Inverse of matrix with use of Doolitle decomposition

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1 Task description

17. Computing $X \approx A^{-1}$ by Doolitle method where A is a pentadiagonal matrix. Use the fact that if $A = LU$, then $A^{-1} = U^{-1}L^{-1}$. Do not use a square array to store matrices A, L, and U(use $5 \times n$ array or 5 vectors for A and analogous structure(s) for L and U).

2 Pentadiagonal matrix and representation in memory

A pentadiagonal matrix is a special case of band matrices. Its only nonzero entries are on the main diagonal and the first two upper and two lower diagonals. [1] Matrix A is in the form:

$$\begin{bmatrix} a_1 & b_1 & c_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ d_2 & a_2 & b_2 & c_2 & \ddots & & & & \vdots \\ e_3 & d_3 & a_3 & b_3 & c_3 & \ddots & & & \vdots \\ 0 & e_4 & d_4 & a_4 & b_4 & c_4 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & e_{n-2} & d_{n-2} & a_{n-2} & b_{n-2} & c_{n-2} \\ \vdots & & & & \ddots & e_{n-1} & d_{n-1} & a_{n-1} & b_{n-1} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & e_n & d_n & a_n \end{bmatrix} \tag{1}$$

In the program matrix (1) is represented as $5 \times n$ matrix. Each row represents a non-zero diagonal. We save some memory and reduced memory complexity of the program to $O(n)$ instead of $O(n^2)$

$$\begin{bmatrix} a_1 & a_2 & a_3 & \dots & \dots & a_{n-1} & a_n \\ b_1 & b_2 & b_3 & \dots & \dots & b_{n-1} & NaN \\ c_1 & c_2 & c_3 & \dots & \dots & NaN & NaN \\ NaN & d_2 & d_3 & \dots & \dots & d_{n-2} & d_{n-1} \\ NaN & NaN & e_3 & \dots & \dots & e_{n-1} & e_n \end{bmatrix} \tag{2}$$

3 Finding LU Decomposition

In order to find LU decomposition of matrix A, I used Doolitle method. As matrix is pentadiagonal there is no need to loop over all zero matrix entries. Algorithm for finding the LU factorization is described in [2]

4 Inverse of L and U matrix

4.1 Finding inverse of lower-triangular matrix L

$$L = \begin{bmatrix} a_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ d_2 & a_2 & \ddots & & & & & \vdots \\ e_3 & d_3 & a_3 & \ddots & & & & \vdots \\ 0 & e_4 & d_4 & a_4 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & e_{n-2} & d_{n-2} & a_{n-1} & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & e_{n-1} & d_{n-1} & a_n \end{bmatrix} \tag{3}$$

$$L^{-1} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & \cdots & \cdots & \cdots & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots & \cdots & \cdots & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & x_{34} & \cdots & \cdots & \cdots & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \vdots & & & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & & & \vdots \\ x_{n1} & x_{n2} & x_{n3} & x_{n4} & \cdots & \cdots & \cdots & \cdots & x_{nn} \end{bmatrix} \tag{4}$$

1. To find the formula for L^{-1} we use definition of matrix inverse:

$$L \cdot L^{-1} = I \tag{5}$$

2. Now we can derive the formula for the inverse L^{-1}. At first we will find formulas for the first row of L^{-1} We have to solve set of equations below:

$$\begin{cases} a_1 \cdot x_{11} = 1 \\ a_1 \cdot x_{12} = 0 \\ a_1 \cdot x_{13} = 0 \\ \text{...} \\ \mathrm{a}_1 \cdot x_{1n} = 0 \end{cases} \implies \begin{cases} \mathrm{x}_{11} = \frac{1}{a_1} \\ \mathrm{x}_{12} = 0 \\ \mathrm{x}_{12} = 0 \\ \text{...} \\ \mathrm{x}_{1n} = 0 \end{cases}$$

3. Next to obtain formulas for the second row of L^{-1} We have to solve following set of equations:

$$\begin{cases} d_2 \cdot x_{12} + \mathrm{a}_2 \cdot x_2 2 = 1 \\ d_2 \cdot x_{13} + \mathrm{a}_2 \cdot x_2 3 = 0 \\ d_2 \cdot x_{14} + a_2 \cdot x_2 4 = 0 \\ ... \\ d_2 \cdot x_{1n} + a_2 \cdot x_2 n = 0 \end{cases} \implies \begin{cases} \mathrm{x}_{11} = \frac{1}{a_2} \\ \mathrm{x}_{23} = 0 \\ \mathrm{x}_{14} = 0 \\ ... \\ \mathrm{x}_{1n} = 0 \end{cases}$$

4. General formulas for the diagonal and elements above it are:

$$\forall i, j : \quad i = j, \qquad \mathrm{x}_{ij} = \frac{1}{a_i}$$
$$\forall i, j : \quad i < j, \qquad \mathrm{x}_{ij} = 0$$

5. Similarly with use set of equations we can get the formula for the diagonal below the main one:

$$x_{i,i-1} = \frac{-L_{ii}}{L_{i-1i-1} \cdot L_{i-1i}} \qquad where \qquad i \in Z \quad \wedge \quad i \leq n \quad \wedge \quad i \geq 2$$

6. Matrix L is a 3 band matrix. Because of that we do not have to use $\sum$ notation.
   Only the multiplication of 3 elements from L row and L^{-1} column contribute to the final formula:

$$x_{j,j-i} = \frac{-L_{j+1j} \cdot x_{j-1,j-i} - L_{j+2j} \cdot x_{j-2,j-i}}{L_{j,j}}$$

   for all pairs (i,j) enumerated by 2 nested loops i=2, 3... n-1 j=i+1,i+2,...n

4.2 Finding inverse of upper-triangular matrix U

Values of elements of U^{-1} are obtained with the same method as inverse of L

$$U = \begin{bmatrix} a_1 & b_1 & c_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & a_2 & b_2 & c_2 & \ddots & & & \vdots \\ \vdots & \ddots & a_3 & b_3 & c_3 & \ddots & & \vdots \\ \vdots & & \ddots & a_4 & b_4 & c_4 & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & \ddots & a_{n-2} & b_{n-2} & c_{n-2} \\ \vdots & & & & & \ddots & a_{n-1} & b_{n-1} \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & a_n \end{bmatrix} \tag{6}$$

$$U^{-1} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & \cdots & \cdots & \cdots & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots & \cdots & \cdots & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & x_{34} & \cdots & \cdots & \cdots & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \vdots & & & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & & & \vdots \\ x_{n1} & x_{n2} & x_{n3} & x_{n4} & \cdots & \cdots & \cdots & \cdots & x_{nn} \end{bmatrix} \tag{7}$$

1. General formula for main diagonal and elements below it:

   $\forall i, j: \quad i = j, \qquad x_{ij} = \frac{1}{a_{ii}}$
   $\forall i, j: \quad i < j, \qquad x_{ji} = 0$

2. First diagonal above the main diagonal:

   $x_{ii+1} = \frac{(U_{i+1i} \cdot x_{i+1i+1})}{U_{i,i}} \qquad where \quad i \in Z \quad \wedge \quad i \leq n-1 \quad \wedge \quad i \geq 1$

3. Rest of the diagonals above the main diagonal:

   $x_{ii+j} = \frac{-U_{ii+1} \cdot x_{i+1i+j} + U_{ii+2} \cdot x_{i+2i+j}}{U_{ii}}$

   for all pairs (i,j) enumerated by 2 nested loops j=2, 3... n-1 i=1,i+2,...n-j

5 Numerical tests

Previously in main_testing I checked for the correctness of solutions. This time main_ function evaluate an error of the solution. (modified version of main_testing attached on Moodle) Below I present the accuracy of the method in form of the table of Euclidean norms:

Size of matrix	**Euclidean norm**
1000	5.5189e-11
2000	6.2482e-10
3000	7.0824e-11
4000	4.0346e-09
5000	2.6983e-09
6000	4.8154e-10
7000	6.6805e-07
8000	1.6769e-09

Table 1: $norm(A^1 \cdot A^{-1} - I)$

6 Efficiency and conclusion

Despite of using of a special structure for pentadiagonal matrix in the memory and adjusting algorithm for pentadiagonal matrix, matlab inv() [3] outperformed my program what is clearly visible for larger matrix sizes. The time complexity is growing exponentially. I measured time for matrix sizes in range between 10 and 10000. I performed this measurements twice:

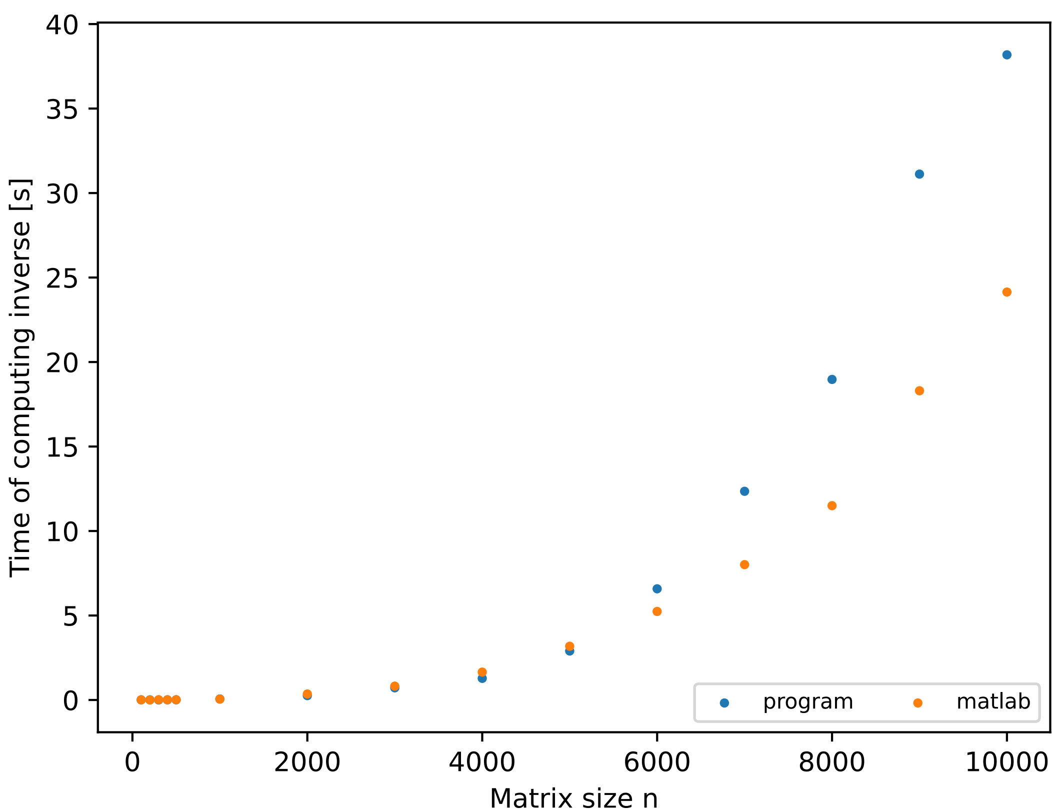

Figure 1: Measurement 1

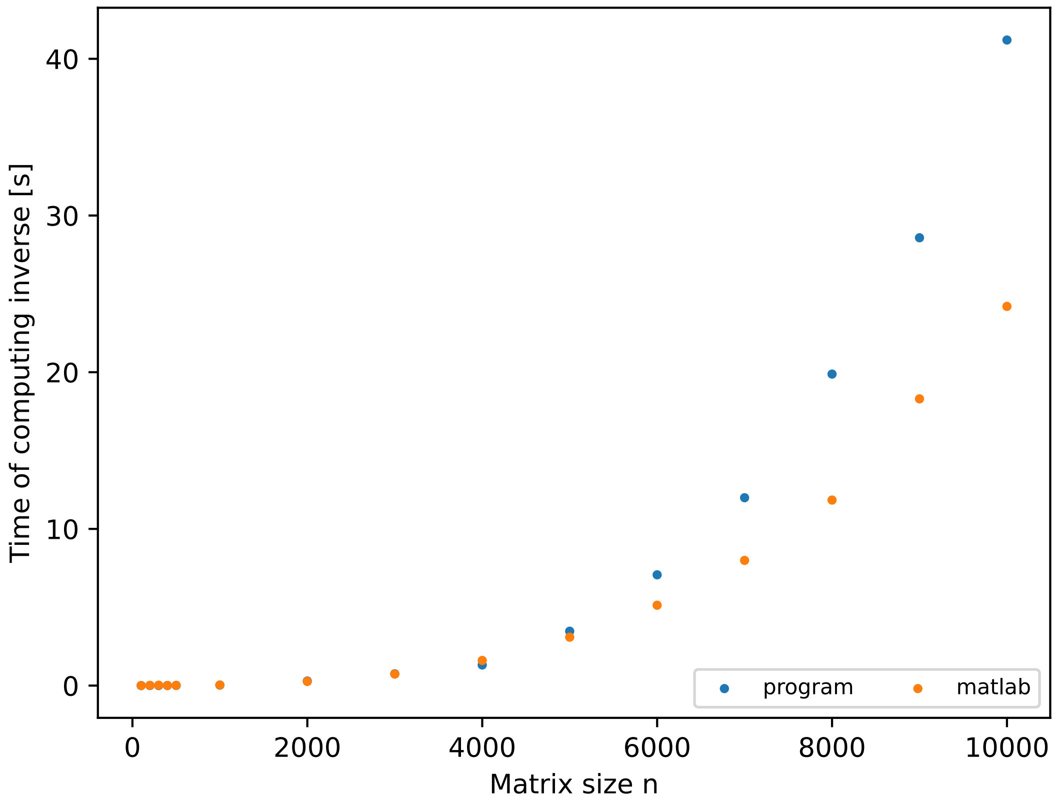

Figure 2: Measurement 2

References

[1] https://en.wikipedia.org/wiki/Pentadiagonal_matrix

[2] https://www.academia.edu/24859989/On_the_inverse_of_a_general_pentadiagonal_matrix

[3] https://ch.mathworks.com/help/matlab/ref/inv.html