Set 31 Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with the same mean μ and variance σ^2, and let $\bar{X}$ denote the sample mean. Use the Central Limit Theorem (CLT, page 156 in our textbox) to establish the following (asymptotic when $n \to \infty$) statements:

(a) The statement

$$|\bar{X} - \mu| > 1.96\frac{\sigma}{\sqrt{n}}$$

holds with probability 0.05.

Using the CLT, as $n \to \infty$

$$|G| = \frac{|\bar{X} - \mu|}{\frac{\sigma}{\sqrt{n}}}$$

Therefore,

$$\mathbb{P}(|G| > 1.96) = 1 - \mathbb{P}(|G| \leq 1.96)$$

But since,

$$\mathbb{P}(|G| \leq 1.96) = 0.95$$

Then,

$$1 - \mathbb{P}(|G| \leq 1.96) = 1 - 0.95$$

$$\mathbb{P}(|G| > 1.96) = 0.05$$

(b) The statement

$$\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}$$

holds with the probability 0.95. NOTE: The statement about give the following (asymptotic when $n \to \infty$) 95% confidence interval

$$\left[\bar{X} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right]$$

Then,

$$\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}$$

$$-\bar{X} + 1.96\frac{\sigma}{\sqrt{n}} \geq -\mu \geq -\bar{X} - 1.96\frac{\sigma}{\sqrt{n}}$$

$$1.96\frac{\sigma}{\sqrt{n}} \geq \bar{X} - \mu \geq -1.96\frac{\sigma}{\sqrt{n}}$$

$$|\bar{X} - \mu| \leq 1.96 \frac{\sigma}{\sqrt{n}}$$

Therefore, using the CLT, as $n \to \infty$

$$|G| = \frac{|\bar{X} - \mu|}{\frac{\sigma}{\sqrt{n}}}$$

And,

$$\mathbb{P}(|G| \leq 1.96) = 0.95$$

(c) The statement

$$\mu - 1.96 \frac{\sigma}{\sqrt{n}} \leq \bar{X} \leq \mu + 1.96 \frac{\sigma}{\sqrt{n}}$$

holds with the probability 0.95.

Then,

$$\mu - 1.96 \frac{\sigma}{\sqrt{n}} \leq \bar{X} \leq \mu + 1.96 \frac{\sigma}{\sqrt{n}}$$

$$-1.96 \frac{\sigma}{\sqrt{n}} \leq \bar{X} - \mu \leq 1.96 \frac{\sigma}{\sqrt{n}}$$

$$|\bar{X} - \mu| \leq 1.96 \frac{\sigma}{\sqrt{n}}$$

Therefore, using the CLT, as $n \to \infty$

$$|G| = \frac{|\bar{X} - \mu|}{\frac{\sigma}{\sqrt{n}}}$$

And,

$$\mathbb{P}(|G| \leq 1.96) = 0.95$$

(d) Let $\omega_1{}^{act}, \dots, \omega_n{}^{act}$ be a sample from a population Ω, and let $X: \Omega \to \mathrm{R}$ be a filter (that is, a random variable in Statistics, or a measurable function in Mathematics) that produces n outputs (called observations in Statistics) $x_1{}^{obs}, \dots, x_n{}^{obs}$ by the formula $x_i{}^{obs} = X(\omega_i{}^{act})$. Let $\bar{x}$ denote the average of $x_1{}^{obs}, \dots, x_n{}^{obs}$. Does the 95% confidence interval

$$\left[\bar{x} - 1.96\,\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\,\frac{\sigma}{\sqrt{n}}\right]$$

cover the (unknown) population mean μ or not?

The confidence interval covers the population mean with a confidence percentage of 95%.