

Horton self-similarity of Kingman's coalescent tree

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Abstract

The paper establishes Horton self-similarity for a tree representation of Kingman's coalescent process. The proof is based on a Smoluchowski-type system of ordinary differential equations for the number of branches of a given Horton-Strahler order in a tree that represents Kingman's N -coalescent process with a constant kernel, in a hydrodynamic limit. We also demonstrate a close connection between the combinatorial Kingman's tree and the combinatorial level set tree of a white noise, which implies Horton self-similarity for the latter. Numerical experiments illustrate the results and suggest that Kingman's coalescent and a white noise also obey a stronger Tokunaga self-similarity.

1 Introduction

In this paper we consider two types of self-similarity for tree graphs related to Horton-Strahler [10, 18] and Tokunaga [19] indexing schemes for the tree branches. These schemes and self-similarity definitions were introduced in hydrology in the mid-20th century to describe the dendritic structure of river networks and have penetrated other areas of sciences since then.

Horton-Strahler indexing assigns positive orders to the tree branches according to their relative importance in the hierarchy. *Horton self-similarity* refers to geometric decay of the number N_k of branches of order k . *Tokunaga self-similarity* is a stronger constraint that addresses *side branching* – merging of branches of distinct orders. A Tokunaga index $T_{i(i+k)}$ is the average number of branches of order $i \geq 1$ that merged a branch of order $(i+k)$, $k > 0$. Having Tokunaga self-similarity implies that different levels of a hierarchical system have the same statistical structure, in the sense $T_{i(i+k)} \equiv T_k = ac^{k-1}$ for a positive pair (a, c) of Tokunaga parameters.

A classical model that exhibits Horton and Tokunaga self-similarity is a critical binary Galton-Watson branching [4, 14, 15], also known in hydrology as Shreve's random topology model for river networks [16]. Recently, Horton and Tokunaga self-similarity were established for the level-set tree representation of a homogeneous symmetric Markov chain and a regular Brownian motion [21].

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This study is a first step towards exploring Horton and Tokunaga self-similarity of trees generated by coalescent processes. We focus on the tree generated by Kingman's coalescent and its finite version, Kingman's N -coalescent process with a constant collision kernel. The main result is a particular form of Horton self-similarity, called here *root-Horton law* for Kingman's coalescent. We also establish a very close relation between the combinatorial trees of Kingman's N -coalescent and a sequence of i.i.d. random variables (referred to as *discrete white noise*), which allows us to extend Horton self-similarity to the level set tree of an infinite i.i.d. sequence. These findings add two important classes of processes to the realm of Horton self-similar systems. Finally, we perform numerical experiments that suggest that Kingman's coalescent, and hence the level-set tree of a white noise, are Horton self-similar in a regular stronger sense as well as asymptotically Tokunaga self-similar.

The paper is organized as follows. Section 2 describes Horton-Strahler ordering of tree branches and the related concept of Horton self-similarity. Kingman's coalescent processes are defined in Sect. 3. The main results are summarized in Sect. 4. Section 5 introduces the Smoluchowski-Horton system of equations that describes the dynamics of Horton-Strahler branches in Kingman's coalescent. A proof of the existence of root-Horton law for Kingman's coalescent is presented in Section 6. Section 7 establishes the connection between the combinatorial tree representation of Kingman's N -coalescent process and combinatorial level-set tree of a discrete white noise. This section also introduces an infinite Kingman's tree as a limit of Kingman's N -coalescent trees viewed from a leaf. Section 8 defines Tokunaga self-similarity and collects numerical results and conjectures. Smoluchowski-Horton system for a general coalescent process with collision kernel is written in Sect. 9. Section 10 concludes.

2 Self-similar trees

This section describes *Horton self-similarity* for rooted binary trees. It is based on Horton-Strahler orders of the tree vertices.

2.1 Rooted trees

A *graph* $\mathcal{G} = (V, E)$ is a collection of vertices $V = \{v_i\}$, $1 \leq i \leq N_V$ and edges $E = \{e_k\}$, $1 \leq k \leq N_E$. In a *simple* undirected graph each edge is defined as an unordered pair of distinct vertices: $\forall 1 \leq k \leq N_E, \exists! 1 \leq i, j \leq N_V, i \neq j$ such that $e_k = (v_i, v_j)$ and we say that the edge k *connects* vertices v_i and v_j . Furthermore, each pair of vertices in a simple graph may have at most one connecting edge. A *tree* is a connected simple graph $T = (V, E)$ without cycles. In a *rooted* tree, one node is designated as a root; this imposes a natural *direction* of edges as well as the parent-child relationship between the vertices. Specifically, of the two connected vertices the one closest to the root is called *parent*, and the other – *child*. Sometimes we consider trees embedded in a plane (*planar trees*), where the children of the same parent are ordered.

A *time-oriented tree* $T = (V, E, S)$ assigns time marks $S = \{s_i\}$, $1 \leq i \leq N_V$ to the tree vertices in such a way that the parent mark is always larger (smaller) than that of its children. A *combinatorial tree* $\text{SHAPE}(T) \equiv (V, E)$ discards the time marks of a time-oriented tree T ,

as well as possible planar embedding, and only preserves its graph-theoretic structure.

We often work with a space \mathcal{T}_N of combinatorial (not labeled, not embedded) rooted binary trees with N leaves, and space \mathcal{T} of all (finite or infinite) rooted binary trees.

2.2 Horton-Strahler orders

Horton-Strahler ordering of the vertices of a finite rooted binary tree is performed in a hierarchical fashion, from leaves to the root [14, 12, 4]: (i) each leaf has order $r(\text{leaf}) = 1$; (ii) when both children, c_1, c_2 , of a parent vertex p have the same order r , the vertex p is assigned order $r(p) = r + 1$; (iii) when two children of vertex p have different orders, the vertex p is assigned the higher order of the two. Figure 1a illustrates this definition. A *branch* is defined as a union of connected vertices with the same order. The order $\Omega(T)$ of a finite tree T is the order of its root. By N_r we denote the total number of branches of order r .

2.3 Horton self-similarity

Let \mathcal{Q}_N be a probability measure on \mathcal{T}_N and $N_k^{(\mathcal{Q}_N)}$ be the number of nodes of Horton-Strahler order k in a tree generated according to the measure \mathcal{Q}_N .

Definition 1. We say that a sequence of probability laws $\{\mathcal{Q}_N\}_{N \in \mathbb{N}}$ has *well-defined asymptotic Horton ratios* if for each $k \in \mathbb{N}$, random variables $(N_k^{(\mathcal{Q}_N)}/N)$ converge in probability, as $N \rightarrow \infty$, to a constant value \mathcal{N}_k , called the *asymptotic ratio* of the branches of order k .

Horton self-similarity implies that the sequence \mathcal{N}_k decreases in a geometric fashion as k goes to infinity.

Definition 2. A sequence $\{\mathcal{Q}_N\}_{N \in \mathbb{N}}$ of probability laws on \mathcal{T} with well-defined asymptotic Horton ratios is said to obey a *root-Horton self-similarity law* if and only if the following limit exists and is finite and positive: $\lim_{k \rightarrow \infty} (\mathcal{N}_k)^{-\frac{1}{k}} = R > 0$. The constant R is called the *Horton exponent*.

3 Coalescent processes, trees

This section reviews finite and infinite Kingman's coalescent processes and introduces a tree representation of Kingman's N -coalescent process.

3.1 Kingman's N -coalescent process

We start by considering a general finite coalescent process defined by a collision kernel [3, 15, 2]. The process begins with N particles (clusters) of mass one. The cluster formation is governed by a collision rate kernel $K(i, j) = K(j, i) > 0$. Namely, a pair of clusters with masses i and j coalesces at the rate $K(i, j)$, independently of the other pairs, to form a new cluster of mass $i + j$. The process continues until there is a single cluster of mass N .

Formally, for a given N consider the space $\mathcal{P}_{[N]}$ of partitions of $[N] = \{1, 2, \dots, N\}$. Let $\Pi_0^{(N)}$ be the initial partition in singletons, and $\Pi_t^{(N)}$ ($t \geq 0$) be a strong Markov process such that $\Pi_t^{(N)}$ transitions from partition $\pi \in \mathcal{P}_{[N]}$ to $\pi' \in \mathcal{P}_{[N]}$ with rate $K(i, j)$ provided that partition π' is obtained from partition π by merging two clusters of π of masses i and j . If $K(i, j) \equiv 1$ for all positive integer masses i and j , the process $\Pi_t^{(N)}$ is known as Kingman's N -coalescent process, or N -coalescent.

3.2 Kingman's coalescent

The infinite Kingman's coalescent $\Pi_t^{(\infty)}$ is a coalescent process defined over the space $\mathcal{P}_{\mathbb{N}}$ of all partitions of $\mathbb{N} = \{1, 2, \dots\}$ such that [3, 15, 2]

- $\Pi_0^{(\infty)}$ is the initial partition of \mathbb{N} into singletons;
- (*consistency*) a Markov process obtained by restricting $\Pi_t^{(\infty)}$ to $[N]$ is equivalent in distribution law to the N -coalescent $\Pi_t^{(N)}$.

In addition to the existence and uniqueness of such process $\Pi_t^{(\infty)}$, which follows from consistency of finite restrictions and Kolmogorov's extension theorem, Kingman [11] also provides an explicit probabilistic construction of the process. In this construction, each cluster at a given time is indexed by the lowest element of \mathbb{N} contained in the cluster. Thus the index of the cluster a particle belongs to is nonincreasing. For each particle, the process tracks the index of the cluster it belongs to throughout time.

3.3 Coalescent tree

A merger history of Kingman's N -coalescent process can be naturally described by a time-oriented binary tree $T_K^{(N)}$ constructed as follows. Start with N leaves that represent the initial N particles and have time mark $t = 0$. When two clusters coalesce (a transition occurs), merge the corresponding vertices to form an internal vertex with a time mark of the coalescent. The final coalescence forms the tree root. The resulting time-oriented tree represents the history of the process. It is readily seen that there is one-to-one map from the trajectories of an N -coalescence process onto the time-oriented trees with N leaves.

Finally, observe that the combinatorial version of the above coalescent tree is invariant under time scaling $t_{\text{new}} = C t_{\text{old}}$, $C > 0$. Thus without loss of generality we let $K(i, j) \equiv 1/N$ in Kingman's N -coalescent process. Slowing the process's evolution N times is natural in Smoluchowski coagulation equations that describe the dynamics of the fraction of clusters of different masses.

4 Statement of results

The main result of this paper is Horton self-similarity for the combinatorial tree $\text{SHAPE} \left(T_K^{(N)} \right)$ of the Kingman's N -coalescent process, as N goes to infinity. Specifically, let N_k denote

the number of branches of Horton-Strahler order k in the tree $T_K^{(N)}$ that describes the N -coalescent. We show in Sect. 5, Lemma 3 that for each k , N_k/N converges in probability to the asymptotic Horton ratio

$$\mathcal{N}_k = \lim_{N \rightarrow \infty} N_k/N.$$

Moreover, these \mathcal{N}_k are finite, and can be expressed as

$$\mathcal{N}_k = \frac{1}{2} \int_0^\infty g_k^2(x) dx,$$

where the sequence $g_k(x)$ solves the following system of ordinary differential equations:

$$g'_{k+1}(x) - \frac{g_k^2(x)}{2} + g_k(x)g_{k+1}(x) = 0, \quad x \geq 0$$

with $g_1(x) = 2/(x+2)$, $g_k(0) = 0$ for $k \geq 2$.

The root-law Horton self-similarity is proven in Section 6 in the following statement.

Theorem 1. *The asymptotic Horton ratios \mathcal{N}_k exist and finite and satisfy the convergence $\lim_{k \rightarrow \infty} (\mathcal{N}_k)^{-\frac{1}{k}} = R$ with $2 \leq R \leq 4$.*

Section 7.1 introduces a *level set tree* $\text{LEVEL}(X)$ that describes the structure of the level sets of a finite time series X_k and provides a one-to-one map between rooted planar time-oriented trees and sequences of the local extrema of a time series. Furthermore, let $X = (X_k)$ be a time series with N local maxima separated by $N - 1$ internal local minima that are independent and identically distributed with a common continuous distribution F ; we call X an *extended discrete white noise*. Section 7.2 establishes the following equivalence.

Theorem 2. *The combinatorial level set tree of an extended discrete white noise X with N local maxima has the same distribution on \mathcal{T}_N as the combinatorial tree generated by Kingman's N -coalescent and a constant collision kernel.*

In Sect. 7.5 we construct an infinite tree for Kingman's coalescent as a limit of Kingman's N -coalescent trees viewed from a leaf as $N \rightarrow \infty$. We also describe two complementary constructions of an infinite tree for an infinite extended discrete white noise viewed from a leaf. Theorem 2 is used then to establish the distributional equivalence of the infinite combinatorial trees for the Kingman coalescent and an infinite extended white noise. This also allows us to naturally interpret the values \mathcal{N}_k as Horton ratios for Kingman's coalescent tree (or, equivalently, tree of a white noise) and leads to the following result.

Theorem 3. *Kingman coalescent and an infinite discrete white noise are root-Horton self-similar with $2 \leq R \leq 4$.*

Finally, numerical experiments in Sect. 8 (i) suggest that Kingman's coalescent and white noise trees obey a stronger, geometric, version of Horton self-similarity, (ii) provide a close estimation of Horton exponent $R = 3.043827\dots$, and (iii) support a conjecture that these trees enjoy a stronger *Tokunaga self-similarity* defined in Sect. 8.2.

5 Smoluchowski-Horton ODEs for Kingman's coalescent

Consider Kingman's N -coalescent process with a constant kernel. In Section 5.1 we informally write Smoluchowski-type ordinary differential equations (ODEs) for the number of Horton-Strahler branches in the coalescent tree $T_K^{(N)}$ and consider the asymptotic version of these equations as $N \rightarrow \infty$. Section 5.2 formally establishes the validity of the hydrodynamic limit.

5.1 Main equation

Recall that we let $K(i, j) \equiv 1/N$ in Kingman's N -coalescent process. Let $|\Pi_t^{(N)}|$ denote the total number of clusters (of any mass) at time $t \geq 0$, and let $\eta_{(N)}(t) := |\Pi_t^{(N)}|/N$ be the total number of clusters relative to the system size N . Then $\eta_{(N)}(0) = N/N = 1$ and $\eta_{(N)}(t)$ decreases by $1/N$ with each coalescence of clusters at the rate of

$$\frac{1}{N} \binom{N \eta_{(N)}(t)}{2} = \frac{\eta_{(N)}^2(t)}{2} \cdot N + o(N), \quad \text{as } N \rightarrow \infty$$

since $1/N$ is the coalescence rate for any pair of clusters regardless of their masses. Informally, this implies that the limit relative number of clusters $\eta(t) = \lim_{N \rightarrow \infty} \eta_{(N)}(t)$ satisfies the following ODE:

$$\frac{d}{dt} \eta(t) = -\frac{\eta^2(t)}{2}. \quad (1)$$

Next, for any $j \in \mathbb{N}$ we define $\eta_{j,N}(t)$ to be the number of clusters of Horton-Strahler order j at time t relative to the system size N . Initially, each particle represents a leaf of Horton-Strahler order 1. Thus, the initial conditions are set to be, using Kronecker's delta notation,

$$\eta_{j,N}(0) = \delta_1(j).$$

We describe now the evolution of $\eta_{j,N}(t)$ using the definition of Horton-Strahler orders.

Observe that at any time t , $\eta_{j,N}(t)$ increases by $1/N$ with each coalescence of clusters of Horton-Strahler order $j-1$ with rate

$$\frac{1}{N} \binom{N \eta_{(j-1),N}(t)}{2} = \frac{\eta_{(j-1),N}^2(t)}{2} \cdot N + o(N).$$

Thus $\frac{\eta_{(j-1),N}^2(t)}{2} + o(1)$ is the instantaneous rate of increase of $\eta_{j,N}(t)$.

Similarly, $\eta_{j,N}(t)$ decreases by $1/N$ when a cluster of order j coalesces with a cluster of order strictly higher than j with rate

$$\eta_{j,N}(t) \left(\eta_{(N)}(t) - \sum_{k=1}^j \eta_{k,N}(t) \right) \cdot N,$$

and it decreases by $2/N$ when a cluster of order j coalesces with another cluster of order j with rate

$$\frac{1}{N} \binom{N \eta_{j,N}(t)}{2} = \frac{\eta_{j,N}^2(t)}{2} \cdot N + o(N).$$

Thus the instantaneous rate of change of $\eta_{j,N}(t)$ is

$$\eta_{j,N}(t) \left(\eta_{(N)}(t) - \sum_{k=1}^j \eta_{k,N}(t) \right) + \eta_{j,N}^2(t) + o(1).$$

Now we can informally write the limit rates-in and the rates-out for the clusters of Horton-Strahler order via the following *Smoluchowski-Horton system* of ODEs:

$$\frac{d}{dt} \eta_j(t) = \frac{\eta_{j-1}(t)}{2} - \eta_j(t) \left(\eta(t) - \sum_{k=1}^{j-1} \eta_k(t) \right) \quad (2)$$

with the initial conditions $\eta_j(0) = \delta_1(j)$. Here we define $\eta_k(t) = \lim_{N \rightarrow \infty} \eta_{k,N}(t)$, provided it exists, and let $\eta_0 \equiv 0$.

Since $\eta_j(t)$ has the instantaneous rate of increase $\frac{\eta_{j-1}^2(t)}{2}$, the relative total number of clusters of Horton-Strahler order j is given by

$$\mathcal{N}_j = \delta_1(j) + \int_0^\infty \frac{\eta_{j-1}^2(t)}{2} dt. \quad (3)$$

It is not hard to compute the first terms of sequence \mathcal{N}_k by solving equations (1) and (2) in the first three iterations:

$$\mathcal{N}_1 = 1, \quad \mathcal{N}_2 = \frac{1}{3}, \quad \text{and} \quad \mathcal{N}_3 = \frac{e^4}{128} - \frac{e^2}{8} + \frac{233}{384} = 0.109686868100941 \dots$$

Hence, we have $\mathcal{N}_1/\mathcal{N}_2 = 3$ and $\mathcal{N}_2/\mathcal{N}_3 = 3.038953879388 \dots$. Our numerical results in Sect. 8 yield, moreover,

$$\lim_{k \rightarrow \infty} (\mathcal{N}_k)^{-\frac{1}{k}} = \lim_{k \rightarrow \infty} \frac{\mathcal{N}_k}{\mathcal{N}_{k+1}} = 3.0438279 \dots$$

5.2 Hydrodynamic limit

This section establishes the existence of the asymptotic ratios \mathcal{N}_k as well as the validity of the equations (1), (2) and (3) in a hydrodynamic limit. We refer to Darling and Norris [5] for a survey of formal techniques for proving that a Markov chain converges to the solution of a differential equation.

Notice that *quasilinearity* of the system of ODEs in (2) implies the existence and uniqueness. Specifically, if the first $j-1$ functions $\eta_1(t), \dots, \eta_{j-1}(t)$ are given, then (2) is a linear equation in $\eta_j(t)$. This makes the following argument less technically involved than the one presented by Norris [13] for the Smoluchowski equations.

5.2.1 Hydrodynamic limit for equation (1)

Lemma 1. *The relative total number $\eta_{(N)}(t)$ of clusters converges in probability, as $N \rightarrow \infty$, to $\eta(t)$ that satisfies equation (1) with the initial condition $\eta(0) = 1$.*

Proof. Take $\delta > 0$. Consider $\eta_{(N)}(t) - \eta_{(N)}(t + \delta)$ given $\eta_{(N)}(t)$. The Chernoff inequality bounds the probability that there is more than $\frac{\delta}{N} \binom{N\eta_{(N)}(t)}{2} + N^{2/3}$ coalescing pairs during $[t, t + \delta]$. Specifically, we consider the probability that a sum of $\frac{\delta}{N} \binom{N\eta_{(N)}(t)}{2} + N^{2/3}$ exponential inter-arrival times with the rate not exceeding $\frac{1}{N} \binom{N\eta_{(N)}(t)}{2}$ adds up to less than δ . Then, applying Chernoff inequality with $s > 0$, we obtain

$$\begin{aligned}
P \left(\eta_{(N)}(t) - \eta_{(N)}(t + \delta) > \frac{\delta}{N} \binom{N\eta_{(N)}(t)}{2} + N^{2/3} \right) \\
&\leq \frac{\exp\{s\delta/N\}}{\left(1 + \frac{s}{\binom{N\eta_{(N)}(t)}{2}}\right)^{\frac{\delta}{N} \binom{N\eta_{(N)}(t)}{2} + N^{2/3}}} \\
&\leq \exp \left\{ \frac{-\frac{1}{2}sN^{2/3} + \delta \frac{s^2}{N}}{\binom{N\eta_{(N)}(t)}{2}} \right\} \\
&\leq \exp \left\{ \frac{-\frac{1}{2}N^{13/6} + \delta N^2}{\binom{N\eta_{(N)}(t)}{2}} \right\} \\
&\quad (\text{taking } s = N^{3/2}) \\
&\leq \exp \{-N^{1/6} + 2\delta\}
\end{aligned}$$

for N large enough, as $\ln(1+x) > x - x^2$ for $x > 0$, and $\ln(1+x) > \frac{1}{2}x$ for $x \in (0, 2)$. In the above inequality, we assume that $\eta_{(N)}(t)$ is bounded from below by a given $\epsilon_0 \in (0, 1)$, and that integer rounding is applied as necessary.

Now that we know with probability exceeding $1 - e^{-N^{1/6} + 2\delta}$ that there are no more than $\frac{\delta}{N} \binom{N\eta_{(N)}(t)}{2} + N^{2/3}$ coalescing pairs during $[t, t + \delta]$, we also know that the exponential rates of inter-arrival times are greater than $\frac{1}{N} \binom{N[\eta_{(N)}(t) - \delta\eta_{(N)}^2(t)/2]}{2}$. Therefore we can use Chernoff inequality to bound the conditional probability that there are fewer than $\frac{\delta}{N} \binom{N[\eta_{(N)}(t) - \delta\eta_{(N)}^2(t)/2]}{2} - N^{2/3}$ coalescing pairs in $[t, t + \delta]$. Specifically, we bound the probability that a sum of $\frac{\delta}{N} \binom{N[\eta_{(N)}(t) - \delta\eta_{(N)}^2(t)/2]}{2} - N^{2/3}$ independent exponential random variables of rate $\frac{1}{N} \binom{N[\eta_{(N)}(t) - \delta\eta_{(N)}^2(t)/2]}{2}$ is greater than δ . Chernoff inequality with $s > 0$ implies

$$\begin{aligned}
P \left(\eta_{(N)}(t) - \eta_{(N)}(t + \delta) < \frac{\delta}{N} \binom{N[\eta_{(N)}(t) - \delta \eta_{(N)}^2(t)/2]}{2} - N^{2/3} \right) \\
\leq \frac{\exp\{-s\delta/N\}}{\left(1 - \frac{s}{\binom{N[\eta_{(N)}(t) - \delta \eta_{(N)}^2(t)/2]}{2}}\right)^{\frac{\delta}{N} \binom{N[\eta_{(N)}(t) - \delta \eta_{(N)}^2(t)/2]}{2} - N^{2/3}}} \\
\leq \exp \left\{ \frac{-sN^{2/3} + \delta \frac{s^2}{N}}{\binom{N[\eta_{(N)}(t) - \delta \eta_{(N)}^2(t)/2]}{2}} \right\} \\
\leq \exp \left\{ \frac{-N^{13/6} + \delta N^2}{\binom{N[\eta_{(N)}(t) - \delta \eta_{(N)}^2(t)/2]}{2}} \right\} \quad (\text{taking } s = N^{3/2}) \\
\leq \exp \{-2N^{1/6} + 2\delta\}
\end{aligned}$$

as $-x - x^2 < \ln(1 - x) < -x$ for $x \in (0, \frac{1}{2})$.

Hence, if we partition $[0, K]$ into K/δ subintervals, then with probability greater than

$$\left(1 - \frac{\delta}{K} e^{-N^{1/6} + 2\delta}\right) \left(1 - \frac{\delta}{K} e^{-2N^{1/6} + 2\delta}\right),$$

for each left partition point t ,

$$\Delta_\delta \eta_{(N)}(t) = -\frac{\eta_{(N)}^2(t)}{2} + \mathcal{E}'(t), \quad (4)$$

where $\Delta_\delta f(x) := \frac{f(x+\delta) - f(x)}{\delta}$ denotes the forward difference, and the error $|\mathcal{E}'(t)| < C_1 (\delta + \delta^{-1} N^{-1/3})$ for some positive C_1 as $0 \leq \psi_N(t) \leq 1$.

Next, we let $N \rightarrow \infty$ and $\delta \rightarrow 0_+$ so that $\delta N^{1/3} \rightarrow \infty$. We use the error propagation in (4) to show $\|\eta_{(N)} - \varphi_\delta\|_{L^2[0, K]} \rightarrow 0$, where φ_δ solves the corresponding difference equation

$$\Delta_\delta \varphi_\delta(t) = -\frac{\varphi_\delta^2(t)}{2} \quad (5)$$

with the initial condition $\varphi_\delta(0) = 1$. Specifically, we partition $[0, K]$ into subintervals of length δ each. The partition points satisfy $t_{j+1} = t_j + \delta$. We consider the error quantities $\varepsilon_j := \eta_{(N)}(t_j) - \varphi_\delta(t_j)$. Then

$$\begin{aligned}
\varepsilon_{j+1} &= \eta_{(N)}(t_{j+1}) - \varphi_\delta(t_{j+1}) \\
&= \left[\varphi_\delta(t_j) + \varepsilon_j - \frac{(\varphi_\delta(t_j) + \varepsilon_j)^2}{2} + \delta \mathcal{E}'(t_j) \right] - \left[\varphi_\delta(t_j) - \frac{\varphi_\delta^2(t_j)}{2} \right] \\
&= (1 - \varphi_\delta(t_j))\varepsilon_j - \frac{\varepsilon_j^2}{2} + \delta \mathcal{E}'(t_j),
\end{aligned}$$

where $0 \leq \varphi_\delta(t_j) \leq 1$. Tracing the above propagation of error results in bounding the error throughout the interval $[0, K]$,

$$|\varepsilon_j| \leq C_2 K (\delta + \delta^{-1} N^{-1/3})$$

for all j and some $C_2 > 0$. Finally, the same error propagation arguments works to show that $\|\eta - \varphi_\delta\|_{L^2[0,K]} \rightarrow 0$ for φ_δ in (5) and η in (1).

The expected time for the first m cluster mergers is

$$\frac{1}{\binom{N}{2}} + \frac{1}{\binom{N-1}{2}} + \cdots + \frac{1}{\binom{N-m+1}{2}} = \frac{2m}{(N-m)N},$$

and by Markov inequality, the probability

$$P(\eta_{(N)}(K) > \varepsilon) \leq \frac{2(1-\varepsilon)}{K} (1 + 2/(\varepsilon N))$$

for any $\varepsilon \in (0, 1)$. Similarly, $\int_K^\infty \eta^2(t)/2 dt = \int_K^\infty \frac{2}{(t+2)^2} dt = \frac{2}{K+2}$, showing that considering K large enough will suffice for the argument. Therefore we have shown that $\|\eta_{(N)} - \eta\|_{L^2[0,\infty)} \rightarrow 0$ in probability. \square

5.2.2 Hydrodynamic limit for equations (2)

Let $G_j(t) = \sum_{i: i \geq j} N_j(t)$ denote the number of clusters of Horton-Strahler order j or higher.

Also let $\eta_{j,N}(t) := N^{-1} N_j(t)$ and $g_{j,N} := N^{-1} G_j(t) = \eta_{(N)}(t) - \sum_{k:k < j} \eta_{k,N}(t)$.

Lemma 2. *For each j , the relative number $\eta_{j,N}(t)$ converges in probability, as $N \rightarrow \infty$, to $\eta_j(t)$ that satisfies system (2).*

Proof. Consider interval $[t, t + \delta] \subset [0, K]$. As it was shown above, there are

$$m = \delta N \left(\frac{\eta_{(N)}^2(t)}{2} + O(\delta + \delta^{-1} N^{-1/3}) \right)$$

transitions (coalescences) within $[t, t + \delta]$ with the probability exceeding

$$(1 - e^{-N^{1/6} + 2\delta})(1 - e^{-2N^{1/6} + 2\delta}) > 1 - 2e^{-N^{1/6} + 2\delta}.$$

For a fixed integer $j > 1$, this gives us the number m of transitions for the vector

$$(\eta_{1,N}(t), \eta_{2,N}(t), \dots, \eta_{j-1,N}(t), g_{j,N}(t))$$

within the time interval, and the upper and lower bounds on the probabilities. We apply Chernoff inequality for m independent Bernoulli random variables of $2j - 1$ outcomes. In particular, the transition that results in decreasing $\eta_{k,N}(t)$ by $2/N$ and increasing $\eta_{k+1,N}(t)$

by $1/N$ happens with the probability bounded above by $\binom{N\eta_{k,N}(t)}{2} / \binom{N\eta_{(N)}(t)-m}{2}$, and bounded below by $\binom{N\eta_{k,N}(t)-m}{2} / \binom{N\eta_{(N)}(t)}{2}$. Similarly, the transition that results in decreasing $\eta_{k,N}(t)$ by $1/N$ happens with the probability bounded from above by $N^2\eta_{k,N}(t)g_{k+1,N}(t) / \binom{N\eta_{(N)}(t)-m}{2}$, and bounded from below by $(N\eta_{k,N}(t) - m)Ng_{k+1,N}(t) / \binom{N\eta_{(N)}(t)}{2}$. We obtain the system of difference equations

$$\begin{aligned}
\Delta_\delta \eta_{1,N}(t) &= -\eta_{1,N}(t)g_{1,N}(t) + \mathcal{E}_1 \\
\Delta_\delta \eta_{2,N}(t) &= \frac{\eta_{1,N}^2(t)}{2} - \eta_{2,N}(t)g_{2,N}(t) + \mathcal{E}_2 \\
&\vdots \\
\Delta_\delta \eta_{j-1,N}(t) &= \frac{\eta_{j-2,N}^2(t)}{2} - \eta_{j-1,N}(t)g_{j-1,N}(t) + \mathcal{E}_{j-1} \\
\Delta_\delta g_{j,N}(t) &= \frac{\eta_{j-1,N}^2(t)}{2} - \frac{g_{j,N}^2(t)}{2} + \mathcal{E}_j
\end{aligned} \tag{6}$$

with the initial conditions

$$(\eta_{1,N}(0), \eta_{2,N}(0), \dots, \eta_{j-1,N}(0), g_{j,N}(0)) = (1, 0, \dots, 0),$$

where each $\mathcal{E}_k = O(\delta + \delta^{-1}N^{-1/3})$. The errors here are obtained using Chernoff inequality. Namely, if S_m is a Binomial random variable that represents m Bernoulli trials with probability $0 < p < 1$ of success, then for $s > 0$,

$$\begin{aligned}
P(S_m \geq mp + m^{2/3}) &\leq (pe^s + 1 - p)^m \exp \left\{ -s(mp + m^{2/3}) \right\} \\
&= \exp \left\{ m \ln(pe^s + 1 - p) - s(mp + m^{2/3}) \right\} \\
&= \exp \left\{ m \ln(pe^{m^{-1/3}} + 1 - p) - m^{2/3}p + m^{1/3} \right\} \quad (\text{taking } s = m^{-1/3}) \\
&\leq \exp \left\{ [p^2 + p - 1]m^{1/3} \right\} = \exp \left\{ -[1 - p - p^2]m^{1/3} \right\}
\end{aligned}$$

The lower bound on $P(S_m \leq mp - m^{2/3})$ follows symmetrically.

Finally, the same error propagation analysis applies to compare the above difference equations (6) to the difference equations that correspond to the following system of ODEs

$$\begin{aligned}
\frac{d}{dt} \eta_1(t) &= -\eta_1(t)g_1(t) \\
\frac{d}{dt} \eta_2(t) &= \frac{\eta_1^2(t)}{2} - \eta_2(t)g_2(t) \\
&\vdots \\
\frac{d}{dt} \eta_{j-1}(t) &= \frac{\eta_{j-2}^2(t)}{2} - \eta_{j-1}(t)g_{j-1}(t) \\
\frac{d}{dt} g_j(t) &= \frac{\eta_{j-1}^2(t)}{2} - \frac{g_j^2(t)}{2},
\end{aligned}$$

where $g_i(t) := \eta(t) - \sum_{k: k < i} \eta_k(t)$. Thus we showed that functions $\eta_{j,N}(t)$ converge in probability to functions $\eta_j(t)$ that solve (2). \square

5.2.3 Hydrodynamic limit for asymptotic Horton ratios

Lemma 3. *The Horton ratios N_k/N converge in probability to a finite constant \mathcal{N}_k given by (3), as $N \rightarrow \infty$.*

Proof. Observe that in the difference equations (6), the number of emerging clusters of Horton-Strahler order j within $[t, t + \delta]$ time interval divided by N is

$$\frac{\eta_{j-1,N}^2(t)}{2} \cdot \delta + O(\delta^2 + N^{-1/3})$$

with probability greater than $1 - 2e^{-N^{1/6} + 2\delta}$.

Hence, for $j \geq 2$, the total number of emerging clusters of Horton-Strahler order j within $[0, K]$ time interval divided by N is

$$\int_0^K \frac{\eta_{j-1,N}^2(t)}{2} dt + O(\delta + \delta^{-1} N^{-1/3})$$

with probability exceeding $1 - 2e^{-N^{1/6} + 2\delta}$.

Therefore, the total number of emerging clusters of Horton-Strahler order j within $[0, \infty)$ time interval divided by N is

$$N_j/N = \int_0^K \frac{\eta_{j-1,N}^2(t)}{2} dt + \mathcal{R}_{\delta,\varepsilon,N},$$

where $\mathcal{R}_{\delta,\varepsilon,N} = O(\delta + \delta^{-1} N^{-1/3} + \varepsilon)$, with probability exceeding

$$\mathcal{P}_{\delta,\varepsilon,K,N} = \left(1 - 2e^{-N^{1/6} + 2\delta}\right) \left(1 - \frac{2(1-\varepsilon)}{K}(1 + 2/(\varepsilon N))\right)$$

as $P(\eta_{(N)}(K) > \varepsilon) \leq \frac{2(1-\varepsilon)}{K}(1 + 2/(\varepsilon N))$ for any $\varepsilon \in (0, 1)$.

So, with probability exceeding $\mathcal{P}_{\delta,\varepsilon,K,N}$,

$$\left| N_j/N - \int_0^\infty \frac{\eta_{j-1}^2(t)}{2} dt \right| \leq \int_0^K \frac{(\eta_{j-1,N}(t) - \eta_j(t))^2}{2} dt + \int_K^\infty \frac{\eta_{j-1}^2(t)}{2} dt + \mathcal{R}_{\delta,\varepsilon,N},$$

where $\int_K^\infty \frac{\eta_{j-1}^2(t)}{2} dt \leq \int_K^\infty \eta^2(t)/2 dt = \int_K^\infty \frac{2}{(t+2)^2} dt = \frac{2}{K+2}$.

Thus, for any $\epsilon > 0$ and any $\alpha \in (0, 1)$, there exist sufficiently small $\delta > 0$ and $\varepsilon > 0$, and sufficiently large K such that the following three inequalities are satisfied for N large enough:

$$P \left(\int_0^K \frac{(\eta_{j-1,N}(t) - \eta_j(t))^2}{2} dt < \epsilon/2 \right) \geq \sqrt{1 - \alpha},$$

$$\frac{2}{K+2} + \mathcal{R}_{\delta,\varepsilon,N} < \epsilon/2,$$

and

$$\mathcal{P}_{\delta,\varepsilon,K,N} \geq \sqrt{1 - \alpha}.$$

Hence

$$P \left(\left| N_j/N - \int_0^\infty \frac{\eta_{j-1}^2(t)}{2} dt \right| < \epsilon \right) \geq 1 - \alpha$$

for any $\epsilon > 0$ and any $\alpha \in (0, 1)$, and N large enough. Thus

$$N_j/N \longrightarrow \delta_1(j) + \int_0^\infty \frac{\eta_{j-1}^2(t)}{2} dt \quad \text{in probability as } N \rightarrow \infty.$$

□

6 The root-Horton self-similarity and related results

We begin this section with preliminary lemmas and propositions, and then proceed to proving Theorem 1.

Let $g_1(t) = \eta(t)$ and $g_j(t) = \eta(t) - \sum_{k: k < j} \eta_k(t)$ be the asymptotic number of clusters of Horton order j or higher at time t . We can rewrite (2) via g_j using $\eta_j(t) = g_j(t) - g_{j+1}(t)$ as follows

$$\frac{d}{dt} g_j(t) - \frac{d}{dt} g_{j+1}(t) = \frac{g_{j-1}(t) - g_j(t)}{2} - (g_j(t) - g_{j+1}(t)) g_j(t)$$

Observe that $g_1(t) \geq g_2(t) \geq g_3(t) \geq \dots$. We now rearrange the terms, obtaining for all $j \geq 2$,

$$\frac{d}{dt} g_{j+1}(t) - \frac{g_j^2(t)}{2} + g_j(t) g_{j+1}(t) = \frac{d}{dt} g_j(t) - \frac{g_{j-1}^2(t)}{2} + g_{j-1}(t) g_j(t). \quad (7)$$

One can readily check that $\frac{d}{dt} g_2(t) - \frac{g_1^2(t)}{2} + g_1(t) g_2(t) = 0$; the above equations hence simplify as follows

$$g'_{j+1}(x) - \frac{g_j^2(x)}{2} + g_j(x) g_{j+1}(x) = 0 \quad (8)$$

with $g_1(x) = \frac{2}{x+2}$, and $g_j(0) = 0$ for $j \geq 2$.

Notice that the above system of ODEs (8) is the quasilinearized Riccati equation $g'(x) = -\frac{g^2(x)}{2}$ with the initial value $g(0) = 0$ that has only a trivial solution.

Next, returning to the asymptotic ratios of the number of order- j branches to N , we observe that (7) implies that

$$\mathcal{N}_j = \int_0^\infty \frac{\eta_{j-1}^2(t)}{2} dt = \int_0^\infty \frac{(g_{j-1}(t) - g_j(t))^2}{2} dt = \int_0^\infty \frac{g_j^2(t)}{2} dt$$

since

$$\frac{(g_{j-1}(t) - g_j(t))^2}{2} = \frac{d}{dt} g_j(t) + \frac{g_j^2(t)}{2},$$

where $\int_0^\infty \frac{d}{dt} g_j(t) dt = g_j(\infty) - g_j(0) = 0$ for $j \geq 2$. Let n_k represent the number of order- k branches relative to the number of order- $(k+1)$ branches:

$$n_k := \frac{\mathcal{N}_k}{\mathcal{N}_{k+1}} = \frac{\int_0^\infty \frac{g_k^2(x)}{2} dx}{\int_0^\infty \frac{g_{k+1}^2(x)}{2} dx} = \frac{\|g_k\|_{L^2[0,\infty)}^2}{\|g_{k+1}\|_{L^2[0,\infty)}^2}.$$

Consider the following limits that represent respectively the root and the ratio asymptotic Horton laws:

$$\lim_{k \rightarrow \infty} (\mathcal{N}_k)^{-\frac{1}{k}} = \lim_{k \rightarrow \infty} \left(\prod_{j=1}^k n_j \right)^{-\frac{1}{k}} \quad \text{and} \quad \lim_{k \rightarrow \infty} n_k = \lim_{k \rightarrow \infty} \frac{\|g_k\|_{L^2[0,\infty)}^2}{\|g_{k+1}\|_{L^2[0,\infty)}^2}.$$

Theorem 1 establishes the existence of the first limit. We expect the second, stronger, limit also to exist and both of them to be equal to $3.043827\dots$ according to our numerical results in Sect. 8. We now establish some basic facts about g_j and n_j .

Proposition 1. *Let $g_j(x)$ be the solutions to the systems of ODEs (8). Then*

- (a) $\int_0^\infty \frac{g_j^2(x)}{2} dx = \int_0^\infty g_j(x) g_{j+1}(x) dx,$
- (b) $\int_0^\infty g_{j+1}^2(x) dx = \int_0^\infty (g_j(x) - g_{j+1}(x))^2 dx,$
- (c) $\lim_{x \rightarrow \infty} x g_j(x) = 2,$
- (d) $n_j = \frac{\|g_j\|_{L^2[0,\infty)}^2}{\|g_{j+1}\|_{L^2[0,\infty)}^2} \geq 2,$

$$(e) \quad n_j = \frac{\|g_j\|_{L^2[0,\infty)}^2}{\|g_{j+1}\|_{L^2[0,\infty)}^2} \leq 4.$$

Proof. Part (a) follows from integrating (8), and part (b) follows from part (a). Part (c) is done by induction, using the L'Hôpital's rule as follows. It is obvious that $\lim_{x \rightarrow \infty} xg_1(x) = 2$. Next, suppose $\lim_{x \rightarrow \infty} xg_j(x) = 2$. Then

$$\lim_{x \rightarrow \infty} xg_{j+1}(x) = \lim_{x \rightarrow \infty} \frac{g_{j+1}(x)}{x^{-1}} = \lim_{x \rightarrow \infty} \frac{\frac{g_j^2(x)}{2} - g_j(x)g_{j+1}(x)}{-x^{-2}} = 2 \lim_{x \rightarrow \infty} xg_{j+1}(x) - 2$$

implying $\lim_{x \rightarrow \infty} xg_{j+1}(x) = 2$. The statement (d) follows from the tree construction process. An alternative proof of (d) using differential equations is given in the following subsection. Part (e) follows from part (a) together with Hölder inequality

$$\frac{1}{2} \|g_j\|_{L^2[0,\infty)}^2 = \int_0^\infty g_j(x)g_{j+1}(x)dx \leq \|g_j\|_{L^2[0,\infty)} \cdot \|g_{j+1}\|_{L^2[0,\infty)},$$

which implies $\frac{\|g_j\|_{L^2[0,\infty)}^2}{\|g_{j+1}\|_{L^2[0,\infty)}^2} \leq 4$. □

6.1 Rescaling to $[0, 1]$ interval

Let

$$h_k(x) = (1-x)^{-1} - (1-x)^{-2}g_{k+1}\left(\frac{2x}{1-x}\right)$$

for $x \in [0, 1]$. Then $h_0 \equiv 0$, $h_1 \equiv 1$, and the system of ODEs (8) rewrites as

$$h'_{k+1}(x) = 2h_k(x)h_{k+1}(x) - h_k^2(x) \quad (9)$$

with the initial conditions $h_k(0) = 1$.

The above system of ODEs (9) is the quasilinearized Riccati equation $h'(x) = h^2(x)$ over $[0, 1]$, with the initial value $h(0) = 1$. Its solution is $h(x) = \frac{1}{1-x}$. Here

$$h_k(x) \rightarrow h(x) = \frac{1}{1-x} \quad \text{and} \quad n_k = \frac{\|1 - h_{k+1}/h\|_{L^2[0,1]}^2}{\|1 - h_k/h\|_{L^2[0,1]}^2}.$$

Observe that $h_2(x) = (1 + e^{2x})/2$, but for $k \geq 3$ finding a closed form expression becomes increasingly hard. Given $h_k(x)$, Eq. (9) is a linear first-order ODE in $h_{k+1}(x)$; its solution is given by $h_{k+1}(x) = \mathcal{H}h_k(x)$ with

$$\mathcal{H}f(x) = \left[1 - \int_0^x f^2(y) e^{-2 \int_0^y f(s) ds} dy \right] \cdot e^{2 \int_0^x f(s) ds}. \quad (10)$$

Hence, the problem we are dealing with concerns the asymptotic behavior of an iterated non-linear functional.

Using the setting of (9), we give an ODE proof to Proposition 1(d). To do so, we first need to prove the following lemma.

Lemma 4.

$$\|1 - h_{k+1}/h\|_{L^2[0,1]} = \|h_{k+1}/h - h_k/h\|_{L^2[0,1]}$$

Proof. Observe that

$$h'_{k+1}(x) + (h_{k+1}(x) - h_k(x))^2 = h_{k+1}^2(x).$$

We now use integration by parts to obtain

$$\begin{aligned} \int_0^1 \frac{(h_{k+1}(x) - h_k(x))^2}{h^2(x)} dx &= \int_0^1 \frac{h_{k+1}^2(x)}{h^2(x)} dx - \int_0^1 \frac{h'_{k+1}(x)}{h^2(x)} dx \\ &= \int_0^1 \frac{h_{k+1}^2(x)}{h^2(x)} dx - \left[\frac{h'_{k+1}(1)}{h^2(1)} - \frac{h'_{k+1}(0)}{h^2(0)} + 2 \int_0^1 \frac{h_{k+1}(x)}{h(x)} dx \right] \\ &= \int_0^1 \frac{h_{k+1}^2(x)}{h^2(x)} dx + 1 - 2 \int_0^1 \frac{h_{k+1}(x)}{h(x)} dx \\ &= \int_0^1 \frac{(1 - h_{k+1}(x))^2}{h^2(x)} dx \end{aligned}$$

since $1/h(x) = 1 - x$. □

Alternative proof of Proposition 1(d). Notice that $h \geq \dots \geq h_{k+1} \geq h_k \geq \dots \geq h_0 \equiv 0$, which follows from $g_1(t) \geq g_2(t) \geq g_3(t) \geq \dots$. The Lemma 4 implies

$$\begin{aligned} \|1 - h_{k+1}/h\|_{L^2[0,1]}^2 &= \|h_{k+1}/h - h_k/h\|_{L^2[0,1]}^2 = \int_0^1 [(1 - h_k/h) - (1 - h_{k+1}/h)]^2 dx \\ &= \|1 - h_{k+1}/h\|_{L^2[0,1]}^2 + \|1 - h_k/h\|_{L^2[0,1]}^2 - 2 \int_0^1 (1 - h_k/h)(1 - h_{k+1}/h) dx \end{aligned}$$

and therefore

$$\begin{aligned} \|1 - h_k/h\|_{L^2[0,1]}^2 &= 2 \int_0^1 (1 - h_k/h)(1 - h_{k+1}/h) dx \\ &= 2 \|1 - h_{k+1}/h\|_{L^2[0,1]}^2 + 2 \int_0^1 (h_{k+1}/h - h_k/h)(1 - h_{k+1}/h) dx \geq 2 \|1 - h_{k+1}/h\|_{L^2[0,1]}^2 \end{aligned}$$

yielding $2 \leq \frac{\|1 - h_k/h\|_{L^2[0,1]}^2}{\|1 - h_{k+1}/h\|_{L^2[0,1]}^2} = n_k$ as in Proposition 1(d). □

It is also true that one can improve Proposition 1(d) to make it a strict inequality since one can check that

$$h(x) > \cdots > h_{k+1}(x) > h_k(x) > \cdots > h_0(x) \equiv 0 \quad \text{for } x \in (0, 1).$$

6.2 Proof of the existence of the root-Horton limit

Here we present the proof of our main Theorem 1. It is based on the following two lemmas, Lemma 5 and Lemma 6, that will be proven in the following two subsections.

Lemma 5. *If the limit $\lim_{k \rightarrow \infty} \frac{h_{k+1}(1)}{h_k(1)}$ exists, then $\lim_{k \rightarrow \infty} (\mathcal{N}_k)^{-\frac{1}{k}} = \lim_{k \rightarrow \infty} \left(\prod_{j=1}^k n_j \right)^{-\frac{1}{k}}$ also exists, and*

$$\lim_{k \rightarrow \infty} (\mathcal{N}_k)^{-\frac{1}{k}} = \lim_{k \rightarrow +\infty} \left(\frac{1}{h_k(1)} \right)^{-\frac{1}{k}} = \lim_{k \rightarrow \infty} \frac{h_{k+1}(1)}{h_k(1)}.$$

Lemma 6. *The limit $\lim_{k \rightarrow \infty} \frac{h_{k+1}(1)}{h_k(1)} \geq 1$ exists, and is finite.*

Theorem 1. *The limit $\lim_{k \rightarrow \infty} (\mathcal{N}_k)^{-\frac{1}{k}} = \lim_{k \rightarrow \infty} \left(\prod_{j=1}^k n_j \right)^{-\frac{1}{k}} = R$ exists. Moreover, $R = \lim_{k \rightarrow \infty} \frac{h_{k+1}(1)}{h_k(1)}$, and $2 \leq R \leq 4$.*

Proof. The existence and finiteness of $\lim_{k \rightarrow \infty} \frac{h_{k+1}(1)}{h_k(1)}$ established in Lemma 6 is the precondition for Lemma 5 that in turn implies the existence and finiteness of the limit $\lim_{k \rightarrow \infty} (\mathcal{N}_k)^{-\frac{1}{k}}$ as needed for the root-Horton law. Finally, $2 \leq R \leq 4$ follows from Proposition 1. \square

6.3 Proof of Lemma 5 and related results

Proposition 2.

$$\|1 - h_{k+1}(x)/h(x)\|_{L^2[0,1]}^2 \leq \frac{1}{h_{k+1}(1)} \leq \|1 - h_k(x)/h(x)\|_{L^2[0,1]}^2.$$

Proof. Integrating from 0 to 1 both sides of the equation

$$\frac{\frac{d}{dx} h_{k+1}(x)}{h_{k+1}^2(x)} = 1 - \frac{(h_{k+1}(x) - h_k(x))^2}{h_{k+1}^2(x)}$$

we obtain $\frac{1}{h_{k+1}(1)} = \int_0^1 \frac{(h_{k+1}(x) - h_k(x))^2}{h_{k+1}^2(x)} dx$ as $h_{k+1}(0) = 1$.

Hence,

$$\frac{1}{h_{k+1}(1)} = \int_0^1 \frac{(h_{k+1}(x) - h_k(x))^2}{h_{k+1}^2(x)} dx \geq \int_0^1 \frac{(h_{k+1}(x) - h_k(x))^2}{h^2(x)} dx = \int_0^1 \left(1 - \frac{h_{k+1}(x)}{h(x)}\right)^2 dx$$

by Proposition 4, proving the first inequality.

Now,

$$\frac{1}{h_{k+1}(1)} = \|1 - h_k(x)/h_{k+1}(x)\|_{L^2[0,1]}^2 \leq \|1 - h_k(x)/h(x)\|_{L^2[0,1]}^2$$

thus completing the proof. \square

Proof of Lemma 5. If the limit $\lim_{k \rightarrow \infty} \frac{h_{k+1}(1)}{h_k(1)}$ exists and is finite, then $\lim_{k \rightarrow \infty} \left(\frac{1}{h_k(1)}\right)^{-\frac{1}{k}}$ must also exist and be finite. Hence the existence and finiteness of

$$\lim_{k \rightarrow \infty} (\mathcal{N}_k)^{-\frac{1}{k}} = \lim_{k \rightarrow \infty} \left(\int_0^1 \left(1 - \frac{h_k(x)}{h(x)}\right)^2 dx \right)^{-\frac{1}{k}}$$

follows from Proposition 2. \square

6.4 Proof of Lemma 6 and related results

In this subsection we use the approach developed by Drmota [8] to prove the existence and finiteness of $\lim_{k \rightarrow \infty} \frac{h_{k+1}(1)}{h_k(1)} \geq 1$. As we observed earlier this result is needed to prove the

existence, finiteness, and positivity of $\lim_{k \rightarrow \infty} (\mathcal{N}_k)^{-\frac{1}{k}} = \lim_{k \rightarrow \infty} \left(\prod_{j=1}^k n_j \right)^{-\frac{1}{k}}$, the root-Horton law.

Definition 3. Given $\gamma \in (0, 1]$. Let

$$V_{k,\gamma}(x) = \begin{cases} \frac{1}{1-x} & \text{for } 0 \leq x \leq 1 - \gamma, \\ \gamma^{-1} h_k\left(\frac{x-(1-\gamma)}{\gamma}\right) & \text{for } 1 - \gamma \leq x \leq 1. \end{cases}$$

Note that sequences of functions $h_k(x)$ and $V_{k,\gamma}(x)$ can be extended beyond $x = 1$. Here are some observations we make about the above defined functions.

Observation 1. $V_{k,\gamma}(x)$ are positive continuous functions satisfying

$$V'_{k+1,\gamma}(x) = 2V_{k+1,\gamma}(x)V_{k,\gamma}(x) - V_{k,\gamma}^2(x)$$

for all $x \in [0, 1] \setminus (1 - \gamma)$, with initial conditions $V_{k,\gamma}(0) = 1$.

Observation 2. Let $\gamma_k = \frac{h_k(1)}{h_{k+1}(1)}$. Then

$$V_{k,\gamma_k}(1) = h_{k+1}(1) \quad (11)$$

and

$$V_{k,\gamma}(1) = \gamma^{-1}h_k(1) \geq h_{k+1}(1) \quad \text{whenever } \gamma \leq \gamma_k. \quad (12)$$

Observation 3.

$$V_{k,\gamma}(x) \leq V_{k+1,\gamma}(x)$$

for all $x \in [0, 1]$ since $h_k(x) \leq h_{k+1}(x)$.

Observation 4. Since $h_1(x) \equiv 1$ and $\gamma_1 = \frac{h_1(1)}{h_2(1)}$,

$$h_2(x) \leq V_{1,\gamma_1}(x) = \begin{cases} \frac{1}{1-x} & \text{for } 0 \leq x \leq 1 - \gamma_1, \\ \gamma_1^{-1} = h_2(1) & \text{for } 1 - \gamma_1 \leq x \leq 1. \end{cases}$$

The above observation generalizes as follows.

Proposition 3.

$$h_{k+1}(x) \leq V_{k,\gamma_k}(x) = \begin{cases} \frac{1}{1-x} & \text{for } 0 \leq x \leq 1 - \gamma_k, \\ \gamma_k^{-1}h_k\left(\frac{x-(1-\gamma_k)}{\gamma_k}\right) & \text{for } 1 - \gamma_k \leq x \leq 1. \end{cases}$$

In order to prove Proposition 3 we will need the following lemma.

Lemma 7. For any $\gamma \in (0, 1)$ and $k \geq 1$, function $V_{k,\gamma}(x) - h_{k+1}(x)$ changes its sign at most once as x increases from $1 - \gamma$ to 1. Moreover, since $V_{k,\gamma}(1 - \gamma) = h(1 - \gamma) > h_{k+1}(1 - \gamma)$, function $V_{k,\gamma}(x) - h_{k+1}(x)$ can only change sign from nonnegative to negative.

Proof. This is a proof by induction with base at $k = 1$. Here $V_{1,\gamma}(x) = \frac{1}{\gamma}$ is constant on $[1 - \gamma, 1]$, while $h_2(x) = (1 + e^{2x})/2$ is an increasing function, and

$$V_{1,\gamma}(1 - \gamma) = h(1 - \gamma) > h_2(1 - \gamma)$$

For the induction step, we need to show that if $V_{k,\gamma}(x) - h_{k+1}(x)$ changes its sign at most once, then so does $V_{k+1,\gamma}(x) - h_{k+2}(x)$. Since both sequences of functions satisfy the same ODE relation (see Observation 1), we have

$$\begin{aligned} & \frac{d}{dx} \left[(V_{k+1,\gamma}(x) - h_{k+2}(x)) \cdot e^{-2 \int_{1-\gamma}^x h_{k+1}(y) dy} \right] \\ &= (2V_{k+1,\gamma}(x) - V_{k,\gamma}(x) - h_{k+1}(x)) \cdot (V_{k,\gamma}(x) - h_{k+1}(x)) \cdot e^{-2 \int_{1-\gamma}^x h_{k+1}(y) dy}, \end{aligned}$$

where $h_{k+1}(x) \leq V_{k+1,\gamma}(x)$ by definition of $V_{k+1,\gamma}(x)$, and $V_{k,\gamma}(x) \leq V_{k+1,\gamma}(x)$ as in Observation 3.

Now, let

$$I(x) := \int_{1-\gamma}^x (2V_{k+1,\gamma}(s) - V_{k,\gamma}(s) - h_{k+1}(s)) \cdot (V_{k,\gamma}(s) - h_{k+1}(s)) \cdot e^{-2 \int_{1-\gamma}^s h_{k+1}(y) dy} ds.$$

Then

$$(V_{k+1,\gamma}(x) - h_{k+2}(x)) \cdot e^{-2 \int_{1-\gamma}^x h_{k+1}(y) dy} = V_{k+1,\gamma}(1-\gamma) - h_{k+2}(1-\gamma) + I(x).$$

The function $2V_{k+1,\gamma}(x) - V_{k,\gamma}(x) - h_{k+1}(x) \geq 0$, and since $V_{k,\gamma}(x) - h_{k+1}(x)$ changes its sign at most once, then $I(x)$ should change its sign from nonnegative to negative at most once as x increases from $1-\gamma$ to 1. Hence

$$V_{k+1,\gamma}(x) - h_{k+2}(x) = (V_{k+1,\gamma}(1-\gamma) - h_{k+2}(1-\gamma) + I(x)) \cdot e^{2 \int_{1-\gamma}^x h_{k+1}(y) dy}$$

should change its sign from nonnegative to negative at most once as

$$V_{k+1,\gamma}(1-\gamma) = h(1-\gamma) > h_{k+2}(1-\gamma).$$

□

Proof of Proposition 3. Take $\gamma = \gamma_k$ in Lemma 7. Then function $h_{k+1}(x) - V_{k,\gamma_k}(x)$ should change its sign from nonnegative to negative at most once within the interval $[1-\gamma_k, 1]$. Hence, $V_{k,\gamma_k}(1-\gamma_k) > h_{k+1}(1-\gamma_k)$ and $h_{k+1}(1) = V_{k,\gamma_k}(1)$ imply $h_{k+1}(x) \leq V_{k,\gamma_k}(x)$ as in the statement of the proposition. □

Now we are ready to prove the monotonicity result.

Lemma 8.

$$\gamma_k \leq \gamma_{k+1} \quad \text{for all } k \in \mathbb{N}.$$

Proof. We prove it by contradiction. Suppose $\gamma_k \geq \gamma_{k+1}$ for some $k \in \mathbb{N}$. Then

$$V_{k,\gamma_k}(x) \leq V_{k,\gamma_{k+1}}(x) = \begin{cases} \frac{1}{1-x} & \text{for } 0 \leq x \leq 1-\gamma_{k+1}, \\ \gamma_{k+1}^{-1} h_k \left(\frac{x-(1-\gamma_{k+1})}{\gamma_{k+1}} \right) & \text{for } 1-\gamma_{k+1} \leq x \leq 1 \end{cases}$$

and therefore

$$h_{k+1}(x) \leq V_{k,\gamma_k}(x) \leq V_{k,\gamma_{k+1}}(x) \leq V_{k+1,\gamma_{k+1}}(x)$$

as $h_{k+1}(x) \leq V_{k,\gamma_k}(x)$ by Proposition 3.

Recall that for $x \in [1-\gamma_{k+1}, 1]$,

$$V'_{k+1,\gamma_{k+1}}(x) = 2V_{k,\gamma_{k+1}}(x)V_{k+1,\gamma_{k+1}}(x) - V_{k,\gamma_{k+1}}^2,$$

where at $1 - \gamma_{k+1}$ we consider only the right-hand derivative. Thus for $x \in [1 - \gamma_{k+1}, 1]$,

$$\frac{d}{dx} \left(V_{k+1, \gamma_{k+1}}(x) - h_{k+2}(x) \right) = A(x) + B(x) \left(V_{k+1, \gamma_{k+1}}(x) - h_{k+2}(x) \right),$$

where $A(x) = 2V_{k+1, \gamma_{k+1}}(x) - V_{k, \gamma_{k+1}}(x) - h_{k+1}(x) \geq 0$, $B(x) = 2h_{k+1}(x) > 0$, and $V_{k+1, \gamma_{k+1}}(1 - \gamma_{k+1}) - h_{k+2}(1 - \gamma_{k+1}) = h(1 - \gamma_{k+1}) - h_{k+2}(1 - \gamma_{k+1}) > 0$. Hence

$$V_{k+1, \gamma_{k+1}}(1) - h_{k+2}(1) \geq V_{k+1, \gamma_{k+1}}(1 - \gamma_{k+1}) - h_{k+2}(1 - \gamma_{k+1}) > 0$$

arriving to a contradiction since $V_{k+1, \gamma_{k+1}}(1) = h_{k+2}(1)$. \square

Corollary. *Limit $\lim_{k \rightarrow \infty} \gamma_k$ exists.*

Proof. Lemma 8 implies γ_k is a monotone increasing sequence, bounded by 1. \square

Proof of Lemma 6. Lemma 6 follows immediately from an observation that $\frac{h_{k+1}(1)}{h_k(1)} = \frac{1}{\gamma_k}$. \square

7 Relation to the tree representation of white noise

This section establishes a close connection between the combinatorial tree of Kingman's N -coalescent and the combinatorial level-set tree of a discrete white noise.

7.1 Level set tree of a time series

We start with recalling basic facts about tree representation of time series; for details and further results see [21]. Consider a finite time series X_k with discrete time index $k = 0, 1, \dots, N$ and values distributed without singularities over \mathbb{R} . Let X_t (also denoted $X(t)$) be the time series with continuous time $t \in [0, N]$ obtained from X_k by linear interpolation of its values. The level set $\mathcal{L}_\alpha(X_t)$ is defined as the pre-image of the function values above α :

$$\mathcal{L}_\alpha(X_t) = \{t : X_t \geq \alpha\}.$$

The level set \mathcal{L}_α for each α is a union of non-overlapping intervals; we write $|\mathcal{L}_\alpha|$ for their number. Notice that $|\mathcal{L}_\alpha| = |\mathcal{L}_\beta|$ as soon as the interval $[\alpha, \beta]$ does not contain a value of local maxima or minima of X_t and $0 \leq |\mathcal{L}_\alpha| \leq n$, where n is the number of the local maxima of X_t .

The *level set tree* $\text{LEVEL}(X_t)$ is a planar time-oriented tree that describes the topology of the level sets \mathcal{L}_α as a function of threshold α , as illustrated in Fig. 2. Namely, there are bijections between (i) the leaves of $\text{LEVEL}(X_t)$ and the local maxima of X_t , (ii) the internal (parental) vertices of $\text{LEVEL}(X_t)$ and the local minima of X_t (excluding possible local minima at the boundary points), and (iii) the pair of subtrees of $\text{LEVEL}(X_t)$ rooted at a local minima $X(t_i)$ and the first positive excursions (or meanders bounded by $t = 0$ or

$t = N$) of $X(t) - X(t_i)$ to right and left of t_i . Each vertex in the tree is assigned a mark equal to the value of the local extrema according to the bijections (i) and (ii) above. This makes the tree time-oriented according to the threshold α . It is readily seen that any function X_t with distinct values of consecutive local minima corresponds to a binary tree $\text{LEVEL}(X_t)$. Please see [21] for a discussion of some subtleties related to this construction as well as for further references.

7.2 Finite case

Let $W_k^{(N)}$, $k = 1, \dots, N-1$, be a *discrete white noise* that is a discrete time series comprised of $N-1$ i.i.d. random variables with a common continuous distribution. Consider now an auxiliary time series $\tilde{W}_i^{(N)}$, $i = 1, \dots, 2N-1$ such that it has exactly N local maxima and $N-1$ internal local minima $\tilde{W}_{2k}^{(N)} = W_k^{(N)}$, $k = 1, \dots, N-1$. We call $\tilde{W}_i^{(N)}$ an *extended white noise*; it can be constructed, for example, as follows:

$$\tilde{W}_i^{(N)} = \begin{cases} W_{i/2}^{(N)}, & \text{for even } i, \\ \max\left(W_{\max(1, \frac{i-1}{2})}^{(N)}, W_{\min(N-1, \frac{i+1}{2})}^{(N)}\right) + 1, & \text{for odd } i. \end{cases} \quad (13)$$

Let $L_W^{(N)} = \text{LEVEL}\left(\tilde{W}_i^{(N)}\right)$ be the level-set tree of $\tilde{W}_i^{(N)}$ and $\text{SHAPE}\left(L_W^{(N)}\right)$ be a (random) combinatorial tree that retains the graph-theoretic structure of $L_W^{(N)}$ and drops its planar embedding as well as the vertex marks. By construction, $L_W^{(N)}$ has exactly N leaves.

Lemma 9. *The distribution of $\text{SHAPE}\left(L_W^{(N)}\right)$ on \mathcal{T}_N is the same for any continuous distribution F of the values of the associated white noise $W_k^{(N)}$.*

Proof. The continuity of F is necessary to ensure that the level set tree is binary with probability 1. By construction, the combinatorial level set tree is completely determined by the ordering of the local minima of the respective time series, independently of the particular values of its local maxima and minima. We complete the proof by noticing that the ordering of $W_k^{(N)}$ is the same for any choice of continuous distribution F . \square

Let $T_K^{(N)}$ be the tree that corresponds to a Kingman's N -coalescent with a constant kernel, and let $\text{SHAPE}\left(T_K^{(N)}\right)$ be its combinatorial version that drops the time marks of the vertices. Both the trees $\text{SHAPE}\left(L_W^{(N)}\right)$ and $\text{SHAPE}\left(T_K^{(N)}\right)$, belong to the space \mathcal{T}_N of binary rooted trees with N leaves.

Theorem 2. *The trees $\text{SHAPE}\left(L_W^{(N)}\right)$ and $\text{SHAPE}\left(T_K^{(N)}\right)$ have the same distribution on \mathcal{T}_N .*

The proof below uses the duality between coalescence and fragmentation processes [1]. Recall that a *fragmentation process* starts with a single cluster of mass N at time $t = 0$. Each existing cluster of mass m splits into two clusters of masses $m-x$ and x at the splitting rate $S_t(m, x)$, $1 < m \leq N$, $1 \leq x < N$. A coalescence process on N particles with time-dependent

collision kernel $K_t(x, y)$, $1 \leq x, y < N$ is equivalent, upon time reversal, to a discrete-mass fragmentation process of initial mass N with some splitting kernel $S_t(m, x)$. See Aldous [1] for further details and the relationship between the dual collision and splitting kernels in general case.

Proof of Theorem 2. We show that both the examined trees have the same distribution as the combinatorial tree of a fragmentation process with mass N and a splitting kernel that is uniform in mass: $S_t(m, x) = S(t)$.

Kingman's N -coalescence with kernel $K(x, y) = 1$ is dual to the fragmentation process with splitting kernel [1, Table 3]

$$S_t(m, x) = \frac{2}{t(t+2)}.$$

This kernel is independent of the cluster mass, which means that the splitting of mass m is uniform among the $m - 1$ possible pairs $\{1, m - 1\}, \{2, m - 2\}, \dots, \{m - 1, 1\}$. The time dependence of the kernel does not affect the combinatorial structure of the fragmentation tree (and can be removed by a deterministic time change.)

The level-set tree $L_W^{(N)}$ can be viewed as a tree that describes a fragmentation process with the initial mass N equal to the number of local maxima of the time series $\tilde{W}_k^{(N)}$. By construction, each subtree of $L_W^{(N)}$ with n leaves corresponds to an excursion (or meander, if we treat one of the boundaries) with n local maxima. This subtree (as well as the corresponding excursion or meander) splits into two by the internal global minimum of $\tilde{W}_k^{(N)}$ at the corresponding time interval.

The global minimum splits the series $\tilde{W}_k^{(N)}$ into two, to the left and right of the minimum, with M_L and $(N - M_L)$ local maxima, respectively. Since the local minima of $\tilde{W}_k^{(N)}$ form a white noise, the distribution of M_L is uniform on $[1, N - 1]$. Next, the internal vertices of the level set tree of the left (or right) time series correspond to its $M_L - 1$ (or $N - M_L - 1$) internal local minima that form a white noise (with the distribution different from that of the initial white noise $W_k^{(N)}$). Hence, the subsequent splits of masses (number of local maxima) continues according to a discrete uniform distribution. And so on down the tree.

Hence, the combinatorial level set tree of $\tilde{W}_k^{(N)}$ has the same distribution as a combinatorial tree of a fragmentation process with uniform mass splitting. This completes the proof. \square

Remark 1. We notice that the dual splitting kernels for multiplicative and additive coalescences [1, Table 3] only differ by their time dependence, and are equivalent as functions of mass. Hence, the combinatorial structure of the respective trees is the same.

7.3 Rooted trees with a selected leaf

To construct an infinite tree that represents Kingman's coalescent viewed from a leaf, we need to introduce some notations. Consider a space Γ of (non-embedded, unlabeled) *rooted binary trees with a selected leaf* γ . For any tree $T \in \Gamma$, let $\gamma = \gamma_T$ be the selected leaf and

$\rho_T = (\gamma_T, \phi_T)$ denote the ancestral path from the selected leaf γ_T to its parent, grandparent, great-grandparent and on towards the tree root ϕ_T , even if ϕ_T is a point at infinity.

The path ρ_T consists of $n_T \leq \infty$ edges and $n_T + 1$ vertices that we index by $i \geq 0$ along the path from the leaf $\gamma_T = \rho_T(0)$ to the root $\phi_T = \rho_T(n_T)$. Each tree $T \in \Gamma$ can be represented as a *forest attached to the line* ρ :

$$T = \{T_i\}_{i \geq 1}, \quad (14)$$

where $T_i \in \mathcal{T}$ for $1 \leq i < n_T$ denotes a subtree of T with the root at $\rho(i)$ and we put $T_i = \emptyset$ for all $i \geq n_T$.

A metric on Γ is defined as

$$\mu(A, B) = \frac{1}{1 + \sup \{n : A_k|n = B_k|n \quad \forall k \leq n\}} \quad (15)$$

for any $A = \{A_i\} \in \Gamma$ and $B = \{B_i\} \in \Gamma$ represented as in (14). Here $T|n \in \bigcup_{i=1}^{2^{n-1}} \mathcal{T}_i$ denotes the restriction of $T \in \mathcal{T}$ to the vertices at the depth less than n from the root.

To show that $\mu(A, B)$ is indeed a metric on Γ , take trees A, B , and C in Γ such that

$$\kappa = \sup \{n : A_k|n = B_k|n \quad \forall k \leq n\} \leq \sup \{n : B_k|n = C_k|n \quad \forall k \leq n\}.$$

Then

$$A_k|\kappa = B_k|\kappa = C_k|\kappa \quad \forall k \leq \kappa,$$

and therefore $\sup \{n : A_k|n = C_k|n \quad \forall k \leq n\} \geq \kappa$. Hence

$$\mu(A, C) \leq \frac{1}{1 + \kappa} = \mu(A, B) \leq \mu(A, B) + \mu(B, C)$$

and the corresponding equality holds only if $B = C$. Also, $\mu(A, B) = 0$ if and only if $\sup \{n : A_k|n = B_k|n \quad \forall k \leq n\} = 0$, i.e. trees A and B are identical.

So μ is a metric on Γ that compares trees in the neighborhood of the selected leaf γ . We denote by Γ_N a subspace of Γ that contains all trees with $N \geq 1$ leaves.

Lemma 10. (Γ, μ) is a compact Polish space.

Proof. The countable dense subset $\Gamma_{<\infty} := \bigcup_{N \geq 1} \Gamma_N$ of finite rooted trees with a selected leaf

makes Γ a Polish space. The compactness is readily shown by selecting a subsequence T^{n_i} of an infinite sequence $T^n = \{T_i^n\}_{i \geq 1}$ of trees such that $\mu(T^{n_i}, A) < i^{-1}$ for some infinite $A \in \Gamma$, which is always possible since for any $i \in \mathbb{N}$ the number of binary trees of depth $\leq i$ is finite. Thus there is a collection of i binary trees S_1, \dots, S_i , each of depth $\leq i$, such that

$$T_k^n|i = S_k \quad \forall k \in \{1, 2, \dots, i\}$$

holds for infinitely many T^n of which we select one T^{n_i} , and use the rest for selecting $T^{n_{i+1}}, T^{n_{i+2}}, \dots$. The resulting limiting tree A of T^{n_i} will satisfy

$$A_k|i = S_k \quad \forall k \in \{1, 2, \dots, i\}.$$

□

7.4 Kingman's coalescent with a selected particle

Recall that Kingman's coalescent is constructed in [11] as a continuous time Markov process over the set $\mathcal{P}_{\mathbb{N}}$ of all partitions of $\mathbb{N} = \{1, 2, \dots\}$. Here we will use the following special property of Kingman's coalescent: it can be restricted consistently to the set $\mathcal{P}_{[N]}$ of all partitions of $\{1, 2, \dots, n\}$. These restrictions correspond to Kingman's N -coalescent processes $\Pi_t^{(N)}$ (e.g., [11, 15]). In the above construction let 1 be the *selected particle*, and the block containing 1 at time t be the *selected block*.

When constructing a combinatorial tree for Kingman's N -coalescent process, each merger history of $\Pi_t^{(N)}$ corresponds to a tree T from Γ_N with the selected leaf $\gamma = \gamma_T$ corresponding to the selected particle 1 and ρ_T corresponding to the merger history of the selected block. Hence the process induces a distribution on Γ_N , which we denote \mathcal{Q}_N^* . The sequence of measures $(\mathcal{Q}_N^*)_{N \in \mathbb{N}}$ is *tight* on Γ due to the space compactness (Lemma 10). The Prokhorov's theorem implies that this sequence is then *relatively compact*, which means that it includes converging subsequences. By construction, each limit measure \mathcal{Q}^* is concentrated on infinite trees $\Gamma_{\infty} = \Gamma \setminus \Gamma_{<\infty}$. The uniqueness of the limit follows from the fact that all finite-dimensional projections of \mathcal{Q}^* are uniquely defined by a consistent set of \mathcal{Q}_N^* . This corresponds to being able to restrict Kingman's coalescent consistently to $\mathcal{P}_{[N]}$. We denote the unique limit measure by \mathcal{Q}^* . It is natural to interpret the space $(\Gamma_{\infty}, \mu, \mathcal{Q}^*)$ as a (random) combinatorial Kingman's infinite tree viewed from a leaf. We emphasize the two features that distinguish our Kingman's tree construction from the others that exist in the literature (e.g., that of Evans [9]): (i) only considering the combinatorial part of a tree and (ii) focusing attention on the vicinity of a selected leaf.

For the extended white noise an infinite tree can be constructed explicitly as a strong limit of finite trees $L_W^{(N)}$, as shown in the next section.

7.5 Infinite case

We describe below two equivalent constructions of an infinite tree for a discrete time series with infinite number of local maxima. The first construction involves limit of finite trees in the space (Γ, μ) ; the second introduces an infinite tree as a random metric space.

7.5.1 Construction 1: Limit of finite trees

An infinite level set tree L_W^{∞} that represents a time series W_k^{∞} with an infinite number of local maxima can be constructed as follows. Fix a trajectory $W(\omega)$ of W_k^{∞} . Choose a local maximum of the $W(\omega)$ closest to $k = 0$ and assign it index $i = 0$. Now index all the local maxima $W_{(i)}$ and the respective time instants $t_{(i)}$ in the order of their appearance to the right or left of this chosen maximum by an integer index i , so that the first local maximum to the right of $i = 0$ is assigned index $i = 1$, the first local maximum to the left $i = -1$, etc. The trajectory on the interval $[t_{(i_1)}, t_{(i_2)}]$ corresponds to a level-set tree $L_W^{[i_1, i_2]}(\omega)$ with $(i_2 - i_1 + 1)$ leaves. Notably, if $[j_1, j_2] \subset [i_1, i_2]$ then $[t_{(j_1)}, t_{(j_2)}] \subset [t_{(i_1)}, t_{(i_2)}]$ and the level set tree $L_W^{[j_1, j_2]}(\omega)$ is a subtree of $L_W^{[i_1, i_2]}(\omega)$.

Consider the trajectory $W(\omega)$ on the interval $[t_{(-i)}, t_{(i)}]$ and take the leaf that corresponds

to $W_{(0)}$ as the *selected leaf* of the tree $L_W^{[-i,i]}(\omega)$, which hence becomes an element of (Γ_{2i+1}, μ) as defined in Sect. 7.3. We define $L_W^\infty(\omega)$ as the limit of $L_W^{[-i,i]}(\omega)$ as $N \rightarrow \infty$ and, in particular,

$$\text{SHAPE}(L_W^\infty(\omega)) := \lim_{i \rightarrow \infty} \text{SHAPE}\left(L_W^{[-i,i]}(\omega)\right).$$

The sequence of increasing trees $\text{SHAPE}\left(L_W^{[-i,i]}(\omega)\right)$, $i > 0$, converges according to measure μ of (15) that is concentrated in the vicinity of the selected leaf, so the limit above is well defined.

7.5.2 Construction 2: Random metric space

The limit tree $L_W^\infty(\omega)$ is in fact the *tree in continuous path* of the function $W(t)$ obtained by linear interpolation of the values W_k on \mathbb{R} . Specifically, recall the following definition [15, Section 7].

Definition 4. A metric space (M, d) is called a *tree* if for each choice of $u, v \in M$ there is a unique continuous path $\sigma_{u,v} : [0, d(u, v)] \rightarrow M$ that travels from u to v at unit speed, and for any simple continuous path $F : [0, T] \rightarrow M$ with $f(0) = u$ and $f(T) = v$ the ranges of f and $\sigma_{u,v}$ coincide.

Let $X(t) \in C(I)$, the space of continuous functions from $I \subset \mathbb{R}$ to \mathbb{R} , and $\underline{X}[a, b] := \inf_{t \in [a, b]} X(t)$, for any $a, b \in I$. We define a *pseudo-metric* on I as

$$d_X(a, b) := (X(a) - \underline{X}[a, b]) + (X(b) - \underline{X}[a, b]), \quad a, b \in I. \quad (16)$$

We write $a \sim_X b$ if $d_X(a, b) = 0$. The points on the interval I with metric d_X form a metric space $(I / \sim_X, d_X)$ [15, Section 7].

Definition 5. We call $(I / \sim_W, d_W)$ the *tree embedded in the continuous path* $W(t)$ on the interval I and denote it by $\text{TREE}_W(I)$. If $W_{(0)} \in I$ and is attained at $t_{(0)}$ then this point is taken to be the selected leaf of $\text{TREE}_W(I)$.

It is readily seen that we have the following equivalence for any $i > 0$:

$$L_W^{[-i,i]}(\omega) = \text{TREE}_W\left([t_{(-i)}(\omega), t_{(i)}(\omega)]\right)$$

and $L_W^\infty(\omega) = \text{TREE}_W(\mathbb{R}, \omega)$, where the second argument emphasizes that this tree corresponds to a particular trajectory $W(\omega)$.

A probability measure is induced on the space of the trees in continuous path with a selected leaf by the probability measure on the space of the time series trajectories. In particular, there exists a probability measure $\mathcal{Q}_{\text{WN}}^*$ on (Γ, μ) that corresponds to a (random) combinatorial tree of an infinite extended white noise viewed from a leaf.

Corollary. $\mathcal{Q}_{\text{WN}}^*(A) = \mathcal{Q}^*(A)$ for any $A \in (\Gamma, \mu)$.

Proof. The statement follows immediately from the equivalence of the finite-dimensional distributions generated by an extended white noise and Kingman's N -coalescent processes on (Γ_N, μ) , as in Theorem 2. \square

Now we can extend the definition of branch statistics relevant to Horton analyzes to the infinite trees of Kingman's coalescent and discrete white noise. Let $B_{k,i}$, $k, i \geq 1$ denotes the number of branches of Horton-Strahler order k in the tree that corresponds to the interval of an infinite extended white noise between $W_{(-i)}$ and $W_{(i)}$. By Theorem 2, it has the same distribution as the number of branches of order k in a combinatorial tree of Kingman's N -coalescent process with a constant kernel and $2i + 1$ particles. Lemma 3 implies that the following limit exists (in probability) and is finite

$$\lim_{i \rightarrow \infty} \frac{B_{k,i}}{(2i + 1)} = \mathcal{N}_k.$$

This suggests an intuitive interpretation of the asymptotic ratios \mathcal{N}_k as the *Horton indices* for an infinite tree of Kingman's coalescent or extended white noise and proves the following result.

Theorem 3. *The random combinatorial infinite trees $(\Gamma_\infty, \mu, \mathcal{Q}^*)$ and $(\Gamma_\infty, \mu, \mathcal{Q}_{\text{WN}}^*)$ are root-Horton self-similar.*

Remark 2. The extended white noise was introduced to show the equivalence of the finite dimensional distributions in Theorem 2. At the same time, the statement of Theorem 3 (Horton self-similarity) applies as well to the infinite combinatorial tree of a discrete white noise with a continuous distribution of the values. This is easily seen if one recalls the operation of tree *pruning* $\mathcal{R}(T) : \mathcal{T} \rightarrow \mathcal{T}$ that cuts the tree leaves and removes possible resulting nodes of degree 2 [4, 21]. By definition, pruning corresponds to index shift in Horton statistics: $N_k \rightarrow N_{k-1}$, $k > 1$. It has been shown in [21] that

$$\mathcal{R} \left[\text{LEVEL} \left(\tilde{W}_k^{(N)} \right) \right] = \text{LEVEL} \left(W_k^{(N)} \right).$$

Hence, Horton self-similarity for one of these time series implies that for the other.

8 Numerical results

This section illustrates the theoretical results of the paper and provides further insight into self-similarity of Kingman's coalescent in a series of numerical experiments.

8.1 Horton self-similarity

Figure 3 shows (by shaded circles) the asymptotic proportion \mathcal{N}_k according to the equation

$$\mathcal{N}_k = \int_0^1 \left(1 - \frac{h_{k-1}(x)}{h(x)} \right)^2 dx. \quad (17)$$

The integral is evaluated using the numeric solution $\hat{h}_k(x)$ to the system (9). The computations are done in Matlab, using the nominal absolute tolerance of $\epsilon = 10^{-16}$ on an irregular time grid with steps decreasing towards 1 and being as small as $\Delta = 10^{-10}$. The values of the

estimated \mathcal{N}_k as well as the consecutive ratios $\mathcal{N}_k/\mathcal{N}_{k+1}$ are reported in Table 1 with 7 digit precision. We notice that the values in the second column differ from the ratios of the values in the first column, because of the round-off effects. The values of \mathcal{N}_k are exponentially decreasing in such a way that the ratio $\mathcal{N}_k/\mathcal{N}_{k+1}$ quickly converges to $R = 3.043827\dots$. This ratio corresponds to the geometric decay

$$\mathcal{N}_k = (3.043827\dots)^{-(k-1)} = 10^{(-0.4834\dots)(k-1)}.$$

A black line in Fig. 3 illustrates a geometric decay with this rate and an arbitrary offset. The observed convergence of ratios is stronger than the root-convergence proven in our Theorem 1. We conjecture that in fact the strongest possible geometric-Horton law also holds for the sequence \mathcal{N}_k , that is $\lim_{k \rightarrow \infty} (\mathcal{N}_k R^k) = N_0 > 0$ for the same Horton exponent R as above.

Table 1 also reports (column 4) the average number N_k of branches of order k relative to N_1 observed in a level-set tree of an extended white noise with $N = 2^{18} = 262,144$ local maxima, taken over 1000 independent realizations. The typical order of such a tree is $\Omega = 11$. The values are reported with 4 significant digits. The agreement between the asymptotic statistics for Kingman's coalescent and those for a finite extended white noise is very good, in accordance with our equivalence Theorem 2. Column 5 of Table 1 shows the empirical coefficient of variation $\rho(N_k) = \sqrt{\text{Var}(N_k)}/\text{E}(N_k)$ for the random variables N_k . The values of $\rho(N_k)$ rapidly decrease with the order difference $\Omega - k$; this implies that the branch statistics for an individual level set tree are very close to the asymptotic Kingman's values, at least for small orders k . This is illustrated in Fig. 3 that shows the values of N_k/N_1 computed in a single realization of an extended white noise. It should be emphasized that the coefficient of variation $\rho(N_k)$ depends on N and decreases for each k as N increases (not illustrated).

Table 1: Statistics of order- k branches for Kingman's coalescence (columns 2, 3) and a finite extended white noise (columns 4, 5).

| k | \mathcal{N}_k | $\mathcal{N}_k/\mathcal{N}_{k+1}$ | N_k/N_1 | $\rho(N_k)$ |
|-----|-----------------|-----------------------------------|------------|-------------|
| 1 | 1.0000000 | 3.0000000 | 1.0000 | 0.001 |
| 2 | 0.3333333 | 3.0389538 | 0.3333 | 0.002 |
| 3 | 0.1096869 | 3.0432806 | 0.1097 | 0.005 |
| 4 | 0.0360423 | 3.0437674 | 0.0360 | 0.008 |
| 5 | 0.0118413 | 3.0438212 | 0.01183 | 0.014 |
| 6 | 0.0038903 | 3.0438271 | 0.003885 | 0.026 |
| 7 | 0.0012781 | 3.0438277 | 0.001273 | 0.044 |
| 8 | 0.0004200 | 3.0438278 | 0.0004156 | 0.074 |
| 9 | 0.0001380 | 3.0438279 | 0.0001342 | 0.142 |
| 10 | 0.0000453 | 3.0438279 | 0.00004148 | 0.311 |
| 11 | 0.0000149 | - | 0.00001105 | 1.591 |

8.2 Tokunaga self-similarity

The *Tokunaga self-similarity* for trees is based on Tokunaga indexing [12, 19, 14], which extends upon Horton-Strahler orders (see Fig. 1b). This indexing focuses on *side-branching*,

which is the merging between branches of different order. Let N_{ij} , $1 \leq i < j \leq \Omega$, be the total number of branches of order i that join branch of order j in a finite tree T of order Ω . Tokunaga index $\tau_{ij} = N_{ij}/N_j$ is the average number of branches of order $i < j$ per branch of order j . By $\tau_{ij}^{(\mathcal{Q}_N)}$ we denote the (random) index τ_{ij} computed for a tree generated according to a measure \mathcal{Q}_N on \mathcal{T}_N .

Definition 6. We say that a sequence of probability laws $\{\mathcal{Q}_N\}_{N \in \mathbb{N}}$ has *well-defined asymptotic Tokunaga indices* if for each $k \in \mathbb{N}$ random variables $\tau_{ij}^{(\mathcal{Q}_N)}$ converge in probability, as $N \rightarrow \infty$, to a constant value T_{ij} , called the *asymptotic Tokunaga index*.

Definition 7. A sequence of measures $\{\mathcal{Q}_N\}_{N \in \mathbb{N}}$ with well-defined Tokunaga indices is said to be *Tokunaga self-similar* with parameters (a, c) if

$$T_{ij} =: T_k = a c^{k-1} \quad a, c > 0, \quad 1 \leq k \leq \Omega - 1.$$

To write out the equations for Tokunaga indexes T_{ij} in Kingman's coalescent, we observe that the asymptotic ratio of N_{ij} to N is given by the original ODE (2) as

$$\mathcal{N}_{ij} = \int_0^\infty \eta_i(t) \eta_j(t) dt = \int_0^\infty (g_i(x) - g_{i+1}(x)) (g_j(x) - g_{j+1}(x)) dx.$$

This can be rewritten using the rescaled equations in (9) as

$$\mathcal{N}_{i-1, j-1} = 2 \int_0^1 \frac{(h_i(x) - h_{i+1}(x))(h_j(x) - h_{j+1}(x))}{h^2(x)} dx.$$

We now use (17) to obtain

$$\begin{aligned} \lim_{N \rightarrow \infty} T_{i-1, j-1} &= \lim_{N \rightarrow \infty} \frac{N_{i-1, j-1}}{N_{j-1}} = \frac{\mathcal{N}_{i-1, j-1}}{\mathcal{N}_{j-1}} \\ &= 2 \frac{\int_0^1 \frac{(h_i(x) - h_{i+1}(x))(h_j(x) - h_{j+1}(x))}{h^2(x)} dx}{\int_0^1 \left(1 - \frac{h_j(x)}{h(x)}\right)^2 dx}. \end{aligned} \tag{18}$$

Table 2 reports Tokunaga indices evaluated numerically using (18). The integrals and functions $h_k(x)$ are evaluated in Matlab with nominal absolute tolerance $\epsilon = 10^{-20}$ and using the same grid as in computing the Horton statistics. The values are reported here with 4 digits precision. We notice that evaluation of Tokunaga indices for larger ij pairs faces numerical problems because of divergence of $h(x)$, and the associated “bursts” of $h_k(x)$, at unity.

The reported values $T_{i, i+k}$ converge to the limit T_k as i increases. The convergence rate is very fast; the limit value, within the reported precision, is achieved for $i \geq 4$ or faster. The reported simulations suggest that the convergence rate increases with k .

Table 2: Asymptotic Tokunaga indices T_{ij} for Kingman's coalescence

| | $j = 2$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---------|---------|--------|--------|--------|--------|--------|--------|--------|
| $i = 1$ | 0.8196 | 0.5687 | 0.2641 | 0.0993 | 0.0342 | 0.0114 | 0.0038 | 0.0012 |
| 2 | | 0.8234 | 0.5720 | 0.2655 | 0.0999 | 0.0344 | 0.0115 | 0.0038 |
| 3 | | | 0.8232 | 0.5724 | 0.2657 | 0.0999 | 0.0344 | 0.0115 |
| 4 | | | | 0.8231 | 0.5724 | 0.2657 | 0.0999 | 0.0344 |
| 5 | | | | | 0.8231 | 0.5724 | 0.2657 | 0.0999 |
| 6 | | | | | | 0.8231 | 0.5724 | 0.2657 |
| 7 | | | | | | | 0.8231 | 0.5724 |
| 8 | | | | | | | | 0.8231 |

Figure 4 shows by shaded circles the “limit” Tokunaga indices T_{i9} , $i = 1, \dots, 8$, as a function of $k = 9 - i$. The figure may suggest that Tokunaga indices form a geometric series, asymptotic in k . Summing up, we conjecture that

$$\lim_{i \rightarrow \infty} T_{i,i+k} =: T_k \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{T_k}{c^{k-1}} = a$$

with $a \approx 3 \pm 0.02$, $c \approx -0.5 \pm 0.02$ obtained by fitting a geometric series to the last couple of “limit” Tokunaga indices. We notice that the existing literature only considers the conventional Tokunaga self-similarity that assumes the geometric form of T_k starting from $k = 1$.

We also show in Fig. 4, by black squares, Tokunaga indices \hat{T}_k estimated from the level-set trees of a finite extended white noises. The values \hat{T}_k are obtained by averaging the empirical Tokunaga indices in 100 trees of size $N = 2^{17} = 131,072$; the typical order of such trees is $\Omega = 11$. The indices are averaged over different trees and over all pairs of subindices $\{i, i+k\}$ with $i \neq 1$. The empirical values match very closely the asymptotic values from Kingman's process, in accordance with our equivalence Theorem 2.

9 General coalescent processes

The ODE approach introduced in this paper can be extended to the coalescent kernels other than $K(i, j) \equiv 1$. For that we need to classify the relative number $\eta_j(t)$ of clusters of order j at time t according to the cluster masses. Namely, let $\eta_{j,k}(t)$ be the average number of clusters of order j and mass $k \geq 2^j$ at time t . Then

$$\eta_j(t) = \sum_{k=2^j}^{\infty} \eta_{j,k}(t).$$

In the case of a symmetric coalescent kernel $K(i, j) = K(j, i)$ the Smoluchowski-Horton

ODEs can be written asymptotically as

$$\begin{aligned}
\frac{d}{dt}\eta_{j,k}(t) &= \sum_{i=1}^{j-1} \sum_{\kappa=2^i}^{k-2^i} \eta_{j,\kappa}(t) \eta_{i,k-\kappa} K(\kappa, k-\kappa) \\
&+ \frac{1}{2} \sum_{\substack{k_1+k_2=k \\ k_1, k_2 \geq 2^{j-1}}} \eta_{j-1,k_1}(t) \eta_{j-1,k_2}(t) K(k_1, k_2) \\
&- \eta_{j,k}(t) \sum_{\tilde{k}=2^i}^{\infty} K(k, \tilde{k}) \left(\sum_{i=1}^{\infty} \eta_{i,\tilde{k}}(t) \right)
\end{aligned} \tag{19}$$

with the initial conditions $\eta_{1,1}(0) = 1$ and $\eta_{j,k}(0) = 0$ for all $(j, k) \neq (1, 1)$.

Observe that when $K(i, j) \equiv 1$, summing the above equations (19) over index k produces the Smoluchowski-Horton ODE (2) for the average relative number of order- j branches $\eta_j(t)$ in Kingman's coalescent process.

10 Discussion

This paper establishes the root-Horton self-similarity (Theorem 1) and states a numerical conjecture about the asymptotic Tokunaga self-similarity (Sect. 8) for Kingman's coalescent process. We also demonstrate (Theorem 2) the distributional equivalence of the combinatorial trees of Kingman's N -coalescent process with a constant collision kernel to that of a discrete extended white noise with N local maxima, hence extending the self-similarity results to a tree representation of an infinite white noise (Sect. 7.5, Remark 2).

Our Theorem 1 establishes a weak root-law convergence of the asymptotic ratios \mathcal{N}_k , while we believe that the stronger (ratio and geometric) forms of convergence are also valid. These stronger Horton laws are usually considered in the literature (e.g., [14, 10, 7, 21]). For instance, a well-known result is that a tree corresponding to a critical binary Galton-Watson process obey the geometric Horton law with $R = 4$; see [16, 20, 14, 6, 4]. It seems important to show rigorously at least the ratio-Horton law ($\lim_{k \rightarrow \infty} \mathcal{N}_k / \mathcal{N}_{k+1} = R > 0$) because the ability to work with asymptotic ratios is necessary to tackle Tokunaga self-similarity, which provides much stricter constraints on a branching structure.

The Smoluchowski-Horton equations (2) that form a core of the presented method and their equivalents (8) and (9) seem to be promising for further more detailed exploration. Indeed, one may hope that the approach that refers explicitly to the Horton-Strahler orders might effectively complement conventional analysis of cluster masses. The analysis of the Smoluchowski-Horton systems can be done within the ODE framework, similarly to the present study, or within the nonlinear iterative system framework (see (10)). The latter approach is still to be explored.

Finally, it is noteworthy that the analysis of multiplicative and additive coalescents according to the general Smoluchowski-Horton system (19) appears, after a certain series of transformations, to follow many of the steps implemented in this paper for Kingman's coalescent, with the ODE system being replaced by a suitable PDE one. These results will be published elsewhere.

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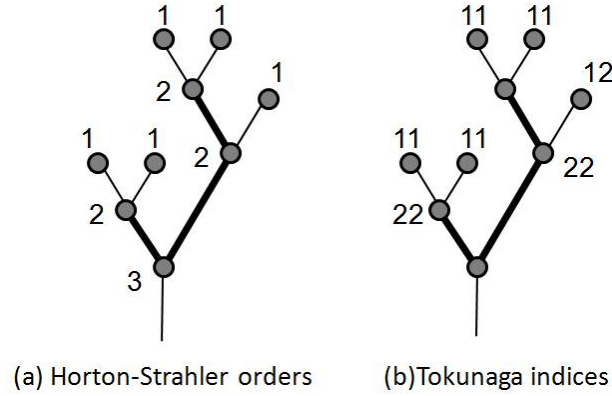


Figure 1: Example of (a) Horton-Strahler ordering, and of (b) Tokunaga indexing. Two order-2 branches are depicted by heavy lines in both panels. Horton-Strahler orders refer, interchangeably, to the tree nodes or to their parent links. Tokunaga indices refer to entire branches, and not to individual vertices.

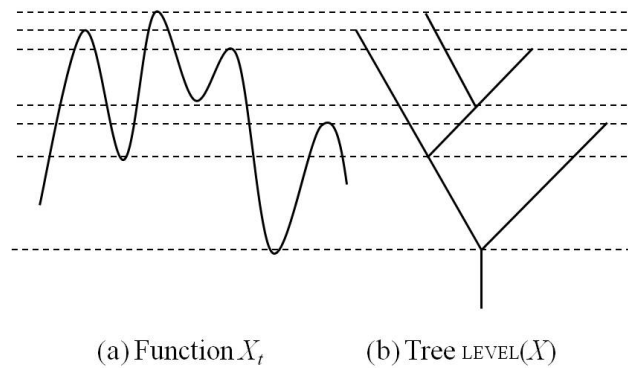


Figure 2: Function X_t (panel a) with a finite number of local extrema and its level-set tree $\text{LEVEL}(X)$ (panel b).

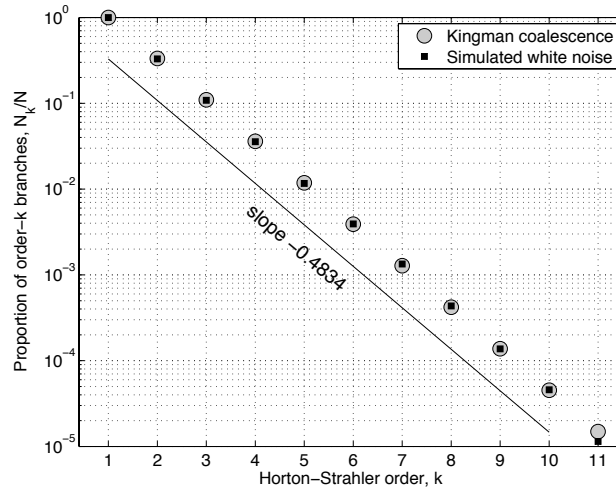


Figure 3: Horton self-similarity. Filled circles: The asymptotic ratios \mathcal{N}_k for Kingman's coalescent. Black squares: The empirical ratios N_k/N_1 in a level-set tree for a single trajectory of an extended white noise with $N = 2^{18}$ local maxima.

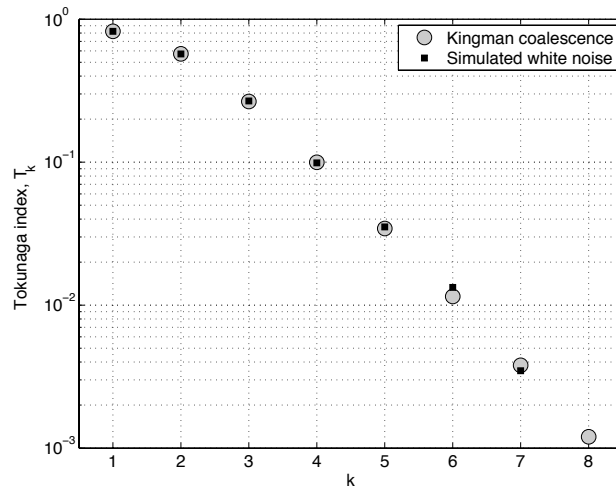


Figure 4: Asymptotic Tokunaga self-similarity. Filled circles: The asymptotic Tokunaga indices T_{i9} in Kingman's coalescent. Black squares: The empirical Tokunaga indices averaged over 100 level-set trees for extended white noises with $N = 2^{17}$ local maxima.