Ramanujan's approximation to factorial

I saw this approximation for the factorial given by Ramanujan as

$$\log(n!) \approx n\log n - n + \frac{\log(n(1+4n(1+2n)))}{6} + \frac{\log(\pi)}{2}$$

in wikipedia, which claims the approximation is superior to Stirling's approximation. I tried to locate the reference but unfortunately I could not.

I would appreciate if someone can throw light on how this asymptotic is obtained and the order of the error.

(reference-request) (factorial)

asked Jun 1 '12 at 6:18
user17762

2 Nobody knows how any of Ramanujan's results were obtained. – Gerry Myerson Jun 1 '12 at 6:20

1 @Marvis: A good thing would be to email Bruce Berndt. – user9413 Jun 1 '12 at 6:42

2 Answers

$$\log(n!) \approx (\ln(n)-1)\,n + \ln(\sqrt{2\pi}) + \frac{\ln(n)}{2} + \frac{1}{12}n^{-1} - \frac{1}{360}n^{-3} + O\left(n^{-5}\right)$$

$$\begin{aligned} n\ln(n) - n + \frac{\ln(n\,(1+4\,n\,(1+2\,n)))}{6} + \frac{\ln(\pi)}{2} \\ = (\ln(n)-1)\,n + \ln(\sqrt{2\pi}) + \frac{\ln(n)}{2} + \frac{1}{12}n^{-1} - \frac{1}{288}n^{-3} + \frac{1}{768}n^{-4} \\ + O\left(n^{-5}\right) \end{aligned}$$

So the error in Ramanujan's approximation is asymptotic to $\left(\frac{1}{288}-\frac{1}{360}\right)n^{-3} = \frac{1}{1440}n^{-3}$.

EDIT: an even better approximation, then, would be

$$n\ln(n) - n + \frac{\ln(1/30 + n\,(1+4\,n\,(1+2\,n)))}{6} + \frac{\ln(\pi)}{2}$$

where the error is asymptotic to $-\frac{11}{11520}n^{-4}$. Thus at $n = 10$ we have $\ln 10! \approx 15.1044125730755$, Ramanujan's approximation ≈ 15.1044119983597 and the improved approximation ≈ 15.1044126589476 .

edited Jun 1 '12 at 7:05

answered Jun 1 '12 at 6:55

Robert Israel
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Nice. Thanks! I suspected that by matching the first few coefficients in the error term of Stirling's and rewriting them as log one could get this approximation. I was wondering if there was an inherently new way of coming up with this asymptotic directly instead of comparing with Stirling and then matching coefficients. – user17762 Jun 1 '12 at 17:11

2 The Ramanujan approximation can equivalently be written as $\frac{n!(e/n)^n}{\sqrt{2\pi n}} = \left(1 + \frac{1}{2n} + \frac{1}{8n^2} + \ldots\right)^{1/6}$, compared to the standard Stirling series $1 + \frac{1}{12n} + \frac{1}{288n^2} + \ldots$. I don't know what is so special about the exponent $1/6$ in this context. – Robert Israel Jun 1 '12 at 18:41

On the other hand, you could write this as $\left(1 + \frac{1}{\sqrt{5}n} + \frac{1}{10n^2} + \ldots\right)^{\sqrt{5}/12}$, which has the advantage that the coefficient of $1/n^3$ is 0. – Robert Israel Jun 1 '12 at 18:49

Very nice answer and comments. Nevertheless, I would still like to know where exactly it can be found in Bruce Berndt's books on Ramanujan's lost notebook. Does anyone know that? – Manolito Pérez Mar 12 '15 at 11:47

Well, I finally found the formula in

S. Ramanujan, The Lost Notebook and other Unpublished Papers. S. Raghavan and S. S. Rangachari, editors. Narosa, New Delhi, 1987.

page 339.

answered Apr 15 '15 at 8:41

Manolito Pérez
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