Chapter 2. Conservation of Energy

Notes:

- *Most of the material in this chapter is taken from Young and Freedman, Chapters 6 and 7.*

2.1 Work

We have discussed forces at length in Chapter 1 and how we can use Newton's Laws to solve problems involving them. In this chapter we introduce new quantities and concepts that are also related to forces and can advantageously be used to understand the dynamics of bodies and physical systems. The first quantity we study here is that of the **work** done on a body. In one-dimension or for rectilinear motion we define the work done on an object as

$$W = F_{\text{net}} s, \tag{2.1}$$

where F_{net} is the net force acting on the body and s its displacement (in meters). The units for work are therefore $\mathrm{N \cdot m}$, or Joule (J; one Joule equals 1 Newton times 1 kg). In general, the work can be expressed as the scalar product of the three-dimensional versions of these two quantities

$$W = \mathbf{F}_{\text{net}} \cdot \mathbf{s}. \tag{2.2}$$

It is thus evident that the work is a scalar quantity. Although it was understood that in equation (2.1) both the force and the displacement are in one and the same orientation (e.g., along the positive and/or the negative x-axis), it should be clear that in equation (2.2) $\mathbf{F}_{\text{net}}$ and $\mathbf{s}$ are vectors that can have completely distinct orientations. If θ is the angle between the orientations of $\mathbf{F}_{\text{net}}$ and $\mathbf{s}$, then

$$W = F_{\text{net}} s \cos(\theta), \tag{2.3}$$

with, this time, $F_{\text{net}} = |\mathbf{F}_{\text{net}}|$ and $s = |\mathbf{s}|$. It follows from this that the work done on an object can either be positive ($-\pi/2 < \theta < \pi/2$), negative ($\pi/2 < \theta < 3\pi/2$), or zero ($\theta = \pm\pi/2$).

2.2 Kinetic Energy and the Work-Energy Theorem

Using equation (2.2) (or (2.3)) in conjunction with Newton's Second Law it becomes obvious that the work is not only linked to the displacement but also to the velocity of the body under consideration. More precisely, since a non-zero net force tends to accelerate (or rather change) the speed of a body in the direction of the force according to

$$\mathbf{F}_{\text{net}} = m\mathbf{a} \tag{2.4}$$

the work will be positive if the object is accelerated, negative if it is decelerated, and zero if its speed remain unchanged. But before we can go any further in studying the more complete relationship between velocity and work we must make a short mathematical interlude…

2.2.1 *Simple Derivatives and Integrals*[1]

We already know the relations between the position, velocity, and acceleration

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt}\\ \mathbf{a} &= \frac{d\mathbf{v}}{dt}\\ &= \frac{d^2\mathbf{r}}{dt^2}.\end{aligned} \tag{2.5}$$

It follows that given a form for the position vector as a function of time (i.e., $\mathbf{r}(t)$) both the velocity and acceleration can be determined. For example, if we know that the position of a body is initially $\mathbf{r}_0$ at $t=0$ and changes linearly with time, then

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{b}t \tag{2.6}$$

with $\mathbf{b}$ a constant and

$$\begin{aligned}\mathbf{v}(t) &= \frac{d}{dt}(\mathbf{r}_0 + \mathbf{b}t)\\ &= \frac{d\mathbf{r}_0}{dt} + \frac{d(\mathbf{b}t)}{dt}\\ &= 0 + \mathbf{b}\frac{dt}{dt}\\ &= \mathbf{b}.\end{aligned} \tag{2.7}$$

It would have in fact been simpler to write

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}t \tag{2.8}$$

to start with. Application of the second of equations (2.5) would reveal the acceleration to be zero since the velocity is constant. It is found that in general

[1] This section contains advanced mathematical concepts on which you will ***not*** be tested. You may therefore skip this section if you desire, except for equations (2.14) and (2.19), which will be used extensively in the future.

$$\frac{d\left(t^n\right)}{dt}=nt^{n-1}, \tag{2.9}$$

and we find that the two derivatives in equation (2.7) correspond to the cases for $n=0$ and $n=1$. Other important derivatives include

$$\begin{aligned}
\frac{d}{dt}\cos(\omega t)&=-\omega\sin(\omega t)\\
\frac{d}{dt}\sin(\omega t)&=\omega\cos(\omega t)\\
\frac{d}{dt}e^{at}&=ae^{at}\\
\frac{d}{dt}\ln(bt)&=\frac{1}{t},
\end{aligned} \tag{2.10}$$

where ω, a, and b are constants. A brief study of equation (2.9) will quickly convince us that a constant acceleration $\mathbf{a}$ requires that $n=2$ since if we write (see equation (1.21) of Chapter 1)

$$\mathbf{r}(t)=\mathbf{r}_0+\mathbf{v}_0t+\frac{1}{2}\mathbf{a}t^2 \tag{2.11}$$

with $\mathbf{x}_0$ and $\mathbf{v}_0$ the initial position and velocity at $t=0$ (i.e., they are constants), then

$$\begin{aligned}
\mathbf{v}(t)&=\frac{d}{dt}\left(\mathbf{r}_0+\mathbf{v}_0t+\frac{1}{2}\mathbf{a}t^2\right)\\
&=\frac{d\mathbf{r}_0}{dt}+\frac{d\left(\mathbf{v}_0t\right)}{dt}+\frac{1}{2}\frac{d\left(\mathbf{a}t^2\right)}{dt}\\
&=0+\mathbf{v}_0\frac{dt}{dt}+\frac{1}{2}\mathbf{a}\frac{d\left(t^2\right)}{dt}\\
&=\mathbf{v}_0+\frac{1}{2}\mathbf{a}(2t)\\
&=\mathbf{v}_0+\mathbf{a}t
\end{aligned} \tag{2.12}$$

and

$$
\begin{aligned}
\mathbf{a} &= \frac{d}{dt}\left(\mathbf{v}_0 + \mathbf{a}t\right) \\
&= \frac{d\mathbf{v}_0}{dt} + \frac{d\left(\mathbf{a}t\right)}{dt} \\
&= 0 + \mathbf{a}\frac{dt}{dt} \\
&= \mathbf{a},
\end{aligned}
\tag{2.13}
$$

as should be expected. This process could be extended to more complicated systems by considering cases where $n > 2$.

One very useful set of equations can be obtained from this constant acceleration case

$$
\begin{aligned}
&\mathbf{a}(t) = \mathbf{a} \quad \text{(i.e., the acceleration is a constant)} \\
&\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}t \\
&\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2
\end{aligned}
\tag{2.14}
$$

by eliminating the time from them. To do so we consider each orientation independently, such that for the x direction

$$
t = \frac{1}{a_x}\left[v_x(t) - v_{0x}\right]. \tag{2.15}
$$

We then insert this equation into the third of equations (2.14)

$$
\begin{aligned}
x(t) &= x_0 + v_{0x}\frac{\left[v(t) - v_{0x}\right]}{a_x} + \frac{1}{2}a_x\frac{\left[v(t) - v_{0x}\right]^2}{a_x^2} \\
&= x_0 + \frac{\left[v(t) - v_{0x}\right]}{a_x}\left\{v_{0x} + \frac{1}{2}\left[v(t) - v_{0x}\right]\right\} \\
&= x_0 + \frac{1}{2a_x}\left[v(t) - v_{0x}\right]\left[v(t) + v_{0x}\right],
\end{aligned}
\tag{2.16}
$$

or

$$
x(t) - x_0 = \frac{v_x^2(t) - v_{0x}^2}{2a_x} \tag{2.17}
$$

and alternatively

$$\frac{1}{2}\left[v_x^2(t)-v_{0x}^2\right]=a_x\left[x(t)-x_0\right]. \tag{2.18}$$

It should also be noted that equations similar to equation (2.18) can be written for the y and z directions such that (by summing the three)

$$\frac{1}{2}\left[v^2(t)-v_0^2\right]=\mathbf{a}\cdot\left[\mathbf{r}(t)-\mathbf{r}_0\right]. \tag{2.19}$$

We will soon return to equation (2.19) and its application to problems involving work.

Studying equations (2.9) and (2.10) we realize that given the result obtained for a derivative we can guess the original function on which the derivative was effected. This "reverse engineering" process is often called the "antiderivative" and is symbolically written as

$$F=\int f(t)\,dt, \tag{2.20}$$

where it is said that the function F is the "antiderivative" or "primitive" of the function $f(t)$ (conversely $f(t)$ is the derivative of F and, as is more commonly the custom, equation (2.20) is referred to as an (indefinite) **integral**). We can therefore "guess" that

$$\begin{aligned}
\int t^n\,dt &= \frac{t^{n+1}}{n+1}+c \\
\int \cos(\omega t)\,dt &= \frac{1}{\omega}\sin(\omega t)+c \\
\int \sin(\omega t)\,dt &= \frac{-1}{\omega}\cos(\omega t)+c \\
\int e^{at}\,dt &= \frac{1}{a}e^{at}+c \\
\int \frac{dt}{t} &= \ln(t)+d \\
&= \ln(bt),
\end{aligned} \tag{2.21}$$

where $\omega, a, c,$ and $d=\ln(b)$ are constants.

There is another important connection between integrals, a primitive, and its derivative. Let us consider again a function $F(t)$ and its derivative $f(t)$, both a function of time as indicated. We seek to evaluate the following difference $F(t+\tau)-F(t)$. Let us further define

$$\tau=n\Delta t, \tag{2.22}$$

where Δt is an infinitesimal time interval and n a very large integer. The equation we seek to evaluate can therefore be written as

$$F(t+\tau)-F(t)=F(t+n\Delta t)-F(t). \tag{2.23}$$

We now transform this equation as follows

$$\begin{aligned}F(t+\tau)-F(t)=&\big[F(t+n\Delta t)-F\big(t+(n-1)\Delta t\big)\big]\\&+\big[F\big(t+(n-1)\Delta t\big)-F\big(t+(n-2)\Delta t\big)\big]\\&+\cdots+\big[F(t+\Delta t)-F(t)\big],\end{aligned} \tag{2.24}$$

which we slightly modify to

$$\begin{aligned}F(t+\tau)-F(t)=&\Big[F\Big(\big(t+(n-1)\Delta t\big)+\Delta t\Big)-F\big(t+(n-1)\Delta t\big)\Big]\\&+\Big[F\Big(\big(t+(n-2)\Delta t\big)+\Delta t\Big)-F\big(t+(n-2)\Delta t\big)\Big]\\&+\cdots+\big[F(t+\Delta t)-F(t)\big].\end{aligned} \tag{2.25}$$

However, we know from the definition of the derivative that

$$f(t)\equiv\lim_{\Delta t\to 0}\frac{F(t+\Delta t)-F(t)}{\Delta t}. \tag{2.26}$$

Inserting equation (2.26) in equation (2.25) we have

$$\begin{aligned}F(t+\tau)-F(t)=\lim_{\Delta t\to 0}\Big\{&f\big[t+(n-1)\Delta t\big]+f\big[t+(n-2)\Delta t\big]\\&+\cdots+f(t)\Big\}\Delta t,\end{aligned} \tag{2.27}$$

or in a more compact form

$$F(t+\tau)-F(t)=\lim_{\Delta t\to 0}\sum_{i=0}^{n-1}f(t+i\Delta t)\Delta t. \tag{2.28}$$

We can appreciate that the summation on the right-hand side of equation (2.28) represents the "area under the curve" traced by the derivative $f(t)$ since $f(t+i\Delta t)\Delta t$ is the area contained in the "column" of height $f(t+i\Delta t)$ and base Δt. We can therefore state the fundamental result:

The area contained between "under" a curve $f(t)$ between the points t and $t+\tau$ is given by the subtraction of the primitive $F(t)$ of $f(t)$ at those same two points, i.e., the area equals $F(t+\tau)-F(t)$.

Furthermore, since we earlier introduce the following notation to express the primitive as a function of its derivative (see equation (2.20))

$$F = \int f(t)\,dt, \tag{2.29}$$

we now write

$$F(t+\tau)-F(t) \equiv \int_t^{t+\tau} f(\lambda)\,d\lambda. \tag{2.30}$$

Equation (2.30) is meant to mathematically convey the exact words of the fundamental result enunciated above. For example, if $f(t)=t$ and we want to determined the area it covers between $t=0$ and $t=\tau$, we first use the first of equations (2.21) with $n=1$ to find that $F(t)=t^2/2$, and then

$$\begin{aligned} \int_0^\tau t\,dt &= \frac{(t=\tau)^2}{2} - \frac{(t=0)^2}{2} \\ &= \frac{\tau^2}{2}. \end{aligned} \tag{2.31}$$

This result is in perfect agreement with the common definition for the area contained within a triangle (i.e., the product of the base and the height divided by 2).

2.2.2 Kinetic Energy

We now return to equation (2.2) and define the displacement vector as the difference between the position at time t and that at $t=0$. That is,

$$\mathbf{s} = \mathbf{r}(t) - \mathbf{r}_0, \tag{2.32}$$

and in turn

$$W = \mathbf{F}_{\text{net}} \cdot \left[\mathbf{r}(t) - \mathbf{r}_0\right]. \tag{2.33}$$

If we now insert equation (2.19) into equation (2.33) we get another important and fundamental result (this one for physics as opposed to mathematics)

$$\mathbf{F}_{\text{net}}\cdot\left[\mathbf{r}(t)-\mathbf{r}_0\right]=m\mathbf{a}\cdot\left[\mathbf{r}(t)-\mathbf{r}_0\right]$$
$$=\frac{1}{2}mv^2(t)-\frac{1}{2}mv_0^2. \tag{2.34}$$

The quantity

$$K\equiv\frac{1}{2}mv^2 \tag{2.35}$$

is called the **kinetic energy** of the body (particle) of mass m. We are now in a position to state the so-called **work-energy theorem**: *the work done by the net force on a body equals the change in the body's kinetic energy*

$$\begin{aligned}W&=K_2-K_1\\&=\Delta K.\end{aligned} \tag{2.36}$$

In equation (2.36) the subscripts "1" and "2" correspond to start and end points between which the net force is applied and does work. We must, however, acknowledge that we arrived at this result through equation (2.19), which is itself based on the assumption that the acceleration, and therefore the net force, is constant. It is reasonable to ask whether this result connecting the work done on an object and the change in its kinetic energy is valid for the more general case where the acceleration is not constant but changes with position?

We answer this question by considering the infinitesimal amount of work done by a net force $\mathbf{F}_{\text{net}}$ when applied on a body at position $\mathbf{r}_1$ over an infinitesimal distance $\Delta\mathbf{r}$

$$\Delta W=\mathbf{F}_{\text{net}}\left(\mathbf{r}_1\right)\cdot\Delta\mathbf{r}. \tag{2.37}$$

Whether the net force is constant or not we can calculate the total work over a macroscopic distance $\mathbf{r}_2=\mathbf{r}_1+n\Delta\mathbf{r}$ with (n is once again a large integer)

$$W=\lim_{\Delta\mathbf{r}\to0}\sum_{i=0}^{n-1}\mathbf{F}_{\text{net}}\left(\mathbf{r}_1+i\Delta\mathbf{r}\right)\cdot\Delta\mathbf{r}. \tag{2.38}$$

This process is illustrated in Figure 1. We can now use the result given by equations (2.28) and (2.30) to write

$$W=\int_{\mathbf{r}_1}^{\mathbf{r}_2}\mathbf{F}_{\text{net}}\cdot d\mathbf{r}. \tag{2.39}$$

That is, *the total work done by a net force between two points equals the area covered by the curve defining the force between these two same points*.

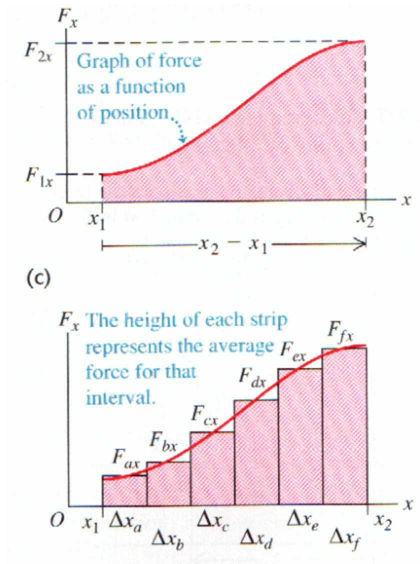

Figure 1 – One-dimensional representation of the process described in equation (2.38). We see that the work done consists of the area under the curve traced by $F(x)$.

To prove that equation (2.36) applied generally we now consider the following relation between the acceleration and the velocity

$$\begin{aligned}\mathbf{a}\cdot d\mathbf{r} &= \frac{d\mathbf{v}}{dt}\cdot d\mathbf{r}\\ &= \frac{d\mathbf{v}}{dt}\cdot \mathbf{v}dt\\ &= \frac{1}{2}\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v})dt,\end{aligned} \tag{2.40}$$

and finally

$$\mathbf{a}\cdot d\mathbf{r} = \frac{1}{2}d\left(v^2\right). \tag{2.41}$$

We can then modify equation (2.39) with

$$\begin{aligned}W &= \int_{\mathbf{r}_1}^{\mathbf{r}_2} m\mathbf{a}\cdot d\mathbf{r}\\ &= \frac{1}{2}m\int_{\mathbf{r}_1}^{\mathbf{r}_2} d\left(v^2\right),\end{aligned} \tag{2.42}$$

which finally proves that

$$
\begin{aligned}
W &= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \\
&= K_2 - K_1 \\
&= \Delta K,
\end{aligned}
\tag{2.43}
$$

with v_1 and v_2 the speed at $\mathbf{r}_1$ and $\mathbf{r}_2$. It is important to emphasize that we arrived at equation (2.43), the **work-energy theorem**, without the requirement of a constant acceleration (or net force).

2.3 Power

Power has units of energy per time, or Joule per second or **Watt** in MKS. We can define the **average power** by considering an amount of work ΔW and the time Δt over which it is accomplished

$$
P_{\text{av}} = \frac{\Delta W}{\Delta t}. \tag{2.44}
$$

By reducing the time interval to an infinitesimally small time dt we can also define the **instantaneous power** with

$$
\begin{aligned}
P &= \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} \\
&= \frac{dW}{dt}.
\end{aligned}
\tag{2.45}
$$

Since the infinitesimal work is defined as (see equation (2.37) in the limit $\Delta\mathbf{r} \to 0$)

$$
dW = \mathbf{F}_{\text{net}} \cdot d\mathbf{r}, \tag{2.46}
$$

we can write

$$
P = \mathbf{F}_{\text{net}} \cdot \mathbf{v} \tag{2.47}
$$

for the instantaneous power (since $\mathbf{v} = d\mathbf{r}/dt$).

2.4 Exercises

1. (Prob. 6.7 in Young and Freedman.) Two blocks are connected by a very light string passing over a massless and frictionless pulley (see Figure 2). Travelling at constant speed, the 20-N block moves 75 cm to the right and the 12-N block moves 75 cm downward. During this process, how much work is done (a) on the 12-N block by (i) gravity and (ii) the tension in the string? (b) On the 20-N block by (i) gravity, (ii) the

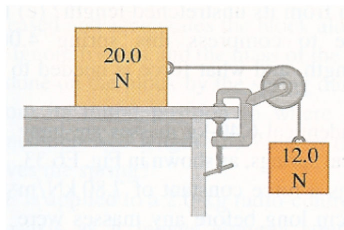

Figure 2 – Blocks-pulley arrangement for Problem 1.

tension in the string, (iii) friction, and (iv) the normal force? (c) Find the total work done on each block.

Solution.

Since the 12-N block moves at constant speed, its acceleration $a = 0$ and the tension T in the string is $T = 12\ \text{N}$. Since the 20-N block moves at constant speed the friction force f_k on it is to the left and $f_k = T = 12\ \text{N}$.

(a) We use $W = Fs\cos(\theta)$ for the work. For gravity on the 12-N block $\theta = 0$ and

$$\begin{aligned} W &= (12\ \text{N})(0.75\ \text{m})\cos(0) \\ &= 9.0\ \text{J}. \end{aligned} \tag{2.48}$$

For the tension the magnitude of the force is the same but $\theta = \pi$ so $W = -9.0\ \text{J}$.

(b) For gravity on the 20-N block $\theta = \pi/2$ and $W = 0$ since $\cos(\pi/2) = 0$. For the tension $\theta = 0$ and

$$\begin{aligned} W &= (12\ \text{N})(0.75\ \text{m})\cos(0) \\ &= 9.0\ \text{J}. \end{aligned} \tag{2.49}$$

For the friction $\theta = \pi$ and since $f_k = T = 12\ \text{N}$ we have $W = -9.0\ \text{J}$. Finally for the normal contact force $\theta = -\pi/2$ and $W = 0$.

(c) The net force on each block is zero, since their acceleration is also zero. It follows that the total work on each block must be zero.

2. (Prob. 6.19 in Young and Freedman.) Use the work-energy theorem to work these problems. Neglect air resistance in all cases. (a) A branch falls from a 95-m tall redwood tree, starting from rest. How fast is it moving when it reaches the ground? (b) A volcano ejects a boulder directly upward 525 m into the air. How fast was the boulder moving just as it left the volcano? (c) A skier moving at 5.0 m/s encounters a long, rough horizontal patch of snow having coefficient of friction 0.22 with her skies. How far does she travel on this patch before stopping? (d) Suppose the rough patch of part (c) is only 2.9 m long.

How fast would the skier be moving when she reached the end of the patch? (e) At the base of a frictionless icy hill that rises at 25° above the horizontal, a toboggan has a speed of 12.0 m/s toward the hill. How far vertically above the base will it go before stopping?

Solution.

The work-energy theorem, equation (2.43), states that $W = K_2 - K_1$ in general, with $K = mv^2/2$. But when the forces are constant we further have $W = \mathbf{F}_{\text{net}} \cdot \mathbf{s}$ (notice that there is no integral present). We will combine these two equations.

(a) For the redwood branch $K_1 = 0$, since it is starting from rest. Because gravity is oriented downward, as is the motion of the branch, we can write

$$\begin{aligned} \frac{W}{m} &= \frac{v_2^2}{2} \\ &= gs \\ &= \left(9.8 \text{ m/s}^2\right)\left(95 \text{ m}\right) \\ &= 931 \text{ J/kg}, \end{aligned} \tag{2.50}$$

which yields $v_2 = 43.2 \text{ m/s}$.

(b) For the boulder $K_2 = 0$ and the displacement is in a direction opposite that or gravity. We have

$$\begin{aligned} \frac{W}{m} &= -\frac{v_1^2}{2} \\ &= -gs \\ &= -\left(9.8 \text{ m/s}^2\right)\left(525 \text{ m}\right) \\ &= -5145 \text{ J/kg}, \end{aligned} \tag{2.51}$$

which gives $v_1 = 101.4 \text{ m/s}$.

(c) The only force acting on the skier is the friction force, which is directed against the direction of motion, and we also have $K_2 = 0$. We can thus write

$$\begin{aligned} \frac{W}{m} &= -\frac{v_1^2}{2} \\ &= -\mu_k gs \end{aligned} \tag{2.52}$$

or

$$\begin{aligned} s &= \frac{v_1^2}{2\mu_k g} \\ &= \frac{25.0\ \text{m}^2/\text{s}^2}{2\cdot 0.22\cdot 9.8\ \text{m/s}^2} \\ &= 5.8\ \text{m}. \end{aligned} \tag{2.53}$$

(d) In this case $K_2 \neq 0$ and

$$\begin{aligned} v_2^2 &= v_1^2 - 2\mu_k g s \\ &= 25\ \text{m}^2/\text{s}^2 - 2\cdot 0.22\cdot 9.8\ \text{m/s}^2\cdot 2.9\ \text{m} \\ &= 12.5\ \text{m}^2/\text{s}^2 \end{aligned} \tag{2.54}$$

or $v_2 = 3.53\ \text{m/s}$.

(e) For the toboggan $K_2 = 0$ and we write

$$\begin{aligned} \frac{W}{m} &= -\frac{v_1^2}{2} \\ &= -gs\sin\left(25^\circ\right) \end{aligned} \tag{2.55}$$

or the distance it will travel on the inclined icy surface is

$$s = \frac{v_1^2}{2g\sin\left(25^\circ\right)}. \tag{2.56}$$

The vertical distance travelled is therefore

$$\begin{aligned} y &= s\sin\left(25^\circ\right) \\ &= \frac{v_1^2}{2g} \\ &= \frac{144\ \text{m}^2/\text{s}^2}{2\left(9.8\ \text{m/s}^2\right)} \\ &= 7.35\ \text{m}. \end{aligned} \tag{2.57}$$

3. (Prob. 6.102 in Young and Freedman.) On a winter day a warehouse worker is shoving boxes up a rough plank inclined at an angle α from the horizontal. The plank is partially covered with ice, with more ice near the bottom of the plank than near the top, so that the coefficient of friction increases with distance x along the plank: $\mu = Ax$, where A is a

positive constant and the bottom of the plank is at $x=0$. (For this plank the coefficients of kinetic and static friction are equal: $\mu_k = \mu_s = \mu$.) The worker shoves a box up the plank so that it leaves the bottom of the plank at speed v_1. Show that when the box first comes to rest, it will remain at rest if

$$v_1^2 \geq \frac{3g\sin^2(\alpha)}{A\cos(\alpha)}. \tag{2.58}$$

Solution.

The component of gravity along the plank and oriented against the direction of motion is $mg\sin(\alpha)$, while the friction force, also oriented against the direction of motion, is $f = \mu mg\cos(\alpha)$. The total work per unit mass is given by

$$\begin{aligned} \frac{W}{m} &= \frac{1}{m}\int_1^2 \mathbf{F}_{\text{net}} \cdot d\mathbf{r} \\ &= -\int_1^2 \left[g\sin(\alpha) + Ag\cos(\alpha)x\right]dx \\ &= -g\left[\sin(\alpha)\int_1^2 dx + A\cos(\alpha)\int_1^2 x\,dx\right] \\ &= -g\left[\sin(\alpha)(x_2 - x_1) + A\cos(\alpha)\frac{1}{2}\left(x_2^2 - x_1^2\right)\right], \end{aligned} \tag{2.59}$$

but since $x_1 = 0$

$$\frac{W}{m} = -g\left[\sin(\alpha)x_2 + \frac{A}{2}\cos(\alpha)x_2^2\right]. \tag{2.60}$$

On the other hand we can also express the total work with

$$\frac{W}{m} = -\frac{v_1^2}{2}, \tag{2.61}$$

and by equating equations (2.60) and (2.61) we have

$$\frac{A}{2}g\cos(\alpha)x_2^2 + g\sin(\alpha)x_2 - \frac{v_1^2}{2} = 0. \tag{2.62}$$

We can solve this equation since it is a quadratic in x_2; the solution is given by

$$x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{2.63}$$

with

$$
\begin{aligned}
a &= \frac{A}{2} g \cos(\alpha) \\
b &= g \sin(\alpha) \\
c &= -\frac{v_1^2}{2}.
\end{aligned}
\tag{2.64}
$$

We therefore have

$$x_2 = \frac{-g\sin(\alpha) \pm \sqrt{g^2 \sin^2(\alpha) + Agv_1^2 \cos(\alpha)}}{Ag\cos(\alpha)}. \tag{2.65}$$

But since it must be that $x_2 > 0$ we choose

$$x_2 = \frac{\sqrt{g^2 \sin^2(\alpha) + Agv_1^2 \cos(\alpha)} - g\sin(\alpha)}{Ag\cos(\alpha)}. \tag{2.66}$$

However, for the box to stay at rest at the end of its travel we must have, from Newton's Second Law,

$$Ax_2 mg\cos(\alpha) - mg\sin(\alpha) \geq 0, \tag{2.67}$$

or

$$x_2 \geq \frac{\sin(\alpha)}{A\cos(\alpha)}. \tag{2.68}$$

Inserting equation (2.68) into equation (2.66) we find that

$$\frac{\sqrt{g^2 \sin^2(\alpha) + Agv_1^2 \cos(\alpha)} - g\sin(\alpha)}{Ag\cos(\alpha)} \geq \frac{\sin(\alpha)}{A\cos(\alpha)} \tag{2.69}$$

or

$$\sqrt{g^2 \sin^2(\alpha) + Agv_1^2 \cos(\alpha)} \geq 2g\sin(\alpha). \tag{2.70}$$

We can further transform this equation (squaring both sides) to get

$$v_1^2 \geq \frac{3g\sin^2(\alpha)}{A\cos(\alpha)}. \tag{2.71}$$

2.5 Gravitational Potential Energy and Conservation of Mechanical Energy

We know from some of the problems we previously worked out that gravity can do work on an object. That is, using equation (2.39), adapted for the gravitational force, we have

$$W_{\mathrm{grav}} = m\int_1^2 \mathbf{g}\cdot d\mathbf{r}. \tag{2.72}$$

We can easily simplify this equation by postulating that gravity is oriented along the negative y-axis and write

$$\begin{aligned} W_{\mathrm{grav}} &= -mg\int_1^2 \mathbf{e}_y \cdot d\mathbf{r} \\ &= -mg\int_1^2 dy \\ &= -mg\left(y_2 - y_1\right). \end{aligned} \tag{2.73}$$

We should note that the above derivation assumes that the gravitational acceleration g is the same (i.e., it is constant) for any position y relative to the earth's surface. This approximation is only valid as long as the distance between y and the earth's surface is insignificant in comparison to the radius of the earth (the mean radius is 6,371 km).

If we define the **gravitational potential energy** at y with

$$U_{\mathrm{grav}} = mgy, \tag{2.74}$$

then we can write for the work done by gravity

$$\begin{aligned} W_{\mathrm{grav}} &= -\left(U_{\mathrm{grav},2} - U_{\mathrm{grav},1}\right) \\ &= -\Delta U_{\mathrm{grav}}. \end{aligned} \tag{2.75}$$

Furthermore, we know from the work-energy theorem that, for cases where gravity is the only force acting on the body,

$$\begin{aligned} W_{\mathrm{grav}} &= K_2 - K_1 \\ &= \Delta K. \end{aligned} \tag{2.76}$$

Therefore combining equations (2.75) and (2.76) we find the fundamental result

$$\Delta K + \Delta U_{\text{grav}} = 0, \tag{2.77}$$

or

$$\begin{aligned} K_1 + U_{\text{grav},1} &= K_2 + U_{\text{grav},2} \\ \frac{1}{2}mv_1^2 + mgy_1 &= \frac{1}{2}mv_2^2 + mgy_2. \end{aligned} \tag{2.78}$$

The sum of the kinetic and gravitational energies $mv^2/2 + mgy$ is called the **total mechanical energy**. It follows from equation (2.78) that, *for such a close system*, the total mechanical energy is a constant

$$\begin{aligned} E &= \frac{1}{2}mv^2 + mgy \\ &= \text{constant}. \end{aligned} \tag{2.79}$$

That is, the total mechanical energy is a *conserved quantity*. This leads to the statement (or theorem) of the **conservation of mechanical energy**: *When only the force of gravity does work on a system, the total mechanical energy is conserved.*

Looking back at some of the problems we previously solved in this chapter that only involved gravity (e.g., the cases of the redwood tree and the boulder), it is clear that we have already been using the notion that the total mechanical energy is conserved to obtain solutions. But what happens when other forces beside gravity are at play? Does the conservation of mechanical energy still apply?

To verify this we can easily generalize equation (2.72) to include any other forces beyond gravity and define the total work with

$$\begin{aligned} W &= \int_1^2 \left(\mathbf{F}_{\text{other}} - m\mathbf{g}\right) \cdot d\mathbf{r} \\ &= \int_1^2 \mathbf{F}_{\text{other}} \cdot d\mathbf{r} - m\int_1^2 \mathbf{g} \cdot d\mathbf{r} \\ &= W_{\text{other}} + W_{\text{grav}} \\ &= W_{\text{other}} + U_{\text{grav},1} - U_{\text{grav},2}. \end{aligned} \tag{2.80}$$

Once again using equation (2.43) (i.e., the work-energy theorem) we find that

$$\begin{aligned} K_1 + U_{\text{grav},1} + W_{\text{other}} &= K_2 + U_{\text{grav},2} \\ \frac{1}{2}mv_1^2 + mgy_1 + W_{\text{other}} &= \frac{1}{2}mv_2^2 + mgy_2. \end{aligned} \tag{2.81}$$

Alternatively, we can write

$$W_{\text{other}} = \Delta K + \Delta U_{\text{grav}}. \tag{2.82}$$

That is, *the work done by all forces other than gravity equals the change in the total mechanical energy of the system*.

2.6 Elastic Potential Energy

It is possible to store energy, or to "acquire" potential energy, in ways different than by raising an object subjected to gravity (in that case, increasing the position y above the earth's surface increases the gravitational potential energy $U_{\text{grav}} = mgy$). For example, experiments show that a (moderate) stretching or compressing of a spring implies the presence of a restoring force

$$F = -kx, \tag{2.83}$$

where k is the spring's **force constant**. At $x = 0$ the spring is neither stretched nor compressed and the force is zero, when $x < 0$ the spring is compressed and exerts a (positive) repulsive force, while when $x > 0$ an attractive (negative) force is present. This linear response of the spring given by equation (2.83), known as **Hooke's Law**, is valid only as long as $|x|$ is small enough. This is probably the simplest example of an **elastic force**. We could, for example, similarly model the response of a rubber band.

It is then straightforward to consider the work done by a spring with

$$\begin{aligned} W_{\text{el}} &= \int_1^2 F\,dx \\ &= -k\int_1^2 x\,dx \\ &= -\frac{1}{2}k\left(x_2^2 - x_1^2\right). \end{aligned} \tag{2.84}$$

We can further associate this work with an energy $kx^2/2$ pertaining to the spring. It should be clear that we are not dealing with a kinetic energy since there is no dependency on any velocity in equation (2.84). Instead this energy is a function of the (square of the) position x, not unlike (although not exactly the same) as is the case for gravity. We therefore define

$$U_{\text{el}} = \frac{1}{2}kx^2 \tag{2.85}$$

as the **elastic potential energy**. As was the case for gravity (see equation (2.75) we write

$$W_{\text{el}} = -\Delta U_{\text{el}}. \tag{2.86}$$

It follows that for a system that includes gravity, a spring (or other media exhibiting elastic responses), and other forces we can generalize equations (2.81)

$$\begin{gathered}K_1 + U_{\text{grav},1} + U_{\text{el},1} + W_{\text{other}} = K_2 + U_{\text{grav},2} + U_{\text{el},2} \\ \frac{1}{2}mv_1^2 + mgy_1 + \frac{1}{2}kx_1^2 + W_{\text{other}} = \frac{1}{2}mv_2^2 + mgy_2 + \frac{1}{2}kx_1^2\end{gathered} \tag{2.87}$$

and equation (2.82) for the theorem of conservation of mechanical energy

$$W_{\text{other}} = \Delta K + \Delta U_{\text{grav}} + \Delta U_{\text{el}}. \tag{2.88}$$

2.6.1 Exercises

4. (Prob. 7.42 in Young and Freedman.) A 2.0-kg block is pushed against a spring of negligible mass and force constant $k = 400$ N/m, compressing it 0.22 m. The block is then released, and moves along a frictionless, horizontal surface then up an incline with slope $\theta = 37.0^\circ$ (see Figure 3). (a) What is the speed of the block as it slides along the horizontal surface after having left the spring? (b) How far does the block travel up the inclined plane before starting to slide back down?

Solution.

There are two forces at play in this problem: the restoring force of the spring and gravity. But these two forces are linked to corresponding potential energies. We can therefore use equation (2.87) (or (2.88)) with $W_{\text{other}} = 0$ to obtain a solution.

(a) If we denote by "1" and "2" the physical conditions of the system before and after the mass is released, respectively, then we have

$$U_{\text{el},1} + U_{\text{grav},1} + K_1 = U_{\text{el},2} + U_{\text{grav},2} + K_2. \tag{2.89}$$

But it should be clear that

$$K_1 = U_{\text{el},2} = 0, \tag{2.90}$$

since $v_1 = x_2 = 0$, and that

$$U_{\text{grav},1} = U_{\text{grav},2}. \tag{2.91}$$

It therefore follows that

$$\frac{1}{2}kx_1^2 = \frac{1}{2}mv_2^2, \tag{2.92}$$

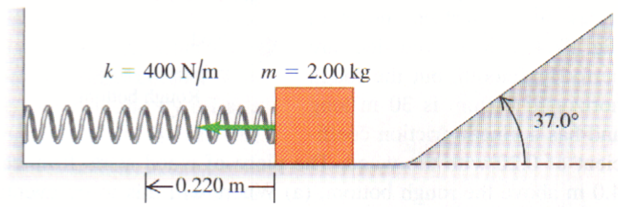

Figure 3 – Set up for Prob. 4.

or

$$v_2 = \sqrt{\frac{k}{m}} x_1 = \sqrt{\frac{400\ \text{N/m}}{2.0\ \text{kg}}} 0.22\ \text{m} = 3.1\ \text{m/s}. \qquad (2.93)$$

(b) We now denote by "3" the conditions when the block reaches its highest elevation on the inclined plane. We therefore write

$$U_{\text{grav},2} + K_2 = U_{\text{grav},3} + K_3, \qquad (2.94)$$

with or since $v_3 = 0$

$$\Delta U_{\text{grav}} = K_2 \qquad (2.95)$$

where $\Delta U_{\text{grav}} = U_{\text{grav},3} - U_{\text{grav},2} = mg\Delta y$ and Δy the change in the block's position in the vertical direction. We thus write

$$\Delta y = \frac{v_2^2}{2g} = \frac{9.68\ \text{m}^2/\text{s}^2}{2 \cdot 9.8\ \text{m/s}^2} = 0.494\ \text{m}. \qquad (2.96)$$

However, we seek to find out how far on the incline the block gets to before coming to a stop. If we define this distance as Δl, then

$$\Delta l = \frac{\Delta y}{\sin(\theta)} = 0.821\ \text{m}. \qquad (2.97)$$

2.7 Conservative and Non-conservative Forces

Whenever an object moves up against gravity or a spring is compressed, energy is stored in what we call potential energy. Since we have conservation of mechanical energy, which is the sum of the potential and kinetic energies, this stored energy can later be used to generate, or be transferred to, kinetic energy. Such forces are called **conservative forces**.

A conservative force has interesting properties. For example, we already know that the work done by such a force is given by the following relation (or definition)

$$W = \int_1^2 \mathbf{F}_\mathrm{c} \cdot d\mathbf{r}, \tag{2.98}$$

where $\mathbf{F}_\mathrm{c}$ is the force under consideration. We also have established that the potential energy associated with the force is defined as (see equation (2.75))

$$U_{\mathrm{c},1} - U_{\mathrm{c},2} = \int_1^2 \mathbf{F}_\mathrm{c} \cdot d\mathbf{r}. \tag{2.99}$$

It should be clear that this equation implies that the change in potential energy due to the action of the force is *completely independent of the path the body traces in space*. That is, *the change in potential energy associated with a conservative force is only a function of the initial and final points of the path traced by the body*. For example, if one raises an object by Δy against gravity, then the increase in gravitational potential energy is $mg\Delta y$ whether the object was lifted in a straight line or through a curved (and perhaps tortuous) path. An obvious consequence of this is that if the initial and final points are the same, then the change in potential energy is zero.

[2]Another very interesting property of conservative forces is found if we consider the following equation (using Cartesian coordinates)

$$\nabla U_\mathrm{c} \equiv \frac{\partial U_\mathrm{c}}{\partial x}\mathbf{e}_x + \frac{\partial U_\mathrm{c}}{\partial y}\mathbf{e}_y + \frac{\partial U_\mathrm{c}}{\partial z}\mathbf{e}_z. \tag{2.100}$$

The function ∇U_c (note that it is a vector) is commonly known as the **gradient** of U_c. Now if we also consider that

$$d\mathbf{r} = dx\,\mathbf{e}_x + dy\,\mathbf{e}_y + dz\,\mathbf{e}_z, \tag{2.101}$$

then we have for the following integral

---

[2] The following discussion on conservative forces is based on advanced mathematical material on which you will ***not*** be tested. You may therefore skip this discussion if you desire, except for equations (2.100) and (2.105).

$$\begin{aligned}\int_1^2 \nabla U_\mathrm{c}\cdot d\mathbf{r} &= \int_1^2\left(\frac{\partial U_\mathrm{c}}{\partial x}\mathbf{e}_x+\frac{\partial U_\mathrm{c}}{\partial y}\mathbf{e}_y+\frac{\partial U_\mathrm{c}}{\partial z}\mathbf{e}_z\right)\cdot\left(dx\,\mathbf{e}_x+dy\,\mathbf{e}_y+dz\,\mathbf{e}_z\right)\\ &= \int_1^2\left(\frac{\partial U_\mathrm{c}}{\partial x}dx+\frac{\partial U_\mathrm{c}}{\partial y}dy+\frac{\partial U_\mathrm{c}}{\partial z}dz\right),\end{aligned}\tag{2.102}$$

since $\mathbf{e}_x\cdot\mathbf{e}_x=\mathbf{e}_y\cdot\mathbf{e}_y=\mathbf{e}_z\cdot\mathbf{e}_z=1$ and $\mathbf{e}_x\cdot\mathbf{e}_y=\mathbf{e}_x\cdot\mathbf{e}_z=\mathbf{e}_y\cdot\mathbf{e}_z=0$.

We should now step back for a moment and ask ourselves what would the total change dU_c be for the function U_c as we go from the point $\mathbf{r}=x\mathbf{e}_x+y\mathbf{e}_y+z\mathbf{e}_z$ to the point $\mathbf{r}+d\mathbf{r}=(x+dx)\mathbf{e}_x+(y+dy)\mathbf{e}_y+(z+dz)\mathbf{e}_z$, if $d\mathbf{r}$ is infinitesimally small? Clearly we would have to look at the rates with which U_c changes along the $x, y,$ and z directions and multiply these rates with $dx, dy,$ and dz, respectively. That is, we would have

$$dU_\mathrm{c}=\frac{\partial U_\mathrm{c}}{\partial x}dx+\frac{\partial U_\mathrm{c}}{\partial y}dy+\frac{\partial U_\mathrm{c}}{\partial z}dz.\tag{2.103}$$

Inserting this relation in equation (2.102) yields

$$\begin{aligned}\int_1^2 \nabla U_\mathrm{c}\cdot d\mathbf{r} &= \int_1^2 dU_\mathrm{c}\\ &= U_{\mathrm{c},2}-U_{\mathrm{c},1}.\end{aligned}\tag{2.104}$$

Comparing equations (2.99) and (2.104) we find the fundamental result

$$\mathbf{F}_\mathrm{c}=-\nabla U_\mathrm{c}.\tag{2.105}$$

That is, *a conservative force equals the negative of the gradient of the corresponding potential energy*.

If, for example, we consider the gravitational force and the spring restoring force we have

$$\begin{aligned}U_\mathrm{grav} &= mgy\\ \mathbf{F}_\mathrm{grav} &= -\nabla U_\mathrm{grav}\\ &= -mg\nabla y\\ &= -mg\left(\frac{\partial y}{\partial x}\mathbf{e}_x+\frac{\partial y}{\partial y}\mathbf{e}_y+\frac{\partial y}{\partial z}\mathbf{e}_z\right)\\ &= -mg\mathbf{e}_y\end{aligned}\tag{2.106}$$

and

$$
\begin{aligned}
U_{\mathrm{el}} &= \frac{1}{2}kx^2 \\
\mathbf{F}_{\mathrm{el}} &= -\nabla U_{\mathrm{el}} \\
&= -\frac{1}{2}k\nabla\left(x^2\right) \\
&= -\frac{1}{2}k\left[\frac{\partial\left(x^2\right)}{\partial x}\mathbf{e}_x + \frac{\partial\left(x^2\right)}{\partial y}\mathbf{e}_y + \frac{\partial\left(x^2\right)}{\partial z}\mathbf{e}_z\right] \\
&= -kx\,\mathbf{e}_x,
\end{aligned}
\tag{2.107}
$$

respectively. We can verify that these results are in perfect agreement with the relations we previously obtained for these conservative forces.

We finally note that there is a fair amount of freedom in defining a potential energy. Let us clarify this statement by assuming that we have $U_{\mathrm{grav}} = mgy$ for the gravitational potential energy of an object. Then according to equations (2.105) and (2.106)

$$
\begin{aligned}
\mathbf{F}_{\mathrm{grav}} &= -\nabla U_{\mathrm{grav}} \\
&= -mg\mathbf{e}_y.
\end{aligned}
\tag{2.108}
$$

Now suppose that we redefine the gravitational potential energy with

$$
\begin{aligned}
U'_{\mathrm{grav}} &= U_{\mathrm{grav}} + A \\
&= mgy + A,
\end{aligned}
\tag{2.109}
$$

where A is a constant. We then find that

$$
\begin{aligned}
\mathbf{F}'_{\mathrm{grav}} &= -\nabla U'_{\mathrm{grav}} \\
&= -\nabla U_{\mathrm{grav}} - \nabla A \\
&= -\nabla U_{\mathrm{grav}} \\
&= \mathbf{F}_{\mathrm{grav}},
\end{aligned}
\tag{2.110}
$$

since the derivative of a constant is zero (i.e., $\nabla A = 0$). Because the forces resulting from U_{grav} and U'_{grav} are the same we see that adding a constant A to the potential energy has absolutely no effect on the dynamics on the system (i.e., it does not cause an additional acceleration on the object). It is therefore said that *a potential energy can only be defined up to a constant value*.

On the other hand, **non-conservative forces** do not share these characteristics. If we consider the kinetic friction force as an example, it is clear that the energy that will be dissipated from the friction as an object slides on a surface will be lost (to heat generation) and will not be available to the system at later times for mechanical work.

Furthermore, the amount of energy dissipated will be highly dependent on the path taken by the object. The longer the path the larger the amount of energy lost for mechanical energy. It therefore follows that we cannot associated a potential energy to a non-conservative force nor can we define it as the gradient of some function.

Nonetheless, non-conservative forces can be incorporated into the law of conservation of energy by incorporating the work they do on a system in the W_{other} term included in equations (2.87) and (2.88). It is also possible to formulate things differently by considering the amount of heat generated by the energy dissipation (for a kinetic friction force, for example). One can then define a new term ΔU_{int}, which accounts for the **internal energy** (this is what heat is, a form of an internal energy). More precisely, we write

$$\Delta U_{\text{int}} = -W_{\text{other}}. \tag{2.111}$$

Equation (2.111) can actually be established experimentally. The **law of conservation of energy** can then be generally written as

$$K_1 + U_1 - \Delta U_{\text{int}} = K_2 + U_2 \tag{2.112}$$

or alternatively

$$\Delta K + \Delta U + \Delta U_{\text{int}} = 0. \tag{2.113}$$

2.7.1 Exercises

5. (Prob. 7.66 in Young and Freedman.) A truck with mass m has a brake failure while going down an icy mountain road of constant downward slope angle α. Initially the truck is moving downhill at speed v_1. After careening downhill a distance L with negligible friction, the truck driver steers the runaway vehicle onto a runaway truck ramp of constant upward slope angle β. The truck ramp has a soft sand surface for which the coefficient of rolling friction is μ_{r}. What is the distance the truck moves up the ramp before coming to a halt? Solve using energy methods.

Solution.

If we denote the initial and final states with "1" and "2", respectively, then we can write

$$\begin{aligned} K_1 + U_{\text{grav},1} + W_{\text{other}} &= K_2 + U_{\text{grav},2} \\ \frac{1}{2}mv_1^2 + mgy_1 + W_{\text{other}} &= \frac{1}{2}mv_2^2 + mgy_2. \end{aligned} \tag{2.114}$$

But we know that $y_1 = L\sin(\alpha)$ and $v_2 = 0$, and if we define d as the distance the truck moves up the ramp before stopping, then $y_2 = d\sin(\beta)$ and $W_{\text{other}} = -\mu_r mg\cos(\beta)d$. We now have

$$\frac{1}{2}mv_1^2 + mgL\sin(\alpha) - \mu_r mgd\cos(\beta) = mgd\sin(\beta), \tag{2.115}$$

and

$$d = \frac{\left(v_1^2/2g\right) + L\sin(\alpha)}{\sin(\alpha) + \mu_r\cos(\beta)}. \tag{2.116}$$

6. (Prob. 7.87 in Young and Freedman.) A proton with mass m moves in one dimension. Its potential energy is given by $U(x) = \alpha/x^2 - \beta/x$, where α and β are positive constants. The proton is released from rest at $x_0 = \alpha/\beta$. (a) Show that the potential energy can be written as

$$U(x) = \frac{\alpha}{x_0^2}\left[\left(\frac{x_0}{x}\right)^2 - \frac{x_0}{x}\right]. \tag{2.117}$$

Graph $U(x)$. Calculate $U(x_0)$ and thereby locate the point x_0 on the graph. (b) Calculate $v(x)$ and give a qualitative description of the motion. (c) For what value of x is the speed of the proton maximum and what is that value? (d) What is the force on the proton when its speed is maximum? (e) Let the proton be released instead at $x_1 = 3\alpha/\beta$. Locate x_1 on the graph of $U(x)$. Calculate $v(x)$ and give a qualitative description of the motion. (f) For each of the release points x_0 and x_1, what are the maximum and minimum values of x reached during the motion?

Solution.

(a) By factoring α/x_0^2 on the right-hand side of equation (2.117), while remembering that $\beta = \alpha/x_0$

$$\begin{aligned} U(x) &= \frac{\alpha}{x_0^2}\frac{x_0^2}{x^2} - \frac{\alpha}{x_0 x} \\ &= \frac{\alpha}{x_0^2}\frac{x_0^2}{x^2} - \frac{\alpha}{x_0^2}\frac{x_0}{x} \\ &= \frac{\alpha}{x_0^2}\left(\frac{x_0^2}{x^2} - \frac{x_0}{x}\right). \end{aligned} \tag{2.118}$$

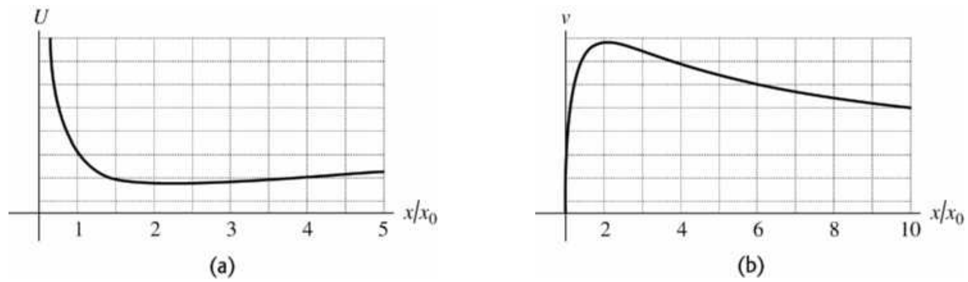

Figure 4 – Curves for $U(x)$ and $v(x)$ in Problem 6.

It is clear that $U(x_0)=0$, while $U(x)$ is positive for $x<x_0$ and negative when $x>x_0$. A graph for the potential energy is shown in Figure 4a.

(b) At x_0 the proton is released from rest, which implies that $v(x_0)=K(x_0)=0$, and since $U(x_0)=0$ we find that the total mechanical energy of the proton is

$$E=K+U=0. \tag{2.119}$$

Since energy is conserved we have

$$K=-U, \tag{2.120}$$

or

$$\begin{aligned} v(x) &= \sqrt{-\frac{2U(x)}{m}} \\ &= \sqrt{\frac{2\alpha}{mx_0^2}\left(\frac{x_0}{x}-\frac{x_0^2}{x^2}\right)}. \end{aligned} \tag{2.121}$$

The proton moves in the positive x-direction, speeding up until it reaches a maximum speed, and then slows down but it never stops. The proton cannot be found at $x<x_0$ since the quantity under the square root would then be negative. It will therefore only be found where $U(x)<0$ when $x>x_0$. A graph of $v(x)$ is shown in Figure 4b.

(c) The velocity will be at a maximum when the kinetic energy will also be maximum, which from equation (2.119) implies that the potential energy will then be at a minimum. A close examination of Figure 4a shows that the minimum for $U(x)$ happens when $dU/dx=0$. We therefore calculate

$$\frac{dU}{dx} = -\frac{\alpha}{x_0^2}\left(\frac{2x_0^2}{x^3} - \frac{x_0}{x^2}\right)$$
$$= -\frac{\alpha}{x_0 x^2}\left(\frac{2x_0}{x} - 1\right), \tag{2.122}$$

which means that $dU/dx = 0$ when $x = 2x_0$. The maximum speed is then

$$v_{\max} = \sqrt{\frac{\alpha}{2mx_0^2}}. \tag{2.123}$$

(d) We can establish from equation (2.105) that

$$\begin{aligned}\mathbf{F} &= -\nabla U \\ &= -\left(\frac{dU}{dx}\mathbf{e}_x\right) \\ &= 0.\end{aligned} \tag{2.124}$$

(e) At $x_1 = 3\alpha/\beta$ the potential energy is

$$U(x_1) = -\frac{2\alpha}{9x_0^2}, \tag{2.125}$$

and from the law of conservation of energy

$$\frac{1}{2}mv^2(x) + U(x) = U(x_1), \tag{2.126}$$

since $v(x_1) = 0$. It follows that

$$\begin{aligned}v(x) &= \sqrt{\frac{2}{m}\left[U(x_1) - U(x)\right]} \\ &= \sqrt{\frac{2\alpha}{mx_0^2}\left(\frac{x_0}{x} - \frac{x_0^2}{x^2} - \frac{2}{9}\right)}.\end{aligned} \tag{2.127}$$

Since at $x_1 > 3x_0$ the quantity under the square root is negative, the proton is confined to $x \leq 3x_0$, where $U(x) \leq U(x_1)$. But we can see in Figure 4a that there is also a minimum value for x below which $U(x) > U(x_1)$; the proton will therefore oscillate between that minimum location and x_1.

(f) When the proton is released at x_0 we have

$$\begin{aligned}-U(x) &= \frac{\alpha}{x_0^2}\left(\frac{x_0^2}{x^2}-\frac{x_0}{x}\right) \\ &= \frac{\alpha}{x_0^2}\frac{x_0}{x}\left(\frac{x_0}{x}-1\right),\end{aligned} \tag{2.128}$$

which means that the speed (and kinetic energy) will be zero at x_0 and at $x \to \infty$. The proton will therefore keep moving to more positive values of x without stopping when released.

When the proton is released at $x_1 = 3x_0$ we note that we can expand

$$\begin{aligned}U(x_1)-U(x) &= \frac{\alpha}{x_0^2}\left(\frac{x_0}{x}-\frac{x_0^2}{x^2}-\frac{2}{9}\right) \\ &= -\frac{\alpha}{x_0^2}\left(\frac{x_0}{x}-\frac{1}{3}\right)\left(\frac{x_0}{x}-\frac{2}{3}\right).\end{aligned} \tag{2.129}$$

The speed (and kinetic energy) will therefore be zero at $x_1 = 3x_0$ (as we already knew) and $x = 3x_0/2$. The proton will then be oscillating between these two points, i.e., $3x_0/2 \le x \le 3x_0$.