STATS 2141 REVIEW

- sample space (Ω): a set consisting of all the possible experimental outcomes
- an event is the subset of a sample space (ω)
  - ↳ is equal to 2^n, where n is # of things in sample space
- probability is the measure of how likely an event is to occur
  - ↳ to measure probability we just divide the number of objects of our intrest by the total number
- the addition rule:
  1. when two events A and B, are mutually exclusive, the probability that A or B will occur is the sum of the probability of each event $P(A \cup B) = P(A) + P(B)$

     (two events are mutually exclusive if they cannot occur at the same time - or, disjoint, which means probability of them both occuring at the same time is 0)
  2. when two events A and B, are non-mutually exclusive, the probability that A or B will occur is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- a random variable is a function that associates a real number to each outcome of an experiment
  - ↳ it transforms events (ω) from a sample space (Ω) into numerical values
  - ↳ not random or a variable
- distribution function always goes up
- reliability function always goes down
- a probability mass function is a function that gives the probability that a discrete random variable is exactly equal to some value
- Bernoulli random variable is a random variable that just takes on two values, 0 and 1
  - ↳ PMF: $\mathbb{P}(X=x) = p^x(1-p)^{1-p}$
  - ↳ expectation: $\mathbb{E}(X) = p$
  - ↳ variance: $Var(x) = p(1-p)$
  - ↳ use this to find probability of any process that only has two possible outcomes

Bernoulli expectation and variance proof:

- Bernoulli outcomes of a Bernoulli random variable are typically labelled 0 and 1, and the random variable is defined by the parameter p, $0 \leq p \leq 1$, which is the probability that the outcome is 1

therefore, the expectation is

$$\mathbb{E}[X] = (0 \times P(X=0)) + (1 \times P(X=1))$$
$$= (0 \times (1-p)) + (1 \times p)$$
$$= p$$

also since,

$$\mathbb{E}[X^2] = (0^2 \times P(X=0)) + (1^2 \times P(X=1)) = p$$

the variance is $Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = p - p^2 = p(1-p)$

- a binomial random variable is a specific type of discrete random variable that counts how often a particular event occurs in a fixed number of tries or trials
  - ↳ PMF: $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

    n = trials, k = successes
  - ↳ expectation: $\mathbb{E}[X] = np$
  - ↳ variance: $Var(X) = np(1-p)$

→ can be used under the conditions that:

1. there are a fixed number of trials (fixed sample size)
2. on each trial, the event either occurs or does not
3. probability of occurance (or not) is the same each trial
4. trials are independent of one another

(variance is a positive quantity that measures the spread of the distribution of the random variable about its mean value. Larger values of the variance indicate that the distribution is more spread out)

- variance and expectation proof:

since the Bernoulli random variables each have an expectation p, the expectation of a $B(n,p)$ random variable is calculated to be

$$E(X) = E(X_1) + \ldots + E(X_n) = p + \ldots + p = np$$

also because the Bernoulli random variables each have a variance of $p(1-p)$ and are independent, the variance of a $B(n,p)$ random variable is

$$Var(X) = Var(X_1) + \ldots + Var(X_n)$$
$$= p(1-p) + \ldots + p(1-p)$$
$$= np(1-p)$$

- poisson random variable is a random variable that counts the total number of occurances or events that occur within certain specified boundaries (during a given time period)
  - ↳ pmf: $P(X=K) = e^{-\lambda} \lambda^{K} / K!$
    - λ = expected number (positive and real) of occurances
    - K = number of occurances of an event
- pmf graph resembles bar graph
- a probability density function $f(x)$ defines the probalistic properties of a continuous random variable
  - ↳ it must satisfy $f(x) \geq 0$ and $\int_{\text{statespace}} f(x)\,dx = 1$
  - ↳ the probability that the random variable lies between two values is obtained by integrating the probability density function between the two values
- an exponential random variable is a variable that yeilds the probability of an event occuring over a continuous interval
  - ↳ PDF: $f_x(x|\lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$ where $\lambda > 0$ is called the rate of distribution
  - ↳ distribution function: $F(x) = \int_0^x \lambda e^{-\lambda y}\,dy = 1 - e^{-\lambda x}$ for $x \geq 0$
  - ↳ reliability function: $R(t) = e^{-\lambda t}$
  - ↳ hazard rate function: $h(t) = \frac{F(t)}{r(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$
    - implies that any two components that are operating successfully are equally likely to suddenly fail, regardless of how much older one component is than the other
    - can be interpreted as the chance that a component that has not failed by time t suddenly fails.
- expected value of an exponential random variable
  - Let X be a continuous random variable with an exponential density function with parameter K
  - integrating by parts with $u = Kx$ and $dv = e^{-Kx}dx$ so that $du = K\,dx$ and $v = \frac{-1}{K}e^{-Kx}$, we find

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} Kxe^{-Kx}\,dx$$
$$= \lim_{r\to\infty} \left[-xe^{-Kx} - \tfrac{1}{K}e^{-Kx}\right]\Big|_0^r$$
$$= \frac{1}{K}$$

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- expected value of an exponential random variable

  Let X be a continuous random variable with an exponential density function with parameter k

  integrating by parts with $u = kx$ and $dv = e^{-kx}dx$ so that $du = kdx$ and $v = -\frac{1}{k}e^{-kx}$, we find

  $$E(x) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} kxe^{-kx} dx$$
  $$= \lim_{r\to\infty} [-xe^{-kx} - \tfrac{1}{k}e^{-kx}]\Big|_0^r$$
  $$= \frac{1}{k}$$

- variance of exponential random variables

integrating by parts with $u = Kx^2$ and $dv = e^{-Kx}dx$ so that $du = 2Kx\,dx$ and $v = \frac{-1}{K}e^{-Kx}$, we have

$$\int_0^\infty x^2 e^{-Kx}\,dx = \lim_{r\to\infty}\left([-x^2e^{-Kx}]\Big|_0^r + 2\int_0^r xe^{-Kx}\,dx\right)$$

$$= \lim_{r\to\infty}\left([-x^2e^{-Kx} - \frac{2}{K}xe^{-Kx} - \frac{2}{K^2}e^{-Kx}]\Big|_0^r\right)$$

$$= \frac{2}{K^2}$$

so, $var(x) = \frac{2}{K^2} - E(x)^2 = \frac{2}{K^2} - \frac{1}{K^2} = \frac{1}{K^2}$

(expected value or expectation of a continuous random variable provides a summary measure of the average value taken by the random variable, and it is also known as the mean of the random variable)

- a distribution function (also called the cumulative distribution function or cumulative frequency function) describes the probability that a random variable takes on a value less or equal to a number)
- the reliability function is the probability that an object of intrest will survive beyond a specified time
- the hazard rate is the instantaneous (conditional) failure rate
- five number summary:
  - gives you a rough idea about what your data set will look like
  - includes: lowest value (the minimum), the highest value (the maximum) Q1 (first quartile, or 25% mark), Q3 (the third quartile, or 75% mark) and the median
- how to find 5 number summary:
  1. write your numbers in ascending order (smallest → largest)
  2. find the maximum and minimum for the data set
  3. find the median (middle number of set)
  4. place paretheses around the numbers above + below the median
  5. Q1 can be found by finding the median of the lower half of the data, Q3 can be found by finding the median of the upper half of the data
  6. write a summary found in the above steps.

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a)

b) $MRL(t) = \mu(t) = \int_0^\infty R(x|t)\,dx = \frac{1}{R(t)} \int_t^\infty R(x)\,dx$

c) Proove

$MRL(t) = \frac{1}{R(t)} \int_t^\infty (x-t)\, f(x)\,dx$.

$MRL(t) = \mathbb{E}[T-x \mid T \geq x] = \frac{\int_t^\infty (x-t)\, f(x)\,dx}{R(t)} = \frac{1}{R(t)} \int_t^\infty (x-t)\, f(x)\,dx$

d)

MRL:

$\mathbb{E}[(T-$

(44) $\mathbb{E}[X] = \mathbb{E}[X|A]\,\mathbb{P}(A) + \mathbb{E}[X|B]\,\mathbb{P}(B)$

Partial Theorem for Expectation

$$\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}(X|A_i)\,\mathbb{P}(A_i)$$

Proof: Note that $\mathbb{E}(X|A_i) = \sum_{x} x\,\mathbb{P}(X=x|A_i)$ This gives:

$$\sum_i \mathbb{E}(X|A_i)P(A_i) = \sum_i \sum_x x\,\mathbb{P}(X=x|A_i)\,\mathbb{P}(A_i)$$

interchanging order of summation (assume absolute convergence)

$$= \sum_x x \left(\sum_i \mathbb{P}(X=x|A_i)\,\mathbb{P}(A_i) \right)$$

then by the partition theorem, we have

$$= \sum_x x\,\mathbb{P}(X=x)$$

$$= \mathbb{E}(X)$$

(45) $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$

The law of Iterated Expectation:

$$\mathbb{E}[\mathbb{E}(X|Y)] = \mathbb{E}[X]$$

$$\mathbb{E}[\mathbb{E}[X|Y]] = \int \left(\int y\, f_{X|Y}(x|y)\,dy \right) f_X(x)\,dx$$

$$= \iint y\, f_{X|Y}(x|y)\, f_X(x)\,dy\,dx$$

$$= \iint y\, f_{X,Y}(x|y)\,dy\,dx \quad (\text{since } f_{X|Y}\, f_Y = f_{X,Y}$$

$$= \int y \left(\int f_{X,Y}(x,y)\,dx \right) dy$$

$$= \int y\, f_Y(y)\,dy \quad \left(\text{since } \int f_{X,Y}(x,y)\,dx = f_Y(y)\right)$$

$$= \mathbb{E}[Y]$$

c) S(

(4.2) wind farm

↳ 20 wind turbines

↳ 5 wind turbines in Location A

↳ 15 wind turbines in Location B

Mean Time To Failure, MTTF = ?

↳ MTTF, location A → 6 months

↳ MTTF, location B → 4 months

$$\underbrace{\mathbb{E}[T]}_{MTTF}, \underbrace{\mathbb{E}[T|A]}_{6}, \underbrace{\mathbb{E}[T|B]}_{4}$$

$$\begin{aligned} \mathbb{E}[T] &= \mathbb{E}[T|A]P(A) + \mathbb{E}[T|B]P(B) \\ &= \mathbb{E}[T|A](5/20) + \mathbb{E}[T|B](15/20) \\ &= 6(5/20) + 4(15/20) \\ &= 4.5 \end{aligned}$$

MTTF of the entire wind farm is 4.5 months.

(43) N : # of shoppers

X_n : amount of money that the n-th shopper spends during one week

each shopper spends on average, \$200 in one week

average # of shoppers / week = 500

S_N : total sum of money spent by all N shoppers in one week

N and X_n are independent

Average $\mathbb{E}[S_N]$ = ?

$$S_N = \sum_{i=1}^{N} x_i$$

$$\mathbb{E}[S_N] = \mathbb{E}\left[\sum_{i=1}^{N} x_i\right] = \sum_i \mathbb{E}[S_N | N = i] P(N=i)$$

$$= \sum_i \mathbb{E}[S_i | N = i] P(N=i)$$

$S_i = x_1 + x_2 + \dots x_i$

↳ $\sum_{k=1}^{i} x_k$

$$= \sum_i \mathbb{E}[S_i | N = i] P(N=i) = \sum (\mathbb{E}[x_1] + \dots + \mathbb{E}[x_i]) P(N=i)$$

$$= \sum (200\, x_i) P(N=i)$$

$$= 200 \sum_i P(N=i)$$

$$= 500\, \mathbb{E}[N]$$

$$= 500(200)$$

the

terva

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a) we do not know whether the interval covers μ

→ population mean is a constant. The probability a constant falls within a given range is 0 or 1

b) Yes, $\bar{x}$ is in the middle of the confidence interval, which is $\bar{x} \pm$ margin of error

39 a) $D = X - Y$ $\bar{d} \pm z$

$d_i = x_i - y_i$

c) when we plug in those observed, actual z's in Fo ①, we get $\hat{\sigma}_z^2$

$\therefore$ we have $\bar{z} \pm 1.96\sqrt{\frac{\hat{\sigma}_z^2}{n}}$

In this case, we want $\mathbb{E}[Y] > \mathbb{E}[X]$ because that means the new roots are working,

So, if the confidence interval covers a positive range (ex. above 0, its good)

what this confidence interval says: If we collect these pairs of data sets between the old gas mileage and new gas mileage, we can be 95% sure the difference between those averages is one of the values within the interval.

37.

Taxi			Taxi	
1	x_1		1	Y_1
2	x_2		2	Y_2
⋮	⋮		⋮	⋮
n	x_n		n	Y_n

Data are INDEPENDENT, not paired

→ From this we get $\bar{x}$ and $\bar{Y}$

→ $(\bar{x} - \bar{Y}) - \underbrace{(\mathbb{E}[X] - \mathbb{E}[Y])}_{\mathbb{E}[Z]}$

$\bar{x}$ $\hat{\sigma}_x^2 = \frac{1}{n_1}\sum_1^{n_1}(x_i - \bar{x})^2$ $\bar{Y}$ $\hat{\sigma}_y^2 = \frac{1}{n_2}\sum_1^{n_2}(Y_i - \bar{Y})^2$

our equation is therefore generalized as

$$\frac{(\bar{x} - \bar{Y}) - \overbrace{(\mathbb{E}[X] - \mathbb{E}[Y])}^{\mathbb{E}[Z]}}{\sqrt{\underbrace{\frac{\sigma_x^2}{n_1}}_{\substack{\sigma_{\bar{x}}^2 \\ \text{variance of } x}} + \underbrace{\frac{\sigma_y^2}{n_2}}_{\substack{\sigma_{\bar{y}}^2 \\ \text{variance of } y}}}}$$

Plugging in observed (actualized) values,

$$\hat{\sigma}_x^2 = \frac{1}{n_1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$\hat{\sigma}_y^2 = \frac{1}{n_2}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

and $\therefore$ a 95% confidence interval for $\mathbb{E}[X] - \mathbb{E}[Y]$ is:

$$(\bar{x} - \bar{Y}) \pm 1.96\sqrt{\frac{\hat{\sigma}_x^2}{n_1} + \frac{\hat{\sigma}_Y^2}{n_2}}$$

could also write

$$\sigma_{\bar{x}-\bar{y}}^2 = \sigma_{\bar{x}}^2 + \sigma_{\bar{y}}^2$$

(28)

b) → conservative

Formula for conservative confidence interval found online:

$$\hat{p} \pm \frac{1}{\sqrt{n}}$$

what zitikis said in class : use 0.5 for the value of p

① $\hat{p} \pm 1.96 \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

→ $0.5 \pm 1.96 \sqrt{\frac{0.5(1-0.5)}{5}} = [0.061, 0.94]$

c) $P(T \leq 10) = \frac{3}{5}$

$\hat{p} \pm \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.6 \pm \sqrt{\frac{0.6(1-0.6)}{5}} = [0.38, 0.81]$

d) conservative

same as b then...

$\hat{p} \pm 1.96 \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

→ $0.5 \pm 1.96 \sqrt{\frac{0.5(1-0.5)}{5}} = [0.061, 0.94]$

(35) → we want an estimate of reliability function at $t=10$ ($R(t)$ at $t=10$)

→ Also want a 95% confidence interval where margin of error ≤ 0.01

→ How many n for

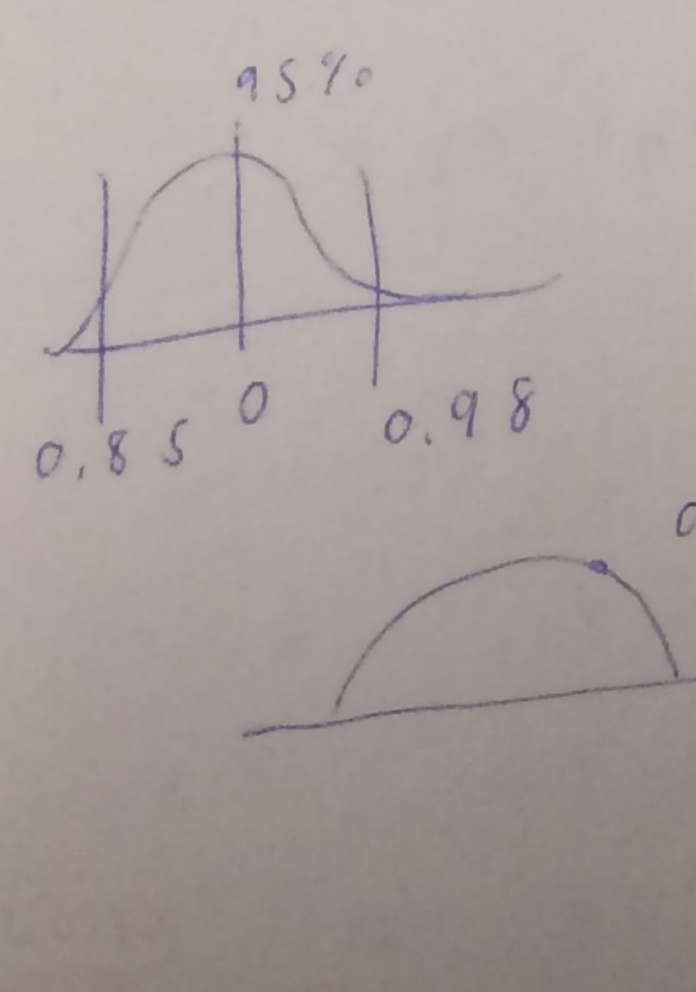
a) No additional info:

$-1.96 \frac{0.5}{\sqrt{n}} \leq 0.01$ (since $\sqrt{0.5(1-0.5)}) = 0.5$

→ we don't know the value of p so we assume the biggest value for the variance

Try : $\frac{1.96(0.5)}{0.01} \leq \sqrt{n}$ and $n = (98)^2 = 9604$

b)



so we take 0.85 for our p value

$1.96 \cdot \frac{\sqrt{0.85(1-0.85)}}{\sqrt{n}} \leq 0.01$

so, $\frac{1.96\sqrt{0.85(1-0.85)}}{0.01} \leq \sqrt{n}$

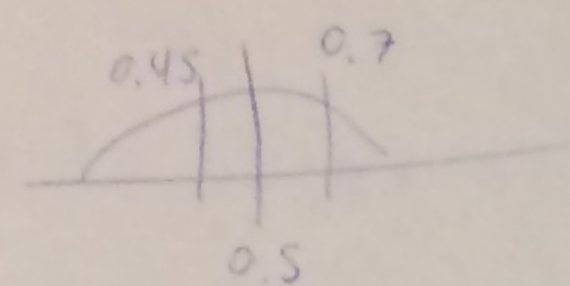
$69.98 \leq \sqrt{n}$

$4898 \leq n$

b) This is one of those trick questions I think

c) Same thing

this time for p



Let's use 0.45

$$1.96 + \frac{\sqrt{p(1-p)}}{\sqrt{n}} \le 0.01$$

$$\therefore 1.96 \frac{\sqrt{0.45(1-0.45)}}{\sqrt{n}} \le 0.01$$

and $1.96 \frac{\sqrt{0.45(1-0.45)}}{0.01} \le \sqrt{n}$ so, $97.5 \le \sqrt{n}$ and $\boxed{9507 \le n}$

36. A long question but it makes sense!

→ This question deals with estimating the difference between 2 proportions

→ $n = 10$ → we got the same set of taxi's being tested

Taxi #	First Day		Second Day
1	$(X_1$	,	$Y_1)$
2	$(X_2$	,	$Y_2)$
⋮	⋮		⋮
n	$(X_n$	,	$Y_n)$

$$\mu_{X-Y} = \mathbb{E}[X-Y]$$
$$= \mathbb{E}[X] - \mathbb{E}[Y]$$

if the confidence interval is

↳ positive (right of 0) ⇒ $\mathbb{E}[X] > \mathbb{E}[Y]$

↳ 0 (covers 0) ⇒ we know nothing!

↳ negative (left of 0) ⇒ $\mathbb{E}[X] < \mathbb{E}[Y]$

So, define Z as $X - Y$

→ we want to make a confidence interval for new RV Z

→ $\bar{Z}$, avg of those observed values

μ_Z

$$\bar{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i$$ → can calculate from $x_1\ Y_1\ \dots\ Z_1$, $x_2\ Y_2\ \dots\ Z_2$

→ we do not know the variance of Z, but we can estimate

$$\sigma_n^2 = \frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z})^2 \quad ①$$

(33) outcomes 1, 0, 0, 1 and 1

Heads → 1's
Tails → 0's

a) $\hat{p} = \frac{3}{5} = 0.6$

b) 95% (asymptotic)

$$\sigma^2 = \frac{1}{5}\left((1-0.6)^2 + (0-0.6)^2 + (0-0.6)^2 + (1-0.6)^2 + (1-0.6)^2\right)$$

$$= 0.24$$

$$\sigma = 0.49$$

$$CI = \hat{p} \pm \frac{1.96\sqrt{\hat{p}(1-\hat{p})}}{\sqrt{n}}$$

$\therefore$ we have $\left[0.6 - \frac{1.96\sqrt{0.6(1-0.6)}}{\sqrt{5}},\ 0.6 + \frac{1.96\sqrt{0.6(1-0.6)}}{\sqrt{5}}\right]$

$$= [0.171, 1.03]$$

c) conservative 95% (asymptotic)

Assume $\hat{p} = 0.5$

$$\left[0.5 - \frac{1.96\sqrt{0.5(1-0.5)}}{\sqrt{5}},\ 0.5 + \frac{1.96\sqrt{0.5(1-0.5)}}{\sqrt{5}}\right]$$

$$= [0.061, 0.938]$$

(34) → create a 95% (asymptotic) confidence interval

a) for functions without a failure for t = 10 months

$FCRM = \left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}},\ \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$, failure times 6, 8, 9, 11, 14

$P(T \geq 10) = \frac{2}{5} = 0.4$ $\therefore$ we have $\hat{p} \pm \left(\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right) = 0.4 \pm \sqrt{\frac{0.4(1-0.4)}{5}}$

so, $\left[0.4 - \sqrt{\frac{0.4(1-0.4)}{5}},\ 0.4 + \sqrt{\frac{0.4(1-0.4)}{5}}\right]$

$$= [0.18, 0.62]$$

d) This is one of those trick questions I think
→ Does 95% confidence interval cover μ or not?

$$\left[\bar{x} - 1.96 \frac{\sigma}{\sqrt{n}},\ \bar{x} + 1.96 \frac{\sigma}{\sqrt{n}} \right]$$ where $\bar{x}$ is the average of x obs.

→ we don't know whether the interval covers μ or not. Population mean is a constant. Probability a constant falls within a given range is either 0 or 1 (see set 35)

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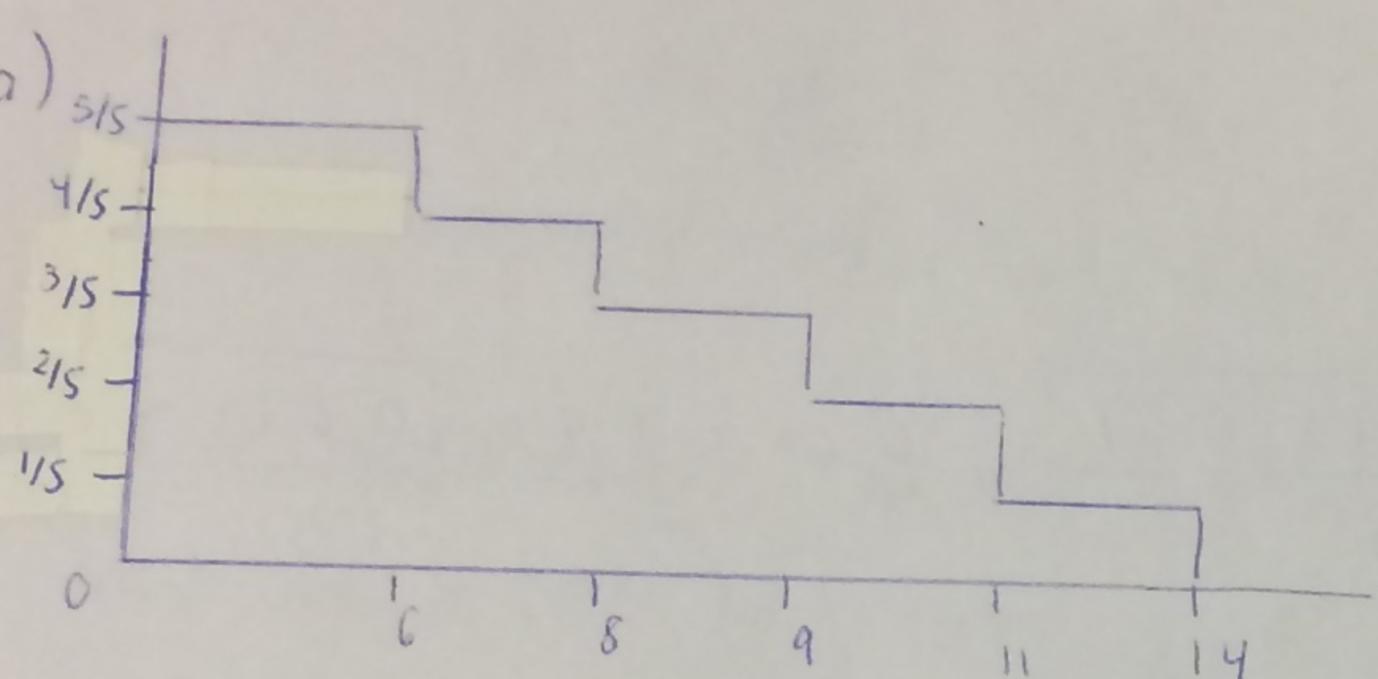
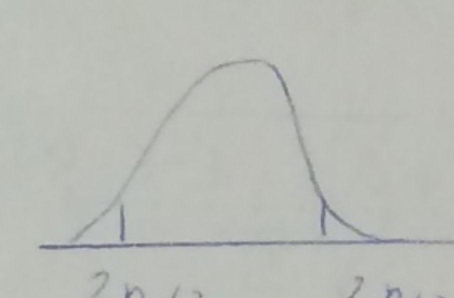
a)



b)
$A = 6(1) + 2(4/5) + 1(3/5) + 2(2/5) + 3(1/5)$
$= 9.6$

c) $\bar{t} = \dfrac{6+8+9+11+14}{5} = 9.6$

d) $\bar{t} = 9.6$



for 95% its 1.96

* confidence interval $\left(\bar{x} - z_{n/2} \frac{\sigma}{\sqrt{n}},\ \bar{x} + z_{\frac{n}{2}} - \frac{\sigma}{\sqrt{n}}\right)$

$= \left(9.6 - 1.96 \frac{\sqrt{n}}{\sqrt{5}},\ 9.6 + 1.96 \frac{\sqrt{n}}{\sqrt{5}}\right)$

$= (6.92, 12.23)$

e) sample variance $= \dfrac{(6-9.6)^2 + (8-9.6)^2 + (9-9.6)^2 + (11-9.6)^2 + (14-9.6)^2}{5}$

$\hat{\sigma}^2 = 7.44$

$\hat{\sigma} = 2.72$

f) $\left(9.6 - 1.96 \frac{\sqrt{7.44}}{\sqrt{5}},\ 9.6 + 1.96 \frac{\sqrt{7.44}}{\sqrt{5}}\right) =$

$= (7.209, 11.991)$

30 a) Prove $P(|\bar{x}-4| > \sqrt{20}\frac{\sigma}{\sqrt{n}}) \le 0.05$

→ we know $P(|\bar{x}-4| > \delta) \le \frac{Var(\bar{x})}{\delta^2}$

and

$var(\bar{x}) = \mathbb{E}[|\bar{x}-\mu|^2] = \frac{\sigma^2}{n}$

in this case, $\delta = \frac{\sqrt{20}\,\sigma}{\sqrt{n}}$ and $\delta^2 = \frac{20\sigma^2}{n}$

so, $\frac{\frac{\sigma^2}{n}}{\frac{20\sigma^2}{n}} = \frac{\cancel{\sigma^2}}{\cancel{n}} \cdot \frac{\cancel{n}}{20\cancel{\sigma^2}} = \frac{1}{20} = 0.05$

$\therefore P(|\bar{x}-4| > \delta) \le 0.05$

b) Prove $P(\bar{x}-\sqrt{20}\frac{\sigma}{\sqrt{n}} \le 4 \le \bar{x}+\sqrt{20}\frac{\sigma}{\sqrt{n}}) > 0.95$ ①

→ since $|\bar{x}-4| = |4-\bar{x}|$ and $-a \le 4-\bar{x} \le a$

and $P(A) + P(\bar{A}) = 1$,

since ① can be rewritten as $P(|\bar{x}-\mu| \le \sqrt{20}\frac{\sigma}{\sqrt{n}}) > 0.95$

and we have already proven $P(|\bar{x}-4| > \sqrt{20}\frac{\sigma}{\sqrt{n}}) \le 0.05$,

① is simply its converse

$P(|\bar{x}-4| \le \sqrt{20}\frac{\sigma}{\sqrt{n}}) > 0.95 \Leftrightarrow P(\bar{x}-\sqrt{20}\frac{\sigma}{\sqrt{n}} \le 4 \le \bar{x}+\sqrt{20}\frac{\sigma}{\sqrt{n}}) > 0.95$

c) we have proven $P(|\bar{x}-4| \le \sqrt{20}\frac{\sigma}{\sqrt{n}}) > 0.95$

$\mu - \sqrt{20}\frac{\sigma}{\sqrt{n}} \le \bar{x} \le 4 + \sqrt{20}\frac{\sigma}{\sqrt{n}} \Leftrightarrow \sqrt{20}\frac{\sigma}{\sqrt{n}} \le \bar{x}-4 \le \sqrt{20} + \frac{\sigma}{\sqrt{n}}$

$|\bar{x}-4| \le \sqrt{20}\frac{\sigma}{\sqrt{n}}$

$\therefore$ if we know

$P(|\bar{x}-4| \le \sqrt{20}\cdot\frac{\sigma}{\sqrt{n}}) > 0.95$, we have already proven it!

(31)

a) → use CLT to prove $P(|\bar{x} - \mu| > 1.96 \frac{\sigma}{\sqrt{n}}) = 0.05$

→ $S_n = x_1 + x_2 + \ldots x_n$ | → $Var(S_n) = n\sigma^2$

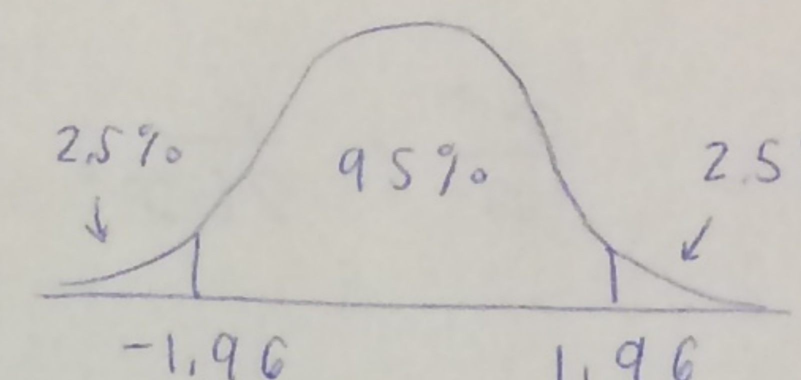
→ $\mathbb{E}[S_n] = n\mu$ | → $SD(S_n) = \sqrt{n\sigma^2} = \sigma\sqrt{n}$

so, if n is large we can say (due to the CLT)

$P(|S_n - \mathbb{E}[S_n]| \leq 1.96\, SD(S_n))$

$= P(|S_n - n\mu| \leq 1.96\, \sigma\sqrt{n})$

$= P(|\bar{x} - \mu| \leq 1.96 \frac{\sigma}{\sqrt{n}}) = 0.95$



∴ the converse, $P(|\bar{x} - \mu| > 1.96 \frac{\sigma}{\sqrt{n}}) = 0.05$ is true!

b) More algebra... so, if we know $P(|\bar{x} - \mu| \leq 1.96 \frac{\sigma}{\sqrt{n}}) = 0.95$
and we want to prove $P(\bar{x} - 1.96 \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + 1.96 \frac{\sigma}{\sqrt{n}}) = 0.95$

$= P(-1.96 \frac{\sigma}{\sqrt{n}} \leq \mu - \bar{x} \leq 1.96 \frac{\sigma}{\sqrt{n}}) = 0.95$

so, this can be written as $P(|\bar{x} - \mu| \leq 1.96 \frac{\sigma}{\sqrt{n}}) = 0.95$

c) we know $P(|\bar{x} - \mu| > 1.96 \frac{\sigma}{\sqrt{n}}) = 0.95$
(same thing as last question)

Prove $P(\mu - 1.96 \frac{\sigma}{\sqrt{n}} \leq \bar{x} \leq \mu + 1.96 \frac{\sigma}{\sqrt{n}}) = 0.95$

$P(-1.96 \frac{\sigma}{\sqrt{n}} \leq \bar{x} - \mu \leq 1.96 \frac{\sigma}{\sqrt{n}}) = 0.95$

This really seems like $-1.96 \frac{\sigma}{\sqrt{n}} \leq \bar{x} - \mu \longleftrightarrow \mu - \bar{x} \leq 1.96 \frac{\sigma}{\sqrt{n}}$

$\bar{x} - \mu \leq 1.96 \frac{\sigma}{\sqrt{n}}$

$P(|\bar{x} - \mu| \leq 1.96 \frac{\sigma}{\sqrt{n}}) = 0.95$

(30) a) Prove $P(|\bar{x} - \mu| > \sqrt{20}\,\frac{\sigma}{\ }) \le \ldots$

→ we kn

29e) using Tchebychev's inequality

$$\delta^2 \mathbb{P}(|R_n(t) - R(t)| > \delta) \le \frac{\mathbb{E}[|R_n(t) - R(t)|^2]}{n}$$

$$\mathbb{P}(|R_n(t) - R(t)| > \delta) \le \frac{\mathbb{E}[|R_n(t) - R(t)|^2]}{n\delta^2}$$

As $n \to \infty$, $\frac{\mathbb{E}[|R_n(t) - R(t)|^2]}{n\delta^2} \to 0$, and $\mathbb{P}(|R_n(t) - R(t)| > \delta) \to 0$

f) using Tchebychev's inequality

$$\delta^2 \mathbb{P}(|F_n(x) - F(x)| > \delta) \le \frac{\mathbb{E}[|F_n(x) - F(x)|^2]}{n}$$

$$\mathbb{P}(|F_n(x) - F(x)| > \delta) \le \frac{\mathbb{E}[|F_n(x) - F(x)|^2]}{n\delta^2}$$

As $n \to \infty$, $\frac{\mathbb{E}[|F_n(x) - F(x)|^2]}{n\delta^2} \to 0$, and $\mathbb{P}(|F_n(x) - F(x)| > \delta) \to 0$

BERNOULLI SD and Variance Derivation

Let X be the Bernoulli random variable. Give the definition of this random variable and derive its variance.

Bernoulli random variable is a RV that yields 1 upon success and 0 upon failure

Derive its variance:

$$\mathbb{E}(X) = (0 \times \mathbb{P}(X=0)) + (1 \times \mathbb{P}(X=1))$$
$$= (0 \times (1-p)) + (1 \times p)$$
$$= p$$

Also since,

$$\mathbb{E}(X^2) = (0^2 \times \mathbb{P}(X=0)) + (1^2 \times \mathbb{P}(X=1)) = p$$

the variance is therefore,

$$\therefore Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = p - p^2 = p(1-p)$$

if it asks standard deviation,

standard deviation is the square root of var

$$\therefore SD = \sqrt{p(1-p)}$$

27 d) $\mathbb{E}[R_n(t)] = \mathbb{E}[I_{T_1 > t}]$

$\mathbb{E}[R_n(t)] = \mathbb{P}(T_1 > t)$

$\mathbb{E}[R_n(t)] = R_1(t)$

$\mathbb{E}[R_n(t)] = R_n(t)$

e) Empirical variance $= \frac{var}{n}$

Therefore,

$var[R_1(t)] = \mathbb{P}(T_1 > t)(1 - \mathbb{P}(T_1 > t))$

$var[R_1(t)] = R_1(t)(1 - R_1(t))$

$var[R_1(t)] = R_1(t) - R_1(t)^2$

And,

$var[R_n(t)] = \frac{R_n(t) - R_n(t)^2}{n}$

28 a) $1_{AK} \begin{cases} 1, & x_K \leq x \\ 0, & \text{otherwise} \end{cases}$

$I_{A_1} = I_{T_1 \leq x}$

$I_{A_1} = \mathbb{P}(X_1 \leq x)$

$I_{A_1} = F_1(x)$

$I_{A_1} = F_n(x)$

b) $F_n(x) = \mathbb{P}(x_K > x)$

$F_n(x) = \mathbb{E}[1_{x_K > x}]$

$F_n(x) = \frac{1}{n} \sum_{K=1}^{n} I_{AK}$

c) $\mathbb{E}[F_n(x)] = \mathbb{E}[1_{x_1 > x}]$

$= \mathbb{P}(x_1 > x)$

$= F_1(x)$

$= F_n(x)$

d) Emperical var = var/n

Therefore,

$var[F_1(x)] = \mathbb{P}(\lambda_1 > x)(1 - \mathbb{P}(\lambda_1 > x))$

$= F_1(x)(1 - F_1(x))$

$= F_1(x) - F_1(x)^2$

And

$var[F_n(x)] = \frac{F_1(x) - F_1(x)^2}{n}$

29. a) since $Y > \delta$ Markov's Inequality

$$\mathbb{E}(Y) = \int_0^\infty y\,p(y)\,dy = \int_0^\delta y\,p(y)\,dy + \int_\delta^\infty y\,p(y)\,dy$$

$$\geq \int_0^\infty y\,p(y)\,dy \geq \delta \int_\delta^\infty p(y)\,dy = \delta\,\mathbb{P}(Y > \delta)$$

$$\delta\,\mathbb{P}(Y > \delta) = \mathbb{E}(Y)$$

$$\mathbb{P}(Y > \delta) = \frac{\mathbb{E}(Y)}{\delta} \quad \checkmark$$

b) Tchebychev's Inequality

$$\mathbb{P}(|Z - \mu| > \delta) \leq \frac{\sigma^2}{\delta^2}$$

$$\mathbb{P}(|Z - \mu| > \delta) = \mathbb{P}(|Z - \mu|^2 > \delta^2) \leq \frac{\mathbb{E}(Z - \mu)^2}{\delta} \leq \frac{\sigma^2}{\delta^2}$$

c) Weak Law of Large Numbers (WLLN)

$$\lim_{n \to \infty} P(|\bar{x} - \mu| > \delta) = 0$$

we make the assumption that $x_1, x_2, \ldots x_n$ have finite variance σ^2

Now,

$$\mathbb{E}[\bar{x}] = \mu$$

$$var[\bar{x}] = \frac{\sigma^2}{n}$$

By Tchebychev's Inequality

$$\mathbb{P}\{|\bar{x} - \mu| \geq \delta\} \leq \frac{\sigma^2}{n\delta^2}$$

As $n \to \infty$, $\frac{\sigma^2}{n\delta^2} \to 0$ and $\mathbb{P}\{|\bar{x} - \mu| \geq \delta\} \to 0$

(squeeze theorem)

d) using Tchebychev's Inequality,

Let $\bar{p} = \frac{1}{n} \sum_{k=1}^{n} (p_k)$

$$\delta^2\,\mathbb{P}(|\bar{p} - p| > \delta) \leq \frac{\mathbb{E}[|\bar{p} - p|^2]}{n}$$

$$\mathbb{P}(|\bar{p} - p| > \delta) \leq \frac{\mathbb{E}[|\bar{p} - p|^2]}{n\delta^2}$$

As $n \to \infty$, $\frac{\mathbb{E}[|\bar{p} - p|^2]}{n\delta^2} \to 0$, and $\mathbb{P}(|\bar{p} - p| > \delta) \to 0$

(25) a) $\mathbb{E}[\bar{x}] = \mu$

$$\mathbb{E}[\bar{x}] = \mathbb{E}[(x_1 + \dots + x_n)/n]$$
$$= \mathbb{E}[x_1 + \dots + x_n]/n$$
$$= (\mathbb{E}[x_1] + \dots + \mathbb{E}[x_n])/n$$

However,

$$\mathbb{E}[\bar{x}] = (\mu_1 + \dots + \mu_n)/n$$
$$\mu = \mu_1 = \dots = \mu_n$$

Therefore,

$$\mathbb{E}[\bar{x}] = (\mu + \dots + \mu)/n$$
$$= (\mu n)/n$$
$$= \mu$$

b) $\operatorname{var}[\bar{x}] = \dfrac{\sigma^2}{n}$

$$\operatorname{var}[\bar{x}] = \operatorname{var}[(x_1 + \dots + x_n)/n]$$
$$\operatorname{var}[\bar{x}] = \operatorname{var}[x_1 + \dots + x_n]/n^2$$
$$\operatorname{var}[\bar{x}] = (\operatorname{var}[x_1] + \dots + \operatorname{var}[x_n])/n^2$$
$$\operatorname{var}[\bar{x}] = (\sigma_1^2 + \dots + \sigma_n^2)/n^2$$

However,

$$\sigma^2 = \sigma_1^2 = \dots = \sigma_n^2$$

Therefore,

$$\operatorname{var}[\bar{x}] = (\sigma^2 + \dots + \sigma^2)/n^2$$
$$\operatorname{var}[\bar{x}] = \frac{n\sigma^2}{n^2}$$
$$\operatorname{var}[\bar{x}] = \frac{\sigma^2}{n}$$

(26) $RSS = \left(\sum_{k=1}^{n} x_k^2\right) - n\bar{x}^2$

$$RSS = \sum_{i=1}^{n} (x_i - \bar{x})^2$$
$$= \sum_{i=1}^{n} (x_i^2 - 2x_i\bar{x} + \bar{x}^2)$$
$$= \sum_{i=1}^{n} x_i^2 - 2\bar{x}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \bar{x}^2$$

However $\bar{x} = (x_1 + \dots + x_n)/n$

$$n\bar{x} = x_1 + \dots + x_n$$

Therefore

$$RSS = \sum_{i=1}^{n} x_i^2 - 2\bar{x}n\bar{x} + n\bar{x}^2$$
$$= \sum_{i=1}^{n} x_i^2 - 2n\bar{x}^2 + n\bar{x}^2$$
$$= \sum_{i=1}^{n} x_i^2 - n\bar{x}^2$$

b) $\mathbb{E}[S^2] = \mathbb{E}\left[\frac{RSS}{n-1}\right]$

$$= \mathbb{E}\left[\frac{1}{n-1}\left(\left(\sum_{k=1}^{n} x_k^2\right) - n\bar{x}^2\right)\right]$$

$$= \frac{1}{n-1}\mathbb{E}\left[\left(\sum_{k=1}^{n} x_k^2\right) - n\bar{x}^2\right]$$

$$= \frac{1}{n-1}\left(\mathbb{E}\left[\sum_{k=1}^{n} x_k^2\right] - n\,\mathbb{E}[\bar{x}^2]\right)$$

$$= \frac{1}{n-1}\left(\sum_{k=1}^{n}\mathbb{E}[x_k^2] - n\,\mathbb{E}[\bar{x}^2]\right)$$

Note that,

$$\mathbb{E}[x_k^2] = \mathbb{E}[x_k]^2 + \mathrm{var}(x_k)$$

$$\mathbb{E}[x_k^2] = \mu^2 + \sigma^2$$

Therefore,

$$\mathbb{E}[S^2] = \frac{1}{n-1}\left(\sum_{k=1}^{n}(\mu^2+\sigma^2) - n\left(\mu^2 + \frac{\sigma^2}{n}\right)\right)$$

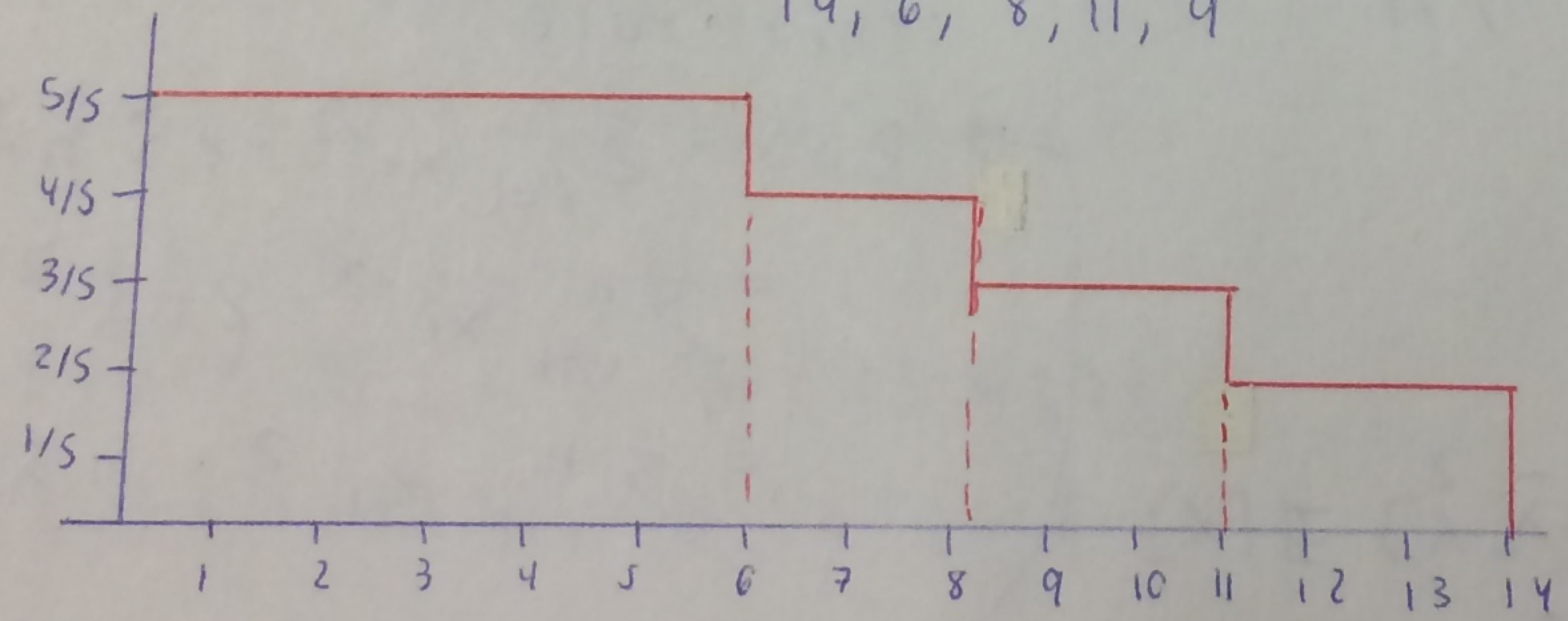
$$= \frac{1}{n-1}(n\mu^2 + n\sigma^2 - n\mu^2 - \sigma^2)$$

$$= \frac{1}{n-1}(n\sigma^2 - \sigma^2)$$

$$= \frac{\sigma^2}{\cancel{n-1}}(\cancel{n-1})$$

$$= \sigma^2$$

2.7 a) 14, 6, 8, 11, 9



b) $I_{B_k} = \begin{cases} 1, & T_k > t \\ 0, & \text{otherwise} \end{cases}$

$I_{B_1} = I_{T_1 > t}$

$I_{B_1} = \mathbb{P}(T_1 > t)$

$I_{B_1} = R_1(t)$

$I_{B_1} = R_n(t)$

c) $R_n(t) = \mathbb{P}(T_k > t)$

$= \mathbb{E}[I_{T_k > t}]$

$= \frac{1}{n}\sum_{k=1}^{n} I_{B_k}$

(25) a) $\mathbb{E}[\bar{x}] = \mu$

f) $h(t) = \dfrac{f(t)}{R(t)}$

$R(t) = 1 - F(t)$

$= 1 - (1 - \exp\{-at^b\})$

$= \exp\{-at^b\}$

$= e^{-at^b}$

$f(t) = F'(t)$

$= abt^{b-1} e^{-at^b}$

$h(t) = \dfrac{abt^{b-1} \cancel{e^{-at^b}}}{\cancel{e^{-at^b}}}$

$= abt^{b-1}$

$= (1)(2)t^{2-1}$

$= 2t^1$

(24) a) $I_c(w) = \begin{cases} 1 & \text{when } w \in C \\ 0 & \text{otherwise} \end{cases}$

~~A and B are independent~~

$\cancel{\mathbb{E}[I_A I_B] = \underbrace{\mathbb{E}[I_A]}_{P(A)} \underbrace{\mathbb{E}[I_B]}_{P(B)}}$

~~If A and B are independents so are A^c and B^c~~

$\cancel{P(A^c \cap B^c) = \mathbb{E}\{(1-I_A)(1-I_B)\}}$

$\cancel{= \mathbb{E}(1 - I_A - I_B - I_A I_B)}$

$\cancel{= 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(B)}$

$\cancel{= \mathbb{P}(A^c)\mathbb{P}(B^c)}$

24 a) $\mathbb{E}[I_A I_B] = \mathbb{P}(A \cap B)$

If A and B are independent then,

$\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B)$

therefore,

$\mathbb{E}[I_A I_B] = \mathbb{P}(A \cap B)$

$= \mathbb{P}(A) \times \mathbb{P}(B)$

$= \mathbb{E}[I_A] \times \mathbb{E}[I_B]$

18

24 b)

$$X = \sum x I\{X = x\} \qquad \mathbb{E}[X] \sum x \mathbb{P}(X = x)$$
$$Y = \sum y I\{Y = y\} \qquad \mathbb{E}[Y] = \sum y \mathbb{P}(Y = y)$$

Then,

$$XY = \sum\sum xy \, I\{X = x\} I\{Y = y\}$$
$$= \sum\sum xy \, I\{X = x ; Y = y\}$$

and,

$$\mathbb{E}(XY) = \sum\sum xy \, \mathbb{P}(X = x ; Y = y)$$
$$= \sum\sum xy \, \mathbb{P}(X = x) \mathbb{P}(Y = y)$$
$$= \left(\sum x \mathbb{P}(X = x)\right)\left(\sum y \mathbb{P}(Y = y)\right)$$
$$= \mathbb{E}(XY)$$

c) $\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

If X and Y are independent, then

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

Therefore,

$$\mathrm{Cov}(X, Y) = \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[X]\mathbb{E}[Y]$$
$$\mathrm{Cov}(X, Y) = 0$$

d) $\mathrm{Var}(X) = \mathrm{Cov}(X, X)$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]\mathbb{E}[X]$$
$$\mathrm{Var}(Y) = \mathrm{Cov}(Y, Y)$$
$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]\mathbb{E}[Y]$$
$$\mathrm{Var}(X) + \mathrm{Var}(Y) = \mathbb{E}[X^2] - \mathbb{E}[X]\mathbb{E}[X] + \mathbb{E}[Y^2] - \mathbb{E}[Y]\mathbb{E}[Y]$$
$$\mathrm{Var}(X + Y) = \mathrm{Cov}(X + Y, X + Y)$$
$$\mathrm{Var}(X + Y) = \mathbb{E}[(X + Y)(X + Y)] - \mathbb{E}[X + Y]\mathbb{E}[X + Y]$$
$$= \mathbb{E}[X^2 + Y^2 + 2XY] - \mathbb{E}[X + Y]\mathbb{E}[X + Y]$$
$$= \mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\mathbb{E}[XY] - \mathbb{E}[X + Y]\mathbb{E}[X + Y]$$
$$= \mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[X]\mathbb{E}[X] - 2\mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[Y]\mathbb{E}[Y]$$
$$= \mathbb{E}[X^2] - \mathbb{E}[X]\mathbb{E}[X] + \mathbb{E}[Y^2] - \mathbb{E}[Y]\mathbb{E}[Y]$$
$$= \mathrm{Var}(X) + \mathrm{Var}(Y)$$

d) $h(n) = \dfrac{P(T=n)}{P(n-1<T)}$

$h(n) = \dfrac{P(T=n)}{R(n-1)}$

$$\begin{aligned} P(T=n) &= h(n)R(n-1) \\ &= h(n)(1-h(n-1))R(n-2) \\ &= h(n)(1-h(n-1))(1-h(n-2))R(n-3) \\ &= h(n)(1-h(n-1))(1-h(n-2)) \times \ldots \times (1-h(1))R(1) \end{aligned}$$

There fore,

$$P(T=n) = (1-h(1))(1-h(2)) \times \ldots \times (1-h(n-1))h(n)$$

e) $$\begin{aligned} h(n) &= \frac{P(T=n)}{P(n-1<T)} \\ &= \frac{(1-c)^{n-1}}{\sum_{i=1}^{n}(1-c)^{i-1}c} \\ &= \frac{(1-c)^{n-1}}{c\sum_{i=1}^{n}(1-c)^{i-1}} \\ &= \frac{(1-c)^{n-1}}{\sum_{i=1}^{n}(1-c)^{i-1}} \\ &= \frac{\cancel{(1-c)^{n-1}}}{\cancel{(1-c)^{n-1}}\sum_{i=1}^{n-1}(1-c)^{i-1}} \\ &= \frac{1}{\sum_{i=1}^{n-1}(1-c)^{i-1}} \\ &= \frac{1}{\frac{(1-c)(1-(1-c)^{n-1-1})}{1-(1-c)}} \\ &= \frac{c}{(1-c)(1-(1-c)^{n-2})} \\ &= \frac{1/2}{(1-c)(1-(1-c)^{n-2})} \\ &= \frac{1/2}{(1-1/2)(1-(1-1/2)^{n-2})} \\ &= \frac{1/2}{1/2(1-(1/2)^{n-2})} \\ &= \frac{1}{1-(\frac{1}{2})^{n-2}} \end{aligned}$$

(23).

$$h(t) = P(t-dt < T \le t \mid t-dt < T)$$

$$h(t) = \frac{P(t-dt < T \le t \cap t-dt < T)}{P(t-dt < T)}$$

$$h(t) = \frac{P(t-dt < T \le t)}{P(t-dt < T)}$$

$$h(t) = \frac{f_T(t)}{R(t)}$$

$$h(t) = \frac{F'(t)}{R(t)}$$

$$h(t) = -\frac{R'(t)}{R(t)}$$

b) $R(t) = \exp\left\{-\int_0^t h(s)\,ds\right\}$

$$= \exp\{-\lambda t\}$$

$$= e^{-\lambda t}$$

c) $h(t) = \frac{f(t)}{R(t)}$

$$h(t)\,R(t) = f(t)$$

$$f(t) = \lambda e^{-\lambda t}$$

d) $h(t) = \frac{f(t)}{R(t)}$

$$h(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}}$$

$$h(t) = \lambda$$

e) $P(T > s+t \mid T > s) = \frac{P(T > s+t)}{P(T > s)}$

$$= \frac{P(T > s+t)}{P(T > s)}$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}}$$

$$= e^{-\lambda t}$$

$$P(T-s > t \mid T > s) = e^{-\lambda t}$$

$$= P(T > t).$$

The lack-of-memory property is a property that eliminates the significance of elapsed time. A RV that has the property does not get effected by elapsed time!

20

$n = 5$ missiles

a) $p = 0.05$

$$f_x(x) = \binom{5}{x}(0.05)^x(1-0.05)^{5-x}$$

$$E[x] = np$$
$$= 5(0.05)$$
$$= 0.25$$

b) $(p > 2 \text{ hits}) < 0.001$

$$p(5) + p(4) + p(3) < 0.001$$

$$\binom{5}{5}p^5(1-p)^0 + \binom{5}{4}p^4(1-p)^1 + \binom{5}{3}p^3(1-p)^2 < 0.001$$

$$p^5 + 5p^4(1-p) + 10p^3(1-p)^2 = 0 \qquad (1-p)(1-p) = 1-2p+p^2$$

$$p^5 + 5p^4 - 5p^5 + 10p^3 - 20p^4 + 10p^5 < 0.001$$

$$6p^5 - 15p^4 + 10p^3 < 0.001$$

$$6p^5 - 15p^4 + 10p^3 - 0.001 < 0$$

c)

d) $$f_x(x) = \sum_{j=x}^{5} \binom{5}{j} p^j (1-p)^{5-j} \binom{j}{x} p^x (1-p)^{j-x}$$

$$f_x(3) + f_x(4) + f_x(5) > 0.001$$

$$0.001 < \sum_{i=3}^{5} \sum_{j=i}^{5} \binom{5}{j} p^j (1-p)^{5-j} \binom{j}{i} p^i (1-p)^{j-1}$$

18

21.

$$P(C \mid H) = \frac{P(H \mid C)\,P(C)}{P(H)}$$

$P(H \mid C) = 0.86$

$P(C) = 0.0975$

$$P(H) = \frac{(324)(0.83) + (35)(0.86)}{359}$$

$= 0.833$

$$P(C \mid H) = \frac{(0.86)(0.0975)}{(0.833)}$$

$= 0.1007$

$\therefore$ 10% probability that the happy person is canadian.

22. a) $h(n) = P(T = n \mid n-1 < T)$

$$= \frac{P(T = n \wedge n-1 < T)}{P(n-1 < T)}$$

$$= \frac{P(T = n)}{P(n-1 < T)}$$

1

b) $h(n) = \dfrac{P(T = n)}{P(n-1 < T)}$

$$= \frac{R(n-1) - R(n)}{R(n-1)}$$

c) $h(n) = \dfrac{R(n-1) - R(n)}{R(n-1)}$

$h(n)R(n-1) = R(n-1) - R(n)$

$h(n)R(n-1) - R(n-1) = -R(n)$

$R(n) = R(n-1) - h(n)R(n-1)$

$R(n) = R(n-1)(1 - h(n))$

$R(n) = R(n-2)(1-h(n-1))(1-h(n))$

$R(n) = R(n-3)(1-h(n-2))(1-h(n-1))(1-h(n))$

$R(n) = R(0) \times \ldots \times (1-h(n-2))(1-h(n-1))(1-h(n))$

Therefore,

$$R(n) = (1-h(1))(1-h(2)) \times \ldots \times (1-h(n-1))(1-h(n))$$

d) $h(n) = \dfrac{\mathbb{P}(T = n)}{\mathbb{P}(T > n-1)}$

$h(n) = \dfrac{\mathbb{P}(T = n)}{R(n-1)}$

$\mathbb{P}(T = n) = h(n)R(n-1)$

$= h(n)(1-h(n-1))R(n-2)$

$= h(n)(1-h(n-1)) \times (1-h(n-2))R(n-3)$

$= h(n)(1-h(n-1))(1-h(n-2)) \times \ldots \times (1-h(1))R(1))$

$\mathbb{P}(T = n) = (1-h(1)) \times (1-h(2)) \times \ldots \times (1-h(n-1))\,h(n)$

c)

d)

e)

f) 2, 1 and 4 years

$\mu = \frac{2+1+4}{3} = \frac{7}{3}$

$\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2}$

$= \sqrt{\frac{1}{3} \left[\left(2 - \frac{7}{3}\right)^2 + \left(1 - \frac{7}{3}\right)^2 + \left(4 - \frac{7}{3}\right)^2 \right]}$

?!

$= 0.609$

17. a) No.

$P(\text{ent} \cap \text{eng}) = \cancel{\frac{100}{200} + \frac{150}{200} - 50}$

b) $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$P(\text{entrepreneur} | \text{eng}) = \frac{P(\text{ent} \cap \text{eng})}{P(\text{eng})}$

$= \frac{50}{200} \Big/ \frac{150}{200}$ 33.33%

$= 1/3$

c) $P(\text{eng} | \text{ent}) = \frac{50}{200} \Big/ \frac{100}{200}$ 50%

$= 1/2$

18. a) Yes

b) $P(A \cap B) = P(A) \times P(B)$

$= \left(\frac{2}{5}\right)\left(\frac{3}{4}\right)$

$= \frac{6}{20} = \frac{3}{10}$ 30%

c) $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

$= \frac{3}{10} / \frac{3}{4}$ 40%

$= 0.4$

d) $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$

$= \frac{3}{10} / \frac{2}{5}$ 75%

$= 0.75$

19. a) $P(F_T) = \frac{90}{150} = 0.6$

b) $P(F_T \mid P_S) = \frac{P(F_T \cap P_S)}{P(P_S)}$?

$= \frac{30}{150} / \frac{70}{150}$ 42.9%

$= 0.429$

c) $P(F_S) = \frac{80}{150} = 0.53$ 53.3%

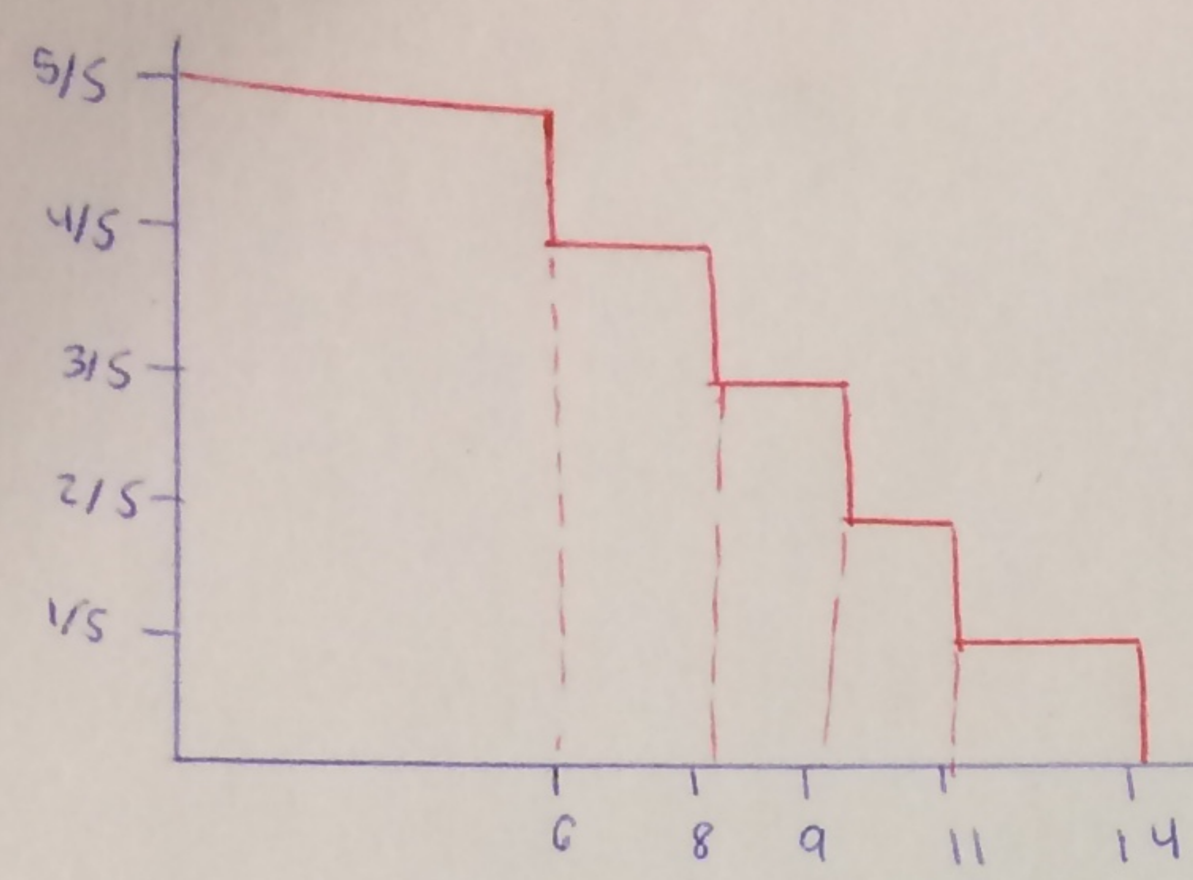
d) $P(F_S \mid P_T) = \frac{P(F_S \cap P_T)}{P(P_T)}$

$= \frac{20}{150} / \frac{60}{150}$ 33.3%

$= 0.333$

15a)

6, 8, 9, 11, 14 } failure times



$A = (5/5)(6) + (4/5)(2) + (3/5)(1) + (2/5)(2) + (1/5)(3)$

$= 9.6$

$\text{Average} = \frac{6+8+9+11+14}{5} = 9.6$

The area and average are the same

b) reliability 0.9

$\mu = np$

$= 2(0.9)$

$= 1.8$

c) $f(t) = (a-b)^{-1}$

$\mathbb{E}[t] = \int_a^b f(t)\,dt$

$= \frac{1}{a-b}\int_a^b t\,dt$

$= \frac{1}{a-b}\,\frac{t^2}{2}\Big|_a^b$

$= \frac{1}{a-b}\left[\frac{b^2}{2} - \frac{a^2}{2}\right]$

$= \frac{1}{a-b}\left[\frac{b^2-a^2}{2}\right]$

$= \frac{1}{\cancel{a-b}}\left[\frac{(\cancel{b-a})(b+a)}{2}\right]$

$= \frac{b+a}{2}$

d) $\mathbb{E}[t] = \lambda^{-1}$

$= 2^{-1}$

$= \frac{1}{2}$

e)

f)

14

16 a) $\sigma^2 = \mathbb{E}[(X-\mu)^2]$ $\mathbb{E}[X^2]-\mu^2$

$\mu = \mathbb{E}[X]$

$(x-\mu)(x-\mu)$

$x^2-2\mu x+\mu^2$ ↑ useful!

$\sigma^2 = \mathbb{E}[(X-\mu)^2]$

$= \mathbb{E}[x^2-2\mu x+\mu^2]$

$= \mathbb{E}[x^2]+\mathbb{E}[-2\mu x]+\mathbb{E}[\mu^2]$

$= \mathbb{E}[x^2]-2\mu\,\mathbb{E}[X]+\mu^2$ sub in $\mu = \mathbb{E}[x]$

$= \mathbb{E}[x^2]-2\,\mathbb{E}[x]\,\mathbb{E}[x]+(\mathbb{E}[x])^2$

$= \mathbb{E}[x^2]-(\mathbb{E}[x])^2$

$= \mathbb{E}[x^2]-\mu^2$ ↑ sub in $\mathbb{E}[x]=\mu$

b) $g(c) = \mathbb{E}[(x-c)^2]$ find the minimum!

$= \mathbb{E}[(x-c)(x-c)]$

$= \mathbb{E}[x^2-2xc+c^2]$

$= \mathbb{E}[-2x+2c]$ $\qquad \phi = -2x+2c,\ x=c$

$\therefore$ when $x=c$ it's at its minimum

13. a) The exponential random variable is a random variable that yields the probability of an event occuring over a continuous interval.

b) $f_x(x) = \{\lambda e^{-\lambda x}, \text{ for } x \geq 0\}$

c) $F(x) = 1 - e^{-\lambda x}$

d) $R(x) = 1 - F(x)$

$= 1 - 1 + e^{-\lambda x}$

$= e^{-\lambda x}$

e) $h(x) = \frac{f_x(x)}{R(x)}$

$= \frac{\lambda \cancel{e^{-\lambda x}}}{\cancel{e^{-\lambda x}}}$

$= \lambda$

f) $f_x(x) = 2e^{-2x}$

$F_x(3) - f_x(1) = 2e^{-2(1)} - 2e^{-2(3)}$

$= 2e^{-2} - 2e^{-6}$

$= 2e^{-2}(1 - e^{-4})$

$= 0.26571$

$\therefore$ The probability that T is between 1 and 3 is approx 26.57%

14. a)

min	Q_1	median	Q_3	max
0	0	1	1	1

0 1 2

1/4 1/2 1/4

0 0 1 1.1 1 2 2

b) 6 8 1 3 9 5 11 2

1 2 3 5 6 8 9 11

min	Q_1	median	Q_3	max
1	2	5	8	11

c)

min	Q_1	med	Q_3	max
0	0.25	0.5	0.75	1

14d) Find the inverse $F(x) = 1 - e^{-\lambda x}$

$$x = 1 - e^{-\lambda F^{-1}(x)}$$

$$-x = -1 + e^{-\lambda F^{-1}(x)}$$

$$1 - x = e^{-\lambda F^{-1}(x)}$$

$$\log_e(1-x) = \log_e e^{-\lambda F^{-1}(x)}$$

$$\log_e(1-x) = -\lambda F^{-1}(x) \log_e e$$

$$\frac{\log_e(1-x)}{\lambda} = -F^{-1}(x) \times 1$$

$$F^{-1}(x) = -\frac{\log(1-x)}{\lambda}$$

$$F^{-1}(x) = \frac{\log_e(1-x)^{-1}}{\lambda}$$

therefore,

$$F^{-1}(0.25) = \frac{\log_e(1-0.25)^{-1}}{1}$$

$$F^{-1}(0.25) = \log_e(0.75)^{-1}$$

$$F^{-1}(0.25) = \log_e(4/3)$$

$$F^{-1}(0.5) = \frac{\log_e(1-0.5)^{-1}}{1}$$

$$F^{-1}(0.5) = \log_e(0.5)^{-1}$$

$$F^{-1}(0.5) = \log_e\left(\frac{2}{1}\right)$$

$$F^{-1}(0.5) = \log_e(2)$$

$$F^{-1}(0.75) = \frac{\log_e(1-0.75)^{-1}}{1}$$

$$F^{-1}(0.75) = \log_e(0.25)^{-1}$$

$$F^{-1}(0.75) = \log_e\left(\frac{4}{1}\right)$$

$$F^{-1}(0.75) = \log_e(4)$$

$$F^{-1}(0.75) = \log_e(2)^2$$

$$F^{-1}(0.75) = 2\log_e(2)$$

min	Q1	median	Q3	max
0	$\log_e(4/3)$	$\log_e(2)$	$2\log_e(2)$	∞

9d)

$$P_x(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

probability of requiring repair = 0.6
probability working = 0.4

$$P_x(5) = \binom{5}{5}(0.4)^5(0.6)^{5-5}$$
$$= 1(0.4)^5$$
$$= 0.01024$$

∴ The probability that the windmills will not need repairs during the upcoming month is 1.024%.

e) At least 4 of them
↳ repair 4 or 5 of them

$$P_x(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$P_x(5) + P_x(4) = \binom{5}{5}(0.6)^5(1-0.6)^{5-5} + \binom{5}{4}(0.6)^4(1-0.6)^{5-4}$$

all five fail <-> 4 fail (need repair)

$$= 1(0.6)^5 + 5(0.6)^4(0.4)^1$$
$$= 0.33696$$

∴ The probability that at least four windmills will need repairing is 33.696%.

10. a) Poisson random variable is a random variable that yields the probability of a given number of events occuring in a fixed interval

b) $P_x(x) = \left\{ \frac{e^{-\lambda}\lambda^x}{x!}, \text{ if } x = 0, 1, 2 \ldots \right\}$

c) $P_x(x) = \frac{e^{-\lambda}\lambda^x}{x!}$

$\lambda = 1$
$x = 0$

$$P_x(0) = \frac{e^{-1}1^0}{0!}$$
$$= 0.3679$$

∴ The probability that no repair will be required during the upcoming month is approximately 36.79%

d) At least two repairs
↳ 2, 3, 4, 5 repairs
↳ easier to say 1 − (1 repair) ← 0 repair

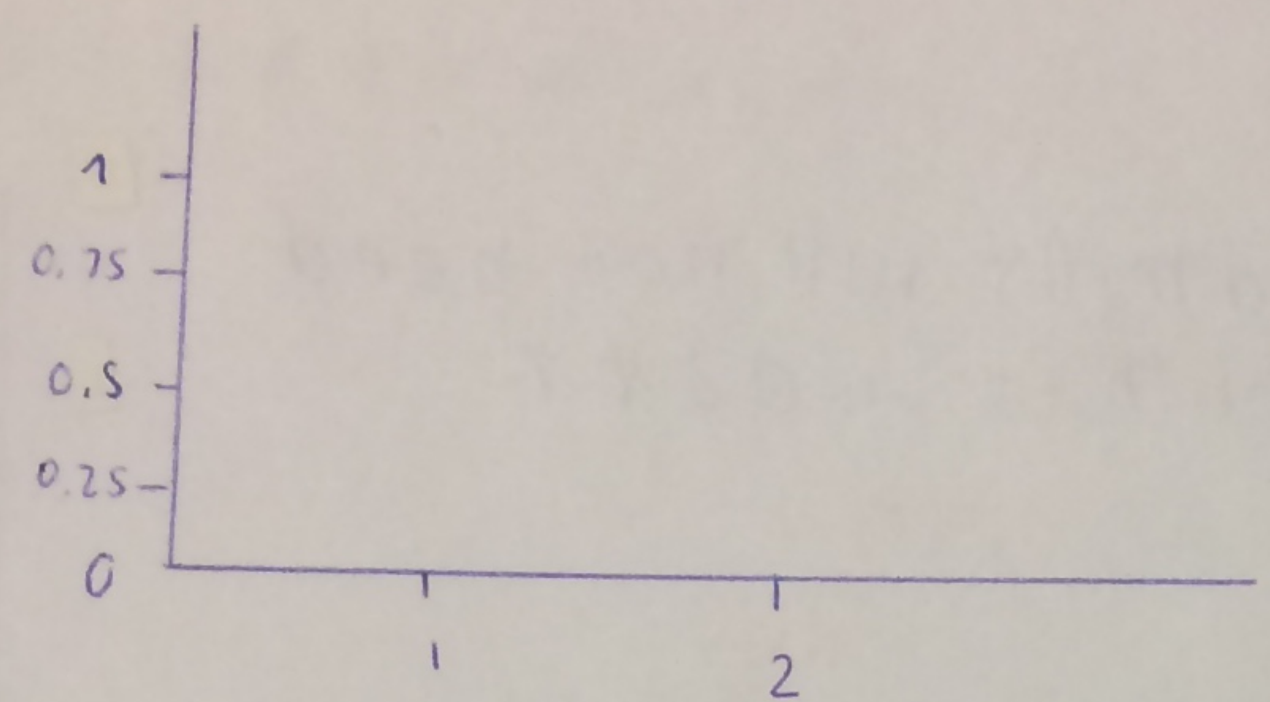
$$P_x(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$1 - P_x(0) - P_x(1) = 1 - \frac{e^{-1}1^0}{0!} - \frac{e^{-1}1^1}{1!}$$
$$= 1 - 0.3678 - 0.3678$$
$$= 0.264$$

∴ The probability is approx 26.42%

11 a) $V(x) = \binom{n}{x} p^x q^{n-x}$ # heads

$= \binom{2}{x} 0.5^x \, 0.5^{2-x}$



b) $W(x) = \binom{n}{x} p^x q^{n-x}$ # tails

$= \binom{2}{x} 0.5^x \, 0.5^{2-x}$

c) $X(x) = \binom{2}{x} 0.9^x \, 0.1^{2-x}$

d) $Y(x) = \binom{2}{x} 0.1^x \, 0.9^{2-x}$

GRAPHS still need to be drawn

12. a) The uniform random variable on [0, 1] is a random variable that is equally probable on the [0, 1] interval

b) $f_x(x) = \{ 1^{-1}, \text{ if } a \le x \le b \}$

c) $\cancel{f_x(x) = (b-a)^{-1}}$

$\cancel{= (1-0)^{-1}}$

$\cancel{= 1}$

$F(x) = \frac{(x-a)}{(b-a)}$

$= \frac{(x-0)}{(1-0)}$

$= x$

d) $\cancel{F(x) = \frac{(x-a)}{(x-b)}}$

$F(x) = \frac{(x-a)}{(b-a)}$

$R(x) = 1 - F(x)$

$= 1 - x$

e) $\text{area} = \frac{8}{b-a}$

$8 = 3/4 - 1/4$

$\text{area} = \frac{1/2}{(1-0)}$

$\text{area} = 1/2$

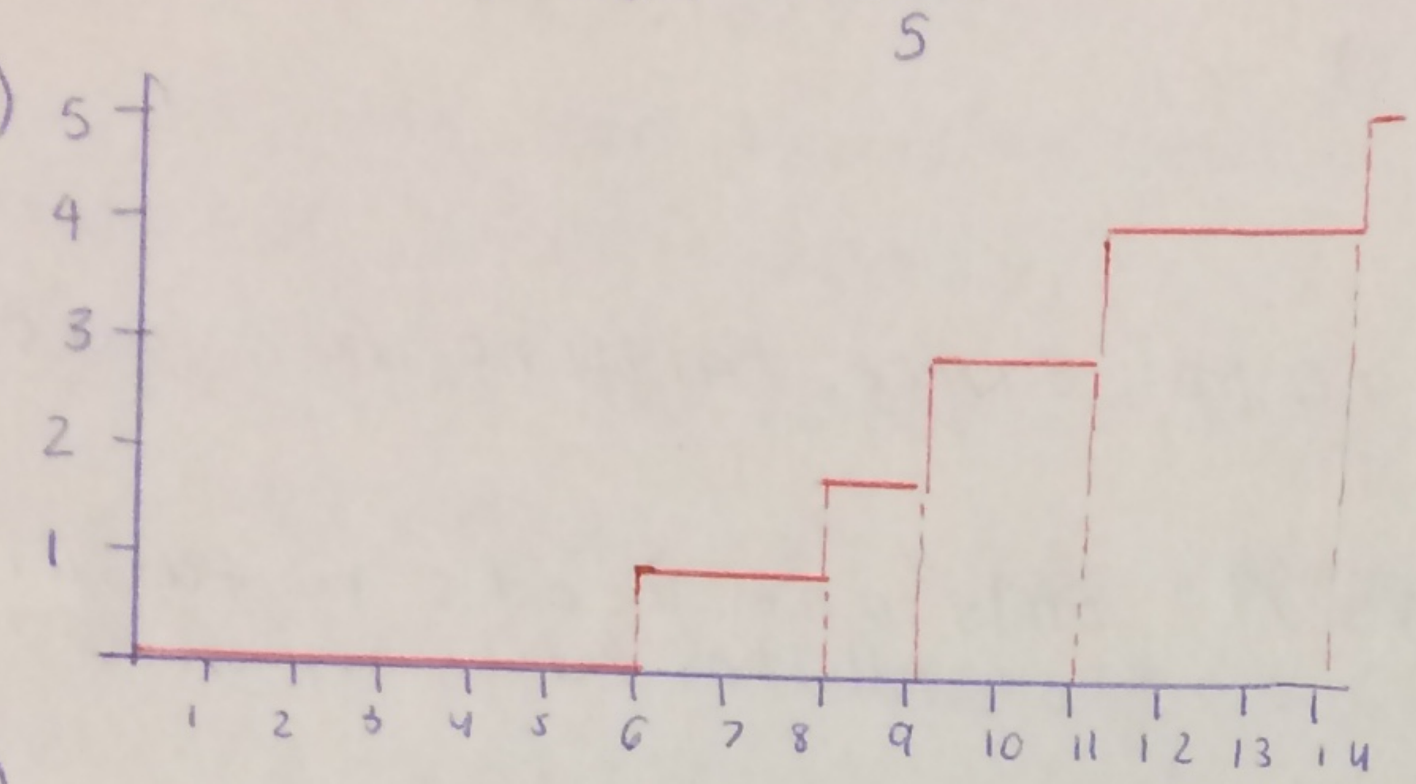
$\therefore$ the probability is 50%.

(7.) Failure times: 6, 8, 9, 11, and 14

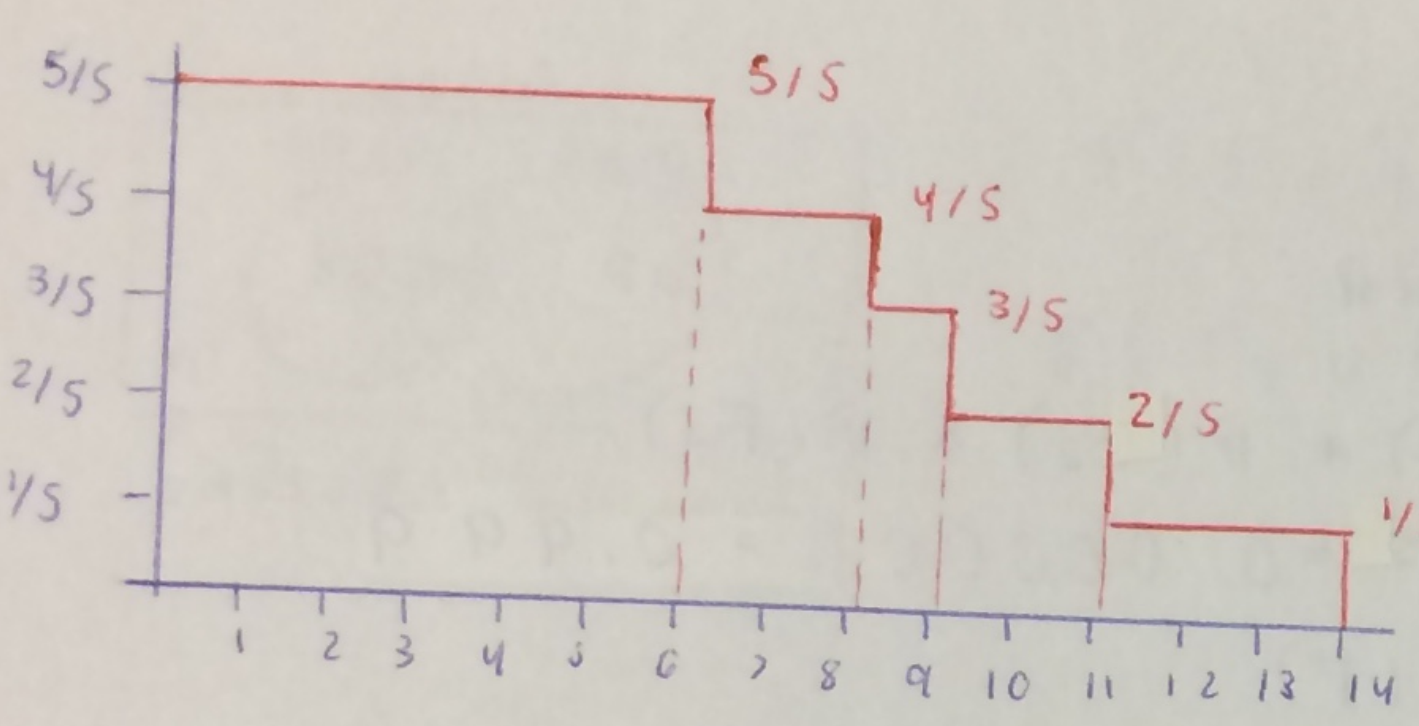
a) $\mathbb{P}(7 \le T \le 11) = \frac{3}{5}$ ← how many numbers fit

b) $\mathbb{P}(7 \le T < 11) = \frac{2}{5}$

c)



d)



(8.) a) Bernoulli random variable is a random variable that yields 1 upon success and 0 upon failure

b) $p_x(x) = \{p, \text{ if } x = 1;\ q, \text{ if } x = 0\}$ where $q = 1-p$

c) Yes.

d) Yes.

(9) a) Binomial random variable is a random variable that yields the number of successes from a sequence of n independent experiments. A binomial random variable w̄ $n = 1$ experiments is the Bernoulli random variable.

b) $p_x(x) = \{\binom{n}{x} p^x (1-p)^{n-x},\ \text{if } x = 0, 1, \dots n\}$

c) The binomial model can be used where there are $1 \le n$ experiments that have same success probability p.

also:

1. Fixed # of trials
2. on each trial the event of interest either occurs or does not occur
3. The probabability of occurence (or not) is the same on each trial
4. Trials are independent of one another

Nov 27

Stats Practice Problems Solutions

Megan Hasegawa

1. a) $\Omega = \{H, T\}$

b) $\Omega = \{W, F\}$

c) $\Omega = \{HHH, TTT, HHT, HTH, THH, TTH, THT, HTT\}$

d) $\Omega = \{WWW, FFF, WWF, WFW, FWW, FFW, FWF, WFF\}$

2. a). $\Omega = \{HH, TT, HT, TH\}$

16 events in total...

$\mathcal{F} = \{ \{\emptyset\}, \{HH\}, \{HT\}, \{TH\}, \{TT\}, \{HH, HT\}, \{HH, TH\}, \{HH, TT\}, \{HT, TH\}, \{HT, TT\}, \{TH, TT\}, \{HH, HT, TH\}, \{HH, HT, TT\}, \{HH, TH, TT\}, \{TH, HT, TT\}, \{\Omega\} \}$

b) ω is a possible outcome of Ω, while ω^{act} is an actual outcome of an experiment.

c) An event happens when $\omega^{act} \in F$

d) $A = \{WWF, WWW, WFW, FWW\}$

3. a)

230 students
200 eng (event E)
10 bus (event B)
20 sci (event S)

disjoint events!!

$$\mathbb{P}(B \cup S) = \mathbb{P}(B) + \mathbb{P}(S)$$
$$= \frac{10}{230} + \frac{20}{230}$$
$$= \frac{30}{230}$$

b) Addition Rule: If A and B are events associated $\bar{c}$ our exp. $\mathcal{E}$, and these events are disjoint (or exclusive) in that it is impossible for A and B to occur simultaneously on any performance of the experiment and if

$A \cup B$ is the event that "A happens or B happens"

~~Bette~~ New Answer:

The addition rule states that the probability of two events is the probability of the intersection of the events subtracted from the addition of the probabilities of the events.

$\mathbb{P}(A)$ can be written as

$$\mathbb{P}(A) = \mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B)$$

$\mathbb{P}(A \cup B)$ can be written as

$$\mathbb{P}(A \cup B) = \mathbb{P}(A \setminus B) + \mathbb{P}(B)$$

Therefore, we can reorder the $\mathbb{P}(A)$ equation

$$\mathbb{P}(A) - \mathbb{P}(A \cap B) = \mathbb{P}(A \setminus B)$$

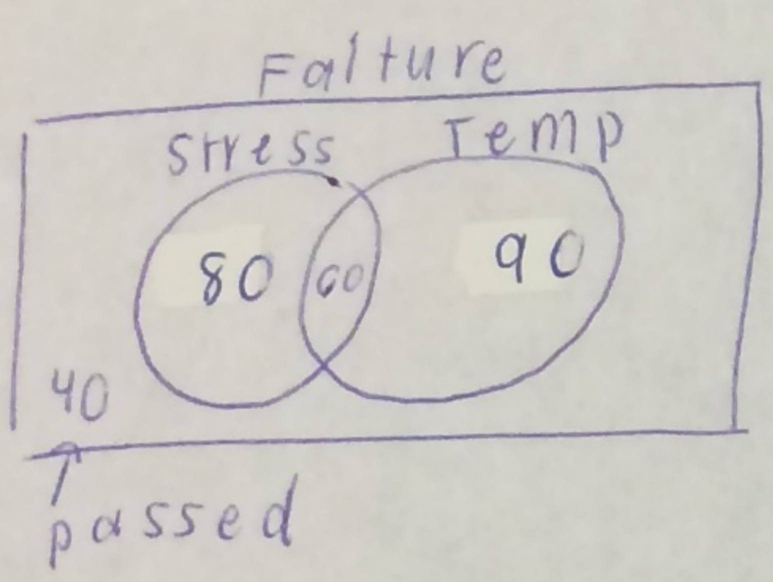
And substitute the value of $\mathbb{P}(A \setminus B)$ in the second eq.

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) - \mathbb{P}(A \cap B) + \mathbb{P}(B)$$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

Hence, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

c)



$\mathbb{P}(F_T \cap F_S) = \frac{60}{150}$ $\therefore$ probability is $\frac{6}{15}$

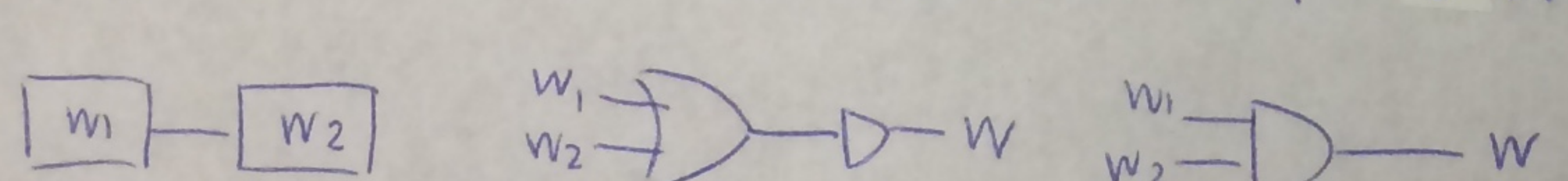
$$\mathbb{P}(F_T \cup F_S) = \mathbb{P}(F_T) + \mathbb{P}(F_S) - \mathbb{P}(F_T \cap F_S)$$

$$= \frac{90}{150} + \frac{80}{150} - \frac{60}{150}$$

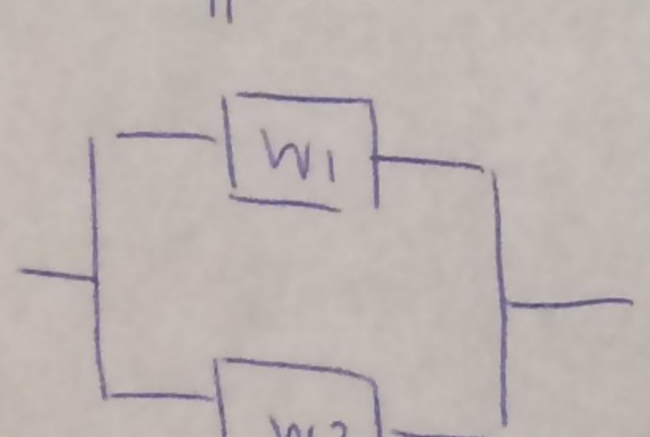
$$= \frac{110}{150}$$ $\therefore$ probability is $\frac{11}{15}$

d) If we assume that the events are independent. Then, $\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B)$. We know that the sets are disjoint, therefore, $A \cap B = \emptyset$ and $\mathbb{P}(A \cap B) = \mathbb{P}(\emptyset) = 0$, which means that $0 = \mathbb{P}(A) \times \mathbb{P}(B)$, For $\mathbb{P}(A) \times \mathbb{P}(B) = 0$ to be true, either $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$ has to be true. However, $A \neq \emptyset$ and $B \neq \emptyset$, hence, $\mathbb{P}(A) \neq 0$ and $\mathbb{P}(B) \neq 0$ therefore, $\mathbb{P}(A) \times \mathbb{P}(B) \neq 0$ then the events can not be independent, and can only be dependent.

4 a) de Morgan's rule: $(W_1 \cap W_2)^c = W_1^c \cup W_2^c$



b) another de Morgan's rule: $(W_1 \cup W_2)^c = W_1^c \cap W_2^c$



c) the distributive rule:

$$W_1 \cap (W_2 \cup W_3) = (W_1 \cap W_2) \cup (W_1 \cap W_3)$$