Chapter 3. Conservation of Linear Momentum

Notes:

- *Most of the material in this chapter is taken from Young and Freedman, Chap. 8.*

3.1 The Impulse

We have already defined the momentum vector $\mathbf{p}$ of a body in Chapter 1 in relation to the net force $\mathbf{F}_{\text{net}}$ acting on it with

$$\mathbf{F}_{\text{net}} = \frac{d\mathbf{p}}{dt}, \tag{3.1}$$

where

$$\mathbf{p} = m\mathbf{v}. \tag{3.2}$$

That is, the force is the time derivative of the momentum. Conversely, it also follows that we can express the momentum as the anti-derivative of the force

$$\mathbf{p} = \int \mathbf{F}_{\text{net}}\, dt. \tag{3.3}$$

Let us simplify things and consider the case of a constant force is that acts on the body for the period of time $\Delta t = t_2 - t_1$. In this case we consider the change in momentum $\Delta\mathbf{p} = \mathbf{p}_2 - \mathbf{p}_1$ that takes place under the action of the force in the interval Δt. Equation (3.3) then yields

$$\begin{aligned}\Delta\mathbf{p} &= \int_{t_1}^{t_2} \mathbf{F}_{\text{net}}\, dt \\ &= \mathbf{F}_{\text{net}} \int_{t_1}^{t_2} dt \\ &= \mathbf{F}_{\text{net}}\left(t_2 - t_1\right),\end{aligned} \tag{3.4}$$

or

$$\Delta\mathbf{p} = \mathbf{F}_{\text{net}}\Delta t. \tag{3.5}$$

The **impulse** $\mathbf{J}$ is defined as this change in momentum. That is

$$\mathbf{J} \equiv \mathbf{p}_2 - \mathbf{p}_1. \tag{3.6}$$

Equation (3.6) is a mathematical statement of the so-called **impulse-momentum theorem**, which is expressed as follows

The change in momentum of a body during a time interval equals the impulse of the net force that acts on the particle during that interval.

Although we focused on the case where a constant force is at play, the same result applies in the more general situation when the force can vary in magnitude and orientation. In that case we write

$$\begin{aligned} \mathbf{J} &= \int_{t_1}^{t_2} \mathbf{F}_{\text{net}}\, dt \\ &= \int_{t_1}^{t_2} \frac{d\mathbf{p}}{dt}\, dt \\ &= \int_{t_1}^{t_2} d\mathbf{p} \\ &= \mathbf{p}_2 - \mathbf{p}_1. \end{aligned} \tag{3.7}$$

Even when the force is variable it is possible, however, to define an average force such that (following equations (3.5) and (3.7))

$$\mathbf{J} = \mathbf{F}_{\text{ave}}\left(t_2 - t_1\right). \tag{3.8}$$

It is interesting to note the similarity between in the respective forms of the work W done by a net force $\mathbf{F}_{\text{net}}$ over a displacement $\Delta\mathbf{r}$ and the impulse $\mathbf{J}$ resulting from that same force over a time interval Δt. That is,

$$\begin{aligned} W &= \int_1^2 \mathbf{F}_{\text{net}} \cdot d\mathbf{r} \\ &= K_2 - K_1 \\ \mathbf{J} &= \int_1^2 \mathbf{F}_{\text{net}}\, dt \\ &= \mathbf{p}_2 - \mathbf{p}_1. \end{aligned} \tag{3.9}$$

Just as we had that the final kinetic energy of the object equals the sum of the initial kinetic energy and the work done by the net force on the object

$$K_2 = K_1 + W, \tag{3.10}$$

the final momentum equals the sum of the initial momentum and the impulse

$$\mathbf{p}_2 = \mathbf{p}_1 + \mathbf{J}. \tag{3.11}$$

3.1.1 Exercises

1. (Prob. 8.7 in Young and Freedman.) A 0.0450 kg golf ball initially at rest is given a speed of 25.0 m/s when a club strikes it. If the club and ball are in contact for 2.00 ms,

what average force acts on the ball? Is the effect of the ball's weight during the time of contact significant? Why or why not?

Solution.

If the direction of the final velocity of the ball is along the positive x-axis, then $v_{x1} = 0$ and $v_{x2} = 25$ m/s. From equation (3.8) we have

$$\begin{aligned} F_{\text{ave}} &= \frac{mv_{x2} - mv_{x1}}{t_2 - t_1} \\ &= \frac{0.0450 \text{ kg} \cdot 25 \text{ m/s}}{2.00 \times 10^{-3} \text{ s}} \\ &= 562 \text{ N}. \end{aligned} \tag{3.12}$$

On the other hand, the weight of the ball is $w = mg = 0.0450 \text{ kg} \cdot 9.80 \text{ m/s}^2 = 0.441 \text{ N}$. The force on the ball therefore exceeds the weight of the ball by a factor of more than 1000; the weight of the ball is utterly insignificant.

2. (Prob. 8.12 in Young and Freedman.) A bat strikes a 0.145 kg baseball. Just before impact, the ball is travelling horizontally to the right at 50.0 m/s, and it leaves the bat travelling to the left at an angle of 30° above horizontal with a speed of 65.0 m/s. If the ball and bat are in contact for 1.75 ms, find the horizontal and vertical components of the average force on the ball.

Solution.

Let $x > 0$ and $y > 0$ be oriented to the right and upward. The impulse components along these directions are

$$\begin{aligned} J_x &= m\left(v_{x2} - v_{x1}\right) \\ &= 0.145 \text{ kg} \cdot \left[-65\cos\left(30^\circ\right) - 50\right] \text{ m/s} \\ &= -15.4 \text{ N} \cdot \text{s} \\ J_y &= m\left(v_{y2} - v_{y1}\right) \\ &= 0.145 \text{ kg} \cdot \left[65\sin\left(30^\circ\right) - 0\right] \text{ m/s} \\ &= 4.71 \text{ N} \cdot \text{s}. \end{aligned} \tag{3.13}$$

The corresponding components of the average force are

$$\begin{aligned} F_{\text{ave},x} &= \frac{J_x}{\Delta t} \\ &= \frac{-15.4 \text{ N}\cdot\text{s}}{1.75\times10^{-3} \text{ s}} \\ &= -8800 \text{ N} \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} F_{\text{ave},y} &= \frac{J_y}{\Delta t} \\ &= \frac{4.71 \text{ N}\cdot\text{s}}{1.75\times10^{-3} \text{ s}} \\ &= 2690 \text{ N.} \end{aligned} \tag{3.15}$$

3.2 Conservation of Linear Momentum

Given that considerations of equation (3.10) led us to the principle of conservation of energy in Chapter 2, and that equation (3.11) for the momentum and impulse has a form that is basically similar to that of equation (3.10), it is then reasonable to expect that we would also have conservation of the linear momentum (for an isolated system).

To show this we consider an *isolated system* for which all the forces involved in the dynamics of the set of particles contained in the system are internal to it. That is, **internal forces** denote interactions between particles (according to Newton's Third Law), as opposed to **external forces**, which act on the system as a whole. For example, let us assume that an isolated system is made of only two particles, which we denote as "1" and "2". According to Newton's Third Law if $\mathbf{F}_{12}$ is the net internal force that particle "1" applies on particle "2", then $\mathbf{F}_{21} = -\mathbf{F}_{12}$ and

$$\mathbf{F}_{12} + \mathbf{F}_{21} = 0. \tag{3.16}$$

If we now use Newton's Second Law as expressed in equation (3.1), then we can write

$$\begin{aligned} \mathbf{F}_{12} + \mathbf{F}_{21} &= \frac{d\mathbf{p}_2}{dt} + \frac{d\mathbf{p}_1}{dt} \\ &= \frac{d}{dt}(\mathbf{p}_2 + \mathbf{p}_1) \\ &= 0. \end{aligned} \tag{3.17}$$

But since the total linear momentum $\mathbf{p}_{\text{tot}}$ of the system is

$$\mathbf{p}_{\text{tot}} = \mathbf{p}_1 + \mathbf{p}_2, \tag{3.18}$$

it follows that

$$\frac{d\mathbf{p}_{\text{tot}}}{dt} = 0 \tag{3.19}$$

or

$$\mathbf{p}_{\text{tot}} = \text{constant}. \tag{3.20}$$

Although we only considered two particles, this result is applicable to any number of particles, and we are lead to the **principle of conservation of linear momentum**

If no net external force is acting on a system (i.e., an isolated system), then the total linear momentum of the system is constant.

For an isolated system containing N we would have

$$\begin{aligned}\mathbf{p}_{\text{tot}} &= \sum_{i=1}^{N} \mathbf{p}_i \\ &= \text{constant}.\end{aligned} \tag{3.21}$$

It should be noted that this result relies entirely on Newton's Third Law, as stated in equation (3.16). Because this equation does not make any requirement on the nature of the internal forces involved (i.e., they do not need to be "central", or being directed at the centers of the particles), it is often called the "weak form" of Newton's Third Law. We will use another stronger version of this law to derive the principle of conservation of angular momentum in the next chapter.

3.2.1 Exercises

3. (Prob. 8.18 in Young and Freedman.) A 68.5-kg astronaut is doing a repair in space on the orbiting space station. She throws a 2.25-kg tool away from her at 3.20 m/s relative to the space station. With what speed and in what direction will she begin to move?

Solution.

Let us assume that the tool is thrown in the positive x direction and that its velocity is denoted by v_{Bx}; the astronaut's velocity is v_{Ax}. Defining by "1" and "2" the initial and final conditions of the astronaut-tool system we have $v_{Ax1} = v_{Bx1} = 0$ and by the principle of conservation of the linear momentum we write

$$\begin{aligned}m_A v_{Ax1} + m_B v_{Bx1} &= m_A v_{Ax2} + m_B v_{Bx2} \\ &= 0.\end{aligned} \tag{3.22}$$

It therefore follows that

$$\begin{aligned} v_{Ax2} &= -\frac{m_B}{m_A} v_{Bx2} \\ &= -\frac{2.25 \text{ kg}}{68.5 \text{ kg}} 3.20 \text{ m/s} \\ &= -0.105 \text{ m/s} \end{aligned} \tag{3.23}$$

and the astronaut starts moving opposite to the direction in which she throws the tool.

4. (Prob. 8.23 in Young and Freedman.) Two identical 1.50-kg masses are pressed against opposite ends of a light spring of force constant 1.75 N/cm, compressing it by 20.0 cm from its normal length. Find the speed of each mass when it has moved free of the spring on a frictionless horizontal table.

Solution.

The initial and final conditions of the system are shown in Figure 1. Since the linear momentum is conserved and $v_{A1} = v_{B1} = 0$, it is clear that

$$\begin{aligned} m_A v_{A1} + m_B v_{B1} &= m_A v_{A2} + m_B v_{B2} \\ &= 0 \end{aligned} \tag{3.24}$$

and $v_{A2} = -v_{B2}$ (because $m_A = m_B$). But the system is also conservative since no energy is dissipated. We can therefore write

$$\frac{1}{2} m v_{A1}^2 + \frac{1}{2} m v_{B1}^2 + \frac{1}{2} k x_1^2 = \frac{1}{2} m v_{A2}^2 + \frac{1}{2} m v_{B2}^2 + \frac{1}{2} k x_2^2 \tag{3.25}$$

or, since $x_2 = 0$,

$$\begin{aligned} \frac{1}{2} k x_1^2 &= \frac{1}{2} m v_{A2}^2 + \frac{1}{2} m v_{B2}^2 \\ &= m v_{A2}^2 . \end{aligned} \tag{3.26}$$

We then find that

$$\begin{aligned} v_{A2} &= \left| x_1 \right| \sqrt{\frac{k}{2m}} \\ &= 0.200 \text{ m} \sqrt{\frac{175 \text{ N/m}}{2 \cdot 1.50 \text{ kg}}} \\ &= 1.53 \text{ m/s}. \end{aligned} \tag{3.27}$$

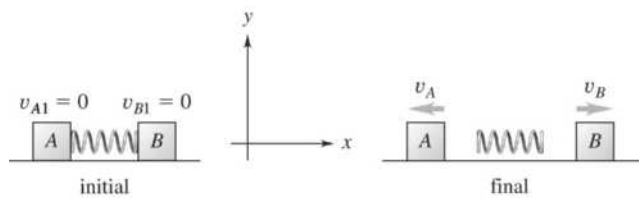

Figure 1 – The initial and final conditions for the system of Problem 4.

3.3 Collisions

Collisions can be thought of as special cases where internal forces are responsible for the strong interactions occurring between the bodies composing the system. There are two different, broad classes of collisions: **elastic** and **inelastic collisions**. The distinction between these two classes resides with the fact that conservation of kinetic energy is achieved in an elastic collision whereas it is not in an inelastic collision. In other words, the internal forces are conservative in the first case, but are not in the other.

It is also essential to remember that experiments show that *linear momentum is always conserved in a collision, irrespective of whether kinetic energy is conserved or not*.

3.3.1 Elastic Collisions

If we consider a collision between two bodies of masses and velocities m_A and $\mathbf{v}_A$, and m_B and $\mathbf{v}_B$, respectively, then we have for an elastic collision

$$\begin{aligned}\frac{1}{2}m_A v_{A1}^2 + \frac{1}{2}m_B v_{B1}^2 &= \frac{1}{2}m_A v_{A2}^2 + \frac{1}{2}m_B v_{B2}^2 \\ m_A\mathbf{v}_{A1} + m_B\mathbf{v}_{B1} &= m_A\mathbf{v}_{A2} + m_B\mathbf{v}_{B2},\end{aligned} \tag{3.28}$$

where the subscripts "1" and "2" denote the physical conditions before and after the collision. We will now make use the fact that we expect that *the laws of physics are the same in any inertial reference frame* (see the discussion in Section 1.1.2 in Chapter 1). We therefore choose a new reference frame that is moving with the initial velocity $\mathbf{v}_{B1}$ of the second object. That is, in that frame the velocity become

$$\begin{aligned}\mathbf{w}_{A1} &= \mathbf{v}_{A1} - \mathbf{v}_{B1} \\ \mathbf{w}_{B1} &= \mathbf{v}_{B1} - \mathbf{v}_{B1} = 0 \\ \mathbf{w}_{A2} &= \mathbf{v}_{A2} - \mathbf{v}_{B1} \\ \mathbf{w}_{B2} &= \mathbf{v}_{B2} - \mathbf{v}_{B1}.\end{aligned} \tag{3.29}$$

That is, the second body is initially at rest in this inertial frame. The equations for the conservations of kinetic energy and linear momentum in that inertial frame are

$$
\begin{aligned}
\frac{1}{2}m_A w_{A1}^2 &= \frac{1}{2}m_A w_{A2}^2 + \frac{1}{2}m_B w_{B2}^2 \\
m_A \mathbf{w}_{A1} &= m_A \mathbf{w}_{A2} + m_B \mathbf{w}_{B2},
\end{aligned}
\tag{3.30}
$$

which can be transformed to

$$
\begin{aligned}
m_B w_{B2}^2 &= m_A \left(w_{A1}^2 - w_{A2}^2 \right) \\
m_B \mathbf{w}_{B2} &= m_A \left(\mathbf{w}_{A1} - \mathbf{w}_{A2} \right).
\end{aligned}
\tag{3.31}
$$

The first of equations (3.31) can further be rewritten as

$$
m_B \mathbf{w}_{B2} \cdot \mathbf{w}_{B2} = m_A \left(\mathbf{w}_{A1} - \mathbf{w}_{A2} \right) \cdot \left(\mathbf{w}_{A1} + \mathbf{w}_{A2} \right), \tag{3.32}
$$

and upon insertion of the second of equations (3.31) on the left-hand side of equation (3.32) becomes

$$
m_A \left(\mathbf{w}_{A1} - \mathbf{w}_{A2} \right) \cdot \mathbf{w}_{B2} = m_A \left(\mathbf{w}_{A1} - \mathbf{w}_{A2} \right) \cdot \left(\mathbf{w}_{A1} + \mathbf{w}_{A2} \right). \tag{3.33}
$$

It therefore follows that

$$
\mathbf{w}_{B2} = \mathbf{w}_{A1} + \mathbf{w}_{A2}, \tag{3.34}
$$

We can in turn insert this result back into equation (3.32) to get

$$
m_B \left(\mathbf{w}_{A1} + \mathbf{w}_{A2} \right) = m_A \left(\mathbf{w}_{A1} - \mathbf{w}_{A2} \right) \tag{3.35}
$$

or an expression for the final velocity for the first body as a function of its initial velocity

$$
\mathbf{w}_{A2} = \frac{m_A - m_B}{m_A + m_B} \mathbf{w}_{A1}. \tag{3.36}
$$

The final velocity of the second body can also be determined by inserting equation (3.36) into equation (3.34) with

$$
\mathbf{w}_{B2} = \frac{2m_A}{m_A + m_B} \mathbf{w}_{A1}. \tag{3.37}
$$

It is interesting to study some limiting cases of collisions. For example, when $m_A \ll m_B$ we find that

$$
\begin{aligned}
\mathbf{w}_{A2} &\simeq -\mathbf{w}_{A1} \\
\mathbf{w}_{B2} &\simeq 0.
\end{aligned}
\tag{3.38}
$$

That is, the light, first body bounces off the heavy second body and reverses it velocity; the heavy body remains stationary. On the other hand, if we have $m_A \gg m_B$, then

$$\begin{aligned} \mathbf{w}_{A2} &\simeq \mathbf{w}_{A1} \\ \mathbf{w}_{B2} &\simeq 2\mathbf{w}_{A1}. \end{aligned} \tag{3.39}$$

The heavy, first body keeps going unimpeded by the light second body, which starts moving at twice the initial velocity of the heavy body. A few moments of thoughts should convince you that these two examples are consistent with one another.

Another interesting case happens when $m_A = m_B$. We then find that

$$\begin{aligned} \mathbf{w}_{A2} &= 0 \\ \mathbf{w}_{B2} &= \mathbf{w}_{A1}. \end{aligned} \tag{3.40}$$

That is, the particles exchange their velocities.

We now come back to equation (3.34) and rewrite it as

$$\mathbf{w}_{A1} = \mathbf{w}_{B2} - \mathbf{w}_{A2}, \tag{3.41}$$

and we now return to our original inertial frame. That is, transform back the velocities with equations (3.29) to get the important result

$$\mathbf{v}_{A1} - \mathbf{v}_{B1} = -\left(\mathbf{v}_{A2} - \mathbf{v}_{B2}\right). \tag{3.42}$$

Equation (3.42) (or equation (3.41), for that matter) establishes the fact *in an elastic collision the relative velocity between the two bodies has the same magnitude (but is reversed) before and after the collision*. One can readily verify that the three cases we just studies (i.e., equations (3.38), (3.39), and (3.40)) all agree with equation (3.42).

3.3.2 Completely Inelastic Collisions

We already mentioned that inelastic collisions do not conserve kinetic energy. A special case can be studied when the two bodies join and stick together after the collision; these are called **completely inelastic collisions**. We now have, using again our inertial frame where the second body is initially at rest,

$$\mathbf{w}_{A2} = \mathbf{w}_{B2} \equiv \mathbf{w}_2. \tag{3.43}$$

And the principle of conservation of linear momentum allows us to write

$$m_A \mathbf{w}_{A1} = \left(m_A + m_B\right)\mathbf{w}_2, \tag{3.44}$$

or

$$\mathbf{w}_2 = \frac{m_A}{m_A + m_B}\mathbf{w}_{A1}. \tag{3.45}$$

We can now verify that kinetic energy is lost because of the collision by calculating the kinetic energies before and after the interaction

$$\begin{aligned} K_1 &= \frac{1}{2} m_A w_{A1}^2 \\ K_2 &= \frac{1}{2}\left(m_A + m_B\right) w_2^2 \\ &= \frac{1}{2}\frac{m_A^2}{m_A + m_B} w_{A1}^2. \end{aligned} \tag{3.46}$$

The loss of kinetic energy is made evident by the ratio

$$\frac{K_2}{K_1} = \frac{m_A}{m_A + m_B} < 1, \tag{3.47}$$

which, as shown here, is always less than unity.

3.3.3 Exercises

5. (Prob. 8.49 in Young and Freedman.) Canadian nuclear reactors use heavy water moderators in which elastic collisions occur between the neutrons and deuterons of mass 2.0 u ('u' is an **atomic mass unit**). (a) What is the speed of a neutron, expressed as a fraction of its original speed, after a head-on, elastic collision with a deuteron that is initially at rest? (b) What is its kinetic energy, expressed a fraction of its original kinetic energy? (c) How many such successive collisions will reduce the speed of a neutron to 1/59,000 of its original value?

Solution.

Let the positive x direction be that of the initial momentum of the neutron. The mass of a neutron is $m_{\mathrm{n}} = 1.0\ \mathrm{u}$ and that of a deuteron $m_{\mathrm{d}} = 2.0\ \mathrm{u}$.

(a) We denote by "1" and "2" the conditions before and after the collision, respectively. We then have from the principles of conservation of energy and linear momentum

$$\begin{aligned} \frac{1}{2} m_{\mathrm{n}} v_{\mathrm{n}x,1}^2 &= \frac{1}{2} m_{\mathrm{n}} v_{\mathrm{n}x,2}^2 + \frac{1}{2} m_{\mathrm{d}} v_{\mathrm{d}x,2}^2 \\ m_{\mathrm{n}} v_{\mathrm{n}x,1} &= m_{\mathrm{n}} v_{\mathrm{n}x,2} + m_{\mathrm{d}} v_{\mathrm{d}x,2}. \end{aligned} \tag{3.48}$$

Solving these equations was accomplished before in three dimensions and shown to yield equation (3.36), from which we use only the x-component

$$\begin{aligned} v_{\mathrm{n}x,2} &= \frac{m_\mathrm{n} - m_\mathrm{d}}{m_\mathrm{n} + m_\mathrm{d}} v_{\mathrm{n}x,1} \\ &= \frac{1.0\ \mathrm{u} - 2.0\ \mathrm{u}}{1.0\ \mathrm{u} + 2.0\ \mathrm{u}} v_{\mathrm{n}x,1} \\ &= -\frac{v_{\mathrm{n}x,1}}{3}. \end{aligned} \tag{3.49}$$

(b) The neutron kinetic energy after the collision is

$$\begin{aligned} K_{\mathrm{n},2} &= \frac{1}{2} m_\mathrm{n} v_{\mathrm{n}x,2}^2 \\ &= \frac{1}{2} m_\mathrm{n} \left(\frac{v_{\mathrm{n}x,1}}{3} \right)^2 \\ &= \frac{K_{\mathrm{n},1}}{9}. \end{aligned} \tag{3.50}$$

(c) For each collision the speed of the neutron is reduced by a factor of three. So for n collisions we have

$$\begin{aligned} \frac{v_{\mathrm{n}x,(n+1)}}{v_{\mathrm{n}x,1}} &= \left(\frac{1}{3} \right)^n \\ &= \frac{1}{59{,}000}. \end{aligned} \tag{3.51}$$

This implies that

$$3^n = 59{,}000, \tag{3.52}$$

or

$$\begin{aligned} \ln\left(3^n\right) &= n \ln(3) \\ &= \ln(59{,}000) \end{aligned} \tag{3.53}$$

and

$$\begin{aligned} n &= \frac{\ln(59{,}000)}{\ln(3)} \\ &= 10.0. \end{aligned} \tag{3.54}$$

3.4 Centre of Mass Motion

Given a system of, say, N particles of potentially different masses, it would be interesting and perhaps useful to define some mean position for the ensemble using the position of each particle. Since more massive particles will carry more momentum for a given velocity it makes sense to weight the position of a particle with its mass. That is, we define the location of the **centre of mass** of the system with

$$\mathbf{r}_{\mathrm{cm}} = \frac{\sum_{i=1}^{N} m_i \mathbf{r}_i}{\sum_{i=1}^{N} m_i}, \tag{3.55}$$

where m_i and $\mathbf{r}_i$ are the mass and position of particle i, respectively. Alternatively, we can write

$$\mathbf{r}_{\mathrm{cm}} = \frac{1}{M} \sum_{i=1}^{N} m_i \mathbf{r}_i, \tag{3.56}$$

with the total mass of the system $M = \sum_{i=1}^{N} m_i$. We can readily verify that when all the particles have the same mass $m = m_i$, for all i, the centre of mass is simply the average of the particles' positions

$$\begin{aligned} \mathbf{r}_{\mathrm{cm}} &= \frac{\sum_{i=1}^{N} m \mathbf{r}_i}{\sum_{i=1}^{N} m} \\ &= \frac{m}{Nm} \sum_{i=1}^{N} \mathbf{r}_i \\ &= \frac{1}{N} \sum_{i=1}^{N} \mathbf{r}_i, \end{aligned} \tag{3.57}$$

as would be expected. For solid or rigid bodies, which are composed of a continuum of microscopic particles (i.e., molecules), the summation of equation (3.56) is replaced by a three-dimensional integral[1]

$$\mathbf{r}_{\mathrm{cm}} = \frac{1}{M} \int \rho \mathbf{r}\, d^3 r, \tag{3.58}$$

[1] The definition of a solid body as a continuum of particles, as well as equations (3.58) and (3.59), are advanced concepts on which you will not be tested.

with ρ mass density in $\mathrm{kg/m^3}$ and the total mass

$$M = \int \rho \, d^3 r. \tag{3.59}$$

It is straightforward to determine the velocity of the centre of mass with

$$\begin{aligned} \mathbf{v}_{\mathrm{cm}} &\equiv \frac{d\mathbf{r}_{\mathrm{cm}}}{dt} \\ &= \frac{1}{M} \sum_{i=1}^{N} \frac{d\left(m_i \mathbf{r}_i\right)}{dt} \\ &= \frac{1}{M} \sum_{i=1}^{N} m_i \frac{d\mathbf{r}_i}{dt} \\ &= \frac{1}{M} \sum_{i=1}^{N} m_i \mathbf{v}_i, \end{aligned} \tag{3.60}$$

or again

$$M\mathbf{v}_{\mathrm{cm}} = \sum_{i=1}^{N} m_i \mathbf{v}_i. \tag{3.61}$$

But if we consider the total linear momentum of the system of particles $\mathbf{P}$ with

$$\mathbf{P} = \sum_{i=1}^{N} \mathbf{p}_i, \tag{3.62}$$

then we can also conveniently write

$$\mathbf{P} = M\mathbf{v}_{\mathrm{cm}}. \tag{3.63}$$

That is, *the total momentum of the system of particles equals the total mass times the velocity of the centre of mass*. The obvious thing to do now is to proceed with the next time derivative using equation (3.60) as a starting point

$$\begin{aligned} \mathbf{a}_{\mathrm{cm}} &\equiv \frac{d\mathbf{v}_{\mathrm{cm}}}{dt} \\ &= \frac{1}{M} \sum_{i=1}^{N} m_i \frac{d\mathbf{v}_i}{dt} \\ &= \frac{1}{M} \sum_{i=1}^{N} m_i \mathbf{a}_i, \end{aligned} \tag{3.64}$$

or in the same manner as before

$$M\mathbf{a}_{\text{cm}} = \sum_{i=1}^{N} m_i \mathbf{a}_i. \tag{3.65}$$

But we clearly could have proceed differently and defined the force acting on the centre of mass from Newton's Second Law and equation (3.63)

$$\begin{aligned}
\mathbf{F}_{\text{cm}} &= \frac{d\mathbf{P}}{dt} \\
&= \frac{d\left(M\mathbf{v}_{\text{cm}}\right)}{dt} \\
&= M\frac{d\mathbf{v}_{\text{cm}}}{dt} \\
&= M\mathbf{a}_{\text{cm}},
\end{aligned} \tag{3.66}$$

which is the same as on the left-hand side of equation (3.65). On the other hand, we have from equation (3.62)

$$\begin{aligned}
\mathbf{F}_{\text{cm}} &= \sum_{i=1}^{N} \frac{d\mathbf{p}_i}{dt} \\
&= \sum_{i=1}^{N} \mathbf{F}_i.
\end{aligned} \tag{3.67}$$

However, $\mathbf{F}_i$ is the total force acting on particle i that consists of the internal force $\mathbf{F}_{\text{int},i}$ and the external force $\mathbf{F}_{\text{ext},i}$ acting on it. We can therefore write

$$\mathbf{F}_{\text{cm}} = \sum_{i=1}^{N} \left(\mathbf{F}_{\text{ext},i} + \mathbf{F}_{\text{int},i}\right). \tag{3.68}$$

But since we know from Newton's Third Law that internal forces come in pairs that cancel each other (see equation (3.16)), it follows that the summation over all particles in equation (3.68) will yield

$$\sum_{i=1}^{N} \mathbf{F}_{\text{int},i} = 0. \tag{3.69}$$

We therefore find that

$$\mathbf{F}_{\text{cm}} = \sum_{i=1}^{N} \mathbf{F}_{\text{ext},i}. \tag{3.70}$$

That is, *a system of particles acted upon by external forces behaves as though all the mass were concentrated at the centre of mass, while the net force acting on the total mass is the sum of the external forces acting on the particles composing the system*.

3.4.1 Exercises

6. (Prob. 8.110 in Young and Freedman.) A 12-kg shell is launched at an angle of 55.0° above the horizontal with an initial speed of 150 m/s. When it is at its highest point, the shell explodes into two fragments, one three times heavier than the other. The two fragments reach the ground at the same time. Assume the air resistance can be ignored. If the heavier fragment lands back at the same point from which the shell was launched, where will the lighter fragment land, and how much energy was released in the explosion?

Solution.

At its highest point, the shell will be moving horizontally; its momentum is then

$$p = m_{\text{shell}} v_0 \cos(\theta), \tag{3.71}$$

since there is no net force acting horizontally ($\theta = 55^\circ$). This momentum will be conserved once the shell explodes and

$$p = m_1 v_1 + m_2 v_2, \tag{3.72}$$

where the subscript "1" is for the heavier fragment; it follows that $m_1 = 3m_2$, and therefore $m_1 = 9\text{ kg}$ and $m_2 = 3\text{ kg}$. If the heavier fragment lands back to the launching point, then

$$v_1 = -v_0 \cos(\theta), \tag{3.73}$$

and

$$\begin{aligned} p &= m_{\text{shell}} v_0 \cos(\theta) \\ &= (m_1 + m_2) v_0 \cos(\theta) \\ &= -m_1 v_0 \cos(\theta) + m_2 v_2. \end{aligned} \tag{3.74}$$

We now have an expression for the speed of the lighter fragment

$$\begin{aligned} v_2 &= \frac{2m_1 + m_2}{m_2} v_0 \cos(\theta) \\ &= 7 v_0 \cos(\theta). \end{aligned} \tag{3.75}$$

The initial vertical speed of the shell is given by

$$v_{0y} = v_0 \sin(\theta), \tag{3.76}$$

and since its vertical speed at its highest point is zero we can determine the time elapsed between the launch and the apex with

$$0 = v_{0y} - gt, \tag{3.77}$$

or

$$t = \frac{v_0}{g}\sin(\theta). \tag{3.78}$$

But this will also be the time it will take for the heavier fragment return to the launch point; the horizontal distance it thus travels in the backward direction is

$$\begin{aligned} x_1 &= v_1 t \\ &= -\frac{v_0^2}{g}\sin(\theta)\cos(\theta) \\ &= -\frac{v_0^2}{2g}\sin(2\theta). \end{aligned} \tag{3.79}$$

Obviously the distance covered by the lighter fragment will be the sum of the horizontal distance travelled by the shell before it exploded and that it travels after the explosion at speed v_2. That is,

$$\begin{aligned} x_2 &= |x_1| + v_2 t \\ &= (1+7)\frac{v_0^2}{2g}\sin(2\theta) \\ &= \frac{4\cdot(150\text{ m/s})^2}{9.80\text{ m/s}^2}\sin(110^\circ) \\ &= 8630\text{ m}. \end{aligned} \tag{3.80}$$

Finally, the energy released by the explosion is the energy that is not contained in the kinetic or potential energies of the fragments right after the explosion

$$\begin{aligned} \Delta K &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{1}{2}m_{\text{shell}} v_0^2 \cos^2(\theta) \\ &= 24 m_2 v_0^2 \cos^2(\theta) \\ &= 5.33\times 10^5\text{ J}. \end{aligned} \tag{3.81}$$