PROBLEM 1. The objective of this problem is to prove that, with respect to the Theorem of Graham & Brent, a greedy scheduler achieves the stronger bound:

$$T_p \leq \frac{T_1 - T_\infty}{p} + T_\infty$$

Let $G = (V, E)$ be the DAG representing the instruction stream for a multithreaded program in the fork-join parallelism model. The sets V and E denote the vertices and edges of G respectively. Let T_1 and T_∞ be the work and span of the corresponding multithreaded program. We assume that G is connected. We also assume that G admits a single source (vertex with no predecessors) denoted by s and a single target (vertex with no successors) denoted by t. Recall that T_1 is the total number of elements of V and T_∞ is the maximum number of nodes on a path from s to t (counting s and t).

Let $S_0 = \{s\}$. For $i \geq 0$, we denote by S_{i+1} the set of the vertices ω satisfying the following two properties:

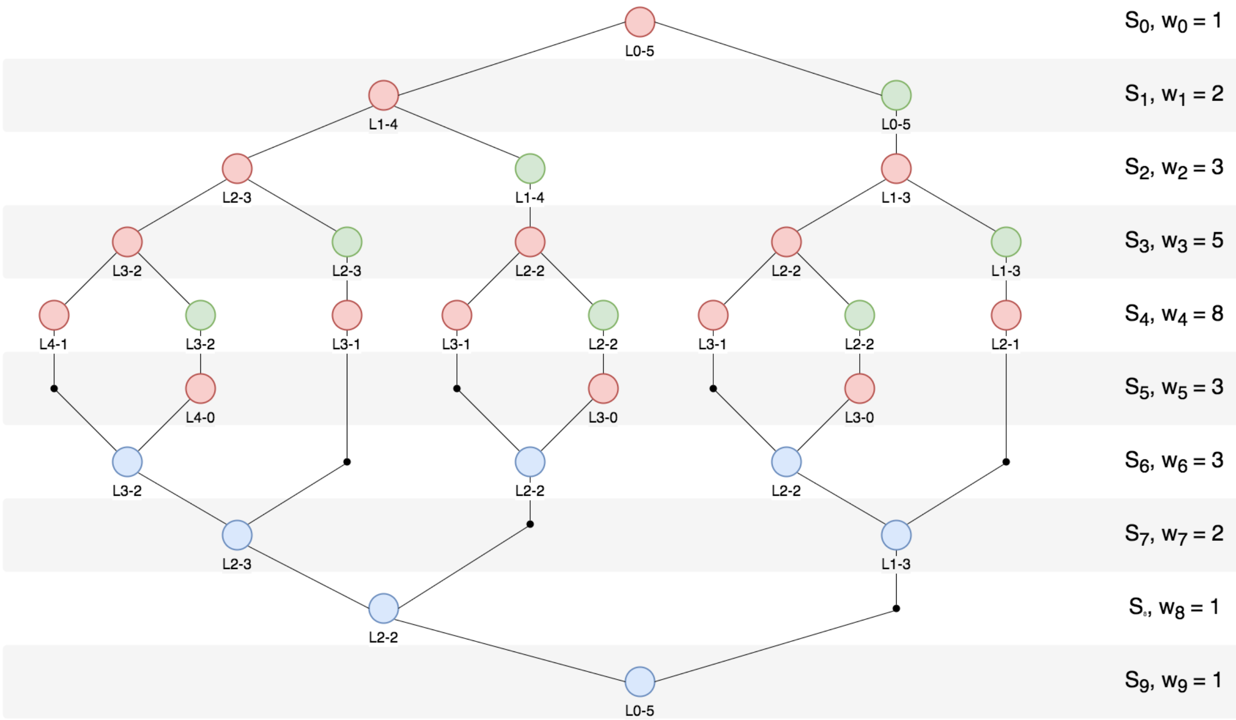
(i) all immediate predecessors of ω belong to $S_i \cup S_{i-1} \cup \ldots \cup S_0$
(ii) at least one immediate predecessor of ω belongs to S_i.

Therefore, the set S_i represents all the unites of work which can be done during the i-th parallel step (and not before that point) on infinitely many processors.

Let $p > 1$ be an integer. For all $i \geq 0$, we denote by ω_i the number of elements in S_i. Let l be the largest integer i such that $\omega_i \neq 0$. Observe that $S_0, S_1 \ldots, S_l$ from a partition of V. Finally, we define the following sequence of integers:

$$c_i = \begin{cases} 0 \text{ if } \omega_i \leq p \\ \left\lceil \frac{\omega_i}{p} \right\rceil - 1 \text{ if } \omega_i > p \end{cases}$$

Question 1. For the computation of the 5-th Fibonacci number (as studied in class) what are $S_0, S_1, S_2, \ldots$?



Question 2. Show that $l + 1 = T_\infty$ and $\omega_0 + \cdots + \omega_i = T_1$ both hold.

T_∞ is the number of nodes on the largest path between $\{s, t\}$. Furthermore, it is observed that $S_0, S_1 \dots, S_i$ form a partition of V by level. It is also observed that for any level S_{i+1} there exists at least one immediate predecessor of ω where $\omega \in S_{i+1}$ that belongs to S_i meaning that the longest path between $\{s, t\}$ must contain at least one vertex of G in every level. Therefore,

$$T_\infty = |S| = |\{S_0, S_1, \dots, S_i\}|$$

Additionally, l is the largest integer i such that $\omega_i \neq 0$. Consequently, l is the i where $S_i = \{t\}$. Hence,

$$|\{S_0, S_1, \dots, S_i\}| = l + 1$$

**The 1 is added as an accomidation for the i counter starting at 0.

$$T_\infty = l + 1$$

T_1 is the number of vertices in the DAG, G. Furthermore, it is observed that $S_0, S_1 \dots, S_i$ form a partition of V, therefore,

$$V = \{S_0, S_1, \dots, S_i\}$$

And,

$$|V| = |S_0| + |S_1| + \cdots + |S_i| = T_1$$

Additionally, since S_i, where $i \geq 0$, represents all the unites of work which can be done during the i-th parallel step on infinitely many processors. Then,

$$|S_0| + |S_1| + \cdots + |S_i| = \omega_0 + \omega_1 + \cdots + \omega_l$$

Hence,

$$\omega_0 + \cdots + \omega_i = T_1$$

Question 3. Show that we have:

$$c_0 + \cdots + c_l \leq \frac{T_1 - T_\infty}{p}$$

Given,

$$c_i = \begin{cases} 0 \text{ if } \omega_i \leq p \\ \left\lceil \frac{\omega_i}{p} \right\rceil - 1 \text{ if } \omega_i > p \end{cases}$$

Upon investigating the equation of c closely, it becomes clear that c calculates the number of cycles required to complete a step upon encountering an incomplete step in a greedy scheduler. Therefore,

$$c_0 + c_1 + \cdots + c_l = T_p - T_\infty$$

Hence,

$$c_0 + c_1 + \cdots + c_l \leq \frac{T_1 - T_\infty}{p}$$

$$T_p - T_\infty \leq \frac{T_1 - T_\infty}{p}$$

Question 4. Prove the desired inequality:

$$T_P \leq \frac{T_1 - T_\infty}{p} + T_\infty$$

With the help of the inequality deduced in the question 3.

$$T_p - T_\infty \leq \frac{T_1 - T_\infty}{p}$$

The inequality could be easily proven via rearrangement

$$T_p - T_\infty \leq \frac{T_1 - T_\infty}{p}$$

$$T_p \leq \frac{T_1 - T_\infty}{p} + T_\infty$$

Question 5. Application; Professor Brown takes some measurements of his (deterministic) multithreaded program, which is scheduled using a greedy scheduler and finds that $T_8 = 80$ seconds and $T_{64} = 20$ seconds. Give lower bound and an upper bound for Professor Brown's computation running time on p processors, for $1 \leq p \leq 100$? Using a plot is recommended.

Using the Theorem of Graham & Brent,

$$T_p \leq \frac{T_1}{p} + T_\infty$$

$$T_p - \frac{T_1}{p} \leq T_\infty$$

The value of p, $1 \leq p \leq 100$, allows us to transform the inequality to an equation because p is small. Therefore, the following system of equations can be used to determine the values of T_1 and T_∞,

$$80 - \frac{T_1}{8} = T_\infty$$

$$20 - \frac{T_1}{64} = T_\infty$$

$$80 - \frac{T_1}{8} = 20 - \frac{T_1}{64}$$

$$80 - \frac{8}{8} \times \frac{T_1}{8} = 20 - \frac{T_1}{64}$$

$$80 - \frac{8 \times T_1}{64} = 20 - \frac{T_1}{64}$$

$$80 - 20 = \frac{8 \times T_1}{64} - \frac{T_1}{64}$$

$$60 = \frac{7 \times T_1}{64}$$

$$\frac{60 \times 64}{7} = T_1$$

$$\frac{3840}{7} = T_1$$

Hence,

$$80-\frac{\frac{3840}{7}}{8}=T_{\infty}$$

$$80-\frac{3840}{7\times 8}=T_{\infty}$$

$$\frac{80\times 7}{7}-\frac{480}{7}=T_{\infty}$$

$$\frac{80\times 7}{7}-\frac{480}{7}=T_{\infty}$$

$$\frac{560}{7}-\frac{480}{7}=T_{\infty}$$

$$\frac{80}{7}=T_{\infty}$$

Finally, to determine the upper and lower bounds of the program plot 100 and 1 for the lower bound and upper bound respectively

$$\min\left(\frac{T_1}{p},T_{\infty}\right)\leq T_p\leq\frac{T_1-T_{\infty}}{p}+T_{\infty}$$

$$\min\left(\frac{\frac{3840}{7}}{p},\frac{80}{7}\right)\leq T_p\leq\frac{\frac{3840}{7}-\frac{80}{7}}{p}+\frac{80}{7}$$

$$\min\left(\frac{\frac{3840}{7}}{p},\frac{80}{7}\right)\leq T_p\leq\frac{\frac{3760}{7}}{p}+\frac{80}{7}$$

$$\min\left(\frac{3840}{7\times p},\frac{80}{7}\right)\leq T_p\leq\frac{3760}{7\times p}+\frac{80}{7}$$

$$\min\left(\frac{3840}{7\times 100},\frac{80}{7}\right)\leq T_p\leq\frac{3760}{7\times 1}+\frac{80}{7}$$

$$\min\left(\frac{3840}{700},\frac{80}{7}\right)\leq T_p\leq\frac{3760}{7}+\frac{80}{7}$$

$$\frac{80}{7}\leq T_p\leq\frac{3840}{7}$$