Physics 1501A – 4th Problem List

1. (Prob. 13.73 in Young and Freedman.) Comets travel around the sun in elliptical orbits with large eccentricities. If a comet has a speed of 2.0×10^4 m/s when at a distance of 2.5×10^{11} m from the centre of the sun, what is its speed when at a distance of 5.0×10^{10} m ?

Solution.

We can simply use the conservation of energy with

$$\begin{aligned} K_1 + U_1 &= K_2 + U_2 \\ \frac{1}{2}mv_1^2 - \frac{Gm_\mathrm{E}m}{r_1} &= \frac{1}{2}mv_2^2 - \frac{Gm_\mathrm{E}m}{r_2}. \end{aligned} \tag{1.1}$$

We find that

$$v_2^2 = v_1^2 - 2Gm_\mathrm{E}\left(\frac{1}{r_1} - \frac{1}{r_2}\right), \tag{1.2}$$

which yields $v_2 = 6.81\times10^4\,\mathrm{m/s}$.

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2. (Prob. 13.77 in Young and Freedman.) Consider a spacecraft in an elliptical orbit around the earth. At the perigee of its orbit, it is 400 km above the earth's surface; at the apogee, it is 4,000 km above the earth's surface. (a) What is the period of the spacecraft's orbit? (b) Using conservation of angular momentum, find the ratio of the spacecraft's speed at perigee to its speed at apogee. (c) Using conservation of energy, find the speed at perigee and the speed at apogee. (d) It is necessary to have the spacecraft escape from the earth completely. If the spacecraft's rockets are fired at perigee, by how much would the speed have to be increased to achieve this? What if the rockets were fired at apogee? Which point in the orbit is more efficient to use?

Solution.

(a) The semi-major axis is

$$\begin{aligned} 2a &= r_\mathrm{p} + r_\mathrm{a} \\ &= \left(R_\mathrm{E} + h_\mathrm{p}\right) + \left(R_\mathrm{E} + h_\mathrm{a}\right) \\ &= 2R_\mathrm{E} + h_\mathrm{p} + h_\mathrm{a} \end{aligned} \tag{2.1}$$

or $a = 8.58\times10^6\,\mathrm{m}$ and using equation (5.51) of the lecture notes we have

$$T^2 = \frac{4\pi^2 a^3}{Gm_\mathrm{E}}, \tag{2.2}$$

which yields $T = 7.91\times10^3\mathrm{s}$.

(b) At perigee or apogee the orbital velocity is perpendicular to the radius connecting the centre of the earth to the spacecraft. The angular momentum is then

$$\begin{aligned} L &= I\omega \\ &= mr^2\frac{v}{r} \\ &= mrv, \end{aligned} \tag{2.3}$$

which yields, because of conservation of angular momentum

$$\frac{r_\mathrm{p}v_\mathrm{p}}{r_\mathrm{a}v_\mathrm{a}} = 1, \tag{2.4}$$

or

$$\begin{aligned} \frac{v_\mathrm{p}}{v_\mathrm{a}} &= \frac{r_\mathrm{a}}{r_\mathrm{p}} \\ &= 1.53. \end{aligned} \tag{2.5}$$

(c) Since energy is conserved

$$\begin{aligned} K_1 + U_1 &= K_2 + U_2 \\ \frac{1}{2}mv_\mathrm{p}^2 - \frac{Gm_\mathrm{E}m}{r_\mathrm{p}} &= \frac{1}{2}mv_\mathrm{a}^2 - \frac{Gm_\mathrm{E}m}{r_\mathrm{a}} \end{aligned} \tag{2.6}$$

which, using equation (2.5), is transformed to

$$\begin{aligned} v_\mathrm{a}^2 - v_\mathrm{p}^2 &= 2Gm_\mathrm{E}\left(\frac{1}{r_\mathrm{a}} - \frac{1}{r_\mathrm{p}}\right) \\ v_\mathrm{a}^2\left(1 - \frac{r_\mathrm{p}^2}{r_\mathrm{a}^2}\right) &= 2Gm_\mathrm{E}\left(\frac{1}{r_\mathrm{a}} - \frac{1}{r_\mathrm{p}}\right), \end{aligned} \tag{2.7}$$

and

$$v_a^2 = 2Gm_E\left(\frac{r_a r_p}{r_p + r_a}\right). \tag{2.8}$$

We find that $v_a = 5.51\times10^3\,\text{m/s}$ and $v_p = 8.43\times10^3\,\text{m/s}$.

(d) The escape speed at perigee is such that

$$\begin{aligned} E &= \frac{1}{2}mv_p^2 - \frac{Gm_E m}{r_p} \\ &= 0, \end{aligned} \tag{2.9}$$

or

$$\begin{aligned} v_p &= \sqrt{\frac{2Gm_E}{r_p}} \\ &= 1.08\times10^4\,\text{m/s}. \end{aligned} \tag{2.10}$$

The same reasoning apply at apogee with

$$\begin{aligned} v_a &= v_p\sqrt{\frac{r_p}{r_a}} \\ &= 8.76\times10^3\,\text{m/s}. \end{aligned} \tag{2.11}$$

The increase in speed required is less at perigee ($\Delta v_p = 2.41\times10^3\,\text{m/s}$) than it is at apogee ($\Delta v_p = 3.25\times10^3\,\text{m/s}$). It is therefore efficient to use the perigee since less work will be required from the rockets.

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3. (Prob. 13.86 in Young and Freedman.) When an object is in a circular orbit of radius r around the earth (mass m_E), the period of the orbit is $T = 2\pi r^{3/2}\big/\sqrt{Gm_E}$ and the orbital speed $v = \sqrt{Gm_E/r}$. Show that when the object is moved into a circular orbit of slightly larger radius $r+\Delta r$, where $\Delta r \ll r$, its period is $T+\Delta T$ and its new orbital speed $v-\Delta v$, where Δr, ΔT, and Δv are all positive quantities, and

$$\begin{aligned} \Delta T &= \frac{3\pi\Delta r}{v} \\ \Delta v &= \frac{\pi\Delta r}{T}. \end{aligned} \tag{3.1}$$

[Hint: Use the expression $\left(1+x\right)^n \simeq 1+nx$ when $\left|x\right| \ll 1$.]

Solution.

When the object moves to the larger orbit we have

$$
\begin{aligned}
T+\Delta T &= \frac{2\pi\left(r+\Delta r\right)^{3/2}}{\sqrt{Gm_{\mathrm{E}}}} \\
&\simeq \frac{2\pi r^{3/2}\left(1+3\Delta r/2r\right)}{\sqrt{Gm_{\mathrm{E}}}} \\
&\simeq \frac{2\pi r^{3/2}}{\sqrt{Gm_{\mathrm{E}}}}+3\pi\Delta r\sqrt{\frac{r}{Gm_{\mathrm{E}}}} \\
&\simeq T+\frac{3\pi\Delta r}{v},
\end{aligned}
\tag{3.2}
$$

and

$$
\Delta T=\frac{3\pi\Delta r}{v}. \tag{3.3}
$$

Finally, we can write

$$
\begin{aligned}
v-\Delta v &= \sqrt{\frac{Gm_{\mathrm{E}}}{r+\Delta r}} \\
&= \sqrt{\frac{Gm_{\mathrm{E}}}{r}}\cdot\frac{1}{\sqrt{1+\Delta r/r}} \\
&\simeq v\cdot\left(1-\frac{\Delta r}{2r}\right) \\
&\simeq v-\frac{\Delta r\sqrt{Gm_{\mathrm{E}}}}{2r^{3/2}} \\
&\simeq v-\frac{\pi\Delta r}{T},
\end{aligned}
\tag{3.4}
$$

and

$$
\Delta v=\frac{\pi\Delta r}{T}. \tag{3.5}
$$

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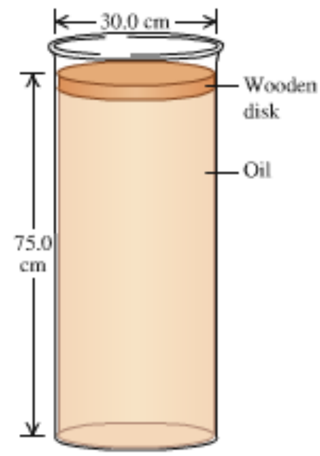
4. (Prob. 12.21 in Young and Freedman.) A cylindrical disk of wood weighing 45.0 N and having a diameter of 30.0 cm floats on a cylinder of oil of density 0.850 g/cm^3 (see the figure on the right). The cylinder of oil is 75.0 cm deep and has a diameter the same as that of the wood. (a) What is the gauge pressure at the top of the oil column? (b) Suppose now that someone puts a weight of 83.0 N on top of the wood, but no oil seeps around the edge of the wood. What is the change in pressure at (i) the bottom of the oil and (ii) halfway down in the oil?



Solution.

(a) The absolute pressure at the top of the oil column is the sum of the atmospheric pressure and the weight of the disk divided by its area. The gauge pressure is defined as the pressure relative to atmospheric pressure, we then have

$$\begin{aligned} p - p_0 &= \frac{W}{\pi R^2} \\ &= \frac{45.0\text{ N}}{\pi (0.150\text{ m})^2} \\ &= 636\text{ Pa.} \end{aligned} \tag{4.1}$$

(b) Since pressure is defined as the net applied force per area, then the increase in pressure *at any point* in the oil cylinder will equal that due to the weight placed on top of the wood

$$\begin{aligned} \Delta p &= \frac{83.0\text{ N}}{\pi (0.150\text{ m})^2} \\ &= 1170\text{ Pa.} \end{aligned} \tag{4.2}$$

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5. (Prob. 12.65 in Young and Freedman.) A piece of wood is 0.600 m long, 0.250 m wide, and 0.080 m thick. Its density is 700 kg/m^3. What volume of lead ($11{,}300\text{ kg/m}^3$) must be fastened underneath it to sink the wood in calm water so that its top is just even with the water level? What is the mass of this volume of lead?

Solution.

We denote with the subscripts "1" and "2" the densities, volumes, and masses of wood and lead, respectively. According to Archimedes' Principle the buoyant force F_b on the wood-lead object is

$$F_b = \rho_{\text{water}}(V_1+V_2)g$$
$$= (m_1+m_2)g \tag{5.1}$$
$$= (\rho_1 V_1 + \rho_2 V_2)g,$$

or

$$V_2 = V_1\left(\frac{\rho_{\text{water}} - \rho_1}{\rho_2 - \rho_{\text{water}}}\right)$$
$$= (0.600 \cdot 0.250 \cdot 0.080 \text{ m}^3)\left(\frac{1{,}000-700}{11{,}300-1{,}000}\right) \tag{5.2}$$
$$= 3.50\times10^{-4} \text{ m}^3.$$

(b) The mass of lead is simply $m_2 = \rho_2 V_2 = 3.95$ kg .

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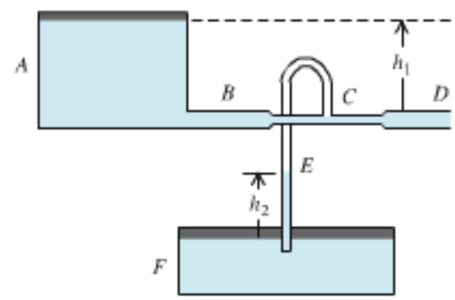
6. (Prob. 12.93 in Young and Freedman.) Two very large open tanks A and F (see the figure on the right) contain the same liquid. A horizontal pipe BCD, having a constriction at C and open to the air at D, leads out of the bottom of tank A, and a vertical pipe E opens into the constriction at C and dips into the liquid in tank F. Assume streamline flow and no viscosity. If the cross-sectional area at C is one-half the area at D and if D is a distance h_1 below the level of the liquid in A, to what height h_2 will liquid rise in pipe E? Express your answer in terms of h_1.



Solution.

Using Bernoulli's equation at the top of tank A (point 1) and at the exit D (point 2), we write

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho v_2^2, \tag{5.3}$$

or

$$v_2 = \sqrt{v_1^2 + \frac{2(p_1-p_2)}{\rho} + 2gh_1}$$
$$\simeq \sqrt{2gh_1}, \tag{5.4}$$

where the last equation arises from the facts that $p_1 = p_2 = p_{\text{atm}}$ and $v_1 \simeq 0$ (since the tanks are very large). Using the continuity equation we then have

$$\begin{aligned} v_C &= v_2 \frac{A_D}{A_C} \\ &= \sqrt{8gh_1}, \end{aligned} \tag{5.5}$$

since $A_D = 2A_C$. Applying Bernoulli's equation between the top of tank A (point 1) and point C yields

$$\begin{aligned} p_C &= p_1 - \rho\left(gh_1 - \frac{v_C^2}{2}\right) \\ &= p_1 - 3\rho gh_1. \end{aligned} \tag{5.6}$$

Also, the pressure at point E is given by

$$p_E = p_1 - \rho gh_2, \tag{5.7}$$

where p_1 is also the pressure at the top of tank F. Since the pressure is the same at points C and E, we then have $h_2 = 3h_1$.
