Assignment #5

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1. You pick a bit string from the set of all bit strings of length 10.

   a. What is the probability that the bit string has exactly two 1s, given the string begins with a 1?

   E = the bit string has exactly two 1s
   F = the string begins with a 1.

   $$p(E|F) = \frac{p(E \cap F)}{p(F)} = \frac{C\binom{10}{2}/2^{10}}{1/2} = \frac{2(45)}{2^{10}} = \frac{90}{2^{10}} \cong 0.087890625$$

   Therfore, the probability that the bit string has exactly two 1s, given the string begins with a 1 is approximately 0.09.

   b. What is the probability that the bit string begins and ends with 0?

   Since the first and last digits have to be 0s, we must find the all the possible ways that the middle 8 bit strings can be formed, and dived it by the total number of combinations of the 10 bit strings.

   $$p = \frac{2^8}{2^{10}} = \frac{1}{2^2} = \frac{1}{4} = 0.25$$

   Therefore, the probability that the bit string begins and ends with 0 is 0.25.

   c. What is the probability that the bit string has more 0s than 1s?

   Therefore we have 5 cases:-

   Case 1: six 0s, four 1s

   $$C\binom{10}{6} = 210$$

Case 2: seven 0s, three 1s

$$C\binom{10}{7} = 120$$

Case 3: eight 0s, two 1s

$$C\binom{10}{8} = 45$$

Case 4: nine 0s, one 1s

$$C\binom{10}{9} = 10$$

Case 5: six 0s, four 1s

$$C\binom{10}{10} = 1$$

Therefore, the probability is equal to the sum of all 5 cases divided by all the possible bit strings.

$$p = \frac{\sum cases}{2^{10}} = \frac{C\binom{10}{6} + p\binom{10}{7} + C\binom{10}{8} + C\binom{10}{9} + C\binom{10}{10}}{2^{10}}$$

$$= \frac{210 + 120 + 45 + 10 + 1}{2^{10}} = \frac{386}{1024} \cong 0.376953125$$

Therefore, the probability that the bit string has more 0s than 1s is approximately 0.38.

d. What is the probability that the bit string has the sum of its digits equal to seven?

We only have one case that fulfills this condition which is when the bit string contains seven 1s. Which is all combinations of 10 choose 7 divided by all the possible combinations.

$$C\binom{10}{7} = 120$$

$$p = \frac{C\binom{10}{7}}{2^{10}} = \frac{120}{1024} \cong 0.1171875$$

Therefore, the probability that the bit string has the sum of its digits equal to seven is approximately 0.12.

e. What is the probability that the bit string begins with 111?

Since the first three digits are 111, we must find the all the possible ways that the last 7 bit strings can be formed and dived it by the total number of possible ways that the 10 bit strings can be written.

$$p = \frac{2^7}{2^{10}} = \frac{1}{2^3} = \frac{1}{8} = 0.125$$

Therefore the probability that the bit string begins with 111 is 0.125.

2. What is the probability of these events when we randomly select a permutation of the 26 lowercase letters of the English alphabet?

a. The first 13 letters in the permutation are in alphabetical order.

In this case we must find the permutation of the 13 letters from 13-26 assuming that the first 13 we're in an alphabetical order.

$$p = \frac{\rho\binom{26}{13}}{\rho\binom{26}{26}} = \frac{26!/13!}{26!/1} = \frac{\cancel{26!}}{\cancel{26!} \times 13!} = \frac{1}{13!} \cong 0.000000001606$$

Therefore, the probability that the first 13 letters in the permutation are in alphabetical order is approximately 0.000.

b. "a" is the first letter of the permutation and "z" is the last letter.

Since the permutation starts with an "a" then the possible number of permutation is 25 choose 25 but since the permutation ends with a "z" then the number of permutation will be 24 choose 24 since the first and last letters are already choosing. Furthermore, if we divide that number by the number all possible permutations of the 26 letters.

$$p = \frac{\rho\binom{24}{24}}{\rho\binom{26}{26}} = \frac{24!/0!}{26!/0!} = \frac{24!/1}{26!/1} = \frac{\cancel{24!}}{26 \times 25 \times \cancel{24!}} = \frac{1}{26 \times 25} \cong 0.0015384615384615$$

Therefore, the probability of having "a" as the first letter of the permutation and "z" as the last letter is approximately 0.002.

c. “a” and “z” are next to each other in the permutation.

This problem is very similar to previous problem, the only difference is that the letters “a” and “z” can be spread over 25 different positions in the string, beginning, middle, end, etc…, multiplied by two because the order matters, za != za

$$= \frac{2\times 25\times \rho\binom{24}{24}}{\rho\binom{26}{26}} = \frac{2\times 25\times {}^{24!}/_{0!}}{{}^{26!}/_{0!}} = \frac{2\times 25\times {}^{24!}/_{1}}{{}^{26!}/_{1}} = \frac{\cancel{2\times 25\times 24!}}{\cancel{26\times 25\times 24!}} = \frac{1}{13} \cong 0.076923076923077$$

Therefore, the probability of that the letters “a” and “z” are next to each other in the permutation is approximately 0.077.

d. “a” and “z” are separated by at least 23 letters in the permutation.

This problem is also similar to the past two problems, the main difference that at least 23 letters have to separate the letters “a” and “z”. Hence, there are 6 cases there first three are if “a” comes first multiplied by the permutations of the 24 other letters, and the second three is when “z” comes before “a”. Which will produce 6 multiplied by the number of permutations of the other 24 letters.

$$p = \frac{6\times \rho\binom{24}{24}}{\rho\binom{26}{26}} = \frac{6\times {}^{24!}/_{0!}}{{}^{26!}/_{0!}} = \frac{6\times {}^{24!}/_{1}}{{}^{26!}/_{1}} = \frac{6\times \cancel{24!}}{26\times 25\times \cancel{24!}} = \frac{6}{26\times 25} \cong 0.00923$$

Therefore, the probability that the two letters “a” and “z” are going to be separated by at least 23 letters in the permutation is approximately 0.009.

e. “z” precedes both “a” and “b” in the permutation.

For this problem we have 23 different cases where z precedes both letters “a” and “b” where z is between the positions 1-24 and “a” and “b” are at the positions succeeding those positions. The sum of these cases will be multiplied by the permutations of the other 23 letters. Hence,

$$p = \frac{\sum cases \times \rho\binom{23}{23}}{\rho\binom{26}{26}} = \frac{\left(p\binom{25}{2} + p\binom{24}{2} + p\binom{23}{2} + \cdots + p\binom{3}{2} + p\binom{2}{2}\right) \times \rho\binom{23}{23}}{\rho\binom{26}{26}}$$

$$= \frac{5{,}200 \cancel{\times 23!}}{26\times 25\times 24 \cancel{\times 23!}} = \frac{5{,}200}{26\times 25\times 24} = \frac{1}{3} \cong 0.\overline{3}$$

Therefore, the probability that the permutation will have the letter “z” preceding the letters “a” and “b” is approximately 0.333.

3. Suppose that 4% of the patients tested in a clinic are infected with avian influenza. Furthermore, suppose that when a blood test for avian influenza is given, 97% of the patients infected with avian influenza test positive and that 2% of the patients not infected with avian influenza test positive. What is the probability that:

   a. A patient testing positive for avian influenza with this test is infected with it.

      Using Bayes' Theorem.

$$p(F|E) = \frac{p(E|F)p(F)}{p(E|F)p(F) + p(E|\bar{F})p(\bar{F})}$$

F $=$ the patient is infected.
E $=$ the patient tested positive.

$p(E|F) = 0.97$, the probability that the patient is infected, and tests positive.
$p(F) = 0.04$, the probability that the patient is infected.
$p(E|\bar{F}) = 0.02$, the probability that the patient is not infected, but tests positive.
$p(\bar{F}) = 0.96$, the probability that the patient is not infected.

$$p(F|E) = \frac{0.97 \times 0.04}{(0.97 \times 0.04) + (0.02 \times 0.96)} = \frac{0.0388}{0.0388 + \ 0.0192} = \frac{0.0388}{0.058} \cong \ 0.66897$$

Therefore, the probability that the patient is positive for avian influenza is approximately 0.669.

   b. A patient testing positive for avian influenza with this test, is not infected with it?

F = the patient is not infected.
E = the patient tested positive.

$p(E|F) = 0.02$, the probability that the patient is not infected, and tests positive.
$p(F) = 0.96$, the probability that the patient is not infected.
$p(E|\bar{F}) = \ 0.97$, the probability that the patient is infected, but tests positive.
$p(\bar{F}) = \ 0.04$, the probability that the patient is infected.

$$p(F|E) = \frac{0.96 \times 0.02}{(0.97 \times 0.04) + (0.02 \times 0.96)} = \frac{0.0192}{0.0388 + \ 0.0192} = \frac{0.0192}{0.058} \cong \ 0.33103$$

Hence, the probability that a patient testing positive for avian influenza with this test is not infected is approximately 0.331.

c. A patient testing negative for avian influenza with this test, is infected with it?

F = the patient is infected.
E = the patient tested negative.

$p(E|F) = 0.03$, the probability that the patient is infected, but tests negative.
$p(F) = 0.04$, the probability that the patient is infected.
$p(E|\bar{F}) = 0.98$, the probability that the patient is not infected, and tests negative.
$p(\bar{F}) = 0.96$, the probability that the patient is not infected.

$$p(F|E) = \frac{0.04 \times 0.03}{(0.04 \times 0.03) + (0.98 \times 0.96)} = \frac{0.0012}{0.0012 + 0.9408} = \frac{0.0012}{0.942} \cong 0.00128$$

Therefore, the probability that a patient testing negative for avian influenza with this test is infected with it is approximately 0.0013.

d. A patient testing negative for avian influenza with this test is not infected with it?

F = the patient is not infected.
E = the patient tested negative.

$p(E|F) = 0.98$, the probability that the patient is not infected, and tests negative.
$p(F) = 0.96$, the probability that the patient is not infected.
$p(E|\bar{F}) = 0.03$, the probability that the patient is infected, but tests negative.
$p(\bar{F}) = 0.04$, the probability that the patient is infected.

$$p(F|E) = \frac{0.98 \times 0.96}{(0.98 \times 0.96) + (0.03 \times 0.04)} = \frac{0.9408}{0.9408 + 0.0012} = \frac{0.9408}{0.942} \cong 0.99872$$

Therefore, the probability that a patient testing negative for avian influenza with this test is not infected with it is approximately 0.999.

4. Suppose that a Bayesian spam filer is trained on a set of 500 spam messages and 200 messages that are not spam. The word "Exciting" appears in 40 spam messages and 25 messages that are not spam. Would an incoming message be rejected as spam if it contains the word "exciting" and the threshold for rejecting spam is 0.9 (Assume, for simplicity, that the message is equally likely to be spam, as it is not to be spam.)

   Using Bayes' Theorem.

$$r(w) = \frac{p(w)}{p(w) + q(w)}$$

   $p(w)$: The probability of the word appearing in a spam message.
   $q(w)$: The probability of the word appearing in a non-spam message.

$$p(w) = \frac{{}^{40}\!/_{500}}{{}^{40}\!/_{500} + {}^{25}\!/_{200}} = \frac{0.08}{0.08 + 0.125} = \frac{0.08}{0.205} \cong 0.3902439$$

   Therefore, since the spam filter result is less than 0.9 the message will not be rejected as a spam if it contains the word "exciting."

5. Let A be the set of all ordered pairs of integers for which the second element of the pair is nonzero. Symbolically,

$$A = Z \times (Z \backslash \{0\}).$$

   Define a binary relation R on A as followers: For all $(a, b), (c, d) \in A$,

$$(a, b)\ R\ (c, d) \Leftrightarrow ad = bc.$$

   a. Is R reflexive?

      Reflexive: $\forall x[x \in A \rightarrow (x, x) \in R]$
      $x = (a, b)\ or\ (c, d)$

      $ab = ba, \qquad cd = dc$

      The relation still holds. Therefore the relation is reflexive.

b. Is R symmetric?

Symmetric: $\forall x \forall y[(x, y) \in R \rightarrow (y, x) \in R]$
$x = (a, b), y = (c, d)\ or\ x = (c, d), y = (a, b)$

$ad = bc\ \rightarrow cb = da$

Since the operating between the variables is multiplication, the order in which the letters appear does not matter. Hence the relation holds and it is symmetric.

c. Is R anti-symmetric?

Anti-symmetric: $\forall x \forall y[(x, y) \in R\ and\ (y, x) \in R \rightarrow x = y]$

Since R is symmetric then (x, y) and (y, x) belong to the relation, but that does not mean that x=y. For examples let x = (2, 4) and y = (1, 2), hence, (2, 4) != (1, 2).

Therefore, the relation does not hold, hence, the relation is not anti-symmetric.

Therefore the relation is symmetric.

d. Is R transitive?

$\forall x \forall y \forall z[(x, y) \in R\ and\ (y, z) \in R \rightarrow (x, z) \in R]$

Let x = (a, b), y = (c, d), z = (e, f), therefore,

$ad = bc$
$cf = de$

Now since, $ad = bc$, then, $d = {}^{bc}/_{a}$

Substitute this equation into, $cf = de$

$cf = \left({}^{bc}/_{a}\right)e$, divide sides by c and multiply both sides by a.

$$\frac{(cf = ({}^{bc}/_{a})e)}{\boldsymbol{c}} = \boldsymbol{a} \times \left(f = {}^{be}/_{a}\right) = (af = be)$$

Therefore, since the relation holds, R is transitive.

e. Is R an equivalence relation, a partial order, neither or both?

   Since the relation was proven to be reflexive, symmetric, and transitive. We can conclude that the relation is an equivalence relation but not a partial order because it is not anti-symmetric.

f. List four distinct elements in the equivalence class [(1, 3)]

   $[(1,3)]_R \in \{(2,6),(3,9),(4,12),(5,15)\}$

g. List four distinct elements in the equivalence class [(2, 5)]

   $[(2,5)]_R \in \{\ (4,10),(6,15),(8,20),(10,25)\}$

h. Describe the equivalence classes of R.

   The equivalence classes of R are multiples of each other, also, the fractions generated by dividing the a and b pairs will equal to the to any fractions generated by dividing a and b pairs of any other equivalence class.