Assignment #4

[redacted]

Name: Zaid Albirawi

UWO email: zalbiraw@uwo.ca

1. Translate each of the following sentences into the language of the predicate calculus, using the indicated letter for predicate symbols.

   $M(x)$: "x is mysterious"
   $S(x)$: "x is a student"
   $B(x)$: "x is beautiful"
   The universe of discourse is the set of all people.

   a. All who aren't student are mysterious.

      $\forall x(\neg S(x) \to M(x))$

   b. Some students are mysterious and beautiful.

      $\exists x(S(x) \land M(x) \land B(x))$

   c. No student who is beautiful is mysterious.

      $\neg\exists x(S(x) \land B(x) \land M(x))$

   d. Not all who are mysterious or beautiful are students.

      $\neg\forall x(M(x) \lor B(x) \to S(x))$

   e. Not all students are mysterious.

      $\exists x(S(x) \land \neg M(x))$

2. Translate the following sentences in the languages of predicate calculus. Use the given predicate and function symbols. The universe of discourse is the positive integers.

   $L(x, y)$: x is less than y
   $E(x)$: x is even
   $P(x)$: x is prime
   $O(x)$: x is odd
   $Pro(x)$: x * y

   a. Every positive integer is less than some positive integer.

      $\forall x \exists y (L(x, y))$

   b. There is a positive integer with no positive integer less than it.

      $\exists x \forall y \neg (L(y, x))$

   c. 2 is even and prime and it is the only positive integer that is both even and prime.

      $\forall x \neg \big(E(x) \wedge P(x)\big), x \neq 2$

   d. The product of any pair of odd positive integers is itself odd.

      $\forall x \forall y ((O(x) \wedge O(y)) \rightarrow O(Pro(x, y)))$

   e. If either of a paint of positive integers are even, their product is even.

      $\forall x \forall y ((E(x) \vee E(y)) \rightarrow E(Pro(x, y)))$

3. For each of the following five sets of formulas, find a domain and interpretation in which the last formula is false, but the other formulas are true.

   a. Premise 1: $\exists x P(f(x), x)$
      Premise 2: $\forall x \forall y \big(Q(x) \wedge Q(y) \rightarrow P(x, y)\big)$
      Premise 3: $\exists x Q\big(f(x)\big)$
      Conclusion: $\exists x Q(x)$

      Let the universe be the set of even numbers.

      Let $Q(x)$ be true when x is an odd number.
      Let $P(x, y)$ be true when $x + y$ is an even number.
      Let $f(x) = {}^{x}/_{2}$

      Premise 1 $\exists x P(f(x), x)$: $f(0) + 0 = 0$
      Premise 2 $\forall x \forall y \big(Q(x) \wedge Q(y) \rightarrow P(x, y)\big)$, since the universe is the even numbers then, $Q(x) \wedge Q(y)$ will always result into 0, therefore, the premise will always be true.
      Premise 3 $\exists x Q\big(f(x)\big)$: $f(6) = {}^{6}/_{2} = 3$
      Conclusion: $\exists x Q(x)$: Since the universe us the set of even numbers, there does not exists a number in this set that is odd, hence the statement cannot be fulfilled.(Invalid question, f(x) is outside the domain)

   b. Premise 1: $\exists x Q(x) \rightarrow \exists x P(x)$
      Premise 2: $P(a)$
      Conclusion: $\forall x \big(Q(x) \rightarrow P(x)\big)$

      Let the universe be the set of ASCII characters.

      Let $Q(x)$ be true when x is a number.
      Let $P(x)$ be true when x is a prime number.
      Let $a = 1$

      Premise 1 $\exists x Q(x) \rightarrow \exists x P(x)$: $x = 7$, 7 is a number that is also prime therefore, true.
      Premise 2 $P(a)$: 1, the number 1 is a prime, therefore, the statement is true.
      Conclusion$\forall x \big(Q(x) \rightarrow P(x)\big)$: 4, since 4 is a number, but not a prime, the statement is false.

   c. Premise 1: $\forall x \big(R(x) \rightarrow \neg P(x)\big)$
      Conclusion: $\exists x \big(R(x) \wedge \neg P(x)\big)$

      Let the universe be the set of even integers.

Let $R(x)$ be true when, x is an odd number.
Let $P(x)$ be true when, x is an even number.

Premise 1 $\forall x\left(R(x) \rightarrow \neg P(x)\right)$: Since the universe is the set of even integers, then $R(x)$ will always be 0. Hence, premise 1 will always be true.
Conclusion $\exists x\left(R(x) \wedge \neg P(x)\right)$: With $R(x)$ equal to zero in this universe, we can conclude that there does not exist a value of x where $R(x) \wedge \neg P(x) = 1$. Therefore, the conclusion will always be false.

d. Premise 1: $\exists x \forall y Q(x, y)$
Conclusion: $\forall x Q(x, f(x))$

Let the universe be the set of positive integers including zero.

Let $Q(x, y)$ be true when, y is divisible for x.
Let $f(x) = x$

Premise 1 $\exists x \forall y Q(x, y)$: $x = 1$, all the positive numbers are divisible by 1, hence there exists a number x, where all the y values are divisible by x.
Conclusion: $\forall x Q(x, f(x))$: $x = 0$, the number 0 is not a divisor of itself. Therefore, there exists a value of x where x is not a divisible by itself.

e. Premise 1: $\forall x(R(x) \rightarrow \neg P(x))$
Premise 2: $\forall x(P(x) \rightarrow Q(x))$
Premise 3: $\exists x(R(x) \wedge \neg Q(x))$
Conclusion: $\forall x(R(x) \rightarrow \neg Q(x))$

Let the universe be the set of positive integers.

Let $R(x)$ be true when x is an even number.
Let $P(x)$ be true when x is an odd number.
Let $Q(x)$ be true when x is divisible by itself.

Premise 1 $\forall x(R(x) \rightarrow \neg P(x))$: This premise will always be true because, if x is an even number, then it can't be odd, $1 \rightarrow \neg 0$. Also if x is odd, $R(x) = 0$, therefore the statement is true.
Premise 2: $\forall x(P(x) \rightarrow Q(x))$: All odd numbers are divisible by themselves, hence, $1 \rightarrow 1$. However, if the number is not odd then $P(x) = 0$, therefore, the statement will always be true.
Premise 3: $\exists x(R(x) \wedge \neg Q(x))$: $x = 0$, since zero is an even number that is not divisible by itself. There exists a number that is even and not divisible by itself.
Conclusion: $\forall x(R(x) \rightarrow \neg Q(x))$: $x = 2$, since 2 is divisible by itself then, $Q(x) = 0$, therefore, the statement is false. Hence, there exists a number that is even and divisible by itself.

4. Translate the following argument in the language of predicate calculus choosing appropriate symbols for constants, variables, predicates, or functions. The universe of discourse is the set of all teenagers. Prove that the conclusion is a logical consequence of premises. Justify your answer.

Premise 1: Teenagers who play tennis are in good health.
Premise 2: Teenagers who play football have team spirit.
Premise 3: Anyone who has team spirit has good health.
Premise 4: All teenagers who go to college either play tennis or football.
Conclusion: If some teenager goes to college then some teenager is in good health.

Let $T(x)$ be a teenager who plays tennis.
Let $F(x)$ be a teenager who plays football.
Let $H(x)$ be a teenager who has good health.
Let $S(x)$ be a teenager who has team spirit.
Let $C(x)$ be a teenager who goes to college.

Translations:

Premise 1: $\forall x\big(T(x) \rightarrow H(x)\big)$
Premise 2: $\forall x\big(F(x) \rightarrow S(x)\big)$
Premise 3: $\forall x(S(x) \rightarrow H(x))$
Premise 4: $\forall x\left(C(x) \rightarrow \big(T(x) \vee F(x)\big) \wedge \big(\neg T(x) \vee \neg F(x)\big)\right)$
Conclusion: $\forall x\big(C(x) \rightarrow H(x)\big)$

Let the premises equal to 1 and the conclusion equal 0.

For the conclusion to be false, $C(x)$ has to equal 1, while $H(x)$ equals 0.
For premise 1 to be true, $T(x)$ must equal to 0 because we already know that $H(x)$ equals 0.
For premise 3 to be true, $S(x)$ must equal to 0 because we already know that $H(x)$ equals 0.
For premise 4 to be true, either $T(x)$ or $F(x)$ must be true, but since $T(x)$'s value is already set to be 0, then $F(x)$ has to be equal to 1.
Finally, for premise 2 to be true and knowing that $F(x)$ equal 1, $S(x)$ must be true. But since $S(x)$ is already set to be equal to 0, we reach a contradiction.

Therefore, the argument is valid.

5. Consider the following argument:

Premise 1: $\forall y \forall x\big(P(x,y) \rightarrow P(y,x)\big)$
Premise 2: $\forall z \forall y \forall x\big(P(x,y) \land P(y,z) \rightarrow P(x,z)\big)$
Conclusion: $\forall y \forall x\big(P(x,y) \rightarrow P(x,x)\big)$

Find out whether or not the argument is valid, i.e. whether or not the conclusion is a logical consequence of the premises. Justify your answer.

Assume that premises 1 and 2 are true and that the conclusion is false.

$\neg\forall y \forall x\big(P(x,y) \rightarrow P(x,x)\big) = \neg 0$
$\exists y \exists x \neg\big(P(x,y) \rightarrow P(x,x)\big) = 1$
$\exists y \exists x \neg\big(\neg P(x,y) \lor P(x,x)\big) = 1$
$\exists y \exists x\big(P(x,y) \land \neg P(x,x)\big) = 1$

Therefore there exists at least one value of x and one value of y that will make the conclusion equal to 1. Let those value be x_0 and y_0.

Therefore, $\exists y \exists x\big(P(x,y) \land \neg P(x,x)\big) = 1$
$P(x_0, y_0) = 1$
$\neg P(x_0, x_0) = 1$

Since $\forall y \forall x\big(P(x,y) \rightarrow P(y,x)\big)$, then, $P(x_0, y_0) \rightarrow P(y_0, x_0)$ has to be true. Since $P(x_0, y_0)$ is already 1, then $P(y_0, x_0)$ has to be equal to 1 for premise 1 to be true.

Furthermore, since $\forall z \forall y \forall x\big(P(x,y) \land P(y,z) \rightarrow P(x,z)\big)$, then $\forall z\big(P(x_0, y_0) \land P(y_0, z) \rightarrow P(x_0, z)\big)$. Also, having the universal quantifier allows us to choose any value for z, $\forall z\big(P(x_0, y_0) \land P(y_0, x_0) \rightarrow P(x_0, x_0)\big)$. With $z = x_0$, the premise is equal to 1 because $P(x_0, y_0) = 1, P(y_0, x_0) = 1, P(x_0, x_0) = 1$.

Finally, since there exists a set of values, y_0 and x_0, that make both premises and conclusion true, then we can conclude that the argument is valid.