Chapter 4. Rotation and Conservation of Angular Momentum

Notes:

- *Most of the material in this chapter is taken from Young and Freedman, Chaps. 9 and 10.*

4.1 Angular Velocity and Acceleration

We have already briefly discussed rotational motion in Chapter 1 when we sought to derive an expression for the centripetal acceleration in cases involving circular motion (see Section 1.4 and equation (1.46)). We will revisit these notions here but with a somewhat broader scope.

We reintroduce some basic relations between an angle of rotation θ about some fixed axis, the radius, and the arc traced by the radius over the angle θ. Figure 1 shows these relationships. First, the natural angular unit is the **radian**, not the degree as one might have expected. The definition of the radian is such that it is the angle for which the radius r and the arc s have the same length (see Figure 1a). The circumference of a circle equals 2π times the radius; it therefore follows that

$$1 \text{ rad} = \frac{360^\circ}{2\pi} = 57.3^\circ. \tag{4.1}$$

Second, as we previously saw in Chap. 1, the angle is expressed with

$$\theta = \frac{s}{r} \tag{4.2}$$

and can be seen from Figure 1b.

We can define an **average angular velocity** as the ratio of an angular change $\Delta\theta$ over a

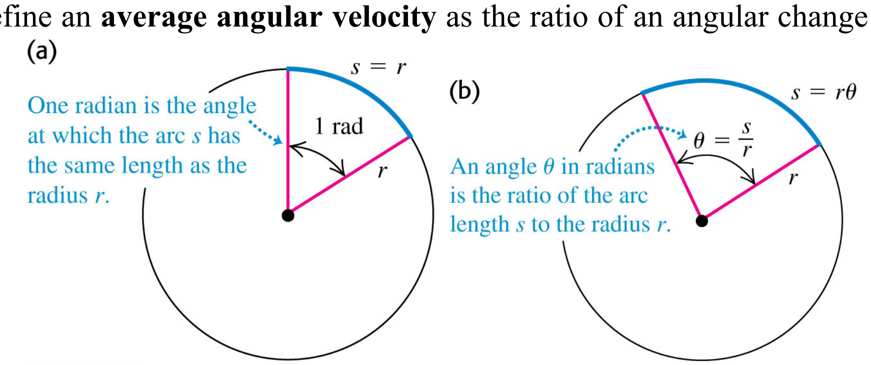


Figure 1 – The relations between an angle of rotation θ about some fixed axis, the radius r, and the arc s traced by the radius over the angle θ.

time interval Δt

$$\omega_{\text{ave},z} = \frac{\Delta\theta_z}{\Delta t}. \tag{4.3}$$

For example, an object that accomplishes one complete rotation in one second has an average angular velocity (also sometimes called **average angular frequency**) of 2π rad/s. If we make these intervals infinitesimal, then we can define the **instantaneous angular velocity** (or **frequency**) with

$$\begin{aligned}\omega_z &= \lim_{\Delta t\to 0}\frac{\Delta\theta_z}{\Delta t}\\ &= \frac{d\theta_z}{dt}.\end{aligned} \tag{4.4}$$

That is, *the instantaneous angular velocity is the time derivative of the angular displacement*. The reason for the presence of the subscript "z" in equations (4.3) and (4.4) will soon be made clearer. It should be noted that an angular displacement $\Delta\theta$ can either be positive or negative; it is a matter of convention how the sign is defined. We will define an angular displacement as positive when it is effected in a counter-clockwise direction, as seen from an observer, when the fixed about which the rotation is done is pointing in the direction of the observer. This is perhaps more easily visualized with Figure 2.

4.1.1 Vector Notation

Since a rotation is defined in relation to some fixed axis, it should perhaps not be too surprising that we can use a vector notation for angular displacements. That is, just as we can define a vector $\Delta\mathbf{r}$ composed of linear displacements along the three independent

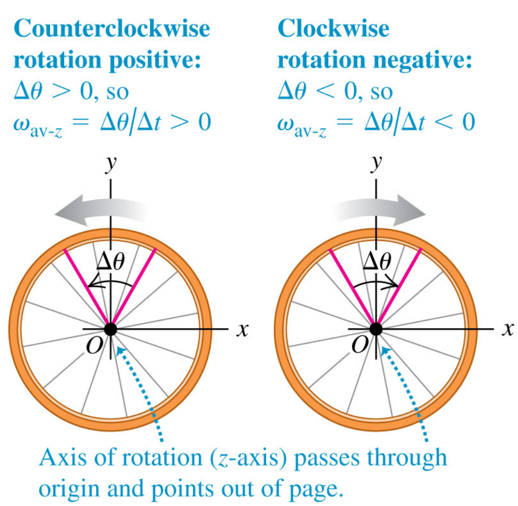


Figure 2 – Convention for the sign of an angle.

axes in Cartesian coordinates with

$$\Delta\mathbf{r} = \Delta x\,\mathbf{e}_x + \Delta y\,\mathbf{e}_y + \Delta z\,\mathbf{e}_z, \tag{4.5}$$

we can do the same for an angular displacement vector $\Delta\boldsymbol{\theta}$ with

$$\Delta\boldsymbol{\theta} = \Delta\theta_x \mathbf{e}_x + \Delta\theta_y \mathbf{e}_y + \Delta\theta_z \mathbf{e}_z. \tag{4.6}$$

It is understood that in equation (4.6) $\Delta\theta_x$ is an angular displacement about the fixed x-axis, etc. The notation used in equations (4.3) and (4.4) is now understood as meaning that the angular displacement and velocity are about the fixed z-axis. An example is shown in Figure 3, along with the so-called right-hand rule, for the angular velocity vector

$$\boldsymbol{\omega} = \frac{d\boldsymbol{\theta}}{dt}. \tag{4.7}$$

The introduction of a vector notation has many benefits and simplifies the form of several relations that we will encounter. A first example is that of the infinitesimal arc vector $d\mathbf{r}$ that results for an infinitesimal rotation vector $d\boldsymbol{\theta}$ of a **rigid body** (please note that we have intentionally replaced s for the finite arc in equation (4.2) with $d\mathbf{r}$ and not $d\mathbf{s}$). Let us consider the special case shown in Figure 4 where an infinitesimal rotation $d\boldsymbol{\theta} = d\theta_z \mathbf{e}_z$ about the z-axis is effected on a vector $\mathbf{r} = r\,\mathbf{e}_x$ aligned along the x-axis. As can be seen from the figure, the resulting infinitesimal arc $d\mathbf{r} = dr\,\mathbf{e}_y$ will be oriented along the y-axis. We know from equation (4.2) that

$$dr = r d\theta, \tag{4.8}$$

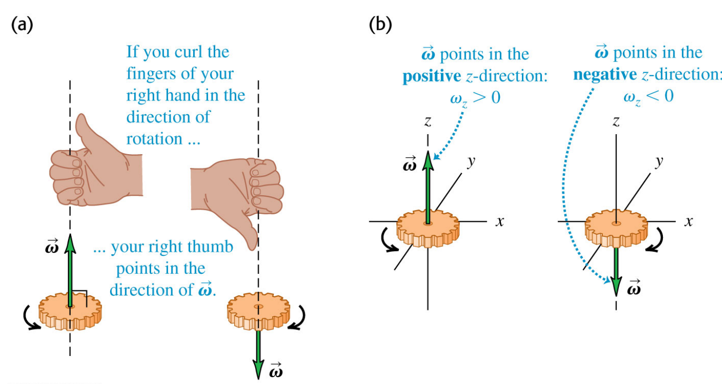


Figure 3 – Shown is the vector representation for an angular velocity about the z-axis, along with the so-called right-hand rule.

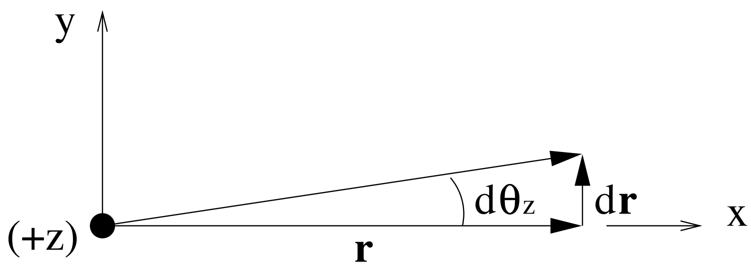


Figure 4 - Infinitesimal rotation of a rigid body about the z-axis .

but how can we mathematically determine the orientation of the infinitesimal arc from that of the rotation and radius? To do so, we must introduce the **cross product** between two vectors.

Let $\mathbf{a}$ and $\mathbf{b}$ two vectors such that

$$\begin{aligned}\mathbf{a} &= a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z \\ \mathbf{b} &= b_x\mathbf{e}_x + b_y\mathbf{e}_y + b_z\mathbf{e}_z.\end{aligned} \tag{4.9}$$

Then we define the cross product

$$\mathbf{a}\times\mathbf{b} = \left(a_yb_z - a_zb_y\right)\mathbf{e}_x + \left(a_zb_x - a_xb_z\right)\mathbf{e}_y + \left(a_xb_y - a_yb_x\right)\mathbf{e}_z. \tag{4.10}$$

It is important to note that

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}. \tag{4.11}$$

It is then straightforward to establish the following

$$\begin{aligned}\mathbf{e}_x \times \mathbf{e}_y &= \mathbf{e}_z \\ \mathbf{e}_y \times \mathbf{e}_z &= \mathbf{e}_x \\ \mathbf{e}_z \times \mathbf{e}_x &= \mathbf{e}_y,\end{aligned} \tag{4.12}$$

and

$$\mathbf{e}_i \times \mathbf{e}_i = 0, \tag{4.13}$$

where $i = x, y, \text{or } z$. Coming back to our simple example of Figure 4, and considering equations (4.8) and (4.12) we find that

$$dr\,\mathbf{e}_y = d\theta_z\mathbf{e}_z \times r\,\mathbf{e}_x. \tag{4.14}$$

Although equation (4.14) results from a special case where the orientation of the different vectors was specified a priori, this relation can be generalized with

$$d\mathbf{r} = d\boldsymbol{\theta} \times \mathbf{r}, \tag{4.15}$$

as could readily verified by changing the orientation of $\mathbf{r}$ and $d\boldsymbol{\theta}$ in Figure 4. We therefore realize that the infinitesimal arc $d\mathbf{r}$ represents the change in the radius vector $\mathbf{r}$ under a rotation $d\boldsymbol{\theta}$; hence the chosen notation. Moreover, we can find a vector generalization of equation (1.44) in Chapter 1 that established the relationship between the linear and angular velocities by dividing by an infinitesimal time interval dt on both sides of equation (4.15). We then find

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}, \tag{4.16}$$

where $\mathbf{v} = d\mathbf{r}/dt$ and $\boldsymbol{\omega} = d\boldsymbol{\theta}/dt$.

Just as we did for the angular velocity in equations (4.3) and (4.4) (but using a vector notation), we can define an **average angular acceleration** over a time interval Δt with

$$\boldsymbol{\alpha}_{\text{ave}} = \frac{\Delta\boldsymbol{\omega}}{\Delta t} \tag{4.17}$$

and an **instantaneous angular acceleration** with

$$\begin{aligned} \boldsymbol{\alpha} &= \lim_{\Delta t \to 0} \frac{\Delta\boldsymbol{\omega}}{\Delta t} \\ &= \frac{d\boldsymbol{\omega}}{dt}. \end{aligned} \tag{4.18}$$

Combining equations (4.7) and (4.18), we can also express the instantaneous angular acceleration as the *second order time derivative* of the angular displacement

$$\begin{aligned} \boldsymbol{\alpha} &= \frac{d\boldsymbol{\omega}}{dt} \\ &= \frac{d}{dt}\left(\frac{d\boldsymbol{\theta}}{dt}\right) \\ &= \frac{d^2\boldsymbol{\theta}}{dt^2}. \end{aligned} \tag{4.19}$$

4.1.2 Constant Angular Acceleration

We have so far observed a perfect correspondence between the angular displacement $d\boldsymbol{\theta}$, velocity $\boldsymbol{\omega}$, and acceleration $\boldsymbol{\alpha}$ with their linear counterparts $d\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$. We also studied in Section 2.1 of Chapter 2 the case of a constant linear acceleration and found that

$$
\begin{aligned}
\mathbf{a}(t) &= \mathbf{a} \quad \text{(constant)} \\
\mathbf{v}(t) &= \mathbf{v}_0 + \mathbf{a}t \\
\mathbf{r}(t) &= \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2,
\end{aligned}
\tag{4.20}
$$

with $\mathbf{r}_0$ and $\mathbf{v}_0$ the initial position and velocity, respectively. We further combined these equations to derive the following relation

$$
\frac{1}{2}\left[v^2(t) - v_0^2\right] = \mathbf{a} \cdot \left[\mathbf{r}(t) - \mathbf{r}_0\right]. \tag{4.21}
$$

Because the relationship between $\boldsymbol{\theta}$, $\boldsymbol{\omega}$, and $\boldsymbol{\alpha}$ is the same as that between $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$, we can write down similar equations for the case of constant angular acceleration

$$
\begin{aligned}
\boldsymbol{\alpha}(t) &= \boldsymbol{\alpha} \quad \text{(constant)} \\
\boldsymbol{\omega}(t) &= \boldsymbol{\omega}_0 + \boldsymbol{\alpha}t \\
\boldsymbol{\theta}(t) &= \boldsymbol{\theta}_0 + \boldsymbol{\omega}_0 t + \frac{1}{2}\boldsymbol{\alpha}t^2
\end{aligned}
\tag{4.22}
$$

and

$$
\frac{1}{2}\left[\omega^2(t) - \omega_0^2\right] = \boldsymbol{\alpha} \cdot \left[\boldsymbol{\theta}(t) - \boldsymbol{\theta}_0\right] \tag{4.23}
$$

without deriving them, since the process would be identical to the one we went through for the constant linear acceleration case.

4.1.3 Linear Acceleration of a Rotating Rigid Body

We previously derived equation (4.16) for the linear velocity of a rotating rigid body. We could think, for example, of a solid, rotating disk and focus on the trajectory of a point on its surface. Since this point, which at a given instant has the velocity $\mathbf{v}$, does not move linearly but rotates, there must be a force "pulling" it toward the centre point of the disk. This leads us to consider, one more time, the concept of the centripetal acceleration discussed in Chapter 1 (see Section 1.4). It is, however, possible to use equation (4.16) to combine two types of accelerated motions. To do so, we take its time derivative

$$
\begin{aligned}
\frac{d\mathbf{v}}{dt} &= \frac{d}{dt}(\boldsymbol{\omega} \times \mathbf{r}) \\
&= \left(\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}\right) + \left(\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt}\right) \\
&= \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v}.
\end{aligned}
\tag{4.24}
$$

On the second line of equation (4.24) we use the fact that the derivative of a product of functions, say, f and g, yields

$$\frac{d}{dt}(fg)=\frac{df}{dt}g+f\frac{dg}{dt}. \tag{4.25}$$

The same result holds for the scalar or cross products between two vectors $\mathbf{a}$ and $\mathbf{b}$. We can verify this as follows

$$\begin{aligned}
\frac{d}{dt}(\mathbf{a}\cdot\mathbf{b}) &= \frac{d}{dt}\left(a_x b_x + a_y b_y + a_z b_z\right)\\
&= \frac{d}{dt}\left(a_x b_x\right)+\frac{d}{dt}\left(a_y b_y\right)+\frac{d}{dt}\left(a_z b_z\right)\\
&= \frac{da_x}{dt}b_x + a_x\frac{db_x}{dt}+\frac{da_y}{dt}b_y + a_y\frac{db_y}{dt}+\frac{da_z}{dt}b_z + a_z\frac{db_z}{dt}\\
&= \frac{da_x}{dt}b_x+\frac{da_y}{dt}b_y+\frac{da_z}{dt}b_z+a_x\frac{db_x}{dt}+a_y\frac{db_y}{dt}+a_z\frac{db_z}{dt}\\
&= \frac{d\mathbf{a}}{dt}\cdot\mathbf{b}+\mathbf{a}\cdot\frac{d\mathbf{b}}{dt}
\end{aligned} \tag{4.26}$$

and if we consider the x component of $\mathbf{a}\times\mathbf{b}$ (see equation (4.10))

$$\begin{aligned}
\frac{d}{dt}(\mathbf{a}\times\mathbf{b})_x &= \frac{d}{dt}\left(a_y b_z - a_z b_y\right)\\
&= \frac{d}{dt}\left(a_y b_z\right)-\frac{d}{dt}\left(a_z b_y\right)\\
&= \left(\frac{da_y}{dt}b_z + a_y\frac{db_z}{dt}\right)-\left(\frac{da_z}{dt}b_y + a_z\frac{db_y}{dt}\right)\\
&= \left(\frac{da_y}{dt}b_z - \frac{da_z}{dt}b_y\right)+\left(a_y\frac{db_z}{dt} - a_z\frac{db_y}{dt}\right)\\
&= \left(\frac{d\mathbf{a}}{dt}\times\mathbf{b}\right)_x+\left(\mathbf{a}\times\frac{d\mathbf{b}}{dt}\right)_x.
\end{aligned} \tag{4.27}$$

If we also consider similar solutions for the y and z, then we have

$$\frac{d}{dt}(\mathbf{a}\times\mathbf{b})=\frac{d\mathbf{a}}{dt}\times\mathbf{b}+\mathbf{a}\times\frac{d\mathbf{b}}{dt}. \tag{4.28}$$

Returning to equation (4.24), we insert equation (4.16) back into it to get

$$\mathbf{a}=\boldsymbol{\alpha}\times\mathbf{r}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}). \tag{4.29}$$

The term within square brackets can be further expanded using the following identity (which could be proven combining equation (4.10) and the expansion for the scalar product)

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}. \tag{4.30}$$

We then find that

$$\begin{aligned}\mathbf{a}&=\boldsymbol{\alpha}\times\mathbf{r}+(\boldsymbol{\omega}\cdot\mathbf{r})\boldsymbol{\omega}-(\boldsymbol{\omega}\cdot\boldsymbol{\omega})\mathbf{r}\\&=\boldsymbol{\alpha}\times\mathbf{r}+(\boldsymbol{\omega}\cdot\mathbf{r})\boldsymbol{\omega}-\omega^2\mathbf{r}.\end{aligned} \tag{4.31}$$

Let us now examine the last two terms on the right-hand side of second of equations (4.31). First, we break down the position vector into two parts

$$\mathbf{r}=\mathbf{r}_{\parallel}+\mathbf{r}_{\perp}, \tag{4.32}$$

where $\mathbf{r}_{\parallel}$ and $\mathbf{r}_{\perp}$ are the parts of $\mathbf{r}$ that are parallel and perpendicular to the orientation of the angular velocity vector $\boldsymbol{\omega}$, respectively. The perpendicular component $\mathbf{r}_{\perp}$ is simply the distance of the point under consideration from the axis of rotation. It follows that we can write for the second term

$$\begin{aligned}(\boldsymbol{\omega}\cdot\mathbf{r})\boldsymbol{\omega}&=(\omega r_{\parallel})\boldsymbol{\omega}\\&=\omega^2\mathbf{r}_{\parallel}.\end{aligned} \tag{4.33}$$

If we know combine this equation with the third term of equation (4.31), then we find that

$$\begin{aligned}(\boldsymbol{\omega}\cdot\mathbf{r})\boldsymbol{\omega}-\omega^2\mathbf{r}&=\omega^2\mathbf{r}_{\parallel}-\omega^2\mathbf{r}\\&=\omega^2(\mathbf{r}_{\parallel}-\mathbf{r})\\&=-\omega^2\mathbf{r}_{\perp}.\end{aligned} \tag{4.34}$$

Equation (4.31) for the linear acceleration of a point on a rigid body becomes

$$\mathbf{a}=\boldsymbol{\alpha}\times\mathbf{r}-\omega^2\mathbf{r}_{\perp}. \tag{4.35}$$

The first term on the right-hand side of this equation is the **tangential acceleration** due to the angular acceleration

$$\mathbf{a}_{\tan}=\boldsymbol{\alpha}\times\mathbf{r}. \tag{4.36}$$

We should note that only the perpendicular component $\mathbf{r}_\perp$ contributes to the magnitude of the tangential acceleration, because of the presence of the cross product. We can verify this as follows

$$\begin{aligned}\boldsymbol{\alpha}\times\mathbf{r} &= \boldsymbol{\alpha}\times\left(\mathbf{r}_\parallel+\mathbf{r}_\perp\right)\\ &= \boldsymbol{\alpha}\times\mathbf{r}_\parallel+\boldsymbol{\alpha}\times\mathbf{r}_\perp,\end{aligned} \tag{4.37}$$

but $\boldsymbol{\alpha}\times\mathbf{r}_\parallel = 0$ since $\boldsymbol{\alpha}$ and $\mathbf{r}_\parallel$ are parallel to one another. For our rigid body, we then rewrite equation (4.35) as

$$\mathbf{a} = \boldsymbol{\alpha}\times\mathbf{r}_\perp - \omega^2\mathbf{r}_\perp, \tag{4.38}$$

while the tangential acceleration becomes

$$\mathbf{a}_{\mathrm{tan}} = \boldsymbol{\alpha}\times\mathbf{r}_\perp \tag{4.39}$$

and

$$a_{\mathrm{tan}} = \alpha r_\perp. \tag{4.40}$$

The second term on the right-hand side of equation (4.35) is the **radial acceleration** due to the angular acceleration

$$\mathbf{a}_{\mathrm{rad}} = -\omega^2\mathbf{r}_\perp. \tag{4.41}$$

This acceleration is nothing more than the vector form of the **centripetal acceleration** discussed in Chapter 1 for circular motions. The minus sign in equation (4.41) indicated that the acceleration is directed toward the origin (or the axis of rotation). The magnitude of the radial acceleration is

$$a_{\mathrm{rad}} = \omega^2 r_\perp, \tag{4.42}$$

which is the same result obtained with equation (1.46). Accordingly, we could have worked out this analysis by first noticing that, for rigid body, equation (4.16) simplifies to

$$\begin{aligned}\mathbf{v} &= \boldsymbol{\omega}\times\mathbf{r}\\ &= \boldsymbol{\omega}\times\left(\mathbf{r}_\parallel+\mathbf{r}_\perp\right)\\ &= \boldsymbol{\omega}\times\mathbf{r}_\perp.\end{aligned} \tag{4.43}$$

It would have then become clear from the onset that only the perpendicular component $\mathbf{r}_\perp$ partakes in the analysis.

4.1.4 Exercises

1. (Prob. 9.17 in Young and Freedman.) A safety device brings the blade of a power mower from an initial angular speed of ω_1 to rest in 1.00 revolution. At the same constant angular acceleration, how many revolutions would it take the blade to come to rest from an initial angular speed ω_2 that was three times as great ($\omega_2 = 3\omega_1$).

Solution.

We use equation (4.23) (in one dimension) to relate the necessary quantities. That is,

$$\frac{1}{2}\omega_1^2 = \alpha \cdot 2\pi, \tag{4.44}$$

where the "2π" corresponds to one revolution (i.e., $\theta_1 = 2\pi$). We therefore have

$$\alpha = \frac{\omega_1^2}{4\pi}. \tag{4.45}$$

For the second angular speed we have

$$\begin{aligned} \frac{1}{2}\omega_2^2 &= \alpha\theta_2 \\ &= \frac{\omega_1^2}{4\pi}\theta_2, \end{aligned} \tag{4.46}$$

or

$$\begin{aligned} \theta_2 &= 2\pi\frac{\omega_2^2}{\omega_1^2} \\ &= 18\pi. \end{aligned} \tag{4.47}$$

It therefore takes 9.00 revolutions to stop the blade.

2. (Prob. 9.22 in Young and Freedman.) You are to design a rotating cylindrical axle to lift 800-N buckets of cement from the ground to a rooftop 78.0 m above the ground. The buckets will be attached to the free end of a cable that wraps around the rim of the axle; as the axle turns the buckets will rise. (a) What should the diameter of the axle be in order to raise the buckets at a steady 2.00 cm/s when it is turning at 7.5 rpm? (b) If instead the axle must give the buckets an upward acceleration of 0.400 m/s^2, what should the angular acceleration of the axle be?

Solution.

(a) We know that the magnitude of the tangential velocity at rim is

$$v = \omega r \tag{4.48}$$

and will also be the speed at which the buckets will rise. We therefore have

$$\begin{aligned} r &= \frac{v}{\omega} \\ &= \frac{0.02\ \text{m/s}}{2\pi \cdot 7.5/60\ \text{rad/s}} \\ &= 2.55\ \text{cm}. \end{aligned} \tag{4.49}$$

(b) The angular acceleration can be evaluated with equation (4.40)

$$\begin{aligned} \alpha &= \frac{a_{\tan}}{r} \\ &= \frac{0.4\ \text{m/s}^2}{0.0255\ \text{m}} \\ &= 15.7\ \text{rad/s}^2. \end{aligned} \tag{4.50}$$

4.2 Moment of Inertia and Rotational Kinetic Energy

We will once again concentrate on a given point on or in our rotating rigid body located at position $\mathbf{r}_i$, the subscript "i" identifies the particle located at that point. We now calculate the kinetic energy associated with this particle with

$$\begin{aligned} K_i &= \frac{1}{2} m_i v_i^2 \\ &= \frac{1}{2} m_i \left(\boldsymbol{\omega} \times \mathbf{r}_i \right)^2, \end{aligned} \tag{4.51}$$

where we used equation (4.16) for the velocity. We will now make use of the following

$$\left(\mathbf{a} \times \mathbf{b} \right) \cdot \left(\mathbf{c} \times \mathbf{d} \right) = \left(\mathbf{a} \cdot \mathbf{c} \right) \left(\mathbf{b} \cdot \mathbf{d} \right) - \left(\mathbf{a} \cdot \mathbf{d} \right) \left(\mathbf{b} \cdot \mathbf{c} \right), \tag{4.52}$$

and equation (4.51) becomes (with $\mathbf{a} = \mathbf{c} = \boldsymbol{\omega}$ and $\mathbf{b} = \mathbf{d} = \mathbf{r}_i$)

$$\begin{aligned} K_i &= \frac{1}{2} m_i \left(\boldsymbol{\omega} \times \mathbf{r}_i \right) \cdot \left(\boldsymbol{\omega} \times \mathbf{r}_i \right) \\ &= \frac{1}{2} m_i \left[\omega^2 r_i^2 - \left(\boldsymbol{\omega} \cdot \mathbf{r}_i \right)^2 \right] \\ &= \frac{1}{2} m_i \left[\omega^2 r_i^2 - \left(\omega r_{i,\parallel} \right)^2 \right], \end{aligned} \tag{4.53}$$

and since, as before,

$$\mathbf{r}_i = \mathbf{r}_{i,\parallel} + \mathbf{r}_{i,\perp} \tag{4.54}$$

while because $\mathbf{r}_\parallel$ and $\mathbf{r}_\perp$ are perpendicular to one another

$$r_i^2 = r_{i,\parallel}^2 + r_{i,\perp}^2, \tag{4.55}$$

we finally find

$$K_i = \frac{1}{2} m_i r_{i,\perp}^2 \omega^2. \tag{4.56}$$

We now define a new quantity

$$I_i = m_i r_{i,\perp}^2 \tag{4.57}$$

and we rewrite equation (4.53) as

$$K_i = \frac{1}{2} I_i \omega^2. \tag{4.58}$$

and we see that I_i serves the same role for the rotational kinetic energy of the particle as the mass does for the kinetic energy due to linear motion. If we now sum over all particles that compose the rigid body, we find for the total **rotational kinetic energy**

$$\begin{aligned} K &= \frac{1}{2}\left(\sum_i I_i\right)\omega^2 \\ &= \frac{1}{2} I \omega^2, \end{aligned} \tag{4.59}$$

where we introduced the **moment of inertia** of the rigid body

$$\begin{aligned} I &= \sum_i I_i \\ &= \sum_i m_i r_{i,\perp}^2. \end{aligned} \tag{4.60}$$

The moment of inertia is a function of the geometry of the rigid body as well as the distribution of the matter within it. And as was mentioned above, its role for rotational motions is similar to that of the mass when dealing with linear motions. This implies, among other things, that the greater the moment of inertia, the harder it is to start the body rotating from rest (or slowing it down when already rotating).

The last of equations (4.60) would only be useful in calculating the moment of inertia for cases where the rigid body is made of a discrete arrangement of particles. For a body made from a continuous distribution of matter, the summation in equation (4.60) is replaced by an integral

$$I = \int r_\perp^2 \, dm, \tag{4.61}$$

where dm is an infinitesimal element of mass located a distance $r_\perp$ away from the axis of rotation. If we define ρ has the **mass density** of the body (in $\mathrm{kg/m^3}$) and dV as the elemental volume where dm is located, then we can write

$$I = \int r_\perp^2 \rho \, dV. \tag{4.62}$$

The calculation of this integral for different shapes and geometries of objects is beyond the scope of our study. But one important aspect that comes out from such analyses is that, *for a given body (e.g., a sphere or a cube), the position of the axes about which the moment of inertia is calculated (i.e., the axis of rotation in our case) will affect the value of the integral in equation (4.62)*. Examples of moments of inertia for a few rigid bodies and axis positions are shown in Figure 5.

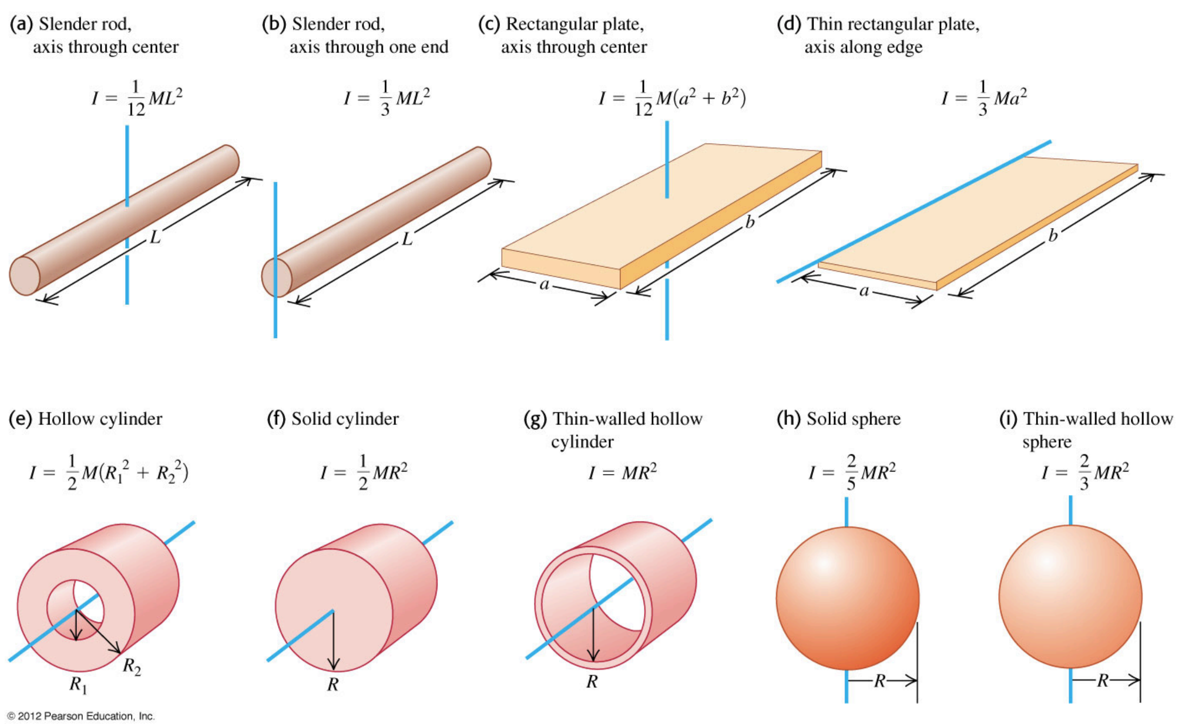


Figure 5 – The moment inertia for different geometries of rigid bodies.

4.3 Gravitational Potential Energy of a Rigid Body

We can proceed in a similar manner to determine the gravitational potential energy of a rigid body as we did for the rotational kinetic energy. That is, we once again concentrate on a given point on or in our rotating rigid body located at position $\mathbf{r}_i$, the subscript "i" identifies the particle located at that point whose gravitational potential energy is

$$U_i = m_i g y_i, \tag{4.63}$$

where as usual

$$\mathbf{r}_i = x_i \mathbf{e}_x + y_i \mathbf{e}_y + z_i \mathbf{e}_z, \tag{4.64}$$

or $y_i = \mathbf{r}_i \cdot \mathbf{e}_y$. We can determine the total gravitational potential energy by summing over all the particles that make the rigid body

$$\begin{aligned} U &= \sum_i U_i \\ &= g \sum_i m_i y_i. \end{aligned} \tag{4.65}$$

Referring to equation (3.56) in Chapter 3, where we defined the centre of mass of a body, we can transform equation (4.65) to

$$U = Mgy_{\mathrm{cm}}, \tag{4.66}$$

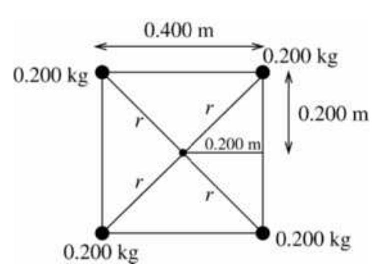
where $M = \sum_i m_i$ is the total mass of the body. *The total gravitational potential energy of an extended, rigid body is calculated as if all its mass was concentrated at its centre of mass*.

4.4 Exercises

3. (Prob. 9.30 in Young and Freedman.) Four small spheres, each of which you can regard as a point of mass 0.200 kg, are arranged in a square 0.400 m on a side and connected by extremely light rods. Find the moment of inertia of the system about an axis (a) through the centre of the square, perpendicular to its plane; (b) bisecting two opposite sides of the square; (c) that passes through the centres of the upper left and lower right spheres.

Solution.

(a) The distance from the centre is the same for the four spheres; denoting it by r, we have

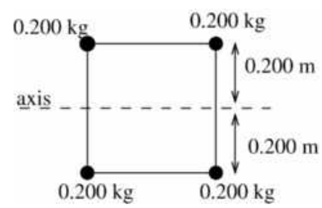


$$r = \sqrt{(0.200)^2 + (0.200)^2} \\ = 0.283 \text{ m.} \tag{4.67}$$

The moment of inertia is then

$$\begin{aligned} I &= r^2 \sum_i m_i \\ &= (0.283 \text{ m})^2 (4 \cdot 0.200 \text{ kg}) \\ &= 0.064 \text{ kg m}^2. \end{aligned} \tag{4.68}$$

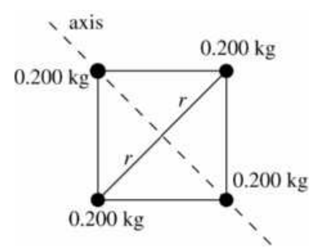
(b) In this case again the distance of the four masses to the axis is the same with $r = 0.200$ m, which implies that



$$\begin{aligned} I &= r^2 \sum_i m_i \\ &= (0.200 \text{ m})^2 (4 \cdot 0.200 \text{ kg}) \\ &= 0.032 \text{ kg m}^2. \end{aligned} \tag{4.69}$$

It is half of the value obtained in (a).

(c) Now, two of the masses are located on the axis and have $r_1 = 0$ and do not contribute to the moment of inertia. The other two have $r_2 = \sqrt{(0.200)^2 + (0.200)^2} = 0.283 \text{ m}^2$. The moment of inertia then becomes



$$\begin{aligned} I &= 2mr_2^2 \\ &= (0.283 \text{ m})^2 (2 \cdot 0.200 \text{ kg}) \\ &= 0.032 \text{ kg m}^2. \end{aligned} \tag{4.70}$$

4. (Prob. 9.47 in Young and Freedman.) A frictionless pulley has the shape of a uniform solid disk of mass 2.50 kg and radius 20.0 cm. A 1.50 kg stone is attached to a very light wire that is wrapped around the rim of the pulley, and the system is released from rest. (a) How far must the stone fall so that the pulley has 4.50 J of kinetic energy? (b) What percent of total kinetic energy does the pulley have?

Solution.

(a) The velocity of the stone $\mathbf{v} = -v\mathbf{e}_y$ (the y-axis is positive-vertical) is related to the angular velocity $\boldsymbol{\omega} = \omega\mathbf{e}_z$ of the disk and its radius $\mathbf{r}$ (located in the xy-plane) with

$$
\begin{aligned}
\mathbf{v} &= \boldsymbol{\omega} \times \mathbf{r} \\
&= \omega r \, \mathbf{e}_y,
\end{aligned} \tag{4.71}
$$

or $v = \omega r$. The conservation of energy before and after the stone starts falling tells us that

$$
mgy_0 = mgy_1 + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2, \tag{4.72}
$$

where m is the mass of the stone and $I = MR^2/2$ is the moment of inertia of the disk, which we determined using Figure 5f. We can then write

$$
mg\left(y_0 - y_1\right) = \frac{1}{2}m\omega^2 R^2 + \frac{1}{4}MR^2\omega^2. \tag{4.73}
$$

But the kinetic energy of the pulley is given by the last term on the right hand-side of equation (4.73), and

$$
\begin{aligned}
\omega^2 &= \frac{4K_{\text{disk}}}{MR^2} \\
&= \frac{4 \cdot 4.50 \text{ J}}{2.50 \text{ kg} \cdot \left(0.200 \text{ cm}\right)^2} \\
&= 180 \text{ rad}^2/\text{s}^2.
\end{aligned} \tag{4.74}
$$

We finally write

$$
\begin{aligned}
y_0 - y_1 &= \frac{\omega^2 R^2}{2g}\left(1 + \frac{M}{2m}\right) \\
&= \frac{180 \text{ rad}^2/\text{s}^2 \cdot 0.04 \text{ m}^2}{2 \cdot 9.80 \text{ m/s}^2}\left(1 + \frac{2.50 \text{ kg}}{2 \cdot 1.50 \text{ kg}}\right) \\
&= 0.673 \text{ m}.
\end{aligned} \tag{4.75}
$$

The stone need to fall 0.673 m for 4.50 J of rotational kinetic energy to be stored in the disk.

(b) The percent of kinetic energy stored in the pulley is

$$
\begin{aligned}
\frac{K_{\text{disk}}}{K_{\text{tot}}} &= \frac{MR^2\omega^2/4}{m\omega^2R^2/2 + MR^2\omega^2/4} \\
&= \frac{1}{1+2m/M} \\
&= \frac{M}{M+2m} \\
&= 45.5\%.
\end{aligned}
\tag{4.76}
$$

It is interesting to note that this figure is a constant of time and only a function of the masses of the pulley and stone.

4.5 Parallel-Axis Theorem

We already stated in Section 4.2 that the moment of inertia of a rigid body depends on the choice of the axis about which it is calculated. There is, however, a useful theorem that links the moment of inertia obtained when using an axis that goes through its centre of mass, we denote it by I_{cm}, and another that is parallel to it but located some distance d away, let us call it I_{p}. We can conveniently, but in all generality, locate the origin of the system of axes at the position of the centre of mass. It follows from this that

$$
\mathbf{r}_{\text{cm}} = 0. \tag{4.77}
$$

We can also, without any loss of generality, choose the z-axis as that going through the centre of mass and about which the moment of inertia is I_{cm}. With this choice, we can write for $d\ \left(=|\mathbf{d}|\right)$ the location of the parallel axis

$$
\mathbf{d} = a\mathbf{e}_x + b\mathbf{e}_y. \tag{4.78}
$$

We now use the second of equations (4.60) for the definition of the moment of inertia

$$
\begin{aligned}
I_{\text{cm}} &= \sum_i m_i r_{i,\perp}^2 \\
&= \sum_i m_i\left(x_i^2 + y_i^2\right).
\end{aligned}
\tag{4.79}
$$

We can also use the same definition for I_{p}, but the distance of a point located at $\mathbf{r}_{i,\perp}$ (relative to the z-axis) is $\mathbf{r}_{i,\perp} - \mathbf{d}$ from the parallel axis. We therefore have, with $M = \sum_i m_i$,

$$
\begin{aligned}
I_{\text{p}} &= \sum_i m_i\left[(x_i-a)^2+(y_i-b)^2\right] \\
&= \sum_i m_i\left[\left(x_i^2+y_i^2\right)-2\left(ax_i+by_i\right)+\left(a^2+b^2\right)\right] \\
&= \sum_i m_i\left[\left(x_i^2+y_i^2\right)-2\left(ax_i+by_i\right)+d^2\right] \\
&= I_{\text{cm}}-2a\sum_i m_i x_i-2b\sum_i m_i y_i+d^2\sum_i m_i \\
&= I_{\text{cm}}-2Max_{\text{cm}}-2Mby_{\text{cm}}+Md^2,
\end{aligned}
\tag{4.80}
$$

or alternatively

$$
I_{\text{p}} = I_{\text{cm}} - 2M\mathbf{d}\cdot\mathbf{r}_{\text{cm},\perp} + Md^2. \tag{4.81}
$$

We know from equation (4.77) that $\mathbf{r}_{\text{cm},\perp}=0$, which then yields the final result for the **parallel-axis theorem**

$$
I_{\text{p}} = I_{\text{cm}} + Md^2. \tag{4.82}
$$

For example, let us use this equation to calculate the moment of inertia for the slender rod of Figure 5b, where the axis used for I_{p} is located at one end of the rod, starting with the result of Figure 5a, where the axis used for I_{cm} goes through the centre of mass. In this case $d=L/2$, then

$$
\begin{aligned}
I_{\text{p}} &= I_{\text{cm}}+\frac{ML^2}{4} \\
&= \frac{ML^2}{12}+\frac{ML^2}{4} \\
&= \frac{ML^2}{3},
\end{aligned}
\tag{4.83}
$$

which is the result expected.

4.5.1 Exercises

5. (Prob. 9.53 in Young and Freedman.) About what axis will a uniform, balsa-wood sphere have the same moment of inertia as does a thin-walled, lead sphere of the same mass and radius, with the axis along a diameter?

Solution.

If we the subscripts "1" and "2" for the uniform and thin-walled spheres, respectively, then from Figure 5h and i for an axis along a diameter

$$
\begin{aligned}
I_{1,\mathrm{cm}} &= \frac{2}{5}MR^2 \\
I_{2,\mathrm{cm}} &= \frac{2}{3}MR^2.
\end{aligned} \tag{4.84}
$$

We need to determine d for

$$
\begin{aligned}
I_{1,\mathrm{p}} &= I_{1,\mathrm{cm}} + Md^2 \\
&= I_{2,\mathrm{cm}}.
\end{aligned} \tag{4.85}
$$

We therefore have

$$
\begin{aligned}
d^2 &= \frac{1}{M}\left(I_{2,\mathrm{cm}} - I_{1,\mathrm{cm}}\right) \\
&= 2R^2\left(\frac{1}{3} - \frac{1}{5}\right) \\
&= \frac{4}{15}R^2,
\end{aligned} \tag{4.86}
$$

or

$$
d = 0.516R. \tag{4.87}
$$

4.6 Angular Momentum and Torque

We now seek to further expand on the correspondence between linear and rotational motions for a rigid body. In what will follow, we will put a further restriction that *the axis rotation about which the rotation takes place is a symmetry axis*. For example, one can think of the long axis of a cylinder, as in Figure 5f. While this simplification will ease our analysis, the results can be shown to be of greater generality, but this demonstration is beyond the scope of our study (see footnote 1 on a later page).

We already know of the correspondences between the following quantities when considering translational and rotational motions:

$$
\begin{aligned}
\mathbf{x} &\leftrightarrow \boldsymbol{\theta} \\
\mathbf{v} &\leftrightarrow \boldsymbol{\omega} \\
\mathbf{a} &\leftrightarrow \boldsymbol{\alpha} \\
m &\leftrightarrow I.
\end{aligned} \tag{4.88}
$$

But we have yet to find or define analogs to the linear momentum $\mathbf{p}$ and the force $\mathbf{F}$. To do so, we start by considering a mass element m_i located at $\mathbf{r}_i$ in the rigid body, and for which

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i. \tag{4.89}$$

We now first multiply by m_i (to get the linear momentum) and then cross-multiply with $\mathbf{r}_i$. That is, using equations (4.30) and (4.32),

$$\begin{aligned}
\mathbf{r}_i \times m_i \mathbf{v}_i &= \mathbf{r}_i \times \left(m_i \boldsymbol{\omega} \times \mathbf{r}_i \right) \\
&= m_i r_i^2 \boldsymbol{\omega} - m_i \left(\boldsymbol{\omega} \cdot \mathbf{r}_i \right) \mathbf{r}_i \\
&= m_i \left[r_i^2 \boldsymbol{\omega} - \left(\boldsymbol{\omega} \cdot \mathbf{r}_i \right) \mathbf{r}_{i,\parallel} \right] - m_i \left(\boldsymbol{\omega} \cdot \mathbf{r}_i \right) \mathbf{r}_{i,\perp} \\
&= m_i \left(r_i^2 \boldsymbol{\omega} - r_{i,\parallel}^2 \boldsymbol{\omega} \right) - \omega m_i r_{i,\parallel} \mathbf{r}_{i,\perp} \\
&= m_i r_{i,\perp}^2 \boldsymbol{\omega} - \omega m_i r_{i,\parallel} \mathbf{r}_{i,\perp}.
\end{aligned} \tag{4.90}$$

(Note that $r_{i,\parallel}$ can be positive or negative in this equation.) We introduce a new vector quantity

$$\begin{aligned}
\mathbf{L}_i &\equiv \mathbf{r}_i \times m_i \mathbf{v}_i \\
&\equiv \mathbf{r}_i \times \mathbf{p}_i,
\end{aligned} \tag{4.91}$$

which we sum all over the volume spanned by the rigid body (i.e., over i). Equation (4.90) then becomes

$$\begin{aligned}
\sum_i \mathbf{L}_i &= \sum_i \left(\boldsymbol{\omega} m_i r_{i,\perp}^2 - \omega m_i r_{i,\parallel} \mathbf{r}_{i,\perp} \right) \\
&= \boldsymbol{\omega} \sum_i m_i r_{i,\perp}^2 - \omega \sum_i m_i r_{i,\parallel} \mathbf{r}_{i,\perp}.
\end{aligned} \tag{4.92}$$

We know from equation (4.60) that the first term on the right-hand side is the product of the moment of inertia and the angular velocity $I\boldsymbol{\omega}$, but the *last term will cancel out when the axis of rotation is an axis of symmetry*. That is,

$$\sum_i m_i r_{i,\parallel} \mathbf{r}_{i,\perp} = 0. \tag{4.93}$$

This is because as one effects the sum about the axis of symmetry, for each term $m_i r_{i,\parallel} \mathbf{r}_{i,\perp}$ there will exist another one such that $m_j r_{j,\parallel} \mathbf{r}_{j,\perp} = -m_i r_{i,\parallel} \mathbf{r}_{i,\perp}$. It follows that the total **angular momentum** $\mathbf{L}$ is defined as

$$\begin{aligned}\mathbf{L} &= \sum_i \mathbf{L}_i \\ &= I\boldsymbol{\omega}\end{aligned} \tag{4.94}$$

when the rigid body is rotating about an axis of symmetry.[1] We then find a new correspondence between linear and angular momentum that is perfectly consistent with equations (4.88), i.e.,

$$\mathbf{p} = m\mathbf{v} \leftrightarrow \mathbf{L} = I\boldsymbol{\omega}. \tag{4.95}$$

Pushing the analogy further we could postulate that the analog of the force for rotational motion must be the time derivative of the angular momentum since according to Newton's Second Law

$$\mathbf{F}_{\text{net}} = \frac{d\mathbf{p}}{dt}. \tag{4.96}$$

We would then define the **torque** with

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}, \tag{4.97}$$

which according to equation (4.94) would yield

$$\begin{aligned}\boldsymbol{\tau} &= \frac{d}{dt}(I\boldsymbol{\omega}) \\ &= I\frac{d\boldsymbol{\omega}}{dt} \\ &= I\boldsymbol{\alpha},\end{aligned} \tag{4.98}$$

since the moment of inertia I is constant for a rigid body. Equations (4.97) and (4.98) are the correct relations that link the torque and angular momentum for a rigid body rotating about an axis of symmetry (although equation (4.97) is true in general). But just as the angular momentum is also related to the linear momentum through equation (4.91), there is a similar connection between the torque and the force. We thus proceed as follows

$$\boldsymbol{\tau} = \frac{d}{dt}\left(\sum_i \mathbf{L}_i\right), \tag{4.99}$$

[1] Although the result $\mathbf{L} = I\boldsymbol{\omega}$ was derived for rotation about symmetry axes and that the more general relation is mathematically different, it is always possible to bring this more general relation into that simpler form through a judicious orientation of the system of axes chosen as a basis for the rigid body (commonly called the **principal axes**).

which, since $\mathbf{v}_i \times m_i \mathbf{v}_i = 0$, becomes

$$
\begin{aligned}
\boldsymbol{\tau} &= \sum_i \frac{d}{dt}\left(\mathbf{r}_i \times \mathbf{p}_i\right) \\
&= \sum_i \frac{d\mathbf{r}_i}{dt} \times \mathbf{p}_i + \sum_i \mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt} \\
&= \sum_i \mathbf{v}_i \times m_i \mathbf{v}_i + \sum_i \mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt} \\
&= \sum_i \mathbf{r}_i \times \mathbf{F}_{\text{net},i} \\
&= \sum_i \boldsymbol{\tau}_i .
\end{aligned}
\tag{4.100}
$$

The total torque on the rigid body is therefore the sum of the all the torques acting on the individual particles that form the rigid body.

We note that the torques on the individual particles result from the net force applied on these particles, which is the sum of the corresponding external and internal forces. But if the internal forces that characterize the interaction between particles are *central forces*, i.e., if they act along the line that joins a given pair of interacting particles, then the torques resulting from these forces will cancel each other (because of Newton's Third Law; see Figure 6). It follows that *only external forces are involved in the determination of the total torque acting on a rigid body*. We can therefore write

$$
\begin{aligned}
\boldsymbol{\tau} &= \sum_i \mathbf{r}_i \times \mathbf{F}_{\text{ext},i} \\
&\sum_i \boldsymbol{\tau}_{\text{ext},i} .
\end{aligned}
\tag{4.101}
$$

We then find our last correspondence between translational and rotational motions

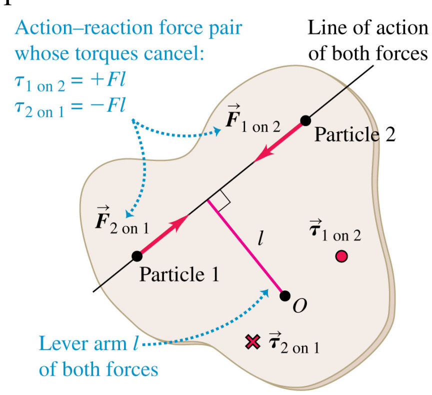


Figure 6 – The cancellation of torques resulting from the internal forces on a pair of particles due to their interaction.

$$\mathbf{F} = m\mathbf{a} \leftrightarrow \boldsymbol{\tau} = I\boldsymbol{\alpha}. \tag{4.102}$$

It is also important to note that the second and third of equations (4.100) are valid in general for a single particle with

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}_{\text{net}}. \tag{4.103}$$

4.6.1 Exercises

6. (Prob. 10.3 in Young and Freedman.) A square metal plate 0.180 m on each side is pivoted about on axis through point O at its centre and perpendicular to the plate. Calculate the net torque about this axis due to the three forces shown in Figure 7 if the magnitude of the forces are $F_1 = 18.0\text{ N}$, $F_2 = 26.0\text{ N}$, and $F_3 = 14.0\text{ N}$. The plate and all forces are in the plane of the page.

Solution.

Using equation (4.103) for the torque we can write

$$\begin{aligned} \boldsymbol{\tau} &= \mathbf{r} \times \mathbf{F}_{\text{net}} \\ &= rF_{\text{net}} \sin(\phi)\mathbf{e}_z, \end{aligned} \tag{4.104}$$

where ϕ is the angle between $\mathbf{r}$ and $\mathbf{F}_{\text{net}}$ and $\mathbf{e}_z$ is directed out of the page. We then have

$$\begin{aligned} \tau_1 &= (18.0\text{ N})\left(\sqrt{2} \cdot 0.090\text{ m}\right)\sin\left(-135^\circ\right) \\ &= -1.62\text{ N} \cdot \text{m}, \end{aligned} \tag{4.105}$$

and since the torque is negative, it is directed into the page (i.e., the plate is moving

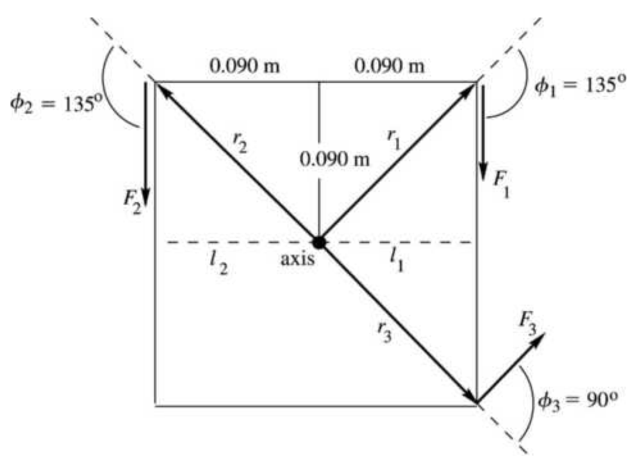


Figure 7 – Torques on the metal plate of Prob. 6.

clockwise. For the second force

$$\begin{aligned}\tau_2 &= (26.0\ \text{N})\left(\sqrt{2}\cdot 0.090\ \text{m}\right)\sin\left(135^\circ\right)\\ &= 2.34\ \text{N}\cdot\text{m},\end{aligned} \tag{4.106}$$

the torque is coming out of the page. Finally, for the last torque

$$\begin{aligned}\tau_3 &= (14.0\ \text{N})\left(\sqrt{2}\cdot 0.090\ \text{m}\right)\sin\left(90^\circ\right)\\ &= 1.78\ \text{N}\cdot\text{m},\end{aligned} \tag{4.107}$$

also coming out of the page.

7. (Prob. 10.17 in Young and Freedman.) A 12.0-kg box resting on a horizontal, frictionless surface is attached to a 5.00-kg weight by a thin, light wire that passes over a frictionless pulley. The pulley has the shape of a uniform solid disk of mass 2.00 kg and diameter 0.500 m. After the system is released, find (a) the tension on the wire on both sides of the pulley and the acceleration of the box, and (b) the horizontal and vertical components of the force that the axle exerts on the pulley.

Solution.

(a) From the first two free-body diagrams shown in Figure 8 we have

$$\begin{aligned}T_1 &= m_1 a\\ m_2 g - T_2 &= m_2 a,\end{aligned} \tag{4.108}$$

or when combining these two equations

$$T_2 - T_1 = m_2 g - \left(m_1 + m_2\right)a. \tag{4.109}$$

From the third free-body diagram of the figure we can also write for the torque acting on the pulley

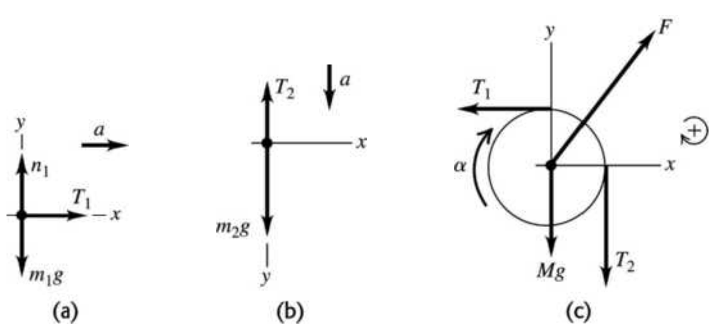


Figure 8 – The free-body diagrams for Prob. 7.

$$
\begin{aligned}
\tau &= r\left(T_2 - T_1\right) \\
&= I\alpha \\
&= I\frac{a}{r},
\end{aligned} \tag{4.110}
$$

since the tangential acceleration is $a = \alpha r$, with r the radius of the pulley (see equation (4.40)). We can transform equation (4.110) to

$$
\begin{aligned}
T_2 - T_1 &= \left(\frac{1}{2}Mr^2\right)\frac{a}{r^2} \\
&= \frac{1}{2}Ma,
\end{aligned} \tag{4.111}
$$

with $I = Mr^2/2$ from Figure 5f. If we now subtract equations (4.109) and (4.111), we then have

$$
\begin{aligned}
a &= \frac{2m_2 g}{2m_1 + 2m_2 + M} \\
&= \frac{2 \cdot 5.00\ \text{kg} \cdot 9.80\ \text{m/s}^2}{\left(2 \cdot 12.0 + 2 \cdot 5.00 + 2.00\right)\text{kg}} \\
&= 2.72\ \text{m/s}^2.
\end{aligned} \tag{4.112}
$$

Inserting the first of equations (4.112) into equations (4.108) we find

$$
\begin{aligned}
T_1 &= \frac{2m_1 m_2 g}{2m_1 + 2m_2 + M} \\
&= \frac{2 \cdot 12.0\ \text{kg} \cdot 5.00\ \text{kg} \cdot 9.80\ \text{m/s}^2}{\left(2 \cdot 12.0 + 2 \cdot 5.00 + 2.00\right)\text{kg}} \\
&= 32.6\ \text{N}
\end{aligned} \tag{4.113}
$$

and

$$
\begin{aligned}
T_2 &= \left(\frac{2m_1 + M}{2m_2 + 2m_1 + M}\right)m_2 g \\
&= \left[\frac{\left(2 \cdot 12.00 + 2.00\right)\text{kg}}{\left(2 \cdot 12.0 + 2 \cdot 5.00 + 2.00\right)\text{kg}}\right] \cdot 5.00\ \text{kg} \cdot 9.80\ \text{m/s}^2 \\
&= 35.4\ \text{N}.
\end{aligned} \tag{4.114}
$$

It is important to note that the reason for having $T_1 \neq T_2$ is that we account the moment of inertia of the pulley with equation (4.111).

(b) The last free-body diagram of Figure 8 shows that three forces are acting on the pulley, which implies that it must react with a force $\mathbf{F}$ that equals minus the resultant of the three forces. That is,

$$\begin{aligned}\mathbf{F} &= -\left(\mathbf{T}_1 + \mathbf{T}_2 + m_2\mathbf{g}\right)\\ &= -\left(-T_1\right)\mathbf{e}_x - \left(-T_2 - m_2 g\right)\mathbf{e}_y\\ &= T_1\mathbf{e}_x + \left(T_2 + m_2 g\right)\mathbf{e}_y\\ &= \left(32.6\,\mathbf{e}_x + 55.0\,\mathbf{e}_y\right)\text{N}.\end{aligned} \tag{4.115}$$

4.7 Combined Translational and Rotational Motions

Consider two frames of reference: one inertial frame of axes x', y', and z' and a rotating frame with axes x, y, and z tied to a rotating rigid body; this is shown in Figure 9. If we choose a point P in the rigid body located at $\mathbf{r}_i$ from the origin of the rotating axes and we further located this origin to that of the inertial frame with $\mathbf{R}$, then we can write

$$\mathbf{r}_i' = \mathbf{R} + \mathbf{r}_i \tag{4.116}$$

for the position of P in the inertial frame. We now inquire about the motion of P as seen by an observer located at the origin of the inertial system with

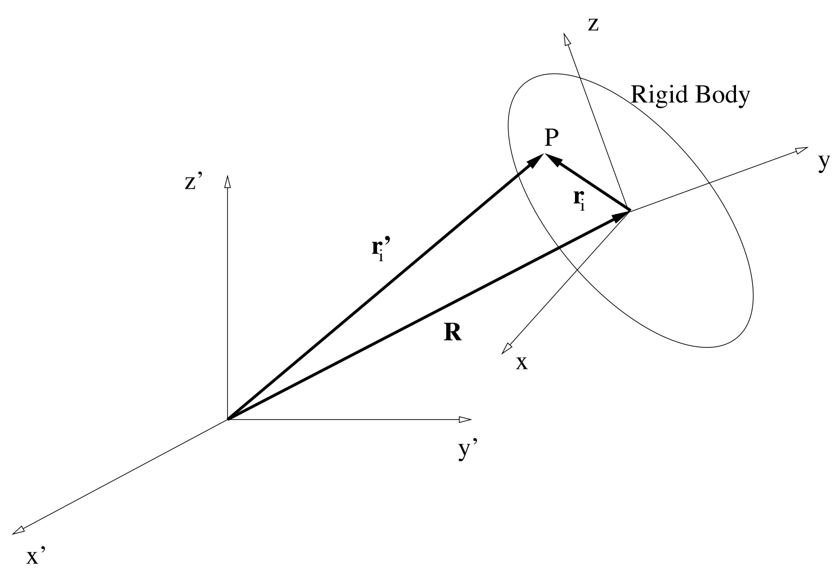


Figure 9 – A fixed inertial frame with axes x', y', and z', and a rotating frame with axes x, y, and z. The vector $\mathbf{R}$ locates the centre of mass of the rotating rigid body.

$$\frac{d\mathbf{r}'_i}{dt} = \frac{d}{dt}\left(\mathbf{R} + \mathbf{r}_i\right) = \frac{d\mathbf{R}}{dt} + \frac{d\mathbf{r}_i}{dt}. \tag{4.117}$$

Upon using equation (4.16) we find that

$$\mathbf{v}'_i = \mathbf{V} + \mathbf{v}_i = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_i, \tag{4.118}$$

where

$$\begin{aligned} \mathbf{v}_i &\equiv \frac{d\mathbf{r}_i}{dt} \\ \mathbf{v}'_i &\equiv \frac{d\mathbf{r}'_i}{dt} \\ \mathbf{V} &\equiv \frac{d\mathbf{R}}{dt}. \end{aligned} \tag{4.119}$$

We therefore find that *the motion of a point on the rigid body can be broken in its rotation motion about a given centre of rotation and the translational motion of that centre relative to an inertial frame of reference*.

4.7.1 With Rotation about the Centre of Mass

We now constrain the rotation to be about an axis that passes through the centre of mass of the rigid body. That is, $\mathbf{r}'$ is now the position of P from the centre of mass and $\mathbf{R}$ the location of the centre of mass to the origin of the inertial frame of reference.

We have already shown in Section 3.4 of Chapter 3 that the total linear momentum $\mathbf{P}$ of the rigid body measured in the inertial frame, obtained by summing over all points such as P, equals the total mass M of the rigid body times the velocity $\mathbf{V}$ of its centre of mass. This is again readily verified with equation (4.118)

$$\begin{aligned} \mathbf{P} &= \sum_i m_i \mathbf{v}'_i \\ &= \sum_i m_i \left(\mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_i\right) \\ &= \mathbf{V}\sum_i m_i + \boldsymbol{\omega} \times \sum_i m_i \mathbf{r}_i \\ &= M\mathbf{V} + \boldsymbol{\omega} \times \sum_i m_i \mathbf{r}_i, \end{aligned} \tag{4.120}$$

which, since by definition $\sum_i m_i \mathbf{r}_i = 0$, becomes

$$\mathbf{P} = M\mathbf{V}. \tag{4.121}$$

The next obvious inquiry to make at this point concerns the total angular moment $\mathbf{L}$ measured in the inertial frame. We do so as follows

$$\begin{aligned}
\mathbf{L} &= \sum_i \mathbf{L}_i \\
&= \sum_i \mathbf{r}_i' \times m_i \mathbf{v}_i' \\
&= \sum_i \left[\left(\mathbf{R} + \mathbf{r}_i \right) \times m_i \left(\mathbf{V} + \mathbf{v}_i \right) \right] \\
&= \sum_i m_i \left(\mathbf{R} \times \mathbf{V} + \mathbf{R} \times \mathbf{v}_i + \mathbf{r}_i \times \mathbf{V} + \mathbf{r}_i \times \mathbf{v}_i \right) \\
&= \mathbf{R} \times \mathbf{P} + \sum_i m_i \left(\mathbf{R} \times \mathbf{v}_i + \mathbf{r}_i \times \mathbf{V} \right) + \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i .
\end{aligned} \tag{4.122}$$

However, the second and third terms can be shown to cancel since

$$\begin{aligned}
\sum_i m_i \left(\mathbf{R} \times \mathbf{v}_i + \mathbf{r}_i \times \mathbf{V} \right) &= \sum_i m_i \left[\frac{d}{dt} \left(\mathbf{R} \times \mathbf{r}_i \right) - \mathbf{V} \times \mathbf{r}_i + \mathbf{r}_i \times \mathbf{V} \right] \\
&= \frac{d}{dt} \left(\mathbf{R} \times \sum_i m_i \mathbf{r}_i \right) - 2\mathbf{V} \times \sum_i m_i \mathbf{r}_i \\
&= 0,
\end{aligned} \tag{4.123}$$

because, once again, $\sum_i m_i \mathbf{r}_i = 0$ by virtue of the definition for the centre of mass. Equation (4.122) then yields the important result

$$\mathbf{L} = \mathbf{R} \times \mathbf{P} + \sum_i \mathbf{r}_i \times \mathbf{p}_i . \tag{4.124}$$

That is, *the total angular momentum of a rigid body about the origin of an inertial frame is the sum of the angular momentum of the centre of mass about that origin and the angular momentum of the rigid body about its centre of mass*.

4.7.2 Energy Relations

We just saw that the overall motion of a rigid body can be advantageously broken down into motions about its centre of mass and of its centre of mass relative to the origin of an inertial frame. Does the same type of relation exist for the kinetic energy? To answer this question we calculate for the point P

$$
\begin{aligned}
K_i &= \frac{1}{2} m_i v_i'^2 \\
&= \frac{1}{2} m_i \left(\mathbf{V} + \mathbf{v}_i\right)^2 \\
&= \frac{1}{2} m_i \left(V^2 + 2\mathbf{V} \cdot \mathbf{v}_i + v_i^2\right),
\end{aligned}
\tag{4.125}
$$

where we used equation (4.118). Summing the energy over the entire rigid body we get

$$
\begin{aligned}
K &= \frac{1}{2} \sum_i m_i \left(V^2 + 2\mathbf{V} \cdot \mathbf{v}_i + v_i^2\right) \\
&= \frac{1}{2} M V^2 + \mathbf{V} \cdot \left(\sum_i m_i \mathbf{v}_i\right) + \frac{1}{2} \sum_i m_i v_i^2 .
\end{aligned}
\tag{4.126}
$$

However, the second term on the right-hand side can be shown to vanish since

$$
\begin{aligned}
\sum_i m_i \mathbf{v}_i &= \frac{d}{dt}\left(\sum_i m_i \mathbf{r}_i\right) \\
&= 0
\end{aligned}
\tag{4.127}
$$

from the definition of the centre of mass, and

$$
K = \frac{1}{2} M V^2 + \frac{1}{2} \sum_i m_i v_i^2 . \tag{4.128}
$$

Using the result already established with equations (4.51) through (4.59) we can finally write

$$
K = \frac{1}{2} M V^2 + \frac{1}{2} I_{\text{cm}} \omega^2 , \tag{4.129}
$$

where the moment of inertia relative to (an axis passing through) the centre of mass of the rigid body I_{cm} is given in equation (4.79). Equation (4.129) answers our earlier question and implies that *the total kinetic energy of the rigid body as seen in an inertial frame is the sum of the kinetic energy of a particle of mass M moving with the velocity of the centre of mass and the kinetic energy of motion (or rotation) of the rigid body about its centre of mass.*

Finally, the potential gravitational energy of a rigid body of total mass M that exhibit translational and rotational motion (about any axis of rotation) remains as was described in Section 4.3 with

$$U = Mgy_{\mathrm{cm}}. \tag{4.130}$$

4.7.3 Exercises

8. (Prob. 10.22 in Young and Freedman.) A hollow, spherical shell with mass 2.00 kg rolls without slipping down a 38.0° slope. (a) Find the acceleration, the friction force, and the minimum coefficient of friction needed to prevent slipping. (b) How would your answers to part (a) change if the mass were doubled to 4.00 kg?

Solution.

We will set the x-axis to be down the incline and let the shell be turning in the positive direction. The free-body diagram is shown in Figure 10.

(a) According to the figure we have for the forces acting on the centre of mass and the torque acting on the sphere

$$\begin{aligned} ma_{\mathrm{cm}} &= mg\sin(\beta) - f_{\mathrm{s}} \\ I_{\mathrm{cm}}\alpha &= f_{\mathrm{s}}R, \end{aligned} \tag{4.131}$$

where we must use the static (and not dynamic) force of friction since the point of the sphere contacting the surface is not moving relative to it. But since the sphere is not slipping we also have that $a_{\mathrm{cm}} = \alpha R$ and from the second of equations (4.131)

$$f_{\mathrm{s}} = \frac{a_{\mathrm{cm}} I_{\mathrm{cm}}}{R^2}, \tag{4.132}$$

which we can insert in the first of equations (4.131) to get

$$a_{\mathrm{cm}} = \frac{g\sin(\beta)}{1 + I_{\mathrm{cm}}/mR^2}. \tag{4.133}$$

Alternatively, we can evaluate

$$f_{\mathrm{s}} = \frac{mg\sin(\beta)}{1 + mR^2/I_{\mathrm{cm}}}. \tag{4.134}$$

For a hollow sphere we have

$$I_{\mathrm{cm}} = 2/3\,mR^2 \tag{4.135}$$

which leads to

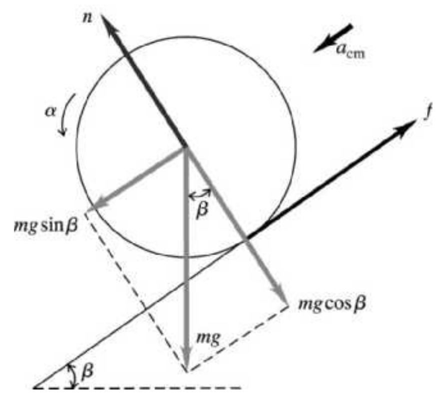


Figure 10 – Free-body diagram for the rolling sphere in Prob. 8.

$$\begin{aligned} a_{cm} &= \frac{3}{5} g \sin(\beta) \\ &= 3.62 \text{ m/s}^2 \\ f_s &= \frac{2}{5} mg \sin(\beta) \\ &= 4.83 \text{ N}. \end{aligned} \tag{4.136}$$

The coefficient of static friction is given by

$$\begin{aligned} \mu_s &= \frac{f_s}{n} \\ &= \frac{f_s}{mg\cos(\beta)} \\ &= \frac{2}{5} \tan(\beta) \\ &= 0.313. \end{aligned} \tag{4.137}$$

This coefficient is required to ensure that the sphere does not slip down the incline.

(b) The only part of (a) that would change if the mass of the sphere was doubled is the magnitude of the static friction force, which would also double to 9.66 N.

9. (Prob. 10.24 in Young and Freedman.) A uniform marble rolls down a symmetrical bowl, starting from rest at the top of the left side. The top of each side is a distance h above the bottom of the bowl. The left half of the bowl is rough enough to cause the marble to roll without slipping, but the right half has no friction because it is coated with oil. (a) How far up the smooth side will the marble go, measured vertically from the bottom? (b) How high would the marble go if both sides were as rough as the left side? (c) How do you account for the fact that the marble goes *higher* with friction on the right than without friction?

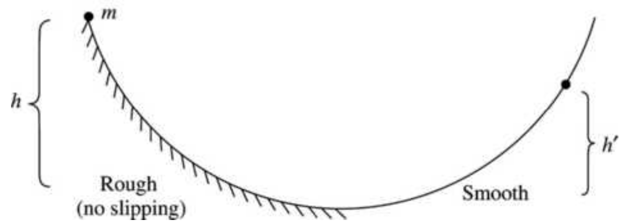


Figure 11 – A marble rolling inside a bowl, in Prob. 9.

Solution.

(a) We will let $y = 0$ at the bottom of the bowl, shown in Figure 11. Since the marble of mass m starts from rest, its kinetic energy at the bottom of the bowl will be given by equation (4.129)

$$\begin{aligned} K &= \frac{1}{2}mv_{\mathrm{cm}}^2 + \frac{1}{2}I_{\mathrm{cm}}\omega^2 \\ &= \frac{1}{2}mv_{\mathrm{cm}}^2 + \frac{1}{2}I_{\mathrm{cm}}\left(\frac{v_{\mathrm{cm}}}{R}\right)^2, \end{aligned} \tag{4.138}$$

where R is the radius of the marble and we imposed the no-slipping condition $v_{\mathrm{cm}} = \omega R$. Because of the principle of conservation of energy, this (change in) kinetic energy equals the change in potential gravitational energy with

$$\frac{1}{2}mv_{\mathrm{cm}}^2 + \frac{1}{2}I_{\mathrm{cm}}\left(\frac{v_{\mathrm{cm}}}{R}\right)^2 = mgh, \tag{4.139}$$

and, since $I_{\mathrm{cm}} = 2/5\,mR^2$,

$$v_{\mathrm{cm}}^2 = \frac{10}{7}gh. \tag{4.140}$$

But at the bottom of the bowl the no-slipping condition does not hold anymore and the rotational kinetic energy will be conserved, as no torque (due to friction) will be acting on the marble. Conservation of energy then dictates that from the bottom of the bowl to the height h' the marble will reach on the right side of the bowl we must have

$$\frac{1}{2}mv_{\mathrm{cm}}^2 + \frac{1}{2}I_{\mathrm{cm}}\omega^2 = \frac{1}{2}I_{\mathrm{cm}}\omega^2 + mgh', \tag{4.141}$$

or

$$\begin{aligned} h' &= \frac{v_{\mathrm{cm}}^2}{2g} \\ &= \frac{5}{7}h, \end{aligned} \tag{4.142}$$

where equation (4.140) was used.

(b) If the right side of the bowl had the same roughness as the left side, then the kinetic energy stored in the rotation of the marble at the bottom of the bowl would be transferred back to potential gravitational energy and equation (4.141) would be replaced by

$$\frac{1}{2}mv_{\mathrm{cm}}^2+\frac{1}{2}I_{\mathrm{cm}}\omega^2 = mgh' \tag{4.143}$$

we would find that $h = h'$ from equation (4.139).

(c) As was stated above, the marble goes higher when the whole bowl has a rough surface because the kinetic energy stored in the rotation marble at the bottom of the bowl is transferred back to potential gravitational energy.

4.8 Work and Power in Rotational Motion

We know from our treatment in Chapter 2 that the infinitesimal work dW done by a force $\mathbf{F}$ acting through an infinitesimal distance $d\mathbf{r}$ is given by

$$dW = \mathbf{F}\cdot d\mathbf{r}. \tag{4.144}$$

But for a rotational motion we know from equation (4.15) that

$$d\mathbf{r} = d\boldsymbol{\theta}\times\mathbf{r}, \tag{4.145}$$

where $d\mathbf{r}$ is now understood to be the infinitesimal arc subtended by the radius $\mathbf{r}$ over an infinitesimal angular displacement vector $d\boldsymbol{\theta}$. Inserting equation (4.145) in equation (4.144) we have

$$dW = \left(d\boldsymbol{\theta}\times\mathbf{r}\right)\cdot\mathbf{F}, \tag{4.146}$$

but since $\left(\mathbf{a}\times\mathbf{b}\right)\cdot\mathbf{c} = \left(\mathbf{c}\times\mathbf{a}\right)\cdot\mathbf{b} = \left(\mathbf{b}\times\mathbf{c}\right)\cdot\mathbf{a}$ we can also write

$$\begin{aligned} dW &= \left(\mathbf{r}\times\mathbf{F}\right)\cdot d\boldsymbol{\theta} \\ &= \boldsymbol{\tau}\cdot d\boldsymbol{\theta}, \end{aligned} \tag{4.147}$$

where $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ is the torque on the system. If we now integrate equation (4.147) through a finite angular displacement $\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1$ we find that the total work done is

$$\begin{aligned} W &= \int_{\theta_1}^{\theta_2} \boldsymbol{\tau} \cdot d\boldsymbol{\theta} \\ &= \int_{\theta_1}^{\theta_2} (\mathbf{r} \times \mathbf{F}) \cdot d\boldsymbol{\theta}. \end{aligned} \tag{4.148}$$

This relation is the analog of equation (2.39) in Chapter 2 for translational motions

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r}. \tag{4.149}$$

It follows from equation (4.148) that for rotational motion a force can only do work if it has a component perpendicular to the radius, as expected. Of course, the work-energy theorem derived in Section 2.2 of Chapter 2 also applies here. For example, if rotation happens about a symmetry axis we have

$$\begin{aligned} \boldsymbol{\tau} \cdot d\boldsymbol{\theta} &= (I\boldsymbol{\alpha}) \cdot d\boldsymbol{\theta} \\ &= I \frac{d\boldsymbol{\omega}}{dt} \cdot d\boldsymbol{\theta} \\ &= I \frac{d\boldsymbol{\omega}}{dt} \cdot \boldsymbol{\omega} dt \\ &= I\, \boldsymbol{\omega} \cdot d\boldsymbol{\omega}, \end{aligned} \tag{4.150}$$

and

$$\begin{aligned} W &= \int_1^2 I\, \boldsymbol{\omega} \cdot d\boldsymbol{\omega} \\ &= \frac{I}{2} \int_1^2 d\left(\omega^2\right) \\ &= \frac{1}{2} I \omega_2^2 - \frac{1}{2} I \omega_1^2 \\ &= \Delta K_{\text{rot}}. \end{aligned} \tag{4.151}$$

It should be noted that this result could also be obtained for the special case of a constant angular acceleration using equation (4.23), in a manner similar to what was done in Chapter 2 for translational motions.

The power associated with the work done by the torque (or the force) can readily be determined with

$$
\begin{aligned}
P &= \frac{dW}{dt} \\
&= \frac{d}{dt}\left(\boldsymbol{\tau}\cdot d\boldsymbol{\theta}\right) \\
&= \boldsymbol{\tau}\cdot\frac{d\boldsymbol{\theta}}{dt}
\end{aligned}
\tag{4.152}
$$

and finally

$$
P = \boldsymbol{\tau}\cdot\boldsymbol{\omega}. \tag{4.153}
$$

Again this result is analogous to the one obtain with equation (2.47) in Chapter 2 for translational motions

$$
P = \mathbf{F}\cdot\mathbf{v}. \tag{4.154}
$$

4.8.1 Exercises

10. (Prob. 10.65 in Young and Freedman.) You connect a light string to a point on the edge of a uniform vertical disk with radius R and mass M. The disk is free to rotate without friction about a stationary horizontal axis through its centre. Initially, the disk is at rest with the string connection at the highest point on the disk. You pull the string with a constant horizontal force $\mathbf{F}$ until the wheel has made exactly one-quarter revolution about the horizontal axis, and then you let go. (a) Use equation (4.148) to find the work done by the string. (b) Use equation (4.149) to find the work done by the string. Do you obtain the same result as in part (a)? (c) Find the angular speed of the disk. (d) Find the maximum tangential acceleration of a point on the disk. (e) Find the maximum radial (centripetal) acceleration of a point on the disk.

Solution.

From Figure 12 we can verify that

$$
\begin{aligned}
\mathbf{F} &= -F\mathbf{e}_x \\
\mathbf{R} &= R\left[-\cos(\theta)\mathbf{e}_x + \sin(\theta)\mathbf{e}_y\right].
\end{aligned}
\tag{4.155}
$$

(a) Since the force and the torque are dependent on the angle θ we write

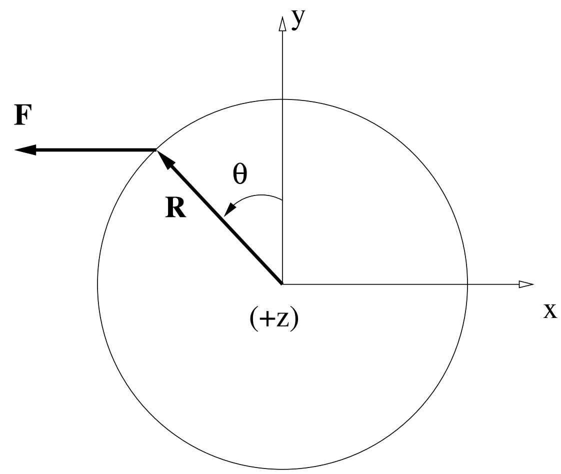


Figure 12 – The string-disk arrangement of Prob. 10.

$$
\begin{aligned}
W &= \int_1^2 \boldsymbol{\tau} \cdot d\boldsymbol{\theta} \\
&= \int_1^2 (\mathbf{R} \times \mathbf{F}) \cdot d\boldsymbol{\theta} \\
&= \int_1^2 \left\{ R\left[-\sin(\theta)\mathbf{e}_x + \cos(\theta)\mathbf{e}_y \right] \times (-F\mathbf{e}_x) \right\} \cdot (\mathbf{e}_z d\theta) \\
&= -FR(\mathbf{e}_y \times \mathbf{e}_x) \cdot \mathbf{e}_z \int_1^2 \cos(\theta) d\theta,
\end{aligned}
\tag{4.156}
$$

but since $\int \cos(\theta) d\theta = \sin(\theta)$ and $\mathbf{e}_y \times \mathbf{e}_x = -\mathbf{e}_z$, we have (with $\theta_1 = 0$ and $\theta_2 = \pi/2$)

$$
\begin{aligned}
W &= FR\left[\sin(\theta_2) - \sin(\theta_1) \right] \\
&= FR.
\end{aligned}
\tag{4.157}
$$

(b) Using equation (4.149) with the arc $d\mathbf{r} = d\boldsymbol{\theta} \times \mathbf{R}$ we have

$$
\begin{aligned}
W &= \int_1^2 \mathbf{F} \cdot (d\boldsymbol{\theta} \times \mathbf{R}) \\
&= \int_1^2 (\mathbf{R} \times \mathbf{F}) \cdot d\boldsymbol{\theta},
\end{aligned}
\tag{4.158}
$$

which equals the second of equations (4.156) (we also used $\mathbf{F} \cdot (d\boldsymbol{\theta} \times \mathbf{R}) = (\mathbf{R} \times \mathbf{F}) \cdot d\boldsymbol{\theta}$). It follows that the two approaches, i.e., this one and the one of part (a), will give the same result.

(c) We know from equation (4.151) that

$$\begin{aligned}\omega_2 &= \sqrt{\frac{2W}{I}} \\ &= \sqrt{\frac{4FR}{MR^2}} \\ &= 2\sqrt{\frac{F}{MR}}.\end{aligned} \tag{4.159}$$

(d) The tangential acceleration is given by

$$\begin{aligned}\mathbf{a}_{\tan} &= \boldsymbol{\alpha} \times \mathbf{R} \\ &= \frac{\boldsymbol{\tau}}{I} \times \mathbf{R}\end{aligned} \tag{4.160}$$

and since $\boldsymbol{\tau}$ and $\mathbf{R}$ are perpendicular to one another

$$\begin{aligned}a_{\tan} &= \frac{\tau R}{I} \\ &= \frac{FR\cos(\theta)R}{MR^2/2} \\ &= \frac{2F}{M}\cos(\theta),\end{aligned} \tag{4.161}$$

which will be maximum with $2F/M$ at $\theta = 0$.

(e) The radial acceleration is given by

$$\begin{aligned}\mathbf{a}_{\text{rad}} &= \omega^2 \mathbf{R} \\ &= \frac{4F\sin(\theta)}{MR}\mathbf{R},\end{aligned} \tag{4.162}$$

where we used the value of W from equation (4.157) at an arbitrary angle θ. The centripetal acceleration will be at a maximum at $\theta = \pi/2$ with $a_{\text{rad}} = 4F/M$.

4.9 The Conservation of Angular Momentum

When deriving the principle of conservation of linear momentum in Chapter 3, we considered an isolated system of particles; this principle only applies for such system. Our derivation rested entirely on Newton's Third Law. We showed that if interactions between pairs of particles happen through internal forces in such a way that

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}, \tag{4.163}$$

where i and j denote a pair of particles, then the total linear momentum of the system was conserved

$$\frac{d\mathbf{p}_{\text{tot}}}{dt} = 0. \tag{4.164}$$

On the other hand, if the system was not isolated but was subjected to a net external force, then the total linear momentum of the system was allowed to change according to Newton's Second Law

$$\mathbf{F}_{\text{ext}} = \frac{d\mathbf{p}_{\text{tot}}}{dt}. \tag{4.165}$$

For the purpose of investigating a similar conservation of angular momentum we must also consider an isolated system; certainly a rigid body satisfies this requirement. From our previous study on the relation between the torque and angular momentum, we have a perfect correspondence between the force-linear momentum and torque-angular momentum pairs. For example, we know from equation (4.97) that

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}. \tag{4.166}$$

For a rigid body not subjected to a net external torque, and therefore isolated from any agent that could change its state of rotation, equation (4.166) tells us that

$$\frac{d\mathbf{L}}{dt} = 0. \tag{4.167}$$

In other words, *when the net torque applied to a system is zero, then the total angular momentum of the system is conserved and remains unchanged*. It is important to note that this principle is a universal conservation law, to the same fundamental level as the principles of conservation of energy and linear momentum.

It is possible that internal torques arise between the components of a system (just as internal forces could be present when investigating the total linear momentum of an isolated system of particles). But if we again call upon Newton's Third Law such that these internal torques arise from interaction forces that satisfy equation (4.163), the sum of all such internal torques can be written as

$$\begin{aligned}\sum_i \boldsymbol{\tau}_{\text{int},i} &= \sum_i \left(\mathbf{r}_i \times \sum_{j \neq i} \mathbf{F}_{ji} \right) \\ &= \sum_{ij \text{ pairs}} \left(\mathbf{r}_i \times \mathbf{F}_{ji} + \mathbf{r}_j \times \mathbf{F}_{ij} \right),\end{aligned} \tag{4.168}$$

where the last summation is on unique pairs of particles i and j (i.e., if we consider i and j then we shouldn't include j and i to avoid double-counting). But from equation (4.163) we can write

$$\sum_i \boldsymbol{\tau}_{\text{int},i} = \sum_{ij \text{ pairs}} \left[\left(\mathbf{r}_i - \mathbf{r}_j \right) \times \mathbf{F}_{ji} \right]. \tag{4.169}$$

If we now enforce the further constraint that the internal forces be *central*, i.e., that they are directed along the straight line joining the two interacting particles (i.e., $\mathbf{F}_{ji}$ is parallel or anti-parallel to $\mathbf{r}_i - \mathbf{r}_j$), then

$$\left(\mathbf{r}_i - \mathbf{r}_j \right) \times \mathbf{F}_{ji} = 0 \tag{4.170}$$

and

$$\sum_i \boldsymbol{\tau}_{\text{int},i} = 0. \tag{4.171}$$

This implies that parts of the system can experience a change in their angular momentum, but the total angular momentum must be conserved when no external torque is applied. Equations (4.168) to (4.171) put on a firm mathematical basis what was discussed in Section 4.6 and illustrated in Figure 6. When an external torque is applied, then the total angular momentum will change in accordance with equation (4.166).

4.9.1 Exercises

11. (Prob. 10.41 in Young and Freedman.) Under some circumstances, a star can collapse into an extremely dense object made mostly of neutrons and called a neutron star. The density of a neutron star is approximately 10^{14} times that of ordinary solid matter. Suppose we represent the star as a uniform, solid, rigid sphere, both before and after the collapse. The star's initial radius was 7.0×10^5 km (comparable to our sun); its final radius is 16 km. If the original star rotated once in 30 days, find the angular speed of the neutron star.

Solution.

The angular momentum must be conserved with

$$\begin{aligned} I_{\text{star}}\omega_1 &= I_{\text{neutron}}\omega_2 \\ \frac{2}{5}MR_{\text{star}}^2\omega_1 &= \frac{2}{5}MR_{\text{neutron}}^2\omega_2, \end{aligned} \tag{4.172}$$

or

$$\begin{aligned}\omega_2 &= \omega_1\left(\frac{R_{\text{star}}}{R_{\text{neutron}}}\right)^2 \\ &= \frac{2\pi}{30\cdot 86400}\text{rad/sec}\cdot\left(\frac{7.0\times 10^5}{16}\right)^2 \\ &= 4640 \text{ rad/s (or 738 rev/s).}\end{aligned} \tag{4.173}$$

12. (Prob. 10.43 in Young and Freedman.) The outstretched hands and arms of a figure skater preparing for a spin can be considered a slender rod pivoting about an axis through its centre. When the skater's hands and arms are brought in and wrapped around his body to execute the spin, the hands and arms can be considered a thin-walled, hollow cylinder. His hands and arms have a combined mass of 8.00 kg. When outstretched, they span 1.80 m; when wrapped, they form a cylinder of radius 25.0 cm. The moment of inertia about the rotation axis or the remainder of the body is constant and equal to $0.400\ \text{kg}\cdot\text{m}^2$. If his original angular speed is 0.400 rev/s, what is his final angular speed?

Solution.

The angular momentum must be conserved through this maneuver, i.e.,

$$\left(I_{\text{arms,1}}+I_{\text{body}}\right)\omega_1 = \left(I_{\text{arms,2}}+I_{\text{body}}\right)\omega_2, \tag{4.174}$$

or

$$\omega_2 = \omega_1\left(\frac{I_{\text{arms,1}}+I_{\text{body}}}{I_{\text{arms,2}}+I_{\text{body}}}\right). \tag{4.175}$$

From Figure 5 we have that

$$\begin{aligned} I_{\text{arms,1}} &= \frac{1}{12}ML^2 \\ I_{\text{arms,2}} &= MR^2, \end{aligned} \tag{4.176}$$

where L and R are the length and radius of the hands and arms when outstretched and brought in, respectively. It follows that

$$\begin{aligned}\omega_2 &= \omega_1\left(\frac{ML^2/12 + I_{\text{body}}}{MR^2 + I_{\text{body}}}\right)\\ &= (2\pi\cdot 0.4)\,\text{rad/s}\cdot\left(\frac{8.0\cdot 1.8^2/12 + 0.4}{8.0\cdot 0.25^2 + 0.4}\right)\\ &= 7.15\text{ rad/s (or 1.14 rev/s).}\end{aligned} \tag{4.177}$$

4.10 Precession

We have all experienced the remarkable dynamical behavior of a spinning top. One intriguing aspect is the observation of an increasing rotational-wobble of the spinning axis as the angular speed of the top (about the spinning axis) is slowing down. This type of motion is called **precession**. We have now developed all the tools necessary to understand this behavior.

Let us consider a flywheel with its symmetry, and spin, axis positioned horizontally as depicted in Figure 13. The flywheel is spinning with an angular velocity ω about its symmetry axis, initially directed along the x direction (i.e., $\boldsymbol{\omega} = \omega\mathbf{e}_x$), and is simultaneously subjected to gravity with its weight $\mathbf{w} = -w\mathbf{e}_z$ pointing downward. The presence of this force located at the centre of mass of flywheel-axis system brings a torque initially pointing along the y-axis (i.e., pointing into the page)

$$\begin{aligned}\boldsymbol{\tau} &= \mathbf{r}\times\mathbf{w}\\ &= \tau\mathbf{e}_y.\end{aligned} \tag{4.178}$$

It would perhaps be intuitive to think that the presence of this torque would start the flywheel rotating about the y-axis and eventually bring it in contact with the ground. This is indeed what would be observed if the flywheel were not spinning about its symmetry axis. But the flywheel's dynamics are much more interesting because of its rotational motion…

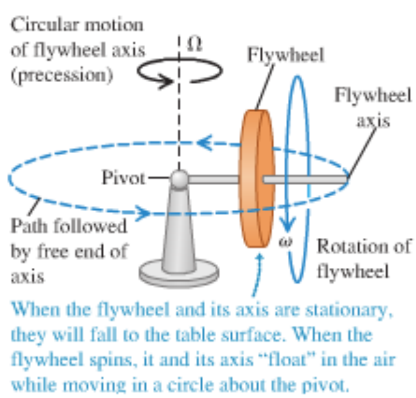


Figure 13 – Precession of a flywheel as it spins about its symmetry axis.

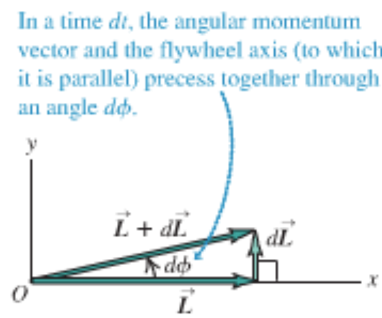


Figure 14 – The angular momentum after an interval dt.

We first investigate the infinitesimal change in angular momentum $d\mathbf{L}$ during a time dt with

$$d\mathbf{L} = \boldsymbol{\tau} dt, \tag{4.179}$$

which is initially oriented along the y-axis. The resulting angular momentum is generally expressed by

$$\mathbf{L} + d\mathbf{L} = I\boldsymbol{\omega} + \boldsymbol{\tau} dt, \tag{4.180}$$

which initially (i.e., after the interval dt) is given by

$$\mathbf{L} + d\mathbf{L} = I\omega\,\mathbf{e}_x + \tau dt\,\mathbf{e}_y. \tag{4.181}$$

This is shown in Figure 14. One might be inclined to think from equation (4.181) that the magnitude of the angular momentum has changed in the process, but this would be misleading. To verify this, let us calculate the amount of work done on the flywheel by the torque during the interval dt. Using equation (4.147) with $d\boldsymbol{\theta} = \boldsymbol{\omega} dt$, we have

$$dW = \boldsymbol{\tau} \cdot \boldsymbol{\omega} dt, \tag{4.182}$$

which initially is

$$\begin{aligned} dW &= \tau\omega\, dt\left(\mathbf{e}_y \cdot \mathbf{e}_x\right) \\ &= 0. \end{aligned} \tag{4.183}$$

We note that equation (4.183) is valid at all times. The fact that no work is done by the torque is fundamentally important for what follows. The main implication we emphasize is that, from equation (4.151), there is also no change of the kinetic energy stored in the flywheel, which also implies that

$$\begin{aligned}\omega &= \text{constant}\\ L = I\omega &= \text{constant}.\end{aligned} \tag{4.184}$$

It is important to stress that there is no conservation of the angular momentum, since it changes direction (see Figure 14), but *the magnitude of the angular momentum remains constant*. More importantly, the flywheel (and angular momentum) moves in xy-plane not downward! Again, the reason for this is that as the flywheel precesses in the xy-plane, the torque due to its weight is always oriented perpendicular to the angular displacement, as exemplified with equation (4.183). *No work is ever being done on the flywheel*.

This behavior is entirely due to the rotation of the flywheel. In the case where the flywheel is not initially spinning about its axis, equation (4.183) does not apply and

$$\begin{aligned}dW &= \boldsymbol{\tau}\cdot d\boldsymbol{\theta}\\ &\neq 0\end{aligned} \tag{4.185}$$

in general. As the flywheel starts its fall downward, it also feels an angular acceleration due to the torque that precipitates its fall (until it reaches the ground).

Let us now see if we can quantify the precession of the spinning flywheel. We start with

$$\begin{aligned}d\mathbf{L} &= \boldsymbol{\tau}dt\\ &= (\mathbf{r}\times\mathbf{w})dt\\ &= \left\{r\left[\cos(\phi)\mathbf{e}_x + \sin(\phi)\mathbf{e}_y\right]\times(-w\mathbf{e}_z)\right\}dt\\ &= rwdt\left[\cos(\phi)\mathbf{e}_y - \sin(\phi)\mathbf{e}_x\right],\end{aligned} \tag{4.186}$$

where ϕ is the precession angle measured from the x-axis in the xy-plane. We now introduce the **precession frequency**

$$\boldsymbol{\Omega} = \frac{d\boldsymbol{\phi}}{dt}. \tag{4.187}$$

The magnitude Ω of this frequency is given by (see Figure 14)

$$\begin{aligned}\Omega &= \frac{d\phi}{dt}\\ &= \frac{|d\mathbf{L}|/|\mathbf{L}|}{dt},\end{aligned} \tag{4.188}$$

which from equations (4.184) and (4.186) yields

$$\Omega = \frac{rw}{I\omega}. \tag{4.189}$$

Incidentally, solving equation (4.186) we find

$$\begin{aligned} \mathbf{L} &= \int d\mathbf{L} \\ &= rw\left[\mathbf{e}_y \int \cos(\Omega t)\,dt - \mathbf{e}_x \int \sin(\Omega t)\,dt\right] \\ &= \frac{wr}{\Omega}\left[\cos(\Omega t)\mathbf{e}_x + \sin(\Omega t)\mathbf{e}_y\right], \end{aligned} \tag{4.190}$$

where we used

$$\begin{aligned} \int \cos(at)\,dt &= \frac{1}{a}\sin(at) \\ \int \sin(at)\,dt &= -\frac{1}{a}\cos(at). \end{aligned} \tag{4.191}$$

Upon inserting equation (4.189) in equation (4.190) we find that

$$\mathbf{L} = I\omega\left[\cos(\Omega t)\mathbf{e}_x + \sin(\Omega t)\mathbf{e}_y\right]. \tag{4.192}$$

This equation makes it clear that the magnitude of the angular momentum remains unchanged at $I\omega$, but that the spin axis rotates at the precession frequency Ω in the xy-plane. Finally, we also demonstrated with equation (4.189) the fact stated at the start that the precession frequency (i.e., the wobbling motion) increases as the rotation speed of the spinning top (the flywheel, in this case) winds down.