Proof of Sufficiency for Euler Circuits

First we will prove a lemma, the use of which will be allowed on future assignments:

1 Lemma Proof

Claim: Every connected graph of two or more vertices has a vertex that can be removed (along with its incident edges) without disconnecting the remaining graph.
Proof: We will prove an even stronger fact by induction on the number of vertices:
$P(n)$ = "A connected graph with n vertices has two distinct vertices, each of which can be removed individually (along with incident edges) without disconnecting the remaining graph"
Base Case: $P(2)$: Either one of the vertices can be removed. The remaining graph is a single vertex in both cases, which is a connected graph.
Inductive Case: $P(2) \wedge \cdots \wedge P(n) \rightarrow P(n+1)$

1. Consider arbitrary connected graph G with $n+1$ vertices.
2. Remove an arbitrary vertex v to get G'.
3. Case 1: G' is connected
    (a) Since $P(n)$ is true (Inductive Hypothesis), there are two distinct vertices w and u that (individually) can be removed from G' without disconnecting it.
    (b) If w were removed from G instead of v, the resulting graph would still be connected.
    (c) Therefore, two distinct vertices can be safely removed from G, and we are done.
4. Case 2: G' is disconnected
5. Then G' is made up of k connected component subgraphs $C_1, \ldots, C_k$.
6. $k \geq 2$ because with at least $n+1$ vertices in G where $n \geq 2$, the remaining n vertices (at least 2) must be in separate components if G' is disconnected.

7. Each component C_i has a vertex v_i that was a neighbor of v in G.
8. Case 2.1: For each C_i, vertex v_i is the only vertex in the component.
   (a) Add v back to G' get G again, and then remove v_1 instead.
   (b) Alternately, since $k \geq 2$, we could have removed v_k instead.
   (c) Now we have removed a vertex from G in two different ways, and each results in a connected graph.
9. Case 2.2: At least one of C_i, call it C_q, has 2 or more vertices.
   (a) The Strong Inductive Hypothesis applies to C_q, so there are two distinct vertices that can be removed from C_q without disconnecting it.
   (b) Since there are two, and they are distinct, at least one of them is not v_q. Call the other one x.
   (c) Add v back to G' to get G again.
   (d) We have already identified one vertex, x, that can be safely removed from G without disconnecting it.
   (e) We also know that there is at least one other C_p where $p \neq q$ because $k \geq 2$.
   (f) If C_p is just one vertex, we can remove it (v_p) as in Case 2.1.
   (g) If C_p is more than one vertex, then the I.H. allows us to find and remove one vertex (not v_p) as in Case 2.2.
   (h) The vertex removed from C_p accounts for the second vertex that we could remove from G without disconnecting it.
10. In all cases, $P(n+1)$ is true because we found two distinct vertices, either of which could be removed without disconnecting G.

Feel free to use this lemma on tests and other assignments. Now we can use this lemma to do the actual proofs from class.
Note: Can you figure out why the stronger fact about two distinct vertices had to be proven instead? All we care about is having one vertex to safely remove, so why was this proof done instead? Discuss this on Piazza.

2 Induction on number of vertices

$P(n) =$ "A connected multi-graph with n vertices, each of even degree, has an Euler circuit"
Base Case: $P(2)$:

1. Because vertex degrees are even, there must be an even number of edges between these two vertices.

2. Call the vertices a and b, and assume there are $2k$ edges.

3. Then going from a to b and then back again to a k times results in an Euler circuit.

Inductive Case: $P(n) \rightarrow P(n+1)$

1. Take arbitrary connect graph G with $n+1$ vertices, each of even degree.

2. By the lemma, we can remove a vertex v (and incident edges) that does not disconnect the graph. Call the result G'.

3. v had an even degree in G, so we can arbitrarily pair up all of v's incident edges.

4. For every such pair of edges (x, v), (v, y) that existed in G, add one edge (x, y) to G'.

5. The degree of each remaining vertex in G' stays the same as in G, so all are still even.

6. The Inductive Hypothesis then indicates that G' has an Euler circuit. Call it C.

7. Add v back to get G again, and restore the edges to their original state.

8. Create a path in G from C: whenever an edge (x, y) is traversed that existed in G', but does not exist in G, traverse (x, v) then (v, y) instead.

9. The resulting path is an Euler circuit in G.

Q.E.D.

3 Induction on number of edges

$P(n) =$ "A connected multi-graph with n edges and all vertices of even degree has an Euler circuit"
Base Case: $P(2)$:

1. Because there are only two edges, and vertex degrees are even, these edges must both be between the same two vertices.

2. Call the vertices a and b: Then (a, b, a) is an Euler circuit.

Inductive Case: $P(n) \rightarrow P(n+1)$:

1. Start with connected graph G with $n+1$ edges and vertices all of even degree.

2. By lemma, pick a vertex v that can be removed without disconnecting the graph.

3. Since the removal of v will not disconnect the graph, no number of edges removed from it will disconnect the graph either.
4. v must have at least 2 edges since the graph is connected and all vertices have even degree.
5. Pick any two such edges (x, v) and (y, v).
6. Remove (x, v) and (y, v) from G, and add an edge (x, y). If v now has no incident edges, remove it as well. Call the result G'.
7. G' is connected, and it has n edges (we removed 2, but then added 1).
8. The Inductive Hypothesis says that G' has an Euler circuit C.
9. Restore the graph to get G again, and traverse C within G.
10. The first time (x, y) is traversed, traverse (x, v) and then (v, y) instead.
11. The result is an Euler circuit in G.

Q.E.D.

4 Proof via maximal path

1. Consider a maximal simple path $W = (a, \ldots, b)$ in connected graph G, where each vertex has even degree.
2. Look at the endpoint a, which like all other vertices has even degree.
3. Because W is maximal, a's neighbors must be in W (otherwise we could extend the path).
4. Moreover, for each neighbor x of a, each edge (x, a) must be in W in order for the path to be maximal.
5. Because a is the starting point, and its even number of edges are all in the path, the only way for all of them to be traversed is for W to end at a.
6. Therefore $a = b$ and W is a (simple) circuit.
7. Now, assume W is not an Euler circuit BWOC.
8. Then there is an edge (x, y) that is not in W.
9. Since G is connected, there is a path from y to some vertex v in W, which is $(y, \ldots, v)$.
10. Crete a path $(x, y, \ldots, v, \ldots, a, \ldots, v)$, which exists because W is a circuit.
11. This path is clearly longer than W, which contradicts the assumption that it is a maximal path.