Set 33 Suppose that you have tossed a coin – which may or may not be balanced – five times and reported the outcomes 1, 0, 0, 1, and 1, with Heads coded 1's and Tails by 0's.

(a) Calculate the sample proportion $\hat{p}$ of Heads.

$$\hat{p} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_{x_i=H}$$

$$\hat{p} = \frac{3}{5}$$

(b) Construct 95% (asymptotic) interval for the population probability of Heads.

The equation for a confidence interval with 95% confidence is,

$$\hat{p} - 1.96\frac{\hat{\sigma}}{\sqrt{n}} \leq p \leq \hat{p} + 1.96\frac{\hat{\sigma}}{\sqrt{n}}$$

And the standard deviation of the population sample is,

$$\hat{\sigma} = \hat{p}(1-\hat{p})$$

Then,

$$\hat{p} - 1.96\frac{\sqrt{\hat{p}(1-\hat{p})}}{\sqrt{n}} \leq p \leq \hat{p} + 1.96\frac{\sqrt{\hat{p}(1-\hat{p})}}{\sqrt{n}}$$

$$\frac{3}{5} - 1.96\frac{\sqrt{\frac{3}{5}\left(1-\frac{3}{5}\right)}}{\sqrt{5}} \leq p \leq \frac{3}{5} + 1.96\frac{\sqrt{\frac{3}{5}\left(1-\frac{3}{5}\right)}}{\sqrt{5}}$$

$$\frac{3}{5} - 1.96\frac{\sqrt{\frac{6}{25}}}{\sqrt{5}} \leq p \leq \frac{3}{5} + 1.96\frac{\sqrt{\frac{6}{25}}}{\sqrt{5}}$$

$$\frac{3}{5} - 1.96\frac{\sqrt{6}}{5\sqrt{5}} \leq p \leq \frac{3}{5} + 1.96\frac{\sqrt{6}}{5\sqrt{5}}$$

Therefore, the 95% confidence interval is

$$\left[\frac{3\sqrt{5} - 1.96\sqrt{6}}{5\sqrt{5}}, \frac{3\sqrt{5} + 1.96\sqrt{6}}{5\sqrt{5}}\right]$$

(c) Construct a conservative 95% (asymptotic) confidence interval for the population probability of Heads.

The conservative assumption is

$$\hat{p} = 0.5$$

Then,

$$\hat{p} - 1.96\frac{\sqrt{\hat{p}(1-\hat{p})}}{\sqrt{n}} \leq p \leq \hat{p} + 1.96\frac{\sqrt{\hat{p}(1-\hat{p})}}{\sqrt{n}}$$

$$\frac{1}{2} - 1.96\frac{\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)}}{\sqrt{5}} \leq p \leq \frac{1}{2} + 1.96\frac{\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)}}{\sqrt{5}}$$

$$\frac{1}{2} - 1.96\frac{\sqrt{\frac{1}{4}}}{\sqrt{5}} \leq p \leq \frac{1}{2} + 1.96\frac{\sqrt{\frac{1}{4}}}{\sqrt{5}}$$

$$\frac{1}{2} - 1.96\frac{1}{2\sqrt{5}} \leq p \leq \frac{1}{2} + 1.96\frac{1}{2\sqrt{5}}$$

Therefore, the 95% conservative confidence interval is

$$\left[\frac{\sqrt{5} - 1.96}{2\sqrt{5}}, \frac{\sqrt{5} + 1.96}{2\sqrt{5}}\right]$$