System of two equations with use of midpoint rule and Huen method

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1 Task description

3. The midpoint method ($Y_{k+1} = Y_{k-1} + 2hF_k$) for a system of 2 ordinary differential equations. Use Heun method to calculate starting value Y_1

2 Formula derivation

Because of the fact that we need Y_{k-1} in the midpoint rule we have to use differrent method on the first step - Heun method [**?**]:

$k_1 = f(x_0, y_0)$
$k_2 = f(x0 + h, y0 + k_1 h)$
$Y_1 = Y_0 + (k1 + k2) * h/2$

Moreover, we calculate F_k for $k > 1$ for the rest of the points with use of midpoint method:

$Y_{k+1} = Y_{k-1} + 2hF_k$

In our case we have a system of two differential equation hence Y = [Y1, Y2] is a vector. Initial values for both functions are given. Similarly a slope of coefficient Y' = [Y1', Y2'] = F(Y1, Y2). The principle is the same as in the method shown above with except that we iteratively evaluate approximations for both functions.

x_0	x_1	x_2	x_3	...	x_{k-1}	x_k
$Y_{1,0}$	$Y_{1,1}$	$Y_{1,2}$	$Y_{1,3}$	...	$Y_{1,k-1}$	$Y_{1,k}$
$Y_{2,0}$	$Y_{2,1}$	$Y_{2,2}$	$Y_{2,3}$	...	$Y_{2,k-1}$	$Y_{2,k}$

As in the set of two differential equations Y_1' may depend on Y_2 it is necessary to evaluate subsequent Y_1 and Y_2 values.
We cannot evaluate $Y_{1,k}$ value at the x_k without knowing $Y_{1,k-1}$ and/or $Y_{2,k-1}$

3 Testing

In this section the method is tested and compared to MATLAB ode45() function [2]. A continuous line is a result of MATLAB method, while dots are results of out method. As we can see the results of Midpoint method and Huen method are close to the solutions of MATLAB function.

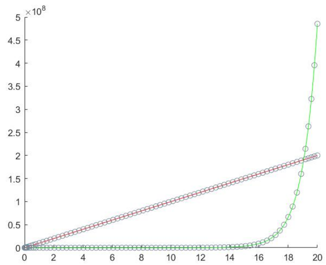

Figure 1: $y_1' = y$ and $y_2' = 10000000$

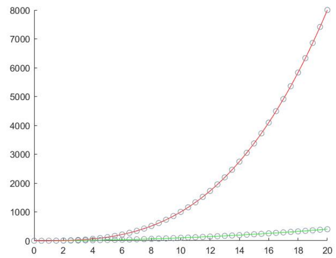

Figure 2: $y_1' = x^2$ and $y_2' = x^3$

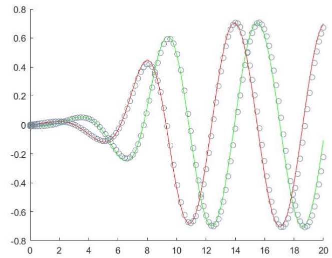

Figure 3: $y_1' = y_2 + y_1 \cdot (0.5 - y_1^2 - y_2^2)$ and $y_2' = -y_1 + y_2 \cdot (0.5 - y_1^2 - y_2^2)$

Furthermore we can compare differences in the results for different step sizes denoted as h. As we can see the accuracy of the method is better as the step size is becoming smaller and smaller.

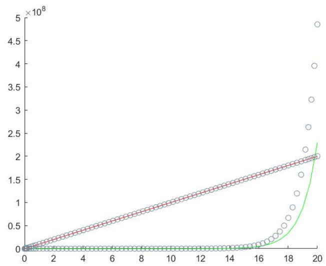

Figure 4: $h = 0.5$

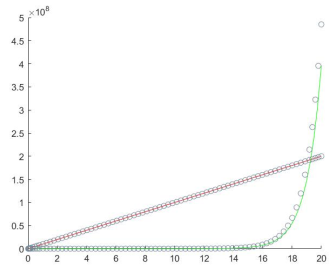

Figure 5: $h = 0.25$

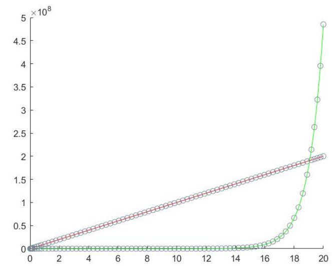

Figure 6: $h = 0.01$

References

[1] https://www.educative.io/answers/what-is-heuns-method

[2] https://www.mathworks.com/help/matlab/ref/ode45.html