paths

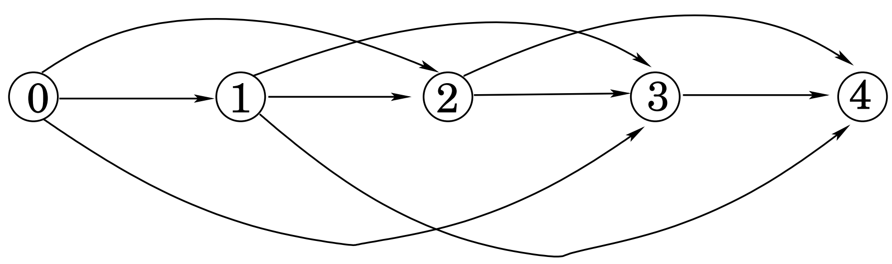
- Shortest path problems arise both as stand-alone models, and as subproblems in more complex settings.
- Obvious applications in telecom and transportation: sending messages or goods quickly and cheaply.
- Urban planning modelers, project management, inventory planning, DNA sequencing.
- Few examples of the use of shortest path useful in modeling subproblems in more complex settings:
  1. Inspection planning in a production line.
  2. Approximating piecewise linear functions.
  3. System of Difference Constraints.
  4. Others....

# Production Line

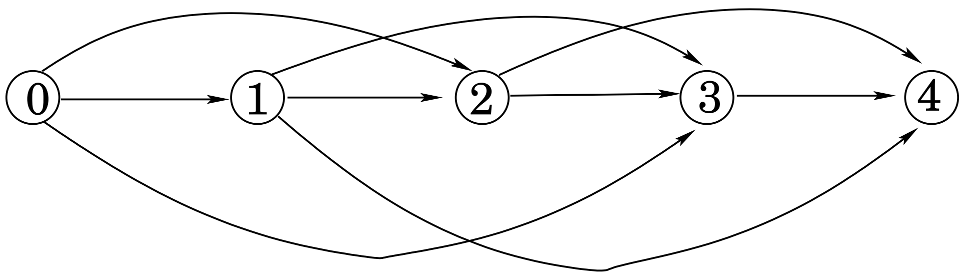
- $n$ stations, each with manufacturing followed by possible inspection stage.
- Product enters stage 1 in $B$-size batches.
- Possible defects introduced at any stage. The prob. of producing a defect in stage $i$ is $\alpha_i$.
- All defects non-repairable, so defective item discarded.
- After any stage, we can either inspect all items, or none (no sampling).
- The $n$th stage must have inspection—can't afford to ship a defective item.
- Design an optimal inspection plan that minimizes the total cost of production and inspection.
- Fewer inspection decrease cost, but would increase production cost because defective items might continue down the production line.

# Production Line

- **Modeling parameters:**
  1. $p_i$**: manufac cost/item in stage** $i$**.**
  2. $f_{ij}$**: fixed inspection cost per batch after stage** $j$**, given the last inspection after stage** $i$**.**
  3. $g_{ij}$**: variable per unit cost for inspection after** $j$**, given last inspection after** $i$**.**
  4. **Inspection after** $j$ **needs to look for defects introduced in stages** $i+1, i+2, \cdots, j$**.**
- **Shortest path problem on nodes** $0, 1, \ldots, n$**.**
- **Put an edge** $(i, j)$ **between all** $i, j$**, with** $i < j$**.**



# Production Line




- **Any path from node $0$ to $n$ is an inspection plan.**
- **E.g. path $0 \rightarrow 2 \rightarrow 4$ means inspection after stage 2 and 4.**
- **Cost of edge $(i, j)$:**

$$c_{ij} \;=\; f_{ij} \;+\; B(i) g_{ij} \;+\; B(i) \sum_{k=i+1}^{j} p_k,$$

**where $B(i) = B \Pi_{k=1}^{i} (1 - \alpha_k)$ denotes the expected number of non-defective units at end of stage $i$.**

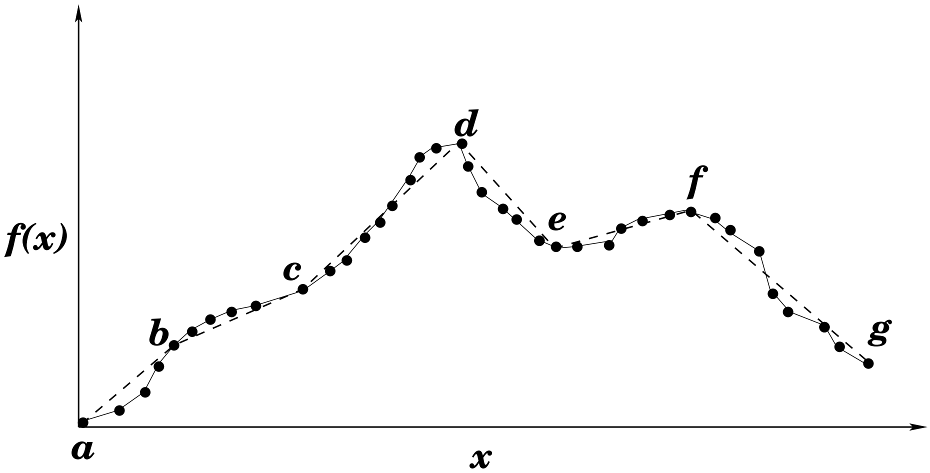
- **Dijkstra's algorithm will solve the problem in $O(n^2)$ time.**

# Piecewise Function Approx

- Scientific data often use piecewise linear functions, contain large number of breakpoints.
- Large data difficult to store, transmit, manipulate, evaluate.
- Use a subset of data—approximation introduces inaccuracies.
- Optimal piecewise approximation? Tradeoff between cost and benefits.
- Model: $f(x)$ a piecewise linear function of scalar $x$.
- Think of tuples $(x_1, f(x_1)), (x_2, f(x_2)), \ldots$ as points in the plane.
- Between two consecutive values, function varies linearly.

# Piecewise Function Approx

- **Assume $n$ is very large, and we want to approximate $f$ by another function $z$ that passes through a smaller subset.**

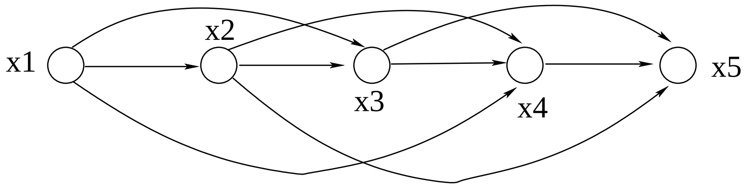



- **Figure shows an example: input of 40 breakpoints (solid) approximated by 7 breakpoints (dashed).**
- **Fixed cost $\alpha$ per breakpoint in storage.**
- **Sum of Squared error: cost of dropping all points between $x_i$ and $x_j$:**

$$\beta \sum_{k=i+1}^{j-1} (f(x_k) - z(x_k))^2$$

# Piecewise Function Approx

- **Shortest path problem on $n$ nodes, $1, 2, \ldots, n$.**




- **Put an edge between each pair $i, j$, with $i < j$. Edge $(i, j)$ signifies that we approximate the piecewise function between $x_i$ and $x_j$ by a single line segment joining $x_i$ to $x_j$.**

- **The cost $c_{ij}$ is**

$$\alpha + \beta \sum_{k=i+1}^{j-1} (f(x_k) - z(x_k))^2$$

- **Each directed path from 1 to $n$ represents an approximating function $z$. The shortest path represents the optimal approximating function.**

# System of Difference Constraints

- **Input:** $n$ **variables** $x()$**, and** $m$ **constraints of the form**

$$x(i) - x(j) \le b(k).$$

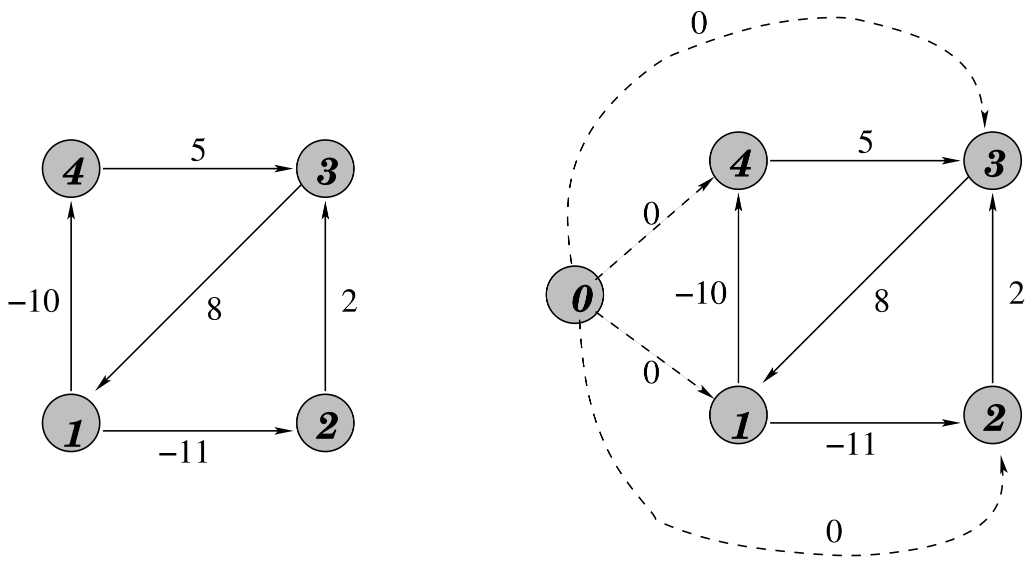
- **Does there exist a feasible solution?**
- **Example:**

$$\begin{aligned} x(3) - x(4) &\le 5 \\ x(4) - x(1) &\le -10 \\ x(1) - x(3) &\le 8 \\ x(2) - x(1) &\le -11 \\ x(3) - x(2) &\le 2 \end{aligned}$$

- **Find real-valued** $x()$ **that satisfy all the constraints?**
- **Example applications: resource allocation, investment management.**

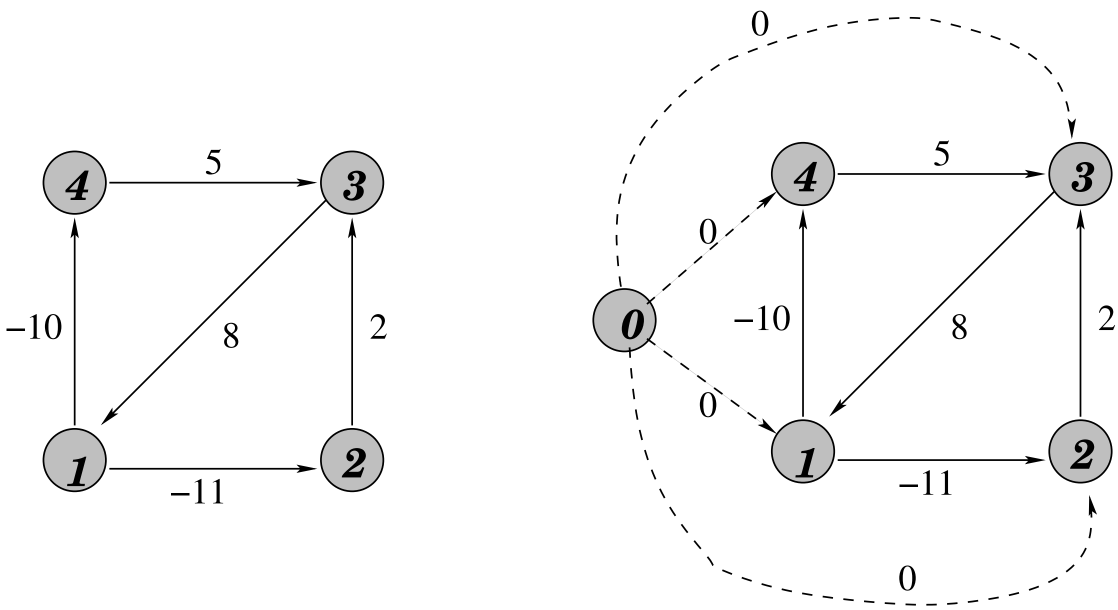
# Difference Constraints (2)

- **Graph Model: variable $\equiv$ nodes, and constraint "$x(i) - x(j)$" $\equiv$ arc $(j, i)$.**




- **Augment the graph with node $0$, with zero-cost edges to all other nodes.**
- **Think of $x()$ as shortest path distances.**
- **By SP Optimality: distance labels $d()$ are shortest path distancs if and only if $d(j) - d(i) \leq c_{ij}$.**
- **Thus, difference constraints satisfied if and only if this graph** *does not have a negative cycle* **(and so shortest path distances exist!).**

# Difference Constraints (3)




- **In this example, 1–2–3 is a negative cycle.**
- **Correspondingly, constraints** $x(1) - x(3) \leq 8$, $x(2) - x(1) \leq -11$, **and** $x(3) - x(2) \leq 2$ **are inconsistent.**

# Dijkstra: Label Setting

**There are four versions of the SP problem.**

1. **[Single Source-Destination Pair] Given a pair of node $s$ and $t$, find a shortest path between them**

2. **[Single-Source] Given a source node $s$, find shortest paths to all other nodes.**

3. **[Single-Destination] Given a node $t$, find shortest paths from all other nodes to $t$.**

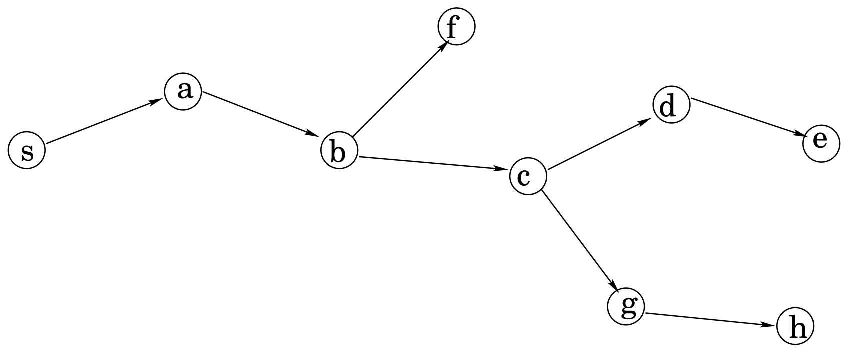
4. **[All-Pair] Find shortest paths between all pairs of nodes in $G$.**

**Observations:**

- $(2) = (3)$ **with edge reversal.**
- **Methods for (1) also solve (2).**
- **For (4), run (1) from each node.**
- **(1) is the central problem.**

# Shortest Path Properties

1. **Shortest Path Tree: union of shortest paths from $s$ to all other nodes.**
2. **Explicitly storing $n-1$ paths can take $\Omega(n^2)$ space.**
3. **Instead, we use the subpath optimality to store the tree.**
4. **If $s \to v_1 \to \cdots \to v_i \to \cdots \to v_j \to \cdots \to v_k$ is a shortest path, then the subpath $v_i \to \cdots \to v_j$ is a SP from $v_i$ to $v_j$.**

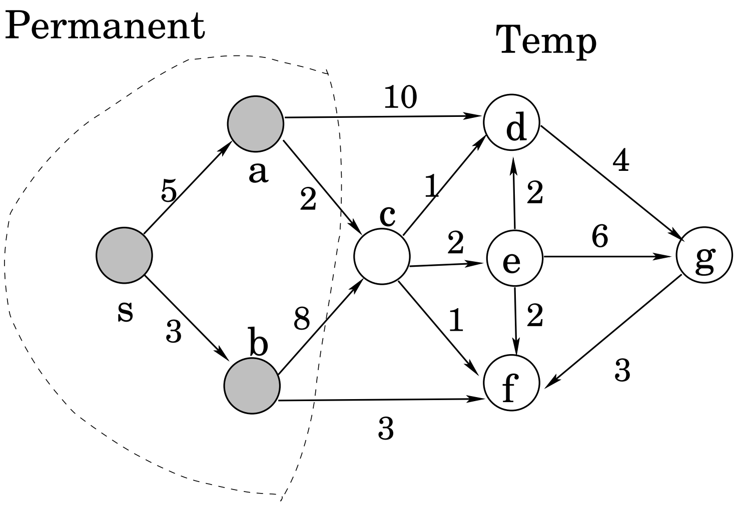



5. **Implicitly storing the shortest path tree:**

| *Node* | s | a | b | c | d | e | f | g | h |
|---|---|---|---|---|---|---|---|---|---|
| *Pred* | $\phi$ | s | a | b | c | d | b | c | g |

# Dijkstra's SP Algorithm

1. **Computes SP tree when the edge costs are non-negative.**
2. **Maintain labels $d(i)$ for each node $i$, current best estimate.**
3. **Some labels** *temporary*, **others** *permanent*.
4. **In each step, choose the node with** *smallest temporary label*. **Make it permanent, and scan its neighbors to update their $d()$.**



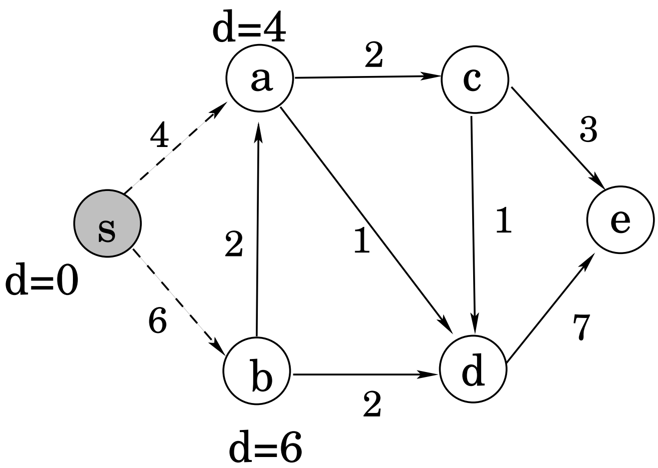

5. **When done, the $d()$ labels are SP dist.**
6. **The predecessors define SP tree.**

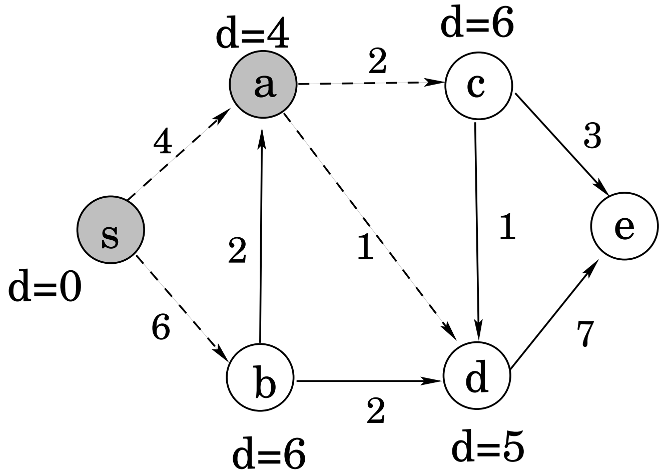
# Dijkstra's SP Algorithm

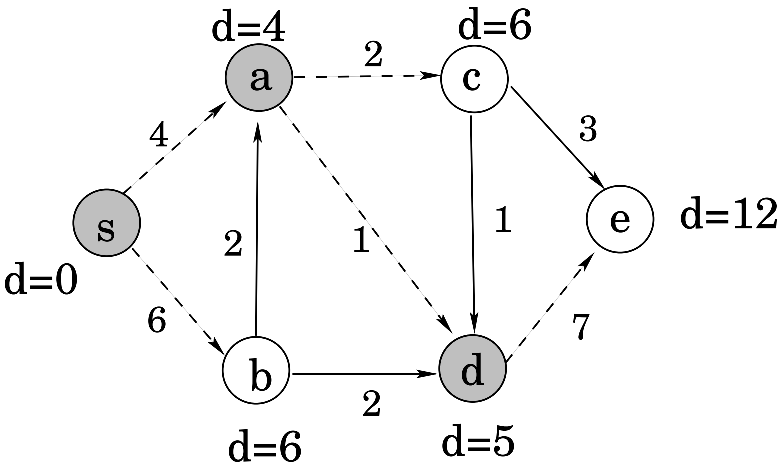
**algorithm Dijkstra**

1. $S = \emptyset$, $T = V$;
2. $d(i) = \infty$ **for each** $i \in T$;
3. $d(s) = 0$, $\text{pred}(s) = 0$;
4. **while** $|S| < |V|$ **do**
5. **Choose** $i \in T$ **with** $d(i) = \min_{j \in T}\{d(j)\}$;
6. $S = S \cup \{i\}$ **and** $T = T - \{i\}$;
7. **for each** $(i, j) \in E$ **do**
8. **if** $d(j) > d(i) + c_{ij}$ **then**
9. $d(j) = d(i) + c_{ij}$ **and** $\text{pred}(j) = i$;
10. **end;**

# Illustrating Dijkstra's Algorithm



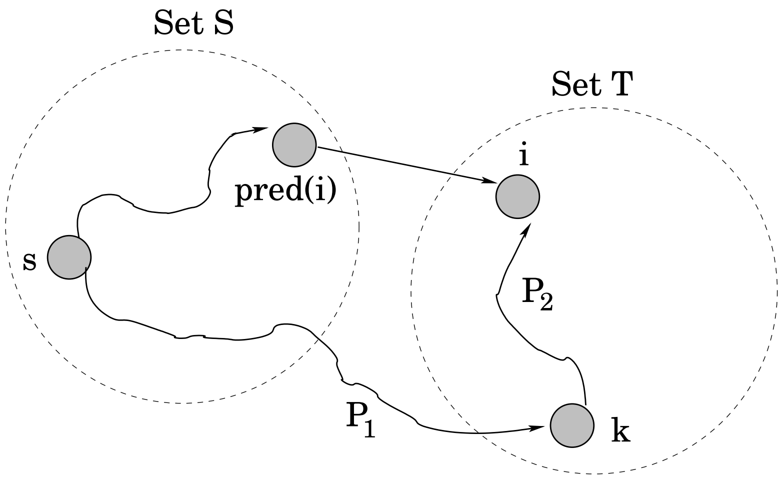
# Correctness of Dijkstra

- **[Invariant 1:]** $d(i)$ **for all** $i \in S$ **is optimal.**
- **[Invariant 2:]** $d(j)$ **for each** $j \in T$ **is the length of any shortest path whose internal nodes lie in** $S$**.**

- **Proof by induction on the cardinality of the set** $S$**.**

# Proof of Invariant 1

- *$d(i)$ for all $i \in S$ is optimal.*

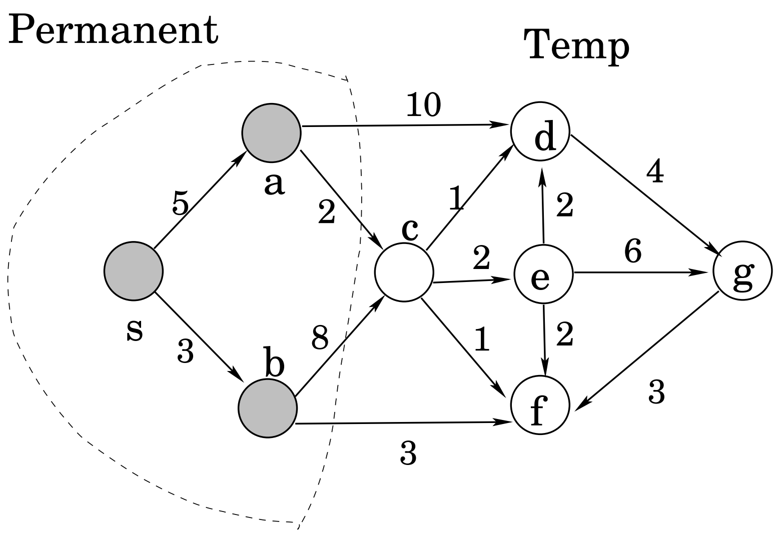



1. **A step transfers node $i$ with smallest label from $T$ to $S$.**
2. **$d(i)$ is SP length over paths not using $T$ nodes. We show that paths using $T$ nodes are at least as long.**
3. **Assume such a shortest path $P$, which can be decomposed into $P_1$ and $P_2$, where $P_1$ has no internal $T$ nodes.**
4. **$i$ has smallest label: $len(P_1) \geq d(k) \geq d(i)$.**
5. **Postive edges costs: $len(P) \geq len(P_1)$.**
6. **Thus, $len(P) \geq d(i)$.**

# Proof of Invariant 2

- *$d(j)$ for each $j \in T$ is the length of any shortest path whose* **internal** *nodes lie in $S$.*

1. **When a new node $i$ is made permanent, $d()$ labels for some nodes in $T$ might decrease. Why?**
2. **Because $i$ might be a new internal node.**




3. **But Dijkstra scans all neighbors of $i$, and updates their distance.**
4. **Thus, $d(j)$ labels are correct for paths that use only $S$ nodes as internal.**

# Time Complexity

- *Node Selection:* **Each selection requires a scan of current temp nodes, and each selection makes one node permanent. So, the total work is** $\sum_{i=n}^{1} i = O(n^2)$**.**

- *Label Update:* **When making** $i$ **permanent, algorithm scans all neighbors of** $i$**. Total work is** $\sum_i |A(i)| = m$**.**

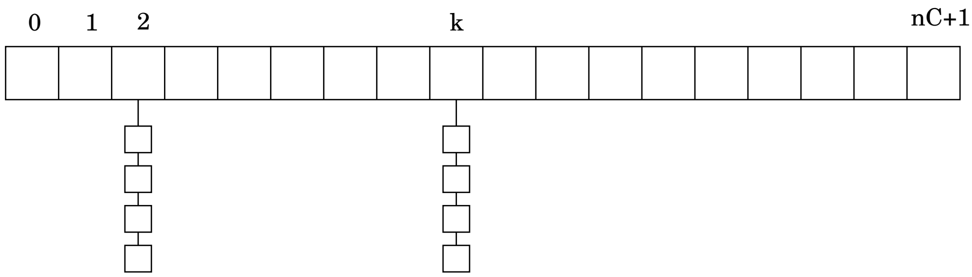
- **Straightforward implementation of Dijkstra takes** $O(n^2)$ **time.**

- **Since the number of edges,** $m$**, is often much smaller than** $n^2$**, can we improve the bound?**

- **One ideas is to improves node selection by keeping** $d$ **labels sorted.**

# Dial's Implementation

- *Monotone Property:* $d$ labels made permanent by Dijkstra are *non-decreasing*.
- Suppose largest edge cost is $C$. Max distance label is $nC$.
- Dial keeps $nC + 1$ buckets, where bucket $k$ stores all nodes with temp distance $k$, in doubly linked list.



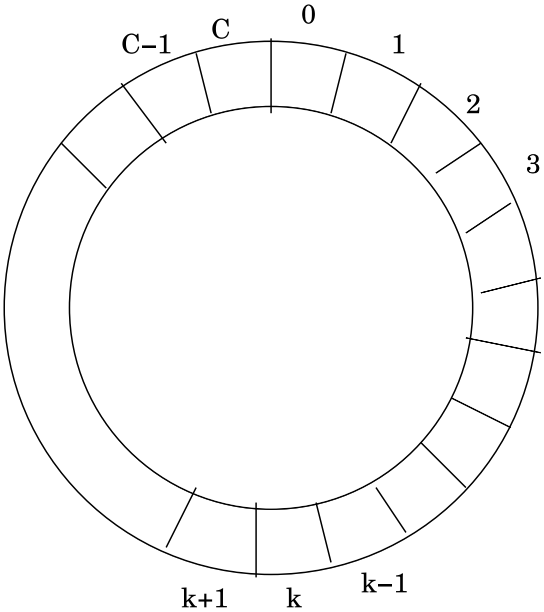

- Find first non-empty bucket, say, $k$. Each node in $k$'s list has smallest temp label. One by one, make them permanent, and scan their neighbors to update their distance.
- When a node's distance is updated from $d_1$ to $d_2$, we move it from bucket $d_1$ to bucket $d_2$.

# Analysis of Dial

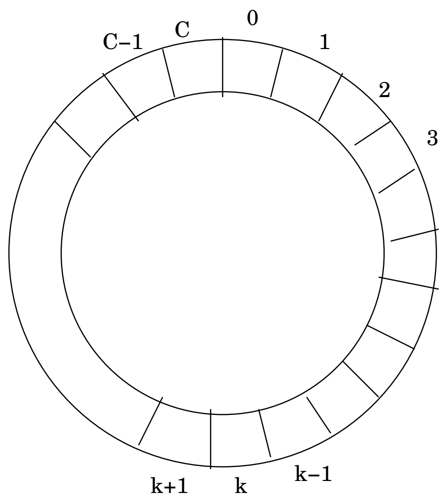
- **By Monotone Property, if the last non-empty bucket was $k$, then the next one must after $k$.**
- **Total time spent in scanning buckets is $nC + 1$.**
- **Adding or removing from a bucket list takes $O(1)$.**
- **Thus, total time is $O(nC + m)$.**
- **When $C$ is small, say, a constant, this is optimal $O(n + m)$.**
- **But if $C$ is large, say, $n^4$ or $2^n$, the bucket scanning can be very slow.**
- **The memory requirement is also large: $O(nC + m)$.**

# Improved Dial

- *Bounded Label Difference:* **At any time, the difference between two (finite) temporary labels is atmost $C$.**
- **Proof. Let $d(i)$ be the last label made permanent.**
- **For any other node $j \in T$, $d(j) = d(l) + c_{l,j}$, for some $l$.**
- **But $d(l) \leq d(i)$, and $d(i) \leq d(j)$.**
- **Thus, nodes in $T$ have labels between $d(i)$ and $d(i) + C$.**



# Improved Dial




- **Store node with temp label $d(j)$ in bucket $d(j)\ mod\ (C+1)$.**
- **Over time, bucket $k$ stores nodes wih distance $k$, $k+C$, $k+2C$ etc. But by Bounded Label Difference, at any time, the bucket holds nodes with the same distance.**
- **Examine buckets sequentially, with wraparound. Start next search from where the last one finished.**
- **Worst-cast time $O(m+nC)$, memory $O(m+C)$.**

# Heap Implementation

- **Heap data structure allows the following operations:**
  1. **CreateHeap (H)**
  2. **FindMin (i, H)**
  3. **Insert (i, H)**
  4. **DecreaseKey (newVal, i, H)**
  5. **DeleteMin (i, H)**
- **Maintain temporary labels in the Heap.**
- **Node Selection done via FindMin and DeleteMin.**
- **Updates handled via DecreaseKey.**
- **Remember Dijkstra's algorithm does $n$ node selections and $m$ upadtes.**

# Different Heaps

- *Binary Heap:* **Standard heap requires $O(\log n)$ time to do insert, decreaseKey, and deleteMin. Other operations take $O(1)$ time.**
- **So, using Binary Heap, Dijkstra takes $O(m \log n)$ time.**
- *d-Heap:* **A $d$-ary heap requires $O(\log_d n)$ time to do insert and decreaseKey, but needs $O(d \log_d n)$ time for each deleteMin. Other operations take $O(1)$ time.**
- **So, using $d$-Heap, for any parameter $d \geq 2$, Dijkstra takes $O(m \log_d n + nd \log_d n)$ time.**
- **How do you pick $d$?**
- **In Sparse graphs, $m = O(n)$, binary heap runs in $O(n \log n)$ time.**
- **In dense graphs, $m = \Omega(n^{1+\varepsilon})$, $d = \lceil m/n \rceil$ makes the algorithm run in $O(m)$ time.**

# More Implementations

- *Fibonacci Heap:* **All operations in $O(1)$ amortized, except deleteMin, which takes $O(\log n)$.**
- **Implementing Dijkstra using F-Heap takes worst-case time $O(m + n \log n)$ time. Best strongly polynomial algorithm.**
- **Empirically, reseachers have found that Dial's implementation is the fastest Label-Setting shortest path algorithm for most networks.**
- **Experiments by Gallo-Pallottino suggest that some implementations of the Label-Correcting algorithms (Thresh1 and Thresh2) have the fastest running time both for positive and negative edge costs.**

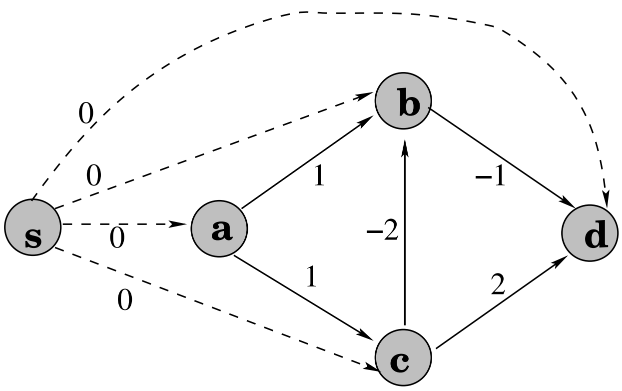
# All-Pairs Shortest Paths

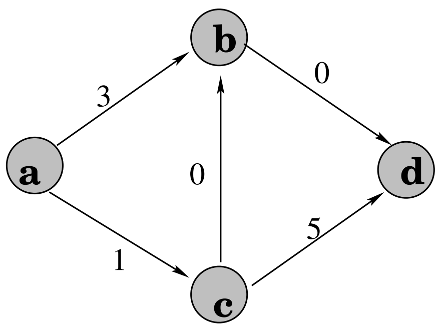
- Given $G = (V, E)$, compute shortest path distances between all pairs of nodes.
- Run single-source shortest path algorithm from each node as root. Total complexity is $O(nS(n, m))$, where $S(n, m)$ is the time for one shortest path iteration.
- If non-negative edges, use Dijkstra's algorithm: $O(m \log n)$ time per iteration.
- With negative edges, need to use Bellman-Ford algorithm: $O(nm)$ time per iteration.
- Edmonds-Karp introduced a trick, which allows one to use Dijkstra after one run of Bellman-Ford.

# All-Pairs Shortest Paths (2)

- **Add a new node $s$; join it to all others with 0-cost edges.**
- **Compute shortest path distances from $s$ to all other nodes, using Bellman-Ford.**
- **E.g.** $d(a) = 0, d(b) = -2, d(c) = 0, d(d) = -3$**.**
- **In the new graph, modify the weight of edge $(u, v)$ as follows:**

$$c'(u, v) = c(u, v) + d(u) - d(v) \qquad (*)$$



# All-Pairs Shortest Paths (3)

- **New costs are non-negative (why?), so use Dijkstra's algorithm, $n$ times, once per node.**
- **Non-negativity follows from Bellman-Ford condition: $d(v) \leq d(u) + c_{uv}$, which implies that $c'_{uv} = c_{uv} + d(u) - d(v) \geq 0$.**
- **Correctness of path computation: for any path $p$ from $x$ to $y$, we have $l'(p) = l(p) + d(x) - d(y)$.**
- **This follows from telescoping of terms. If $p$ breaks into two subpaths, $p_1$ from $x$ to $z$, and $p_2$ from $z$ to $y$, then by induction: $l'(p) = l'(p_1) + l'(p_2) = (l(p_1) + d(x) - d(z)) + (l(p_2) + d(z) - d(y)) = l(p) + d(x) - d(y)$.**
- **Thus, total time is $O(nm + nm \log n)$.**

# All-Pairs Shortest Paths (4)

- **In worst-case, for even positive-weight graphs, All-Pairs algorithm takes $\Theta(n^3 \log n)$ time.**
- **For negative-weight graphs, complexity is $O(n^4)$, but Edmonds-Karp heuristic improves it $O(n^3 \log n)$. But does require implementing both Dijkstra and Bellman-Ford.**
- **Floyd-Warshall is a simple algorithm, with no data structure at all, for computing all pair shortest paths.**
- **The algorithm is always $O(n^3)$.**

# Floyd-Warshall Algorithm

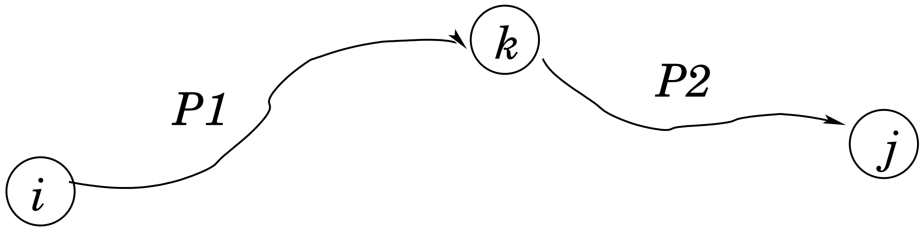
- $G = (V, E)$ **has vertices** $\{1, 2, \ldots, n\}$**.**
- $W$ **is the cost adjacency matrix of graph** $G$**.**
- **Matrix** $D$ **encodes the pair-wise distances. Assume all entries initialized to 0.**

**algorithm Floyd-Warshall**

```
D = W;
for k = 1 to n
    for i = 1 to n
        for j = 1 to n
            d_ij = min{d_ij,  d_ik + d_kj}
return D.
```

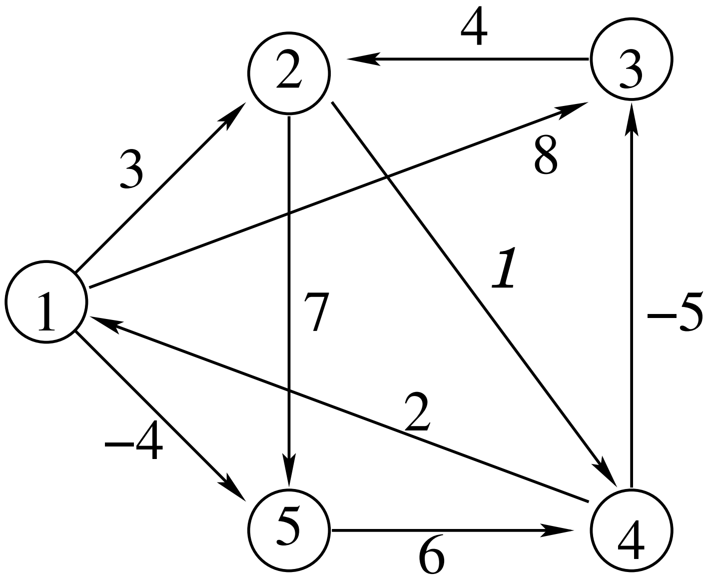
# Correctness of Floyd-Warshall

- $P_{ij}^k$ : **shortest path whose intermediate nodes are in** $\{1, 2, \ldots, k\}$**.**
- **Goal is to compute** $P_{ij}^n$**, for all** $i, j$**.**




- **Use Dynamic Programming. Two cases:**
  1. **Vertex** $k$ **not on** $P_{ij}^k$**. Then,** $P_{ij}^k = P_{ij}^{k-1}$**.**
  2. **Vertex** $k$ **is on** $P_{ij}^k$**. Then, neither** $P_1$ **nor** $P_2$ **uses** $k$ **as an intermediate node. in its interior. (Simplicity of** $P_{ij}^k$**.) Thus,** $P_{ij}^k = P_{ik}^{k-1} + P_{kj}^{k-1}$
- **Recursive formula for** $P_{ij}^k$**:**
  1. **If** $k = 0$**,** $P_{ij}^k = c_{ij}$**.**
  2. **If** $k > 0$**,** $d_{ij}^k = \min\{d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}\}$

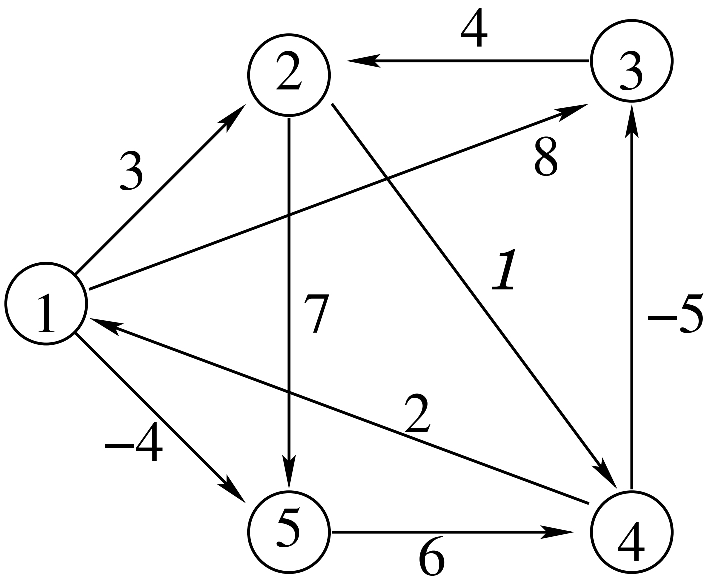
# Example




- **Matrices $D_0$ and $D_1$:**

$$\begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

# Example




- **Matrices $D_2$ and $D_5$:**

$$\begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix}$$