Assignment #2

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1. Use the laws of propositional calculus to find a formula in disjunctive normal for tautologically equivalent to the formula:

$$(p \to q \ \wedge \neg r) \to (p \vee r \to \neg q \wedge r)$$

State all laws of propositional calculus that you use. Simplify your result as must as possible.

$(p \to q \ \wedge \neg r) \to (p \vee r \to \neg q \wedge r)$	Elimination of implication
$\neg(p \to (q \ \wedge \neg r)) \vee ((p \vee r) \to (\neg q \wedge r))$	Elimination of implication
$\neg(\neg p \vee (q \ \wedge \neg r)) \vee (\neg(p \vee r) \vee (\neg q \wedge r))$	Elimination of implication
$(p \wedge \neg(q \ \wedge \neg r)) \vee ((\neg p \wedge \neg r) \vee (\neg q \wedge r))$	Double-negation law
$(p \wedge (\neg q \ \vee r)) \vee ((\neg p \wedge \neg r) \vee (\neg q \wedge r))$	Distributive law
$(p \wedge \neg q) \vee (p \wedge r) \vee (\neg p \wedge \neg r) \vee (\neg q \wedge r)$	

Therefore, $(p \to q \ \wedge \neg r) \to (p \vee r \to \neg q \wedge r)$ is equivalent to the disjunctive formula $(p \wedge \neg q) \vee (p \wedge r) \vee (\neg p \wedge \neg r) \vee (\neg q \wedge r)$

2. Use the laws of propositional calculus to find a formula in conjunctive normal for tautologically equivalent to the formula:

$$\big((\neg A \leftrightarrow B) \vee C\big) \wedge (\neg B \wedge C)$$

State all laws of propositional calculus that you use. Simplify your result as must as possible.

$\big((\neg A \leftrightarrow B) \vee C\big) \wedge (\neg B \wedge C)$	Elimination of implication
$(((\neg A \to B) \wedge (\neg A \leftarrow B)) \vee C) \wedge (\neg B \wedge C)$	Distributive law
$\big(\big((\neg A \to B) \vee C\big) \wedge \big((\neg A \leftarrow B) \vee C\big)\big) \wedge (\neg B \wedge C)$	Elimination of implication
$\big((\neg A \to B) \vee C\big) \wedge \big((B \to \neg A) \vee C\big) \wedge (\neg B \wedge C)$	Elimination of implication
$\big((A \vee B) \vee C\big) \wedge \big((\neg B \vee \neg A) \vee C\big) \wedge (\neg B \wedge C)$	Associative law
$(A \vee B \vee C) \wedge (\neg B \vee \neg A \vee C) \wedge \neg B \wedge C$	Commutative law
$\big((A \vee B \vee C) \wedge C\big) \wedge \big((\neg A \vee \neg B \vee C) \wedge C\big) \wedge \neg B \wedge C$	Idempotent law
$C \wedge C \neg B \wedge C$	Absorption law
$\neg B \wedge C$	Idempotent law

Therefore, $\big((\neg A \leftrightarrow B) \vee C\big) \wedge (\neg B \wedge C)$ is equivalent to the conjunctive formula $\neg B \wedge C$

3. Use the truth table method to find the disjunctive normal for of the formula in **question 1**, and the conjunctive normal form of the formula in **question 2**. How do the resulting formula compare to the ones you obtained in **questions 1 and 2**? Justify your answer.

$(p \to q \ \land \neg r) \to (p \lor r \to \neg q \land r)$

p	q	r	$\neg q$	$\neg r$	$q \ \land \neg r$	$p \to q \ \land \neg r$	$p \lor r$	$\neg q \land r$	$p \lor r \to \neg q \land r$	$(p \to q \ \land \neg r) \to (p \lor r \to \neg q \land r)$
0	0	0	1	1	0	1	0	0	1	1
0	0	1	1	0	0	1	1	1	1	1
0	1	0	0	1	1	1	0	0	1	1
0	1	1	0	0	0	1	1	0	0	0
1	0	0	1	1	0	0	1	0	0	1
1	0	1	1	0	0	0	1	1	1	1
1	1	0	0	1	1	1	1	0	0	0
1	1	1	0	0	0	0	1	0	0	1

Therefore, since $(p \to q \ \land \neg r) \to (p \lor r \to \neg q \land r)$ is true when the following values are true,

$\neg p, \neg q, \neg\, r$
$\neg p, \neg q, r$
$\neg p, q, \neg\, r$
$p, \neg q, \neg\, r$
$p, \neg q, r$
p, q, r

We can conclude that the disjunctive formula is,

$(\ \neg p \land \neg q \land \neg\, r) \lor (\neg p \land \neg q \land r) \lor (\neg p \land q \land \neg\, r) \lor (p \land \neg q \land \neg\, r) \lor (p \land \neg q \land r) \lor (p \land q \land r)$

$\big((\neg A \leftrightarrow B) \vee C\big) \wedge (\neg B \wedge C)$

A	B	C	$\neg A$	$\neg B$	$\neg A \leftrightarrow B$	$(\neg A \leftrightarrow B) \vee C$	$\neg B \wedge C$	$\big((\neg A \leftrightarrow B) \vee C\big) \wedge (\neg B \wedge C)$
0	0	0	1	1	0	0	0	0
0	0	1	1	1	0	1	1	1
0	1	0	1	0	1	1	0	0
0	1	1	1	0	1	1	0	0
1	0	0	0	1	1	1	0	0
1	0	1	0	1	1	1	1	1
1	1	0	0	0	0	0	0	0
1	1	1	0	0	0	1	0	0

Therefore, since $\big((\neg A \leftrightarrow B) \vee C\big) \wedge (\neg B \wedge C)$ is false when the following values are true,

$\neg A, \neg B, \neg C$
$\neg A, B, \neg C$
$\neg A, B, C$
$A, \neg B, \neg C$
$A, B, \neg C$
A, B, C

We can conclude that the disjunctive formula for $\neg\big((\neg A \leftrightarrow B) \vee C\big) \wedge (\neg B \wedge C)$ is,

$$(\neg A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge B \wedge \neg C) \vee (\neg A \wedge B \wedge C) \vee (A \wedge \neg B \wedge \neg C) \vee (A \wedge B \wedge \neg C) \vee (A \wedge B \wedge C)$$

and that the conjunctive formula for $\big((\neg A \leftrightarrow B) \vee C\big) \wedge (\neg B \wedge C)$ is,

$$\neg(\neg A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge B \wedge \neg C) \vee (\neg A \wedge B \wedge C) \vee (A \wedge \neg B \wedge \neg C) \vee (A \wedge B \wedge \neg C) \vee (A \wedge B \wedge C)$$

$$(A \vee B \vee C) \wedge (A \vee \neg B \vee C) \wedge (A \vee \neg B \vee \neg C) \wedge (\neg A \vee B \vee C) \wedge (\neg A \vee \neg B \vee C) \wedge (\neg A \vee \neg B \vee \neg C)$$

The resulting formulas compare the formulas in questions 1 and 2 by the fact that they are equal. Furthermore, the formulas in question 3 include every case produced by the truth table, but the results in questions 1 and 2 include a more compressed version of these results, thanks to the propositional laws of calculus.

4. Connectives.
    a. Show that the set of connectives $\{\wedge, \leftrightarrow, \ \oplus\}$ is adequate, where $\oplus$ is define the truth table:

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

During the class discussion we conclude that any set of connectives is adequate if it can generate a negation and has one of the following connectives $\{\wedge, \vee, \rightarrow\}$

Therefore, we use a truth table to find an expression that is equivalent to a negation.

p	$\neg p$	$(p \oplus p) \leftrightarrow p$
0	1	1
1	0	0

Therefore, since $p \oplus p \leftrightarrow p$ is logically equivalent to $\neg p$. We can conclude that this set $\{\wedge, \leftrightarrow, \ \oplus\}$, is an adequate set, because the set $\{\wedge, \neg\}$ is an adequate set that can be extracted from $\{\wedge, \leftrightarrow, \ \oplus\}$.

    b. Show that the set of connectives $\{\wedge, \leftrightarrow\}$ is not adequate.

To prove that this set is an inadequate set, the set must be able to generate a formula that only uses one variable and is able to equal the negation of that variable. If we look at the truth tables of $\{\wedge, \leftrightarrow\}$ we find that there does not exist a function that uses one variable and is able to generate a negation of that variable. This is because the $\leftrightarrow$ will always return a true whenever its variables are the same, and if the $\wedge$ will return 1 if its values are 1 and 0 otherwise. Any combination of these two connectives will either result in a constant value of 1, or 0, or the same value, p.

However, the formula $0 \leftrightarrow p$ is a valid formula. This formula is logically equivalent to the negation of p. In this case the set of connectives $\{\wedge, \leftrightarrow\}$ is not adequate.

5. Give formal proofs that the following arguments are valid using only the 11 rules of formal deduction and the theorems proved in class. State each rule you use.

   a. Argument:

      $(A \vee B) \wedge \neg C$
      $\neg C \rightarrow (D \wedge \neg A)$
      $B \rightarrow (A \vee E)$
      $Con: E \vee F$

      $(A \vee B) \wedge \neg C, \neg C \rightarrow (D \wedge \neg A), B \rightarrow (A \vee E) \vdash E \vee F$

      Let $(A \vee B) \wedge \neg C, \neg C \rightarrow (D \wedge \neg A), B \rightarrow (A \vee E) = \sum$

      Proof: $\sum \vdash E \vee F$

1. $\sum \vdash \neg C \rightarrow (D \wedge \neg A)$	$(\in)$
2. $\sum, \neg C \vdash D \wedge \neg A$	$(-\rightarrow, 1)$
3. $\sum, \neg C \vdash \neg A$	$(-\wedge, 2)$
4. $\sum, \neg C \vdash B \rightarrow (A \vee E)$	$(\in)$
5. $\sum, \neg C, B \vdash A \vee E$	$(-\rightarrow, 4)$
6. $\sum, \neg C, B \vdash E$	$(Disjunctive\ solly., 5, 3)$
7. $\sum, \neg C \vdash B$	$(-)$
8. $\sum, \neg C \vdash A \vee B$	$(+\vee, 7)$
9. $\sum \vdash \neg C$	$(-)$
10. $\sum \vdash (A \vee B) \wedge \neg C,$	$(+\wedge, 9, 8)$
11. $\sum \vdash E \vee F$	$(+\vee, 6)$

      Therefore, proven.

b. Argument:

$(O \wedge G) \rightarrow V$
$V \rightarrow \neg M$
$\neg J \rightarrow M$
$Con: G \rightarrow (O \rightarrow J)$

$(O \wedge G) \rightarrow V, V \rightarrow \neg M, \neg J \rightarrow M \vdash G \rightarrow (O \rightarrow J)$

Let $(O \wedge G) \rightarrow V, V \rightarrow \neg M, \neg J \rightarrow M = \sum$

Proof: $\sum, G, O \vdash G \rightarrow (O \rightarrow J)$

1. $\sum, G, O \vdash O \wedge G$	$(\wedge +)$
2. $\sum, G, O \vdash (O \wedge G) \rightarrow V$	$(\in)$
3. $\sum, G, O \vdash V$	$(\rightarrow -, \wedge -)$
4. $\sum, G, O \vdash V \rightarrow \neg M$	$(\in)$
5. $\sum, G, O \vdash \neg M$	$(3, \rightarrow -)$
6. $\sum, G, O \vdash \neg J \rightarrow M$	$(\in)$
7. $\sum, G, O \vdash J$	$(Modus\ Tollens, 5, 6)$
8. $\sum, G \vdash O \rightarrow J$	$(\rightarrow +)$
9. $\sum \vdash G \rightarrow (O \rightarrow J)$	$(\rightarrow +)$

Therefore $\sum \vdash G \rightarrow (O \rightarrow J)$.