A NON-MESSING-UP PHENOMENON FOR POSETS

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Abstract. We classify finite posets with a particular sorting property, generalizing a result for rectangular arrays. Each poset is covered by two sets of disjoint saturated chains such that, for any original labeling, after sorting the labels along both sets of chains, the labels of the chains in the first set remain sorted. We also characterize posets with more restrictive sorting properties.

1. Introduction

The so-called Non-Messing-Up Theorem is a well known sorting result for rectangular arrays. In [6], Donald E. Knuth attributes the result to Hermann Boerner, who mentions it in a footnote in Chapter V, §5 of [1]. Later, David Gale and Richard M. Karp include the phenomenon in [3] and in [4], where they prove more general results about order preservation in sorting procedures. The first use of the term "non-messing-up" seems to be due to Gale and Karp, as suggested in [5]. One statement of the result is as follows.

Theorem 1. *Let $A = (a_{ij})$ be an m-by-n array of real numbers. Put each row of A into non-decreasing order. That is, for each $1 \leq i \leq m$, place the values $\{a_{i1}, \ldots, a_{in}\}$ in non-decreasing order (henceforth denoted* row-sort*). This yields the array $A' = (a'_{ij})$. Column-sort A'. Each row in the resulting array is in non-decreasing order.*

Proof. Following the solution submitted by J. L. Pietenpol in [2], first row-sort A to obtain A'. Column-sort A' by first permuting the rows to order the first column; to order the second column, permute the rows without their first entries; and so on. The rows remain sorted at each step of the procedure. □

Applying the theorem to the transpose of the array A, the sorting can also be done first in the columns, then in the rows, and the columns remain sorted.

Example.

$$\begin{array}{cccc} 4 & 9 & 7 & 8 \\ 12 & 5 & 1 & 10 \\ 2 & 6 & 11 & 3 \end{array} \xrightarrow{\text{row-sort}} \begin{array}{cccc} 4 & 7 & 8 & 9 \\ 1 & 5 & 10 & 12 \\ 2 & 3 & 6 & 11 \end{array} \xrightarrow{\text{column-sort}} \begin{array}{cccc} 1 & 3 & 6 & 9 \\ 2 & 5 & 8 & 11 \\ 4 & 7 & 10 & 12 \end{array}$$

Answering a question posed by Richard P. Stanley, the author's thesis advisor, this paper defines a notion of non-messing-up for posets and Theorem 7 generalizes Theorem 1 by characterizing all posets with this property.

Standard terminology from the theory of partially ordered sets will be used throughout the paper. A good reference for these terms and other information about posets is Chapter 3 of [8].

Date: October 17, 2005.