Context-Free and Noncontext-Free Languages

Chapter 13

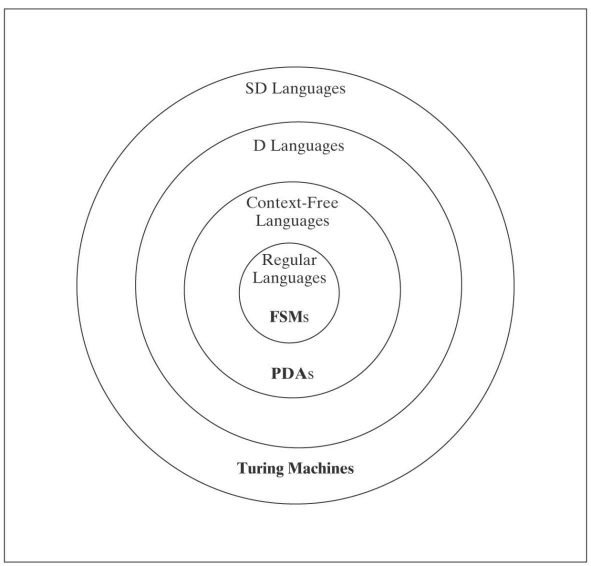
Languages That Are and Are Not Context-Free

a^*b^* is regular.

$A^nB^n = \{a^nb^n : n \geq 0\}$ is context-free but not regular.

$A^nB^nC^n = \{a^nb^nc^n : n \geq 0\}$ is not context-free.

Languages and Machines



The Regular and the CF Languages

Theorem: The regular languages are a proper subset of the context-free languages.

Proof: In two parts:

- Every regular language is CF.
  - Every regular grammar is context-free.

  or
  - Every FSM is a PDA (that is ignoring its stack).
- There exists at least one language that is CF but not regular.
  - A^nB^n

How Many Context-Free Languages Are There?

Theorem: There is a countably infinite number of CFLs.

Proof:

- Upper bound: we can lexicographically enumerate all the CFGs.

- Lower bound: {a}, {aa}, {aaa}, … are all CFLs.

So there must exist some languages that are not context-free:

$\{a^n b^n c^n : n \geq 0\}$

Showing that L is Context-Free

Techniques for showing that a language *L* is context-free:

1. Exhibit a context-free grammar for *L*.

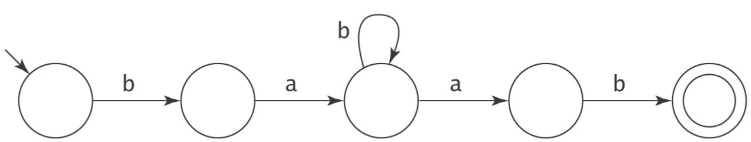
2. Exhibit a PDA for *L*.

3. Use the closure properties of context-free languages.

   Unfortunately, these are weaker than they are for regular languages.

Showing that *L* is Not Context-Free

Remember the pumping argument for regular languages:



A Review of Parse Trees

A parse tree, derived by a grammar $G = (V, \Sigma, R, S)$, is a rooted, ordered tree in which:

- Every leaf node is labeled with an element of $\Sigma \cup \{\varepsilon\}$,

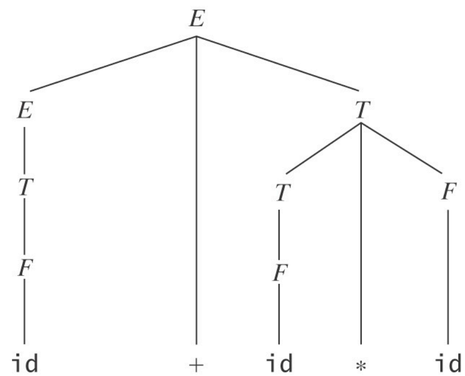
- The root node is labeled S,

- Every other node is labeled with some element of $V - \Sigma$,

- If m is a nonleaf node labeled X and the children of m are labeled $x_1, x_2, \ldots, x_n$, then the rule $X \rightarrow x_1, x_2, \ldots, x_n$ is in R.

Example

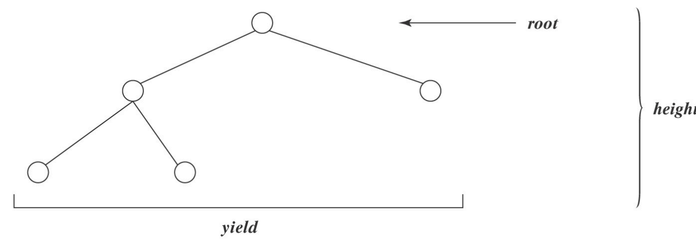
$E \rightarrow E + T$
$E \rightarrow T$
$T \rightarrow T * F$
$T \rightarrow F$
$F \rightarrow (E)$
$F \rightarrow \text{id}$



Some Tree Basics

The ***height*** of a tree is the length of the longest path from the root to any leaf.

The ***branching factor*** of a tree is the largest number of sons associated with any node in the tree.



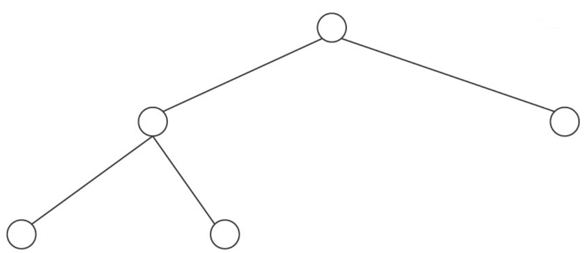
Theorem: The length of the yield of any tree T with height h and branching factor b is $\leq b^h$.

From Grammars to Trees

Given a context-free grammar G:

- Let n be the number of nonterminal symbols in G.
- Let b be the branching factor of G

Suppose that T is generated by G and no nonterminal appears more than once on any path:



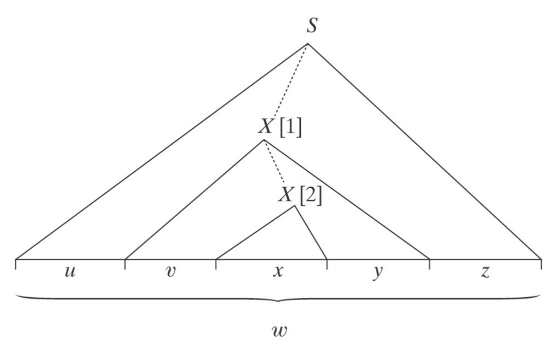
The maximum height of T is:

The maximum length of T's yield is:

The Context-Free Pumping Theorem

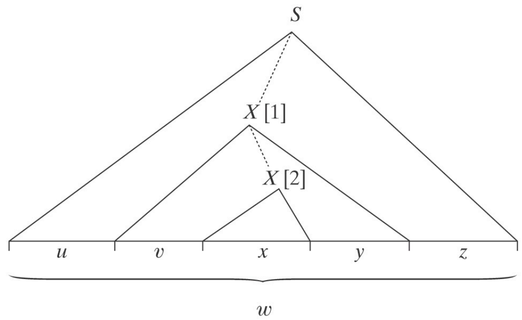
This time we use parse trees, not automata as the basis for our argument.

If w is "long", then its parse trees must look like:



Choose one such tree such that there's no other with fewer nodes.

The Context-Free Pumping Theorem



We have the derivations:

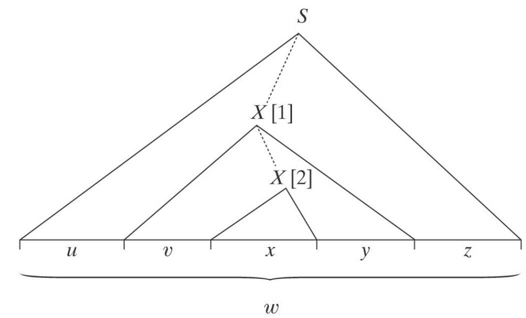
$S \Rightarrow^* uXz \Rightarrow^* uxz \in L(G)$

$S \Rightarrow^* uXz \Rightarrow^* uvXyz \Rightarrow^* uvvXyyz \Rightarrow^* uvvxyyz \in L(G)$

We can similarly derive all the strings: uv^2xy^2z, uv^3xy^3z, … $\in L(G)$

Thus: $\forall q \geq 0,\ uv^qxy^qz \in L(G)$

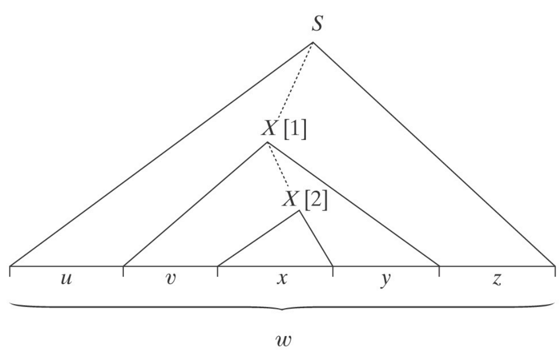
The Context-Free Pumping Theorem



***vy* ≠ ε**

Proof: If $vy = \varepsilon$, then the derivation $S \Rightarrow^* uXz \Rightarrow^* uxz$ would also yield w and it would create a parse tree with fewer nodes. But that contradicts the assumption that we started with a tree with the smallest possible number of nodes.

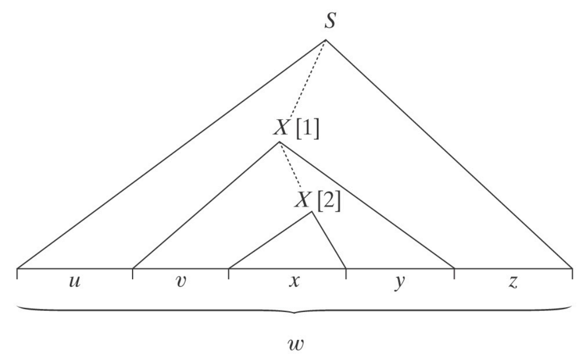
The Context-Free Pumping Theorem



The height of the subtree rooted at [1] is at most: $n + 1$

So $|vxy| \leq b^{n+1}$.

The Context-Free Pumping Theorem



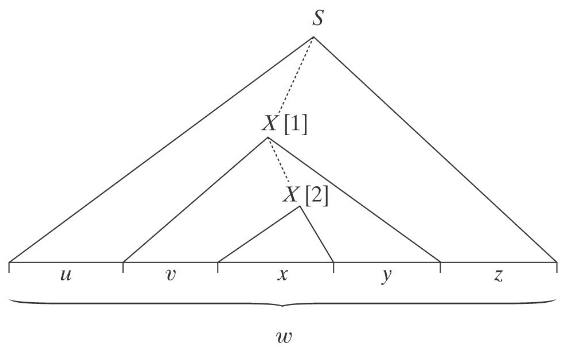
If L is a context-free language, then $\exists k \geq 1$ such that any $w \in L$ with $|w| \geq k$ can be written as $w = uvxyz$, for some $u, v, x, y, z \in \Sigma^*$, such that

- $vy \neq \varepsilon$,
- $|vxy| \leq k$ and
- $\forall q \geq 0,\ uv^qxy^qz \in L$

What Is k?

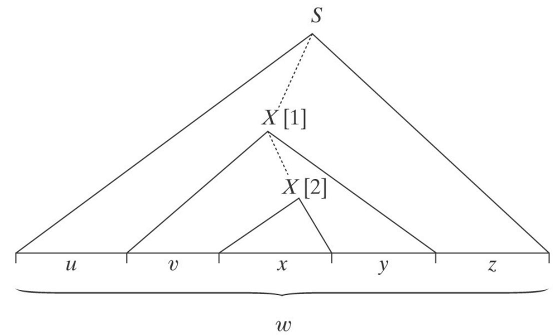
k serves two roles:

- How long must w be to guarantee it is pumpable?
- What's the bound on $|vxy|$?



Let n be the number of nonterminals in G.
Let b be the branching factor of G.

How Long Must w be?

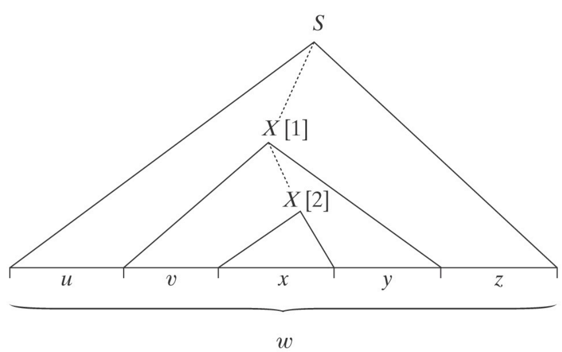


If $height(T) > n$, then some nonterminal occurs more than once on some path. So T is pumpable.

If $height(T) \leq n$, then $|uvxyz| \leq b^n$.

So if $|w| = |uvxyz| > b^n$, $w = uvxyz$ must be pumpable.

What's the Bound on $|vxy|$?



Assume that we are considering the bottom-most two instances of a repeated nonterminal. Then the yield of the upper one has length at most b^{n+1}. That is, $|vxy| \leq b^{n+1}$.

So, we need $k = \max(b^n, b^{n+1})$; let $\mathbf{k = b^{n+1}}$.

An Example of Pumping: $A^nB^nC^n$

$A^nB^nC^n = \{a^nb^nc^n, n \geq 0\}$

Choose $w = a^k b^k c^k = uvxyz$
1 | 2 | 3

If v or y spans regions, then for $q = 2$ the order of the letters changes.

If v and y each contain only one letter, then for q to 2 the number of letters are not the same.

An Example of Pumping: $\{a^{n^2}: n \geq 0\}$

$L = \{a^{n^2}: n \geq 0\}$.

For $n = k^2$ we have $w = a^{k^4}$ = uvxyz

$vy = \mathtt{a}^p$, for some nonzero p.

For q=2, a^{k^4+p} must be in L but it is too short.

The next longer string in L, for $n = k^2+1$, is $a^{(k^2+1)^2} = a^{k^4+2k^2+1}$

That means, $p = 2k^2+1$ but $p = |vy| \leq k$.

Nested and Cross-Serial Dependencies

PalEven = $\{ww^R : w \in \{\mathtt{a}, \mathtt{b}\}^*\}$

a a b b a a

The dependencies are nested – is context-free

WW = $\{ww : w \in \{\mathtt{a}, \mathtt{b}\}^*\}$

a a b a a b

Cross-serial dependencies – is not context-free

WW = $\{ww : w \in \{\mathtt{a}, \mathtt{b}\}^*\}$

Let $\mathtt{a}^{k+1}\mathtt{b}^{k+1}\mathtt{a}^{k+1}\mathtt{b}^{k+1} = uvxyz$

For $q=0$, we have that $uxz \in$ WW, that is, $uxz = ww$ for some w.

Observe that uxz still has the shape $\mathtt{a}^+\mathtt{b}^+\mathtt{a}^+\mathtt{b}^+$ (because $|vxy| \leq k$ so deleting v and y cannot remove more than k symbols). Moreover, only one or two (adjecent) same-letter regions are affected. Also, $uxz = ww$ means that w must have the shape $\mathtt{a}^+\mathtt{b}^+$.

If one region only is decreased, then ww is not in WW.

If two (adjacent) regions are decreased, then ww is not in WW.

Closure Theorems for Context-Free Languages

The context-free languages are closed under:

- Union
- Concatenation
- Kleene star
- Reverse

Closure Under Union

Let $G_1 = (V_1, \Sigma_1, R_1, S_1)$, and
$G_2 = (V_2, \Sigma_2, R_2, S_2)$.

Assume that G_1 and G_2 have disjoint sets of nonterminals, not including S.

Let $L = L(G_1) \cup L(G_2)$.

We can show that L is CF by exhibiting a CFG for it:

$$G = (V_1 \cup V_2 \cup \{S\}, \Sigma_1 \cup \Sigma_2, R_1 \cup R_2 \cup \{S \rightarrow S_1, S \rightarrow S_2\}, S)$$

Closure Under Concatenation

Let $G_1 = (V_1, \Sigma_1, R_1, S_1)$, and
$G_2 = (V_2, \Sigma_2, R_2, S_2)$.

Assume that G_1 and G_2 have disjoint sets of nonterminals, not including S.

Let $L = L(G_1)L(G_2)$.

We can show that L is CF by exhibiting a CFG for it:

$$G = (V_1 \cup V_2 \cup \{S\}, \Sigma_1 \cup \Sigma_2, R_1 \cup R_2 \cup \{S \rightarrow S_1\ S_2\}, S)$$

Closure Under Kleene Star

Let $G = (V, \Sigma, R, S_1)$.

Assume that G does not have the nonterminal S.

Let $L = L(G)^*$.

We can show that L is CF by exhibiting a CFG for it:

$$G = (V_1 \cup \{S\}, \Sigma_1, R_1 \cup \{S \rightarrow \varepsilon, S \rightarrow S\ S_1\}, S)$$

Closure Under Reverse

$L^R = \{w \in \Sigma^* : w = x^R \text{ for some } x \in L\}$.

Let $G = (V, \Sigma, R, S)$ be a context-free grammar.

Every rule in G is of the form $X \rightarrow \alpha$, for $\alpha \in V^*$.

Construct, from G, a new grammar G', such that $L(G') = L^R$:
$G' = (V_G, \Sigma_G, R', S_G)$, where R' is constructed as follows:

- For every rule $X \rightarrow \alpha$ in G, add $X \rightarrow \alpha^R$.

Closure Under Intersection

The context-free languages are not closed under intersection:

The proof is by counterexample. Let:

$L_1 = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^m : n, m \geq 0\}$ /* equal a's and b's.
$L_2 = \{\mathtt{a}^m\mathtt{b}^n\mathtt{c}^n : n, m \geq 0\}$ /* equal b's and c's.

Both L_1 and L_2 are context-free, since there exist straightforward context-free grammars for them.

But now consider:

$$L = L_1 \cap L_2 = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n : n \geq 0\}$$

Closure Under Complement

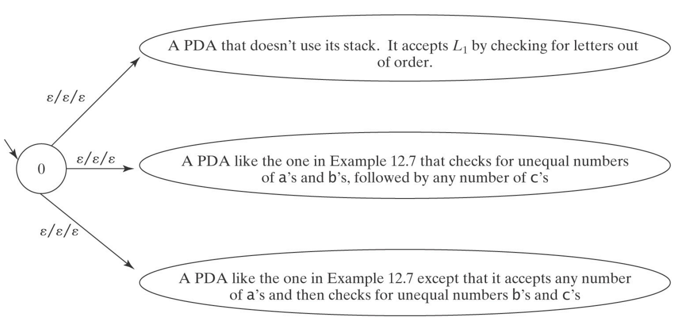
The context-free languages are not closed under complement:

$$L_1 \cap L_2 = \neg(\neg L_1 \cup \neg L_2)$$

The context-free languages are closed under union, so if they were closed under complement, they would be closed under intersection (which they are not).

Closure Under Complement An Example

$\neg A^nB^nC^n$ is context-free:



But $\neg(\neg A^nB^nC^n) = A^nB^nC^n$ is not context-free.

Closure Under Difference

The context-free languages are not closed under difference:

$$\neg L = \Sigma^* - L.$$

Σ^* is context-free. So, if the context-free languages were closed under difference, the complement of any context-free language would necessarily be context-free. But we just showed that that is not so.

The Intersection of a Context-Free Language and a Regular Language is Context-Free

$L = L(M_1)$, a PDA = $(K_1, \Sigma, \Gamma_1, \Delta_1, s_1, A_1)$.
$R = L(M_2)$, a deterministic FSM = $(K_2, \Sigma, \delta, s_2, A_2)$.

We construct a new PDA, M_3, that accepts $L \cap R$ by simulating the parallel execution of M_1 and M_2.

$M = (K_1 \times K_2, \Sigma, \Gamma_1, \Delta, (s_1, s_2), A_1 \times A_2)$.

Insert into Δ:

For each rule $((q_1, a, \beta), (p_1, \gamma))$ in Δ_1,
and each rule (q_2, a, p_2) in δ,
$(((q_1, q_2), a, \beta), ((p_1, p_2), \gamma))$.

For each rule $((q_1, \varepsilon, \beta), (p_1, \gamma)$ in Δ_1,
and each state q_2 in K_2,
$(((q_1, q_2), \varepsilon, \beta), ((p_1, q_2), \gamma))$.

This works because: we can get away with only one stack.

The Difference between a Context-Free Language and a Regular Language is Context-Free

Theorem: The difference $(L_1 - L_2)$ between a context-free language L_1 and a regular language L_2 is context-free.

Proof: $L_1 - L_2 = L_1 \cap \neg L_2$.

If L_2 is regular then so is $\neg L_2$.

If L_1 is context-free, so is $L_1 \cap \neg L_2$.

Using intersection with a regular language

$L = \{w \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^* : \#_{\mathtt{a}}(w) = \#_{\mathtt{b}}(w) = \#_{\mathtt{c}}(w)\}$

If L were context-free, then $L' = L \cap \mathtt{a}^*\mathtt{b}^*\mathtt{c}^*$ would also be context-free.

But $L' = \mathtt{A}^n\mathtt{B}^n\mathtt{C}^n$

So neither is L.